Relativistic Dipole Dynamic Polarizabilities of Lowest $ns_{1/2}$-States in Hydrogen-Like Atoms

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Abstract. A novel closed-form exact analytical expression for the linear response relativistic wave function of the hydrogenic $ns_{1/2}$-level exposed to external dipole radiation is reported. This result is derived using the method due to Podolsky, that is, by means of direct analytic solving of the appropriate systems of inhomogeneous differential equations, thus requiring no prior knowledge of the relativistic Coulomb Green's function. As an important application of the formulas obtained, new expression for the relativistic dipole dynamic polarizabilities of lowest hydrogenic $ns_{1/2}$-levels are calculated.

1 Introduction

Dipole dynamic polarizability (DDP) is a fundamental atomic characteristic which enables, in the first non-vanishing order of the perturbation theory, various physical phenomena involving second order "atom–photon" or "atom–electron" interactions to be conveniently described and analyzed. Among such phenomena are, for example, quadratic AC- and DC-Stark shifts, splitting, ionization power broadening and two-photon (de)excitation of atomic levels exposed to external laser radiation (see [1,2,3] and references therein), elastic (Rayleigh) and inelastic scattering cross sections of light by atoms [4], as well as two-photon *bremsstrahlung* [5]. Not surprisingly, therefore, that the problem of the DDP calculations for various states of single and many-electron atoms (and molecules), in a wide range of frequencies ω, attracted considerable attention over the years [3,6,7,8]. Particular theoretical progress has been made in studies of hydrogenic states, where exact analytic expressions for the *nonrelativistic* DDP of an arbitrary nl-level have been recently obtained [9]. In contrast with most work on the subject, however, the present article is focused on the analytic study of the DDP *outside the nonrelativistic approximation*. In particular, we derive an exact and yet unknown closed-form analytic expression for the linear (i.e., first-order) response *relativistic* wave function $\delta\Psi_{ns}(\boldsymbol{r},\omega)$ of the hydrogenic $ns_{1/2}$-level subject to a linearly polarized electromagnetic field of the angular frequency ω. In addition, as an interesting and important application of this result, a new formula to define the *relativistic* dynamic dipole polarizability (RDDP) of the lowest hydrogenic $ns_{1/2}$-states, $\alpha_{1s_{1/2}}(\omega)$ and $\alpha_{2s_{1/2}}(\omega)$, is reported.

Although interesting in its own right, the present study acquires additional strong motivation due to its direct applicability to highly accurate numerical simulations of (de)excitation and ionization probabilities as well as line profiles in various resonant multiphoton processes occurring with $ns_{1/2}$-states in

simplest atoms. Currently, such processes are of conspicuous interest for ultra high precision Doppler-free spectroscopy of the fundamental bounded systems such as *hydrogen* and its isotopes [10,11,12,13,14,15], *positronium* [16,17], denoted $(e^+ - e^-)$, *muonium* [18,19,17,20,21,22], denoted $(\mu^+ - e^-)$, and the helium atom [23]. In many instances, newly available experimental data pose a significant challenge to theory, thus urging further developments intended to describe the well established photon-induced processes with much higher precision than theoretical technology used so far was able to provide. In particular, the new formula for the RDDP reported here can be used efficiently for a very accurate calculation of the 2+1 resonant 3-photon ionization process with muonium, $1S \xrightarrow{2\hbar\omega} 2S \xrightarrow{\hbar\omega} \varepsilon P$ (see [24] and the paper [25] by V. Yakhontov and K. Jungmann in this volume). Alternative conceivable applications of the new expressions for the RDDP and $\delta\Psi_{ns}(\boldsymbol{r},\omega)$ comprise a wide class of physical processes involving E1-transitions, real or virtual, in the presence of the Coulomb field.

To introduce necessary notations and to recapitulate briefly the underlying physics, it is instructive to revisit the process of inelastic (Raman, Stokes or anti-Stokes) photon scattering by a hydrogen-like atom residing in the state $|i\rangle$. As a most general situation, we assume that the atom is formed by two 1/2-spin particles with the charges $-|e|$ and $Z|e|$ and the masses m and M, $e = -|e|$ being the charge of the electron. The total amplitude $M_{f,i}$ of this reaction can be readily obtained by taking into account the "atom–photon field" interaction $\widehat{V}$ to the second order of the conventional perturbation theory [4,26]. This results in the sum of two amplitudes $M_{f,i}^{(a)}$ and $M_{f,i}^{(b)}$ that are shown diagrammatically in Fig. 1(a)–(b). Apparently, the whole class of the aforementioned two-photon phenomena can be described by just considering different channels of the process in question, while possibly allowing for $\omega' \equiv \omega$ and $|f\rangle \equiv |i\rangle$. Furthermore, let us assume that the electric component $\boldsymbol{E}(\boldsymbol{r},t)$ of the photon field acting on the atom has the form of a standing wave:

$$\boldsymbol{E}(\boldsymbol{r},t) = E_0 \Re\{\boldsymbol{\epsilon} \exp\{\mathrm{i}\,(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\}\ , \tag{1}$$

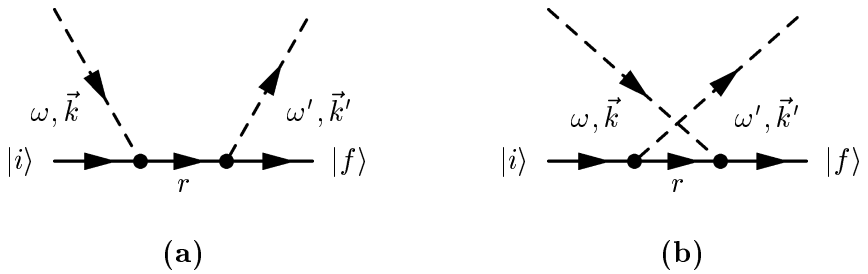

Fig. 1. "Time-direct" – (**a**) – and "time-reverse" – (**b**) – diagrams describing the process of inelastic scattering of a photon with the energy $\hbar\omega$ and the wave vector $\boldsymbol{k}$ by an atom residing in the state $|i\rangle$: $(\omega, \boldsymbol{k}, |i\rangle) \longrightarrow (\omega', \boldsymbol{k}', |f\rangle)$. Solid lines stand for atomic states, dashed lines denote photons in the initial/final states, and filled circles designate the vertices of the "electron–photon" interaction $\widehat{V}$

$$|\boldsymbol{k}| = \omega/c\,, \qquad \boldsymbol{\epsilon}\cdot\boldsymbol{k} = 0\,, \qquad \boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* = 0\,,$$

where E_0, $\boldsymbol{\epsilon}$, ω, $\boldsymbol{k}$ are the field's amplitude, polarization vector, angular frequency, and wave vector, respectively, and c is the speed of light. The amplitudes $M_{f,i}^{(a)}$ and $M_{f,i}^{(b)}$ can be cast in a form known as the Kramers-Heisenberg matrix element [27], i.e. expressed in terms of the RDDP, if $\boldsymbol{k}\cdot\boldsymbol{r} \simeq \omega\lambda_{\rm char}/c \ll 1$, $\lambda_{\rm char}$ being a characteristic spatial size of an atom in the state $|i\rangle$. The latter condition constitutes the dipole approximation [26], and it is well satisfied unless photons of high frequencies ω, ω' and/or highly excited (Rydberg) states $|i\rangle$, $|f\rangle$ are involved:

$$\frac{\omega}{\omega_I} \ll \frac{2}{\alpha Z}\,, \qquad \hbar\omega_I \equiv \frac{Z^2}{2n^2}\frac{e^2}{a_0^*}. \tag{2}$$

