

Spatio-temporal geostatistical analyses of runoff and precipitation

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Abstract

We have analysed spatio-temporal variograms of catchment precipitation and runoff. The sample variograms were calculated on the basis of 20 years of quarter hourly precipitation and runoff data from about 600 catchments in Austria. We have fitted three spatio-temporal variogram models independently to the spatio-temporal sample variograms of catchment precipitation and runoff for three different catchment size classes, small, medium and large. The models were an exponential model, a model proposed by Cressie and Huang, and a product-sum model. We found that the exponential model and the product-sum model provided similarly good fits both for runoff and precipitation. The exponential model was slightly better for runoff, while poorer for precipitation. The model of Cressie and Huang was not suitable for modelling precipitation, and was also significantly poorer for modelling runoff.

We have also fitted regularised variograms jointly to the spatio-temporal sample variograms of the three catchment size classes. From this we back-calculated a spatio-temporal variogram of point runoff. Here, all the three models provided equally good fits.

We are planning to use the back-calculated variograms of point runoff for interpolating runoff time series to catchments without runoff measurements.

1. Introduction

Geostatistical analyses in hydrology have generally focused on either spatial or temporal analyses, while joint space-time analyses have been rare. At the same time, most processes of interest to hydrologists have variability both in time and space. Skøien et al. (2003) analysed the characteristics of variograms in space and time separately for precipitation, runoff, groundwater and soil moisture. Espe-

cially, they examined the effect of different catchment sizes on the spatial and temporal variograms of precipitation and runoff independently.

Here, we examine precipitation and runoff jointly in space and time. We analyse variograms in terms of their support, and suggest that runoff also has a temporal support that must be accounted for in the analyses. We analyse precipitation and runoff in the spatio-temporal domain, and explain differences between sample variograms of runoff from different catchment sizes.

A large number of different spatio-temporal correlation models have appeared in the literature. Most of them are functions of the correlations in time and space. De Cesare et al. (2001) and De Iaco et al. (2001) extended some of the earlier models into a product-sum model. Cressie and Huang (1999) proposed a series of non-separable models.

2. Data

The data used in this paper stem from a comprehensive hydrographical data set of Austria. We have used data series of precipitation and runoff from approximately the same stations as Skøien et al. (2003), but with a much higher temporal resolution than what they used in their analyses. We used quarter hourly runoff data from 590 stream gauges, and hourly catchment precipitation from 756 catchments. Most of the series consisted of approximately 20 years of data.

3. Theory and methods

We can always attach a measurement scale to our measurements. Blöschl and Sivapalan (1995) described the measurement scale as a scale triplet, the distance between measurements (spacing), the total area over which we have measured (extent) and the area or volume that each measurement represents (support). We will in this paper focus on the support of measurements.

Skøien et al. (2003) analysed how variograms of daily precipitation and runoff changed as a function of catchment size. Although they showed that it was mathematically possible to explain the aggregation behaviour of runoff through spatial aggregation, they suggested that runoff has a spatio-temporal support, that has to be considered in the analyses. Runoff at the outlet is not only a spatially averaged value of the instantaneous runoff at every point in space, the variance is further smoothed as the runoff at the outlet consists of runoff generated at different times and in different parts of the catchment. The precipitation data also has a temporal support, but it is the same for all catchments, the 60 minutes over which precipitation is summed in the time series used.

The runoff time series at the outlet can be considered as a convolution of the runoff generation process with a different unit hydrograph for every point (similar to Woods and Sivapalan 1999):

$$Q_i(t) = \iint_{A_i} \int_{T_i}^t W(x, y, t - \tau) u(x, y, t - \tau) d\tau dx dy \quad (1)$$

$Q_i(t)$ is the runoff from catchment i at time t , $W(x, y, t)$ is the point runoff generation process at point (x, y) and time t , while $u(x, y, t)$ is a unit hydrograph for each point in space and time. A_i refers to the catchment area and T_i is the time interval that influences the output, with τ as the integration variable. The specific runoff (runoff per areal unit) can then be found as a weighted average of the runoff generated within each point in the catchment to each time step, where the weighting function can be considered as a unit hydrograph. The unit hydrograph is unknown, and likely to be a complicated function in space and time, where the temporal variable also includes the different responses to different precipitation and evapotranspiration intensities. We assume the simplest possible unit hydrograph, a block hydrograph, that gives constant runoff of $1/T_i$ during a runoff time interval T_i , and 0 at all other times. This gives us the specific runoff at the outlet as the arithmetic average of the runoff point process within the catchment with area A_i , and during the runoff time interval T_i , i.e.:

$$q_i(t) = \frac{1}{A_i T_i} \iint_{A_i} \int_{t-T_i}^t W(x, y, t - \tau) d\tau dx dy \quad (2)$$

