P. Starič, E. Margan

Wideband Amplifiers

Part 1

The Laplace Transform

There is nothing more practical than a good theory!
(William Thompson, Lord Kelvin)

About Transforms

The Laplace transform can be used as a powerful method of solving linear differential equations. By using a time domain integration to obtain the frequency domain transfer function and a frequency domain integration to obtain the time domain response, we are relieved of a few nuisances of differential equations, such as defining boundary conditions, not to speak of the difficulties of solving high order systems of equations.

Although Laplace had used integrals of exponential functions for this same purpose already at the beginning of the 19^{th} century, the method we now attribute to him was effectively developed some 100 years later, based on the Heaviside's operational calculus.

The method is applicable to a variety of physical systems (and even some non physical ones, too!) involving transport of energy, storage and transform, but we are going to use it in a relatively narrow field of calculating the time domain response of amplifier filter systems, starting from a known frequency domain transfer function.

As for any tool, the transform tools, be they Fourier, Laplace, Hilbert, etc., have their limitations. Since the parameters of electronic systems can vary over the widest of ranges, it is important to be aware of these limitations in order to use the transform tool correctly.

Contents:

List of Tables:

List of Figures:

1.0 Introduction

With the advent of television and radar during the Second World War, the behavior of wideband amplifiers in the time domain has become very important [Ref. 1.1]. In today's digital world this is even more the case. It is a paradox that designers and troubleshooters of **digital** equipment still depend on oscilloscopes, which — at least in their fast and low level input part — consist of **analog** wideband amplifiers. So the calculation of the time domain response of wideband amplifiers has become even more important than the frequency, phase, and time delay response.

The emphasis of this book is on the amplifier's time domain response. Therefore a thorough knowledge of time related calculus, explained in Part 1, is a necessary prerequisite for understanding all other parts of this book where wideband amplifier networks are discussed.

The time domain response of an amplifier can be calculated by two main methods: The first one is based on **differential equations** and the second uses the **inverse Laplace transform ($\mathcal{L}^{-1}$ transform)**. The differential equation method requires the calculation of boundary conditions, which — in case of high order equations — means an unpleasant and time consuming job. Another method, which also uses differential equations, is the so called *state variable* calculation, in which a differential equation of order n is split into n differential equations of the first order, in order to simplify the calculations. The state variable method also allows the calculation of non linear differential equations. We will use neither of these, for the simple reason that the Laplace transform and its inverse are based on the system poles and zeros, which prove so useful for network calculations in the frequency domain in the later parts of the book. So most of the data which are calculated there is used further in the time domain analysis, thus saving a great deal of work. Also the use of the $\mathcal{L}^{-1}$ transform does not require the calculation of boundary conditions, giving the result directly in the time domain.

In using the $\mathcal{L}^{-1}$ transform most engineers depend on tables. Their method consists firstly of splitting the amplifier transfer function into partial fractions and then looking for the corresponding time domain functions in the $\mathcal{L}$ transform tables. The sum of all these functions (as derived from partial fractions) is then the result. The difficulty arises when no corresponding function can be found in the tables, or even at an earlier stage, if the mathematical knowledge available is insufficient to transform the partial fractions into such a form as to correspond to the formulae in the tables.

In our opinion an amplifier designer should be self-sufficient in calculating the time domain response of a wideband amplifier. Fortunately, this can be almost always derived from simple rational functions and it is relatively easy to learn the $\mathcal{L}^{-1}$ transforms for such cases. In Part 1 we show how this is done generally, as well as for a few simple examples. A great deal of effort has been spent on illustrating the less clear relationships by relevant figures. Since engineers seek to obtain a first glance insight of their subject of study, we believe this approach will be helpful.

This part consists of four main sections. In the first, the concept of harmonic (e.g., sinusoidal) functions, expressed by pairs of counter-rotating complex conjugate phasors, is explained. Then the Fourier series of periodic waveforms are discussed to obtain the discrete spectra of periodic waveforms. This is followed by the Fourier integral to obtain continuous spectra of non-repetitive waveforms. The convergence problem of the Fourier integral is solved by introducing the complex frequency variable $s = \sigma + j\omega$, thus coming to direct Laplace transform ($\mathcal{L}$ transform).

The second section shows some examples of the $\mathcal{L}$ transforms. The results are useful when we seek the inverse transforms of simple functions.

The third section deals with the theory of functions of complex variables, but only to the extent that is needed for understanding the inverse Laplace transform. Here the line and contour integrals (Cauchy integrals), the theory of residues, the Laurent series and the $\mathcal{L}^{-1}$ transform of rational functions are discussed. The existence of the $\mathcal{L}^{-1}$ transform for rational functions is proved by means of the Cauchy integral.

Finally, the concluding section deals with some aspects of the $\mathcal{L}^{-1}$ transforms and the convolution integral. Only two standard problems of the $\mathcal{L}^{-1}$ transform are shown, because all the transient response calculations (by means of the contour integration and the theory of residues) of amplifier networks, presented in Parts 2–5, give enough examples and help to acquire the necessary know-how.

It is probably impossible to discuss Laplace transform in a manner which would satisfy both engineers and mathematicians. Professor *Ivan Vidav* said: "*If we mathematicians are satisfied, you engineers would not be, and vice versa*". Here we have tried to achieve the best possible compromise: to satisfy electronics engineers and at the same time not to 'offend' the mathematicians. But, as late colleague, the physicist *Marko Kogoj*, used to say: "*Engineers never know enough of mathematics; only mathematicians know their science to the extent which is satisfactory for an engineer, but they hardly ever know what to do with it!*" Thus successful engineers keep improving their general knowledge of mathematics — far beyond the text presented here.

After studying this part the readers will have enough knowledge to understand all the time domain calculations in the subsequent parts of the book. In addition, the readers will acquire the basic knowledge needed to do the time-domain calculations by themselves and so become independent of $\mathcal{L}$ transform tables. Of course, in order to save time, they will undoubtedly still use the tables occasionally, or even make tables of their own. But they will be using them with much more understanding and self-confidence, in comparison with those who can do $\mathcal{L}^{-1}$ transform only via the partial fraction expansion and the tables of basic functions.

Those readers who have already mastered the Laplace transform **and its inverse**, can skip this part up to Sec. 1.14, where the $\mathcal{L}^{-1}$ transform of a two pole network is dealt with. From there on we discuss the basic examples, which we use later in many parts of the book; the content of Sec. 1.14 should be understood thoroughly. However, if the reader notices any substantial gaps in his/her knowledge, it is better to start at the beginning.

In the last two parts of this book, Part 6 and 7, we derive a set of computer algorithms which reduce the circuit's time domain analysis, performance plotting and pole layout optimization to a pure routine. However attractive this may seem, we nevertheless recommend the study of Part 1: a good engineer must understand the tools he/she is using in order to use them effectively.

1.1 Three Different Ways of Expressing a Sinusoidal Function

We will first show how a sinusoidal function can be expressed in three different ways. The most common way is to express the instantaneous value a of a sinusoid of amplitude A and angular frequency $\omega_1 = 2\pi f_1$, ($f_1 =$ frequency) by the well known formula:

$$a = f(t) = A \sin \omega_1 t \tag{1.1.1}$$

The reason that we have appended the index '1' to ω will become apparent very soon when we will discuss complex signals containing different frequency components. The amplitude vs. time relation of this function is shown in Fig. 1.1.1a. This is the most familiar display seen by using any sinewave oscillator and an oscilloscope.

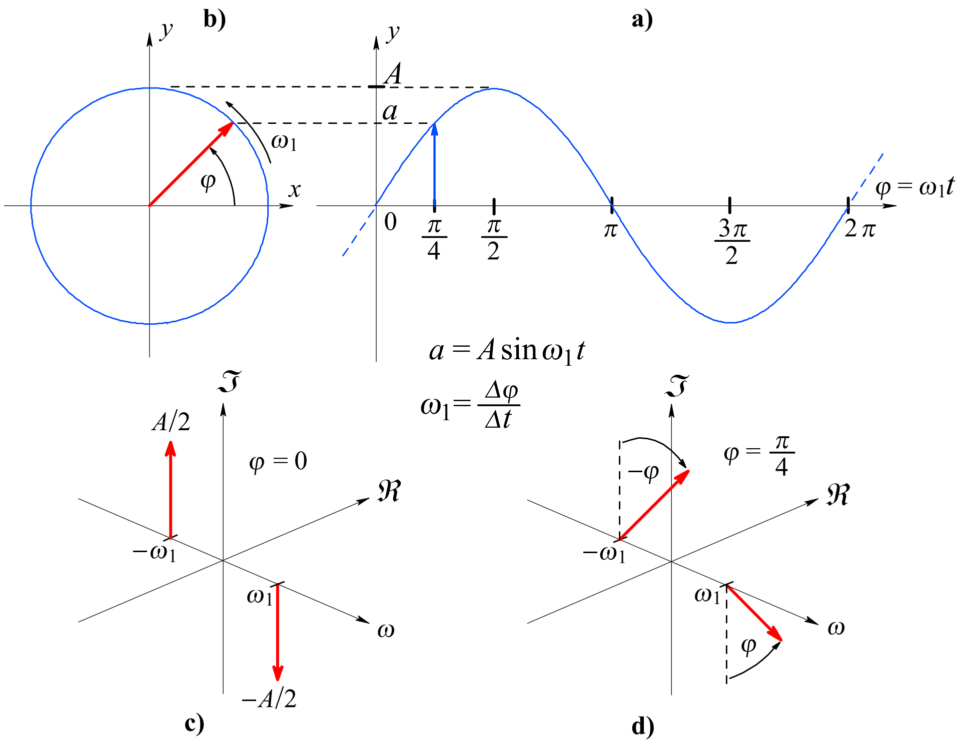

Fig. 1.1.1: Three different presentations of a sine wave: **a)** amplitude in time domain; **b)** a phasor of length A, rotating with angular frequency ω_1; **c)** two complex conjugate phasors of length $A/2$, rotating in opposite directions with angular frequency ω_1, at $\omega_1 t = 0$; **d)** the same as c), except at $\omega_1 t = \pi/4$. To see the movie click on this link: sinemovie.avi.