Here, $a_0^* = \hbar^2/(m^*e^2)$ is the reduced Bohr radius ($m^* = \frac{mM}{m+M}$), n is a principal quantum number of the hydrogenic state $|i\rangle$, and $\alpha = e^2/(\hbar c) \approx 1/137$ is the fine structure constant. In the following discussion, (2) will be the only restriction imposed on values of physical parameters. In particular, no distinction will be made as to whether the photon energy $\hbar\omega$ is less or greater than the ionization potential I_i of the state $|i\rangle$. Note that $I_i = \hbar\omega_I$ holds true only in the nonrelativistic approximation, whereas in general $I_i \neq \hbar\omega_I$.

In the dipole approximation, the relativistic "atom–photon field" interaction $\widehat{V}(\boldsymbol{r},t)$ takes the form [4]

$$\widehat{V}(\boldsymbol{r},t) = -e\boldsymbol{\alpha}\cdot\boldsymbol{A}(\boldsymbol{r},t) = \mathrm{i}\frac{|e|c}{2\omega}E_0\boldsymbol{\alpha}\cdot\boldsymbol{\epsilon}e^{\mathrm{i}\omega t} + c.c.\,, \tag{3}$$

to result eventually in the following expression for the amplitude $M_{f,i}$ (cf. [4,28]):

$$M_{f,i} = M_{f,i}^{(a)} + M_{f,i}^{(b)} = \frac{1}{4}E_0^2\langle f|\widehat{Q}^{(2)}(\omega,\omega')|i\rangle\,, \tag{4}$$

$$\widehat{Q}^{(2)}(\omega,\omega') \equiv \frac{e^2c^2}{\omega\omega'}\left\{\boldsymbol{\epsilon}^*\cdot\boldsymbol{\alpha}\frac{1}{E_i - \hbar\omega - \widehat{H}_0}\boldsymbol{\epsilon}\cdot\boldsymbol{\alpha} + \boldsymbol{\epsilon}\cdot\boldsymbol{\alpha}\frac{1}{E_i + \hbar\omega' - \widehat{H}_0}\boldsymbol{\epsilon}^*\cdot\boldsymbol{\alpha}\right\}. \tag{5}$$

Here, $\boldsymbol{\alpha}$ is the Dirac α-matrix, $\boldsymbol{A}(\boldsymbol{r},t)$ is the vector potential of the field ($\mathrm{div}\boldsymbol{A} = 0$), and $\widehat{H}_0$ is the Hamiltonian of an isolated atom whose energy in the initial state of the reaction is denoted by E_i: $\widehat{H}_0|i\rangle = E_i|i\rangle$. Equation (5) defines the *effective two-photon operator* $\widehat{Q}^{(2)}(\omega,\omega')$ which has the dimension of L^3 and is a straightforward relativistic generalization of its nonrelativistic counterpart that has been first introduced in [28] to describe the process of two-photon absorption. Generally, the matrix elements of $Q_{f,i}^{(2)}(\omega,\omega')$ can be expressed explicitly as

$$\langle f|\widehat{Q}^{(2)}(\omega,\omega')|i\rangle = \frac{e^2c^2}{\omega^2}\sum_r\left\{\frac{\langle f|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\alpha}|r\rangle\langle r|\boldsymbol{\epsilon}\cdot\boldsymbol{\alpha}|i\rangle}{E_i - \hbar\omega - E_r} + \frac{\langle f|\boldsymbol{\epsilon}\cdot\boldsymbol{\alpha}|r\rangle\langle r|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\alpha}|i\rangle}{E_i + \hbar\omega' - E_r + \mathrm{i}0}\right\} \tag{6}$$

where the summation spans all Coulomb eigenstates of the Hamiltonian $\widehat{H}_0$ which are allowed by the selection rules, including those belonging to the positive ($E_r > mc^2$) and negative ($E_r < -mc^2$) continuum. Equation (6) takes especially

simple form if $\omega' = \omega$, that is, for elastic two-photon scattering, in which case the diagonal matrix element

$$\langle i|\widehat{Q}^{(2)}(\omega,\omega)|i\rangle \equiv -\sum_{k,l}(\alpha_{kl}(\omega))_{i,i}e_k e_l^* \tag{7}$$

defines the RDDP $(\alpha_{kl}(\omega))_{i,i}$. Here, e_k and e_l^* are the Cartesian components of the polarization vectors $\boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}^*$. Following the standard prescription [4], the infinitesimal positive imaginary constant has been added in the denominator of the second term in (6) to prevent the occurrence of a singularity at the simple pole $E_r = E_i + \hbar\omega'$, provided that $\hbar\omega' > I_i \equiv m^*c^2 - E_i > 0$. For such ω', $Q^{(2)}_{f,i}(\omega,\omega')$ acquires a non-zero imaginary part, $\Im Q^{(2)}_{f,i}(\omega,\omega')$, whose sign is determined uniquely by the (positive) sign of the infinitesimal constant. In the nonrelativistic limit ($\boldsymbol{\alpha} \to \widehat{\boldsymbol{p}}/m^*c$), $Q^{(2)}_{i,i}(\omega,\omega)$ can be shown [4] to be identical with the Kramers-Heisenberg matrix element [27].

