Testing independence for spatial processes through spectral analysis

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Abstract

The determination of a possible spatial dependence in crucial in many environmental problems where spatial statistics has a role to play. In this paper we present a simple test to detect spatial independence for a two-dimensional process measured over a regular grid. The statistical test is based on the analysis of the spectral representation of such a spatial process. We analyze and present the statistical properties of the test based on Type I error and power under several practical scenarios.

Keywords: Matérn process, Spatial dependence, Spatial processes, Spectral representations, White-noise process.

1 Introduction

The need to analyze spatial processes features prominently in disciplines as diverse as astronomy, ecology, forestry, geography, meteorology, soil sciences and in general environmental sciences. In any serious analysis, the determination of the underlying spatial structure is needed.
Spectral analysis of stationary processes is particularly advantageous in the analysis of large data sets and in studying properties of multivariate processes. Geostatistical data are usually collected over a large region, and handling, large data sets is often problematic for the commonly used techniques: inversion of a large covariance matrix to compute the likelihood function may not be possible or may require a long time in computation. The use of a Fast Fourier transform (FFT) algorithm for spectral densities can be a good solution for these problems. However, FFT can be applied only to regularly gridded data, though this disadvantage is not that important as there are theoretical connections between the estimators of the spectral densities in both the regular lattice and irregular spaced data (Renshaw, 2002). The periodogram, a nonparametric estimate of the spectral density, is a

powerful tool for studying the properties of stationary processes observed on a two-dimensional lattice (Stein, 1999).
In this paper we present a test for detecting spatial dependence/independence for a two-dimensional process Z measured at $N = n_1 x\, n_2$ regularly spaced data. We test the existence of a white noise process against a Matérn spatial process in terms of the corresponding spectral density. We present a complete simulation study to evaluate and compare the significance level and power of the proposed test under a variety of practical scenarios.

The plan of the paper is as follows. Section 2 presents the necessary spectral methodology needed to define the statistical test. The independence test is presented in Section 3, where a complete simulation study is shown. The paper ends with some conclusions and future lines of research.

2 Spectral methodology

2.1 Stationary processes

Let Z be a stationary process observed on a region D. The two-dimensional random field Z can then be represented in the form of the following Fourier-Stieltjes integral (Cressie, 1993)

$$Z(x) = \int_{\Re^2} exp(i\omega^T x) dY(\omega) \tag{1}$$

where Y are random functions with *uncorrelated increments* and ω are the frequencies. The representation (1) is called *spectral representation* of Z. The spectral representation describes the harmonic analysis of a general stationary process, i.e. its representation in a form of a superposition of harmonic oscillations of sine and cosine waves of different frequencies ω.
Let the function F be a positive finite spectral measure for Z, defined by $E|Y(\omega)|^2 = F(\omega)$. If F has a density with respect to the Lebesgue measure, this density is the *spectral density f*, defined as the Fourier transform of the autocovariance function C,

$$f(\omega) = \frac{1}{(2\pi)^2} \int_{\Re^2} exp(-i\omega^T x) C(x) dx \tag{2}$$

By Bochner’s theorem, the function C is an autocovariance if and only if can be represented as in (2), where F is a positive, finite measure. Thus, the spatial structure of Z could be analyzed with a spectral approach or equivalently by estimating the autocovariance function.

A class of practical variograms and autocovariance functions for a two-dimensional process Z, can be obtained from the Matérn class of spectral densities

$$f(\omega) = \phi(\alpha^2 + |\omega|^2)^{(-\nu-1)} \quad (3)$$

with the vector of positive covariance parameters given by $\theta = (\phi, \nu, \alpha)$. The parameter α^{-1} can be interpreted as the autocorrelation range. The parameter ν measures the degree of smoothness of the process, and $\phi = \sigma^2 \alpha^{2\nu}$, a ratio of the variance and the range. The smoothness of a random field in the Matérn class plays a critical role in interpolation problems. A number of commonly used models for the covariance structure, including spherical, exponential and Gaussian structures assume that the smoothness parameter is known a priori. This will not be our case.

Following Stein (1999) and Fuentes (2001), if the goal is spatial interpolation, it is preferable to work on the spectral domain and focus on high frequency values. An approximate expression of (3) for high frequency values is obtained by letting $|\omega|$ go to ∞

$$f(\omega) = \phi(|\omega|^2)^{(-\nu-1)} \quad (4)$$

Thus, the degree of smoothness, ν, and ϕ are the critical parameters. Note that from (4), and working in the log scale, we can fit the following linear model

$$\log f(\omega) = \beta_0 + \beta_1 \log(|\omega|^2) \quad (5)$$

where $\beta_0 = \log(\phi)$ and $\beta_1 = -\nu - 1$.

