2.2 Beam characterization

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2.2.1 Introduction

The success of almost any laser application depends mainly on the power density distributions in a certain area of the laser beam, usually the focal region. It is the aim of laser beam characterization to describe and predict the profiles a beam takes on under free-space propagation or behind optical systems.

The attributes of a power density distribution in a plane transverse to the direction of propagation can be divided into *size* and *shape*. Under free-space propagation the size of the power density profile is always changing with the distance from the source, whereas the shape of the profile may vary or not. Examples for shape-invariant laser beams are the well-known Gaussian, Laguerre-Gaussian, Hermite–Gaussian, and Gauss-Schell model beams.

A complete characterization of laser beams would allow the prediction of power density distributions, including size and shape, behind arbitrary optical systems as far as they are sufficiently known. Admittedly for such detailed characterization a huge amount of data and sophisticated measurement procedures are necessary. But for many applications the knowledge and prediction of the transverse extent of the laser beam profile might be sufficient. Restriction to nearly aberration-free optical systems then enables beam characterization by only ten or less parameters.

In the following the validity of the paraxial approximation will be presumed. In practical this means that the full divergence angle of the beam should not exceed 30 degrees. Furthermore, any polarization effects are neglected. Beam characterization methods based on the considerations presented in this chapter have recently become an international standard, published as ISO 11146 [99ISO].

2.2.2 The Wigner distribution

A complete description of partially coherent radiation fields (within the restrictions stated above) can be given by a two-point-correlation integral of the field in a transverse plane at location z [99Bor]:

$$\tilde{\Gamma}\left(\boldsymbol{r}_1, \boldsymbol{r}_2, z, \tau\right) = \frac{1}{T} \int\limits_{t_0}^{t_0+T} E^*\left(\boldsymbol{r}_1, z, t\right) \, E\left(\boldsymbol{r}_2, z, t+\tau\right) \, \mathrm{d}t \,, \tag{2.2.1}$$

where $E\left(\boldsymbol{r}, z, t\right)$ is the electrical field, z the coordinate along the direction of propagation, $\boldsymbol{r} = (x, y)^{\mathsf{T}}$ a transverse spatial vector (see Fig. 2.2.1), and T the integration time which shall be large enough to ensure that the integration results are independent of the starting time t_0. The temporal Fourier transform of this correlation integral is known as the cross-spectral density or the (mutual) power spectrum:

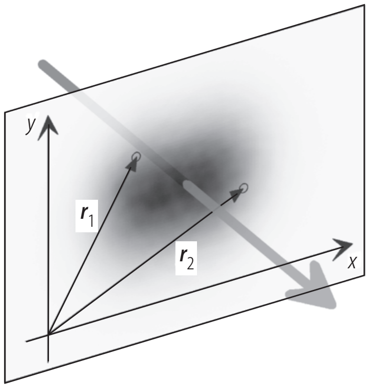

Fig. 2.2.1. Spatial coordinates $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of a pair of points in a plane transverse to the direction of propagation.

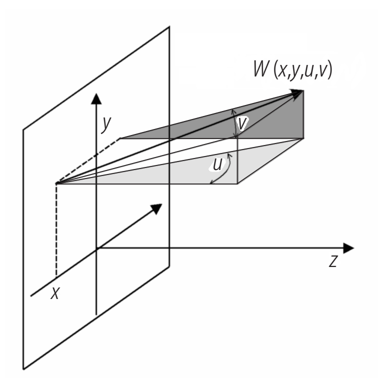

Fig. 2.2.2. The phase space coordinates of the Wigner distribution. x and y are spatial transverse coordinates, u and v are the corresponding angular coordinates.

$$\Gamma\left(\boldsymbol{r}_1, \boldsymbol{r}_2, z, \omega\right) = \int \tilde{\Gamma}\left(\boldsymbol{r}_1, \boldsymbol{r}_2, z, \tau\right) \mathrm{e}^{\mathrm{i}\omega\tau} \, \mathrm{d}\,\tau \; . \tag{2.2.2}$$

Since laser beams in general can be considered as quasi-monochromatic, the frequency dependency will be dropped in the following:

$$\Gamma\left(\boldsymbol{r}_1, \boldsymbol{r}_2, z, \omega_0\right) \rightarrow \Gamma\left(\boldsymbol{r}_1, \boldsymbol{r}_2, z\right) \; . \tag{2.2.3}$$

From the cross-spectral density in a transverse plane at location z the power density in that plane can easily be obtained by

$$I\left(\boldsymbol{r}, z\right) = \Gamma\left(\boldsymbol{r}, \boldsymbol{r}, z\right) \; . \tag{2.2.4}$$

Given the cross-spectral density at an entry plane the further propagation through arbitrary, but well-defined optical systems can be calculated by several methods and hence the power density distribution in the output plane of the systems predicted [99Bor].

The Wigner distribution $W\left(\boldsymbol{r}, \boldsymbol{q}, z\right)$ of partially coherent beams is defined as the Fourier transform of the cross spectral density with respect to the separation vector $\boldsymbol{s}$ [78Bas]:

$$W\left(\boldsymbol{r}, \boldsymbol{q}, z\right) = \int \Gamma\left(\boldsymbol{r} + \tfrac{1}{2}\,\boldsymbol{s}, \boldsymbol{r} - \tfrac{1}{2}\,\boldsymbol{s}, z\right) \mathrm{e}^{-\mathrm{i}k\boldsymbol{q}\,\boldsymbol{s}} \, \mathrm{d}\boldsymbol{s} \; . \tag{2.2.5}$$

The Wigner distribution contains the same information as the cross-spectral density, but in a different, more descriptive manner. Considering $\boldsymbol{q} = (u, v)^{\mathsf{T}}$ as an angular vector with respect to the z-axis (Fig. 2.2.2), the Wigner distribution gives the part (amount) of the radiation power which passes the plane at z through the point $\boldsymbol{r}$ in the direction given by $\boldsymbol{q}$. Within this picture the Wigner distribution might be considered as a generalization of the geometric optical radiance, although this analogy is limited. E.g. the Wigner distribution may take on negative values.

The power density distribution in a transverse plane is obtained by integration over the angles of direction,

$$I(\boldsymbol{r}, z) = \int W(\boldsymbol{r}, \boldsymbol{q}, z)\, \mathrm{d}\boldsymbol{q}\ , \tag{2.2.6}$$

and the far-field power density distribution by integration over the spatial coordinates,

$$I_{\mathrm{F}}(\boldsymbol{q}) = \int W(\boldsymbol{r}, \boldsymbol{q}, z)\, \mathrm{d}\boldsymbol{r}\ . \tag{2.2.7}$$

The Wigner distribution represents the beam in a transverse plane at location z. As the beam propagates in free space or through an optical system the Wigner distribution changes. This is reflected in the z-dependency of the Wigner distribution in the equations above. In the following equations this z-dependency will be dropped wherever appropriate.