In electrical engineering, another presentation of a sinusoidal function is often used, coming from the vertical axis projection of a rotating phasor A, as displayed in Fig. 1.1.1b, for which the same Eq. 1.1.1 is valid. Here both axes are real, but one of the axes may also be imaginary. In this case the corresponding mathematical presentation is:

$$f(t) = \widehat{A} = A\,\mathrm{e}^{j\omega_1 t} \tag{1.1.2}$$

where $\widehat{A}$ is a complex quantity and $\mathrm{e} = 2.718\,281\ldots$ is the basis of natural logarithms. However, we can also obtain the real quantity a by expressing the sinusoidal function by **two complex conjugate phasors** of length $A/2$ which rotate in opposite directions, as displayed in a three-dimensional presentation in Fig. 1.1.1c. Here both phasors are shown at $\omega t = 0$ (or $\omega t = 2\pi,\ 4\pi, \dots$). The sum of both phasors has the instantaneous value a, which is **always real**. This is ensured because both phasors rotate with the same angular frequency $+\omega_1$ and $-\omega_1$, starting as shown in Fig. 1.1.1c, and therefore they are always complex conjugate at any instant. We express a by the well-known *Euler* formula:

$$\boxed{a = f(t) = A \sin \omega_1 t = \frac{A}{2j}\left(\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}\right)} \tag{1.1.3}$$

The j in the denominator means that both phasors are imaginary at $t = 0$. The sum of both rotating phasors is then zero, because:

$$f(0) = \frac{A}{2j}\,\mathrm{e}^{j\omega_1 0} - \frac{A}{2j}\,\mathrm{e}^{-j\omega_1 0} = 0 \tag{1.1.4}$$

Both phasors in Fig. 1.1.1c and 1.1.1d are placed on the frequency axis at such a distance from the origin as to correspond to the frequency $\pm\omega_1$. Since the phasors rotate with time the Fig. 1.1.1d, which shows them at $\varphi = \omega_1 t = \pi/4$, helps us to acquire the idea of a three-dimensional presentation. The understanding of these simple time-frequency relations, presented in Fig. 1.1.1c and 1.1.1d and expressed by Eq. 1.1.3, is essential for understanding both the Fourier transform and the Laplace transform.

Eq. 1.1.3 can be changed to the **cosine** function if the phasor with $+\omega_1$ is multiplied by $j = \mathrm{e}^{j\pi/2}$ and the phasor with $-\omega_1$ by $-j = \mathrm{e}^{-j\pi/2}$. The first multiplication means a **counterclockwise** rotation by 90° and the second a **clockwise** rotation by 90°. This causes both phasors to become real at time $t = 0$, their sum again equaling A:

$$f(t) = j\frac{A}{2j}\,\mathrm{e}^{j\omega_1 t} - (-j)\frac{A}{2j}\,\mathrm{e}^{-j\omega_1 t} = A\cos\omega_1 t \tag{1.1.5}$$

In general a sinusoidal function with a non-zero phase angle φ at $t = 0$ is expressed as:

$$A\sin(\omega t + \varphi) = \frac{A}{2j}\left[\mathrm{e}^{j(\omega t+\varphi)} - \mathrm{e}^{-j(\omega t+\varphi)}\right] \tag{1.1.6}$$

The need to introduce the frequency axis in Fig. 1.1.1c and 1.1.1d will become apparent in the experiment shown in Fig. 1.1.2. Here we have a unity gain amplifier with a poor loop gain, driven by a sinewave source with frequency ω_1 and amplitude A_1, and loaded by the resistor R_L. If the resistor's value is too low and the amplitude of the input signal is high the amplifier reaches its maximum output current level, and the output signal $f(t)$ becomes distorted (we have purposely kept the same notation A as in the previous figure, rather than introducing the symbol V for voltage). The distorted output signal contains not just the original signal with the same fundamental frequency ω_1, but also a third harmonic component with the amplitude $A_3 < A_1$ and frequency $\omega_3 = 3\,\omega_1$:

$$f(t) = A_1 \sin\omega_1 t + A_3 \sin 3\,\omega_1 t = A_1 \sin\omega_1 t + A_3 \sin\omega_3\, t \tag{1.1.7}$$

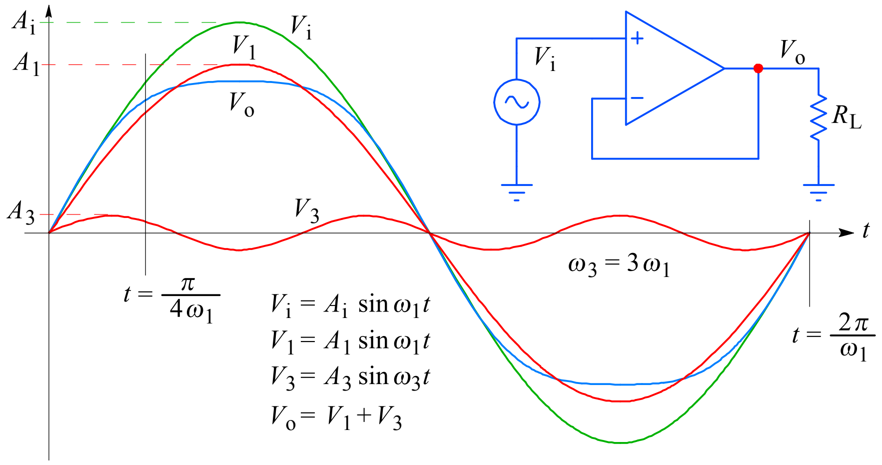

Fig. 1.1.2: The amplifier is slightly overdriven by a pure sinusoidal signal, V_i, with a frequency ω_1 and amplitude A_i. The output signal V_o is distorted, and it can be represented as a sum of two signals, $V_1 + V_3$. The fundamental frequency of V_1 is ω_1 and its amplitude A_1 is somewhat lower. The frequency of V_3 (the third harmonic component) is $\omega_3 = 3\,\omega_1$ and its amplitude is A_3.

Now let us draw the output signal in the same way as we did in Fig. 1.1.1c,d. Here we have two pairs of harmonic components: the first pair of phasors $A_1/2$ rotating with the fundamental frequency $\pm\omega_1$, and the second pair $A_3/2$ rotating with the third harmonic frequency $\pm\omega_3$, which are three times more distant from the origin than ω_1. This is shown in Fig. 1.1.3a, where all four phasors are drawn at time $t = 0$. Fig. 1.1.3b shows the phasors at time $t = \pi/4\,\omega$. Because the third harmonic phasor pair rotates with an angular frequency three times higher, they rotate up to an angle $\pm 3\pi/4$ in the same time.

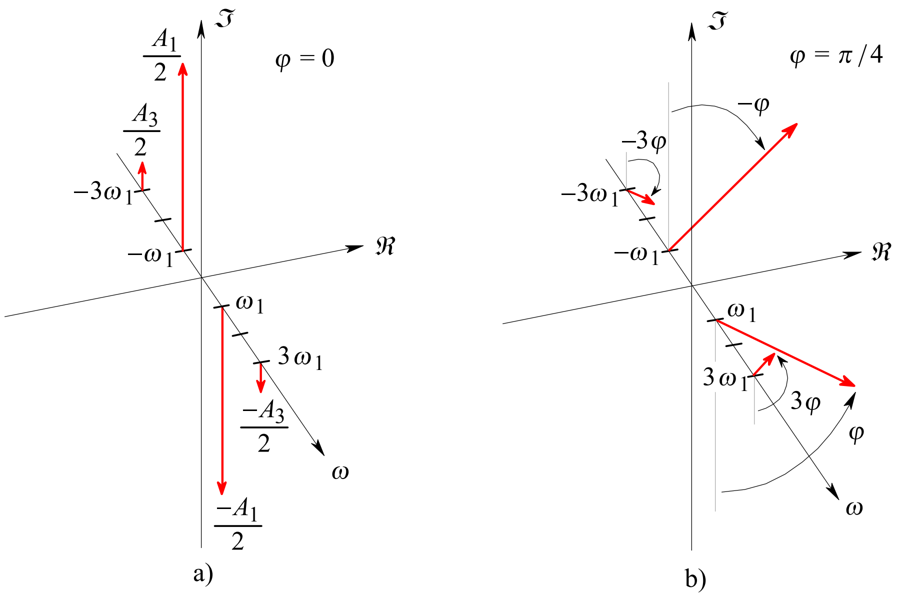

Fig. 1.1.3: The output signal of the amplifier in Fig. 1.1.2, expressed by two pairs of complex conjugate phasors: **a)** at $\omega_1 t = 0$; **b)** at $\omega_1 t = \pi/4$.

Mathematically Eq. 1.1.7, according to Fig. 1.1.2 and 1.1.3, can be expressed as:

$$f(t) = A_1 \sin\omega_1 t + A_3 \sin\omega_3 t$$

$$= \frac{A_1}{2j}\left(e^{j\omega_1 t} - e^{-j\omega_1 t}\right) + \frac{A_3}{2j}\left(e^{j\omega_3 t} - e^{-j\omega_3 t}\right) \tag{1.1.8}$$

The amplifier output obviously cannot exceed either its supply voltage or its maximum output current. So if we keep increasing the input amplitude the amplifier will clip the upper and lower peaks of the output waveform (some input protection, as well as some internal signal source resistance must be assumed if we want the amplifier to survive in these conditions), thus generating more harmonics. If the input amplitude is very high and if the amplifier loop gain is high as well, the output voltage $f(t)$ would eventually approach a square wave shape, such as in Fig. 1.2.1b in the following section. A true mathematical square wave has an infinite number of harmonics; since no amplifier has an infinite bandwidth, the number of harmonics in the output voltage of any practical amplifier will always be finite.

In the next section we are going to examine a generalized harmonic analysis.

1.2 The Fourier Series

In the experiment shown in Fig. 1.1.2 we have **composed** the output time-function $f(t)$ from sinusoidal waveforms of amplitudes A_1 and A_3. Now, if we have a square wave, as in Fig. 1.2.1b, we would have to deal with many more discrete frequency components. We intend to calculate their amplitudes, assuming that the time function of the square wave is known. This means a **decomposition** of the time function $f(t)$ into the corresponding harmonic frequency components. To do so we will examine the *Fourier* series, named after the French mathematician *Jean Baptiste Joseph de Fourier*[1].

The square wave time function is periodic. A function is periodic if it acquires the same value after its characteristic period $T_1 = 2\pi/\omega_1$, at any instant t:

$$f(t) = f(t + T_1) \tag{1.2.1}$$

Consequently the same is true for $f(t) = f(t + n\,T_1)$, where n is an integer. According to Fourier this square wave can be expressed as a sum of harmonic components with frequencies $f_n = \pm n/T_1$. If $n = 1$ we have the fundamental frequency f_1 with a phasor $A_1/2$, rotating counterclockwise. The phasor f_{-1} with the same length $A_1/2 = A_{-1}/2$ rotates clockwise and forms a complex conjugate pair with the first one. A true square wave would have an infinite number of odd-order harmonics (all even-order harmonics are zero).