In practice, the spectral density has to be estimated. The spatial periodogram is a nonparametric estimate of the spectral density, and a powerful tool for studying the properties of random field observed on a lattice. It is the modulus-squared of a finite Fourier transform for the observed region of the process, introduced to search for hidden periodicities of processes. The periodogram itself is not a consistent estimator of the spectral density, but consistency can be achieved by applying linear smoothing filters to the periodogram. Smoothing the periodogram, as a frequently done in time series does not remove large edge-effects in two or more dimensions. The sidelobes (subsidiary peaks) occurring on smoothing filters cause unnecessary large values of the periodogram ordinates for high frequencies and results in substantial bias. This phenomenon is called leakage. Instead of smoothing biased periodogram estimates, direct filtering of the data with a data taper before computing the periodogram can also provide a consistent estimate of the spectral density. The information lost through powerful frequencies by smoothing the periodogram can be recovered by data tapers (Fuentes, 2001, 2002).

Consider the spatial stationry process $Z(\cdot)$ with covariance parameter θ which is assumed here to be known. We observe the process at *N* equally spaced locations in a regular grid $D(n_1 \times n_2)$, where $N = n_1 n_2$.

We define $I_N(\omega)$ to be the periodogram at a frequency ω,

$$I_N(\omega) = (2\pi)^{-2}(n_1 n_2)^{-1} \left| \sum_{x_1=1}^{n_1} \sum_{x_1=1}^{n_1} Z(x) \exp\{-i\boldsymbol{x}^T \omega\} \right|^2 \tag{6}$$

In practice, the periodogram estimate for ω is computed in the set of Fourier frequencies $2\pi f / n$ where $f / n = \left(\frac{f_1}{n_1}, \frac{f_2}{n_2} \right)$, and $f \in J_N$, for

$$J_N = \{\lfloor -(n_1 - 1)/2 \rfloor, \ldots, n_1 - \lfloor n_1 / 2 \rfloor\} \times \{\lfloor -(n_2 - 1)/2 \rfloor, \ldots, n_2 - \lfloor n_2 / 2 \rfloor\} \tag{7}$$

where $\lfloor n \rfloor$ denotes the largest integer less or equal that *n*.

The expected value of the spatial periodogram is not *f(ω)*, but a weighted integral of *f(ω)*. In terms of an increasing density asymptotics, it is asymptotically unbiased, its asymptotic variance is $f^2(\omega)$, and the periodogram values $I_N(\omega)$ and $I'_N(\omega)$ for $\omega \neq \omega'$, are asymptotically uncorrelated (Fuentes, 2002). The asymptotic independence of the periodogram estimates is one of the big advantages of the spectral analysis, and this facilitates the use of techniques such as non-linear least squares (NLS) to fit a theoretical spectral model to the periodogram values.

In the space-time domain, empirical variogram estimates are most commonly used to estimate the correlation structure of a process. When a parametric variogram model is fit to empirical variogram estimates, frequently used techniques such as non-linear least squares or restricted maximum likelihood (REML) approaches generally do not take into account any correlation between estimated variogram values. The same data points are used to estimate the variogram at different lags, and the resulting variogram estimates are more correlated than the observations of the underlying process. Ignoring such correlation can mislead data analyses. Techniques such as the nonlinear least squares method can be naturally applied to those independent estimates. In this case, nonparametric smoothing of the tapered periodogram estimates might give better results than nonparametric variogram estimates.

2.1 Extension to nonstationary processes

Let Z be a nonstationary process observed on a region D. Supose D is covered by welldefined subregions $S_1,...,S_k$, and consequently, Z can be written as weighted average of orthogonal local stationary processes Z_i for i=1,…,k, with $\text{cov}(Z_i(x),Z_j(y))=0$ for $i \neq j$. Where have

$$Z(x)=\sum_{i=1}^{k} Z_i(\boldsymbol{x})K_i(x) \tag{8}$$

where Z_i is a local stationary process in the subregion S_i, $K_i(x)$ is a positive kernel function centered at the centroid of S_i. The weighted average (8) is the discrete representation of the process Z, but we could write this average as an integral to obtain a continous representation.