The propagation of the Wigner distribution through aberration-free first-order optical systems (combinations of parabolic elements and free-space propagation) is very similar to that of geometric-optical rays. Such rays are specified by their position $\boldsymbol{r}$ and direction $\boldsymbol{q}$. After propagation through an aberration-free optical system position and direction will change according to

$$\begin{pmatrix} \boldsymbol{r}_{\mathrm{out}} \\ \boldsymbol{q}_{\mathrm{out}} \end{pmatrix} = \mathsf{S} \cdot \begin{pmatrix} \boldsymbol{r}_{\mathrm{in}} \\ \boldsymbol{q}_{\mathrm{in}} \end{pmatrix}\ , \tag{2.2.8}$$

where S is a 4×4-matrix representing the optical system, the system matrix (see Chap. 3.1). Considering the Wigner distribution as a density distribution of geometric optical rays, its propagation law is given by ray tracing [78Bas]:

$$W_{\mathrm{out}}(\boldsymbol{r}_{\mathrm{out}}, \boldsymbol{q}_{\mathrm{out}}) = W_{\mathrm{in}}(\boldsymbol{r}_{\mathrm{in}}, \boldsymbol{q}_{\mathrm{in}}) \quad \text{with} \quad \begin{pmatrix} \boldsymbol{r}_{\mathrm{in}} \\ \boldsymbol{q}_{\mathrm{in}} \end{pmatrix} = \mathsf{S}^{-1} \cdot \begin{pmatrix} \boldsymbol{r}_{\mathrm{out}} \\ \boldsymbol{q}_{\mathrm{out}} \end{pmatrix}\ . \tag{2.2.9}$$

2.2.3 The second-order moments of the Wigner distribution

From the Wigner distribution smaller sets of data can be derived, which can be associated to certain physical properties of the beams. These sets of data are the so-called moments of the Wigner distribution [86Bas]:

$$\left\langle x^k y^\ell u^m v^n \right\rangle = \frac{\int W(x, y, u, v)\; x^k\, y^\ell\, u^m\, v^n\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}v}{\int W(x, y, u, v)\; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}v} \quad \text{with} \quad k\,, \ell\,, m\,, n \geq 0\ , \tag{2.2.10}$$

where

$$W(x, y, u, v) = W(\boldsymbol{r}, \boldsymbol{q}) \quad \text{with} \quad \boldsymbol{r} = (x, y)^{\mathsf{T}}\ ,\ \ \boldsymbol{q} = (u, v)^{\mathsf{T}}\ . \tag{2.2.11}$$

The order of the moments is defined by the sum of the exponents, $k + \ell + m + n$. There are four first-order moments, $\langle x \rangle$, $\langle y \rangle$, $\langle u \rangle$, and $\langle v \rangle$, which together specify position and direction of propagation of the beam profile's centroids within the given coordinate system.

The *centered* moments of the Wigner distribution are defined to be independent of the coordinate system:

$$\left\langle x^k y^\ell u^m v^n \right\rangle_{\mathrm{c}} = \frac{\int W(x, y, u, v)\; (x - \langle x \rangle)^k\; (y - \langle y \rangle)^\ell\; (u - \langle u \rangle)^m\; (v - \langle v \rangle)^n\; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}v}{\int W(x, y, u, v)\; \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}u\, \mathrm{d}v}\ . \tag{2.2.12}$$

There are ten centered second-order moments, specified by $k + \ell + m + n = 2$. Three pure spatial moments, $\langle x^2 \rangle_c$, $\langle y^2 \rangle_c$, $\langle xy \rangle_c$, three pure angular moments, $\langle u^2 \rangle_c$, $\langle v^2 \rangle_c$, $\langle uv \rangle_c$, and four mixed moments, $\langle x\,u \rangle_c$, $\langle y\,v \rangle_c$, $\langle x\,v \rangle_c$, and $\langle y\,u \rangle_c$. The centered second-order moments are associated to the beam extents in the near and far field and to the propagation of beam widths as will be discussed in the next section.

Only the three pure spatial moments can directly be measured since they can be obtained from the power density distribution in the observation plane by

$$\left\langle x^k y^\ell \right\rangle_c = \frac{1}{P} \int I\,(x, y)\; (x - \langle x \rangle)^k\; (y - \langle y \rangle)^\ell\; \mathrm{d}x\,\mathrm{d}y \tag{2.2.13}$$

with

$$\langle x \rangle = \frac{1}{P} \int I\,(x, y)\; x\,\mathrm{d}x\,\mathrm{d}y\;, \tag{2.2.14}$$

$$\langle y \rangle = \frac{1}{P} \int I\,(x, y)\; y\,\mathrm{d}x\,\mathrm{d}y\;, \tag{2.2.15}$$

and

$$P = \int I\,(x, y)\;\mathrm{d}x\,\mathrm{d}y\;. \tag{2.2.16}$$

As the beam propagates through optical systems the Wigner distribution changes and consequently the moments change, too. A simple propagation law for the centered second-order moments through aberration-free optical systems can be derived from the propagation law of the Wigner distribution (2.2.9). Combining the ten moments in a symmetric 4×4-matrix, the variance matrix

$$\mathsf{P} = \begin{pmatrix} \langle x^2 \rangle_c & \langle xy \rangle_c & \langle xu \rangle_c & \langle xv \rangle_c \\ \langle xy \rangle_c & \langle y^2 \rangle_c & \langle yu \rangle_c & \langle yv \rangle_c \\ \langle xu \rangle_c & \langle yu \rangle_c & \langle u^2 \rangle_c & \langle uv \rangle_c \\ \langle xv \rangle_c & \langle yv \rangle_c & \langle uv \rangle_c & \langle v^2 \rangle_c \end{pmatrix}\;, \tag{2.2.17}$$

delivers the propagation law

$$\mathsf{P}_{\mathrm{out}} = \mathsf{S} \cdot \mathsf{P}_{\mathrm{in}} \cdot \mathsf{S}^{\mathsf{T}}\;, \tag{2.2.18}$$

where P_{in} and $\mathsf{P}_{\mathrm{out}}$ are the variance matrices in the input and output planes of the optical system, respectively, and S is the system matrix.

2.2.4 The second-order moments and related physical properties

In this section the relations between the centered second-order moments and some more physical properties are discussed.

2.2.4.1 Near field

The three spatial-centered second-order moments are related to the spatial extent of the power density in the reference plane as can be derived from (2.2.13). For example, the centered second-order moments $\langle x^2 \rangle_c$, defined by

$$\langle x^2 \rangle_c = \frac{1}{P} \int I(x,y)\,(x - \langle x \rangle)^2 \,\mathrm{d}x\,\mathrm{d}y\,, \tag{2.2.19}$$

can be considered as the intensity-weighted average of the squared distances in x-direction of all points in the plane from the beam-profile center. Obviously, this quantity increases with increasing beam extent in x-direction. A beam width in x-direction can be defined as

$$d_x = 4\sqrt{\langle x^2 \rangle_c}\,. \tag{2.2.20}$$

The factor of 4 in this equation has been chosen by convention to adapt this beam-width definition to the former $1/\mathrm{e}^2$-definition for the beam radius of Gaussian beams. For an aligned elliptical Gaussian beam profile,

$$I(x,y) \propto \mathrm{e}^{-2\frac{x^2}{w_x^2}} \cdot \mathrm{e}^{-2\frac{y^2}{w_y^2}}\,, \tag{2.2.21}$$

where w_x and w_y are the $1/\mathrm{e}^2$-beam radii in x- and y-direction, respectively, the relation

$$d_x = 2\,w_x$$

holds. Similar, a beam width in y-direction can be defined as

$$d_y = 4\sqrt{\langle y^2 \rangle_c}\,. \tag{2.2.22}$$

The beam width along an arbitrary azimuthal direction enclosing an angle of α with the x-axis can be derived from a rotation of the coordinate system delivering

$$d_\alpha = 4\sqrt{\langle x^2 \rangle_c \cos^2\alpha + 2\,\langle xy \rangle_c \sin\alpha\cos\alpha + \langle y^2 \rangle_c \sin^2\alpha}\,. \tag{2.2.23}$$

In general, the beam width considered as a function of the azimuthal direction α has unique maximum and minimum. The related directions are orthogonal to each other and define the principal axes of the beam. The signed angle between the x-axis and that principal axis which is closer to the x-axis is given by

$$\varphi = \frac{1}{2}\,\mathrm{atan}\left(\frac{2\,\langle xy \rangle_c}{\langle x^2 \rangle_c - \langle y^2 \rangle_c}\right)\,. \tag{2.2.24}$$

The beam width along that principal axis which is closer to the x-axis is determined by

$$d'_x = 2\sqrt{2}\left\{\left(\langle x^2 \rangle_c + \langle y^2 \rangle_c\right) + \varepsilon\left[\left(\langle x^2 \rangle_c - \langle y^2 \rangle_c\right)^2 + 4\,\langle xy \rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{2.2.25}$$

with

$$\varepsilon = \mathrm{sgn}\left(\langle x^2 \rangle_c - \langle y^2 \rangle_c\right)\,. \tag{2.2.26}$$

Correspondingly, the beam width along the principal axis closer to the y-axis is given by

$$d'_y = 2\sqrt{2}\left\{\left(\langle x^2 \rangle_c + \langle y^2 \rangle_c\right) - \varepsilon\left[\left(\langle x^2 \rangle_c - \langle y^2 \rangle_c\right)^2 + 4\,\langle xy \rangle_c^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\,. \tag{2.2.27}$$

Hence, the three spatial-centered second-order moments define the size and orientation of the so-called variance ellipse as the representation of a beam profile's extent (Fig. 2.2.3).

