

# Relativistic Recoil Corrections to the Atomic Energy Levels

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**Abstract.** The quantum electrodynamic theory of the nuclear recoil effect in atoms to all orders in  $\alpha Z$  and to first order in  $m/M$  is considered. The complete  $\alpha Z$ -dependence formulas for the relativistic recoil corrections to the atomic energy levels are derived in a simple way. The results of numerical calculations of the recoil effect to all orders in  $\alpha Z$  are presented for hydrogenlike and lithiumlike atoms. These results are compared with analytical results obtained to lowest orders in  $\alpha Z$ . It is shown that even for hydrogen the numerical calculations to all orders in  $\alpha Z$  provide most precise theoretical predictions for the relativistic recoil correction of first order in  $m/M$ .

## 1 Introduction

In the non-relativistic quantum mechanics the nuclear recoil effect for a hydrogenlike atom is easily taken into account by using the reduced mass  $\mu = mM/(m + M)$  instead of the electron mass  $m$  ( $M$  is the nuclear mass). It means that to account for the nuclear recoil effect to first order in  $m/M$  we must simply replace the binding energy  $E$  by  $E(1 - m/M)$ .

Let us consider now a relativistic hydrogenlike atom. In the infinite nucleus mass approximation a hydrogenlike atom is described by the Dirac equation ( $\hbar = c = 1$ )

$$(-i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m + V_C(\mathbf{x}))\psi(\mathbf{x}) = \varepsilon\psi(\mathbf{x}), \quad (1)$$

where  $V_C$  is the Coulomb potential of the nucleus. For the point-nucleus case, analytical solution of this equation yields the well known formula for the energy of a bound state:

$$\varepsilon_{nj} = \frac{mc^2}{\sqrt{1 + \frac{(\alpha Z)^2}{[n - (j+1/2) + \sqrt{(j+1/2)^2 - (\alpha Z)^2}]^2}}}, \quad (2)$$

where  $n$  is the principal quantum number and  $j$  is the total angular momentum of the electron. The main problem we will discuss in this paper is the following: what is the recoil correction to this formula?

It is known that to the lowest order in  $\alpha Z$  the relativistic recoil correction to the energy levels can be derived from the Breit equation. Such a derivation was made by Breit and Brown in 1948 [1] (see also [2]). They found that the relativistic recoil correction to the lowest order in  $\alpha Z$  consists of two terms. The first term

reduces the fine structure splitting by the factor  $(1 - m/M)$ . The second term does not affect the fine structure splitting and is equal to  $-(\alpha Z)^4 m^2 / (8Mn^4)$ . Calculations of the recoil effect to higher orders in  $\alpha Z$  demand using QED beyond the Breit approximation. In quantum electrodynamics a two-body system is generally treated by the Bethe-Salpeter method [3] or by one of versions of the quasipotential method proposed first by Logunov and Tavkhelidze [4]. In Ref. [5] (see also [6]), using the Bethe-Salpeter equation, Salpeter calculated the recoil correction of order  $(\alpha Z)^5 m^2 / M$  to the energy levels of a hydrogenlike atom. This correction gives a contribution of 359 kHz to the  $2s - 2p_{1/2}$  splitting in hydrogen. The current uncertainties of the Lamb and isotopic shift measurements are much smaller than this value [7] and, therefore, calculations of the recoil corrections of higher orders in  $\alpha Z$  are required. In addition, for the last decade a great progress was made in high precision measurements of the Lamb shifts in high-Z few-electron ions [8,9,10]. In these systems, the parameter  $\alpha Z$  is not small and, therefore, calculations of the relativistic recoil corrections to all orders in  $\alpha Z$  are needed.