Focusing now on the nonstationary process Z, defined as a mixture of the stationary processes $Z_1,...,Z_k$ as in (8), the spectral representation of Z is

$$Z(x)=\int_{\Re^2} exp(i\omega^T x)dY(\omega) \tag{9}$$

with $Y(\omega)=\sum_{i=1}^{k} \overline{K}_i * Y_i(\omega)$ for $\overline{K}_i$ the Fourier transform of K_i and * denotes the convolution. The covariance of Z can be defined in terms of the covariance of the orthogonal local stationary processes Z_i, defining a valid nonstationary covariance. The corresponding spectral density is given by $f(\omega_1,\omega_2)=\sum_{i=1}^{k} \hat{f} * \left[\overline{K}i(\omega_1)\overline{K}i(\omega_2)\right]$ where $\overline{K}$ is the FT of K.

3 Testing independence

Suppose we wish to test whether a two-dimensional process Z exhibits spatial dependence. Further, suppose Z is measured at $N=n_1 \times n_2$ regularly spaced data. At this point, it is worth noting that: (a) in practice, we can have missing data at several locations in the lattice, (b) The source data could be sampled over an irregular grid. In this case, and taking into account the results in Renshaw (2002), we could define an appropriate regular grid to approximate the irregular data locations, and proceed normally with or proposed test.

Suppose the spectral density of the process Z belongs to the Matérn class, with the vector of positive parameters given by $\theta = (\phi, \nu, \alpha)$. Focusing on high frequency values, and working now in the log scale, we can fit the model given in equation (5) with $\beta_0 = \log(\phi)$ and $\beta_1 = -\nu - 1$. In practice, *f(ω)* is estimated by the corresponding (tapered) periodogram $I_N(\omega)$ given in equation (6). Taking into account that the periodogram values are asymptotic independent, we can use regression techniques to estimate the values of intercept and slope.

Taking into account that a white-noise process, a process with no spatial dependence, has a constant spectral density, the null hypothesis of our test is that of constant spectral density or equivalently the existence of no linear dependence between the random variables $\log(f(\omega))$ and $\log(|\omega|^2)$. The alternative hypothesis of our test is based on the existence of a spatial process belonging to the Matérn class. Thus, we make use of the Kendall correlation coefficient to build a simple statistical test for the existence of spatial dependence. The test checks for a null correlation coefficient and is defined as

$$t_{n-2} = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \qquad (10)$$

where r stands for the correlation coefficient. Under the null hypothesis of zero correlation between $\log(f(\omega))$ and $\log(|\omega|^2)$ the process exhibits no spatial dependence or equivalently it is a white-noise process. On the other way, a significative rejection of this hypothesis will mean that our spatial process belongs to the Matérn family.

In Figure 1 we show a plot of $\log(f(\omega))$ against $\log(|\omega|^2)$ for both cases a pure-nugget or white-noise process and a Matérn process. We observe a null linear relationship in the first case and a clear linear structure (with negative slope) in the latter. And these are the kind of linear structures our test tries to explore.

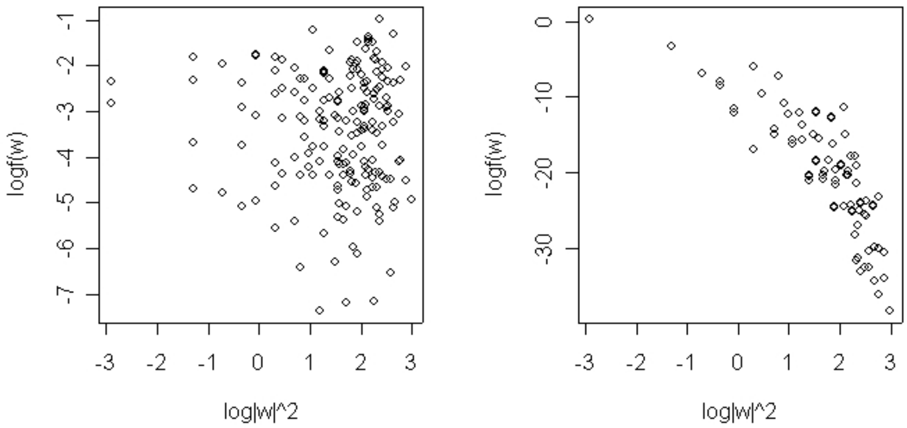

Figure 1: Plot of $\log(f(\omega))$ against $\log(|\omega|^2)$. *Left plot*: Case of a pure-nugget or white-noise process. *Right plot*: Case of a Matérn process

We now present a complete simulation study to evaluate and compare the significance level and power of the proposed test under a variety of practical scenarios. Suppose we have a two-dimensional spatial process Z measured at $N = n_1 \times n_2$ regularly spaced data. If Z is nonstationary, then we can decompose Z into a sum of k local stationary processes, following equation (8), for k subregions covering the region of interest, say D. The number of regions k can be found using an AIC criterium (Fuentes, 2001), or by experimentation depending, for example, on the a priori knowledge of the physical characteristics of the region D. However, the number k is restricted by the number of original sampled locations N.

