1.1 Fundamentals of the semiclassical laser theory

V.A. Lopota, H. Weber

A rigorous description of light–matter interaction requires a fully quantized system of field equations, which is the content of quantum optics [70Hak, 95Wal, 97Scu, 95Man, 01Vog]. This theory is well developed and the results are confirmed perfectly by many experiments (see Chap. 5.1). But most problems of laser design and laser technology can be solved in a satisfactory way by applying the semiclassical theory. This means a non-relativistic quantum-mechanical approach for the electronic system and a non-quantized, classical electromagnetic field.

Non-relativistic means that the velocity of the interacting electrons is small compared with the velocity of light. This holds for the outer shell electrons of the atoms and molecules, which are relevant in laser physics. It is not true for the free-electron laser and for the interaction of strong fields with plasmas, which demand a relativistic treatment.

A non-quantized electromagnetic field implies that the photon is neglected. In laser technology the photon flux in most applications is extremely high and the granulation of light beams is of no importance. It is of significance for metrology, where the lower limit of detectability is partly given by photon statistics. There are some other effects, which are not covered by the semiclassical theory:

- Planck's law, related to photon statistics,
- squeezed states,
- entangled photons,
- zero-point energy effects,
- spontaneous emission,

and some spectral line shifts (Lamb-shift [47Lam]), of minor importance for laser technology, although of great experimental interest for the confirmation of the fundamental theory. The spontaneous emission of excited atoms/molecules is responsible for the lower limit of laser line width [74Sar, 95Man] and for the on-set of laser oscillation. Therefore, spontaneous emission has to be included in the semiclassical theory by a phenomenological term as shown in Fig. 1.1.1.

It is the intention of this chapter to compile the relevant relations of laser dynamics, their application in laser design and to discuss the limitations and approximations. The mathematical derivations can be taken from the references.

1.1.1 The laser oscillator

The laser oscillator is based on the principle of the feed-back amplifier, a principle invented by A. Meissner 1913 and patented 1919 [19Mei]. All coherent electromagnetic waves are generated by such self-sustained oscillators, from radio frequencies to microwaves and finally lasers. Basov and Prokhorov published 1954 a theoretical paper on masers [54Bas], Schawlow and Townes in 1958 [58Sch] a theoretical paper discussing the possibility of masers in the visible range of the spectrum, and Maiman realized 1960 the first laser [60Mai].

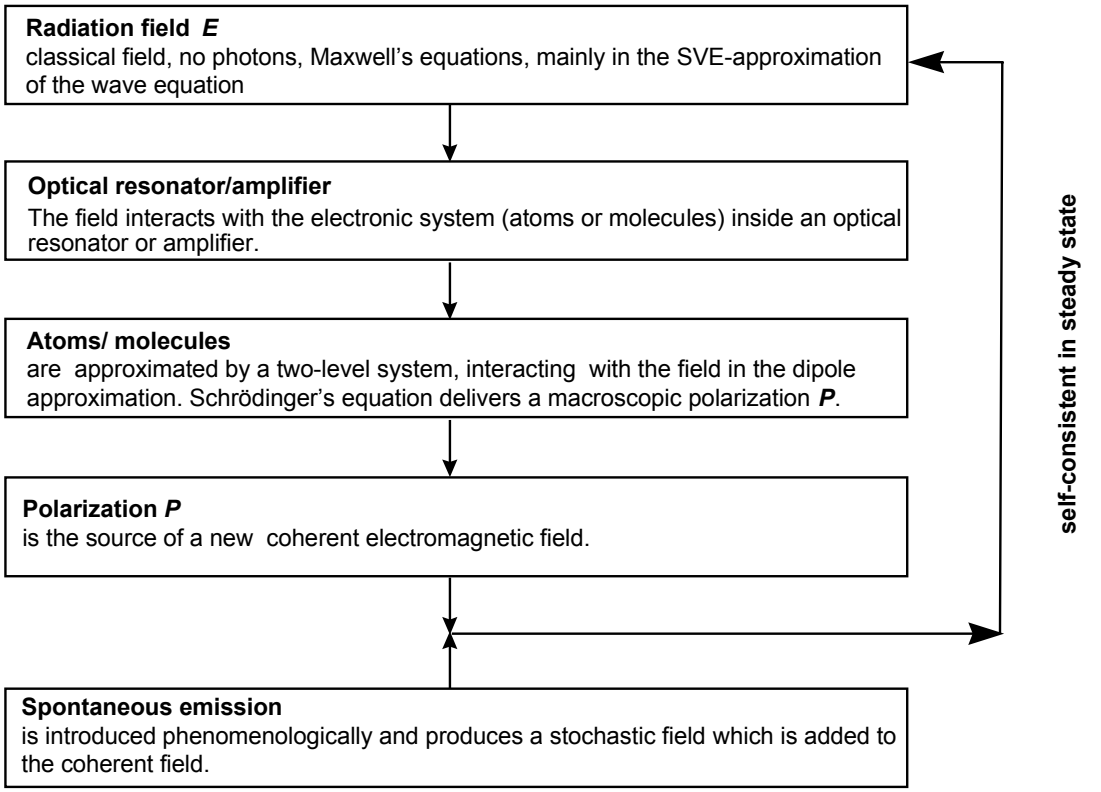

Fig. 1.1.1. The semiclassical laser theory (SVE-approximation: Slowly Varying Envelope approximation, see Sect. 1.1.2.2.3).

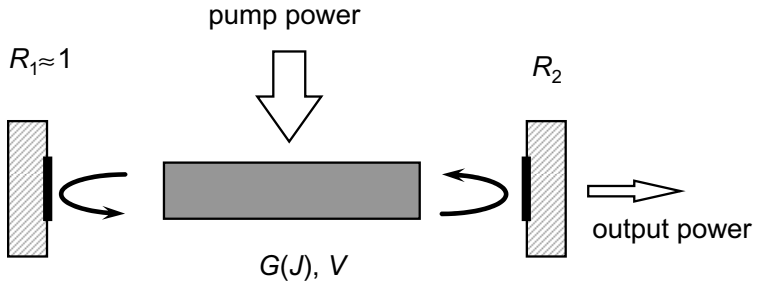

Fig. 1.1.2. Schematic set-up of a laser oscillator.

The principle set-up of a laser oscillator is plotted in Fig. 1.1.2. Light is amplified by induced emission in an active medium (gas discharge, doped crystals or liquids, pn-transitions). The active medium is characterized by an intensity- and frequency-dependent gain factor $G(J)$ (with J: intensity). The beam bounces forth and back between the two mirrors of an optical resonator. On-set of laser oscillation requires a gain factor exceeding the total losses per round trip:

$$G_0 RV > 1 \quad \text{(threshold condition)} \tag{1.1.1}$$

with

G_0: small-signal gain factor for the intensities,
$R = \sqrt{R_1 R_2}$: average reflection factor of the mirrors,
V: internal loss factor of the resonator.

With increasing intensity J the gain decreases due to saturation of the amplifier

$G(J) \leq G_0$.

In steady state the gain has to compensate the losses:

$$G(J)RV = 1 \quad \text{(steady-state condition)} . \tag{1.1.2}$$

If the relation $G(J)$ is known, depending on the specific amplifier, (1.1.2) gives the internal intensity of the laser system in steady state.

The wavelength of the field is determined by the resonance condition. After one round trip the phase shift $\Delta\varphi$ of the field must be

$$\Delta\varphi = 2\pi p\ ,\ p = 1, 2, 3, \ldots \quad \text{(resonance condition)} , \tag{1.1.3}$$

otherwise the field would be reduced by destructive interference. The resonator is mainly responsible for the mode structure of the output field and can be described by a non-quantized field. Details are given in Chap. 8.1. For the interaction field–amplifier a plane wave is assumed and diffraction is neglected.

1.1.2 The electromagnetic field

Light is a special case of propagating electromagnetic waves, as was predicted by Maxwell 1856 [54Max] and confirmed experimentally by Hertz [88Her]. The electromagnetic field is characterized by the electric/magnetic vector fields $\boldsymbol{E}$, $\boldsymbol{H}$. *In this section the propagation of quasi-monochromatic waves with frequency* ω *and wavelength* λ *is investigated. The wavelength range from the infrared (*$\lambda \approx$ *some 10* μm*) to the UV (*$\lambda \approx 0.1$ μm*) is normally called light.*

1.1.2.1 Maxwell's equations

The electromagnetic field is used in the classical representation, neglecting the quantization. The materials equations, based on quantum mechanics, are introduced phenomenologically. The final result is a wave equation, describing the propagation of electromagnetic waves.

The classical electromagnetic field is completely described by Maxwell's equations:

$$\text{curl}\ \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} , \tag{1.1.4}$$

$$\text{curl}\ \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{j} , \tag{1.1.5}$$

$$\text{div}\ \boldsymbol{D} = \rho , \tag{1.1.6}$$

$$\text{div}\ \boldsymbol{B} = 0 \tag{1.1.7}$$

with

$\boldsymbol{E}$: electric field (SI-unit: V/m),
$\boldsymbol{H}$: magnetic field (SI-unit: A/m),
$\boldsymbol{D}$: electric displacement (SI-unit: As/m^2),
$\boldsymbol{B}$: magnetic induction (SI-unit: Vs/m^2),
$\boldsymbol{j}$: current density (SI-unit: A/m^2),
ρ: density of electric charges (SI-unit: As/m^3).

For all quantities the complex notation is used [99Bor], the real quantities are $Q_{\mathrm{real}} = \frac{1}{2}(Q+Q^*)$. The relations between $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{B}$, $\boldsymbol{H}$ are given by the material equations. Under the action of an external electric/magnetic field atomic or molecular electric/magnetic dipoles are generated in matter. The dipole moment per unit volume is called the electric or magnetic polarization $\boldsymbol{P}(\boldsymbol{E},\boldsymbol{H})$ or $\boldsymbol{J}(\boldsymbol{E},\boldsymbol{H})$, respectively. The resulting material quantities are the electric displacement $\boldsymbol{D}$ and the magnetic induction $\boldsymbol{B}$ given as:

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}(\boldsymbol{E},\boldsymbol{H}) = \varepsilon_0\, \varepsilon(\boldsymbol{E},\boldsymbol{H}) \cdot \boldsymbol{E}\ , \tag{1.1.8}$$

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{J}(\boldsymbol{E},\boldsymbol{H}) = \mu_0\, \mu(\boldsymbol{E},\boldsymbol{H}) \cdot \boldsymbol{H} \tag{1.1.9}$$

with

$\boldsymbol{P} = \varepsilon_0 \chi_{\mathrm{e}}(\boldsymbol{E},\boldsymbol{H})\boldsymbol{E}$: electric polarization (SI-unit: As/m^2),
$\boldsymbol{J} = \mu_0 \chi_{\mathrm{m}}(\boldsymbol{E},\boldsymbol{H})\boldsymbol{H}$: magnetic polarization (SI-unit: Vs/m^2),

$\chi_{\mathrm{e}}(\boldsymbol{E},\boldsymbol{H})$, $\chi_{\mathrm{m}}(\boldsymbol{E},\boldsymbol{H})$: electric/magnetic susceptibility, in general a tensor and a function of the fields,
$\varepsilon = 1 + \chi_{\mathrm{e}}$, $\mu = 1 + \chi_{\mathrm{m}}$: permittivity/permeability number, in general tensors, 1 : unit tensor,

$\varepsilon_0 = 8.8542 \times 10^{-12}$ As/Vm: electric constant,
$\mu_0 = 4\pi \times 10^{-7}$ Vs/Am: magnetic constant.

The current inside a medium is caused by the electric field and Ohm's law holds

$$\boldsymbol{j} = \sigma_{\mathrm{e}} \boldsymbol{E} \tag{1.1.10}$$

with

σ_{e}: electric conductivity, in general a tensor and function of the field, (SI-unit: A/Vm).

Electric and magnetic polarization depend in general on both generating fields, $\boldsymbol{E}$ and $\boldsymbol{H}$. In many cases this relation is linear, but quite often a very complicated relation occurs, as in non-linear optics, ferro-magnetism or ferro-electricity. The material equations can only be evaluated by quantum mechanics. In the following non-conducting ($\sigma_{\mathrm{e}} = 0$), charge-free ($\rho = 0$) and non-magnetic ($\chi_{\mathrm{m}} = 0$, $\mu = 1$) media are assumed, which holds for dielectrics. The magnetic field can be eliminated and a wave equation results from Maxwell's equations:

$$\text{grad div}\ \boldsymbol{E} - \Delta \boldsymbol{E} + \frac{1}{c_0^2} \frac{\partial^2}{\partial t^2} \left(\boldsymbol{E} + \frac{1}{\varepsilon_0} \boldsymbol{P} \right) = 0\ , \tag{1.1.11}$$

$$\text{div}\ \boldsymbol{D} = 0 \tag{1.1.12}$$

with

$$c_0 = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.99792458 \times 10^8\ \text{m/s}\ :\ \text{vacuum velocity of light}\ .$$

Equation (1.1.11) is the fundamental equation, describing the propagation of optical fields. It includes diffraction as well as amplification of light and non-linear effects. It has now to be adapted and simplified for the different applications in optics and laser technology.

1.1.2.2 Homogeneous, isotropic, linear dielectrics

The propagation of light in homogeneous media as gases, liquids, glasses or cubic crystals is investigated. These materials are assumed to be homogeneous (permittivity ε does not depend on the

spatial coordinates), isotropic (ε does not depend on the polarization of light), and linear (ε does not depend on the intensity of the field). The last assumption holds for low-intensity fields only.

The permittivity ε is a scalar and (1.1.11)/(1.1.12) reduces to the *standard wave equation*:

$$\Delta \boldsymbol{E} - \frac{\varepsilon}{c_0^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0 \,, \tag{1.1.13}$$

$$\text{div}\ \boldsymbol{E} = 0 \,. \tag{1.1.14}$$

Simple solutions are the plane and the spherical waves.