## 2 Relativistic formula for the recoil correction

First attempts to derive formulas for the relativistic recoil corrections to all orders in  $\alpha Z$  were undertaken in [11,12]. As a result of these attempts, only a part of the desired expressions was found in [12] (see Ref. [13] for details). The complete  $\alpha Z$ -dependence formula for the relativistic recoil effect in the case of a hydrogenlike atom was derived in [14]. The derivation of [14] was based on using a quasipotential equation in which the heavy particle is put on the mass shell [15,16]. According to [14], the relativistic recoil correction to the energy of a state  $a$  is the sum of a lower-order term  $\Delta E_L$  and a higher-order term  $\Delta E_H$ :

$$\Delta E = \Delta E_L + \Delta E_H, \quad (3)$$

$$\Delta E_L = \frac{1}{2M} \langle a | \left( \mathbf{p}^2 - \frac{\alpha Z}{r} \left( \boldsymbol{\alpha} + \frac{(\boldsymbol{\alpha} \cdot \mathbf{r}) \mathbf{r}}{r^2} \right) \cdot \mathbf{p} \right) | a \rangle, \quad (4)$$

$$\begin{aligned} \Delta E_H = & \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \langle a | \left( \mathbf{D}(\omega) - \frac{[\mathbf{p}, V_C]}{\omega + i0} \right) \\ & \times G(\omega + \varepsilon_a) \left( \mathbf{D}(\omega) + \frac{[\mathbf{p}, V_C]}{\omega + i0} \right) | a \rangle. \end{aligned} \quad (5)$$

Here  $|a\rangle$  is the unperturbed state of the Dirac electron in the Coulomb field  $V_C(r) = -\alpha Z/r$ ,  $\mathbf{p} = -i\nabla$  is the momentum operator,  $G(\omega) = [\omega - H(1 - i0)]^{-1}$  is the relativistic Coulomb-Green function,  $H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m + V_C$ ,  $\alpha_l$  ( $l = 1, 2, 3$ ) are the Dirac matrices,  $\varepsilon_a$  is the unperturbed Dirac-Coulomb energy,

$$D_m(\omega) = -4\pi\alpha Z\alpha_l D_{lm}(\omega), \quad (6)$$

$D_{lm}(\omega)$  is the transverse part of the photon propagator in the Coulomb gauge. In the coordinate representation it is

$$D_{ik}(\omega, \mathbf{r}) = -\frac{1}{4\pi} \left\{ \frac{\exp(i|\omega|r)}{r} \delta_{ik} + \nabla_i \nabla_k \frac{(\exp(i|\omega|r) - 1)}{\omega^2 r} \right\}. \quad (7)$$

The scalar product is implicit in the equation (5). In Refs. [17,18], the formulas (3)-(5) were rederived by other methods and in [17] it was noticed that  $\Delta E$  can be written in the following compact form:

$$\Delta E = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \langle a | [\mathbf{p} - \mathbf{D}(\omega)] G(\omega + \varepsilon_a) [\mathbf{p} - \mathbf{D}(\omega)] | a \rangle. \quad (8)$$

However, the representation (3)-(5) is more convenient for practical calculations.

The term  $\Delta E_L$  can easily be calculated by using the virial relations for the Dirac equation [19,20]. Such a calculation gives [14]

$$\Delta E_L = \frac{m^2 - \varepsilon_a^2}{2M}. \quad (9)$$

This simple formula contains all the recoil corrections within the  $(\alpha Z)^4 m^2/M$  approximation. The term  $\Delta E_H$  taken to the lowest order in  $\alpha Z$  gives the Salpeter correction [5]. Evaluation of this term to all orders in  $\alpha Z$  will be discussed below.

The complete  $\alpha Z$ -dependence formulas for the nuclear recoil corrections in high  $Z$  few-electron atoms were derived in Ref. [21]. As it follows from these formulas, within the  $(\alpha Z)^4 m^2/M$  approximation the nuclear recoil corrections can be obtained by averaging the operator

$$H_M^{(L)} = \frac{1}{2M} \sum_{s,s'} \left( \mathbf{p}_s \cdot \mathbf{p}_{s'} - \frac{\alpha Z}{r_s} \left( \boldsymbol{\alpha}_s + \frac{(\boldsymbol{\alpha}_s \cdot \mathbf{r}_s) \mathbf{r}_s}{r_s^2} \right) \cdot \mathbf{p}_{s'} \right) \quad (10)$$

with the Dirac wave functions. This operator can also be used for relativistic calculations of the nuclear recoil effect in neutral atoms. An independent derivation of this operator was done in [22]. The operator (10) was employed in [23] to calculate the  $(\alpha Z)^4 m^2/M$  corrections to the energy levels of two- and three-electron multicharged ions.