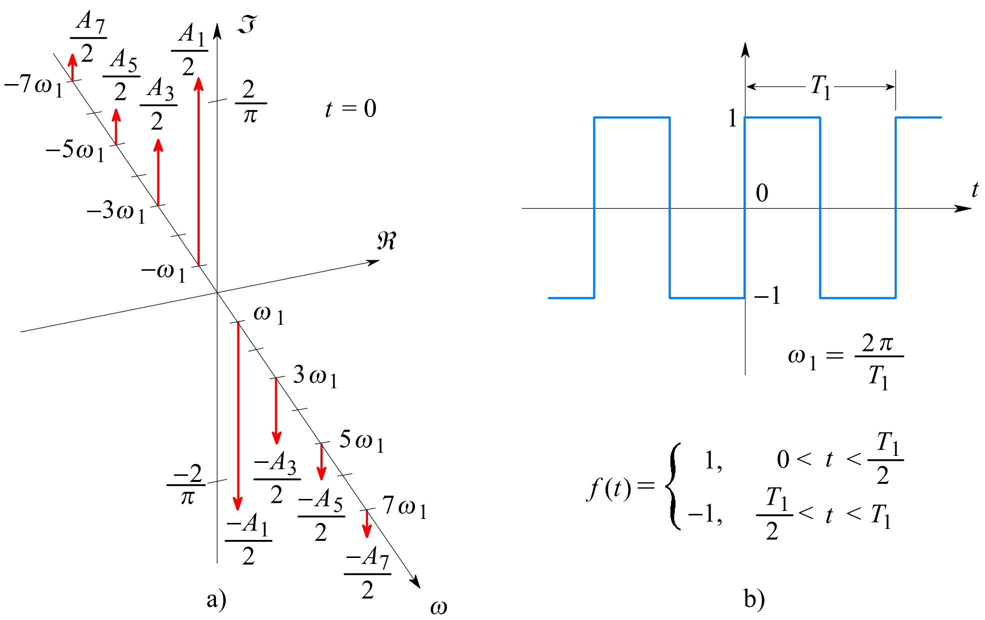

Fig. 1.2.1: A square wave, as shown in **b)**, has an infinite number of odd-order frequency components, of which the first 4 complex-conjugate phasor pairs are drawn in **a)** at time $t = (0 \pm 2\,n\,\pi)/\omega_1$, where n is an integer representing the number of the period.

[1] It is interesting that Fourier developed this method in connection with thermal engineering. As a general in the Napoleon's army he was concerned with gun deformation by heat. He supposed that one side of a straight metal bar is heated and then bent, joining the ends, to form a thorus. Then he calculated the temperature distribution along the circle so formed, in such a way that it would be the sum of sinusoidal functions, each having a different amplitude and a different angular frequency.

In Fig. 1.2.1, we have drawn the complex-conjugate phasor pairs of the first 4 harmonics. Because all the phasor pairs are always complex-conjugate, the sum of any pair, as well as their total sum, is always real. The phasor pairs rotate with different speeds and in opposite directions. Fig. 1.2.2a shows them at time $T_1/8$ to help the reader's imagination. Although this figure looks confusing, the phasors shown have an exact inter-relationship. Looking at the positive ω axis, the phasor with the amplitude $A_1/2$ has rotated in the counterclockwise direction by an angle of $\pi/4$. During the same interval of $T_1/8$ the remaining phasors have rotated: $A_3/2$ by $3\pi/4$; $A_5/2$ by $5\pi/4$; $A_7/2$ by $7\pi/4$; etc. The corresponding complex conjugate phasors on the negative ω axis rotate likewise, but in the opposite (clockwise) direction. The sum of all phasors at any instant t is the instantaneous amplitude of the time domain function. In general, the time function with the fundamental frequency ω_1 is expressed as:

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{A_n}{2}\, \mathrm{e}^{j n \omega_1 t} \\
&= \cdots + \frac{A_{-n}}{2}\, \mathrm{e}^{-j n \omega_1 t} + \cdots + \frac{A_{-2}}{2}\, \mathrm{e}^{-j 2 \omega_1 t} + \frac{A_{-1}}{2}\, \mathrm{e}^{-j \omega_1 t} \\
&\quad + A_0 + \frac{A_1}{2}\, \mathrm{e}^{j \omega_1 t} + \frac{A_2}{2}\, \mathrm{e}^{j 2 \omega_1 t} + \cdots + \frac{A_n}{2}\, \mathrm{e}^{j n \omega_1 t} + \cdots
\end{aligned}
\tag{1.2.2}
$$

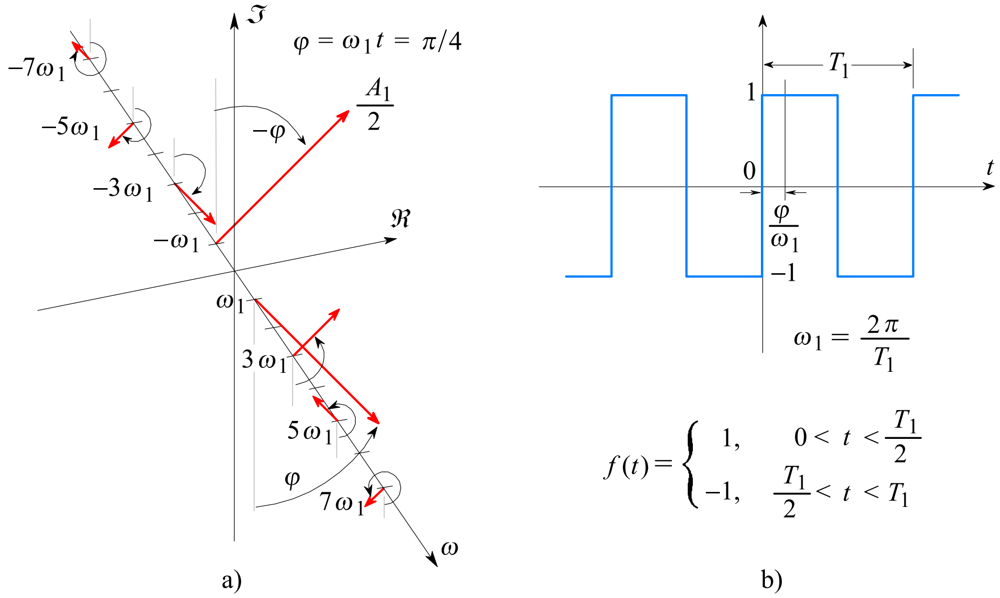

Fig. 1.2.2: As in Fig. 1.2.1, but at an instant $t = (\pi/4 \pm 2n\pi)/\omega_1$; **a)** the spectrum, expressed by complex conjugate phasor pairs, corresponds to the instant $t = \varphi/\omega_1$ in **b)**. To see the movie over one period click this link: harmovie.avi. (we apologize for the bad quality resulting from the poor conversion from vector graphics to the bitmap format; readers who have access to Matlab are invited to run the 'harmovie.m' file).

Note that for the square wave all the even frequency components are missing. For other types of waveforms the even coefficients can be non-zero. In general A_{i} may also be complex, thus containing some non-zero initial phase angle φ_{i}. In Eq. 1.2.2 we have also introduced A_0, the DC component, which did not exist in our special case. The meaning of A_0 can be understood by examining Fig. 1.2.3a, where the so-called saw tooth waveform is shown, with no DC component. In Fig. 1.2.3b, the waveform has a DC component of magnitude A_0.

Eq. 1.2.2 represents the *complex spectrum* of the function $f(t)$, whilst Fig. 1.2.1 represents the corresponding most significant part of the complex spectrum of a square wave. The next step is the calculation of the magnitudes of the rotating phasors.

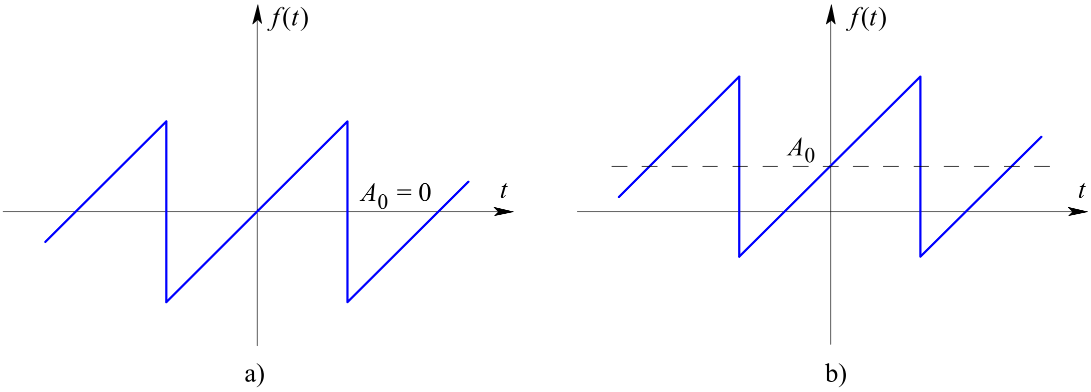

Fig. 1.2.3: a) A waveform without a DC component; **b)** with a DC component A_0.

If we want to measure safely and accurately the diameter of a wheel of a working machine, we must first stop the machine. Something similar can be done with our Eq. 1.2.2, except that here we can mathematically stop the rotation of any single phasor. Suppose we have a phasor $A_\mathrm{k}/2$, rotating counterclockwise with frequency $\omega_\mathrm{k} = \mathrm{k}\,\omega_1$ with an initial phase angle φ_k (at $t = 0$), which is expressed as:

$$\frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j(\omega_\mathrm{k} t + \varphi_\mathrm{k})} = \frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\omega_\mathrm{k} t}\,\mathrm{e}^{j\varphi_\mathrm{k}} \tag{1.2.3}$$

Now we multiply this expression by a unit amplitude, clockwise rotating phasor $\mathrm{e}^{-j\omega_\mathrm{k}t}$ (having the same angular frequency ω_k) to cancel the $\mathrm{e}^{j\omega_\mathrm{k}t}$ term, [Ref. 1.2]:

$$\frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\varphi_\mathrm{k}}\,\mathrm{e}^{j\omega_\mathrm{k} t}\,\mathrm{e}^{-j\omega_\mathrm{k} t} = \frac{A_\mathrm{k}}{2}\,\mathrm{e}^{j\varphi_\mathrm{k}} \tag{1.2.4}$$

and obtain a non-rotating component which has the magnitude $A_\mathrm{k}/2$ and phase angle φ_k **at any time**. With this in mind let us attack the whole time function $f(t)$. The duration of the multiplication must last exactly one whole period and the corresponding expression is:

$$\frac{A_\mathrm{k}}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,\mathrm{e}^{-j\omega_\mathrm{k} t}\,dt \tag{1.2.5}$$

Since we have integrated over the whole period T in order to get the average value of that harmonic component, the result of the integration must be divided by T, as in Eq. 1.2.5. If there is a DC component (with $\omega = 0$) in the spectrum, the calculation of it is simply:

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt \tag{1.2.6}$$

To return to Eq. 1.2.5, let us explain the meaning of the integration Eq. 1.2.5 by means of Fig. 1.2.4.