Beam profiles having approximately equal beam widths in both principal planes, $d'_x \approx d'_y$, may be considered as circular and a beam diameter may be defined by

$$d = 2\sqrt{2}\,\sqrt{\langle x^2 \rangle + \langle y^2 \rangle}\,. \tag{2.2.28}$$

Sometimes this is an useful definition even for non-circular beam profiles, denoted then as "generalized beam diameter".

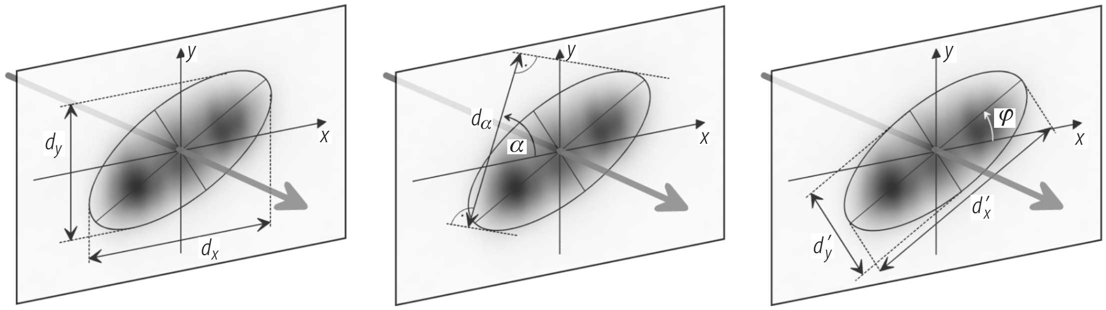

Fig. 2.2.3. Widths and variance ellipse of a power density profile. Left: widths d_x and d_y along the coordinate axes, middle: width d_α along an arbitrary direction, right: widths d'_x and d'_y along the principal axes.

2.2.4.2 Far field

The three angular-centered second-order moments are related to the beam-profile extent in the far field, far away from the reference plane, or in the focal plane of a focusing lens. From the propagation law of the second-order moments, (2.2.18), the dependency of the spatial moments on the propagation distance z from the reference plane can be derived:

$$\begin{aligned} \langle x^2\rangle_\mathrm{c}(z) &= \langle x^2\rangle_{\mathrm{c},0} + 2\,z\,\langle xu\rangle_{\mathrm{c},0} + z^2\,\langle u^2\rangle_{\mathrm{c},0}\ , \\ \langle xy\rangle_\mathrm{c}(z) &= \langle xy\rangle_{\mathrm{c},0} + z\,\left(\langle xv\rangle_{\mathrm{c},0} + \langle yu\rangle_{\mathrm{c},0}\right) + z^2\,\langle uv\rangle_{\mathrm{c},0}\ , \\ \langle y^2\rangle_\mathrm{c}(z) &= \langle y^2\rangle_{\mathrm{c},0} + 2\,z\,\langle yv\rangle_{\mathrm{c},0} + z^2\,\langle v^2\rangle_{\mathrm{c},0}\ . \end{aligned} \tag{2.2.29}$$

For large distances z the spatial moments depend only on the angular moments in the reference plane:

$$\begin{aligned} \langle x^2\rangle_\mathrm{c}(z) &\approx z^2\,\langle u^2\rangle\ , \\ \langle xy\rangle_\mathrm{c}(z) &\approx z^2\,\langle uv\rangle\ , \\ \langle y^2\rangle_\mathrm{c}(z) &\approx z^2\,\langle v^2\rangle\ . \end{aligned} \tag{2.2.30}$$

The azimuthal angle φ_F of that principal axis in the far field, which is closer to the x-axis is then obtained by

$$\varphi_\mathrm{F} = \lim_{z\to\infty}\frac{1}{2}\,\mathrm{atan}\left(\frac{2\,\langle xy\rangle_\mathrm{c}(z)}{\langle x^2\rangle_\mathrm{c}(z) - \langle y^2\rangle_\mathrm{c}(z)}\right) = \frac{1}{2}\,\mathrm{atan}\left(\frac{2\,\langle uv\rangle_\mathrm{c}}{\langle u^2\rangle_\mathrm{c} - \langle v^2\rangle_\mathrm{c}}\right)\ , \tag{2.2.31}$$

and the (full) divergence angles along the principal axes of the far field might be defined as

$$\theta'_x = \lim_{z\to\infty}\frac{d'_x(z)}{z} = 2\sqrt{2}\left\{\left(\langle u^2\rangle_\mathrm{c} + \langle v^2\rangle_\mathrm{c}\right) + \eta\left[\left(\langle u^2\rangle_\mathrm{c} - \langle v^2\rangle_\mathrm{c}\right)^2 + 4\,\langle uv\rangle_\mathrm{c}^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}\ , \tag{2.2.32}$$

$$\theta'_y = \lim_{z\to\infty}\frac{d'_y(z)}{z} = 2\sqrt{2}\left\{\left(\langle u^2\rangle_\mathrm{c} + \langle v^2\rangle_\mathrm{c}\right) - \eta\left[\left(\langle u^2\rangle_\mathrm{c} - \langle v^2\rangle_\mathrm{c}\right)^2 + 4\,\langle uv\rangle_\mathrm{c}^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{2.2.33}$$

with

$$\eta = \mathrm{sgn}\,\left(\langle x^2\rangle_\mathrm{c} - \langle y^2\rangle_\mathrm{c}\right)\ . \tag{2.2.34}$$

The generalized beam divergence angle might be defined as

$$\theta = 2\,\sqrt{2}\,\sqrt{\langle u^2\rangle_\mathrm{c} + \langle v^2\rangle_\mathrm{c}}\ . \tag{2.2.35}$$

The azimuthal orientation of the far field may differ from the orientation of the near field.

2.2.4.3 Phase paraboloid and twist

The four mixed moments $\langle xu\rangle_{\mathrm{c}}$, $\langle xv\rangle_{\mathrm{c}}$, $\langle yu\rangle_{\mathrm{c}}$, and $\langle yv\rangle_{\mathrm{c}}$ are closely related to the phase properties of the beam in the reference plane. Together with the three spatial moments they determine the radii of curvature and azimuthal orientation of the best-fitting phase paraboloid. Although the phase properties of partially coherent beams might be quite complicated, it is always possible to find a best-fitting phase function being quadratic (bilinear) in x and y:

$$\Phi(x,y) = k\left(a\,x^2 + 2\,b\,x\,y + c\,y^2\right) \,. \tag{2.2.36}$$

The best-fitting parameters a, b, c are defined by minimizing the generalized divergence angle, (2.2.35), if a phase function according to (2.2.36) would be subtracted from the actual phase distribution in the reference plane (e.g. by introducing a cylindrical lens) resulting in

$$a = \frac{\left\langle y^2\right\rangle \langle xu\rangle \left(\left\langle x^2\right\rangle + \left\langle y^2\right\rangle\right) - \langle xy\rangle^2 \left(\langle xu\rangle - \langle yv\rangle\right) - \langle xy\rangle \left\langle y^2\right\rangle \left(\langle xv\rangle + \langle yu\rangle\right)}{\left(\langle x^2\rangle + \langle y^2\rangle\right)\left(\langle x^2\rangle \langle y^2\rangle - \langle xy\rangle^2\right)} \,, \tag{2.2.37}$$

$$b = \frac{\left\langle x^2\right\rangle \left\langle y^2\right\rangle \left(\langle xv\rangle + \langle yu\rangle\right) - \langle xy\rangle \left(\left\langle x^2\right\rangle \langle yv\rangle + \left\langle y^2\right\rangle \langle xu\rangle\right)}{\left(\langle x^2\rangle + \langle y^2\rangle\right)\left(\langle x^2\rangle \langle y^2\rangle - \langle xy\rangle^2\right)} \,, \tag{2.2.38}$$

$$c = \frac{\left\langle x^2\right\rangle \langle yv\rangle \left(\left\langle x^2\right\rangle + \left\langle y^2\right\rangle\right) + \langle xy\rangle^2 \left(\langle xu\rangle - \langle yv\rangle\right) - \langle xy\rangle \left\langle x^2\right\rangle \left(\langle xv\rangle + \langle yu\rangle\right)}{\left(\langle x^2\rangle + \langle y^2\rangle\right)\left(\langle x^2\rangle \langle y^2\rangle - \langle xy\rangle^2\right)} \,. \tag{2.2.39}$$