1.1.2.2.1 The plane wave

The infinite, monochromatic wave with a plane phase front and constant amplitude reads:

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[\mathrm{i}(\omega t - n\boldsymbol{k}_0\boldsymbol{r})] \,, \tag{1.1.15}$$

$$\boldsymbol{H} = \boldsymbol{H}_0 \exp[\mathrm{i}(\omega t - n\boldsymbol{k}_0\boldsymbol{r})] \,; \tag{1.1.16}$$

$$\boldsymbol{H}_0 = \frac{[\boldsymbol{k}_0 \times \boldsymbol{E}_0]}{k_0 Z} \,.$$

It is a transversely polarized field with $\boldsymbol{E} \perp \boldsymbol{H} \perp \boldsymbol{k}_0$, as plotted in Fig. 1.1.3.

$$n = \sqrt{\varepsilon} = \sqrt{1 + \chi_\mathrm{e}} \ : \quad \text{the refractive index of the medium, in general complex,} \tag{1.1.17}$$

$k_0 = 2\pi/\lambda_0$: wave number in vacuum,
$\boldsymbol{k}_0$: wave vector, direction of propagation,
λ_0: wavelength in vacuum,

$$Z = \sqrt{\frac{\mu\mu_0}{\varepsilon\varepsilon_0}} \ : \quad \text{impedance,} \quad Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega \ : \quad \text{vacuum impedance.}$$

The Poynting vector or energy flux is a real quantity with

$$\boldsymbol{S} = [\boldsymbol{E}_\mathrm{real} \times \boldsymbol{H}_\mathrm{real}] \quad (\text{SI-unit: W/m}^2).$$

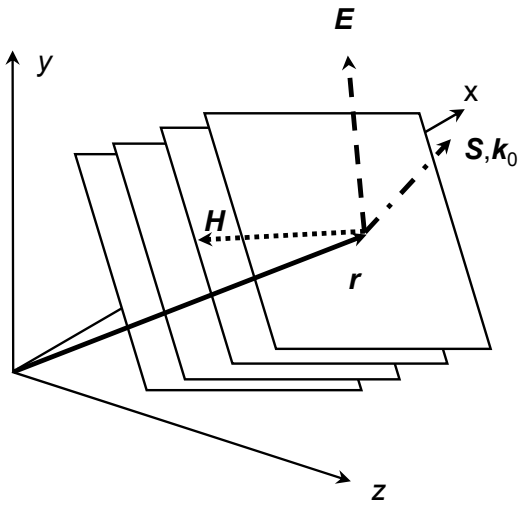

Fig. 1.1.3. The plane wave in a homogeneous, isotropic medium.

Table 1.1.1. Values of refractive index n_r and absorption coefficient α at wavelength λ_0 [85Pal, 82Gra, 78Dri].

Material	λ_0 [μm]	n_r	α [m^{-1}]
Fused quartz	0.54	1.46	very small
Sapphire	0.50	1.765/1.764	very small
Water	0.54	1.332	0.8
Water	1	1.328	80
Copper	0.54	0.7	11.6×10^6
Gold	0.54	0.3	11.1×10^6
Iron	0.54	2.4	16.4×10^6

The intensity is the time average over one period $T = 2\pi/\omega$ and results in:

$$J = \langle \boldsymbol{S} \rangle_T = \frac{1}{4}\left(\frac{1}{Z} + \frac{1}{Z^*}\right) \langle \boldsymbol{E}_0 \boldsymbol{E}_0^* \rangle \ . \tag{1.1.18}$$

For dielectrics without losses ($\mu = 1$, $n = n_r$ is real), (1.1.18) reduces to

$$J = \frac{1}{2} c_0 n_r \varepsilon_0 \left|E_0\right|^2 \tag{1.1.19}$$

with both quantities, $\boldsymbol{E}_0$ and J, inside the medium. For vacuum applies

$$J_{W/m^2} = 1.33 \times 10^{-3} \left|E_{0,V/m}\right|^2 \ , \quad \left|E_{0,V/m}\right| = 27.4 \sqrt{J_{W/m^2}} \ .$$

For a homogeneous dielectric, low-absorbing medium the complex refractive index is given by [99Bor, p. 739]:

$$\hat{n} = n_r - \mathrm{i}\frac{\alpha}{2k_0} \ , \quad \alpha \ll k_0 \tag{1.1.20}$$

with

n_r: real part of the refractive index,
α: absorption coefficient, in general the non-resonant broad-band absorption.

For a field propagating in z-direction (1.1.15)/(1.1.20) deliver an exponentially damped amplitude:

$$\boldsymbol{E}(z,t) = \boldsymbol{E}_0 \exp\left[\mathrm{i}(\omega t - n_r k_0 z) - \frac{\alpha z}{2}\right] \ . \tag{1.1.21}$$

Some numbers of n_r, α are compiled in Table 1.1.1.

1.1.2.2.2 The spherical wave

One solution of the wave equation (1.1.13) in spherical coordinates is the quasi-spherical wave, generated by an oscillating dipole (Hertz's dipole), see Fig. 1.1.4. The far field reads [99Jac]:

$$\boldsymbol{E}\,(r,\vartheta,t) = \frac{\lambda_0 \boldsymbol{E}_\vartheta}{r} \exp\left[\mathrm{i}\,(\omega t - \hat{n}\boldsymbol{k}_0\boldsymbol{r})\right] \sin\vartheta \ , \quad \left|\boldsymbol{E}_\vartheta\right| = \frac{\left|\boldsymbol{\mu}\right| \ 4\pi^2 k_0^3}{\varepsilon_0} \ , \quad r \gg \lambda_0$$

with μ the dipole moment and ϑ the angle between the dipole axis and beam propagation $\boldsymbol{k}_0$. In the paraxial approach ($\vartheta \cong \pi/2$, $\theta \ll 1$) the well-known spherical wave, useful for applying Huygens' principle, results:

$$\boldsymbol{E}(z,t) \cong \frac{\lambda_0}{r}\, \boldsymbol{E}_0 \exp\left[\mathrm{i}\,(\omega t - \hat{n}\boldsymbol{k}_0\boldsymbol{r})\right] \ , \quad \theta \ll 1 \ , \tag{1.1.22}$$

where $\boldsymbol{E}$ is approximately parallel to the dipole axis.

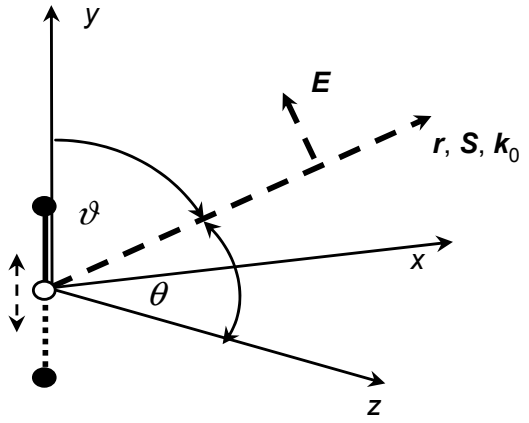

Fig. 1.1.4. A quasi-spherical wave, emitted by an oscillating dipole.

1.1.2.2.3 The slowly varying envelope (SVE) approximation

In the Slowly Varying Envelope approximation (1.1.11) is solved approximately with the ansatz of a quasi-monochromatic, quasi-plane wave

$$\boldsymbol{E} = \boldsymbol{E}_0(x,y,z,t)\exp[\mathrm{i}(\omega t - n_\mathrm{r}k_0 z)]\ , \quad \boldsymbol{P} = \boldsymbol{P}_0(x,y,z,t)\exp[\mathrm{i}(\omega t - n_\mathrm{r}k_0 z)]\ . \tag{1.1.23}$$

The wave propagates mainly in z-direction and the amplitude is slowly varying with x, y, z, t, which means:

- slowly varying in time (quasi-monochromatic): $\partial|\boldsymbol{E}_0|/\partial t \ll \omega|\boldsymbol{E}_0|$, or spectral bandwidth $\Delta\omega \ll \omega$,
- slowly varying in space (quasi-plane wave): $\partial|\boldsymbol{E}_0|/\partial z \ll k_0|\boldsymbol{E}_0|$, which means low divergence of the beam $\Delta\theta \ll 1$ (paraxial approach), and a smooth transverse profile,
- slowly varying polarization $\partial|\boldsymbol{P}_0|/\partial t \ll \omega|\boldsymbol{P}_0|$,
- slowly varying electric susceptibility $\partial|\chi_\mathrm{e}|/\partial t \ll \omega|\chi_\mathrm{e}|$ and $|\mathrm{grad}\ \chi_\mathrm{e}| \ll k_0|\chi_\mathrm{e}|$.

Then second order terms can be neglected and the SVE-approximations are obtained [84She, p. 47], [66War, 86Sie].

1.1.2.2.4 The SVE-approximation for diffraction

Steady-state propagation in vacuum means $\partial|\boldsymbol{E}_0|/\partial t = 0$ and $\boldsymbol{P} = 0$. Equation (1.1.11) delivers with the ansatz (1.1.23) and neglecting $\partial^2\boldsymbol{E}_0/\partial t^2$ the SVE-approximation used in diffraction theory, also called the *Schrödinger equation of optics*:

$$\left(\Delta_\mathrm{tr} - 2\mathrm{i}k_0\frac{\partial}{\partial z}\right)\boldsymbol{E}_0 = 0\ , \quad \mathrm{div}\ \boldsymbol{E} = 0\ . \tag{1.1.24a}$$

Δ_tr is the transverse delta-operator, which in rectangular coordinates reads

$$\Delta_\mathrm{tr} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\ .$$

The field in (1.1.24a) is a vector field, and the Δ-operator in cylinder coordinates is rather complicated, because the unit-vectors are no longer constant [99Jac], especially for non-uniform polarization in circular birefringent media [82Fer, 93Wit]. In most cases (except birefringence) the

scalar version of the SVE-approximation is sufficient. It reads in rectangular/cylindrical coordinates

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-2\mathrm{i}k_0\frac{\partial}{\partial z}\right)E_0=0\,, \tag{1.1.24b}$$

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}-2\mathrm{i}k_0\frac{\partial}{\partial z}\right)E_0=0\,. \tag{1.1.24c}$$

This is the fundamental equation in paraxial diffraction optics. It gives the Fresnel-integral and the eigenmodes of free propagation (Gauss-Hermite/Gauss-Laguerre polynomials, see Chaps. 3.1 and 8.1). Equations (1.1.24a)/(1.1.24b)/(1.1.24c) hold for a homogeneous medium, but can be extended to quadratic index media [86Sie].

1.1.2.3 Propagation in doped media

The active medium of a laser amplifier consists of a host material, doped with the active atoms (molecules). Host and doping interact differently with the laser radiation.

A plane wave without transverse structure interacts with active atoms or molecules and induces a polarization $\boldsymbol{P}_\mathrm{A}$. In most cases the active atoms are embedded in a host medium (glass, crystal, liquid, gas), which is also polarized by the field, generating an additional polarization $\boldsymbol{P}_\mathrm{H}$. The total polarization is:

$$\boldsymbol{P}=\boldsymbol{P}_\mathrm{A}+\boldsymbol{P}_\mathrm{H}=(\boldsymbol{P}_\mathrm{A0}+\boldsymbol{P}_\mathrm{H0})\exp[\mathrm{i}(\omega t-n_\mathrm{r}k_0z)]\,. \tag{1.1.25}$$

The response of the host medium is in most cases very fast (10^{-12} ... 10^{-14} s), no transient behavior occurs and nonlinear effects are assumed to be small. Then the host polarization is proportional to the applied field:

$$\boldsymbol{P}_\mathrm{H}=\varepsilon_0\chi_\mathrm{H}\boldsymbol{E}\,.$$

χ_H is the complex susceptibility of the host material and is related to the refractive index n_r and the loss coefficient α according to (1.1.17)/(1.1.20) [99Ber]:

$$\chi_\mathrm{H}=(n_\mathrm{r}^2-1)-\mathrm{i}\,\frac{n_\mathrm{r}\alpha}{k_0}\,,\quad \alpha\ll k_0\,. \tag{1.1.26}$$

The imaginary part of χ_H is called extinction coefficient. Some values of refractive indices n_r and absorption coefficients α are given in Table 1.1.1. For the polarization of the active atoms one has

$$\boldsymbol{P}_\mathrm{A}=\varepsilon_0\chi_\mathrm{A}(\boldsymbol{E}_0)\boldsymbol{E}\,, \tag{1.1.27}$$

where χ_A depends on the field and has to be evaluated quantum-mechanically. Neglecting first and second order derivations of $\boldsymbol{P}_\mathrm{A0}$ and second order derivations of $\boldsymbol{E}_0$, the SVE-approximation for the interaction is obtained, assuming a plane wave without transverse structure:

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}+\frac{\alpha}{2}\right)E_0=-\mathrm{i}\,\frac{k_0}{2\varepsilon_0 n_\mathrm{r}}\boldsymbol{P}_\mathrm{A0}\,(\boldsymbol{E}_0)\,,\quad \mathrm{div}\,\boldsymbol{E}=0 \tag{1.1.28}$$

(SVE-approximation for the amplitude of a plane wave in an active medium)

with $c=c_0/n_\mathrm{r}$ the phase velocity of the wave in the host medium. The above equation describes the amplification/attenuation of cw-fields and pulsed radiation by an active medium. It provides also the widely used rate-equation approach, as will be shown in Sect. 1.1.5.1. It fails for fields

with amplitudes varying very rapidly in time or space (fs-pulses). If the intensity J (1.1.19) and the susceptibility of the active medium (1.1.27) are introduced, (1.1.28) reduces to:

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right) J+\left(\alpha-\frac{k_0}{n_\mathrm{r}}\,\mathrm{Im}\,\chi_\mathrm{A}\right) J=0\ . \tag{1.1.29}$$

The active atoms enhance or reduce the losses of the medium, depending on the sign of the imaginary part $\mathrm{Im}\,\chi_\mathrm{A}$ of the susceptibility, which is a function of the intensity. In steady state and for constant χ_A, which holds for low intensities, (1.1.29) can be integrated and delivers for the intensity

$$J(z)=J(0)\exp\left[-\alpha+\frac{k_0}{n_\mathrm{r}}\,\mathrm{Im}\,(\chi_\mathrm{A})\right] z\ .$$

The amplifying factor is called the small-signal gain factor G_0 of the medium and the exponent the small-signal gain coefficient g_0:

$$G_0=\exp\left[\frac{k_0}{n_\mathrm{r}}\,\mathrm{Im}(\chi_\mathrm{A})z\right]=\exp\left[g_0 z\right]\ ,\quad g_0=\frac{k_0}{n_\mathrm{r}}\,\mathrm{Im}\,(\chi_\mathrm{A})\ . \tag{1.1.30}$$

Some typical values of g_0 are compiled in Table 1.1.4.

1.1.3 Interaction with two-level systems

Most quantum systems as atoms or molecules have an infinite number of energy levels. To demonstrate the essential features of light–matter interaction, a simplified model with only two levels is presented.

1.1.3.1 The two-level system

The relevant parameters are the energy difference ΔE of the two levels, the inversion Δn, the dipole moment $\boldsymbol{\mu}$, and the polarization $\boldsymbol{P}_\mathrm{A}$.

The two-level system can be part of an atom, ion, molecule, or something more complicated. A monochromatic electric field $\boldsymbol{E}$ of frequency ω in the SVE-approximation according to (1.1.23) acts via the Coulomb force on the bound electrons of the active medium. In linear systems (parabolic potential) the negative electrons will oscillate sinusoidally, whereas the heavy positive nucleus remains more or less at rest. An oscillating dipole is induced with a dipole moment $\boldsymbol{\mu}(t)$, which is given by

$$\boldsymbol{\mu}=-e\boldsymbol{x} \tag{1.1.31}$$

with

e: electron charge,
$\boldsymbol{x}$: displacement of the electron.

