Questions and Answers

Chapter 1

Questions

(Q1) *Based on the classification defined in this chapter, define all the possible classifications of the following registration techniques:*

(a) *Registration by maximization of Mutual Information.*

(b) *Surface Signature registration.*

(c) *Grid Closest Point registration.*

(Q2) *Using the formulation for the Surface Signature images, derive and draw the signature image for the following parametric shapes. (choose points of interest on each shape)*
(a) a sphere of radius r

$$Sph(u, v) = \begin{bmatrix} r \quad cos(2\pi u) \\ r \quad sin(2\pi u)cos(2\pi v) \\ r \quad sin(2\pi u)sin(2\pi v) \end{bmatrix} \tag{1}$$

(b) a cylinder of radius r and height h

$$Cyl(u, v) = \begin{bmatrix} r \quad cos(2\pi u) \\ r \quad sin(2\pi u) \\ v/h \end{bmatrix} \tag{2}$$

(Q3) *Describe how the Genetic Algorithm technique can be used as an optimizer for the MI registration technique.*

(Q4) *Derive*

$$I(R(x), F(T(x))) \equiv h(R(x)) + h(F(T(x))) - h(R(x), F(T(x))). \tag{3}$$

and

$$I(R(x), F(T(x))) \equiv \sum_{R(x), F(T(x))} p\,(r\,(x), F(T(x))) \cdot \log_2 \frac{p\,(r\,(x), F(T(x)))}{p_{R(x)}(r\,(x) \cdot p_{F(T(x))}(F(T(x)))}. \tag{4}$$

Answers

(A1) (a) for the MI technique, the classifications are as follows:
Nature of **registration basis**: intrinsic, voxel-based.
Nature of Transformation: rigid.
Interaction: semi-automatic.
Optimization procedure: parameters are looked for.
Modalities involved: can be used in all the four classes.
Subject: can be used in all classes.

(b) for the Signature registration, the classifications are as follows:
Nature of registration basis: intrinsic, landmarks.
Nature of Transformation: rigid.
Interaction: automatic.
Optimization procedure: parameters computed directly.
Modalities involved: Only surface registration.
Subject: Only surface registration.

(c) for the GCP, the classifications are as follows:
Nature of registration basis: intrinsic, geometric.
Nature of Transformation: rigid.
Interaction: semi-automatic.
Optimization procedure: parameters looked for.
Modalities involved: can work for all four classes.
Subject: can work for all classes.

(A2) A point P on a parametric surface is a function of two parameters u and v. For each pair of values, we generate the point

$$P(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix} \tag{5}$$

For example, the parametric representation of a sphere of radius r is

$$Sph(u,v) = \begin{bmatrix} r & cos(2\pi u) \\ r & sin(2\pi u)cos(2\pi v) \\ r & sin(2\pi u)sin(2\pi v) \end{bmatrix} \tag{6}$$

Also, a cylinder of radius r and height h can be parameterized as

$$Cyl(u,v) = \begin{bmatrix} r & cos(2\pi u) \\ r & sin(2\pi u) \\ r & v/h \end{bmatrix} \tag{7}$$

We will use these parametric surfaces to derive a parametric Signature (u,v) image.

We will start by taking a point $P(u_i, v_i)$ on a parametric surface. The normal of the surface at this point can be easily obtained by finding the tangent plane to the parametric surface at this point. This can be done by differentiating the parametric representation relative to both u and v and then get the cross product of the resulting vectors.

$$U_p(u_i, v_i) = \begin{bmatrix} \frac{\delta x}{\delta y}(u_i, v_i) \\ \frac{\delta y}{\delta u}(u_i, v_i) \\ \frac{\delta z}{\delta u}(u_i, v_i) \end{bmatrix} \times \begin{bmatrix} \frac{\delta x}{\delta v}(u_i, v_i) \\ \frac{\delta y}{\delta v}(u_i, v_i) \\ \frac{\delta z}{\delta v}(u_i, v_i) \end{bmatrix} \tag{8}$$