A phase distribution as given in (2.2.36) can be considered as a rotated phase paraboloid, with

$$\varphi_{\mathrm{P}} = \frac{1}{2}\,\mathrm{atan}\left(\frac{2\,b}{a-c}\right) \tag{2.2.40}$$

as the signed angle between the x-axis and that principal axis of the phase paraboloid, which is closer to the x-axis, and with

$$R'_x = \frac{2}{(a+c) + \mu\sqrt{(a-c)^2 + 4\,b^2}} \tag{2.2.41}$$

and

$$R'_y = \frac{2}{(a+c) - \mu\sqrt{(a-c)^2 + 4\,b^2}} \tag{2.2.42}$$

with

$$\mu = \mathrm{sgn}\;(a-c) \tag{2.2.43}$$

as the radii of curvature along that principal axis of the phase paraboloid, which is closer to the x- and y-axis, respectively. The radii of curvature R'_x and R'_x independently may be positive or negative or infinite, the later indicating a plane phase front along that azimuthal direction. The azimuthal orientation of the phase paraboloid's principal axes may differ from the orientation of the near field and/or far field.

If the radii of phase curvature along both principal axes are approximately equal, $R'_x \approx R'_y$, a generalized phase curvature of the best-fitting rotational symmetric phase paraboloid is defined by

$$R = \frac{\left\langle x^2\right\rangle_{\mathrm{c}} + \left\langle y^2\right\rangle_{\mathrm{c}}}{\langle xu\rangle_{\mathrm{c}} + \langle yv\rangle_{\mathrm{c}}} \,. \tag{2.2.44}$$

Another phase-related parameter is the so-called twist, defined as

$$t_{\mathrm{w}} = \langle xv \rangle - \langle yu \rangle \ . \tag{2.2.45}$$

The twist parameter is proportional to the orbital angular momentum transferred by the beam [93Sim].

2.2.4.4 Invariants

From the ten centered second-order moments two basic quantities can be derived, that are invariant under propagation through aberration-free first-order optics [03Nem].

The effective beam propagation ratio is defined as

$$M_{\mathrm{eff}}^2 = \frac{4\pi}{\lambda} \left(\det(\mathsf{P})\right)^{\frac{1}{4}} \geq 1 \tag{2.2.46}$$

and can be considered as a measure of the focusability of a beam. The lower limit holds only for coherent Gaussian beams.

The intrinsic astigmatism a, given by

$$\begin{aligned} a = \frac{8\pi^2}{\lambda^2} \Big[& \left(\langle x^2 \rangle_{\mathrm{c}} \langle u^2 \rangle_{\mathrm{c}} - \langle xu \rangle_{\mathrm{c}}^2 \right) + \left(\langle y^2 \rangle_{\mathrm{c}} \langle v^2 \rangle_{\mathrm{c}} - \langle yv \rangle_{\mathrm{c}}^2 \right) \\ & +2 \left(\langle xy \rangle_{\mathrm{c}} \langle uv \rangle_{\mathrm{c}} - \langle xv \rangle_{\mathrm{c}} \langle yu \rangle_{\mathrm{c}} \right) \Big] - \left(M_{\mathrm{eff}}^2 \right)^2 \geq 0 \ , \end{aligned} \tag{2.2.47}$$

is related to the visible and hidden astigmatism of the beam (see below).

2.2.4.5 Propagation of beam widths and beam propagation ratios

Under free-space propagation any directional beam width d_α, as well as the generalized beam diameter d, obeys an hyperbolic propagation law:

$$d_\alpha(z) = d_{0,\alpha} \sqrt{1 + \left(\frac{z - z_{0,\alpha}}{z_{\mathrm{R},\alpha}} \right)^2} = \sqrt{d_{0,\alpha}^2 + \theta_\alpha^2 \left(z - z_{0,\alpha} \right)^2} \ , \tag{2.2.48}$$

where $z_{0,\alpha}$ is the z-position of the smallest width, the waist position, $d_{0,\alpha}$ is the waist width, θ_α the divergence angle, and $z_{\mathrm{R},\alpha}$ the Rayleigh length, i.e. the distance from the waist position, where the width has grown by factor of $\sqrt{2}$. For the width along the x-direction, $\alpha = 0$, see Fig. 2.2.4, the parameters can be obtained by

$$z_0 = -\frac{\langle xu \rangle_{\mathrm{c}}}{\langle u^2 \rangle_{\mathrm{c}}} \ , \tag{2.2.49}$$

$$d_0 = 4 \sqrt{\langle x^2 \rangle_{\mathrm{c}} - \frac{\langle xu \rangle_{\mathrm{c}}^2}{\langle u^2 \rangle_{\mathrm{c}}}} \ , \tag{2.2.50}$$

and

$$z_{\mathrm{R}} = \sqrt{\frac{\langle x^2 \rangle_{\mathrm{c}}}{\langle u^2 \rangle_{\mathrm{c}}} - \frac{\langle xu \rangle_{\mathrm{c}}^2}{\langle u^2 \rangle_{\mathrm{c}}^2}} \ . \tag{2.2.51}$$

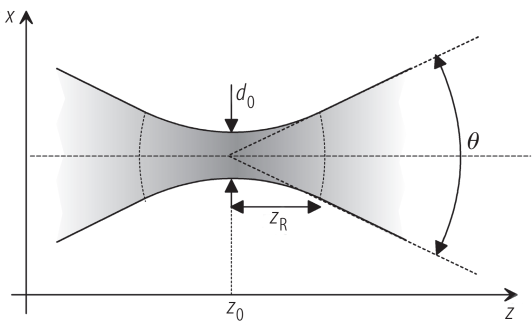

Fig. 2.2.4. Free-space propagation of beam widths with the beam waist position z_0, the beam waist width d_0, the Rayleigh length z_R, and the full divergence angle θ.

For other azimuthal directions α the same equations apply with the following substitutions:

$$\begin{aligned}&\left\langle x^2\right\rangle_\mathrm{c} \to \left\langle x^2\right\rangle_\mathrm{c}\cos^2\alpha + 2\left\langle xy\right\rangle_\mathrm{c}\cos\alpha\sin\alpha + \left\langle y^2\right\rangle_\mathrm{c}\sin^2\alpha\ ,\\ &\left\langle xu\right\rangle_\mathrm{c} \to \left\langle xu\right\rangle_\mathrm{c}\cos^2\alpha + 2\left(\left\langle xv\right\rangle_\mathrm{c} + \left\langle yu\right\rangle_\mathrm{c}\right)\cos\alpha\sin\alpha + \left\langle yv\right\rangle_\mathrm{c}\sin^2\alpha\ ,\\ &\left\langle u^2\right\rangle_\mathrm{c} \to \left\langle u^2\right\rangle_\mathrm{c}\cos^2\alpha + 2\left\langle uv\right\rangle_\mathrm{c}\cos\alpha\sin\alpha + \left\langle v^2\right\rangle_\mathrm{c}\sin^2\alpha\ .\end{aligned} \tag{2.2.52}$$