Despite its considerable age, there have been quite few attempts to evaluate the sum (6) with $\omega, \omega' \neq 0$ analytically, even in the simplest case of a hydrogen-like atom considered here. In particular, the majority of accurate relativistic calculations of the DP [29,30,31,32,33,34,35,36], both analytic and numerical, have been performed for the ground state of hydrogen-like ions and in the static limit only, i.e. at $\omega = \omega' = 0$ (see also [37,38,39]). This is unlike the nonrelativistic version of (6) where the use of the closed-form expression for the nonrelativistic Coulomb Green's function [40,41,42] enabled exact formulas valid for arbitrary states f, i and in the whole range of ω to be obtained [9] (see also [43,44,45,46] and references therein). In the relativistic case, however, the utility of the appropriate Coulomb Green's function [47,48,30,49] is hampered by considerable technical difficulties, often requiring alternative analytic and/or numerical [50,51,34] methods to be used instead. Thus, for example, the RDDP of the ground state of a hydrogenic ion has been calculated analytically in [52,53] using a new algebraic representation for the relativistic Coulomb Green's function. The final result of [53] has a form, however, of a many-fold infinite series involving Γ-functions, thus making it hardly suitable for accurate numerical evaluation both at $\hbar\omega < I_{1s} = m^*c^2(1 - \sqrt{1-\alpha^2Z^2})$ and $\hbar\omega > I_{1s}$. In what follows, $Q^{(2)}_{f,i}(\omega,\omega')$ and $(\alpha_{kl}(\omega))_{i,i}$ will be calculated for the lowest $ns_{1/2}$-levels by a direct method [54,55,56,57] requiring no prior knowledge of the Coulomb Green's function. Similar to our nonrelativistic result [57], this enabled us to cast the final expression for the RDDP of such states in a form which admits error-free numerical evaluation at photon energies lying both below and above ionization threshold I_i.

2 Theory

2.1 Basic Equations

As s starting point of our consideration, we rewrite (6) in terms of the linear response wave function (the reduced atomic system of units, $\hbar = e^2 = m^* = 1$,

is used throughout this section) ,

$$\delta\Psi_i(\boldsymbol{r},E) \equiv \frac{1}{\alpha}\frac{1}{E-\widehat{H}_0}\boldsymbol{\alpha}|i\rangle = \frac{1}{\alpha}\sum\!\!\!\!\!\!\int_r \frac{|r\rangle\langle r|\boldsymbol{\alpha}|i\rangle}{E-E_r} \ , \tag{8}$$

as

$$\langle f|\widehat{Q}^{(2)}(\omega,\omega')|i\rangle = \frac{1}{\alpha\omega\omega'}\sum_{\nu,\mu}(-1)^{\mu}\epsilon^*_{\nu}\epsilon_{-\mu} \tag{9}$$
$$\times\left[\langle f|\alpha_\nu|\delta\Psi_\mu(\boldsymbol{r},E_i-\hbar\omega)\rangle + \langle f|\alpha_\nu|\delta\Psi_\mu(\boldsymbol{r},E_i+\hbar\omega')\rangle\right] ,$$

where ϵ_μ and $\delta\Psi_\mu(\boldsymbol{r},E)$ denote the spherical components of $\boldsymbol{\epsilon}$ and $\delta\Psi_i(\boldsymbol{r},E)$. Hereafter, the subscript i of the function $\delta\Psi_i(\boldsymbol{r},E)$, indicating the reference state $|i\rangle$, will be dropped to simplify notations.

According to (10), $\langle f|\widehat{Q}^{(2)}(\omega,\omega')|i\rangle$ is determined uniquely by the function $\delta\Psi(\boldsymbol{r},E)$. In the following discussion, $\delta\Psi(\boldsymbol{r},E)$ is found analytically using the relativistic version of a method first suggested by Podolsky as far back as 1928. As was shown in his original paper [54], the nonrelativistic function $\delta\Psi(\boldsymbol{r},E)$ can be constructed by solving a certain partial inhomogeneous differential equation. Later, this idea was extended on a wider, mostly nonrelativistic, class of problems by Sternheimer [55], Dalgarno and Lewis [56], and Schwartz [60]. Despite the seeming simplicity of the Podolsky's method, there still remains in many cases a formidable problem of obtaining analytic solution to the appropriate differential equation in a practically useful form. This is evidenced by the use of numerical rather than analytic methods in most DDP-calculations employing this approach [61,62,63,64]. To appreciate the complexity of this analytic problem, it is worthwhile recalling that the *nonrelativistic* closed-form expression for $\delta\Psi_{1s}(\boldsymbol{r},E)$ has been first derived by Luban and Nudler-Blum [58] only 20 years ago. A generalization of their result on arbitrary hydrogenic nl-states has been obtained, in the static limit, in [59], whereas for arbitrary ns-states and $\omega \neq 0$ the problem has been recently solved in [57]. Until now no successful attempts were made, to our knowledge, to derive the relativistic counterparts of these results.

In the spirit of the Podolsky's method, we act with $E-\widehat{H}_0$ on both sides of (8) to get the equation obeyed by $\Psi_\mu(\boldsymbol{r},E)$:

$$(E-\widehat{H}_0)\Psi_\mu(\boldsymbol{r},E) = \alpha_\mu|i\rangle \ . \tag{10}$$

In the following discussion, we restrict ourselves to the case of the $ns_{1/2}$-states:

$$|i\rangle = |n,l=0,j=\frac{1}{2},m\rangle = \begin{pmatrix} f_{n,-1}(r)\Omega_{1/2,0,m}(\boldsymbol{n}) \\ \mathrm{i}\alpha g_{n,-1}(r)\Omega_{1/2,1,m}(\boldsymbol{n}) \end{pmatrix} \ , \tag{11}$$

where $f_{n,-1}(r)$, $g_{n,-1}(r)$ and $\Omega_{1/2,l,m}(\boldsymbol{n})$ are the usual radial and spin-angular parts of the bispinor [4]. To separate the angular variables in (10) we write:

$$\Psi_\mu(\mathbf{r},E) = \Psi^{(1)}_\mu(\mathbf{r},E) + \Psi^{(2)}_\mu(\mathbf{r},E) \ , \tag{12}$$

where

$$\Psi^{(1)}_{\mu}(\boldsymbol{r},E) = (-1)^{1/2+\mu+m}\sqrt{\frac{2}{3}}\frac{1}{r}\begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} \\ -\mu-m & \mu & m \end{pmatrix} \times \begin{pmatrix} \mathrm{i}\xi_1(r)\Omega_{1/2,1,-\mu-m}(\boldsymbol{n}) \\ \alpha\eta_1(r)\Omega_{1/2,0,-\mu-m}(\boldsymbol{n}) \end{pmatrix}, \tag{13}$$

$$\Psi^{(2)}_{\mu}(\boldsymbol{r},E) = (-1)^{1/2+\mu+m}\frac{4}{\sqrt{3}}\frac{1}{r}\begin{pmatrix} \frac{3}{2} & 1 & \frac{1}{2} \\ -\mu-m & \mu & m \end{pmatrix} \times \begin{pmatrix} \mathrm{i}\xi_2(r)\Omega_{3/2,1,-\mu-m}(\boldsymbol{n}) \\ \alpha\eta_2(r)\Omega_{3/2,2,-\mu-m}(\boldsymbol{n}) \end{pmatrix}, \tag{14}$$

with (:::) being the $3j$-symbols. Auxiliary functions $\xi_1(r)$, $\eta_1(r)$ and $\xi_2(r)$, $\eta_2(r)$ entering (13)–(14) satisfy the following system of inhomogeneous differential equations of the first order:

$$\xi_1'(r) + \frac{1}{r}\xi_1(r) + \left(2+\alpha^2\varepsilon+\frac{Z\alpha^2}{r}\right)\eta_1(r) = 3rf_{n,-1}(r) \tag{15a}$$

$$\eta_1'(r) - \frac{1}{r}\eta_1(r) - \left(\varepsilon+\frac{Z}{r}\right)\xi_1(r) = rg_{n,-1}(r) \tag{15b}$$

$$\xi_2'(r) - \frac{2}{r}\xi_2(r) + \left(2+\alpha^2\varepsilon+\frac{Z\alpha^2}{r}\right)\eta_2(r) = 0 \tag{16a}$$

$$\eta_2'(r) + \frac{2}{r}\eta_2(r) - \left(\varepsilon+\frac{Z}{r}\right)\xi_2(r) = -rg_{n,-1}(r)\,. \tag{16b}$$