By multiplying the function $f(t)$ by $e^{-j\omega_k t}$ we have stopped the rotating phasor $A_k/2$, while during the time interval of integration all the other phasors have rotated through an angle of $n\,2\pi$ (where n is an integer), **including** the DC phasor A_0, because it is now multiplied by $e^{-j\omega_k t}$. The result of the integration for all these rotating phasors is zero, as indicated in Fig. 1.2.4a, whilst the phasor $A_k/2$ has stopped, integrating eventually to its full amplitude; the integration for this phasor only is shown in Fig. 1.2.4b.

The understanding of the described effect resulting from the multiplication of $f(t)$ by $e^{-j\omega t}$ is essential for the understanding of the basic principles behind the Fourier series, the Fourier integral and the Laplace transform.

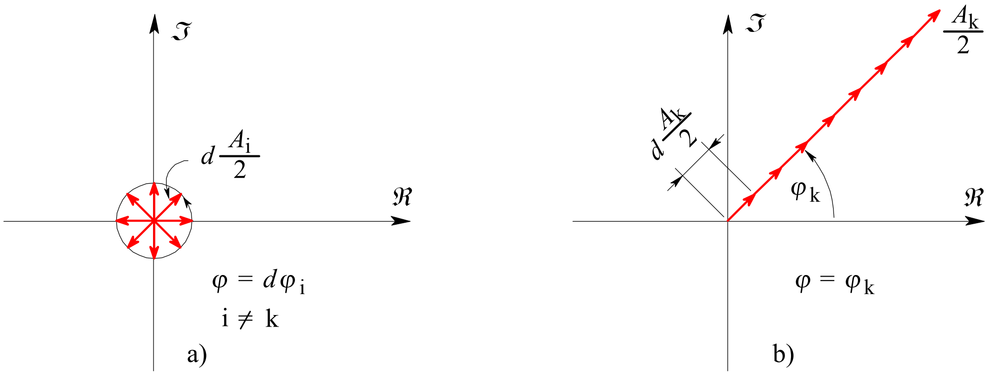

Fig. 1.2.4: a) The integral over the full period T of a rotating phasor is zero; **b)** the integral over a full period T of a non-rotating phasor $d\,A_k/2$, gives its amplitude, $A_k/2$, (the symbol d stands for dt/T — in this figures $dt \rightarrow \Delta t$ such that $\Delta t\,\omega_k = \pi/4$). Note that a stationary phasor retains its initial angle φ_k.

For us the Fourier series represents only a transitional station on the journey towards the Laplace transform. So we will drive through it with a moderate speed "via the Main Street", without investigating some interesting things in the side streets. Nevertheless, it is useful to make a practical example. Since we have started with a square wave, shown in Fig. 1.2.5, let us calculate its complex spectrum components $A_n/2$, assuming that the square wave amplitude is $A = 1$.

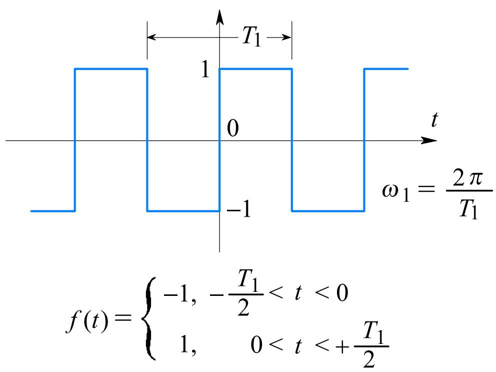

Fig. 1.2.5: A square wave signal.

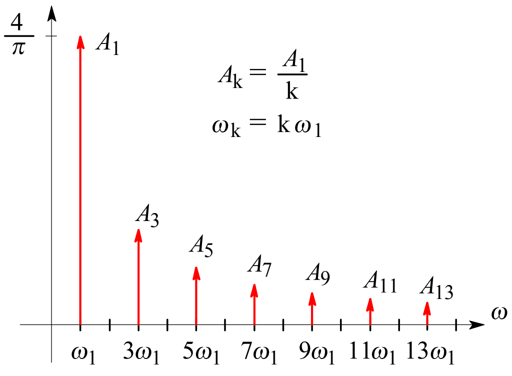

Fig. 1.2.6: The frequency spectrum of a square wave, expressed by **real** values (magnitudes) only.

For a **single period** the corresponding mathematical expression of this function is:

$$f(t) = \begin{cases} -1 & \text{for} \quad -T/2 < t < 0 \\ +1 & \text{for} \quad 0 < t < T/2 \end{cases}$$

According to Eq. 1.2.5 we calculate:

$$\frac{A_n}{2} = \frac{1}{T}\left[\int_{-T/2}^{0}(-1)\,\mathrm{e}^{-j2\pi nt/T}\,dt + \int_{0}^{T/2}(+1)\,\mathrm{e}^{-j2\pi nt/T}\,dt\right]$$

$$= \frac{1}{T}\left(\frac{T}{j2\pi n}\,\mathrm{e}^{-j2\pi nt/T}\Big|_{-T/2}^{0} + \frac{T}{-j2\pi n}\,\mathrm{e}^{-j2\pi nt/T}\Big|_{0}^{T/2}\right)$$

$$= \frac{1}{j2\pi n}\left(1 - \mathrm{e}^{j\pi n} - \mathrm{e}^{-j\pi n} + 1\right) = \frac{-1}{j\pi n}\left(\frac{\mathrm{e}^{j\pi n} + \mathrm{e}^{-j\pi n}}{2} - 1\right)$$

$$= \frac{-1}{j\pi n}(\cos\pi n - 1) \tag{1.2.7}$$

The result is zero for $n = 0$ (the DC component A_0) and for any even n. For any odd n the value of $\cos\pi n = -1$, and for such cases the result is:

$$\frac{A_n}{2} = \frac{2}{j\pi n} = \frac{-2j}{\pi n} \tag{1.2.8}$$

The factor $-j$ in the numerator means that for any positive n the phasor is negative and imaginary (also for $t = 0,\ 2\pi,\ 4\pi,\ 6\pi, \dots$), whilst for negative n it is positive and imaginary. This is evident from Fig. 1.2.1a.

Let us calculate the first eight phasors by using Eq. 1.2.8. The lengths of phasors in Fig. 1.2.1a and 1.2.2.b correspond to the values reported in Table 1.2.1. All the phasors form complex conjugate pairs and their total sum **always gives a real value**.

Table 1.2.1: The first few harmonics of a square wave

$\pm n$	0	1	3	5	7	9	11	13
$\mp A_n/2$	0	$2j/\pi$	$2j/3\pi$	$2j/5\pi$	$2j/7\pi$	$2j/9\pi$	$2j/11\pi$	$2j/13\pi$

However, a spectrum can also be shown with real values only, e.g., as it appears on the cathode ray tube screen of a spectrum analyzer. To obtain this, we simply sum the corresponding complex conjugate phasor pairs (e.g., $|A_n/2| + |A_{-n}/2| = A_n$) and place them on the abscissa of a two-dimensional coordinate system, as shown in Fig. 1.2.6. Such a non-rotating spectrum has only the positive frequency axis. Although such a presentation of spectra is very useful in the analysis of signals containing several (or many) frequency components, we will continue calculating with the complex spectra, because the phase information is also important. And, of course, the Laplace transform, which is our main goal, is based on a complex variable.

Now let us recompose the waveform using only the harmonic frequency components from Table 1.2.1, as shown in Fig. 1.2.7a. The waveform resembles the square wave but it has an exaggerated overshoot $\delta \simeq 18\,\%$ of the nominal amplitude.

The reason for the overshoot δ is that we have abruptly cut off the higher harmonic components from a certain frequency upwards. Would this overshoot be lower if we take more harmonics? In Fig. 1.2.7b we have increased the number of harmonic components three times, but the overshoot remained the same. No matter how many, yet for any **finite** number of harmonic components, used to recompose the waveform, the overshoot would stay the same (only its duration becomes shorter if the number of harmonic components is increased, as is evident from Fig. 1.2.7a and 1.2.7b).

This is the *Gibbs*' phenomenon (*Josiah Willard Gibbs*, 1839–1903, [Ref. 1.24]). It tells us that we should not cut off the frequency response of an amplifier abruptly if we do not wish to add an undesirably high overshoot to the amplified pulse. Fortunately, real amplifiers can not have an infinitely steep high frequency roll off, so a gradual decay of high frequency response is always ensured. However, as we shall explain in Part 2 and 4, the overshoot may increase as a result of other effects.

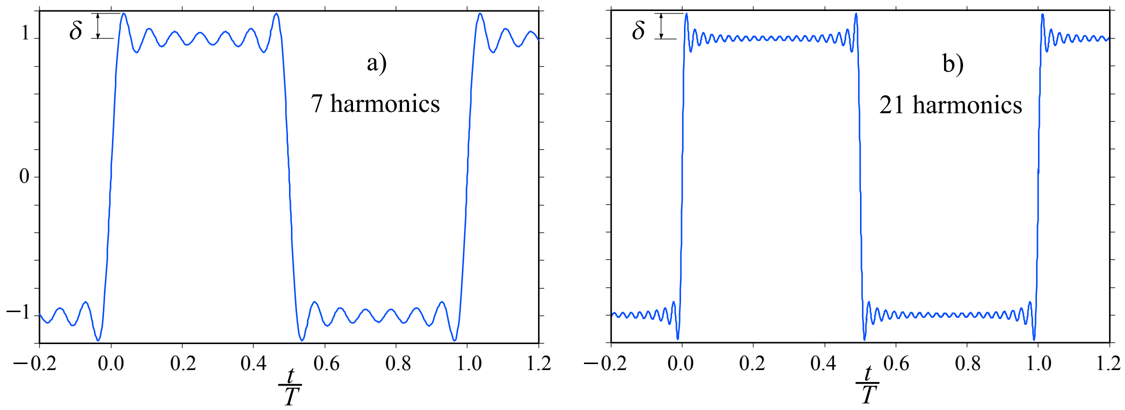

Fig. 1.2.7: The Gibbs' phenomenon; **a)** A signal composed of the first seven harmonics of a square wave spectrum from Table 1.2.1. The overshoot is $\delta \simeq 18\,\%$ of the nominal amplitude; **b)** Even if we take three times more harmonics the overshoot δ is nearly equal in both cases.

In a similar way to that for the square wave, **any periodic signal of finite amplitude and with a finite number of discontinuities within one period, can be decomposed into its frequency components.** As an example the waveform in Fig. 1.2.8 could also be decomposed, but we will not do it here. Instead in the following section we will analyze another waveform which will allow us to generalize the method of frequency analysis.