### 3 Simple approach to the recoil effect in atoms

As was shown in [13], to include the relativistic recoil corrections in calculations of the energy levels, we must add to the standard Hamiltonian of the electron-positron field interacting with the quantized electromagnetic field and with the Coulomb field of the nucleus  $V_C$  an additional term. In the Coulomb gauge, this term is given by

$$H_M = \frac{1}{2M} \int d\mathbf{x} \psi^\dagger(\mathbf{x}) (-i\nabla_{\mathbf{x}}) \psi(\mathbf{x}) \int d\mathbf{y} \psi^\dagger(\mathbf{y}) (-i\nabla_{\mathbf{y}}) \psi(\mathbf{y}) - \frac{eZ}{M} \int d\mathbf{x} \psi^\dagger(\mathbf{x}) (-i\nabla_{\mathbf{x}}) \psi(\mathbf{x}) \mathbf{A}(0) + \frac{e^2 Z^2}{2M} \mathbf{A}^2(0). \quad (11)$$

This operator acts only on the electron-positron and electromagnetic field variables. The normal ordered form of  $H_M$  taken in the interaction representation must be added to the interaction Hamiltonian. It gives additional elements to the

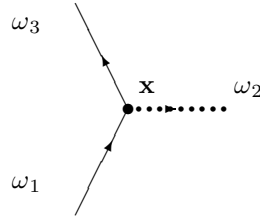
Feynman rules for the Green function. In the Furry picture, in addition to the standard Feynman rules in the energy representation (see [24,13]), the following vertices and lines appear (we assume that the Coulomb gauge is used)

1. *Coulomb contribution.*

An additional line ("Coulomb-recoil" line) appears to be

$$\begin{array}{ccc} \omega & & \\ \bullet \cdots \cdots \bullet & & \frac{i}{2\pi} \frac{\delta_{kl}}{M} \int_{-\infty}^{\infty} d\omega . \\ \mathbf{x} & & \mathbf{y} \end{array}$$

This line joins two vertices each of which corresponds to

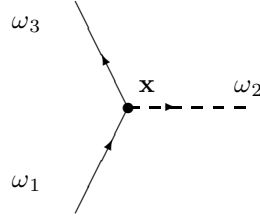


$$-2\pi i \gamma^0 \delta(\omega_1 - \omega_2 - \omega_3) \int d\mathbf{x} p_k ,$$

where  $\mathbf{p} = -i\nabla_{\mathbf{x}}$  and  $k = 1, 2, 3$ .

2. *One-transverse-photon contribution.*

An additional vertex on an electron line appears to be



$$-2\pi i \gamma^0 \delta(\omega_1 - \omega_2 - \omega_3) \frac{eZ}{M} \int d\mathbf{x} p_k ,$$

The transverse photon line attached to this vertex (at the point  $\mathbf{x}$ ) is

$$\begin{array}{ccc} \omega & & \\ \bullet \cdots \cdots \bullet & & \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega D_{kl}(\omega, \mathbf{y}) . \\ \mathbf{x} & & \mathbf{y} \end{array}$$

At the point  $\mathbf{y}$  this line is to be attached to an usual vertex in which we

have  $-2\pi i e \gamma^0 \alpha_l 2\pi \delta(\omega_1 - \omega_2 - \omega_3) \int d\mathbf{y}$ , where  $\alpha_l$  ( $l = 1, 2, 3$ ) are the usual Dirac matrices.

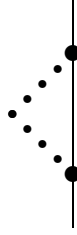
3. *Two-transverse-photon contribution.*

An additional line ("two-transverse-photon-recoil" line) appears to be

$$\begin{array}{c} \omega \\ \text{---} \bullet \text{---} \\ \mathbf{x} \qquad \mathbf{y} \end{array} \quad \frac{i}{2\pi} \frac{e^2 Z^2}{M} \int_{-\infty}^{\infty} d\omega D_{il}(\omega, \mathbf{x}) D_{lk}(\omega, \mathbf{y}).$$

This line joins usual vertices (see the previous item).