This can be readily shown by acting on $\Psi^{(1)}_{\mu}(\boldsymbol{r},E)$ and $\Psi^{(2)}_{\mu}(\boldsymbol{r},E)$ with $E-\widehat{H}_0$, where $\widehat{H}_0$ is given by [65]

$$\widehat{H}_0 = \frac{\mathrm{i}}{\alpha}\boldsymbol{\alpha}\cdot\boldsymbol{n}\left[-\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)+\frac{1}{r}(\boldsymbol{\sigma}\cdot\boldsymbol{l}+1)\right]+\frac{\beta}{\alpha^2}-\frac{Z}{r}\,. \tag{17}$$

Here and above, $\varepsilon = E - 1/\alpha^2$ and β, $\boldsymbol{\sigma}$ are the Dirac and Pauli matrices. By taking into account the following explicit form of the final state wave function,

$$|f\rangle = |n', l=0, j=\frac{1}{2}, m'\rangle = \begin{pmatrix} f_{n',-1}(r)\Omega_{1/2,0,m'}(\boldsymbol{n}) \\ \mathrm{i}\alpha g_{n',-1}(r)\Omega_{1/2,1,m'}(\boldsymbol{n}) \end{pmatrix}, \tag{18}$$

and assuming for simplicity the linear polarization $\boldsymbol{\epsilon}$ of light, we arrive at the relation:

$$\langle f|(\boldsymbol{\epsilon}^*\cdot\boldsymbol{\alpha})\boldsymbol{\epsilon}|\delta\Psi(\boldsymbol{r},E)\rangle = \sum_{\nu,\mu}(-1)^{\mu}\epsilon^*_{\nu}\epsilon_{-\mu}\langle f|\alpha_{\nu}|\delta\Psi_{\mu}(\boldsymbol{r},E)\rangle \tag{19}$$

$$= \frac{\alpha}{9}\int_0^{\infty} r\big[3\eta_1(r)f_{n',-1}(r) - \xi_1(r)g_{n',-1}(r) + 8\xi_2(r)g_{n',-1}(r)\big]\mathrm{d}r\,.$$

Plugging this identity into (10) immediately yields a desired expression for $\langle f|\widehat{Q}^{(2)}(\omega,\omega')|i\rangle$ in terms of $\xi_1(r)$, $\eta_1(r)$, $\xi_2(r)$, and $\eta_2(r)$. In the following section, the system (15a)–(16b) will be solved to find these four functions explicitly

for $n = 1, 2$, in which case the radial parts $f_{n,-1}(r), g_{n,-1}(r)$ are of the form:

$$\begin{bmatrix} f_{n,-1}(r) \\ g_{n,-1}(r) \end{bmatrix} = r^{\gamma_0 - 1} \exp\left(-\frac{Zr}{N_i}\right) \begin{bmatrix} F_{00} + F_{01} r \\ G_{00} + G_{01} r \end{bmatrix} , \tag{20}$$

where $\gamma_0 = \sqrt{1 - (Z\alpha)^2}$, $N_i = 1, \sqrt{2(1+\gamma_0)}$ for $n = 1, 2$, respectively, and the coefficients F_0, F_1, G_0, G_1 are given, in accordance with our definition (11), by [4]

$$\begin{pmatrix} F_{00} & F_{01} \\ G_{00} & G_{01} \end{pmatrix} = \frac{2^{\gamma_0} Z^{\gamma_0 + 1/2}}{\sqrt{(1+\gamma_0)\Gamma(2\gamma_0+1)}} \begin{pmatrix} 1+\gamma_0 & 0 \\ -Z & 0 \end{pmatrix} , \qquad \text{if } n = 1 \tag{21}$$

$$\begin{pmatrix} F_{00} & F_{01} \\ G_{00} & G_{01} \end{pmatrix} = \frac{2^{\gamma_0 - 1/2} Z^{\gamma_0 + 1/2} N_i^{-5/2-\gamma_0}}{\sqrt{(N_i+1)(1+\gamma_0+N_i)\Gamma(2\gamma_0+2)}} \tag{22}$$

$$\times \begin{pmatrix} N_i^2(2\gamma_0+1)(1+\gamma_0+N_i) & -2Z(1+\gamma_0+N_i)(N_i+1) \\ -ZN_i(N_i+2)(2\gamma_0+1) & 2Z^2(N_i+1) \end{pmatrix} , \quad \text{if } n = 2 .$$

As will become clear from our treatment, a generalization of the final expressions (see (52)–(53)) on $n \geq 3$ appears to be straightforward. Appropriate results will be the subject of a separate publication.

2.2 Solutions to Equations (15a)–(16b)

Equations (15a)–(16b) can be uncoupled using the substitutions:

$$\xi_1(r) = \varrho^{\gamma_0} e^{-\varrho/2} \, (P_1(\varrho) + Q_1(\varrho)) \tag{23a}$$

$$\eta_1(r) = -\frac{\varepsilon}{\lambda} \varrho^{\gamma_0} \, e^{-\varrho/2} \, (P_1(\varrho) - Q_1(\varrho)) \tag{23b}$$

$$\xi_2(r) = \varrho^{\gamma_1} e^{-\varrho/2} \, (P_2(\varrho) + Q_2(\varrho)) \tag{24a}$$

$$\eta_2(r) = -\frac{\varepsilon}{\lambda} \varrho^{\gamma_1} \, e^{-\varrho/2} \, (P_2(\varrho) - Q_2(\varrho)) \ , \tag{24b}$$

where

$$\varrho = 2\lambda r \, , \qquad \lambda = \sqrt{-\varepsilon \, (2 + \alpha^2 \varepsilon)} \, , \qquad \gamma_1 = \sqrt{4 - (Z\alpha)^2} \, . \tag{25}$$

Some details of the derivation are provided below only for the system (16a)–(16b), since for (15a)–(15b) these are quite analogous. Final formulas to determine $\xi_1(\varrho)$ and $\eta_1(\varrho)$ are constructed from that for $\xi_2(\varrho)$ and $\eta_2(\varrho)$ by setting $\gamma_3 = 0$ and a trivial change of notations.