The dipole moment per volume is the macroscopic polarization $\boldsymbol{P}_\mathrm{A}$ of the active medium. As all single dipoles are aligned by the electric field, the resulting polarization reads:

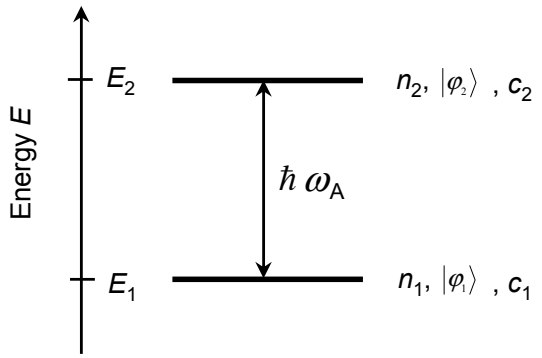

Fig. 1.1.5. The two-level system.

$$\boldsymbol{P}_\mathrm{A} = n_0 \boldsymbol{\mu} \tag{1.1.32}$$

with

n_0: dipole density (m^{-3}),
$\boldsymbol{\mu}$: expectation value of the dipole moment (Asm).

In this section the induced dipole moment will be evaluated quantum-mechanically, which requires some simplifications. It is not the intention to discuss in detail the mathematics, but only to summarize briefly the main results of interest for laser technology and to emphasize the approximations and the range of validity. A consistent presentation of the interaction light–matter, starting from first principles, is given in many textbooks [61Mes, 68Sch, 77Coh, 95Man].

From the infinite number of energy levels of an electronic system only two, E_1 and E_2, are taken into account for the interaction [75All, 89Yar, 69Are], see Fig. 1.1.5. This is a reasonable approach if the field is nearly resonant with the transition from E_1 to E_2. In this case the other levels of the system will not or only very weakly interact with the field.

It applies

$$|\omega_\mathrm{A} - \omega| \ll \Delta\omega_\mathrm{A}$$

with

ω_A: resonance frequency of the transition,
$\Delta\omega_\mathrm{A}$: bandwidth of the transition,
ω: frequency of the radiation field,
$\hbar = 1.0546 \times 10^{-34}$ Ws^2: Planck's constant.

1.1.3.2 The dipole approximation

The oscillating electric field $\boldsymbol{E}$ *deforms the electron cloud of the two-level system and generates a complicated, oscillating charge distribution. A first order approximation is an oscillating dipole. The interaction of this dipole with a monochromatic wave is evaluated quantum-mechanically.*

1.1.3.2.1 Inversion density and polarization

The interaction of an electromagnetic field with a two-level system was first investigated by Bloch [46Blo] and extensively discussed by Allen and Eberly [75All]. It is characterized by its dipole moment and the population densities in the two levels:

n_1, n_2 : density of states (atoms, molecules) in the lower/upper level,
$\Delta n = n_2 - n_1$: inversion density,
$n_0 = n_1 + n_2$: total density, const.

The following assumptions are made:

- Non-relativistic interaction. The velocity of the electrons is small compared with the velocity of light. This does not hold for inner-shell electrons, hot plasmas and free-electron lasers.
- The wavelength of the light is large compared with the diameter of the atoms/molecules. It means that in the domain of the atomic wave function the electromagnetic field is locally constant. Bohr's radius with $r_B = 5.3 \times 10^{-5}$ μm is a typical atomic dimension. The wavelength in the visible range of the spectrum is about 0.5 μm, thus this condition is fulfilled in the visible and UV-part of the spectrum. It is called the dipole approximation [97Scu].
- The permanent dipole moments of the two-level system $\boldsymbol{\mu}_{11} = \boldsymbol{\mu}_{22}$ are zero. Even if larger molecules have a permanent dipole moment, their response to the high-frequency field is small. Only for very strong fields are the permanent dipole moments of importance (see Part 4 on nonlinear optics). A dipole moment exists only for the transition from level 1 to 2 and vice versa. Non-degenerated levels are assumed with $\boldsymbol{\mu} = \boldsymbol{\mu}_{12} = \boldsymbol{\mu}_{21}$.

The two-level system is completely described by its state vector $|\varphi\rangle$, which in general is time-dependent:

$$|\varphi\rangle = c_1(t)\,|\varphi_1\rangle\,\exp\left(-\mathrm{i}\frac{E_1 t}{\hbar}\right) + c_2(t)\,|\varphi_2\rangle\,\exp\left(-\mathrm{i}\frac{E_2 t}{\hbar}\right)\ , \tag{1.1.33}$$

with $|\varphi_1\rangle$, $|\varphi_2\rangle$ the eigenfunctions and E_1, E_2 the energy eigenstates. The eigenfunctions are normalized, orthogonal and depend on the position vector $\boldsymbol{r}$:

$$\int \varphi_i^* \varphi_j \mathrm{d}r = \langle \varphi_1 \varphi_2 \rangle = \delta_{ij}\ . \tag{1.1.34}$$

The state vector has to fulfill the time-dependent Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial\,|\varphi\rangle}{\partial t} = (H_0 + H_{\mathrm{int}})\,|\varphi\rangle\ , \tag{1.1.35}$$

with H_0 the Hamilton operator of the undisturbed system ($H_{\mathrm{int}} = 0$) and H_{int} the interaction energy. For the undisturbed system holds [89Yar]:

$$H_0\,|\varphi_i\rangle = E_i\,|\varphi_i\rangle\ , \quad i = 1,\ 2\ , \tag{1.1.36}$$

which follows directly from (1.1.35) by replacing $|\varphi\rangle$ by $|\varphi_i\rangle \exp(-\mathrm{i}E_i t/\hbar)$. The parameters of interest, the inversion density $\Delta n = n_2 - n_1$ and the macroscopic polarization

$$\boldsymbol{P}_{\mathrm{A}} = n_0\,\boldsymbol{\mu} \tag{1.1.37}$$

are determined by the coefficients c_1, c_2. The probability of the system to be in the lower/upper state is given by $|c_1|^2$, $|c_2|^2$, respectively, which requires:

$$|c_1|^2 + |c_2|^2 = 1\ . \tag{1.1.38}$$

The number of atoms in the lower/upper level is then given by:

$$n_1 = n_0\,|c_1|^2\ , \quad n_2 = n_0\,|c_2|^2\ , \quad n_1 + n_2 = n_0$$

and hence the inversion density :

$$\Delta n = n_0 \left(|c_2|^2 - |c_1|^2 \right) \ . \tag{1.1.39}$$

The expectation value of the dipole moment $\langle \boldsymbol{\mu} \rangle = -e \langle \varphi \boldsymbol{r} \varphi \rangle$ is obtained from (1.1.33). Using the afore mentioned assumptions:

$$\langle \boldsymbol{\mu}_{11} \rangle = -e \langle \varphi_1 \boldsymbol{r} \varphi_1 \rangle = 0 \ , \quad \langle \boldsymbol{\mu}_{22} \rangle = -e \langle \varphi_2 \boldsymbol{r} \varphi_2 \rangle = 0$$

one obtains finally for the polarization from (1.1.33), (1.1.34), (1.1.38)

$$\boldsymbol{P}_\mathrm{A} = n_0 \left\{ \langle \boldsymbol{\mu}_{12} \rangle \ c_1^* c_2 \ \exp\left(-\mathrm{i}\,\omega_\mathrm{A} t\right) + \langle \boldsymbol{\mu}_{21} \rangle \ c_1 c_2^* \ \exp\left(+\mathrm{i}\,\omega_\mathrm{A} t\right) \right\} \tag{1.1.40}$$

with $\langle \boldsymbol{\mu}_{12} \rangle$, $\langle \boldsymbol{\mu}_{21} \rangle$ the dipole moment of the transition $E_1 \leftrightarrow E_2$ and vice versa. For non-degenerated transitions one has $\langle \boldsymbol{\mu}_{12} \rangle = \langle \boldsymbol{\mu}_{21} \rangle = \boldsymbol{\mu}_\mathrm{A}$. In the following only $\boldsymbol{\mu}_\mathrm{A}$ will be used, which is a characteristic parameter of the specific transition:

$$\boldsymbol{\mu}_\mathrm{A} = -e \langle \varphi_1 \boldsymbol{r} \varphi_2 \rangle \ . \tag{1.1.41}$$

1.1.3.2.2 The interaction with a monochromatic field

The interaction Hamiltonian for a non-quantized real field $\boldsymbol{E}_\mathrm{real}$ corresponds to the classical energy of an electric dipole in an electric field. It reads [97Scu]:

$$H_\mathrm{int} = \boldsymbol{\mu}_\mathrm{A} \boldsymbol{E}_\mathrm{real} = \boldsymbol{\mu}_\mathrm{A} \frac{(\boldsymbol{E} + \boldsymbol{E}^*)}{2} \ . \tag{1.1.42}$$

Substitution of (1.1.42) into (1.1.35), using the orthogonality (1.1.34) and (1.1.41) provides two differential equations for the coefficients c_1, c_2 of the state vector:

$$\begin{aligned} \frac{\mathrm{d}c_1}{\mathrm{d}t} &= \frac{\mathrm{i}}{\hbar} c_2 \ \exp\left(-\mathrm{i}\omega_\mathrm{A} t\right) \boldsymbol{\mu}_\mathrm{A} \frac{(\boldsymbol{E} + \boldsymbol{E}^*)}{2} \ , \\ \frac{\mathrm{d}c_2}{\mathrm{d}t} &= \frac{\mathrm{i}}{\hbar} c_1 \ \exp\left(+\mathrm{i}\omega_\mathrm{A} t\right) \boldsymbol{\mu}_\mathrm{A} \frac{(\boldsymbol{E} + \boldsymbol{E}^*)}{2} \ . \end{aligned} \tag{1.1.43}$$

The time dependence of inversion density and polarization is obtained from (1.1.39), (1.1.40) by differentiating and applying (1.1.43). After some simple mathematics the following two equations for the macroscopic parameters of the two-level-system result are obtained:

$$\frac{\partial \Delta n}{\partial t} = \frac{\mathrm{i}}{\hbar} \left\{ (\boldsymbol{E} + \boldsymbol{E}^*) \left(\boldsymbol{P}_\mathrm{A} - \boldsymbol{P}_\mathrm{A}^* \right) \right\} \ , \tag{1.1.44a}$$

$$\frac{\partial \boldsymbol{P}_\mathrm{A}}{\partial t} = \mathrm{i} \left\{ \omega_\mathrm{A} \boldsymbol{P}_\mathrm{A} + \frac{\mu_\mathrm{A}}{\hbar} \langle \boldsymbol{\mu}_\mathrm{A} (\boldsymbol{E} + \boldsymbol{E}^*) \rangle \, \Delta n \right\} \ . \tag{1.1.44b}$$

For $\boldsymbol{E}$ and $\boldsymbol{P}_\mathrm{A}$ the SVE-approximations of (1.1.23), (1.1.25) are used. Then in (1.1.44a), (1.1.44b) terms with the frequency 2ω appear, which are neglected. This approach is called the rotating-wave approximation [97Scu, 72Cou]. The above equations simplify to

$$\frac{\partial \Delta n}{\partial t} = \frac{\mathrm{i}}{2\hbar} \left\{ \boldsymbol{E}_0^* \boldsymbol{P}_\mathrm{A0} - \boldsymbol{E}_0 \boldsymbol{P}_\mathrm{A0}^* \right\} \ , \tag{1.1.45a}$$

$$\frac{\partial \boldsymbol{P}_\mathrm{A0}}{\partial t} = -\mathrm{i}\,\delta \boldsymbol{P}_\mathrm{A0} + \frac{\mathrm{i}\,\boldsymbol{\mu}_\mathrm{A}}{\hbar} \langle \boldsymbol{\mu}_\mathrm{A} \boldsymbol{E}_0 \rangle \, \Delta n \ , \quad \delta = \omega - \omega_\mathrm{A} \tag{1.1.45b}$$

(rotating-wave approximation)
with

$\boldsymbol{\mu}_\mathrm{A}$: electric dipole moment of the transition,
ω: frequency of the interacting field,
ω_A: resonance frequency of the two-level system,
$\hbar = 1.0546 \times 10^{-34}$ Ws^2: Planck's constant.

Some typical values of dipole moments are given in Table 1.1.2.

Table 1.1.2. Typical values of dipole moments [01Men].

Transition			$\lvert\boldsymbol{\mu}_A\rvert$ [As m]
Bohr's radius × electron charge			10^{-29}
Hydrogen	1s – 2p	$\lambda_0 = 121$ nm	0.8×10^{-29}
	4f – 5g	$\lambda_0 = 4053$ nm	8.3×10^{-29}
Chromium ions in ruby	$4A_2(3/2)$ – E levels	$\lambda_0 = 694$ nm	10^{-29}

1.1.3.3 The Maxwell–Bloch equations

The idealized rotating-wave approximation is adapted to the real situation and combined with the SVE wave equation. Incoherent perturbations by the environment are taken into account.

So far the interaction of the two-level system with the electromagnetic field is purely coherent, no perturbations by external influences on the system are considered. Stochastic processes will modify the interaction considerably. Here only a very basic description is presented. A detailed analysis of these statistical processes is given in [70Hak, 97Scu].

1.1.3.3.1 Decay time T_1 of the upper level (energy relaxation)

Three incoherent processes reduce or increase the upper-level population and have to be considered in (1.1.45a), (1.1.45b):

- spontaneous emission,
- interaction with the host material (collisions, lattice vibrations),
- increase of the population by pumping (light, electron collisions, or other processes).

1.1.3.3.1.1 Spontaneous emission

The two-level system is coupled to the modes of the optical resonator or to the free-space modes. Spontaneous emission into these modes reduces the upper-level population. Moreover, by each spontaneous emission process the phase relation between the field and the two-level eigenfunction is destroyed. If the dimensions of the resonator are large compared with the wavelength, the decay is given by $\partial n_2/\partial t = -n_2/T_{sp}$, with $A_{21} = 1/T_{sp}$, the Einstein coefficient of spontaneous emission. If the resonator dimensions are comparable with the wavelength, spontaneous emission is strongly influenced by the resonator geometry, it can be enhanced or reduced (see Chap. 8.1).

1.1.3.3.1.2 Interaction with the host material

This interaction reduces the population density. Energy is transferred to the host material and converted into heat. A simple approach for this decay is again an exponential ansatz $\partial n_2/\partial t = -n_2/T_H$. This decay time together with the spontaneous decay time delivers a resulting decay T_1 of the upper-level population, also called energy relaxation time or longitudinal relaxation time.

1.1.3.3.1.3 Pumping process

The dynamics of upper-level excitation depend on the special pumping scheme and are discussed in Sect. 1.1.5.3 and in Vol. VIII/1B, "Solid-state laser systems". In any case the pump produces in steady state and without a coherent field ($\boldsymbol{E}_0 = 0$) an inversion density Δn_0.

These three processes are included into (1.1.45a) by the term:

$$\frac{\partial \Delta n}{\partial t} = -\frac{\Delta n - \Delta n_0}{T_1} \tag{1.1.46}$$

with

T_1: the resulting time constant.

1.1.3.3.2 Decay time T_2 of the polarization (entropy relaxation)

An external field $\boldsymbol{E}$ induces dipoles, which generate the macroscopic polarization $\boldsymbol{P}_\mathrm{A}$. If the external field is switched off, the polarization will disappear for several reasons:

The energy of the two-level system decays with T_1, which means that the polarization disappears at least with the same time constant.