In this simulation report, we kept fixed the grid size to $N = 20 \times 20$ as this size reported the best results when analyzing these kind of spatial processes observed in regular regions (Mateu and Juan, 2004). We varied the smoothing parameter ($\nu = 0.5,1,1.5,2,3,4$) to cover several processes with different types of differentiability. Also, several combinations for the sill and range parameters were considered ((range,sill)=(10,2),(10,2.9),(166,2.9),(200,2.9),(300,2.9)), focusing on those cases for which there are both small and big differences between the ranges. In practice, $f(\omega)$ is estimated by the corresponding periodogram, $I_N(\omega)$, which can be tapered or not. We focused on three cases (a) no taper, (b) multiplicative and (c) rounded taper and used 400 Fourier frequencies for the periodogram evaluation.

Under the null hypothesis, we simulated a pure-nugget or white-noise model (i.e. a process with no spatial dependence) and under the alternative hypothesis a Matérn spatial process with any parameter combination considered above was simulated. For each single combination of possibilities we used 1000 repetitions to perform the significance level and power of the proposed test. Moreover, we analyzed the performance of the test under the presence of a nugget effect equal to 2 to compare with the case when there was no nugget effect.

The results of these simulations are reported in Figures 2 to 4 and in the corresponding Tables 1 and 2.

The significance level of the test is well-behaved in general for any kind of tapering used, taking values less than 10%. Multiplicative taper provided the best results showing the lower significance levels. Tapering provides reasonable results if ε parameter (the bandwidth-type parameter) is small. In particular $\varepsilon = 10$ shows big significance levels. In general, in presence of a nugget effect, there is a clear increase in the significance values.

Regarding the power of the test, this power is mostly 100% when no nugget effect is present and independently of the type of tapering used. However, when a nugget effect is observed things change dramatically, lowering down the power values. In this case, the biggest power records were obtained when using a small value of range (10) and small values of the smoothing parameter in the Matérn family.

4 Conclusions and further research

We have presented a test to check for spatial dependence/independence in a two-dimensional spatial process measured over a regular grid. The test provided very good results in terms of significance levels particularly when using a multiplicative taper with a rather small bandwidth parameter. In any case, provided values of power close to 100%. When a nugget effect is present in the spatial data, and the smoothing and range parameter of the Matérn family are big enough, the test provides misleading results.

This statistical test is very easy to use in practice and can help researchers in an initial step of their work with spatial data.

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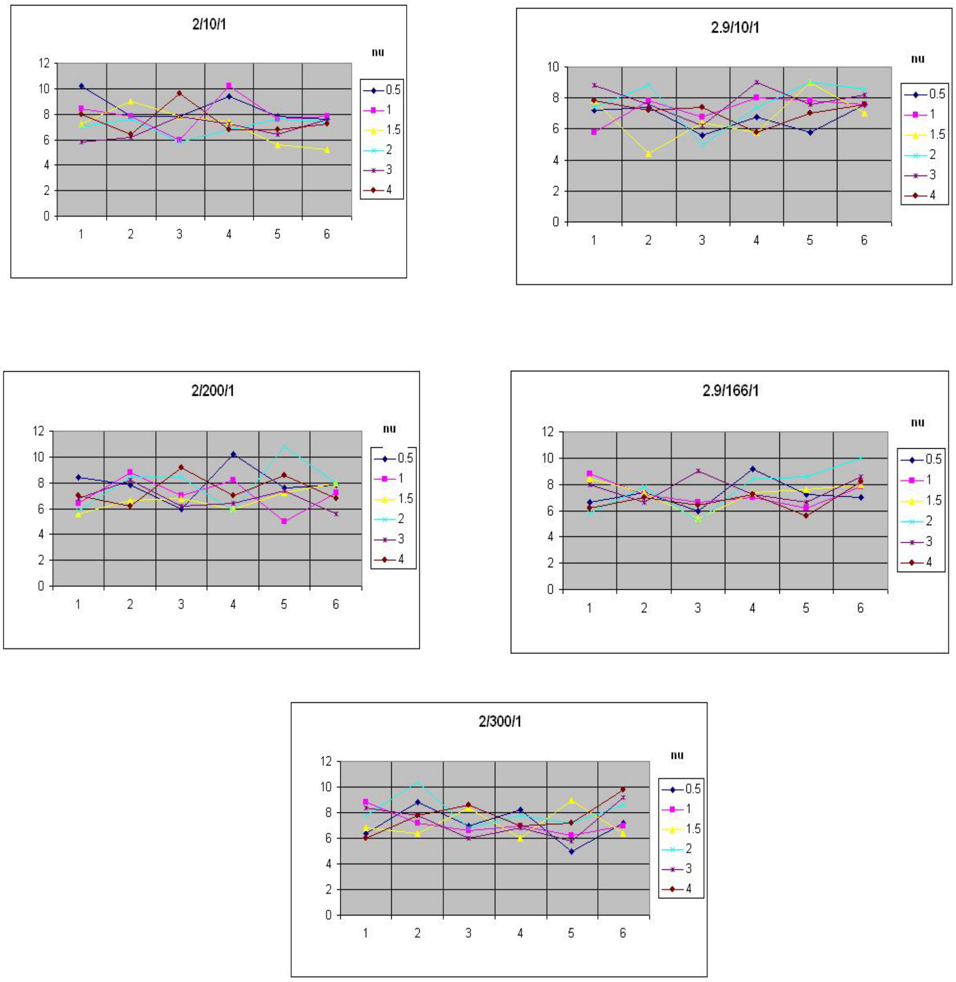