Although T_i may be either a constant or a function, depending on the catchment size and its other properties, for readability and practical reasons, we will use the notation T , as if it were a constant. We assume a simple relationship between this runoff time interval and catchment area:

$$T = \alpha A^\beta \quad (3)$$

3.1 Spatio-temporal variogram models

We estimated the spatio-temporal sample variograms for both precipitation and runoff by:

$$\hat{\gamma}_{st}(h_s, h_t) = \frac{1}{2 \sum_{j=1}^{m(h_s)} n_j(h_t)} \sum_{j=1}^{m(h_s)} \sum_{i=1}^{n_j(h_t)} (u(\mathbf{x}_j + \mathbf{h}_s, t_i + h_t) - u(\mathbf{x}_j, t_i))^2 \quad (4)$$

where $h_s = |\mathbf{h}_s|$ is the space lag, h_t is the time lag, $u(\mathbf{x}_j, t_i)$ is the value of the variable (precipitation or specific runoff) at time t_i and spatial location \mathbf{x}_j of station j , $m(h_s)$ is the number of stations with distance h_s , and $n_j(h_t)$ is the number of pairs with time lag h_t . Not all data values were sampled, to reduce computation time.

Different authors have suggested a large number of spatio-temporal variogram models. We have compared three of these models in this paper, a spatio-temporal exponential model, a model proposed by Cressie and Huang (1999) and the product-sum model (De Cesare et al. 2001; De Iaco et al. 2001).

The first model we examined is an exponential model, partly following the Taylor hypothesis.

$$\gamma_{1st}(h_s, h_t) = a_1(1 - \exp(-(c_1 h_t + h_s)/d_1)^{e_1})) \quad (5)$$

a_1 will be equal to the variance for a homogenous process, c_1 is a scaling parameter for time, d_1 is a spatio-temporal correlation length and e_1 is defining the slope of the short distance part of the variogram.

The second model we examined was proposed by Cressie and Huang (1999). They presented a method for developing classes of non-separable, spatio-temporal stationary covariance functions. We have tested several of the models they suggested from this method, and focus in this paper on the following model:

$$\gamma_{2st}(h_s, h_t) = a_2(1 - \frac{c_2 h_t + 1}{((c_2 h_t + 1)^2 + b_2^2 h_s^2)^{3/2}}) \quad (6)$$

$a_2 \geq 0$ is a scaling parameter for the variogram, equal to the variance for a homogenous process, $b_2 \geq 0$ is a scaling parameter for space and $c_2 \geq 0$ is a scaling parameter for time.

The third model we examined is the product-sum model, which is derived from a covariance model combining products and sums (De Cesare et al. 2001; De Iaco et al. 2001):

$$\gamma_{3st}(h_s, h_t) = \gamma_{st}(h_s, 0) + \gamma_{st}(0, h_t) - k \gamma_{st}(h_s, 0) \gamma_{st}(0, h_t) \quad (7)$$

where γ_{st} is the spatio-temporal variogram, and k is a parameter, partly dependent on the variances. $\gamma_{st}(h_s, 0)$ and $\gamma_{st}(0, h_t)$ represent the spatial and temporal variograms, respectively. The product-sum-model was based on the following one-dimensional variogram model both in spatial and temporal directions:

$$\gamma(h) = a_3(1 - \exp(-(h/d_3)^{e_3})) \quad (8)$$

where the parameters are similar to Eq. 5

Skøien et al. (2003) showed that daily precipitation can be regarded as stationary in time, daily mean runoff is almost stationary in time, while neither of the processes can be regarded as stationary in space within the spatial extent of the data. The models above are all describing stationary processes. The correct way to use them as a starting point for non-stationary models would be to add a fractal part. But this will make it difficult to fit the short distance part of the variograms. We have therefore chosen to multiply it with a fractal part in both spatial and temporal directions as:

$$\gamma_{ist}^u = \gamma_{ist} h_s^a h_t^b \quad (9)$$

u is referring to the non-stationary variogram model, i to the variogram model, and with the approximation that $h_s^a = 1$ if $h_s^a < 1$ and $h_t^b = 1$ when $h_t^b < 1$. When using a variogram model for interpolation, it needs to be positive definite (Journal and Huijbregts 1978). Christakos (1984) has developed a series of tests for positive definiteness, but these are difficult to apply to the above variograms. If we