Due to incoherent interaction with the host material (collisions), the single dipoles are disoriented in their direction or dephased. The resulting polarization becomes zero, although the single dipole still exists. This process can be much faster than T_1 (see Table 1.1.6) and is characterized by a time constant T_2. This decay strongly depends on the interaction process. The simplest approach is :

$$\frac{\partial \boldsymbol{P}_\mathrm{A0}}{\partial t} = -\frac{\boldsymbol{P}_\mathrm{A0}}{T_2} \ , \tag{1.1.47}$$

and (1.1.45b) has to be completed by (1.1.47). T_2 is called the transverse relaxation time, the entropy time constant or the dephasing time. Finally, the two-level equations together with the SVE-approximation, (1.1.28), of the wave equation read:

$$\frac{\partial \Delta n}{\partial t} = -\frac{\mathrm{i}}{2\hbar}\left(\boldsymbol{E}_0^* \boldsymbol{P}_\mathrm{A0} - \boldsymbol{E}_0 \boldsymbol{P}_\mathrm{A0}^*\right) - \frac{\Delta n - \Delta n_0}{T_1} \ , \tag{1.1.48a}$$

$$\frac{\partial \boldsymbol{P}_\mathrm{A0}}{\partial t} = -\left(\mathrm{i}\,\delta + \frac{1}{T_2}\right)\boldsymbol{P}_\mathrm{A0} + \mathrm{i}\frac{\boldsymbol{\mu}_\mathrm{A}\left\langle \boldsymbol{\mu}_\mathrm{A} \boldsymbol{E}_0 \right\rangle}{\hbar}\Delta n \ , \quad \delta = \omega - \omega_\mathrm{A} \ , \tag{1.1.48b}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t} + \frac{\alpha}{2}\right)\boldsymbol{E}_0 = -\mathrm{i}\frac{k_0}{2\varepsilon_0 n_\mathrm{r}}\boldsymbol{P}_\mathrm{A0} \tag{1.1.48c}$$

(Maxwell–Bloch equations).

They describe the propagation of radiation in two-level systems and are called Maxwell–Bloch equations. Equation (1.1.48c) holds, if the transition frequency ω_A for all two-level atoms is the same (homogeneous system). In inhomogeneous systems (see Sect. 1.1.6.3, Fig. 1.1.13) different groups of atoms exist with center frequencies ω_A of each group and a center frequency ω_R of the ensemble. Therefore (1.1.48c) has to be replaced by [81Ver]:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\boldsymbol{E}_0 = -\mathrm{i}\frac{k_0}{2\varepsilon_0 n_\mathrm{r}}\int h(\omega_\mathrm{A}, \omega_\mathrm{R})\boldsymbol{P}_\mathrm{A0}(\boldsymbol{E}_0, \omega_\mathrm{A})\mathrm{d}\omega_\mathrm{A} \ . \tag{1.1.48d}$$

$h(\omega, \omega_\mathrm{A})$ is the spectral density of atoms with the transition frequency ω_A according to (1.1.92)/(1.1.93). For the solution of these equations, three different regimes are distinguished:

Steady-state equations The temporal variations of the radiation field are slow compared with T_1.	$\frac{\partial \Delta n}{\partial t} = 0$	$\frac{\partial \boldsymbol{P}_{\mathrm{A0}}}{\partial t} = 0$
Adiabatic equations no transient effects of the atom, $T_2 \ll T_1$.	$\frac{\partial \Delta n}{\partial t} \neq 0$	$\frac{\partial \boldsymbol{P}_{\mathrm{A0}}}{\partial t} = 0$
Coherent equations The width τ of the interacting pulses is short compared with T_1, T_2; (1.1.45a), (1.1.45b) can be applied.	$\frac{\partial \Delta n}{\partial t} \neq 0$	$\frac{\partial \boldsymbol{P}_{\mathrm{A0}}}{\partial t} \neq 0$

1.1.4 Steady-state solutions

In steady state inversion density Δn_0, polarization $\boldsymbol{P}_{\mathrm{A0}}$, and intensity J of the field are constant in time, but may depend on the spatial coordinates.

1.1.4.1 Inversion density and polarization

The stationary solutions of (1.1.48a), (1.1.48b) are obtained immediately:

$$\Delta n = \frac{\Delta n_0}{1 + (J/J_{\mathrm{s}})\, f(\omega)} \quad (\textit{inversion density}, \text{ homogeneously broadened}), \tag{1.1.49}$$

$$\chi_{\mathrm{A}} = \frac{n_{\mathrm{r}}\sigma}{k_0}\left[\frac{\omega - \omega_{\mathrm{A}}}{\Delta\omega_{\mathrm{A}}/2} + \mathrm{i}\right]\Delta n \quad (\textit{susceptibility}), \tag{1.1.50}$$

$$\boldsymbol{P}_{\mathrm{A0}} = \varepsilon_0 \chi_{\mathrm{A}} \boldsymbol{E}_0 \quad (\textit{polarization}) \tag{1.1.51}$$

with

$$J = \frac{1}{2}\,\varepsilon_0 c_0 n_{\mathrm{r}} |\boldsymbol{E}_0|^2 \quad \text{(intensity of the field)}, \tag{1.1.52}$$

$$J_{\mathrm{s}} = \frac{\hbar\omega_{\mathrm{A}}}{2\sigma_0 T_1} \quad \text{(saturation intensity of the two-level transition)}, \tag{1.1.53}$$

$$\sigma = \sigma_0 f(\omega, \omega_{\mathrm{A}}) \quad \text{(frequency-dependent cross section of the transition)}, \tag{1.1.54}$$

$$\sigma_0 = \frac{|\boldsymbol{\mu}_{\mathrm{A}}|^2\, \omega_{\mathrm{A}} T_2}{\varepsilon_0 c_0 n_{\mathrm{r}} \hbar} \quad \text{(cross section in resonance)}, \tag{1.1.55}$$

$$f_{\mathrm{L}}(\omega, \omega_{\mathrm{A}}) = \frac{(\Delta\omega_{\mathrm{A}}/2)^2}{(\omega_{\mathrm{A}} - \omega)^2 + (\Delta\omega_{\mathrm{A}}/2)^2} \quad \text{(spectral line shape, Lorentzian)}, \tag{1.1.56}$$

$$\Delta\omega_{\mathrm{A}} = 2/T_2 \quad \text{(line width of the transition)}, \tag{1.1.57}$$

$$g_{\mathrm{h}}(\omega,\omega_{\mathrm{A}}) = \Delta\, n\, \sigma = \frac{\Delta\, n_0 \sigma_0 f(\omega,\omega_{\mathrm{A}})}{1+(J/J_{\mathrm{s}})\, f(\omega,\omega_{\mathrm{A}})} \quad \text{(gain coefficient, homogeneously broadened),} \tag{1.1.58a}$$

$$g_{\mathrm{inh}}(\omega,\omega_{\mathrm{R}}) = \frac{\Delta\, n_0 \sigma_0}{\sqrt{1+J/J_{\mathrm{s}}}} h(\omega,\omega_{\mathrm{R}}) \frac{\Delta\omega_{\mathrm{A}}}{2} \quad \text{(gain coefficient, inhomogeneously broadened, see Sect. 1.1.6.3).} \tag{1.1.58b}$$

In Table 1.1.3 some numbers of relevant laser transitions are compiled, in Table 1.1.4 some typical values of the small-signal gain coefficient in resonance are given. The susceptibility strongly depends on the frequency as shown in Fig. 1.1.6. According to (1.1.26) the real part of χ_{A} produces an additional refractive index, and the imaginary part absorption or amplification:

$$\mathrm{Re}\ \chi_{\mathrm{A}} = n_{\mathrm{r}}^2 - 1 = \frac{n_{\mathrm{r}}\sigma}{k_0}\left(\frac{\omega-\omega_{\mathrm{A}}}{\Delta\omega_{\mathrm{A}}/2}\right)\Delta n\ , \tag{1.1.59a}$$

$$\mathrm{Im}\ \chi_{\mathrm{A}} = -n_{\mathrm{r}}\alpha k_0 = \frac{n_{\mathrm{r}}\sigma}{k_0}\Delta n\ . \tag{1.1.59b}$$

The steady-state propagation of the electric field is obtained from (1.1.48c):

$$\frac{\mathrm{d}\boldsymbol{E}_0}{\mathrm{d}z} = \left[-\frac{\alpha}{2} + \frac{\sigma\Delta\, n}{2} + \mathrm{i}\sigma\Delta\, n\frac{\omega-\omega_{\mathrm{A}}}{\Delta\,\omega_{\mathrm{A}}}\right]\boldsymbol{E}_0\ , \tag{1.1.60}$$

where $\Delta\, n$ is a function of the field or the intensity.

Table 1.1.3. Examples of resonance wavelength λ_0, resonance cross section σ_0, upper-level lifetime T_1 and saturation intensity J_{s}. The simple relation (1.1.53) for the saturation intensity holds for two-level systems only and is not applicable in general [01Men].

	λ_0 [μm]	σ_0 [m^2]	T_1 [s]	J_{s} [$\mathrm{W/m}^2$]
Amplifiers				
CO_2-gas (1300 Pa)	10.6	10^{-20}	10^{-5}	2×10^5
Neodymium-ion in glass	1.06	4×10^{-24}	3×10^{-4}	$8\ldots12\times10^7$
Neodymium-ion in YAG	1.06	5×10^{-23}	2×10^{-4}	2×10^7
Chromium-ion in Al_2O_3 (ruby, $T=300$ K)	0.69	2×10^{-24}	3×10^{-3}	2.4×10^7
Neon (25 Pa)	0.63	3×10^{-17}	10^{-8}	5.3×10^5
Rhodamine 6G in ethanol	0.57	4×10^{-20}	5×10^{-9}	10^9
Absorbers				
SF_6	10.6	8×10^{-22}	4×10^{-4}	2.5×10^5
KODAK dye 9860	1.06	4×10^{-20}	$\sim10^{-11}$	5.6×10^{11}
KODAK dye 9740	1.06	6×10^{-20}	$\sim10^{-11}$	4×10^{11}
Cryptocyanine-dye in methanol	0.7	5×10^{-20}	5×10^{-10}	2×10^{10}

Table 1.1.4. Typical values of the small-signal gain coefficient $g_0 = \Delta n_0 \sigma_0$ in resonance. The exact values depend on pumping, doping, and other parameters of operation [01Men].

System	λ_0 [nm]	g_0 [m^{-1}]
He/Ne laser	632.8	0.1
Nd-doped glass	1060	5
Nd-doped YAG	1060	50
GaAs-diode	880	4×10^3

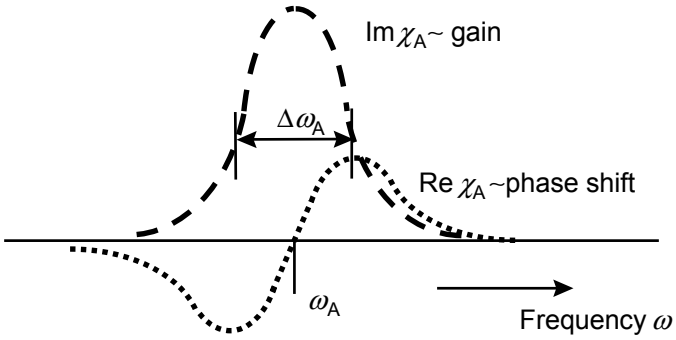

Fig. 1.1.6. Real and imaginary part of the susceptibility vs. frequency.

1.1.4.2 Small-signal solutions

The solutions for low intensities are discussed. Low means that the intensity J is small compared with the characteristic parameter J_s of the system (see Table 1.1.3).

At low intensities $J \ll J_s$, the inversion density is not affected by the intensity,

$$\Delta n = \Delta n_0 \ ,$$

and (1.1.60) can be integrated. Together with (1.1.23), the complete field is obtained:

$$\boldsymbol{E}(z) = \boldsymbol{E}_0(0) \exp[\mathrm{i}(\omega t - n_t k_0 z) - \frac{1}{2}(\alpha - \Delta n_0 \sigma) z] \tag{1.1.61}$$

with a total refractive index n_t

$$n_t = n_r \left(1 + \frac{\sigma \Delta n_0}{n_r k_0} \frac{\omega - \omega_A}{\Delta\omega_A}\right) \ . \tag{1.1.62}$$

The active atoms of the two-level system cause an additional phase shift or refractive index and an additional absorption or amplification, depending on the sign of Δn_0. The small-signal gain factor according to (1.1.30)/(1.1.50) is:

$$G_0 = \exp[\sigma(\omega) \Delta n_0 z] \ . \tag{1.1.63}$$

Amplification, $G_0 > 1$, requires inversion $\Delta n_0 > 0$. The complex amplitude transmission factor A is defined as the ratio of the monochromatic field amplitudes and can be written:

$$A = \frac{E_0(z)}{E_0(0)} = \exp\left[\mathrm{i} \frac{\sigma_0 \Delta n_0}{2} \frac{\Delta\omega_A/2}{(\omega - \omega_A) + \mathrm{i}\,\Delta\omega_A/2}\right] z \ . \tag{1.1.64}$$

It depends on the frequency of the field, which means dispersion. Time-dependent fields and especially short pulses are distorted by the amplifying system, pulse broadening and chirping occur.

1.1.4.3 Strong-signal solutions

The steady-state solutions are discussed for intensities which saturate the inversion, see Fig. 1.1.7.

The inversion now depends on the intensity. For the propagation of the intensity, (1.1.48c) gives in steady state

$$\frac{\mathrm{d}J}{\mathrm{d}z} = (g(J) - \alpha)\ J\ , \tag{1.1.65}$$

where $g(J)$ is the saturated gain coefficient of (1.1.58a), (1.1.58b). For a homogeneously broadened transition and without losses ($\alpha = 0$) this equation can be can be integrated and provides a transcendental relation for the gain factor G:

$$\frac{G_0}{G} = \exp\left[\frac{J(0)}{J_\mathrm{s}} f(\omega)\ (G-1)\right] \tag{1.1.66}$$

with G_0 the small-signal gain factor of (1.1.62) and G the ratio of output/input intensities

$$G = J(z)/J(0)\ .$$

For inhomogeneously broadened transitions a more complicated relation is obtained [81Ver].

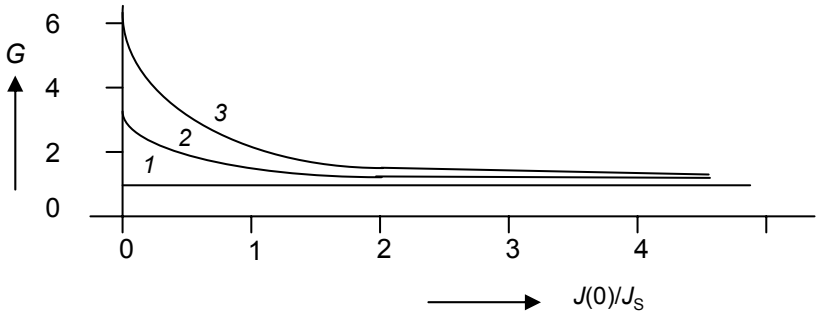

Fig. 1.1.7. Saturation of the gain factor G for a homogeneously and inhomogeneously broadened transition. *1*: $G_0 = 1$, *2*: $G_0 = 4$, *3*: $G_0 = 6$.