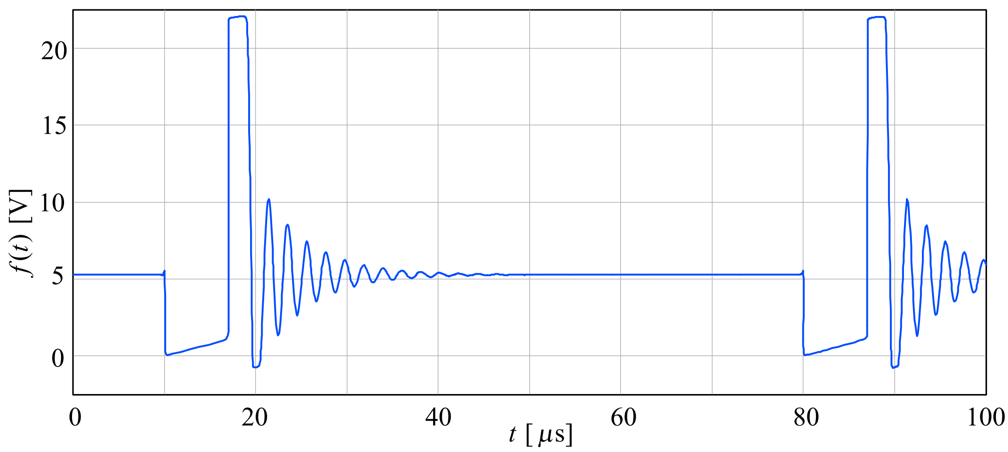

Fig. 1.2.8: An example of a periodic waveform (a typical flyback switching power supply), having a finite number of discontinuities within one period. Its frequency spectrum can also be calculated using the Fourier transform, if needed (e.g., to analyze the possibility of electromagnetic interference at various frequencies), in the same way as we did for the square wave.

1.3 The Fourier Integral

Suppose we have a function $f(t)$ composed of square waves with the duration τ and repeating with a period T, as shown in Fig. 1.3.1. For this function we can also calculate the Fourier series (the corresponding spectrum is shown in Fig. 1.3.2) in the same way as for the continuous square wave case in the previous section.

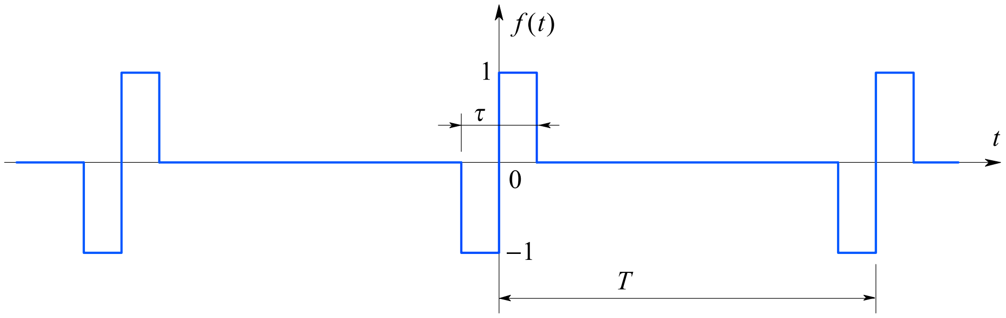

Fig. 1.3.1: A square wave with duration τ and period $T = 5\tau$

The difference between the continuous square wave spectrum and the spaced square wave in Fig. 1.3.1 is that the integral of this function can be broken into two parts, one comprising the length of the pulse, τ, and the zero-valued part between two pulses of a length $T - \tau$. The reader can do this integration for himself, because it is fairly simple. We will only write the result:

$$\frac{A_n}{2} = -j\tau\,\frac{\sin^2\left[\,n\,\omega_1(\tau/4)\,\right]}{n\,\omega_1(\tau/4)} \tag{1.3.1}$$

where $\omega_1 = 2\pi/T$, assuming that the pulse amplitude is 1 (if the amplitude were A it would simply multiply the right hand side of the equation). For the conditions in Fig. 1.3.1, where $T = 5\tau$ and $A = 1$, the spectrum has the form shown in Fig. 1.3.2, with $\omega_\tau = 2\pi/\tau$.

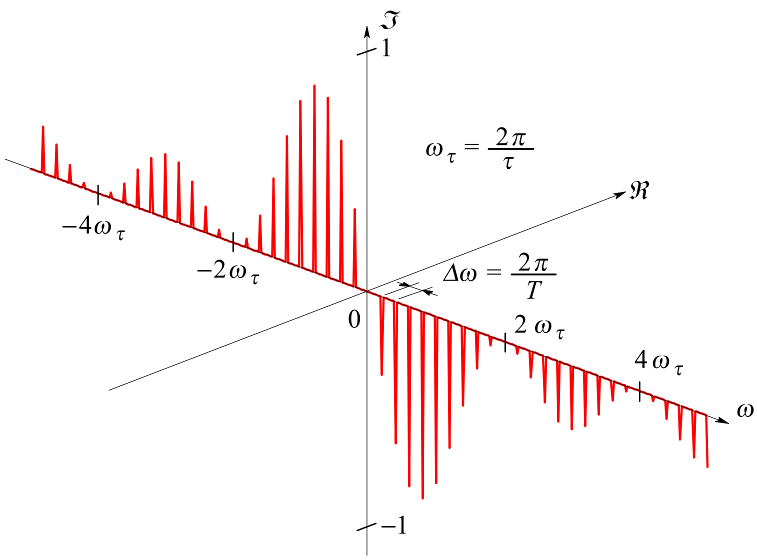

Fig. 1.3.2: Complex spectrum of the waveform in Fig. 1.3.1.

A very interesting question is that of what would happen to the spectrum if we let the period $T \to \infty$? In general a function $f(t)$ can be recomposed by adding all its harmonic components:

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A_n}{2}\, \mathrm{e}^{jn\omega_1 t} \tag{1.3.2}$$

where A_n may also be complex, thus containing the initial phase angle φ_i. Again, as in the previous section, each discrete harmonic component can be calculated with the integral:

$$\frac{A_n}{2} = \frac{1}{T} \int_{-T/2}^{T/2} f(t)\, \mathrm{e}^{-jn\omega_1 t}\, dt \tag{1.3.3}$$

For the case in Fig. 1.3.1 the integration should start at $t = 0$ and the integral has the form:

$$\frac{A_n}{2} = \frac{1}{T} \int_{0}^{T} f(t)\, \mathrm{e}^{-jn\omega_1 t}\, dt \tag{1.3.4}$$

Insert this into Eq. 1.3.2:

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{0}^{T} f(\tau)\, \mathrm{e}^{-jn\omega_1 \tau}\, d\tau \right] \mathrm{e}^{jn\omega_1 t} \tag{1.3.5}$$

Here we have introduced a dummy variable τ in the integral, in order to distinguish it from the variable t outside the brackets. Now we express the integral inside the brackets as:

$$\int_{0}^{T} f(\tau)\, \mathrm{e}^{-jn\omega_1 \tau}\, d\tau = \int_{0}^{T} f(\tau)\, \mathrm{e}^{-j2\pi n\tau/T}\, d\tau = F\!\left(\frac{2n\pi}{T}\right) = F(n\omega_1) \tag{1.3.6}$$

Thus:

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T}\, F(n\omega_1)\, \mathrm{e}^{jn\omega_1 t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{2\pi}{T}\, F(n\omega_1)\, \mathrm{e}^{jn\omega_1 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \omega_1 F(n\omega_1)\, \mathrm{e}^{jn\omega_1 t} \end{aligned} \tag{1.3.7}$$

where $2\pi/T = \omega_1$. If we let $T \to \infty$ then ω_1 becomes infinitesimal, and we call it $d\omega$. Also $n\omega_1$ becomes a continuous variable ω. So in Eq. 1.3.7 the following changes take place:

$$\sum_{n=-\infty}^{\infty} \Rightarrow \int_{-\infty}^{\infty} \qquad\qquad \omega_1 \Rightarrow d\omega \qquad\qquad n\omega_1 \Rightarrow \omega$$

With all these changes Eq. 1.3.7 is transformed into Eq. 1.3.8:

$$\boxed{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{j\omega t}\, d\omega} \tag{1.3.8}$$

Consequently Eq. 1.3.6 also changes, obtaining the form:

$$\boxed{F(\omega) = \int_0^{\infty} f(t)\,\mathrm{e}^{-j\omega t}\,dt} \tag{1.3.9}$$

In Eq. 1.3.9 $F(\omega)$ has no discrete frequency components but it forms a **continuous** spectrum. Since $T \to \infty$ the DC part vanishes (as it would for **any** pulse shape, not just symmetrical shapes), according to Eq. 1.2.6:

$$A_0 = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\,dt = 0 \tag{1.3.10}$$

Eq. 1.3.8 and 1.3.9 are called *Fourier integrals*. Under certain (usually rather limited) conditions, which we will discuss later, it is possible to use them for the calculation of transient phenomena. The second integral (Eq. 1.3.9) is called the *direct Fourier transform*, which we express in a shorter way:

$$\mathcal{F}\{f(t)\} = F(\omega) \tag{1.3.11}$$

The first integral (Eq. 1.3.8) represents the *inverse Fourier transform* and it is usually written as:

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) \tag{1.3.12}$$

In Eq. 1.3.8, $F(\omega)$ means a **firm** spectrum and the factor $\mathrm{e}^{j\omega t}$ means the rotation of each of the corresponding infinite spectrum components contained in $F(\omega)$ with its angular frequency ω, which is a continuous variable. In Eq. 1.3.9 $f(t)$ means the complete time function, containing an infinite number of **rotating** phasors and the factor $\mathrm{e}^{-j\omega t}$ means the rotation 'in the opposite direction' to stop the rotation of the corresponding rotating phasor $\mathrm{e}^{j\omega t}$ contained in $f(t)$, at its particular frequency ω.

Let us now select a suitable time function $f(t)$ and calculate its continuous spectrum. Since we have already calculated the spectrum of a periodic square wave, it would be interesting to display the spectrum of a single square wave as shown in Fig. 1.3.3b. We use Eq. 1.3.9:

$$F(\omega) = \int_{-\tau/2}^{\infty} f(t)\,\mathrm{e}^{-j\omega t}\,dt = \int_{-\tau/2}^{0} (-1)\,\mathrm{e}^{-j\omega t}\,dt + \int_0^{\tau/2} (+1)\,\mathrm{e}^{-j\omega t}\,dt \tag{1.3.13}$$

Here we have a single square wave with a 'period' T from $t = -\tau/2$ to $+\infty$. However, we need to integrate only from $t = -\tau/2$ to $t = \tau/2$, because $f(t)$ is zero outside this interval. It is important to note that at the discontinuity where $t = 0$, we have started the second integral. For a function with more discontinuities, between each of them we must write a separate integral. Thus it is obvious that the function $f(t)$ must have a **finite** number of discontinuities for it to be possible to calculate its spectrum.