Let us apply this formalism to the case of a single level  $a$  in a one-electron atom. To find the Coulomb nuclear recoil correction we have to calculate the contribution of the diagram shown in Fig. 1. A simple calculation of this diagram



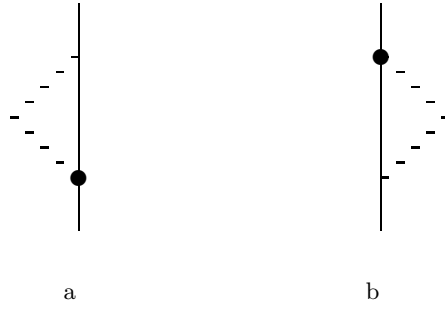
**Fig. 1.** Coulomb nuclear recoil diagram

yields (see Ref. [13] for details)

$$\Delta E_C = \frac{1}{M} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_n \frac{\langle a | p_i | n \rangle \langle n | p_i | a \rangle}{\omega - \varepsilon_n(1 - i0)}. \quad (12)$$

The one-transverse-photon nuclear recoil correction corresponds to the diagrams shown in Fig. 2. One easily obtains

$$\begin{aligned} \Delta E_{tr(1)} = & \frac{4\pi\alpha Z}{M} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_n \left\{ \frac{\langle a | p_i | n \rangle \langle n | \alpha_k D_{ik}(\varepsilon_a - \omega) | a \rangle}{\omega - \varepsilon_n(1 - i0)} \right. \\ & \left. + \frac{\langle a | \alpha_k D_{ik}(\varepsilon_a - \omega) | n \rangle \langle n | p_i | a \rangle}{\omega - \varepsilon_n(1 - i0)} \right\}. \end{aligned} \quad (13)$$



**Fig. 2.** One-transverse-photon nuclear recoil diagrams

The two-transverse-photon nuclear recoil correction is defined by the diagram shown in Fig. 3. We find



**Fig. 3.** Two-transverse-photon nuclear recoil diagram

$$\Delta E_{tr(2)} = \frac{(4\pi\alpha Z)^2}{M} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_n$$

$$\times \frac{\langle a | \alpha_i D_{il}(\varepsilon_a - \omega) | n \rangle \langle n | \alpha_k D_{lk}(\varepsilon_a - \omega) | a \rangle}{\omega - \varepsilon_n(1 - i0)}. \quad (14)$$

The sum of the contributions (12)-(14) is

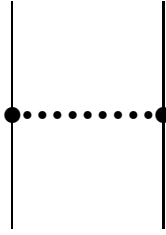
$$\begin{aligned} \Delta E = \frac{1}{M} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega & \langle a | (p_i + 4\pi\alpha Z \alpha_l D_{li}(\omega)) \\ & \times G(\omega + \varepsilon_a) (p_i + 4\pi\alpha Z \alpha_m D_{mi}(\omega)) | a \rangle. \end{aligned} \quad (15)$$

This exactly coincides with formula (8).

Consider now a high- $Z$  two-electron atom. For simplicity, we will assume that the unperturbed wave function is a one-determinant function

$$u(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{\sqrt{2}} \sum_P (-1)^P \psi_{Pa}(\mathbf{x}_1) \psi_{Pb}(\mathbf{x}_2). \quad (16)$$

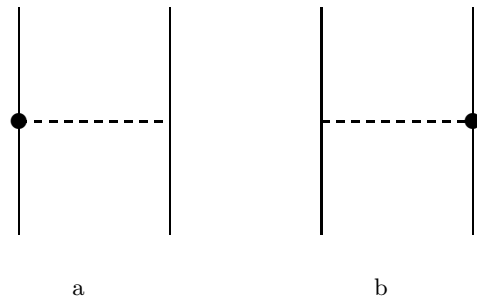
The nuclear recoil correction is the sum of the one-electron and two-electron contributions. The one-electron contribution is the sum of the expressions (15) for the  $a$  and  $b$  states. The two-electron contributions are defined by the diagrams shown in Figs. 4-6. A simple calculation of these diagrams yields



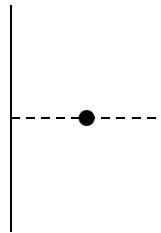
**Fig. 4.** Two-electron Coulomb nuclear recoil diagram

$$\begin{aligned} \Delta E^{(\text{int})} = \frac{1}{M} \sum_P (-1)^P & \langle Pa | p_i + 4\pi\alpha Z \alpha_l D_{li}(\varepsilon_{Pa} - \varepsilon_a) | a \rangle \\ & \times \langle Pb | p_i + 4\pi\alpha Z \alpha_m D_{mi}(\varepsilon_{Pb} - \varepsilon_b) | b \rangle. \end{aligned} \quad (17)$$

The formula (17) was first derived by the quasipotential method in [21].