suppose that the stationary parts of the variograms are valid, we assume that the multiplication with a fractal part with a small fractality should not change this. One important condition for variograms to be valid though, is the condition that $\gamma(\mathbf{h})$ must grow more slowly than $\|\mathbf{h}\|^2$ (Cressie 1991; Christakos 1984). The lag \mathbf{h} will in the spatio-temporal context be $\mathbf{h}=(h_s, h_t)$. Skøien et al. (2003) found that the empirical variograms of runoff did not have a higher slope than one on any of the axes. The highest slope of precipitation is found to be 1.6. We therefore assume that Eq. 9 gives valid variograms for small a and b .

3.2 Spatio-temporal regularisation

We assume that the variance of catchment runoff is also dependent on the temporal scale. We have therefore extended the spatial regularisation, described, e.g., in Skøien et al. (2003) and Western and Blöschl (1999), and use the following equation for space-time regularisation:

$$\gamma_{st}(h_s|a, h_t|T) = \int_0^{T \max} \int_0^{R \max} \gamma_{st}(r, \tau) f_{2s}(r|h_s, a) f_{2t}(\tau|h_t, T) dr d\tau - \int_0^{T \max} \int_0^{R \max} \gamma_{st}(r, \tau) f_{1s}(r|a) f_{1t}(\tau|T) dr d\tau \quad (10)$$

f_{2s} and f_{1s} are the pdfs of spatial distances between points in two catchments with separation distance h_s and catchment area S , and two points within a catchment with area S , respectively. f_{2t} and f_{1t} are the pdfs of temporal separations between two points in time, f_{2t} for points within different catchments and/or different starting times for the runoff time interval T , f_{1t} for points within the same runoff time interval T .

A correct way of performing spatio-temporal regularisation of runoff would be to work with real spatio-temporal pdfs of distances in time and space between points within or between catchments. This would be the result of using realistic unit hydrographs, and then finding the pdfs analytically (which is impossible for most catchment geometries and unit hydrographs) or numerically. To focus on the regularisation, we have assumed a block hydrograph, as described in Eq. 2. A second simplification is that we allow for catchments of equal size to be partly overlapping. Natural catchments will always be either non-overlapping, or the larger one will completely cover the smaller. The simplification of assuming catchments to be of a certain size, and allowing them to be partly overlapping, may have some influence on the estimated variograms at short-distance lags.

3.3 Analyses

We have separated our analyses into two parts. In the first part we have fitted theoretical spatio-temporal variogram models to our sample variograms of small,

medium and large catchments independently. In the second part we have jointly fitted the variogram models to the sample variograms from the three catchment size classes. Using the regularisation procedure described in 3.2, this allowed us to back-calculate spatio-temporal variograms for point runoff.

We use the weighted least-squares (WLS) method (see e.g. Cressie 1991) to estimate the optimal parameters by minimizing the objective function:

$$\Phi(A_l) = \sum_{i=1}^{ns} \sum_{j=1}^{nt} w(i, j) \cdot \left[\frac{(\gamma_{l\text{mod}}(h_{si}, h_{tj})}{(\hat{\gamma}_{lst}(h_{si}, h_{tj}))} - 1 \right]^2 \quad (11)$$

Where A_l is the average area of catchment size class l , $\gamma_{l\text{mod}}(h_{si}, h_{tj})$ is the modelled variogram for a spatial lag h_{si} and temporal lag h_{tj} , $\hat{\gamma}_{lst}(h_{si}, h_{tj})$ is the estimated sample variogram for the same spatial and temporal lag, for the same catchment size class. ns and nt are the number of bins in space and time. $w(i, j)$ is a weight for the error of each bin, with the indices i and j in spatial and temporal directions, respectively. We increased the weights along the axes to make sure that the regularised variograms would be well fitted to the univariate spatial and temporal variograms. We used the SCEUA-method (Duan et al. 1992) in the search for the best parameter set.

The spatio-temporal regularisation was only performed for runoff variograms, as all of our precipitation series have the same temporal support. Due to the assumptions above, the variograms of precipitation from different catchment size classes can be found by spatial regularisation of a spatial variogram of point precipitation. This was done for daily precipitation series by Skøien et. al (2003).