Figure 2: Evaluation of the significance level of the test under several values of the smoothing, range and sill parameters. For the rounded and multiplicative cases we used $\varepsilon = 1$. The labels for the numbers in the x-axis are: *1*=no taper with nugget=0, *2*=no taper with nugget=2, *3*=multiplicative taper with nugget=0, *4*=multiplicative taper with nugget=2, *5*=rounded taper with nugget=0, *6*=rounded taper with nugget=2.

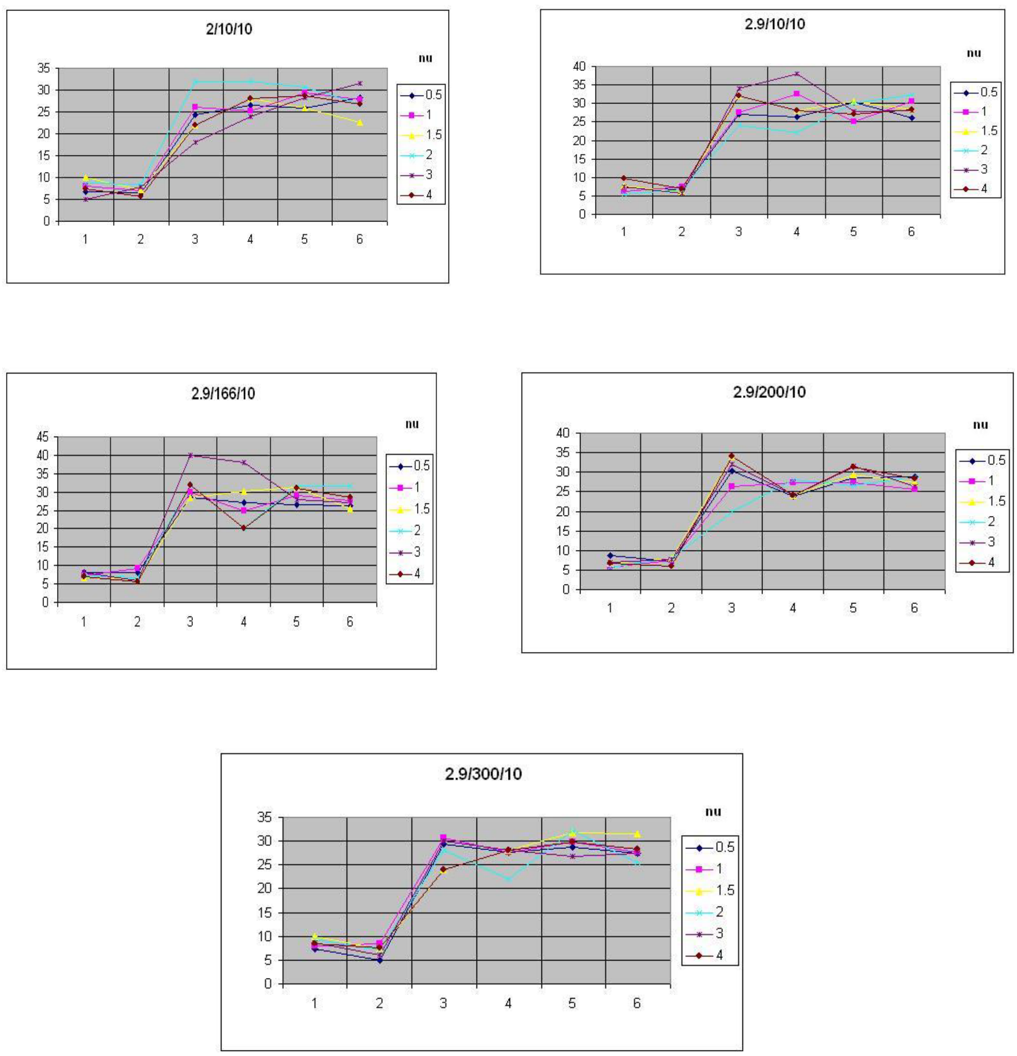

Figure 3: Evaluation of the significance level of the test under several values of the smoothing, range and sill parameters. For the rounded