Electron $g - 2$ and High Precision Determination of α

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Recent progress in the Penning trap measurement of the magnetic moment anomaly a of the electron and positron has enabled Dehmelt and coworkers to determine a to a precision of 4×10^{-9}, providing the strongest challenge to date to the validity of QED. Calculation of a up to the order α^4, where α is the fine structure constant, has now reached the point where the intrinsic theoretical uncertainty is comparable to that of the measurements. Unfortunately rigorous test of QED itself must be postponed until a better value of α becomes available. Pending improved measurement of α, however, one can determine α from theory and the experimental value of a to a precision better than 1×10^{-8}, which is much more accurate than the α's determined from the ac Josephson effect, the quantized Hall effect, or the hyperfine structure of the muonium ground state.

1. Review of Previous Work

My involvement in the study of the lepton magnetic moment anomaly began in 1966 when, inspired by the beautiful muon $g - 2$ experiment then in progress at CERN, I developed a method for determining the coefficients of the $\ln(m_\mu/m_e)$ terms in the α^3 contribution to the muon anomaly [1]. This was the first practical application of the mass singularity theorem [2] and the renormalization group technique. Eventually this work evolved into a full scale numerical evaluation of the α^3 terms of the muon and electron anomalies. It was completed by 1974 when our results became more precise than the experimental data then available [3]. The superiority of theory over experiment, however, was short-lived. It was shattered by the remarkable breakthrough by Dehmelt and coworkers who succeeded in measuring a of an individual electron suspended in a Penning trap [4]. Their precision demanded knowledge of the α^4 term. Our work on this term started in 1977 [5]. Only now it is coming to a conclusion. In this talk I will concentrate on the electron anomaly.

Before describing the present status of the theory, let me first review the previous results. The QED prediction for a can be written as a power series in α/π:

$$a_e(\text{QED}) = C_1(\alpha/\pi) + C_2(\alpha/\pi)^2 + C_3(\alpha/\pi)^3 + C_4(\alpha/\pi)^4 + \ldots\,, \qquad (1)$$

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where C_1 and C_2 are known analytically and given by [6]

$$C_1 = 0.5\ ,$$

$$C_2 = -\ 0.328\ 478\ 965\ \ldots\ . \tag{2}$$

Analytic evaluation of C_3 (which consists of 72 Feynman diagrams) is not yet complete [7] and its value

$$C_3 = 1.176\ 5\ (13) \tag{3}$$

depends partly on numerical integration [3, 5], where the error represents the estimated accuracy (90% confidence limit) of the numerical integration.

As was mentioned already, one must also calculate C_4 to match the high precision of experimental results [4]. This is a formidable task requiring the evaluation of 891 Feynman diagrams. It is made somewhat easier by a technique developed in [3] by which Feynman diagrams having similar structure are combined into one with the help of the Ward-Takahashi identity. Together with time reversal and charge conjugation symmetries this reduces the number of integrals to be separately evaluated to 86 as is indicated below. The diagrams may naturally be classified into five groups:

Group I. Second-order vertex diagrams containing vacuum polarization loops of up to sixth order. This group consists of 25 diagrams. They are reduced to 10 independent integrals by time reversal and charge conjugation symmetries.

Group II. Fourth-order vertex diagrams containing second and fourth order vacuum polarization loops. This group contains 54 diagrams (reduced to 8 integrals by the Ward-Takahashi identity and time reversal and charge conjugation symmetries).

Group III. Sixth-order vertex diagrams containing a second order vacuum polarization loop. There are 150 diagrams (reduced to 8 integrals) in this group.

Group IV. Vertex diagrams containing a photon-photon scattering subdiagram with further radiative corrections. This group consists of 144 diagrams (reduced to 13 integrals).

Group V. Vertex diagrams containing no vacuum polarization loop. This group is comprised of 518 diagrams (reduced to 47 integrals).

Because of the enormous complexity only a handful of these diagrams have been evaluated analytically thus far [8]. We have adopted a purely numerical approach [5]. Each (combined) Feynman amplitude is represented by an integral over a multi-dimensional space. Integrands are generated by the algebraic program SCHOONSCHIP [9]. The integration, over a hypercube of up to 10 dimensions, is carried out using the adaptive Monte Carlo integration routines VEGAS [10] and RIWIAD [11].