4. Results

4.1 Spatio-temporal sample variograms of precipitation and runoff

Figure 1 shows the spatio-temporal sample variograms of precipitation. The total variance of precipitation is similar in time and space, with the measurement scales of our data set. However, the variogram increases faster in the temporal direction than in the spatial direction. The variograms are well-defined for short to medium distances both in space and time, but are a little more noisy for large spatial and temporal distances. There is not much difference between the spatio-temporal sample variograms of precipitation from the different catchment size classes.

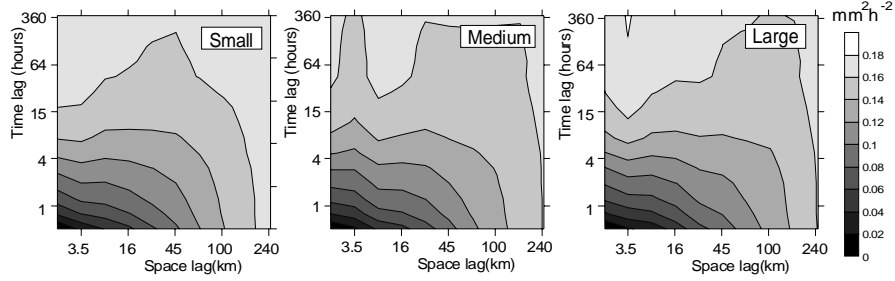


Fig. 1. Spatio-temporal sample variograms of catchment precipitation. Catchment size classes are small, medium and large

Figure 2 shows the spatio-temporal sample variograms of runoff. The figure shows that there is a higher variance in space than in time, with the measurement scales of our data set. The variogram increases faster in the spatial direction than in the temporal direction, opposite to precipitation. The spatio-temporal variograms are well-defined for all spatial and temporal distances, and better defined for long spatial and temporal scales than the variograms of precipitation. There is a large difference between the variograms from catchments of different sizes. It is obvious that the catchment size has an efficient smoothing effect. All of these observations are coherent with the findings of Skøien et al. (2003).

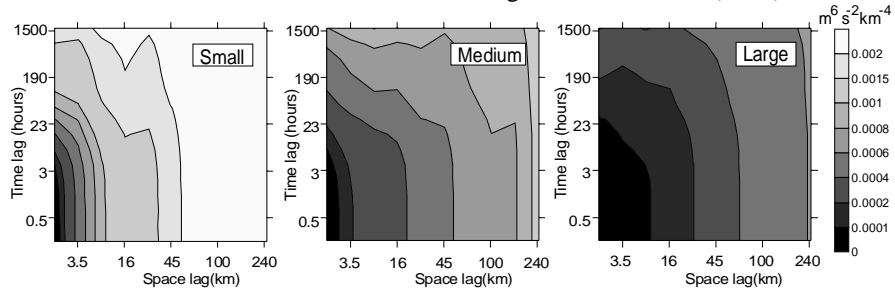


Fig. 2. Spatio-temporal sample variograms of runoff. Catchment size classes are small, medium and large.

4.2 Fitting of spatio-temporal variogram models

We have fitted the spatio-temporal variogram models to the spatio-temporal sample variograms. This has been done independently for the variograms from each catchment size class. In Figure 3, we see how the three different variogram models have been fitted to variograms of precipitation from small catchments. We see that the fitted models fit quite good to the sample variogram. The fits are best for the shorter spatial and temporal distances, the deviations are larger for the longer distances. This is also because the sample variograms are more noisy for larger spatial or temporal distances.

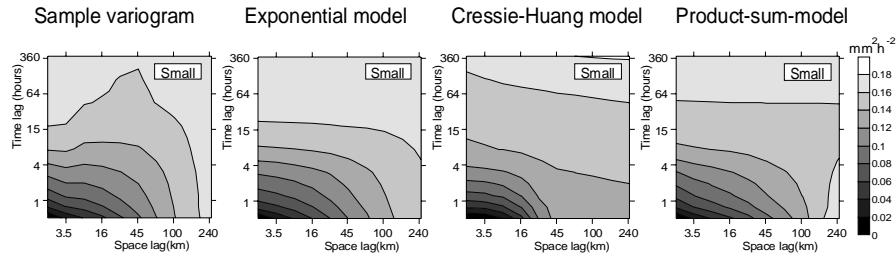


Fig. 3. Spatio-temporal sample variograms of precipitation for small catchments, along with the fitted variogram models.