A substitution (24a)–(24b) for $\xi_2(r), \eta_2(r)$ in (16a)–(16b) yields the following coupled system of equations for $P_2(\varrho)$ and $Q_2(\varrho)$:

$$\varrho P_2'(\varrho) - \varrho Q_2'(\varrho) + \left(\frac{Z\lambda}{\varepsilon} + \gamma_1 + 2\right) P_2(\varrho) + \left(\varrho + \frac{Z\lambda}{\varepsilon} - \gamma_1 - 2\right) Q_2(\varrho) \quad (26)$$
$$= \frac{2G_{00}\lambda + G_{01}\varrho}{2^{\gamma_0+2}\lambda^{\gamma_0+1}\varepsilon}\, \varrho^{1-\gamma_3} \exp\left(\frac{(\lambda N_i - Z)}{2\lambda N_i}\, \varrho\right)$$
$$\varrho P_2'(\varrho) + \varrho Q_2'(\varrho) + \left(\gamma_1 - 2 - \frac{\varepsilon Z\alpha^2}{\lambda}\right) P_2(\varrho)$$
$$+ \left(-\varrho + \frac{\varepsilon Z\alpha^2}{\lambda} + \gamma_1 - 2\right) Q_2(\varrho) = 0\,, \quad (27)$$

where $\gamma_3 = \gamma_1 - \gamma_0 > 1$. By adding and subtracting (26)–(27) term by term one finally ends up with the following uncoupled system of inhomogeneous differential equations of the hypergeometric type:

$$\varrho P_2''(\varrho) + (a_2 - \varrho)\, P_2'(\varrho) - b_2 P_2(\varrho) = \varrho^{-\gamma_3} e^{\beta\varrho} \left(C_2^{(2)}\varrho^2 + C_1^{(2)}\varrho + C_0^{(2)}\right) \quad (28)$$
$$\varrho Q_2''(\varrho) + (a_2 - \varrho)\, Q_2'(\varrho) - (b_2 + 1) Q_2(\varrho) \quad (29)$$
$$= \varrho^{-\gamma_3} e^{\beta\varrho} \left(B_2^{(2)}\varrho^2 + B_1^{(2)}\varrho + B_0^{(2)}\right)\,. \quad (30)$$

Here,

$$a_2 = 1 + 2\gamma_1\,, \quad (31)$$
$$b_2 = \gamma_1 - \frac{Z(1+\alpha^2\varepsilon)}{\lambda}\,, \quad (32)$$
$$\beta = \frac{\lambda N_i - Z}{2\lambda N_i}\,, \quad (33)$$
$$C_2^{(2)} = -\frac{(Z + \lambda N_i) G_{01}}{N_i \varepsilon 2^{\gamma_0+4}\lambda^{\gamma_0+2}}\,, \quad (34a)$$
$$C_1^{(2)} = \frac{N_i(\varepsilon Z\alpha^2 + \lambda\gamma_0)G_{01} - \lambda(Z + \lambda N_i)G_{00}}{N_i \varepsilon 2^{\gamma_0+3}\lambda^{\gamma_0+2}}\,, \quad (34b)$$
$$C_0^{(2)} = \frac{\big(\varepsilon Z\alpha^2 - \lambda(1-\gamma_0)\big)G_{00}}{N_i \varepsilon \lambda^{\gamma_0+1} 2^{\gamma_0+2}}\,, \quad (34c)$$
$$B_2^{(2)} = \frac{(Z - \lambda N_i) G_{01}}{N_i \varepsilon 2^{\gamma_0+4}\lambda^{\gamma_0+2}}\,, \quad (35a)$$
$$B_1^{(2)} = \frac{N_i(\varepsilon Z\alpha^2 - \lambda\gamma_0)G_{01} + \lambda(Z - \lambda N_i)G_{00}}{N_i \varepsilon 2^{\gamma_0+3}\lambda^{\gamma_0+2}}\,, \quad (35b)$$
$$B_0^{(2)} = \frac{\big(\varepsilon Z\alpha^2 + \lambda(1-\gamma_0)\big)G_{00}}{\varepsilon \lambda^{\gamma_0+1} 2^{\gamma_0+2}}\,. \quad (35c)$$

Similar algebra shows that the corresponding equations obeyed by $P_1(\varrho)$ and $Q_1(\varrho)$ can be inferred from (28)–(29) by means of the substitutions:

$$\gamma_1 \to \gamma_0\,, \quad a_2 \to a_1\,, \quad b_2 \to b_1\,, \quad C_k^{(2)} \to C_k^{(1)}\,, \quad B_k^{(2)} \to B_k^{(1)}\,,$$

so that

$$\gamma_3 = \gamma_1 - \gamma_0 \to 0 \,, \tag{36}$$

$$a_1 = 2\gamma_0 + 1 \,, \tag{37}$$

$$b_1 = \gamma_0 - \frac{Z(1+\alpha^2\varepsilon)}{\lambda} \,, \tag{38}$$

and the coefficients $C_k^{(1)},\ k = 0, 1, 2$, are of the form:

$$C_2^{(1)} = -\frac{(Z+\lambda N_i)(3\varepsilon F_{01} - \lambda G_{01})}{N_i \varepsilon 2^{\gamma_0+4}\lambda^{\gamma_0+3}} \,, \tag{39a}$$

$$C_1^{(1)} = -\frac{1}{N_i \varepsilon 2^{\gamma_0+3}\lambda^{\gamma_0+2}}\Big[N_i\big(\varepsilon Z\alpha^2 + \lambda(3+\gamma_0)\big)G_{01} + (\lambda N_i + Z)(3\varepsilon F_{00} - \lambda G_{00}) - 3N_i\big(\varepsilon(\gamma_0+1) - \lambda Z\big)F_{01}\Big] \,, \tag{39b}$$

$$C_0^{(1)} = -\frac{3(\lambda Z - \varepsilon\gamma_0)F_{00} + \big(\varepsilon Z\alpha^2 + \lambda(2+\gamma_0)\big)G_{00}}{\varepsilon\lambda^{\gamma_0+1}2^{\gamma_0+2}} \,, \tag{39c}$$

$$B_2^{(1)} = \frac{(\lambda N_i - Z)(3\varepsilon F_{01} + \lambda G_{01})}{N_i \varepsilon 2^{\gamma_0+4}\lambda^{\gamma_0+3}} \,, \tag{40a}$$

$$B_1^{(1)} = -\frac{1}{N_i \varepsilon 2^{\gamma_0+3}\lambda^{\gamma_0+2}}\Big[N_i\big(\varepsilon Z\alpha^2 - \lambda(3+\gamma_0)\big)G_{01} + (Z - \lambda N_i)(3\varepsilon F_{00} + \lambda G_{00}) - 3N_i\big(\varepsilon(\gamma_0+1) + \lambda Z\big)F_{01}\Big] \,, \tag{40b}$$

$$B_0^{(1)} = \frac{3(\lambda Z + \varepsilon\gamma_0)F_{00} + \big(\lambda(2+\gamma_0) - \varepsilon Z\alpha^2\big)G_{00}}{\varepsilon\lambda^{\gamma_0+1}2^{\gamma_0+2}} \,. \tag{40c}$$