For the generalized diameter d the propagation parameters are obtained by

$$z_0 = -\frac{\left\langle xu\right\rangle_\mathrm{c} + \left\langle yv\right\rangle_\mathrm{c}}{\left\langle u^2\right\rangle_\mathrm{c} + \left\langle v^2\right\rangle_\mathrm{c}}\ , \tag{2.2.53}$$

$$d_0 = 2\sqrt{2}\sqrt{\left(\left\langle x^2\right\rangle_\mathrm{c} + \left\langle y^2\right\rangle_\mathrm{c}\right) - \frac{\left(\left\langle xu\right\rangle_\mathrm{c} + \left\langle yv\right\rangle_\mathrm{c}\right)^2}{\left\langle u^2\right\rangle_\mathrm{c} + \left\langle v^2\right\rangle_\mathrm{c}}}\ , \tag{2.2.54}$$

and

$$z_\mathrm{R} = \sqrt{\frac{\left\langle x^2\right\rangle_\mathrm{c} + \left\langle y^2\right\rangle_\mathrm{c}}{\left\langle u^2\right\rangle_\mathrm{c} + \left\langle v^2\right\rangle_\mathrm{c}} - \left(\frac{\left\langle xu\right\rangle_\mathrm{c} + \left\langle yv\right\rangle_\mathrm{c}}{\left\langle u^2\right\rangle_\mathrm{c} + \left\langle v^2\right\rangle_\mathrm{c}}\right)^2}\ . \tag{2.2.55}$$

It should be noted that beam widths along the principal axes, d'_x and d'_y, do *not* obey the hyperbolic propagation law in the case of a general astigmatic beam with rotating variance ellipse (see next section).

The product of the (directional) beam waist diameter d, d_α and the corresponding far-field divergence angle θ, θ_α is called the beam parameter product. Due to diffraction the beam parameter product has a lower limit given by

$$d_0 \cdot \theta = \frac{d_0^2}{z_\mathrm{R}} \geq 4\frac{\lambda}{\pi}\ , \quad d_{0,\alpha} \cdot \theta_\alpha = \frac{d_{0,\alpha}^2}{z_{\mathrm{R},\alpha}} \geq 4\frac{\lambda}{\pi}\ . \tag{2.2.56}$$

Normalization to this lower limit delivers the so-called beam parameter ratios

$$M^2 = \frac{\pi}{\lambda}\,\frac{d_0 \cdot \theta}{4}\ , \quad M_\alpha^2 = \frac{\pi}{\lambda}\,\frac{d_{0,\alpha} \cdot \theta_\alpha}{4}\ . \tag{2.2.57}$$

The beam parameter ratios M^2 and M_α^2 are invariant in stigmatic aberration-free first-order optical systems (combinations of perfect spherical lenses). In contrast to the effective beam parameter ratio M_eff^2 , they may change under propagation through cylindrical lenses.

2.2.5 Beam classification

Lasers beams can be classified according to their propagation behavior. The classification is based on the discrimination between circular and non-circular power density profiles and the azimuthal orientation of the non-circular profiles. A beam profile is considered circular if the beam widths along both principal axes are approximately equal, or, in practice, if

$$\frac{\min\left(d'_x, d'_y\right)}{\max\left(d'_x, d'_y\right)} > 0.87\ . \tag{2.2.58}$$

In this sense a homogeneous profile with square footprint is regarded circular, see Fig. 2.2.5.

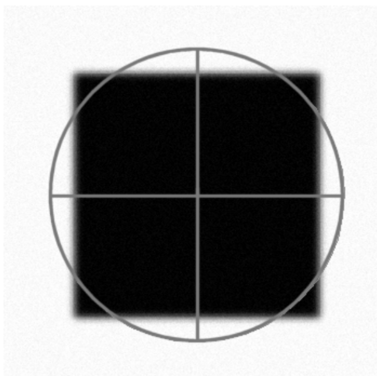

Fig. 2.2.5. Within the concept of second-order-moment beam characterization a square top-hat profile is considered circular: Its width is independent of the azimuthal direction.

2.2.5.1 Stigmatic beams

A laser beam is considered stigmatic if all its profiles under free-space propagation are circular and if all non-circular profiles behind an arbitrary cylindrical lens, inserted somewhere in the beam, have the same azimuthal orientation as the lens. The system matrix P_{st} of a perfectly stigmatic beam has only three independent parameters:

$$\mathsf{P}_{\mathrm{st}} = \begin{pmatrix} \left\langle x^2\right\rangle_{\mathrm{c}} & 0 & \left\langle xu\right\rangle_{\mathrm{c}} & 0 \\ 0 & \left\langle x^2\right\rangle_{\mathrm{c}} & 0 & \left\langle xu\right\rangle_{\mathrm{c}} \\ \left\langle xu\right\rangle_{\mathrm{c}} & 0 & \left\langle u^2\right\rangle_{\mathrm{c}} & 0 \\ 0 & \left\langle xu\right\rangle_{\mathrm{c}} & 0 & \left\langle u^2\right\rangle_{\mathrm{c}} \end{pmatrix}\ . \tag{2.2.59}$$

Physical parameters of a stigmatic beam are the beam diameter in the reference plane

$$d = 4\sqrt{\left\langle x^2\right\rangle_{\mathrm{c}}} \tag{2.2.60}$$

and the full divergence angle

$$\theta = 4\sqrt{\left\langle u^2\right\rangle}\ . \tag{2.2.61}$$

Since the properties of a stigmatic beam are independent of the azimuthal direction, it has a unique waist position

$$z_0 = -\frac{\left\langle xu\right\rangle_{\mathrm{c}}}{\left\langle u^2\right\rangle_{\mathrm{c}}} \tag{2.2.62}$$

with a waist diameter of

$$d_0 = 4\sqrt{\left\langle x^2 \right\rangle_\mathrm{c} - \frac{\left\langle xu \right\rangle_\mathrm{c}^2}{\left\langle u^2 \right\rangle_\mathrm{c}}} \,. \tag{2.2.63}$$

The Rayleigh length z_R is the distance from the waist position where the diameter has grown by a factor of $\sqrt{2}$, given by

$$z_\mathrm{R} = \sqrt{\frac{\left\langle x^2 \right\rangle_\mathrm{c}}{\left\langle u^2 \right\rangle_\mathrm{c}} - \frac{\left\langle xu \right\rangle_\mathrm{c}^2}{\left\langle u^2 \right\rangle_\mathrm{c}^2}} \,. \tag{2.2.64}$$

Finally, the phase paraboloid is of rotational symmetry with the radius of curvature being

$$R = \frac{\left\langle x^2 \right\rangle_\mathrm{c}}{\left\langle xu \right\rangle_\mathrm{c}} \,. \tag{2.2.65}$$

2.2.5.2 Simple astigmatic beams

A laser beam is classified as simple astigmatic if at least some of the power density profiles the beam takes on under free-space propagation are non-circular, but all non-circular profiles have the same azimuthal orientation. In practice, the orientations of two non-circular beam profiles are regarded as equal, if the azimuthal angles differ by less than 10 degrees. A simple astigmatic beam whose principal axes are parallel to the x- and y-axis is called aligned simple astigmatic. The variance matrix P_asa of a perfect aligned simple astigmatic beam has six independent parameters:

$$\mathsf{P}_\mathrm{asa} = \begin{pmatrix} \left\langle x^2 \right\rangle_\mathrm{c} & 0 & \left\langle xu \right\rangle_\mathrm{c} & 0 \\ 0 & \left\langle y^2 \right\rangle_\mathrm{c} & 0 & \left\langle yv \right\rangle_\mathrm{c} \\ \left\langle xu \right\rangle_\mathrm{c} & 0 & \left\langle u^2 \right\rangle_\mathrm{c} & 0 \\ 0 & \left\langle yv \right\rangle_\mathrm{c} & 0 & \left\langle v^2 \right\rangle_\mathrm{c} \end{pmatrix} . \tag{2.2.66}$$