**Fig. 5.** Two-electron one-transverse-photon nuclear recoil diagrams



**Fig. 6.** Two-electron two-transverse-photon nuclear recoil diagram



## 4 Numerical results

### 4.1 Hydrogenlike atoms

According to equations (3)-(5) the recoil correction is the sum of the low-order and higher-order terms. The low-order term  $\Delta E_L$  is given by equation (9). The higher order term  $\Delta E_H$  was calculated to all orders in  $\alpha Z$  in [25,26,27]. The results of these calculations expressed in terms of the function  $P(\alpha Z)$  defined as

$$\Delta E_H = \frac{m^2}{M} \frac{(\alpha Z)^5}{\pi n^3} P(\alpha Z) \quad (18)$$

are presented in Table 1. To the lowest order in  $\alpha Z$  the function  $P(\alpha Z)$  is given

**Table 1.** The results of the numerical calculation of the function  $P(\alpha Z)$  for low-lying states of hydrogenlike atoms

| $Z$ | $1s$       | $2s$       | $2p_{1/2}$ | $2p_{3/2}$ |
|-----|------------|------------|------------|------------|
| 1   | 5.42990(3) | 6.15483(5) | -0.30112   | -0.3013(4) |
| 5   | 4.3033(4)  | 5.0335(2)  | -0.2692    | -0.2724(1) |
| 10  | 3.7950(1)  | 4.5383(1)  | -0.2277    | -0.2379    |
| 20  | 3.2940(1)  | 4.0825     | -0.1393    | -0.1726    |
| 30  | 3.0437(1)  | 3.9037     | -0.0421    | -0.1107    |
| 40  | 2.9268(1)  | 3.8900     | 0.0685     | -0.0517    |
| 50  | 2.9137(1)  | 4.0228(1)  | 0.2000     | 0.0050     |
| 60  | 3.0061(2)  | 4.3248(2)  | 0.3655     | 0.0597     |
| 70  | 3.2334(4)  | 4.8656(5)  | 0.5894     | 0.1125     |
| 80  | 3.672(1)   | 5.807(2)   | 0.9214(2)  | 0.1638     |
| 90  | 4.519(8)   | 7.557(9)   | 1.481(1)   | 0.2138     |
| 100 | 6.4(1)     | 11.4(2)    | 2.63(2)    | 0.2625     |

by Salpeter's expressions:

$$P_S^{(1s)}(\alpha Z) = -\frac{2}{3} \log(\alpha Z) - \frac{8}{3} 2.984129 + \frac{14}{3} \log 2 + \frac{62}{9}, \quad (19)$$

$$P_S^{(2s)}(\alpha Z) = -\frac{2}{3} \log(\alpha Z) - \frac{8}{3} 2.811769 + \frac{187}{18}, \quad (20)$$

$$P_S^{(2p_{1/2})} = P_S^{(2p_{3/2})} = \frac{8}{3} 0.030017 - \frac{7}{18}. \quad (21)$$

Comparing the function  $P(\alpha Z)$  from Table 1 with the lowest order contributions (19)-(21) shows that for high  $Z$  the complete  $\alpha Z$ -dependence results differ considerably from Salpeter's ones.