Table 1 shows the calculated values of the objective function for the best fitted variogram models. The table includes both the objective functions for each independently fitted variogram, and the sum of the objective functions of all the catchment size classes. Table 1 indicates that the product-sum model gives slightly better fits to the sample variograms of precipitation than the exponential model. The Cressie-Huang model is not suited to model spatio-temporal variograms of precipitation. Its errors are largest for short to medium distances along the spatial axis.

Table 1. Objective function for variogram models fitted independently to spatio-temporal sample variograms of precipitation for small, medium and large catchment size classes. Total relates to the sum of the objective functions for the three catchment size classes.

Variogram model	Small catch-ments	Medium catch-ments	Large catch-ments	Total
Exponential model	0.66	0.48	0.79	1.93
Cressie-Huang model	13.8	15.0	13.0	41.8
Product-sum model	0.40	0.40	0.63	1.43

For runoff, the variogram models provide similarly good fits to the sample variograms, although the objective function was higher than for precipitation. Figure 4 shows the spatio-temporal sample variogram of runoff from small catchments, together with the best fitted variogram models. The fits are better for the short spatial and temporal distances than for the longer distances.

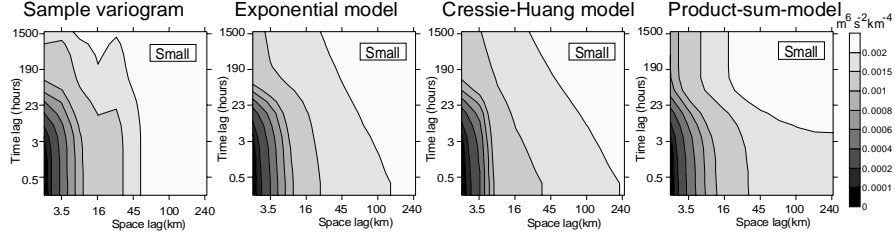


Fig. 4. Spatio-temporal sample variograms of runoff for small catchments, along with the fitted variogram models.

Table 2 shows the calculated values of the objective function for the fitted variogram models. The table includes both the objective function for each independently fitted variogram, and the sum of the objective functions of all the catchment size classes. The exponential model provide a slightly better fit to the sample variograms of runoff than the product-sum model. The Cressie Huang model, again, performs slightly poorer than the two other models.

We would actually expect a non-separable model, such as the Cressie-Huang model, to perform better than the product-sum model for variograms of runoff. Cressie and Huang (1999) formulates the problem with separable variograms as:

$$C^0(h_{s1}; h_t) \propto C^0(h_{s2}; h_t) \quad (12)$$

i.e., the shape of the correlation function in the temporal direction will be equal for different spatial distances. It might be that our choice of a separable non-stationary part has changed the goodness-of-fit of the models. It can also be that both the exponential model and the product-sum model are so much better suited for the spatial and temporal variograms, that they also perform better for the spatio-temporal variograms.

Table 2. Objective function for variogram models fitted independently to spatio-temporal sample variograms of runoff for small, medium and large catchment size classes. Total relates to the sum of the objective functions for the three catchment size classes.

Variogram model	Small catch-ments	Medium catch-ments	Large catch-ments	Total
Exponential model	4.8	6.7	4.6	16.1
Cressie-Huang model	7.5	17.8	24.1	49.4
Product-sum model	9.1	7.2	6.9	23.2

4.3 Joint fitting of spatio-temporal variograms

We have fitted the regularised spatio-temporal variograms to the spatio-temporal sample variograms. This has now been done jointly for the variograms from all

catchment size classes. This allows us to back-calculate a variogram for point runoff. Figure 5 shows the three variogram models fitted to the sample variograms. The Cressie Huang model, again, performs poorer than the two other models.

Table 3 shows the objective functions from the fitting procedure for the different models. As the variograms of different catchment size classes have been fitted jointly, only the total objective function is given. All the three spatio-temporal variogram models give practically equally good fits as a spatio-temporal variogram of point runoff. The regularised variograms from the Cressie-Huang model give better fits to the sample variograms than when we were fitting the model directly.

The back-calculated point variogram models exhibit significantly shorter spatial correlation lengths than any of the catchment scale variogram models. The correlation lengths are on the scale of a kilometre or less, while the small catchments showed correlation lengths of 10-20 kilometres. This is plausible.

The back-calculated point variograms based on the exponential model and the product-sum model exhibit slightly shorter temporal correlation lengths than the small catchment scale variograms, while the Cressie-Huang model gives longer temporal correlation lengths.