For all others pair of u and v, a point $P(u,v)$ on the surface is generated and the two signature parameters D and α are derived. However they are not discrete anymore and in fact they become parameterized by u and v and we can define now $D(u,v)$ and $\alpha(u,v)$. Also the curvature value at each $D(u,v)$ and $\alpha(u,v)$ can be theoretically calculated using the

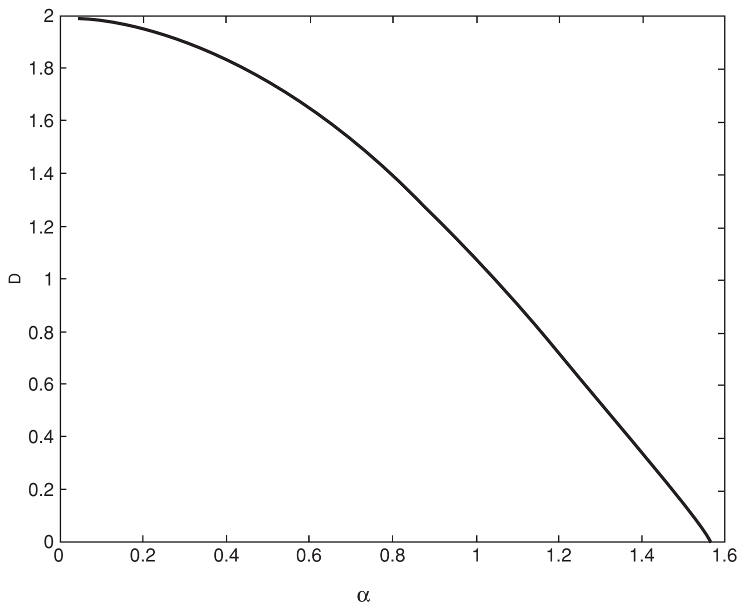

Figure 1: An example of a parametric Signature generated for a point on a parametric sphere. We have to notice that any and all points on the sphere will have the same signature. Again this is to be expected as there exists no unique registration solution to a sphere.

shape operator.

$$D(u, v) = \sqrt{(x(u, v) - x(u_i, v_i))^2 + (y(u, v) - y(u_i, v_i))^2 + (z(u, v) - z(u_i, v_i))^2} \tag{9}$$

$$\alpha(u, v) = cos^{-1}\left(\begin{bmatrix} x(u_i, v_i) \\ y(u_i, v_i) \\ z(u_i, v_i) \end{bmatrix} \cdot \begin{bmatrix} x(u, v) - x(u_i, v_i) \\ y(u, v) - y(u_i, v_i) \\ z(u, v) - z(u_i, v_i) \end{bmatrix}\right) \tag{10}$$

Figure 1 shows an example of a parametric signature generated for a point on a parametric sphere. We have to notice that any and all points on the sphere will have the same signature. Again this is to be expected as there exists no unique registration solution to a sphere.

Figures 2 and 3 show two parametric surfaces, namely, a cylinder, and a torus and corresponding parametric signatures generated at different