The result of the above integration is:

$$F(\omega) = \frac{1}{-j\,\omega}\left(-1 + e^{j\omega\tau/2} + e^{-j\omega\tau/2} - 1\right) = \frac{2}{j\,\omega}\left(1 - \frac{e^{j\omega\tau/2} + e^{-j\omega\tau/2}}{2}\right)$$

$$= \frac{-2\,j}{\omega}\left(1 - \cos\frac{\omega\,\tau}{2}\right) = \frac{-2\,j}{\omega}\left(2\sin^2\frac{\omega\,\tau}{4}\right) = \frac{-4\,j}{\omega}\sin^2\frac{\omega\,\tau}{4}$$

$$= -j\,\tau\,\frac{\sin^2\dfrac{\omega\,\tau}{4}}{\dfrac{\omega\,\tau}{4}} \tag{1.3.14}$$

A three-dimensional display of a spectrum, corresponding to this result, is shown in Fig. 1.3.3a. Here the frequency scale has been altered with respect to Fig. 1.2.1a in order to display the spectrum better.

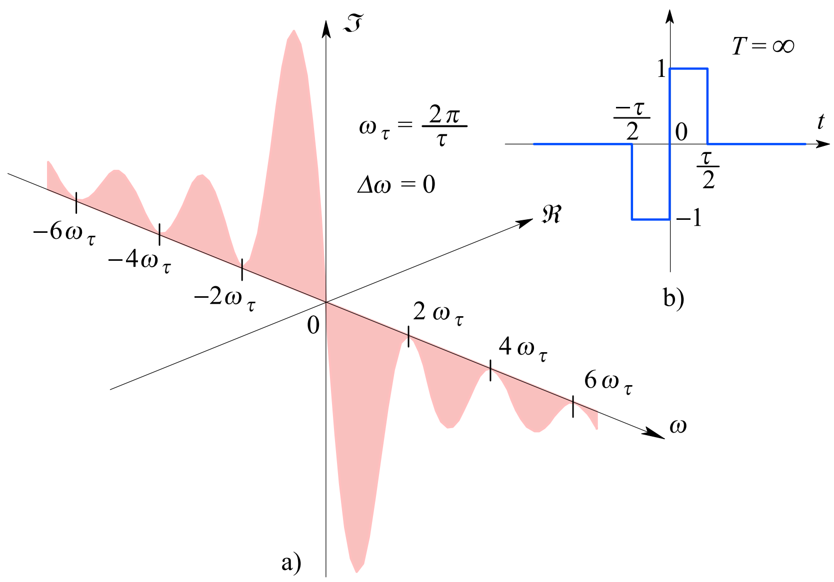

Fig. 1.3.3: a) The frequency spectrum of a single square wave is expressed by complex conjugate phasors. Since the phasors are infinitely many, they merge in a continuous planar form. Also the spectrum extends to $\omega = \pm\infty$. The corresponding waveform is shown in **b)**. Note that all the even frequency components $2\pi n/\tau$ are missing (n is an integer).

By comparing Fig. 1.2.1a and 1.3.3a we may draw the following conclusions:

1. Both spectra contain no even frequency components, e.g., at $\pm 2\,\omega_\tau$, $\pm 4\,\omega_\tau$, etc., where $\omega_\tau = 2\,\pi/\tau$;

2. In both spectra there is no DC component A_0;

3. By comparing Fig. 1.3.2 and 1.3.3 we note that the envelope of both spectra can be expressed by Eq. 1.3.14;

4. By comparing Eq. 1.3.1 and 1.3.14 we note that the discrete frequency $n\,\omega_1$ from the first equation is replaced by the continuous variable ω in the second equation. Everything else has remained the same.

In the above example we have decomposed an aperiodic waveform (also called a *transient*), expressed as $f(t)$, into a **continuous complex** spectrum $F(\omega)$. Before discussing the functions which are suitable for the application of the Fourier integral let us see some common periodic and non-periodic signals. A sustained tone from a trumpet we consider to be a periodic signal, whilst a beat on a drum is a non-periodic signal (in a strict mathematical sense, both signals are non-periodic, because the first one also started out of silence). The transition from silence to sound we call the *transient*. In accordance with this definition, of the waveforms in Fig. 1.3.4 only a) and b) show a periodic waveform, whilst c) and d) display transients.

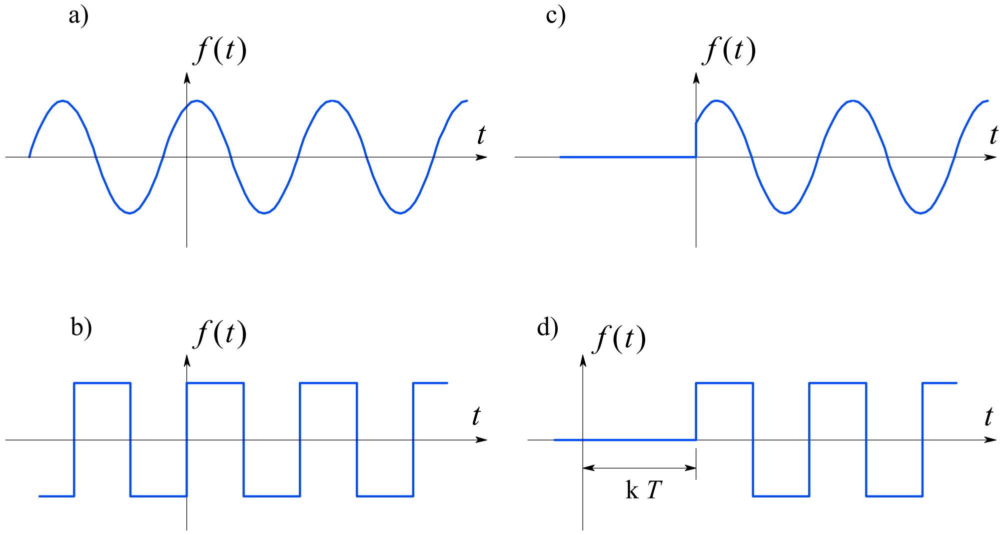

Fig. 1.3.4: a) and **b)** periodic functions, **c)** and **d)** aperiodic functions

The question arises of whether it is possible to calculate the spectra of the transients in Fig. 1.3.4c and 1.3.4d by means of the Fourier integral using Eq. 1.3.8?

The answer is **no**, because the integral in Eq. 1.3.8 does not converge for any of these two functions. The integral is also non-convergent for the most simple step signal, which we intend to use extensively for the calculation of the step response of amplifier networks.

This inconvenience can be avoided if we multiply the function $f(t)$ by a suitable convergence factor, e.g., e^{-ct}, where $c > 0$ and its magnitude is selected so that the integral in Eq. 1.3.2 remains finite when $t \to \infty$. In this way, the problem is solved for $t \geq 0$. In doing so, however, the integral becomes divergent for $t < 0$, because for negative time the factor e^{-ct} has a positive exponent, causing a rapid increase to infinity. But this, too, can be avoided, if we assume that the function $f(t)$ is zero for $t < 0$. In electrical engineering and electronics we can always assume that a circuit is dead until we switch the power on or we apply a step voltage signal to its input and thus generate a transient. The transform where $f(t)$ must be zero for $t < 0$ is called a *unilateral transform*.

For functions which are suitable for the unilateral Fourier transform the following relation must hold [Ref. 1.3]:

$$\lim_{T \to \infty} \int_0^T |f(t)|\, \mathrm{e}^{-ct} dt \; < \; \infty \tag{1.3.15}$$

where $f(t)$ is a single-valued function of t and c is positive and real.

If so, we can write the direct transform:

$$F(c, \omega) = \int_{0}^{\infty} \left[f(t)\, \mathrm{e}^{-ct} \right] \mathrm{e}^{-j\omega t}\, dt \tag{1.3.16}$$

If we want this integral to converge to some finite value for $t \to \infty$, the real constant must be $c \geq \sigma_\mathrm{a}$, where σ_a is the *abscissa of absolute convergence*. The magnitude of σ_a depends on the nature of the function $f(t)$. I.e., if $f(t) = 1$, then $\sigma_\mathrm{a} = 0$, and if $f(t) = \mathrm{e}^{-\alpha t}$ then $\sigma_\mathrm{a} = -\alpha$, where $\alpha > 0$. By applying the convergence factor e^{-ct}, the inverse Fourier transform obtains the form:

$$f(t)\, \mathrm{e}^{-ct} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(c, \omega)\, \mathrm{e}^{j\omega t}\, d\omega \qquad \text{for } t \geq 0 \tag{1.3.17}$$

Here we must add all the complex-conjugate phasors with frequencies from $\omega = -\infty$ to $+\infty$. Although the direct Fourier transform in our case was unilateral, **the inverse transform is always bilateral**. Because in Eq. 1.3.16 we have deliberately introduced the convergence factor e^{-ct} we must limit $c \to 0$ after the integral is solved in order to get the required $F(\omega)$.

Since our final goal is the Laplace transform we will stop the discussion of the Fourier transform here. We will, however, return to this topic later in Part 6, where we will discuss the solving of system transfer functions and transient responses using numerical methods, suitable for machine computation. There we will discuss the application of the very efficient Fast Fourier Transform (FFT) algorithm to both frequency and time domain related problems.

1.4 The Laplace Transform

By a slight change of Eq. 1.3.16 and 1.3.17 we may arrive at a general complex Fourier transform [Ref. 1.3]. This is done so that we join the kernel $\mathrm{e}^{-j\omega t}$ and the convergence factor e^{-ct}. In this way Eq. 1.3.16 is transformed into:

$$F(c+j\omega) = \int_0^\infty \left[f(t) \right] \mathrm{e}^{-(c+j\omega)\,t}\, dt \qquad \text{where} \quad c \geq \sigma_\mathrm{a} \tag{1.4.1}$$

The formula for an inverse transform is derived from Eq. 1.3.17 if both sides of the equation are multiplied by e^{ct}. In addition, the simple variable ω is now replaced by a new one: $c + j\omega$. By doing so we obtain:

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(c+j\omega)\, \mathrm{e}^{(c+j\omega)t}\, d(c+j\omega) \qquad \text{for } t \geq 0 \text{ and } c \geq \sigma_\mathrm{a} \tag{1.4.2}$$

If in Eq. 1.4.1 and 1.4.2 the *constant* c becomes a *real variable* σ, both equations are transformed into the form called *Laplace transform*. The name is fully justified, since the French mathematician *Pierre Simon de Laplace* had already introduced this transform in 1779, whilst Fourier published his transform 43 years later.