In the case of hydrogen, the difference  $\Delta P = P - P_S$  amounts to -0.01616(3), -0.01617(5), and 0.00772 for the  $1s$ ,  $2s$ , and  $2p_{1/2}$  states, respectively. Table 2 displays the relativistic recoil corrections, beyond the Salpeter ones, to the

hydrogen energy levels. These values include also the corresponding correction from the low-order term (9) which is calculated by

$$\Delta' E_L^{(1s)} = 0, \quad (22)$$

$$\Delta' E_L^{(2s)} = \Delta' E_L^{(2p_{1/2})} = \frac{(\alpha Z)^6}{64} \frac{2[3 + \sqrt{1 - (\alpha Z)^2}]}{[1 + \sqrt{1 - (\alpha Z)^2}]^3} \frac{m^2}{M}. \quad (23)$$

The results of Refs. [25,27] which are exact in  $\alpha Z$  are compared with the related corrections obtained to the lowest order in  $\alpha Z$ . In [28,29] it was found that the  $(\alpha Z)^6 \log(\alpha Z) m^2/M$  corrections cancel each other. The  $(\alpha Z)^6 m^2/M$  correction was derived in [18] for  $s$ -states and in [30] for  $p$ -states. The  $(\alpha Z)^7 \log^2(\alpha Z) m^2/M$  correction was recently evaluated in Refs. [31,32]. The uncertainty of the calcu-

**Table 2.** The values of the relativistic recoil correction to hydrogen energy levels beyond the Salpeter contribution, in kHz. The values given in the second and third rows include the  $(\alpha Z)^6 m^2/M$  contribution and all the contributions of higher orders in  $\alpha Z$ . In the last row the sum of the  $(\alpha Z)^6 m^2/M$  and  $(\alpha Z)^7 \log^2(\alpha Z) m^2/M$  contributions is given

| State   | 1s       | 2s        | 2p <sub>1/2</sub> |
|---|----------|-----------|-------------------|
| To all orders in $\alpha Z$ , Ref. [25]               | -7.1(9)  | -0.73(6)  | 0.59              |
| To all orders in $\alpha Z$ , Ref. [27]               | -7.16(1) | -0.737(3) | 0.587             |
| $(\alpha Z)^6 m^2/M$ , Refs. [18,30]                  | -7.4     | -0.77     | 0.58              |
| $(\alpha Z)^7 \log^2(\alpha Z) m^2/M$ , Refs. [31,32] | -0.4     | -0.05     |                   |
| The sum of the low-order terms                        | -7.8     | -0.82     |                   |

lation based on the expansion in  $\alpha Z$  is defined by uncalculated terms of order  $(\alpha Z)^7 m^2/M$  and is expected to be about 1 kHz for the 1s state. It follows that the results of the complete  $\alpha Z$ -dependence calculations are in a good agreement with the results obtained to lowest orders in  $\alpha Z$  but are of much higher accuracy.

As it follows from Ref. [13], the formulas (3)- (5) will incorporate partially the nuclear size corrections to the recoil effect if  $V_C(r)$  is taken to be the potential of an extended nucleus. In particular, this replacement allows one to account for the nuclear size corrections to the Coulomb part of the recoil effect. In Ref. [33], where the calculations of the recoil effect for extended nuclei were performed, it was found that, in the case of hydrogen, the leading relativistic nuclear size correction to the Coulomb low-order part is comparable with the total value of the  $(\alpha Z)^6 m^2/M$  correction but is cancelled by the nuclear size correction to the Coulomb higher-order part.

One of the main goals of the calculations of Refs. [25,26,33] was to evaluate the nuclear recoil correction for highly charged ions. In the case of the ground state of hydrogenlike uranium these calculations yield -0.51 eV for the point nucleus case [25] and -0.46 eV for the extended nucleus case [33]. This correction

is big enough to be included in the current theoretical prediction for the 1s Lamb shift in hydrogenlike uranium [34] but is small compared with the present experimental uncertainty which amounts to 13 eV [10]. However, a much higher precision was obtained in experiments with heavy lithiumlike ions [8,9]. In this connection in Refs. [25,26] the nuclear recoil corrections for lithiumlike ions were calculated as well.