1.1.5 Adiabatic equations

If the polarization is in equilibrium with the applied field, without transient oscillations of the electronic system, the interaction is called adiabatic.

1.1.5.1 Rate equations

The field is replaced by the intensity, most spectral effects are neglected and the rate equations are obtained. They represent an energy balance.

T_2 is the time constant, which characterizes the transient behavior of the polarization. In most cases (see Table 1.1.6) T_2 is much smaller than T_1, and the transient oscillations of the electrons can be neglected. In (1.1.48a) the polarization is replaced by its steady-state value (1.1.50)/(1.1.51) and the rate equations are obtained. They have to be completed by the time-dependent pump term, here labeled as Δn_0. It depends on the specific pump scheme (see Sect. 1.1.5.3). The rate equations are widely used in laser design to evaluate output power, spiking behavior and Q-switching dynamics. The spontaneous emission contributes to the intensity of the interacting field, but only with a very

small amount and is neglected here. Nevertheless it is important, because the laser is started by spontaneous emission and in the lower limit it determines the laser band width (Chap. 5.1).

With these approximations the field equations (1.1.48a)/(1.1.48b)/(1.1.48c) for the interaction with a monochromatic field reduce to one equation for the inversion density and a transport equation for the intensity:

$$\frac{\partial \Delta n}{\partial t} = -\frac{J\, f(\omega)}{J_s T_1}\Delta n - \frac{(\Delta n - \Delta n_0)}{T_1}\ , \tag{1.1.67}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) J = (\Delta n\, \sigma_0 f(\omega))\ J \tag{1.1.68}$$

(rate equations for a homogeneously broadened two-level system and a plane monochromatic wave)

with

$J(z,t)$: local intensity,
J_s: saturation intensity, depends on the level system (2,3, or 4 levels), see Sects. 1.1.4.1/1.1.5.3,
$\Delta n(z,t)$: local inversion density.

1.1.5.2 Thermodynamic considerations

So far the interaction with a monochromatic field of intensity $J(\omega)$ was discussed. Now the intensity is replaced by the spectral energy density ρ_ω of black-body radiation, providing the Einstein coefficients of spontaneous and induced emission.

Einstein published in 1917 [17Ein] his famous work on the quantum theory of radiation, where for the first time induced emission was introduced, the cornerstone of laser physics. He discussed the two-level system in equilibrium with thermal radiation of spectral energy density ρ_ω (energy per volume and spectral range $\mathrm{d}\,\omega$). The density is given by Planck's law [61Mor]:

$$\rho_\omega = \frac{\hbar\,\omega^3}{\pi^2 c^3}\frac{1}{\exp\left[\hbar\,\omega/\kappa T\right] - 1} \qquad \left[\frac{\mathrm{VAs}^2}{\mathrm{m}^3}\right] \tag{1.1.69}$$

with

$\kappa = 1.38 \times 10^{-23}$ VAs/K: Boltzmann's constant.

In thermal equilibrium the levels $|\varphi_1\rangle$, $|\varphi_2\rangle$ are populated according to Boltzmann's law [61Mor]:

$$\frac{n_2}{n_1} = \exp\left[-\hbar\,\omega_\mathrm{A}/\kappa T\right]\ . \tag{1.1.70}$$

These two fundamental laws can only be fulfilled, if induced emission is introduced, and Einstein postulated the following equation in steady state for the interaction of thermal radiation with a two-level system:

$$B_{12}\,\rho_\omega\, n_1 = B_{21}\,\rho_\omega\, n_2 \qquad + A_{21}\, n_2 \tag{1.1.71}$$

(absorption = induced emission + spontaneous emission)

with

B_{12}, B_{21}, A_{21}: Einstein coefficients of induced and spontaneous emission.

The transition of atoms from the lower level to the upper level by absorption of radiation must be balanced by induced emission and spontaneous emission from the upper level. This equation was

derived by thermodynamical considerations. The quantum-mechanical equation (1.1.67) delivers in steady state, replacing Δn by $n_2 - n_1$ and n_0 by $n_1 + n_2$, and furthermore taking into account that for steady state without interaction holds $\Delta n_0 = -n_0$:

$$J\frac{\sigma}{\hbar\omega_\mathrm{A}}n_1 = J\frac{\sigma}{\hbar\omega_\mathrm{A}}n_2 + \frac{n_2}{T_1}\ . \tag{1.1.72}$$

This equation has the same structure as the Einstein equation. If the monochromatic intensity $J(\omega)$ is replaced by the spectral density ρ_ω and integration over the full spectral range is performed, a relation between the Einstein coefficients and the atomic parameters is obtained. These relations read in general for degenerated levels with weighting factors g_1, g_2 (degeneracies) [92Koe, 81Ver, 00Dav]:

$$B_{12} = \frac{g_2}{12\pi}\frac{|\mu_\mathrm{A}|^2}{\hbar^2\varepsilon\varepsilon_0}\ , \tag{1.1.73a}$$

$$B_{21} = \frac{g_1}{12\pi}\frac{|\mu_\mathrm{A}|^2}{\hbar^2\varepsilon\varepsilon_0}\ , \tag{1.1.73b}$$

$$A_{21} = \frac{1}{T_1} = \frac{g_1}{3}|\mu_\mathrm{A}|^2\frac{\omega_\mathrm{A}^3}{\pi\varepsilon\varepsilon_0\hbar c^3}\ , \tag{1.1.74}$$

$$\mu_\mathrm{A} = \mu_{12} = \mu_{21}\ ,$$

$$\sigma_{21}(\omega) = \frac{\lambda^2}{4}A_{21}h(\omega)\ , \tag{1.1.75}$$

$$\sigma_{12}(\omega) = \frac{g_2}{g_1}\sigma_{21}(\omega)\ , \tag{1.1.76}$$

$$\sigma_{21}(\omega_\mathrm{A}) = \frac{\lambda^2}{4\pi}\frac{T_2}{T_1} \le \frac{\lambda^2}{4} \qquad \text{(holds for Lorentzian line shape)}, \tag{1.1.77}$$

$$B_{12}g_1 = B_{21}g_2\ , \tag{1.1.78}$$

$$\frac{A_{21}}{B_{21}} = \frac{2\hbar\omega_\mathrm{A}^3}{\pi c^3}\ . \tag{1.1.79}$$

The above relations were derived for isotropic media. Anisotropic media are discussed in [86Sie]. Equation (1.1.80) holds for all dipole transitions, as long as the quantum system is coupled to a large number of modes (free space or a resonator with dimensions large compared with the wavelength). With these equations the gain coefficient can be related to the Einstein coefficient of spontaneous emission [92Koe]:

$$g(\omega) = \frac{\lambda^2}{4}h(\omega,\omega_\mathrm{A})\left[n_2 - \frac{g_2}{g_1}n_1\right]A_{21} \tag{1.1.80}$$

with

$h(\omega,\omega_\mathrm{A})$: the spectral line shape, depending on the type of broadening (see Sect. 1.1.6).

1.1.5.3 Pumping schemes and complete rate equations

The fundamental methods to obtain inversion are presented, discussing the idealized 3- and 4-level system.

Till now a two-level system was discussed, assuming a steady-state inversion Δn_0, which is always negative. To obtain positive inversion $\Delta n = n_2 - n_1 > 0$ and gain, additional levels are necessary.

$\Delta n > 0$ is a state of non-equilibrium. To support this state, energy has to be pumped into the system. This pumping energy can be incoherent light, kinetic energy of electrons/ions, chemical energy or electric energy. The pumping schemes can become very complicated, and in most cases many energy levels are involved. To understand the principal process for the generation of inversion, two idealized pumping schemes will be discussed.

1.1.5.3.1 The three-level system

The simplified diagram of the three-level system is shown in Fig. 1.1.8. The level E_3 is excited by absorption of light or by electron collisions, depending on the specific system. The decay from E_3 to E_2, the upper laser level, is very fast. Nearly all excited atoms are transferred into this level, which has a very long life time. If the pumping power is sufficiently high to overcome the decay of level E_2, atoms will be accumulated and finally n_2 is larger than n_1. The adiabatic rate equations give for the upper-level population without induced emission between the two levels ($J = 0$):

$$\frac{\mathrm{d}\, n_2}{\mathrm{d}\, t} = W\,(n_0 - n_2) - \frac{n_2}{T_1}\ . \tag{1.1.81}$$

W is the pumping rate, the product of the cross-section σ_{13} and the specific pump parameters. T_1 is the upper laser-level lifetime. This holds under the assumption that the population of level E_3 is zero and that $n_1 + n_2 = n_0$. Equation (1.1.81) reads with the inversion density $\Delta n = n_2 - n_1$:

$$\frac{\mathrm{d}\Delta\, n}{\mathrm{d}\, t} = W\,(n_0 - \Delta\, n) - \frac{n_0 - \Delta\, n}{T_1} \tag{1.1.82}$$

and in steady state one obtains:

$$\frac{\Delta\, n_{\mathrm{steady},3}}{n_0} = \frac{WT_1 - 1}{WT_1 + 1}\ . \tag{1.1.83}$$

The relation between the inversion density and the pump rate is shown in Fig. 1.1.9. Inversion occurs for $WT_1 > 1$. With increasing pump rate the inversion increases also and approaches finally one, all atoms are in the upper level. To obtain $\Delta\, n_{\mathrm{steady},3} > 0$ requires at least 50 % of the active atoms to be pumped into the upper level, high pump rates are necessary and the efficiency is low. Equation (1.1.82) has to be completed by the coherent interaction term of (1.1.67). The complete rate equation for the three-level system with pump rate W, interacting with a monochromatic field of intensity J is given in (1.1.84). For the intensity (1.1.48c), (1.1.48d) hold, depending on the type of line-broadening (Sect. 1.1.6).

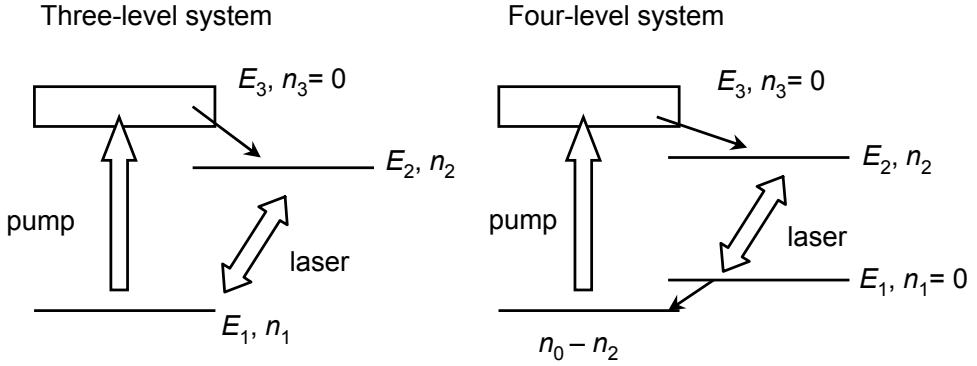

Fig. 1.1.8. The idealized three- and four-level system.

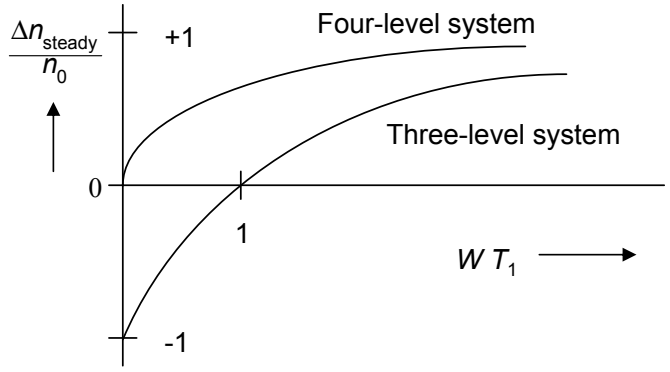

Fig. 1.1.9. Inversion density vs. pump rate for a three- and four-level system.

$$\frac{\partial \Delta n}{\partial t} = -\frac{J}{J_\mathrm{s}} \frac{f(\omega)}{T_1} \Delta n + W (n_0 - \Delta n) - \frac{n_0 + \Delta n}{T_1} \tag{1.1.84}$$

(rate equation of a three-level system).

1.1.5.3.2 The four-level system

The commonly used pump scheme, due to its high efficiency, is the four-level system as shown in Fig. 1.1.8. The two laser levels are E_2 and E_1, where the lower level E_1 has a very short lifetime and its population n_1 is nearly zero. This requires that the energy $E_1 - E_0$ is much larger than the thermal energy κT. The pump level E_3 decays very rapidly to the upper laser level E_2 and its population is again nearly zero. The inversion density now is $\Delta n = n_2 - n_1 \approx n_2$. Then the rate equation for the pump process reads:

$$\frac{\partial \Delta n}{\partial t} = W (n_0 - \Delta n) - \frac{\Delta n}{T_1} \tag{1.1.85}$$

with the steady-state solution (without coherent interaction):

$$\frac{\Delta n_{\mathrm{steady},4}}{n_0} = \frac{WT_1}{1 + WT_1} \, . \tag{1.1.86}$$

Inversion is reached now at very small pump-power levels as shown in Fig. 1.1.9. The efficiency of such systems is much higher than of three-level systems. The complete rate equation for pumping and interaction with a field of intensity J is obtained by taking into account the corresponding term of (1.1.67). It has to be considered that $n_1 = 0$, and therefore the saturation intensity is higher by a factor of 2.

$$\frac{\partial \Delta n}{\partial t} = -\frac{J}{J_{\mathrm{s},4}} \frac{f(\omega)}{T_1} \Delta n + W(n_0 - \Delta n) - \frac{\Delta n}{T_1} \tag{1.1.87}$$

(rate equation of a four-level system)

with

$J_{\mathrm{s},4} = \dfrac{\hbar \omega_\mathrm{A}}{\sigma_0 T_1}$: saturation intensity of the four-level system.

1.1.5.4 Adiabatic pulse amplification

The amplification and shaping of light pulses by saturable two-level systems is presented.

The pulse is adiabatic if its width τ is small compared with T_1 and large compared with T_2. Then the variation of the upper-level population due to spontaneous emission and pump can be neglected and this term can be neglected. If such a pulse travels through an active medium of length ℓ, it depletes the upper level, is amplified and shaped as depicted in Fig. 1.1.10. The initial conditions at $t = -\infty$ are:

Inversion density:	$\Delta n(z) = \Delta n_0$,	$0 \le z \le \ell$.
Input intensity:	J_0 ,	$z = 0$.
Input energy:	E_{in} ,	$z = 0$.