To choose the proper solutions uniquely, $P_k(\varrho), Q_k(\varrho),\ k = 1, 2$ must be subject to the following boundary conditions:

$$P_k(\varrho), Q_k(\varrho) = \begin{cases} o\left(\varrho^{-1-\gamma_0-\gamma_{k-1}}\right) \,, & \text{if}\, \varrho \to 0 \\ o\left\{\exp\left(\frac{\lambda N_i + Z}{2\lambda N_i}\varrho\right)\right\} \,, & \text{if}\, \varrho \to \infty \end{cases} \,. \tag{41}$$

As in the nonrelativistic case [57,58], (41) follows directly from (23a)–(24b) and a natural requirement that (19) should remain finite *for all* $\varepsilon \neq \varepsilon_n$. Here, $1/\alpha^2 + \varepsilon_n$ are the eigenvalues of the Hamiltonian $\widehat{H}_0$, i.e. the values of E whereupon the poles of the resolvent in (8) occur. This argument as well as (41) are consistent with a pair of orthogonality relations resulting directly from (15a)–(16b):

$$\int_0^\infty r\big(\xi_1(r)f_{n,1}(r) - \alpha^2\eta_1(r)g_{n,1}(r)\big)\mathrm{d}r = (\varepsilon_{n,1} - \varepsilon)^{-1}\int_0^\infty r^2\big(3f_{n,-1}(r)g_{n,1}(r) + g_{n,-1}(r)f_{n,1}(r)\big)\mathrm{d}r \,, \tag{42}$$

$$\int_0^\infty r\big(\xi_2(r)f_{n,-2}(r) - \alpha^2\eta_2(r)g_{n,-2}(r)\big)\mathrm{d}r \qquad (43)$$
$$= (\varepsilon - \varepsilon_{n,-2})^{-1}\int_0^\infty r^2 g_{n,-1}(r)f_{n,-2}(r)\mathrm{d}r\ .$$

Here, $f_{n,1}(r), g_{n,1}(r)$ and $f_{n,-2}(r), g_{n,-2}(r)$ are, respectively, the radial parts of the "big" and "small" components of the Coulomb Dirac wave functions with $j = 1/2$ and $j = 3/2$; $\varepsilon_{n,1}$ and $\varepsilon_{n,-2}$ are the energies of these states [4].

Due to the apparent similarity in the mathematical structure of both equation (28)–(29), it is only the first one that needs to be considered. The solution $Q_2(\varrho)$ can be obtained then from $P_2(\varrho)$ by the trivial changes: $b_2 \to b_2 + 1$, $C_k^{(2)} \to B_k^{(2)}$. Similar replacements have to be applied in order to generate $P_1(\varrho)$ and $Q_1(\varrho)$ as well. To simplify notations, $P_2(\varrho)$, a_2, b_2, and $C_k^{(2)}, k = 0\ldots 2$, are denoted below as simply $P(\varrho)$, a, b, and C_k.

The general solution to (28) has the form:

$$P(\varrho) = D_1\Phi(b,a;\varrho) + D_2U(b,a;\varrho) + P_0(\varrho)\ ,$$

where $\Phi(b,a;\varrho)$ and $U(b,a;\varrho)$ are the regular and irregular Kummer functions [66], and $P_0(\varrho)$ denotes a particular solution to the inhomogeneous hypergeometric equation (28). Together with the constants D_1 and D_2, $P_0(\varrho)$ must be chosen so as to satisfy (41) for the entire solution. Using asymptotic representations for the confluent hypergeometric functions [66], it is easy to show that the required behavior at $\varrho \to \infty$ cannot be met unless we set $D_1 = 0$ in order to remove $\Phi(b,a;\varrho) \propto e^{\varrho}$ from $P(\varrho)$. This leaves us with the general solution,

$$P(\varrho) = D_2U(b,a;\varrho) + P_0(\varrho)\ , \qquad (44)$$

involving only one indeterminate constant D_2. To this end we note that any particular solution $P_0(\varrho)$ to (28) is related to $P_{\mathrm{st}}(\varrho,\beta)$, a solution to the equation with a "standard" right-hand side,

$$\varrho P''_{\mathrm{st}}(\varrho,\beta) + (a-\varrho)\,P'_{\mathrm{st}}(\varrho,\beta) - bP_{\mathrm{st}}(\varrho,\beta) = \varrho^{-\gamma_3}e^{\beta\varrho}\ , \qquad (45)$$

as

$$P_0(\varrho) = \sum_{k=0}^{2} C_k\frac{\partial^k P_{\mathrm{st}}(\varrho,\beta)}{\partial\beta^k}\ . \qquad (46)$$

The extra argument is explicitly introduced in $P_{\mathrm{st}}(\varrho,\beta)$ to indicate its dependence on the parameter β (33), the exponent in the right-hand sides of (28) and (45). Accordingly, we will first solve (45) and then use (46) to construct $P_0(\varrho)$ and to fix D_2 in (44).

We seek $P_{\mathrm{st}}(\varrho,\beta)$ in the form

$$P_{\mathrm{st}}(\varrho,\beta) = \frac{1}{2\pi\mathrm{i}}\oint_{C'} q(t)e^{\varrho t}\mathrm{d}t \qquad (47)$$

where the integral is taken along some path C' in the complex t-plane such that $q(t)$ returns back to its initial value after passing along C'. The equation for $q(t)$,

$$t(1-t)q'(t) + \big(1-b+t(a-2)\big)q(t) = \Gamma(1-\gamma_3)(t-\beta)^{\gamma_3-1}\,, \qquad (48)$$

can be deduced by plugging (47) into (45) and using the Hankel's contour integral representation for the Γ-function [66] to represent the right-hand side of (45):

$$\varrho^{-\gamma_3}e^{\beta\varrho} = \frac{\Gamma(1-\gamma_3)}{2\pi\mathrm{i}}\oint_{C_\mathrm{H}} e^{\varrho t}(t-\beta)^{\gamma_3-1}\mathrm{d}t\,.$$

Here, the path C_H runs from $t=-\infty$ along the lower edge of the cut $t\in(-\infty,\beta]$, encircles the point $t=\beta$ in the counterclockwise sense, and runs back to $t=-\infty$ along the upper edge of the same cut. The contour C' in (47) was assumed to be topologically equivalent to this path C_H. This is permissible because C_H complies with all above requirements imposed of C' and enables, at the same time, (41) to be satisfied by solutions obtained thereby (see (52)–(53) below).