#### 4.2 Lithiumlike ions

In lithiumlike ions, in addition to the one-electron contributions, we must evaluate the two-electron contributions. In the case of one electron over the  $(1s)^2$  shell the total two-electron contribution to the zeroth order in  $1/Z$  is given by the expression

$$\Delta E^{\text{int}} = -\frac{1}{M} \sum_{\varepsilon_n = \varepsilon_{1s}} \langle a | \mathbf{p} - \mathbf{D}(\varepsilon_a - \varepsilon_n) | n \rangle \langle n | \mathbf{p} - \mathbf{D}(\varepsilon_a - \varepsilon_n) | a \rangle, \quad (24)$$

where  $\mathbf{D}$  is defined by equation (6). Calculation of this term causes no problem [25,26]. For the  $2p_{1/2}$  and  $2p_{3/2}$  states, the results of this calculation expressed in terms of the function  $Q(\alpha Z)$  defined by

$$\Delta E^{\text{int}} = -\frac{2^9 m^2}{3^8 M} (\alpha Z)^2 Q(\alpha Z) \quad (25)$$

are presented in Table 3. For the  $s$ -states the two-electron contribution is equal zero. To the lowest orders in  $\alpha Z$  the function  $Q(\alpha Z)$  is given by [23]

**Table 3.** The results of the numerical calculation of the function  $Q(\alpha Z)$  for low-lying states of lithiumlike ions

| $Z$ | $(1s)^2 2p_{1/2}$ | $(1s)^2 2p_{3/2}$ |
|-----|-------------------|-------------------|
| 10  | 0.99741           | 0.99810           |
| 20  | 0.98959           | 0.99239           |
| 30  | 0.97645           | 0.98281           |
| 40  | 0.95776           | 0.96926           |
| 50  | 0.93313           | 0.95165           |
| 60  | 0.90195           | 0.92988           |
| 70  | 0.86320           | 0.90390           |
| 80  | 0.81529           | 0.87362           |
| 90  | 0.75570           | 0.83896           |
| 100 | 0.68041           | 0.79951           |

$$Q_L^{(2p_{1/2})}(\alpha Z) = 1 + (\alpha Z)^2 \left( -\frac{29}{48} + \log \frac{9}{8} \right), \quad (26)$$

$$Q_L^{(2p_{3/2})}(\alpha Z) = 1 + (\alpha Z)^2 \left( -\frac{13}{48} + \frac{1}{2} \log \frac{27}{32} \right). \quad (27)$$

The expressions (26)-(27) serve as a good approximation for the  $Q(\alpha Z)$  function even for very high  $Z$ .

For low  $Z$ , in addition to the corrections considered here, the Coulomb inter-electronic interaction effect on the non-relativistic nuclear recoil correction must be taken into account. It contributes on the level of order  $(1/Z)(\alpha Z)^2 m^2/M$ .

To date, the highest precision in experiments with heavy ions was obtained for the  $2p_{3/2} - 2s$  transition in lithiumlike bismuth [9]. The transition energy measured in this experiment amounts to  $(2788.14 \pm 0.04)$  eV. In [8] the energy of the  $2p_{1/2} - 2s$  transition in lithiumlike uranium was measured to be  $(280.59 \pm 0.10)$  eV. In both cases the recoil correction amounts to -0.07 eV and, therefore, is comparable with the experimental uncertainty. At present, the uncertainty of the theoretical predictions for these transition energies is defined by uncalculated contributions of second order in  $\alpha$  (see Refs. [34,35]). When calculations of these contributions are completed, it will be possible to probe the recoil effect in high- $Z$  few-electron systems. This will provide a unique possibility for testing the quantum electrodynamics in the region of strong coupling ( $\alpha Z \sim 1$ ) beyond the external field approximation since in calculations of all other QED corrections in heavy ions the nucleus is considered only as a stationary source of the classical electromagnetic field.

## 5 Conclusion

In this paper the relativistic theory of the recoil effect in atoms is considered. It is shown that the complete  $\alpha Z$ -dependence calculation of the recoil correction provides the highest precision even in the case of hydrogen. The recoil corrections to the energy levels of highly charged ions contribute on the level of the present experimental accuracy. It provides good perspectives for testing the quantum electrodynamics in the region of strong coupling ( $\alpha Z \sim 1$ ) beyond the external field approximation.

## Acknowledgments

The author wants to express his thanks to A.N. Artemyev, T. Beier, G. Plunien, G. Soff, and V.A. Yerokhin for stimulating collaboration. Valuable conversations with S.G. Karshenboim, P.J. Mohr, and A.S. Yelkhovsky are gratefully acknowledged.

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