The equations (1.1.67)/(1.1.68) can be solved for a loss-free-medium ($\alpha = 0$) with a four-level system and yield for the output intensity [63Fra]:

$$J_{\mathrm{out}}(t) = J_{\mathrm{in}}(t - \ell/c) \frac{G_0}{G_0 - (G_0 - 1)\exp\left[-\dfrac{1}{E_{\mathrm{s}}} \displaystyle\int_{-\infty}^{t-\ell/c} J_{\mathrm{in}}(t')\mathrm{d}t'\right]} . \tag{1.1.88}$$

The total output energy density E_{out} of the pulse is

$$E_{\mathrm{out}} = E_{\mathrm{s}}\, \ell\, n\, [1 + G_0\, (\exp\,(E_{\mathrm{in}}/E_{\mathrm{s}}) - 1)] \tag{1.1.89}$$

with the two limiting cases

$$E_{\mathrm{out}} = \begin{cases} G_0 E_{\mathrm{in}} , & E_{\mathrm{in}} \ll E_{\mathrm{s}} , \\ E_{\mathrm{in}} + E_{\mathrm{s}}\, \ell\, n\, G_0 = E_{\mathrm{in}} + \dfrac{\Delta n_0\, \ell\, \hbar\omega_{\mathrm{A}}}{2} , & E_{\mathrm{in}} \gg E_{\mathrm{s}} \end{cases} \tag{1.1.90}$$

with

G_0 : small-signal gain factor, (1.1.63),
$E_{\mathrm{s}} = J_{\mathrm{s},4} T_1$: saturation energy density,
$E_{\mathrm{in,out}}$: input/output energy density.

Equations (1.1.88)–(1.1.90) also hold for saturable absorbers with $G_0 < 1$. The pulse will be shaped in any case and the peak velocity will differ from the phase- and group velocities.

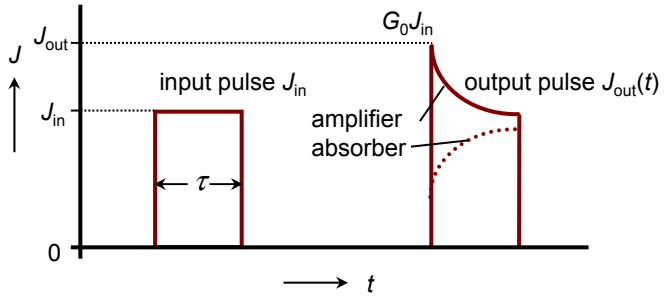

Fig. 1.1.10. Pulse amplification and shaping by a saturable amplifier/absorber.

1.1.5.5 Rate equations for steady-state laser oscillators

In the oscillator system, two counter-propagating traveling waves J^+, J^- appear, see Fig. 1.1.11, which are amplified by an intensity- and z-dependent gain coefficient according to (1.1.58a), (1.1.58b):

$$\frac{\mathrm{d}J^+}{\mathrm{d}z} = [g(J) - \alpha]\ J^+ \ , \tag{1.1.91a}$$

$$\frac{\mathrm{d}J^-}{\mathrm{d}z} = -\,[g(J) - \alpha]\ J^- \ . \tag{1.1.91b}$$

For the two traveling waves the boundary conditions at the mirrors are:

$$J^+(z=0) = J^-(z=0)R_1 \ ,$$

$$J^-(z=\ell) = J^+(z=\ell)R_2 \ .$$

The combination of (1.1.91a) and (1.1.91b) yields [81Ver]:

$$J^+(z)J^-(z) = \text{const.} \ ,$$

a useful relation for analytical solutions. The gain coefficient is saturated by both waves. In steady state (1.1.84)/(1.1.87) hold with $J = J^+ + J^-$, depending on the level system and on the type of broadening. For homogeneous broadening a solution is given in [81Ver]. In general, numerical calculations are necessary. For optimization a diagram is offered in [92Koe]. The intensity rate equations are very useful for laser design and optimization, but deliver no spectral effects such as line width [58Sch, 74Sar, 95Man], mode competition [86Sie, 00Dav], mode hopping [86Sie, 64Lam, 74Sar], or intensity-dependent frequency shifts (Lamb dip) [64Lam]. Multimode oscillation can be described by rate equations with restrictions [64Sta, 63Tan, 93Sve].

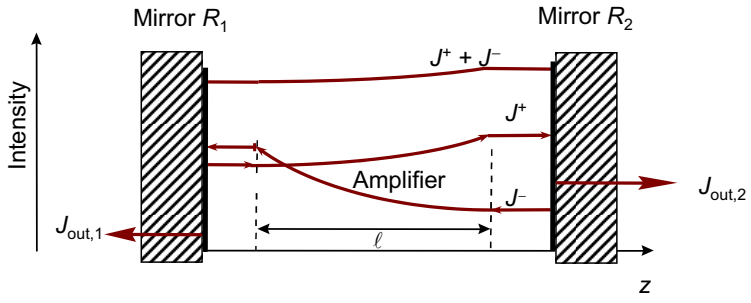

Fig. 1.1.11. The laser oscillator with two counter-propagating waves.

1.1.6 Line shape and line broadening

Shape and width of the spectral response of the two-level system depend on the special stochastic perturbation processes, in detail discussed by [81Ver, 86Eas]. An easy-to-read introduction is given by [86Sie, 00Dav].

1.1.6.1 Normalized shape functions

Normalized line shapes are introduced, which determine the relative strength of interaction.

The line shape depends on the specific interaction process. Two standard line shapes, easy to handle, are the Lorentzian and the Gaussian profiles [92Koe], shown in Fig. 1.1.12. They can be normalized differently.

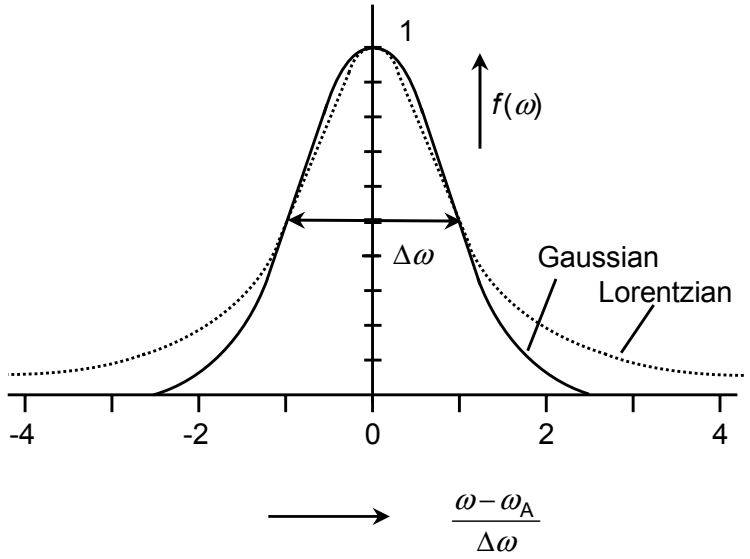

Fig. 1.1.12. Gaussian and Lorentzian line shape.

1.1.6.1.1 Lorentzian line shape

$$f_{\mathrm{L}}(\omega,\omega_{\mathrm{A}}) = \frac{(\Delta\,\omega_{\mathrm{A}}/2)^2}{(\omega-\omega_{\mathrm{A}})^2 + (\Delta\,\omega_{\mathrm{A}}/2)^2}\;, \qquad h_{\mathrm{L}}(\omega,\omega_{\mathrm{A}}) = \frac{2}{\pi\Delta\,\omega_{\mathrm{A}}} f_{\mathrm{L}}(\omega,\omega_{\mathrm{A}})\;. \tag{1.1.92}$$

1.1.6.1.2 Gaussian line shape

$$f_{\mathrm{G}}(\omega) = \exp\left[-\left(\frac{\omega-\omega_{\mathrm{A}}}{\Delta\,\omega_{\mathrm{A}}/2}\right)^2 \ln 2\right]\;, \qquad h_{\mathrm{G}}(\omega) = \sqrt{\frac{\ln 2}{\pi}}\,\frac{2}{\Delta\,\omega_{\mathrm{A}}} f_{\mathrm{G}}(\omega)\;. \tag{1.1.93}$$

1.1.6.1.3 Normalization of line shapes

$$f_{\mathrm{G,L}}(\omega=\omega_{\mathrm{A}}) = 1\;, \quad f_{\mathrm{G,L}}(\omega=\omega_{\mathrm{A}}\pm\Delta\,\omega_{\mathrm{A}}/2) = 0.5\;, \quad \int_{-\infty}^{+\infty} h_{\mathrm{G,L}}(\omega,\omega_{\mathrm{A}})\mathrm{d}\,\omega = 1\;. \tag{1.1.94}$$

1.1.6.2 Mechanisms of line broadening

1.1.6.2.1 Spontaneous emission

The spontaneous emission decay time T_{sp} of quantum dot lasers can be influenced by the geometry [97Scu], but for all macroscopic laser systems it is equal to the free-atom decay and related to the dipole moment (see Sect. 1.1.5.2). The line width of the power spectrum is $\Delta\omega = 1/T_{\mathrm{sp}}$. The line shape is Lorentzian for undisturbed systems.

1.1.6.2.2 Doppler broadening

In thermal equilibrium the particles in a gas have a Maxwellian velocity distribution of the velocity v:

$$h(v) = \sqrt{\frac{m_{\mathrm{A}}}{2\pi\,\kappa T}}\,\exp\left[-\frac{m_{\mathrm{A}}v^2/2}{\kappa T}\right] \tag{1.1.95}$$

with

m_{A} : atomic mass,
κT : thermal energy of the particles.

The resonance frequency of a transition is shifted by the Doppler effect

$$\Delta\omega = \omega_{\mathrm{A}} v/c_0 \ .$$

Replacing the velocity in (1.1.76) by the frequency, delivers for the resulting spectral distribution a Gaussian line shape (1.1.74) with the width

$$\frac{\Delta\omega_{\mathrm{D}}}{\omega_{\mathrm{A}}} = \sqrt{\frac{8\,\kappa T\ln 2}{m_{\mathrm{A}}c_0^2}}\ . \tag{1.1.96}$$

Some numbers are compiled in Table 1.1.5.

Table 1.1.5. Doppler and collision broadening for a thermal energy of $\kappa T = 1$ eV. The Doppler broadening refers to $\omega_{\mathrm{A}} = 10^{15}$ s^{-1}, the collision broadening holds for a pressure of $p = 133$ Pa (1 torr) [81Ver, 01Men].

Gas	Doppler broadening $\Delta\omega_{\mathrm{D}}$ [10^{10} s^{-1}]	Collision broadening $\Delta\omega_{\mathrm{C}}$ [10^{7} s^{-1}]
H_2	5.6	2.8
He	4	1.3
Ne	1.8	0.8
CO_2	1.2	1.2
Ar	1.5	9

1.1.6.2.3 Collision or pressure broadening

Elastic collisions between radiating atoms imply no energy loss, but a discontinuous jump in the phase of the emitted field. The average temporal length of the wave trains, in the undisturbed case

given by the spontaneous life time T_{sp}, is reduced to the collision time τ. The Fourier transform of these shortened waves gives a Lorentzian line shape with the spectral width $\Delta\omega_C = 2/\tau$ or

$$\Delta\omega_C = \frac{32\,\sigma_C\,p}{\sqrt{\pi\,m_A\,\kappa T}} \tag{1.1.97}$$

with

σ_C : collision cross section of the atom,
p : pressure of the gas.

The collision broadening is proportional to the gas pressure. Some numbers are given in Table 1.1.5.

1.1.6.2.4 Saturation broadening

A strong field of intensity J, comparable with the saturation intensity J_s, depletes the upper laser level. The gain is reduced according to (1.1.58a), (1.1.58b) and the gain profile becomes flatter and broader with the spectral width (see Fig. 1.1.13) [81Ver]:

$$\Delta\omega_S = \Delta\omega_A\sqrt{1 + J/J_s}\,.$$

1.1.6.3 Types of broadening

The interaction of the field depends strongly on the type of broadening. Two idealized cases are the homogeneous and the inhomogeneous broadening [00Dav].

1.1.6.3.1 Homogeneous broadening

All transitions have the same resonance frequency ω_A. The gain is saturated for all atoms in the same way as given by (1.1.58a) and shown in Fig. 1.1.13. Examples for this type of broadening are:

- spontaneous emission,
- collision broadening,
- saturation broadening,
- thermal broadening in crystals by interaction with the lattice vibrations.

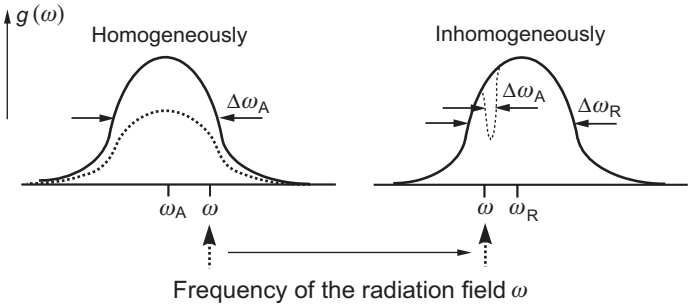

Fig. 1.1.13. Saturation of homogeneously and inhomogeneously broadened systems by a radiation field of frequency ω.

1.1.6.3.2 Inhomogeneous broadening

Groups of atoms with spectral density $h(\omega_R, \omega_A)$ and different frequencies ω_A produce a resulting line profile with center frequency ω_R and width $\Delta\omega_R$ as shown in Fig. 1.1.14. A strong monochromatic field of frequency ω interacts mainly with the group $\omega_A = \omega$ and saturates this particular group. A dip appears in the profile, which is called spectral hole-burning. Examples of inhomogeneous broadening are:

- Doppler broadening,
- Stark broadening in crystals due to statistical local crystalline fields.

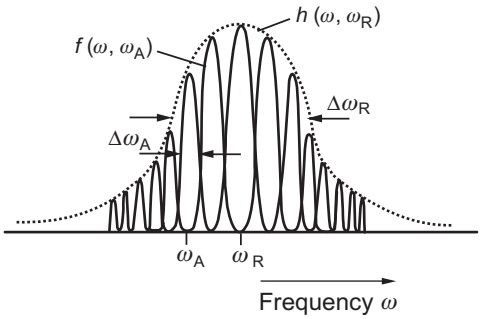

Fig. 1.1.14. An inhomogeneously broadened profile.

The resulting line profile is a convolution of the individual group profiles and the broadening process, which results in complicated integrals. The saturation process for inhomogeneously broadened lines is quite different, as will be shown by a simple example. In this case (1.1.58a) holds only for one group of atoms with the spectral density $h(\omega_A, \omega_R)$. Integration over all groups results in the total gain coefficient g_{inh}:

$$g_{inh}(\omega, \omega_R) = \int_{-\infty}^{+\infty} f(\omega, \omega_A) h(\omega_A, \omega_R) \mathrm{d}\,\omega_A \ . \tag{1.1.98}$$

If the width $\Delta\omega_A$ is much smaller than the total width $\Delta\omega_R$, the function $h(\omega_A, \omega_R)$ can be taken outside of the integral at $\omega_A = \omega$. Assuming a Lorentzian profile for the single group, (1.1.98) becomes:

$$g_{inh}(\omega) = \Delta n_0 \sigma_0 h(\omega, \omega_R) \int \frac{f(\omega, \omega_A)}{1 + (J/J_s)\, f(\omega, \omega_A)} \mathrm{d}\,\omega_A$$

and can be integrated:

$$g_{inh}(\omega) = \frac{\Delta n_0 \sigma_0}{\sqrt{1 + J/J_s}}\, h(\omega, \omega_R) \frac{\pi \Delta\omega_A}{2} = \frac{\Delta n_0\, \sigma_0}{\sqrt{1 + J/J_s}}\, f(\omega) \frac{\Delta\omega_A}{\Delta\omega_R} \ . \tag{1.1.99}$$

The gain saturates slower than in the case of homogeneous broadening, but the maximum gain is lower by the ratio of the line widths. Inhomogeneous gain profiles can also be caused by spatial hole burning in solid-state laser systems. The standing waves between the mirrors produce an inversion grating and holes in the spectral gain profile [86Sie].