Equation (48) can be readily solved to result, after setting the appropriate integration constant equal to 0, in

$$P_\mathrm{st}(\varrho,\beta) = -\frac{\Gamma(1-\gamma_3)}{2\pi\mathrm{i}}\oint_{C_\mathrm{H}} e^{\varrho t}t^{b-1}(1-t)^{a-b-1}\mathrm{d}t \qquad (49)$$
$$\times\int_t^\infty s^{-b}(1-s)^{b-a}(s-\beta)^{\gamma_3-1}\mathrm{d}s$$

where the integral $\int_t^\infty\ldots\mathrm{d}t$ is implied to be taken along any path in the s-plane connecting $s=t\neq\{0,1\}$ with $s=\infty$. For the contour C_H in question, $P_\mathrm{st}(\varrho,\beta)$ of (49) can be simplified further. This is achieved by formally reducing the contour integrals to the ordinary ones and interchanging the order of integration:

$$P_\mathrm{st}(\varrho,\beta) = \frac{1}{\Gamma(\gamma_3)}\int_{-\beta}^\infty s^{-b}(1+s)^{b-a}(s+\beta)^{\gamma_3-1}\mathrm{d}s \qquad (50)$$
$$\times\int_s^\infty e^{-\varrho t}t^{b-1}(1+t)^{a-b-1}\mathrm{d}t\,.$$

To enforce the boundary condition (41) at $\varrho\to 0$ and/or (42)–(43) for the general solution (44)–(45), we must set in (44)

$$D_2 = -\frac{\Gamma(b)}{\Gamma(\gamma_3)}\int_{-\beta}^\infty s^{-b}(1+s)^{b-a}(s+\beta)^{\gamma_3-1}\mathrm{d}s\,. \qquad (51)$$

This choice is unique, and it can be readily established by removing the singular part of $P_\mathrm{st}(\varrho,\beta)$ in (50) using the integral representation for the function $U(b,a;\varrho)$ [66]. By rewriting the resultant expression again as a contour integral, we finally get ($a\equiv a_2$, $b\equiv b_2$):

$$P_2(\varrho) = \frac{\Gamma(1-\gamma_3)}{2\pi\mathrm{i}}e^{-\mathrm{i}\pi\gamma_3}\left(\sum_{k=0}^2 C_k^{(2)}\frac{\partial^k}{\partial\beta^k}\right) \qquad (52)$$
$$\times\oint_C s^{-b}(1+s)^{b-a}(s+\beta)^{\gamma_3-1}\mathrm{d}s\int_0^s e^{-\varrho t}t^{b-1}(1+t)^{a-b-1}\mathrm{d}t\,,$$

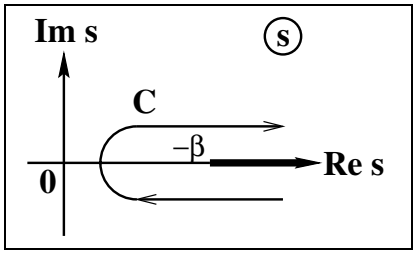

Fig. 2. Path of integration in (52)

where the path C is shown in Fig. 2. As was noted above, $Q_2(\varrho)$ is obtainable by setting $C_k^{(2)} = B_k^{(2)}$. Similarly, expressions for $P_1(\varrho)$, $Q_1(\varrho)$ can be established by putting, respectively, $C_k^{(2)}$ equal to $C_k^{(1)}$ or $B_k^{(1)}$, $a = a_1$, $b = b_1$, and $\gamma_3 = 0$. The latter condition enables (52) to be simplified further, since the contour integral reduces in this case to $(-2\pi\mathrm{i})$ times the residue of the integrand at the simple pole $s = -\beta$. The final result for $P_1(\varrho)$ reads:

$$P_1(\varrho) = -\left(\sum_{k=0}^{2} C_k^{(1)} \frac{\partial^k}{\partial \beta^k}\right) \left\{(-\beta)^{-b_1}(1-\beta)^{b_1-a_1} \times \int_0^{-\beta} e^{-\varrho t} t^{b_1-1}(1+t)^{a_1-b_1-1}\mathrm{d}t\right\} . \tag{53}$$

So far it was implicitly assumed that $\Re(b_1), \Re(b_2) > 0$, in which case no modifications of (52)–(53) are necessary. An analytic continuation of these formulas on $\Re(b_1), \Re(b_2) < 0$ is effectuated by the replacement:

$$\int_0^s e^{-\varrho t} t^{b-1}(1+t)^{a-b-1}\mathrm{d}t \longrightarrow \left(1 - e^{2\pi\mathrm{i}b}\right)^{-1} \oint_{0^-} e^{-\varrho t} t^{b-1}(1+t)^{a-b-1}\mathrm{d}t \, , \tag{54}$$

where the contour starts at $t = s$ lying on the lower edge of the cut $[0, +\infty)$, encircles $t = 0$ in the clockwise sense, and ends at $t = s$ lying on the upper edge of the same cut. The nonrelativistic limits of (52)–(53) formerly obtained in [58,57] can be retrieved by taking $\gamma_3 \to 1$, calculating analytically the derivatives with respect to β, and finally putting $\alpha = 0$ for all variables involved.

2.3 Relativistic Dipole Dynamic Polarizabilities

In this section, (23a)–(24b) together with (52)–(53) and appropriate formulas for $Q_1(\varrho)$ and $Q_2(\varrho)$ are applied to calculate RDDP of the $1s_{1/2}$- and $2s_{1/2}$-levels. The use of (10) and (19) make these calculations quite straightforward, requiring only a one-fold trivial integration over ϱ. Appropriate details will be published

elsewhere, and we present here only the final results. For the $1s_{1/2}$-state ($n'=1$), for example, the part of (19) involving ξ_1, η_1 can be cast into the form:

$$\int_0^\infty r\big(3\eta_1(r)f_{1,-1}(r) - \xi_1(r)g_{1,-1}(r)\big)\mathrm{d}r \tag{55}$$
$$= A\,{}_2F_1\left(a_1+1, b_1+1; b_1+2; \left(\frac{\lambda-Z}{\lambda+Z}\right)^2\right) + D\,,$$

where ${}_2F_1$ is the hypergeometric function [4] and A, D are certain constants, such that in the nonrelativistic limit (55) reduces to

$$384\frac{\nu^5}{(1+\nu)^8(\nu-2)}\,{}_2F_1\left(4, 2-\nu; 3-\nu; \left(\frac{1-\nu}{1+\nu}\right)^2\right) + \frac{9}{2}\,. \tag{56}$$