All the physical parameters given for stigmatic beams can be assigned separately for each principal axis of a simple astigmatic beam. The diameters in x- and y-direction are

$$d_x = 4\sqrt{\left\langle x^2 \right\rangle_\mathrm{c}} \,, \quad d_y = 4\sqrt{\left\langle y^2 \right\rangle_\mathrm{c}} \tag{2.2.67}$$

and the according full divergence angle

$$\theta_x = 4\sqrt{\left\langle u^2 \right\rangle} \,, \quad \theta_y = 4\sqrt{\left\langle v^2 \right\rangle} \,. \tag{2.2.68}$$

Aligned simple astigmatic beams have in general two different waist positions for each principal axis:

$$z_{0,x} = -\frac{\left\langle xu \right\rangle_\mathrm{c}}{\left\langle u^2 \right\rangle_\mathrm{c}} \,, \quad z_{0,y} = -\frac{\left\langle yv \right\rangle_\mathrm{c}}{\left\langle v^2 \right\rangle_\mathrm{c}} \tag{2.2.69}$$

with the associated waist diameters

$$d_{0,x} = 4\sqrt{\left\langle x^2 \right\rangle_\mathrm{c} - \frac{\left\langle xu \right\rangle_\mathrm{c}^2}{\left\langle u^2 \right\rangle_\mathrm{c}}} \,, \quad d_{0,y} = 4\sqrt{\left\langle y^2 \right\rangle_\mathrm{c} - \frac{\left\langle yv \right\rangle_\mathrm{c}^2}{\left\langle v^2 \right\rangle_\mathrm{c}}} \,. \tag{2.2.70}$$

Similarly, two Rayleigh lengths are defined by

$$z_{\mathrm{R},x} = \sqrt{\frac{\left\langle x^2 \right\rangle_\mathrm{c}}{\left\langle u^2 \right\rangle_\mathrm{c}} - \frac{\left\langle xu \right\rangle_\mathrm{c}^2}{\left\langle u^2 \right\rangle_\mathrm{c}^2}} \,, \quad z_{\mathrm{R},y} = \sqrt{\frac{\left\langle y^2 \right\rangle_\mathrm{c}}{\left\langle v^2 \right\rangle_\mathrm{c}} - \frac{\left\langle yv \right\rangle_\mathrm{c}^2}{\left\langle v^2 \right\rangle_\mathrm{c}^2}} \,, \tag{2.2.71}$$

and the radii of phase curvature are

$$R_x = \frac{\langle x^2 \rangle_{\rm c}}{\langle xu \rangle_{\rm c}}\,, \quad R_y = \frac{\langle y^2 \rangle_{\rm c}}{\langle yv \rangle_{\rm c}}\,. \tag{2.2.72}$$

The propagation laws for the beam diameters along both principal axes are:

$$d_x\,(z) = d_{0,x}\sqrt{1+\left(\frac{z-z_{0,x}}{z_{{\rm R},x}}\right)^2} = \sqrt{d_{0,x}^2 + \theta_x^2\,(z-z_{0,x})^2} \tag{2.2.73}$$

and

$$d_y\,(z) = d_{0,y}\sqrt{1+\left(\frac{z-z_{0,y}}{z_{{\rm R},y}}\right)^2} = \sqrt{d_{0,y}^2 + \theta_y^2\,(z-z_{0,y})^2}\,. \tag{2.2.74}$$

For non-aligned simple astigmatic beams similar relations hold.

2.2.5.3 General astigmatic beams

All other beams are classified as general astigmatic. Usually all ten second-order moments are necessary to describe a general astigmatic beam.

2.2.5.4 Pseudo-symmetric beams

Pseudo-symmetric beams are general astigmatic but "look like" stigmatic or simple astigmatic under free-space propagation. They possess an inner astigmatism which is hidden under free propagation and propagation through stigmatic (isotropic) optical systems (i.e. combinations of spherical lenses). Pseudo-symmetric beams differ from real stigmatic or simple astigmatic beams by a non-vanishing twist parameter, $t_{\rm w} \neq 0$.

The variance matrix $\mathsf{P}_{\rm pst}$ of pseudo-stigmatic beams is therefore

$$\mathsf{P}_{\rm pst} = \begin{pmatrix} \langle x^2 \rangle_{\rm c} & 0 & \langle xu \rangle_{\rm c} & \frac{t}{2} \\ 0 & \langle x^2 \rangle_{\rm c} & -\frac{t}{2} & \langle xu \rangle_{\rm c} \\ \langle xu \rangle_{\rm c} & -\frac{t}{2} & \langle u^2 \rangle_{\rm c} & 0 \\ \frac{t}{2} & \langle xu \rangle_{\rm c} & 0 & \langle u^2 \rangle_{\rm c} \end{pmatrix}\,. \tag{2.2.75}$$

Under free-space propagation there is no difference between a real stigmatic beam, $t_{\rm w} = 0$, and the corresponding pseudo-stigmatic one, $t_{\rm w} \neq 0$, (2.2.29). The difference can be uncovered by inserting an arbitrary cylindrical lens somewhere in the beam path. The stigmatic beam is converted into a simple astigmatic beam with non-rotating variance ellipse while the pseudo-stigmatic one is turned into a general astigmatic beam with rotating variance ellipse. Figure 2.2.6 illustrates the different behaviors.

The variance matrix $\mathsf{P}_{\rm psa}$ of aligned pseudo-simple astigmatic beams is given by

$$\mathsf{P}_{\rm psa} = \begin{pmatrix} \langle x^2 \rangle_{\rm c} & 0 & \langle xu \rangle_{\rm c} & \frac{t}{2} \\ 0 & \langle y^2 \rangle_{\rm c} & -\frac{t}{2} & \langle yv \rangle_{\rm c} \\ \langle xu \rangle_{\rm c} & -\frac{t}{2} & \langle u^2 \rangle_{\rm c} & 0 \\ \frac{t}{2} & \langle yv \rangle_{\rm c} & 0 & \langle v^2 \rangle_{\rm c} \end{pmatrix}\,. \tag{2.2.76}$$

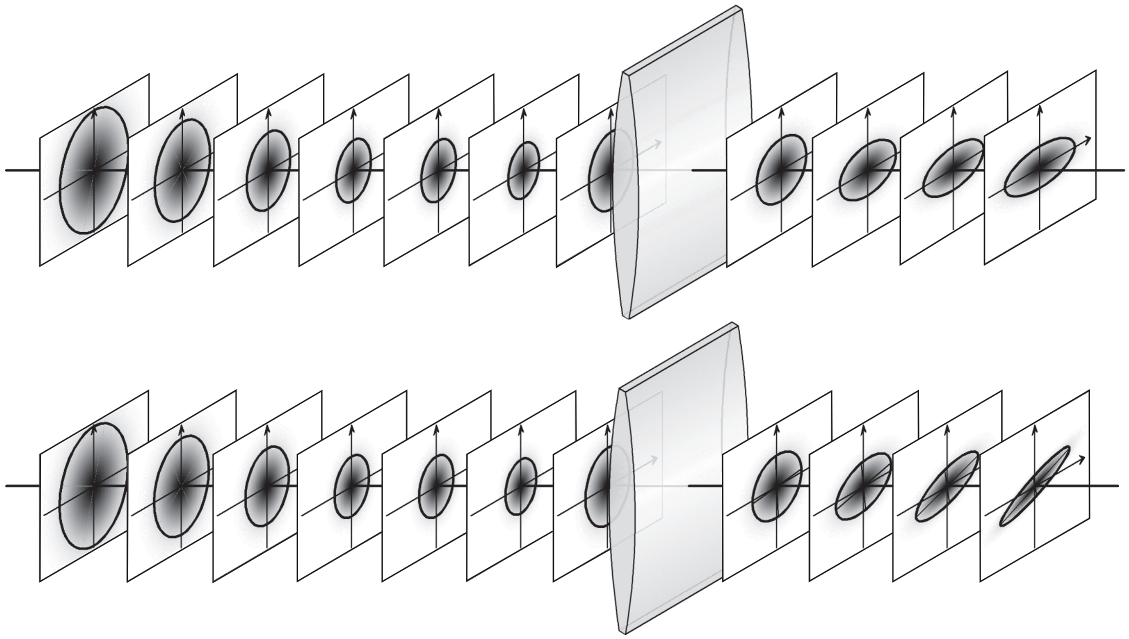

Fig. 2.2.6. Propagation of a stigmatic (top) and pseudo-stigmatic (bottom) laser beam. In free-space propagation both beams are indistinguishable. But a cylindrical lens transforms the stigmatic beam into a simple astigmatic one, whereas the pseudo-stigmatic beam becomes general astigmatic with rotating variance ellipse.