The spectral characteristics of lasers depend strongly on the type of broadening, see Fig. 1.1.15. In steady state the gain compensates losses and the gain profile saturates to fulfill the condition $GRV = 1$. A homogeneously broadened gain profile saturates till the steady-state condition is fulfilled for the central frequency. The bandwidth $\Delta\omega_{L,h}$ is very small and depends on the thermal and

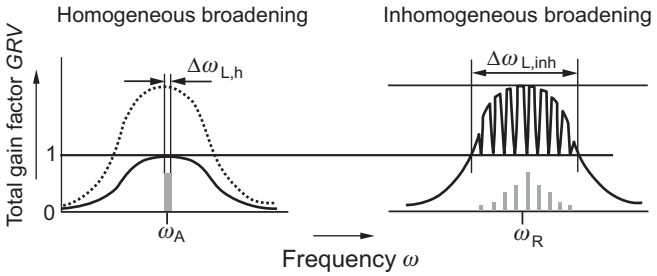

Fig. 1.1.15. Spectrum of an inhomogeneously and homogeneously broadened laser transition in steady state. Total gain factor GRV vs. frequency of the radiation field.

mechanical fluctuations [00Dav]. In the case of solid-state lasers spatial hole burning will influence the spectral behavior and can produce even for homogeneous transitions multi-mode oscillation [63Tan, 66Men]. In the case of inhomogeneous broadening each spectral group of atoms saturates separately and many modes will oscillate, which produces a large lasing bandwidth $\Delta\omega_{\mathrm{L,inh}}$. If single-mode operation is enforced by suitable frequency selecting elements, the left $\rightarrow$ right and the right $\rightarrow$ left traveling waves produce two symmetric holes, due to the Doppler effect. This effect can be used for frequency stabilization (Lamb dip [64Lam]).

1.1.6.4 Time constants

The line profile of a real laser transition is in most cases a mixture of homogeneous and inhomogeneous profiles, depending on the temperature and the pressure. The following time constants are used in literature:

T_{sp} : spontaneous life time,
T_1 : upper-laser-level life time (energy relaxation time, longitudinal relaxation time),
T_2' : Stochastic processes broaden the line homogeneously. The inverse of the line width is the dephasing time T_2'.
T_2^* : The line is broadened inhomogeneously. The inverse of this line width $\Delta\omega_{\mathrm{R}}$ is the dephasing time T_2^*.
T_2 : For the resulting dephasing time (transverse relaxation time, entropy time constant), approximately holds (depends on the line profiles):

$$\frac{1}{T_2^2} \approx \frac{1}{T_2^{*\,2}} + \frac{1}{T_2^{\prime\,2}} \ .$$

Some examples of decay times are given in Table 1.1.6.

1.1.7 Coherent interaction

Radiation field and two-level system are two coupled oscillators. Without stochastic perturbations the stored energy is permanently exchanged between these two systems.

If the interaction time of the radiation field with the two-level system is small compared with all relaxation times, including the pump term, the stochastic processes can be neglected and

Table 1.1.6. Spontaneous life time T_{sp}, upper-laser-level life time T_1, transverse relaxation time T_2, homogeneous relaxation time T_2' and inhomogeneous relaxation time T_2^* [01Iff, 92Koe, 86Sie], [01Men, Chap. 6].

	T_{sp} [s]	T_1 [s]	T_2 [s]	T_2' [s]	T_2^* [s]
Neon-atom (He/Ne-laser), $\lambda_0 = 632.8$ nm, He ($p = 130$ Pa), Ne ($p = 25$ Pa)	10^{-8}	10^{-8}	3×10^{-9}	10^{-8}	4×10^{-9}
Chromion-ion, $\lambda_0 = 694.3$ nm, R_1-transition in ruby					
$T = 300$ K	3×10^{-3}	3×10^{-3}	10^{-12}	10^{-12}	2×10^{-7}
$T = 4$ K	4×10^{-3}	4×10^{-3}	2×10^{-7}	3×10^{-3}	2×10^{-7}
SF_6-molecule, $\lambda_0 = 10.5$ µm, $p = 0.4$ Pa	10^{-3}	10^{-3}	6×10^{-9}	7×10^{-6}	6×10^{-9}
Rhodamin-molecule in ethanol, singlet-transition, $\lambda_0 = 570.0$ nm	5×10^{-9}	5×10^{-9}	10^{-12}	10^{-12}	–
Neodymium-ion in YAG-crystal, $\lambda_0 = 1060$ nm, $T = 300$ K	5×10^{-4}	2.3×10^{-4}	7×10^{-12}	–	–

(1.1.45a)/(1.1.45b) hold. This kind of coherent interaction is of strong interest in nonlinear spectroscopy [84She, 86Sie, 71Lam, 72Cou], [01Men, Chap. 7] and confirmed by many experiments. Examples of nonlinear coherent interaction are transient response of atoms, optical nutation, photon echoes, $n\,\pi$-pulses and quantum beats. Here only some very simple examples will be presented. A more detailed treatment is given in [95Man].

1.1.7.1 The Feynman representation of interaction

Feynman introduced a very elegant representation of interaction, which enables an easy-to-understand visualization.

A very compact description of the two-level interaction was given by Feynman [57Fey]. The real electric field is

$$\boldsymbol{E}_{\text{real}} = \frac{1}{2}\left\{\boldsymbol{E}_0 \exp\left[\mathrm{i}\,(\omega t - kz)\right] + \boldsymbol{E}_0^* \exp\left[-\mathrm{i}\,(\omega t - kz)\right]\right\} \ .$$

It generates a real polarization, (1.1.23), shifted in phase against the field:

$$\begin{aligned}\boldsymbol{P}_{\text{A,real}} &= \frac{1}{2}\left\{\boldsymbol{P}_{\text{A0}} \exp\left[\mathrm{i}(\omega t - kz)\right] + \boldsymbol{P}_{\text{A0}}^* \exp\left[-\mathrm{i}(\omega t - kz)\right]\right\} \\ &= \boldsymbol{C}\,\cos\left(\omega t - kz\right) + \boldsymbol{S}\,\sin\left(\omega t - kz\right)\end{aligned} \tag{1.1.100}$$

with $\boldsymbol{C}$, $\boldsymbol{S}$ real vectors:

$$\boldsymbol{C} = \frac{1}{2}\left(\boldsymbol{P}_{\text{A0}} + \boldsymbol{P}_{\text{A0}}^*\right) \ , \quad \boldsymbol{S} = \frac{1}{2}\,\mathrm{i}\,\left(\boldsymbol{P}_{\text{A0}} - \boldsymbol{P}_{\text{A0}}^*\right) \ .$$

In the following an isotropic medium is assumed. Then $\boldsymbol{\mu}_{\text{A}}$, $\boldsymbol{P}_{\text{A}}$ and $\boldsymbol{E}$ are parallel and can be treated as scalars. With these new real quantities the equations of interaction (1.1.45a), (1.1.45b) become:

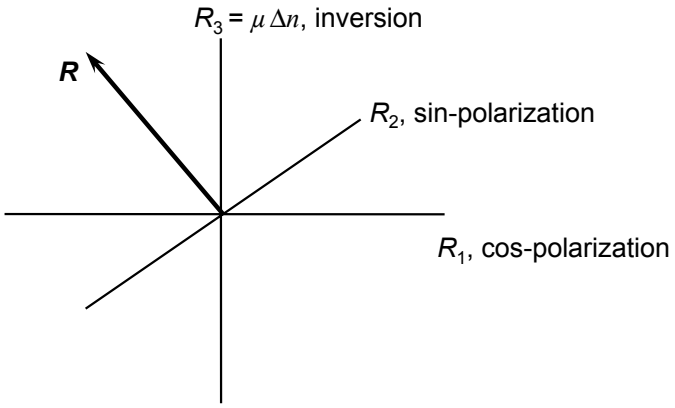

Fig. 1.1.16. In the case of coherent interaction, the system is characterized by its $\boldsymbol{R}$-vector which rotates in the polarization/inversion space with constant length.

$$\frac{\partial C}{\partial t} = -\delta S + \mathrm{i}\,\mu_\mathrm{A} \cdot \Delta\, n \left(\frac{\Lambda - \Lambda^*}{2}\right) , \tag{1.1.101a}$$

$$\frac{\partial S}{\partial t} = \delta C - \mu_\mathrm{A} \cdot \Delta\, n \left(\frac{\Lambda + \Lambda^*}{2}\right) , \tag{1.1.101b}$$

$$\mu_\mathrm{A} \frac{\partial \Delta\, n}{\partial t} = -\mathrm{i}\, C \left(\frac{\Lambda - \Lambda^*}{2}\right) + S \left(\frac{\Lambda + \Lambda^*}{2}\right) , \tag{1.1.101c}$$

where Λ is a complex quantity. Its modulus is called the Rabi frequency:

$$\Lambda(z,t) = \frac{\mu_\mathrm{A}\, E_0}{\hbar} , \quad |\Lambda| \,:\, \text{Rabi frequency} . \tag{1.1.102}$$

Two vectors $\boldsymbol{R}$, $\boldsymbol{F}$ are introduced:

$$\boldsymbol{R} = (C, S, \mu_\mathrm{A}\, \Delta\, n) = (R_1, R_2, R_3) , \quad \boldsymbol{F} = \left(\frac{\Lambda + \Lambda^*}{2}, \mathrm{i}\,\frac{\Lambda - \Lambda^*}{2}, \delta\right) = (F_1, F_2, F_3) .$$

The $\boldsymbol{R}$-vector characterizes the state of the two-level system and can be depicted in an inversion/polarization space, as shown in Fig. 1.1.16. $\boldsymbol{R}$ corresponds to the Bloch vector of the spin-1/2 system [46Blo]. The equations (1.1.101a), (1.1.101b) of interaction can be condensed to:

$$\frac{\partial \boldsymbol{R}}{\partial t} = [\boldsymbol{F} \times \boldsymbol{R}] \quad \text{(coherent interaction)} . \tag{1.1.103}$$

Scalar multiplication of this equation with $\boldsymbol{R}$ results in:

$$\left\langle \boldsymbol{R}\, \frac{\partial \boldsymbol{R}}{\partial t} \right\rangle = \langle \boldsymbol{R}\, [\boldsymbol{F} \times \boldsymbol{R}] \rangle = 0 ,$$

which means that the length of the vector is constant during interaction:

$$|C|^2 + |S|^2 + |\mu_\mathrm{A}\, \Delta\, n|^2 = |\boldsymbol{R}_0|^2 . \tag{1.1.104}$$

The tip of the vector moves on a sphere in the inversion/polarization space with complicated trajectories [69McC, 74Sar, 69Ics]. The incoherent relaxation and pumping of the system can be included in (1.1.103) by an additional relaxation term [72Cou].

1.1.7.2 Constant local electric field

If the amplitude $\boldsymbol{E}_0$ of the electric field is assumed to be constant, a very simple solution of the rotating-wave equations is obtained with one main parameter, the Rabi frequency Λ.

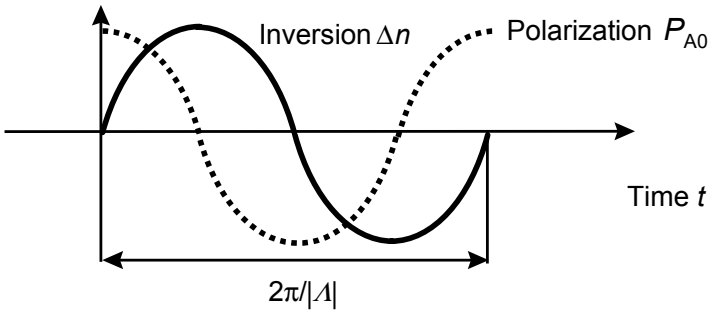

Fig. 1.1.17. Oscillation of inversion density Δn and polarization amplitude $\boldsymbol{P}_{\mathrm{A0}}$ in resonance for a constant local electric field.

For a constant electric field at a fixed position z the rotating-wave approximation has a periodic solution. Inversion and polarization with the initial condition $t = 0$, $\Delta n = n_0$, $\boldsymbol{P}_{\mathrm{A0}} = 0$ are:

$$\frac{\Delta n}{n_0} = \frac{\delta^2 + |\Lambda|^2 \cos \beta t}{\beta^2} , \quad \Lambda = \frac{\mu_{\mathrm{A}} E_0}{\hbar} , \tag{1.1.105}$$

$$P_{\mathrm{A0}} = n_0 \frac{\mu_{\mathrm{A}} \Lambda}{\beta} \left[\frac{\delta}{\beta} (1 - \cos \alpha t) + \mathrm{i} \sin \alpha t \right] , \quad \beta = \sqrt{\delta^2 + |\Lambda|^2} . \tag{1.1.106}$$

In resonance $\omega = \omega_{\mathrm{A}}$, $\delta = 0$, the inversion density Δn and the amplitude $\boldsymbol{P}_{\mathrm{A0}}$ of the polarization oscillate with this frequency, see Fig. 1.1.17. The real polarization $\boldsymbol{P}_{\mathrm{A,real}}$ of (1.1.100) contains the frequencies $\omega_{\mathrm{A}} \pm |\Lambda|$. Some values of dipole moments are given in Table 1.1.2 to estimate $|\Lambda|$. Off resonance the temporal behavior of inversion and polarization is more complicated (optical nutation) [72Cou]. If at $t = 0$ all atoms are in the lower level ($\Delta n_0 = -n_0$) a complete inversion is produced at $t = \pi/|\Lambda|$ by a coherent field. It is called pulse inversion [60Vuy]. At $t = 2/|\Lambda|$, all atoms are again in the lower level, no energy transfer has taken place.

1.1.7.3 Propagation of resonant coherent pulses

For short pulses, $\tau < T_2$, the perturbations can be neglected. The solution of the complete interaction equations (1.1.101a)–(1.1.101c) for a propagating resonant pulse is rather simple.