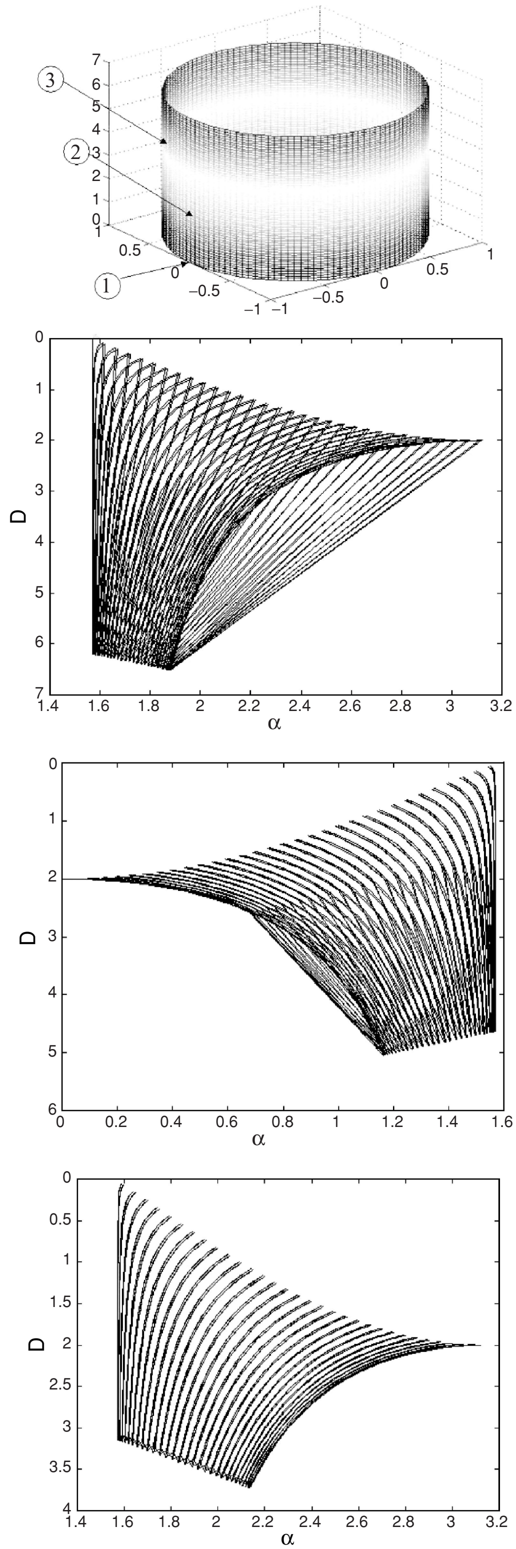

Figure 2: The parametric signature images for different points on a cylinder.