While the differences of the point variograms between the models may be significant, they may be not important for interpolation applications, as the catchments of interest are of similar size as the catchments in the data set used here.

Table 3. Objective function of jointly fitted variograms from all three catchment classes, by regularising variogram models of point runoff.

Variogram model	Total
Exponential model	44.9
Cressie-Huang model	46.2
Product-sum model	46.0

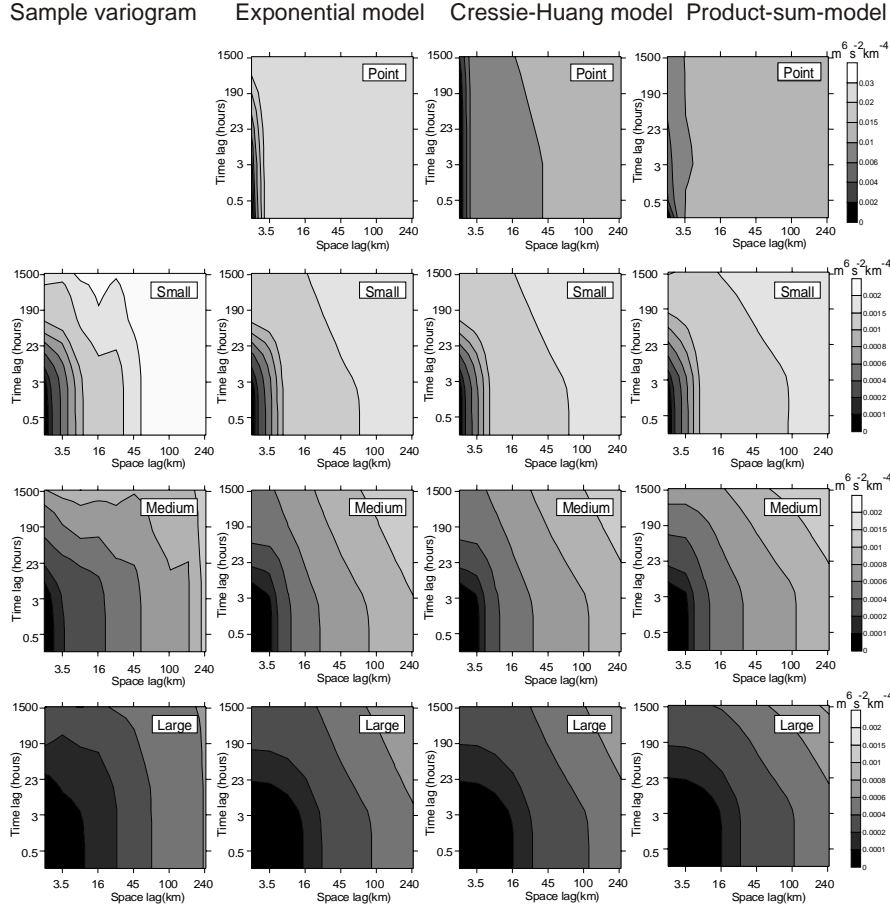


Fig. 5. Spatio-temporal sample variograms (left) of runoff from small, medium and large catchment size classes, along with regularised variograms and the back-calculated variograms of point runoff.

Conclusions

We have fitted three spatio-temporal variogram models independently to the spatio-temporal sample variograms of catchment precipitation and runoff for three different catchment size classes, small, medium and large. The models were an exponential model, a model proposed by Cressie and Huang, and a product-sum model. We found that the exponential model and the product-sum model provided

similarly good fits both for runoff and precipitation. The exponential model was slightly better for runoff, while poorer for precipitation. The model of Cressie and Huang was not suitable for modelling precipitation, and was also significantly poorer for modelling runoff. We think this is related to the fact that both spatial and temporal variograms of runoff can be very well modelled by exponential variograms. It might also be that we have not been able to introduce a proper inseparable non-stationary part in the variogram models, and that we have to search for better non-stationary spatio-temporal variogram models.

We have also fitted regularised variograms jointly to the spatio-temporal sample variograms of the three catchment size classes. From this we back-calculated a spatio-temporal variogram of point runoff. Here, all the three models provided equally good fits.

We are planning to use the back-calculated variograms of point runoff for interpolating runoff time series to catchments without runoff measurements. We found that the back-calculated variograms differed depending on the variogram model used, but for interpolation applications these differences are likely not important as the catchments of interest are of similar size as the catchments in the data set used here.

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