It is a custom to denote the complex variable $\sigma + j\omega$ by a single symbol s, which we also call the *complex frequency* (in some, mostly mathematical, literature this variable is also denoted as p). With this new variable Eq. 1.4.1 can be rewritten:

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, \mathrm{e}^{-st}\, dt} \tag{1.4.3}$$

and this is called the *direct Laplace transform*, or $\mathcal{L}$ *transform*. It represents the complex spectrum $F(s)$. The above integral is valid for functions $f(t)$ such that the factor e^{-st} keeps the integral convergent. If we now insert the variable s in Eq. 1.4.2, we have:

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, \mathrm{e}^{st}\, ds} \tag{1.4.4}$$

This integral is called the *inverse Laplace transform*, or $\mathcal{L}^{-1}$ *transform*.

Like the inverse Fourier transform, Eq. 1.4.4 is a **bilateral transform** too. In the integral Eq. 1.4.3 it is assumed that $f(t) = 0$ for $t < 0$, thus that equation represents the *unilateral transform*. In addition, the real part of the variable s satisfies $\Re\{s\} = \sigma \geq \sigma_\mathrm{a}$, where σ_a is the *abscissa of absolute convergence*, as we have already discussed for Eq. 1.3.16 and 1.3.17 [see also Ref. 1.23]. In the integral Eq. 1.4.4 $t \geq 0$, so here too we must have $\sigma \geq \sigma_\mathrm{a}$.

The path of integration is parallel with the imaginary axis, as shown in Fig. 1.4.1. The constant c in the integration limits must be properly chosen, in order to ensure the convergence of the integral.

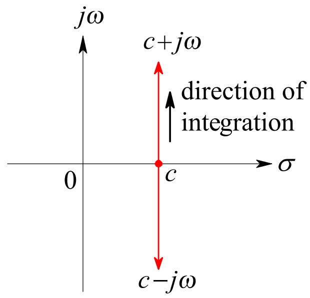

Fig. 1.4.1: The abscissa of absolute convergence — the integration path for Eq. 1.4.4.

The factor e^{-st} in Eq. 1.4.3 is needed to stop the rotation of the corresponding phasor e^{st}; there are infinitely many such phasors in the time function $f(t)$. As our variable is now complex, $s = \sigma + j\omega$, the factor e^{-st} does not mean a simple rotation, but a **spiral rotation** in which the radius is exponentially **decreasing** with t because of σ, the real part of s. This is necessary to cancel the corresponding rotation e^{st}, contained in $f(t)$, with a radius, which, in an exactly equal manner, **increases** with t [Ref. 1.23].

Since in Eq. 1.4.4 the factor e^{st} becomes divergent if the exponent $\Re\{s\,t\} > 1$, the above conditions for the variable σ (and for the constant c) must be met to ensure the convergence of the integral. In the analysis of passive networks these conditions can always be met, as we will show in many examples in the subsequent sections.

Now, because we have reached our goal, the Laplace transform and its inverse, we may ask ourselves what we have accomplished by doing all this hard work.

For the time being we can claim that we have transformed the function of a *real variable* t into a function of a *complex variable* s. This allows us to calculate, using the $\mathcal{L}$ transform, the spectrum function $F(s)$ of a finite transient, defined by the function $f(t)$. Or, more important for us, by means of the $\mathcal{L}^{-1}$ transform we can calculate the time domain function, if the frequency domain function $F(s)$ is known.

Later we will show how we can transform **linear differential equations** in the time domain, by means of the $\mathcal{L}$ transform, into **algebraic equations** in the s domain. Since the algebraic equations are much easier to solve than the differential ones, this means one has a great facility. Once our calculations in the s domain are completed, then by means of the $\mathcal{L}^{-1}$ transform we obtain the corresponding time domain function. In this way we avoid solving directly the differential equations and the calculation of boundary conditions.

1.5 Examples of Direct Laplace Transform

Now let us put our new tools to use and calculate the $\mathcal{L}$ transform of several simple functions. The results may also be used for the $\mathcal{L}^{-1}$ transform and the reader is encouraged to learn the most basic of them by heart, because they are used extensively in the other parts of the book and, of course, in the analysis of the most common electronics circuits.

1.5.1 Example 1

Most of our calculations will deal with the step response of a network. To do so our excitation function will be a simple unit step $h(t)$, or the *Heaviside* function (after *Oliver Heaviside*, 1850–1925) as is shown in Fig. 1.5.1. This function is defined as:

$$f(t) = h(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t > 0 \end{cases}$$

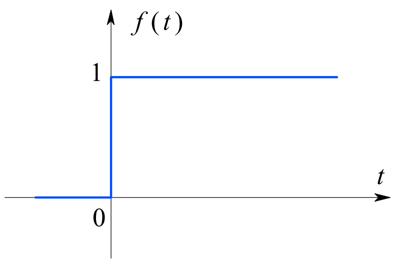

Fig. 1.5.1: Unit step function.

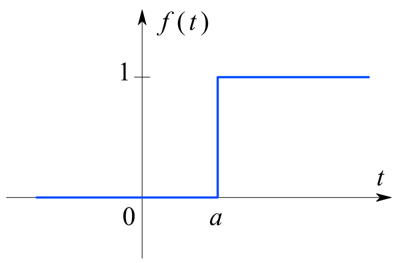

Fig. 1.5.2: Unit step function starting at $t = a$.

As we agreed in the previous section, $f(t) = 0$ for $t < 0$ for all the following functions, and we will not repeat this statement in further examples. At the same time let us mention that for our calculations of $\mathcal{L}$ transform it is not important what is the actual value of $f(0)$, providing it is finite [Ref. 1.3].

The $\mathcal{L}$ transform for the unit step function $f(t) = h(t)$ is:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty [1]\, \mathrm{e}^{-st}\, dt = \frac{1}{-s}\, \mathrm{e}^{-st} \Big|_{t=0}^{t=\infty} = \frac{1}{s} \tag{1.5.1}$$

1.5.2 Example 2

The function is the same as in Example 1, except that the step does not start at $t = 0$ but at $t = a > 0$ (Fig. 1.5.2):

$$f(t) = \begin{cases} 0 & \text{for} \quad t < a \\ 1 & \text{for} \quad t > a \end{cases}$$

Solution:

$$F(s) = \int_a^\infty [1]\, \mathrm{e}^{-st}\, dt = \frac{1}{-s}\, \mathrm{e}^{-st} \Big|_{t=a}^{t=\infty} = \frac{1}{s}\, \mathrm{e}^{-as} \tag{1.5.2}$$

1.5.3 Example 3

The exponential decay function is shown in Fig. 1.5.3; its mathematical expression:

$$f(t) = \mathrm{e}^{-\sigma_1 t}$$

is defined for $t > 0$, as agreed, and σ_1 is a constant.
Solution:

$$F(s) = \int_0^\infty \mathrm{e}^{-\sigma_1 t}\,\mathrm{e}^{-st}\,dt = \int_0^\infty \mathrm{e}^{-(\sigma_1+s)t}\,dt$$

$$= \frac{-1}{\sigma_1+s}\,\mathrm{e}^{-(\sigma_1+s)t}\Bigg|_{t=0}^{t=\infty} = \frac{1}{\sigma_1+s} \tag{1.5.3}$$

Later, we shall meet this and the following function and also their product very often.

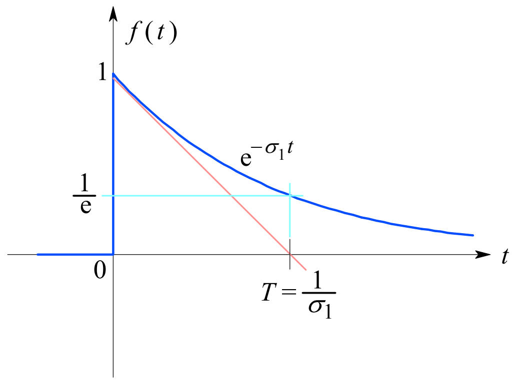

Fig. 1.5.3: Exponential function.

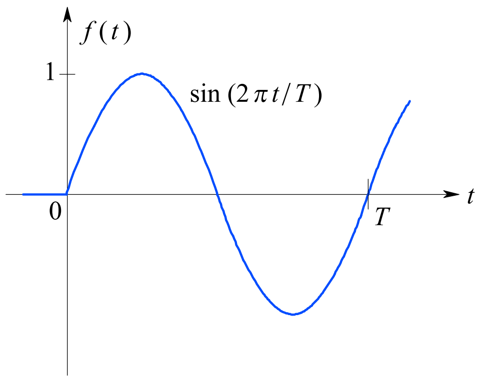

Fig. 1.5.4: Sinusoidal function.

1.5.4 Example 4

We have a sinusoidal function as in Fig. 1.5.4; its corresponding mathematical expression is:

$$f(t) = \sin\omega_1 t$$

where the constant $\omega_1 = 2\pi/T$.
Solution: its $\mathcal{L}$ transform is:

$$F(s) = \int_0^\infty (\sin\omega_1\,t)\,\mathrm{e}^{-st}\,dt \tag{1.5.4}$$

To integrate this function we substitute it using Euler's formula:

$$\sin\omega_1 t = \frac{1}{2j}\left(\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}\right) \tag{1.5.5}$$

Then we have:

$$F(s) = \frac{1}{2j}\left(\int_0^\infty \mathrm{e}^{j\omega_1 t}\,\mathrm{e}^{-st}\,dt - \int_0^\infty \mathrm{e}^{-j\omega_1 t}\,\mathrm{e}^{-st}\,dt\right)$$

$$= \frac{1}{2j}\left[\int_0^\infty \mathrm{e}^{-(s-j\omega_1)t}\,dt - \int_0^\infty \mathrm{e}^{-(s+j\omega_1)t}\,dt\right] \tag{1.5.6}$$

The solution of this integral is, in a way, similar to that in the previous example:

$$F(s) = \frac{1}{2j}\left(\frac{1}{s - j\omega_1} - \frac{1}{s + j\omega_1}\right)$$

$$= \frac{1}{2j} \cdot \frac{s + j\omega_1 - s + j\omega_1}{s^2 + \omega_1^2} = \frac{\omega_1}{s^2 + \omega_1^2} \tag{1.5.7}$$

This is a typical function of a continuous wave (CW) sinusoidal oscillator, with a frequency ω_1.