The following parameterization is assumed here:

$$\varepsilon = \varepsilon_{1s_{1/2}} + \omega = -\frac{Z^2}{2\nu^2}\left(\nu^2 + \frac{1-\gamma_0}{1+\gamma_0}\right)\,, \quad \omega \equiv \frac{Z^2(\nu^2-1)}{2\nu^2}\,, \tag{57}$$

so that $\nu_0 = 1$ and $\nu_I = \mathrm{i}\sqrt{2}/(1-\gamma_0)$ correspond to the static limit ($\omega = 0$) and the exact ionization threshold ($\omega = Z^2/(1+\gamma_0)$). Note that if $\alpha \to 0$, then $\nu_I \to \infty$, $\varepsilon \to -Z^2/2$, and $\omega \to Z^2/2$, the nonrelativistic ionization potential. The expression for $\varepsilon_{1s_{1/2}} - \omega$ is obtained by replacing ν in (57) with $\nu/\sqrt{2\nu^2-1}$ so that $\omega \to -\omega$. The corresponding result for the $2s_{1/2}$-state can be recovered from (55) by changing the constants A, D appropriately and using the substitution:

$$\frac{\lambda-Z}{\lambda+Z} \longrightarrow \frac{\lambda\sqrt{2(1+\gamma_0)}-Z}{\lambda\sqrt{2(1+\gamma_0)}+Z} \underset{\alpha\to 0}{\longrightarrow} \frac{2-\nu}{2+\nu}\,, \qquad \omega \equiv \frac{Z^2(\nu^2-4)}{8\nu^2}\,.$$

Although the ξ_2-depending part of (19) appears to be not so compact as (55), yet its form is very tight as well. For the $1s_{1/2}$-state, for example, appropriate contribution can be expressed as:

$$8\int_0^\infty r\xi_2(r)g_{1,-1}(r)\mathrm{d}r \tag{58}$$
$$= \frac{4G_{00}}{(2\lambda)^{\gamma_0}(2\lambda+Z)}\left[(1-\beta)I_1 + \left(1-\gamma_0-\frac{Z\alpha^2\varepsilon}{\lambda}\right)I_0\right]$$
$$-\frac{\Gamma(2\gamma_0+2)G_{00}^2}{\varepsilon(2\lambda)^{2\gamma_0}(2\lambda+Z)(1-2\beta)^{2\gamma_0+2}}\,, \tag{59}$$

where $I_k, k=0,1$, are defined as

$$I_k \equiv -\frac{\Gamma(\gamma_0+\gamma_1+1+k)}{\Gamma(\gamma_1-\gamma_0)}(1-\beta)^{-\gamma_1-\gamma_0-1-k}\left(C_1^{(2)}\frac{\partial}{\partial\beta} + C_0^{(2)}\right)$$
$$\times\int_{-\beta}^\infty (1+s)^{-a_2}(s+\beta)^{\gamma_1-\gamma_0-1}\mathrm{d}s \tag{60}$$
$$\times\int_0^1 t^{b_2-1}\left(1-\frac{st}{1+s}\right)^{\gamma_0-\gamma_1+k}\left(1+\frac{\beta st}{(1-\beta)(1+s)}\right)^{-\gamma_0-\gamma_1-1-k}\mathrm{d}t\,.$$

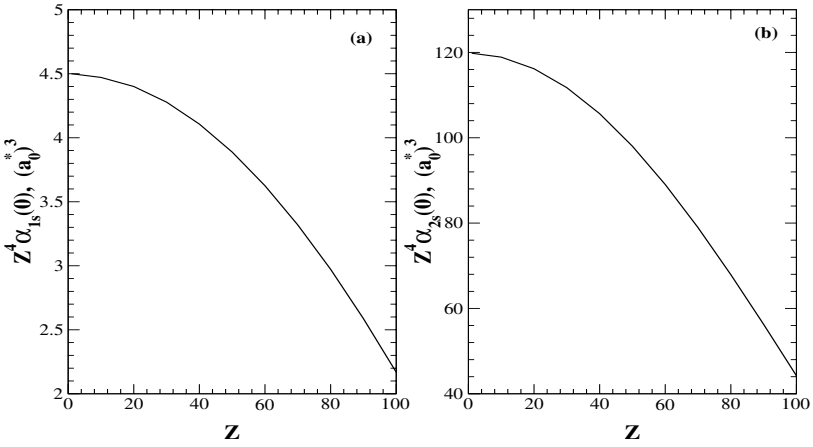

Fig. 3. Static RDP of the hydrogenic $1s_{1/2}$ – **(a)** – and $2s_{1/2}$-state – **(b)**

If desired, the two-fold integral here can be further reduced to a one-fold one by introducing the hypergeometric function F_1 of two variables [67]. Expression (58) is already convenient, however, both for the investigation of general properties as well as for the accurate numerical evaluation. In particular, the nonrelativistic limit of (58) reads:

$$768\frac{\nu^5}{(1+\nu)^8(\nu-2)}\,{}_2F_1\left(4, 2-\nu; 3-\nu; \left(\frac{1-\nu}{1+\nu}\right)^2\right)\,, \tag{61}$$

to result eventually, after adding with (56) and multiplying by $1/(9\omega^2)$, in the well known expression first obtained by Gavrila [68] for the DDP of the hydrogenic ground state. In addition, by expanding all quantities in powers of $(\alpha Z)^2$ and taking the limit $\omega \to 0$, we retrieve the following expansion for the static RDP of the $1s_{1/2}$-level originally derived in [30] (conventional units):

$$\alpha_{1s_{1/2}}(0) = \frac{9(a_0^*)^3}{2Z^4}\left(1 - \frac{28}{27}(\alpha Z)^2 + \frac{2\pi^2+31}{432}(\alpha Z)^4 + \ldots\right)\,. \tag{62}$$

The $2s_{1/2}$-state, for which the counterpart of (62) is currently unknown, can be treated similarly. The new results of these calculations will be published elsewhere. We close our discussion by presenting in Fig. 3 numerical results of our calculations of the static RDP of the $1s_{1/2}$- and $2s_{1/2}$-levels for nucleus charges $Z = 1 \ldots 100$. For the $1s_{1/2}$-state our data are in excellent agreement with the RDP calculations by Szmytkowski [34] and Le Anh Thu et al. [52,53] using relativistic Sturmian expansion and the (algebraic) Coulomb Green's function. This sensitive test strongly supports the utility of our current approach which can be efficiently used for much more sophisticated numerical calculations. In particular, the formulas derived above have been employed in our study to simulate the $2+1$ resonant photoionization of muonium by a CW laser signal of high

intensity, where the photoionization widths of lowest $ns_{1/2}$-levels are needed to be calculated (see paper [25] by V. Yakhontov and K. Jungmann in this volume).

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