Again, under free-space propagation there is no difference between a real simple astigmatic beam, $t_{\mathrm{w}} = 0$, and the corresponding pseudo-simple astigmatic one, $t_{\mathrm{w}} \neq 0$, (2.2.29). Inserting an aligned cylindrical lens somewhere in the beam pass unveils the difference. The simple astigmatic beam keeps being simple astigmatic while the pseudo-simple astigmatic one is turned into a general astigmatic beam with rotating variance ellipse. Figure 2.2.7 illustrates the different behaviors.

2.2.5.5 Intrinsic astigmatism and beam conversion

Applying astigmatic (anisotropic) optical systems (including cylindrical lenses) may convert beams from one class to another. But only beams with vanishing intrinsic astigmatism a, (2.2.47), can be converted into stigmatic ones [94Mor]. In practice, beams with

$$\frac{a}{\left(M_{\mathrm{eff}}^2\right)^2} < 0.039 \tag{2.2.77}$$

are considered intrinsic stigmatic, all others intrinsic astigmatic (the limit of 0.039 is a consequence of (2.2.58)). Intrinsic astigmatic beams can always be converted into pseudo-stigmatic or simple astigmatic ones.

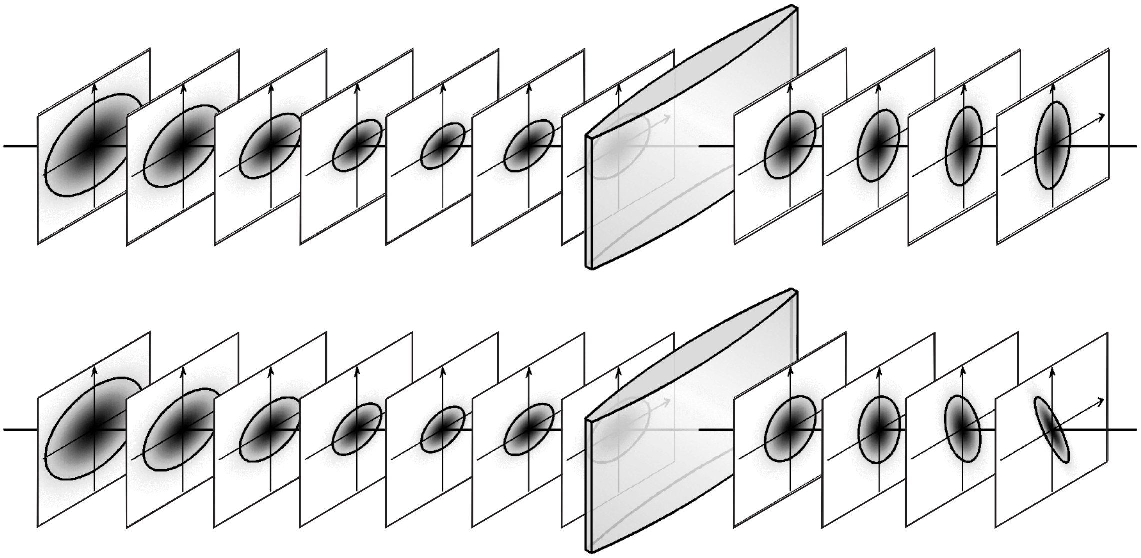

Fig. 2.2.7. Propagation of a simple astigmatic (top) and a pseudo-simple astigmatic (bottom) laser beam. In free-space propagation both beams are indistinguishable. But an aligned cylindrical lens transforms the simple astigmatic beam into a simple astigmatic one, whereas the pseudo-simple astigmatic beam becomes general astigmatic with rotating variance ellipse.

2.2.6 Measurement procedures

Only the three pure spatial moments out of the ten second-order moments are accessible for direct measurement. The other seven moments are retrieved indirectly based on the propagation law of the spatial moments (2.2.29).

The measurement method is based on the acquisition of a couple of power density profiles at different z-locations near the generalized beam waist, (2.2.53), e.g. by means of CCD cameras or similar devices (Fig. 2.2.8, left). From the measured profiles the spatial moments at each measurement plane are calculated. Fitting parabolas with three free parameters to the curve of each spatial moment delivers nine independent quantities: the moments $\langle x^2\rangle_{\mathrm{c},0}$, $\langle xy\rangle_{\mathrm{c},0}$, $\langle y^2\rangle_{\mathrm{c},0}$, $\langle xu\rangle_{\mathrm{c},0}$, $\langle yv\rangle_{\mathrm{c},0}$, $\langle u^2\rangle_{\mathrm{c},0}$, $\langle uv\rangle_{\mathrm{c},0}$, $\langle v^2\rangle_{\mathrm{c},0}$ and the sum of the crossed mixed moments $\langle xv\rangle_{\mathrm{c},0} + \langle yu\rangle_{\mathrm{c},0}$. If the waist of the beam is not accessible, an artificial waist has to be created by inserting an almost aberration-free focusing lens into the beam path. Approximately half of the profiles should be acquired close to the waist within one generalized Rayleigh length, the rest outside two Rayleigh lengths. This ensures balanced accuracy for all parameters of the fitting process.

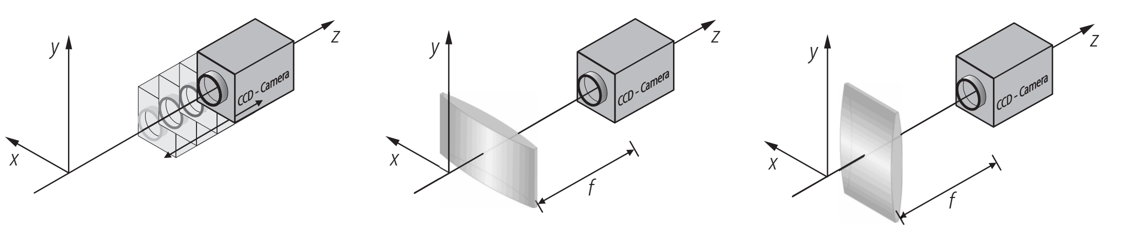

Fig. 2.2.8. Determination of the ten second-order moments in three steps. First step is a z-scan measurement (left), in the second step the CCD camera is placed in the focal plane behind a horizontally oriented cylindrical lens (middle), in the third step the lens is rotated by 90 degrees (right).

At least one cylindrical lens is needed for the measurement of the missing difference of the crossed mixed moments $\langle xv\rangle_{c,0} - \langle yu\rangle_{c,0}$. To retrieve it, a cylindrical lens with focal length f is inserted into the beam path at an arbitrary position in the beam waist region. Firstly, this cylindrical lens shall be aligned with the x-axis and the spatial moment $\langle xy\rangle_1$ is measured in the focal distance behind the lens (Fig. 2.2.8, middle). Next, the lens is rotated by 90 degrees and the spatial moment $\langle xy\rangle_2$ is again measured in the focal distance from the lens (Fig. 2.2.8, right). The missing difference of the crossed mixed moments of the reference plane is then given by

$$\langle xv\rangle_{c,0} - \langle yu\rangle_{c,0} = \frac{\langle xy\rangle_2 - \langle xy\rangle_1}{f} \ . \tag{2.2.78}$$