The propagation of pulses in a two-level system is described by the rotating-wave approximation, (1.1.45a)/(1.1.45b), and by the wave equation in the SVE approximation (1.1.28). The set of these three non-linear equations is difficult to solve, only special cases will be discussed here. At $t = 0$ the electric field E_0 is assumed to be real, $\Lambda = \Lambda^*$. In case of resonance, $\delta = 0$, (1.1.101a) delivers $C = 0$, $R_1 = 0$. The interaction equations (1.1.101b), (1.1.101c) reduce to

$$R_1 = 0 ,$$

$$\frac{\partial R_2}{\partial t} = -\Lambda R_3 ,$$

$$\frac{\partial R_3}{\partial t} = \Lambda R_2 .$$

The $\boldsymbol{R}$-vector moves in the R_2-R_3-plane, see Fig. 1.1.18. If the angle θ with the R_3-axis is introduced, one solution of the above equations is:

$$R_2 = R_0 \sin \theta ,$$

$$R_3 = -R_0 \cos \theta$$

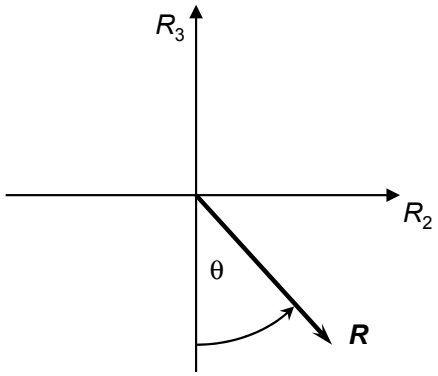

Fig. 1.1.18. In resonance, $\delta = 0$, the $\boldsymbol{R}$-vector of the two-level system rotates in the R_2-R_3-plane.

with

$$\Lambda = \frac{\partial\theta}{\partial t} = \frac{\mu_\mathrm{A}\, E_0}{\hbar}\ . \tag{1.1.107}$$

R_0 is given by the initial conditions at $t = 0$. The SVE-approximation of (1.1.28) then becomes:

$$\left(\frac{\partial^2}{\partial t\,\partial z} + \frac{1}{c}\frac{\partial^2}{\partial t^2}\right)\theta = -\frac{\alpha}{2}\frac{\partial\theta}{\partial t} + \frac{\gamma}{2}\,R_0\,\sin\,\theta\ , \quad \gamma = \frac{\mu_\mathrm{A}\, k_0}{n_\mathrm{r}\,\varepsilon_0\,\hbar}\ . \tag{1.1.108}$$

From θ the amplitude E_0 of the electric field can be calculated with (1.1.107)/(1.1.105).

1.1.7.3.1 Steady-state propagation of $n\pi$-pulses

Steady state means that a pulse is propagating with velocity v and constant pulse envelope $\boldsymbol{E}_0(t, z) = \boldsymbol{E}_0(t - z/v)$. The amplitude depends on one parameter w only:

$$w = t - z/v$$

and (1.1.108) becomes:

$$\left(1 - \frac{c}{v}\right)\frac{\mathrm{d}^2\theta}{\mathrm{d}\,w^2} + \frac{\alpha c}{2}\frac{\mathrm{d}\,\theta}{\mathrm{d}\,w} = c\,\frac{\gamma}{2}\,R_0\,\sin\,\theta\ . \tag{1.1.109}$$

This equation is equivalent to the equation of the pendulum with friction in a gravitational field. In the following examples two different initial conditions are assumed:

$$R_0 = \mu_\mathrm{A}\,\Delta\, n_0 \begin{cases} > 0 & \text{(amplifier)}\ , \\ < 0 & \text{(absorber)}\ , \end{cases}$$

which corresponds to the pendulum up or down at $t = 0$.

1.1.7.3.1.1 2π-pulse in a loss-free medium

A medium without losses ($\alpha = 0$) interacts with a coherent pulse in resonance ($\delta = 0$). The initial condition is $\Delta\, n_0(t = -\infty) = +\Delta\, n_0$ ($\Delta\, n_0 < 0$, absorber). One steady-state solution is the 2π-pulse, see Fig. 1.1.19, which corresponds to a local field of duration $\tau = 2\pi/\Lambda$. The leading edge of the pulse produces an inversion and energy is transferred to the atomic system, the amplitude is reduced. The trailing part of the pulse is then amplified by this inversion. In total the pulse

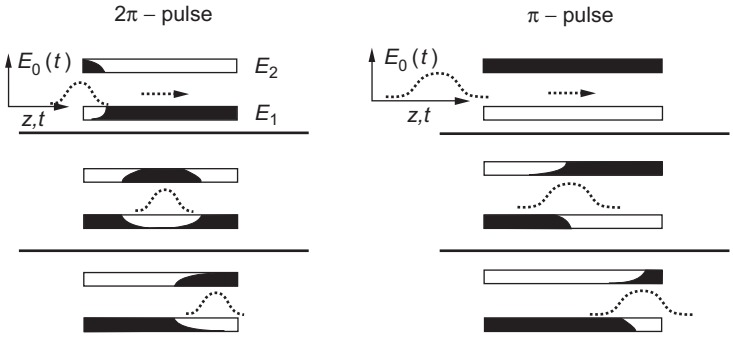

Fig. 1.1.19. Propagation of 2π- and π-pulses in a two-level system.

has lost no energy, but is delayed in time. Such a pulse is only stable, if the broadband losses are negligible and if the initial inversion is negative. The steady-state solution is:

$$E = E_{\text{peak}} \frac{\exp\left[\mathrm{i}\,\omega\,(t - z/c)\right]}{\cosh\left[(t - z/v)\,/\tau\right]} \quad \text{(field)}\,, \tag{1.1.110}$$

$$E_{\text{peak}} = 2\sqrt{\hbar\,\omega}\sqrt{\frac{\Delta\,n_0}{\varepsilon_0(1 - c/v)}} \quad \text{(peak amplitude)}\,, \tag{1.1.111}$$

$$J_{\text{peak}} = \frac{2\,\hbar\,\omega\Delta\,n_0 c}{1 - c/v} \quad \text{(peak intensity)}\,, \tag{1.1.112}$$

$$T_{2\pi} = 2\,\tau = 2\sqrt{\frac{(1 - c/v)\,T_2}{g_0 c}} \quad \text{(pulse duration)}\,, \tag{1.1.113}$$

$$v = \frac{c}{1 - g_0 c\,\tau^2/T_2} \quad \text{(pulse peak velocity)} \tag{1.1.114}$$

with

$g_0 = \Delta\,n_0\sigma < 0$: small-signal absorption coefficient,
c: phase velocity in the medium,
v: pulse peak velocity.

This two-level system is the most simple model of a saturable absorber, which in the case of incoherent interaction absorbs the radiation. But the coherent 2π-pulse transmits the absorber without losing energy. Therefore this effect is called self-induced transparency [75Kri]. The pulse is characterized by three parameters: peak velocity v, peak amplitude E_{peak} and the width $T_{2\pi}$. One of these parameters can be chosen arbitrarily, the other two result from (1.1.112)/(1.1.113)/(1.1.114). But the interaction is coherent only as long as $T_{2\pi} \ll T_2$.

1.1.7.3.1.2 π-pulse in an amplifying medium

A steady-state solution in an amplifying medium, initial condition $\Delta\,n(t = -\infty) = \Delta\,n_0 > 0$, with broadband losses ($\alpha \neq 0$) is the π-pulse [74Loy], see Fig. 1.1.19:

$$E = E_{\text{peak}} \frac{\exp\left[\mathrm{i}\,\omega\,(t - z/c)\right]}{\cosh\left[(t - z/c)/\tau)\right]} \quad \text{(field)}\,, \tag{1.1.115}$$

$$E_{\text{peak}} = \frac{\hbar}{\tau\,\mu} \quad \text{(peak amplitude)}\,, \tag{1.1.116}$$

$$J_{\text{peak}} = \frac{\hbar\,\omega}{2\sigma_0 T_2}\left[\frac{g_0}{\alpha}\right]^2 \quad \text{(peak intensity)}\,, \tag{1.1.117}$$

$$T_\pi = 2\,\tau = 2\;T_2\,\frac{\alpha}{g_0} \quad \text{(pulse duration)}\,. \tag{1.1.118}$$

The pulse propagates approximately with c, depletes at each position the upper level, and converts this energy via the broadband losses α into heat. The saturated gain just compensates the losses. The pulse is only stable for $\alpha > 0$ and $g_0 > 0$.

So far solutions of the steady-state SVE-equation were presented, assuming resonance and a homogeneously broadened two-level system. Off-resonance interaction and inhomogeneously broadened systems are much more complicated and are discussed in detail in the literature [74Sar, 69Ics, 72Cou]. Moreover, the stability of the pulses with respect to small perturbations was not yet mentioned. It is controlled by the area theorem [67McC, 74Sar].

1.1.7.3.2 Superradiance

The spontaneous emission was neglected in the coherent interaction. An initial state, $\boldsymbol{R} = (0, 0, \mu\,\Delta\,n)$, complete inversion, without external field $\boldsymbol{F}$ would be stable according to the interaction equations (1.1.103). But due to spontaneous emission and amplified spontaneous emission, the $\boldsymbol{R}$-vector will be pushed a bit out of equilibrium and decay into the stable position $\boldsymbol{R} = (0, 0, -\mu\,\Delta\,n)$. This phenomenon is called superradiance and discussed in detail in Chap. 6.2.

1.1.8 Notations

Symbol	Unit	Meaning
A_{21}	s^{-1}	Einstein coefficient of spontaneous emission
$\boldsymbol{B}$	$\mathrm{Vs/m^2}$	magnetic induction
B_{12}, B_{21}	$\mathrm{m^3/VAs^3}$	Einstein coefficient of induced emission
$\boldsymbol{C}$	$\mathrm{As/m^2}$	component of the Feynman vector $\boldsymbol{R}$
c_0	m/s	vacuum velocity of a plane wave
c	m/s	phase velocity of light in a medium
$c_{1,2}$	–	coefficients of the eigenvector
$\boldsymbol{D}$	$\mathrm{As/m^2}$	electric displacement
$\boldsymbol{E}$	V/m	electric field
$\boldsymbol{E}_0$	V/m	electric-field amplitude
$E_{1,2}$	VAs	energy eigenstates of the two-level system
E_{in}	VAs	amplifier input energy
E_{out}	VAs	amplifier output energy
E_{S}	$\mathrm{VAs/m^2}$	amplifier saturation energy density
$f(\omega, \omega_{\text{A}})$	–	line shape factor
G	–	gain factor
G_0	–	small-signal gain factor
g	m^{-1}	gain coefficient
g_0	m^{-1}	small-signal gain coefficient
$g_{1,2}$	–	degeneracies of lower/upper laser level

g_h	m^{-1}	gain coefficient of a homogeneously broadened transition
g_inh	m^{-1}	gain coefficient of an inhomogeneously broadened transition
$\boldsymbol{H}$	A/m	magnetic field
$\boldsymbol{H}_0$	A/m	magnetic-field amplitude
H_0	VAs	Hamilton operator of the undisturbed transition
H_int	VAs	Hamilton operator of interaction
$h(\omega, \omega_\mathrm{A})$	s	line shape factor
$\boldsymbol{j}$	$\mathrm{A/m}^2$	current density
$\boldsymbol{J}$	$\mathrm{Vs/m}^2$	magnetic polarization
J	$\mathrm{VA/m}^2$	intensity
J^+, J^-	$\mathrm{VA/m}^2$	intensity inside the resonator
J_s, J_s4	$\mathrm{VA/m}^2$	saturation intensity of 2-, 3- and 4-level system
k	m^{-1}	wave number
$\boldsymbol{k}$	m^{-1}	wave vector inside the medium
$\boldsymbol{k}_0$	m^{-1}	wave vector in vacuum
ℓ	m	geometrical length of the active medium
n	–	complex refractive index
n_r	–	real refractive index
n_0	m^{-3}	density of active atoms
$n_{1,2}$	m^{-3}	density of lower/upper population
$\boldsymbol{P}_\mathrm{A,real}$	$\mathrm{As/m}^2$	real polarization of the active atoms
$\boldsymbol{P}_\mathrm{A}$	$\mathrm{As/m}^2$	complex polarization of the active atoms
$\boldsymbol{P}_\mathrm{A0}$	$\mathrm{As/m}^2$	amplitude of the complex polarization
$\boldsymbol{P}_\mathrm{H}$	$\mathrm{As/m}^2$	complex polarization of the host material
$\boldsymbol{R}$	$\mathrm{As/m}^2$	Feynman vector
R	–	$= \sqrt{R_1 R_2}$, average mirror reflectivity
$R_{1,2}$	–	reflectivity of mirror 1, 2
$\boldsymbol{r}$	m	position vector
$\boldsymbol{S}$	$\mathrm{VA/m}^2$	Poynting vector
T_1	s	upper-laser-level life time
T_2'	s	dephasing time due to homogeneous broadening
T_2^*	s	dephasing time due to inhomogeneous broadening
T_2	s	resulting dephasing time
T_sp	s	spontaneous decay time
T_π , $T_{2\,\pi}$	s	pulse duration of π-, 2 π-pulses
V	–	resonator loss factor per transit
v	m/s	pulse peak velocity
Z	V/A	impedance
Z_0	V/A	vacuum impedance
α	m^{-1}	absorption coefficient
χ_A	–	susceptibility of the active atoms
χ_e	–	electric susceptibility
χ_H	–	susceptibility of the host material
χ_m	–	magnetic susceptibility
δ	s^{-1}	detuning
Δn	m^{-3}	inversion density
Δ_tr	m^{-2}	transverse delta-operator
$\Delta\omega_\mathrm{A}$	s^{-1}	line width of homogeneous broadening
$\Delta\omega_\mathrm{C}$	s^{-1}	line width of collision broadening

$\Delta\omega_{\mathrm{R}}$	s^{-1}	line width of inhomogeneous broadening
$\Delta\omega_{\mathrm{S}}$	s^{-1}	line width of saturation broadening
$\Delta\omega_{\mathrm{L,inh}}$, $\Delta\omega_{\mathrm{L,h}}$	s^{-1}	lasing bandwidth of inhomogeneous/homogeneous transitions
ε	–	permittivity
ε_0	8.8542×10^{-12} As/Vm	electric constant
$\|\varphi\rangle$	–	state vector of the two-level system
$\|\varphi_{1,2}\rangle$	–	eigenfunctions of the two-level system
κ	1.38×10^{-23} $\mathrm{VAs}^2/\mathrm{K}$	Boltzmann's constant
λ_0	m	vacuum wavelength
Λ	s^{-1}	Rabi frequency
μ	–	permeability
μ_0	$4\pi \times 10^{-7}$ Vs/Am	magnetic constant
$\boldsymbol{\mu}_{12}$, $\boldsymbol{\mu}_{21}$	Asm	$= \boldsymbol{\mu}_{\mathrm{A}}$, dipole moment of the two-level transition
$\boldsymbol{\mu}_{\mathrm{A}}$	Asm	dipole moment of the two-level transition
θ	–	beam divergence, slope of the Feynman vector
ρ_ω	$\mathrm{VAs}^2/\mathrm{m}^3$	spectral energy density (per $\mathrm{d}\omega$)
$\sigma(\omega)$	m^2	cross section of the two-level system
σ_{e}	A/Vm	electric conductivity
σ_0	m^2	cross section of the two-level system in resonance
τ	s	pulse width
ω	s^{-1}	frequency of the radiation field
ω_{A}	s^{-1}	resonance frequency of the homogeneously broadened transition
ω_{R}	s^{-1}	resonance frequency of the inhomogeneously broadened transition