A typical integrand of the first three groups is a rational function consisting of up to 2,000 terms, each term being a product of up to 8 or 9 factors. Its FORTRAN source code has size of up to 50 kilobytes. Numerical evaluation of these integrals is relatively straightforward and was completed in 1979. The results are [12]

$$C_4^{I} = 0.076\,6\ (6)\ ,$$

$$C_4^{II} = -\ 0.523\,8\ (10)\ ,$$

$$C_4^{III} = 1.419\ (16)\ . \tag{4}$$

By 1981 we had written and debugged the FORTRAN source codes for all integrals in groups IV and V. These groups require much larger code (100 to 500 kilobytes), and numerical integration is much more difficult and time-consuming. Due to the limited computing power then available it was not possible to explore these integrals thoroughly. Nevertheless we managed to show that our program worked as expected, and obtained very crude and preliminary results [12]

$$C_4^{IV} = -\ 0.78\ (48)\ ,$$

$$C_4^{V} = -\ 1.0\ (2.4)\ . \tag{5}$$

From (4) and (5) one finds

$$C_4 = -\ 0.8\ (2.5)\ . \tag{6}$$

If one uses, for example, the value of the (inverse) fine structure constant α obtained by analyzing several pre-1986 quantized Hall effect measurements [13]

$$\alpha^{-1} = 137.035\,994\,3\ (127) \qquad (0.093\ \text{ppm})\ , \tag{7}$$

the QED predictions (2), (3), and (6) lead to

$$a(\text{QED}) = 1\,159\,652\,188\ (74)(108) \times 10^{-12}\ . \tag{8}$$

To this one must add contributions from other known sources. They include the contributions of the muon loop, τ meson loop, and hadronic effect, as well as the effect of the weak interaction (in the standard Weinberg-Salam-Glashow model) [14]:

$$a(\text{muon}) = 2.8 \times 10^{-12} ,$$

$$a(\tau \text{ meson}) = 0.01 \times 10^{-12} ,$$

$$a(\text{hadron}) = 1.6(2) \times 10^{-12} ,$$

$$a(\text{weak}) = 0.05 \times 10^{-12} . \tag{9}$$

Adding (9) to (8) we arrive at the theoretical prediction

$$a(\text{theory}) = 1\,159\,652\,192\,(74)(108) \times 10^{-12} , \tag{10}$$

where the first error is from theory while the second is due to the measurement error of α in (7).

The value (10) is in good agreement with the latest measurements of a for the electron and positron [15]:

$$a(e^-) = 1\,159\,652\,188.4\,(4.3) \times 10^{-12} ,$$

$$a(e^+) = 1\,159\,652\,187.9\,(4.3) \times 10^{-12} , \tag{11}$$

where the experimental error arises from several sources:

$$\text{statistical error} = 0.62 \times 10^{-12} ,$$

$$\text{error due to microwave power shift} = 1.3 \times 10^{-12} ,$$

$$\text{error due to cavity shift} = 4 \times 10^{-12} . \tag{12}$$

Clearly the cavity shift correction, which results from the radiative interaction of an electron with the metallic walls of the Penning trap that surrounds it and exhibits a complicated resonance behavior, is the largest source of uncertainty at present [16].

The value of C_4 in (6) is rather crude because of very limited integrand sampling, even though it represents an outcome of more than 300 hours of computing on a CDC-7600. The primary significance of the result (6) is not in its precision but in the establishment of bounds on C_4, namely the confirmation that the renormalization of QED in fact gives a convergent result to order α^4, a nontrivial, even though expected, result in view of several thousands of divergent terms that must cancel out completely.

In spite of its crudeness the intrinsic theoretical error given in (10) is smaller than that of the α used in the calculation. This means that the study of a provides a very powerful tool for obtaining a highly accurate value of α. Indeed we find from the experimental result (11) and theory that

$$\alpha^{-1}(g - 2) = 137.035\ 994\ 2\ (89) \quad (0.065\ \text{ppm})\ , \tag{13}$$

which is more accurate than (7).

2. Recent Developments

In order to obtain accurate and statistically reliable results it was estimated that at least 100 times more computation (more than 30000 hours on CDC-7600) was required, which was prohibitively expensive at that time. It is only in the last few years that such a calculation has become feasible in terms of time and cost. This is the primary reason why we have not been able to finish our work more quickly.