2.2.7 Beam positional stability

2.2.7.1 Absolute fluctuations

For various reasons a laser beam may fluctuate in position and/or direction. The positional fluctuations in a transverse plane may be measured by the variance of the first-order spatial moments of the beam profile:

$$\langle x^2\rangle_s = \frac{1}{N}\sum_{i=1}^{N}\langle x\rangle_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}\langle x\rangle_i\right)^2 , \tag{2.2.79}$$

$$\langle xy\rangle_s = \frac{1}{N}\sum_{i=1}^{N}\langle x\rangle_i\langle y\rangle_i - \frac{1}{N}\sum_{i=1}^{N}\langle x\rangle_i\,\frac{1}{N}\sum_{i=1}^{N}\langle y\rangle_i , \tag{2.2.80}$$

$$\langle y^2\rangle_s = \frac{1}{N}\sum_{i=1}^{N}\langle y\rangle_i - \left(\frac{1}{N}\sum_{i=1}^{N}\langle y\rangle_i\right)^2 , \tag{2.2.81}$$

where $\langle x\rangle_i$ and $\langle y\rangle_i$ are the first-order moments determined in N individual measurements and $\bar{x} = \frac{1}{N}\sum_{i=1}^{N}\langle x\rangle_i$, $\bar{y} = \frac{1}{N}\sum_{i=1}^{N}\langle y\rangle_i$ define the long-term average beam position. Obviously, the positional fluctuations are different from plane to plane. It can be shown that, under some reasonable assumptions, the positional fluctuations can be characterized closely analogous to the characterization of the beam extent based on the second-order moments of the Wigner distribution [94Mor, 96Mor]. Within this concept, the fluctuation properties of a laser beam are completely determined by ten different parameters, arranged in a symmetric 4×4 matrix

$$\mathsf{P}_s = \begin{pmatrix} \langle x^2\rangle_s & \langle xy\rangle_s & \langle xu\rangle_s & \langle xv\rangle_s \\ \langle xy\rangle_s & \langle y^2\rangle_s & \langle yu\rangle_s & \langle yv\rangle_s \\ \langle xu\rangle_s & \langle yu\rangle_s & \langle u^2\rangle_s & \langle uv\rangle_s \\ \langle xv\rangle_s & \langle yv\rangle_s & \langle uv\rangle_s & \langle v^2\rangle_s \end{pmatrix} , \tag{2.2.82}$$

obeying the same simple propagation law as the centered second-order moments:

$$\mathsf{P}_{s,out} = \mathsf{S} \cdot \mathsf{P}_{s,in} \cdot \mathsf{S}^\mathsf{T} \ . \tag{2.2.83}$$

The elements of the beam fluctuation matrix may be considered as the centered second-order moments of a probability distribution $p\,(x, y, u, v)$ giving the probability that the fluctuation beam

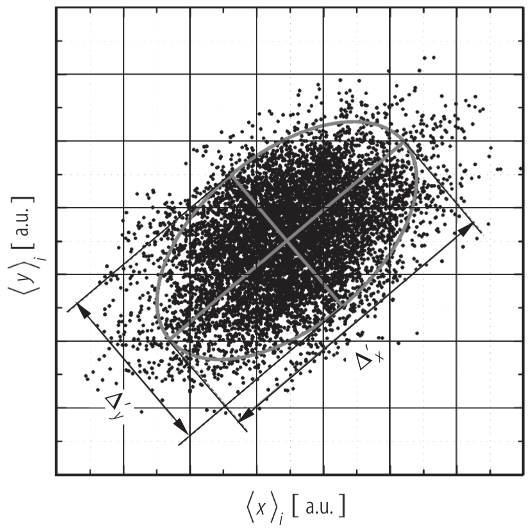

Fig. 2.2.9. Centroid coordinates of fluctuating beam and corresponding variance ellipse characterizing the fluctuations.

has a position (x, y) and direction (u, v) at a random measurement. Similar to the second-order moments of the Wigner distribution, only the three spatial moments are directly measurable. The complete set can be obtained from a z-scan measurement as described in the section above, by acquiring a couple of power density distributions in any measurement plane, calculating the first-order spatial moments from each profile, derive the three variances according to (2.2.79)–(2.2.81), and obtaining the second-order fluctuation moments in the reference plane from a fitting process. Again, measurements behind a cylindrical lens are necessary to achieve all ten parameters.

Fluctuation widths can be derived from the second-order fluctuation moments. In analogy to the beam width definitions, the fluctuation widths are

$$\Delta'_x = 2\sqrt{2}\left\{\left(\left\langle x^2\right\rangle_\mathrm{s} + \left\langle y^2\right\rangle_\mathrm{s}\right) + \tau\left[\left(\left\langle x^2\right\rangle_\mathrm{s} - \left\langle y^2\right\rangle_\mathrm{s}\right)^2 + 4\left\langle xy\right\rangle_\mathrm{s}^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}, \tag{2.2.84}$$

$$\Delta'_y = 2\sqrt{2}\left\{\left(\left\langle x^2\right\rangle_\mathrm{s} + \left\langle y^2\right\rangle_\mathrm{s}\right) - \tau\left[\left(\left\langle x^2\right\rangle_\mathrm{s} - \left\langle y^2\right\rangle_\mathrm{s}\right)^2 + 4\left\langle xy\right\rangle_\mathrm{s}^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{2.2.85}$$

with

$$\tau = \operatorname{sgn}\left(\left\langle x^2\right\rangle_\mathrm{s} - \left\langle y^2\right\rangle_\mathrm{s}\right), \tag{2.2.86}$$

where Δ'_x and Δ'_y are the beam fluctuation widths along the principal axes of the beam positional fluctuations and where

$$\beta = \frac{1}{2}\,\mathrm{atan}\left(\frac{2\left\langle xy\right\rangle_\mathrm{s}}{\left\langle x^2\right\rangle_\mathrm{s} - \left\langle y^2\right\rangle_\mathrm{s}}\right) \tag{2.2.87}$$

is the signed angle between the x-axis and that principal axis of the beam fluctuation which is closer to the x-axis (Fig. 2.2.9). The principal axes of the beam positional fluctuations may not coincide with the principal axes of the power density distribution.

The width of the positional fluctuations along an arbitrary direction, given by the azimuthal angle α, is given by

$$\Delta_\alpha = 4\sqrt{\left\langle x^2\right\rangle_\mathrm{s}\cos^2\alpha + 2\left\langle xy\right\rangle_\mathrm{s}\sin\alpha\,\cos\alpha + \left\langle y^2\right\rangle_\mathrm{s}\sin^2\alpha}\,. \tag{2.2.88}$$

2.2.7.2 Relative fluctuations

For many applications the widths of the positional fluctuations compared to the momentary beam profile width might be more relevant than the absolute fluctuation widths. The relative fluctuation along an arbitrary direction, given by the azimuthal angle α, is defined by

$$\Delta_{\mathrm{rel},\alpha} = \sqrt{\frac{\langle x^2\rangle_{\mathrm{s}}\,\cos^2\alpha + 2\,\langle xy\rangle_{\mathrm{s}}\sin\alpha\,\cos\alpha + \langle y^2\rangle_{\mathrm{s}}\sin^2\alpha}{\langle x^2\rangle_{\mathrm{c}}\,\cos^2\alpha + 2\,\langle xy\rangle_{\mathrm{c}}\sin\alpha\,\cos\alpha + \langle y^2\rangle_{\mathrm{c}}\sin^2\alpha}}\,. \tag{2.2.89}$$

The effective relative fluctuation may by specified by

$$\Delta_{\mathrm{rel}} = \sqrt{\frac{\langle x^2\rangle_{\mathrm{s}} + \langle y^2\rangle_{\mathrm{s}}}{\langle x^2\rangle_{\mathrm{c}} + \langle y^2\rangle_{\mathrm{c}}}}\,. \tag{2.2.90}$$

2.2.7.3 Effective long-term beam widths

For applications with response times much longer than the typical fluctuation durations the time-averaged intensity distribution rather than the momentary beam profile determines the process results:

$$\bar{I}\,(x,y) = \frac{1}{T}\int\limits_{t_0}^{t_0+T} I\,(x,y,t)\;\mathrm{d}\,t\,. \tag{2.2.91}$$

The effective width of the time-averaged power density profile along an azimuthal direction enclosing an angle of α with the x-axis can be obtained from the widths of the momentary beam profile and the fluctuation width by

$$d_{\mathrm{eff},\alpha} = \sqrt{d_\alpha^2 + \Delta_\alpha^2}\,. \tag{2.2.92}$$