and multiplicative cases we used $\varepsilon = 10$. The labels for the numbers in the x-axis are: *1*=no taper with nugget=0, *2*=no taper with nugget=2, *3*=multiplicative taper with nugget=0, *4*=multiplicative taper with nugget=2, *5*=rounded taper with nugget=0, *6*=rounded taper with nugget=2.

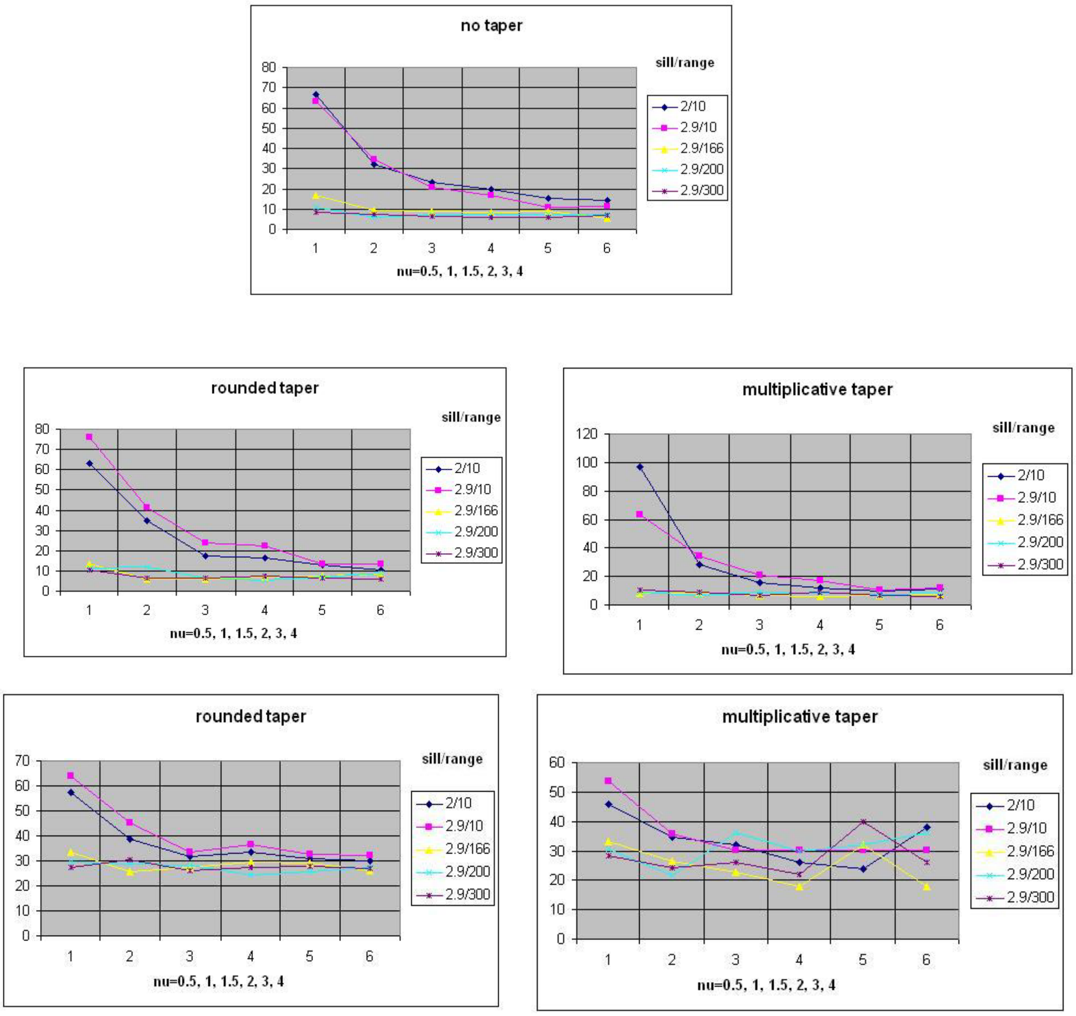

Figure 4: Evaluation of the power of the test under several values of the smoothing, range and sill parameters. The first three plots correspond to the case $\varepsilon = 1$, and the last two to the case $\varepsilon = 10$. All the cases are under a presence of a nugget =2.