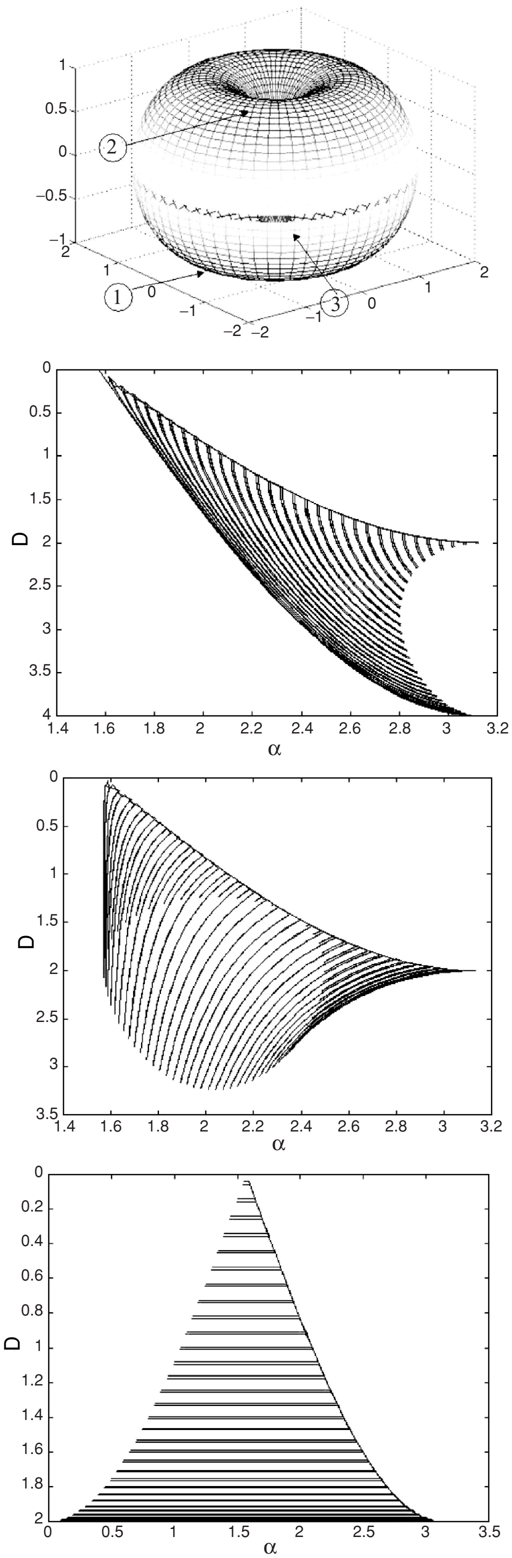

Figure 3: The parametric signature images for different points on a torus.

points on these surfaces. As we can notice, the signatures do reflect the properties of the object seen from the point of generation.

(A3) GA can be used in finding the global Maximization for the MI metric defined in equation in Chapter 1, Volume III in the same way it was used for the GCP, expect in this case the payoff function is the direct calculation of equation in Chapter 1, Volume III.

(A4) Relative entropy, also known as the Kullbak Leibler distance, between two probability mass functions $p(x)$ and $q(x)$, is defined as the quantity $D(p\|q)$ where

$$D(p\|q) = \sum_{x} p(x) \, . \, log_2 \frac{p(x)}{q(x)} \tag{11}$$

For two discrete random variables X and Y, with marginal probability mass functions $p_X(x)$ and $P_Y(y)$, and a joint distribution $p(x, y)$, the mutual information function $I(x, y)$ is the relative entropy between the joint distribution $p(x, y)$, and the distribution $p_X(x) \, . \, p_Y(y)$, the joint distribution when X and Y are independent random variables. Thus, the mutual information of X and Y can be formulated as follows:

$$I(X, Y) = D(p(x, y) \| p_X(x) \, . \, p_Y(y)) = \sum_{X,Y} p(x, y) \, . \, log_2 \frac{p(x, y)}{P_X(x) \, . \, p_Y(y)} \tag{12}$$

If X and Y are independent random variables, then $p(x, y)$ is given by the product of the marginals $p_X(x)$ and $P_Y(y)$, and therefore, the quantity that is the argument of the logarithm function is one. As the logarithm of one is zero, the mutual information $I(X, Y)$, when X and Y are statistically independent, is zero. There is a close relationship between entropy, a measure of information content, and the mutual information quantity. The entropy $H(X)$ of a discrete random variable X. The joint entropy $H(X, Y)$ and the condiutional entropy $H(Y \mid X)$ of the discrete random variables X and Y are defined as follows;

$$H(X) = \sum_{X} p(x) \, . \, log_2 p(x) \tag{13}$$

$$H(X, Y) = \sum_{X,Y} p(x, y) \, . \, log_2 p(x, y) \tag{14}$$

$$H(X \mid Y) = \sum_{X,Y} p(x, y) \, . \, log_2 p(y \mid x) \tag{15}$$

Entropy is a common measurement of information content. Information content is increased as entropy is increased, and decreased as entropy is decreased. The less concentrated a probability density or mass function is, the more information content that is encoded in the random variable. For example, consider a continuous random variable X. If X is distributed as a uniform random variable, the information content of X is greatest, because over the range of X, the probability of X taking on a value x_1 is equal in all cases to X taking on a value of x_2. For X distributed as a Gaussian random variable, with a mean μ and variance σ^2, X encodes less information, as the values of X around μ are more likely than values far from μ, with the spread or concentration of probability around μ quantified by the variance σ^2. By simple algebraic manipulation, and using the definition of conditional probabilities, the mutual information function $I(X, Y)$ can be related in a number of ways to the entropy quantities given above. These relationships are given as follows;

$$I(X, Y) = H(X) - H(X|Y) = H(Y) - H(Y|X) \tag{16}$$

$$I(X, Y) = H(X) + H(Y) - H(X, Y) \tag{17}$$