Before starting such an extensive computation, I felt that it was necessary to make sure that all of the FORTRAN code was completely free of errors. Thus in 1984-85 I generated new code from scratch in a form different from the original, and tested against each other numerically. I now have complete confidence in the entire program.

One might think that the remaining task is simply a matter of number crunching. Unfortunately it is more complicated than that for several reasons. One is the sheer size of the integrands. In order that an integration routine give a reliable result the number N of randomly chosen points where the integrand is sampled in each iteration must be sufficiently large. Otherwise the integration routine may not be able to explore the integrand closely, resulting in deceptively optimistic error estimates. In the early evaluation of the integrals we were unable to choose N larger than 120,000. Our subsequent work indicated that this N was far too small for some integrals of Group IV and most integrals of Group V. It is only when we ran our programs in vectorized form on the HITAC S-810 at Tokyo University and subsequently at KEK, with N in the range of 4- to 20 million, that we were finally able to confirm the $N^{-1/2}$ behavior expected for statistically satisfactory samplings.

Another problem we encountered is deeply rooted in our particular approach to the removal of ultraviolet and infrared divergences [17]. In our method, terms of the integrand generated by SCHOONSCHIP actually diverge on some boundaries of the integration domain and the integral is kept finite by point-by-point cancellation of divergences by carefully tailored counter terms. Because of the insufficient numerical precision of double precision arithmetic, however, round-off errors sometimes force a breakdown of divergence cancellation mechanism, causing undesirable fluctuations. Fortunately this problem can be reduced to a manageable level by switching to quadruple precision arithmetic, which slows down the (unvectorized) computation by a factor of 5 to 6. In practice quadruple precision is needed only in the neighborhood of singularities. One can evaluate the bulk of the integral in double precision resorting to quadruple precision only where it is absolutely needed.

Further complication arose when we tried to check the renormalization procedure of some diagrams of Group IV (which contain an internal light-by-light scattering subdiagram) by comparing it with a parallel scheme which, while keeping Pauli-Villars regularization to insure convergence, avoids explicit renormalization taking advantage of an identity derived from current conservation. What we found after a very extensive and time-consuming numerical experiment was that VEGAS was not able to handle this scheme adequately when regulator masses were too large. However, the results obtained using smaller regulator masses were good enough to convince us that the renormalization was correctly implemented in these diagrams.

Our calculation of C_4 is now close to completion, although further refinement and consistency check are being made on several integrals of Group V. Actually the error on C_4 has already been reduced to less than that on C_3, forcing us to do further work on C_3. The present (not yet final) values are

$$C_3 = 1.175\ 62\ (56)\ ,$$

$$C_4 = -\ 1.472\ (152)\ , \tag{14}$$

which supersede the results (3) and (6). If one uses the value in (7) for α, this leads to

$$a\,(\text{theory}) = \ 1\ 159\ 652\ 164\ (108) \times 10^{-12}\ . \tag{15}$$

Very recently, however, two new measurements of α have been reported [18, 19]. One is based on the equation

$$\alpha^{-1} = 2R_H/\mu_0 c \tag{16}$$

relating α to the quantized Hall resistance R_H, and the other is based on

$$\alpha^{-2} = (c/4R_\infty \gamma_p')(\mu_p'/\mu_B)(2e/h) \tag{17}$$

which relates α to the measurements of the Josephson frequency and the proton gyromagnetic ratio γ_p'. It is also possible to determine α from the formula

$$\alpha^{-3} = (R_H/2\mu_0 R_\infty \gamma_p')(\mu_p'/\mu_B)(2e/h) \tag{18}$$

which is obtained by combining (16) and (17).