Table 1: Significance levels (S) and power (P) of the test under several scenarios when $\varepsilon = 1$. The labels are: t1= S with no taper, t2= S with multiplicative taper, t3= S with rounded taper, t4= P with no taper, t5= P with multiplicative taper, t6= P with rounded taper. The cases in parenthesis are those corresponding to a presence of a nugget effect.

Case: (range = 10, sill = 2.0)

υ	t1	t2	t3	t4	t5	t6
0.5	10.2 (7.8)	7.8 (9.4)	7.8 (7.6)	100 (66.8)	100 (96.6)	100 (63.2)
1.0	8.4 (7.8)	6 (10.2)	7.6 (7.8)	100 (32)	100 (28.4)	100 (34.6)
1.5	7.2 (9)	7.8 (7.4)	5.6 (5.2)	100 (23.4)	100 (15.4)	99.8 (17.2)
2.0	7 (7.6)	5.8 (6.8)	7.6 (7.4)	100 (19.8)	100 (12)	99.4 (16.4)
3.0	5.8 (6.2)	7.8 (7.2)	6.4 (7.6)	100 (15.2)	100 (9.6)	99.6 (13)
4.0	8 (6.4))	9.6 (6.8)	6.8 (7.2)	100 (14.4)	100 (11.4)	99.8 (10.2)

Case: (range = 10, sill = 2.9)

υ	t1	T2	t3	t4	t5	t6
0.5	7.2 (7.4)	5.6 (6.8)	5.8 (7.6)	100 (78.4)	100 (63.4)	100 (76)
1.0	5.8 (7.8)	6.8 (8)	7.8 (7.6)	100 (51.6)	100 (34.4)	100 (41.4)
1.5	7.8 (4.4)	6.4 (5.8)	9 (7)	100 (31.8)	100 (20.8)	100 (24)
2.0	7.4 (8.8)	5 (7.4)	9 (8.6)	100 (27)	100 (17)	99.4 (22.2)
3.0	8.8 (7.6)	6.2 (9)	7.6 (8.2)	100 (20)	100 (10.8)	99 (13.2)
4.0	7.8 (7.2)	7.4 (5.8)	7 (7.6)	100 (14.2)	100 (11.6)	98.4 (13.4)

Case: (range = 166, sill = 2.9)

υ	t1	T2	t3	t4	t5	t6
0.5	6.6 (7.4)	6 (9.2)	7.2 (7)	100 (16.8)	100 (8.4)	100 (13.4)
1.0	8.8 (7.2)	6.6 (7)	6.2 (7.8)	100 (9.6)	100 (8.4)	100 (6)
1.5	8.4 (7.4)	5.4 (7.4)	7.6 (8)	100 (9)	100 (6.8)	100 (5.8)
2.0	5.8 (7.8)	5.2 (8.4)	8.6 (10)	100 (8.4)	100 (6.2)	100 (7)
3.0	8 (6.6)	9 (7.2)	6.6 (8.6)	100 (8.8)	100 (6.4)	100 (7.4)
4.0	6.2 (7)	6.4 (7.2)	5.6 (8.2)	100 (5.6)	100 (7.8)	100 (8.2)

Case: (range = 200, sill = 2.9)

υ	t1	t2	t3	t4	t5	t6
0.5	8.4 (7.8)	6 (10.2)	7.6 (7.8)	100 (10.8)	100 (8.8)	100 (11.4)
1.0	6.4 (8.8)	7 (8.2)	5 (7.2)	100 (5.8)	100 (6.4)	100 (11.8)
1.5	5.6 (6.6)	6.8 (6)	7.2 (8)	100 (7.4)	100 (8.6)	100 (6.6)
2.0	5.8 (8.4)	8.4 (5.8)	10.8 (8)	100 (7.6)	100 (8)	100 (5.4)
3.0	6.8 (8.2)	6.2 (6.4)	7.4 (5.6)	100 (7.4)	100 (7.6)	100 (6.8)
4.0	7 (6.2)	9.2 (7)	8.6 (6.8)	100 (7.4)	100 (8.2)	100 (8.8)

Case: (range = 300, sill = 2.9)

υ	t1	t2	t3	t4	t5	t6
0.5	6.4 (8.8)	7 (8.2)	5 (7.2)	100 (8.2)	100 (10.8)	100 (10.4)
1.0	8.8 (7.2)	6.6 (7)	6.2 (7)	100 (7.4)	100 (8.6)	100 (6.6)