The new values of α are [18]

$$\alpha^{-1}(\mathrm{QHE}) = 137.035\,997\,9\,(33) \quad (0.024\ \mathrm{ppm})\,, \tag{19}$$

and [19]

$$\alpha^{-1}(\mathrm{acJ}\ \&\ \gamma_p') = 137.035\,976\,9\,(77) \quad (0.056\ \mathrm{ppm})\,. \tag{20}$$

Corresponding to (18) one finds

$$\alpha^{-1}(\mathrm{Eq.}(18)) = 137.035\,983\,9\,(51) \quad (0.037\ \mathrm{ppm})\,, \tag{21}$$

which is derived from (19) and (20). Although (19) and (20) are the most accurate values of the respective measurements reported thus far, it should be kept in mind that they are still preliminary having been obtained in a hurry to beat the deadline of a conference. Thus values may move and errors may change before the final reports are written. Both (19) and (20) are measured using a calculable capacitor as a reference and thus sensitive to its uncertainty. In (21), on the other hand, such a dependency cancels out and its error is mainly due to that of γ_p'.

If one accepts (19), which is the best of the three at present, as the value for α, the theoretical prediction for a becomes

$$a(\mathrm{theory}) = 1\,159\,652\,133\,(29) \times 10^{-12}\,, \tag{22}$$

where

$$\begin{aligned} &\text{error due to } C_3 = 7.1 \times 10^{-12}\,, \\ &\text{error due to } C_4 = 4.5 \times 10^{-12}\,, \\ &\text{error due to } \alpha = 28 \times 10^{-12}\,. \end{aligned} \tag{23}$$

Thus the theoretical value (22) and measured value (11) of a are nearly two standard deviations apart. The values of a obtained using (20) and (21) are somewhat less accurate. As is seen from (23) the error on a is still dominated by that on α. This means that a sharper test of QED must be postponed until better value of α is found. Pending improved measurement of α, however, one can calculate α from theory and the experimental value of a :

$$\alpha^{-1}(g - 2) = 137.035\,991\,4\,(11) \quad (0.0081 \text{ ppm}) \tag{24}$$

where

$$\text{error due to experiment} = 0.003\,7 \text{ ppm}\,,$$

$$\text{error due to theory} = 0.007\,2 \text{ ppm}\,. \tag{25}$$

Clearly the result (24) is more precise than those of (19), (20), (21), or one obtained from the muonium hfs [20, 21] :

$$\alpha^{-1}(\mu\text{hfs}) = 137.035\,992\,5\,(224) \quad (0.17 \text{ ppm})\,. \tag{26}$$

In comparison with the results (19), (20), (21), and (24), the precision of $\alpha(\mu\text{hfs})$ in (26), which has not had a major improvement for some time, looks rather poor. Its error is due partly to the uncertainty in the muon mass but mostly to radiative recoil corrections involving two virtual photons which is yet to be calculated. New experiment for a more accurate measurement of the muonium hfs and muon mass is in preparation [22]. There seems to be no obstacle in improving the theoretical prediction by at least an order of magnitude although it will certainly require an extensive calculation. It must be emphasized that the comparison of α obtained from the improved muonium hfs and a is particularly important for checking the internal consistency of QED.

It is generally believed that α's determined by the ac Josephson effect and quantized Hall effect have no *theoretical* uncertainty, although this has never been proved. The difference of 21×10^{-6} between (19) and (20), which is about 3 times larger than the error in (20), might be the first indication that this belief is not above scrutiny. Before pursuing such a question, however, one must examine carefully the errors in the measurements, in particular those of the calculable capacitor and γ_p'.

An advantage of the composite result (21) is that, being independent of calibration of the calculable capacitor, its error comes mainly from the measurement of γ_p'. Thus it is really (21) that should be compared with the $g-2$ value (24) although the present error on (21) is not as good as it should be reflecting the wide separation of (19) and (20).

If the difference between (19) and (20) persists even after the experimental errors are fully understood and improved, it may become necessary to examine the theoretical basis of (16) and (17) closely and evaluate correction terms, if any, to (16) and/or (17). Strictly speaking, it is not possible to test the validity of QED at the level of 10^{-8} until the apparent internal inconsistency of condensed matter physics is resolved. The theoretical situation of QED is under better control. One may be able to compute a unambiguously at least to the level of 10^{-13}.

Sooner or later theory and experiment of a will be pushed to the point where α is determined to a precision of 10^{-9} or better. Together with improved determinations of α by other means, this will provide a far more stringent test of QED as well as the theoretical basis of condensed matter physics. At the same time it will impose strong constraints on theoretical speculations such as possible internal structure of the electron, supersymmetric theories, and superstring theories.

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