1.5	6.8 (6.4)	8.4 (6)	9 (6.4)	100 (6.6)	100 (7)	100 (6.6)
2.0	7.8 (10.4)	6.8 (7.8)	7.2 (8.6)	100 (6)	100 (9)	100 (7.6)
3.0	8.4 (7.8)	6 (6.8)	5.8 (9.2)	100 (6)	100 (6.6)	100 (6.6)
4.0	6 (7.8)	8.6 (7)	7.2 (9.8)	100 (6.8)	100 (6.2)	100 (5.8)

Table 2: Significance levels (S) and power (P) of the test under several scenarios when $\varepsilon = 10$. The labels are: t1= S with no taper, t2= S with multiplicative taper, t3= S with rounded taper, t4= P with no taper, t5= P with multiplicative taper, t6= P with rounded taper. The cases in parenthesis are those corresponding to a presence of a nugget effect.

Case: (range = 10, sill = 2.0)

υ	t1	t2	t3	t5	t6
0.5	6.8 (6.6)	24.4(26.6)	25.8 (28.2)	100 (45.8)	100 (57.2)
1.0	8 (7)	26 (25.2)	29.4 (27.8)	100 (34.8)	100 (38.6)
1.5	10 (7.2)	22 (28)	25.8 (22.6)	100 (32)	100 (31.8)
2.0	8.8 (8.4)	32 (32)	30.6 (26.8)	100 (26)	100 (33.4)
3.0	5 (7.8)	18 (24)	28.2 (31.6)	100 (24)	100 (30.8)
4.0	7.4 (5.6)	22 (28)	28.8 (26.8)	100 (38)	100 (29.8)

Case: (range = 10, sill = 2.9)

υ	t1	t2	t3	t5	t6
0.5	7.2 (6.6)	27.2 (26.4)	30.4 (26)	100 (53.6)	100 (63.8)
1.0	6.2 (7.4)	27.6 (32.6)	25.2 (30.8)	100 (35.6)	100 (45.2)
1.5	7.6 (6)	32 (28)	30.6 (28.2)	100 (30.2)	100 (33.6)
2.0	5.4 (6.8)	24 (22)	30 (32.2)	100 (30.2)	100 (36.4)
3.0	7.4 (5.6)	34 (38)	27.8 (28)	100 (30.2)	100 (32.6)
4.0	9.8 (7)	32 (28)	27.2 (28.2)	100 (30.2)	100 (32)

Case: (range = 166, sill = 2.9)

υ	t1	t2	t3	t5	t6
0.5	8 (8)	28.4 (27)	26.6 (26.4)	100 (33)	100 (33.6)
1.0	7.4 (9.2)	29.8 (25)	29.2 (27.8)	100 (26.4)	100 (25.8)
1.5	6.8 (6.2)	28.6 (30.2)	31.2 (25.4)	100 (22.8)	100 (27.6)
2.0	7.2 (6.6)	32 (20)	31.6 (31.6)	100 (18)	100 (29.6)
3.0	8 (6)	40 (38)	28 (27)	100 (32)	100 (29)
4.0	7 (5.6)	32 (20)	31 (28.4)	100 (18)	100 (26.2)

Case: (range = 200, sill = 2.9)

υ	t1	t2	t3	t5	t6
0.5	8.6 (7.2)	30.4 (23.8)	28.6 (28.8)	100 (30.6)	100 (29.4)
1.0	5.6 (7.4)	26.4 (27.4)	27.4 (25.6)	100 (21.6)	100 (28.6)
1.5	6.8 (8)	34 (24)	29.4 (27.4)	100 (36)	100 (28.4)
2.0	5.8 (8)	20 (34)	26.6 (28.8)	100 (30)	100 (24.4)

3.0	7 (7.8)	32 (28)	31.6 (26.4)	100 (32)	100 (25.8)
4.0	6.6 (6)	34 (24)	31.2 (28.4)	100 (36)	100 (27.2)

Case: (range = 300, sill = 2.9)

υ	t1	t2	t3	t5	t6
0.5	7.4 (5)	29.4(27.6)	28.8 (27.4)	100 (28.2)	100 (27.4)
1.0	8 (8.4)	30.6 (27.6)	29.8 (27.6)	100 (24.4)	100 (30.4)
1.5	10 (7.4)	24 (28)	31.8 (31.6)	100 (26)	100 (26)
2.0	9.2 (7.2)	28 (22)	32.2 (25.6)	100 (22)	100 (27.4)
3.0	8.6 (6)	30 (28)	26.8 (27.5)	100 (40)	100 (28)
4.0	8.4 (7.6)	24 (28)	29.8 (28.4)	100 (26)	100 (27)