1.5.5 Example 5

Here we have the cosine function as in Fig. 1.5.5, expressed as:

$$f(t) = \cos\omega_1 t$$

Solution: the $\mathcal{L}$ transform of this function is calculated in a similar way as for the sine. According to Euler's formula:

$$\cos\omega_1 t = \frac{1}{2}\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right) \tag{1.5.8}$$

Thus we obtain:

$$F(s) = \frac{1}{2}\left[\int_0^\infty e^{-(s - j\omega_1)t}\,dt + \int_0^\infty e^{-(s + j\omega_1)t}\,dt\right] = \frac{1}{2}\left(\frac{1}{s - j\omega_1} + \frac{1}{s + j\omega_1}\right)$$

$$= \frac{1}{2} \cdot \frac{s + j\omega_1 + s - j\omega_1}{s^2 + \omega_1^2} = \frac{s}{s^2 + \omega_1^2} \tag{1.5.9}$$

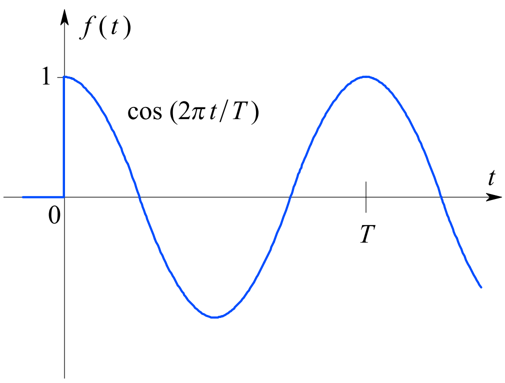

Fig. 1.5.5: Cosine function.

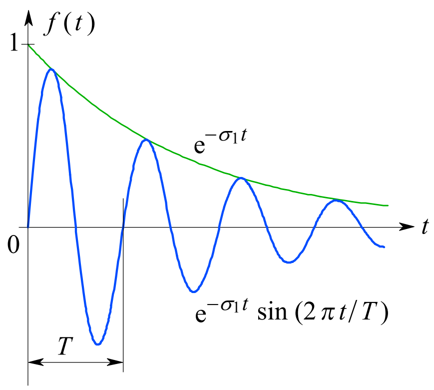

Fig. 1.5.6: Damped oscillations.

1.5.6 Example 6

In Fig. 1.5.6 we have a damped oscillation, expressed by the formula:

$$f(t) = e^{-\sigma_1 t}\sin\omega_1 t$$

Solution: we again substitute the sine function, according to Euler's formula:

$$F(s) = \frac{1}{2j}\int_0^{\infty} \mathrm{e}^{-(s+\sigma_1)t}\left(\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}\right) dt$$

$$= \frac{1}{2j}\int_0^{\infty}\left[\mathrm{e}^{-(s+\sigma_1-j\omega_1)t} - \mathrm{e}^{-(s+\sigma_1+j\omega_1)t}\right] dt$$

$$= \frac{1}{2j}\left(\frac{1}{s+\sigma_1-j\omega_1} - \frac{1}{s+\sigma_1+j\omega_1}\right) = \frac{\omega_1}{(s+\sigma_1)^2+\omega_1^2} \tag{1.5.10}$$

An interesting similarity is found if this formula is compared with the result of Example 4. There, for a CW we have in the denominator s^2 alone, whilst here, because the oscillations are damped, we have $(s+\sigma_1)^2$ instead, and σ_1 is the damping factor.

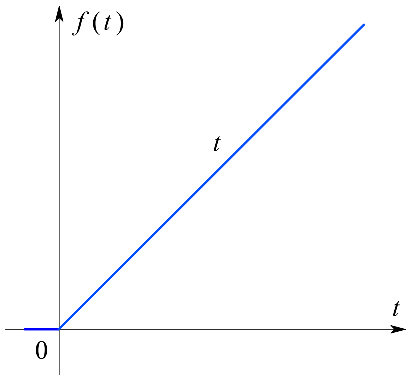

Fig. 1.5.7: Linear ramp $f(t) = t$.

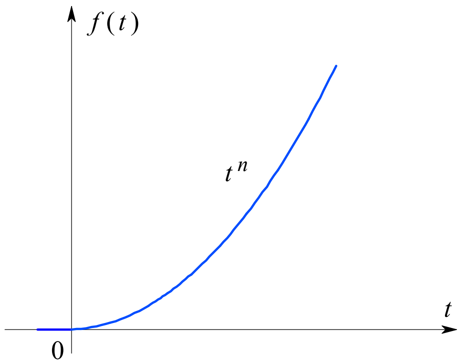

Fig. 1.5.8: Power function $f(t) = t^n$.

1.5.7 Example 7

A linear ramp, as shown in Fig. 1.5.7, is expressed as:

$$f(t) = t$$

Solution: we integrate by parts according to the known relation:

$$\boxed{\int u\, dv = u\, v - \int v\, du}$$

and we assign $t = u$ and $\mathrm{e}^{-s\,t}\, dt = dv$ to obtain:

$$F(s) = \int_0^{\infty} t\,\mathrm{e}^{-st}\, dt = \left.\frac{t\,\mathrm{e}^{-st}}{-s}\right|_{t=0}^{t=\infty} + \frac{1}{s}\int_0^{\infty}\mathrm{e}^{-st}\, dt$$

$$= 0 - 0 - \left.\frac{1}{s^2}\,\mathrm{e}^{-st}\right|_{t=0}^{t=\infty} = \frac{1}{s^2} \tag{1.5.11}$$

1.5.8 Example 8

Fig. 1.5.8 displays a function which has a general analytical form:

$$f(t) = t^n$$

Solution: again we integrate by parts, decomposing the integrand $t^n\,\mathrm{e}^{-st}$ into:

$$u = t^n \qquad du = n\,t^{n-1} dt \qquad v = \frac{1}{-s}\,\mathrm{e}^{-st} \qquad dv = \mathrm{e}^{-st}\,dt$$

With these substitutions we obtain:

$$F(s) = \int_0^\infty t^n\,\mathrm{e}^{-st}\,dt = \frac{t^n\,\mathrm{e}^{-st}}{-s}\bigg|_{t=0}^{t=\infty} + \frac{n}{s}\int_0^\infty t^{n-1}\,\mathrm{e}^{-st}\,dt$$

$$= \frac{n}{s}\int_0^\infty t^{n-1}\,\mathrm{e}^{-st}\,dt \tag{1.5.12}$$

Again integrating by parts:

$$\frac{n}{s}\int_0^\infty t^{n-1}\,\mathrm{e}^{-st}\,dt = \frac{t^{n-1}\,\mathrm{e}^{-st}}{-s}\bigg|_{t=0}^{t=\infty} + \frac{n\,(n-1)}{s^2}\int_0^\infty t^{n-2}\,\mathrm{e}^{-st} dt$$

$$= \frac{n\,(n-1)}{s^2}\int_0^\infty t^{n-2}\,\mathrm{e}^{-st}\,dt \tag{1.5.13}$$

By repeating this procedure n times we finally arrive at:

$$F(s) = \int_0^\infty t^n\,\mathrm{e}^{-st}\,dt = \frac{n\,(n-1)(n-2)\cdots 3\cdot 2\cdot 1}{s^n}\int_0^\infty t^0\,\mathrm{e}^{-st}\,dt = \frac{n!}{s^{n+1}} \tag{1.5.14}$$

1.5.9 Example 9

The function shown in Fig. 1.5.9 corresponds to the expression:

$$f(t) = t\,\mathrm{e}^{-\sigma_1 t}$$

Solution: by integrating by parts we obtain:

$$F(s) = \int_0^\infty t\,\mathrm{e}^{-\sigma_1 t}\,\mathrm{e}^{-st}\,dt = \int_0^\infty t\,\mathrm{e}^{-(\sigma_1+s)t}\,dt = \frac{1}{(\sigma_1+s)^2} \tag{1.5.15}$$

1.5.10 Example 10

Similarly to Example 9, except that here we have $t^{\,n}$, as in Fig. 1.5.10:

$$f(t) = t^{\,n}\,\mathrm{e}^{-\sigma_1 t}$$

Solution: we apply the procedure from Example 8 and Example 9:

$$F(s) = \int_0^\infty t^{\,n}\,\mathrm{e}^{-\sigma_1 t}\,\mathrm{e}^{-st}\,dt = \int_0^\infty t^{\,n}\,\mathrm{e}^{-(\sigma_1+s)t}\,dt = \frac{n!}{(\sigma_1+s)^{n+1}} \tag{1.5.16}$$

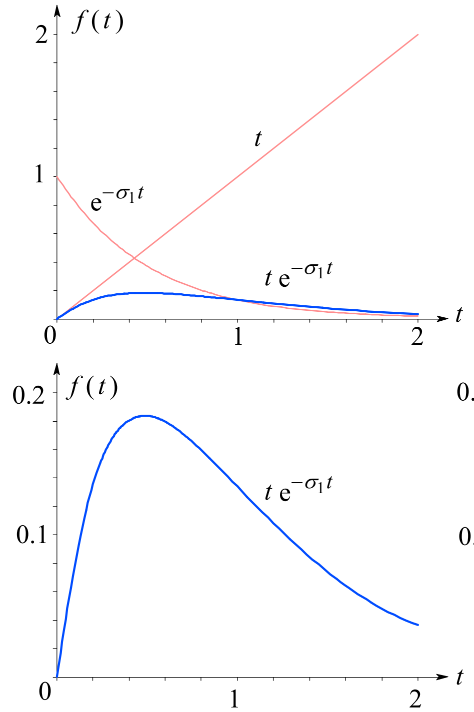

Fig. 1.5.9: Function $f(t) = t\,e^{-\sigma_1 t}$.

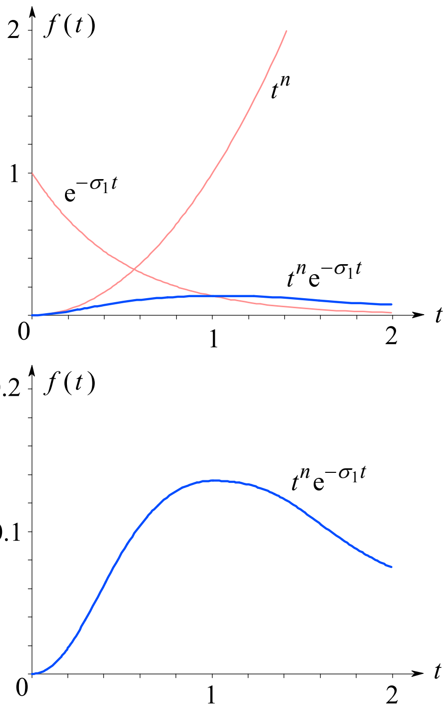

Fig. 1.5.10: Function $f(t) = t^n\,e^{-\sigma_1 t}$.

These ten examples, which we frequently meet in practice, demonstrate that the calculation of an $\mathcal{L}$ transform is not difficult. Since the results derived are used often, we have collected them in Table 1.5.1.

Table 1.5.1: Ten frequently met $\mathcal{L}$ transform examples

No.	$f(t)$	$F(s)$		No.	$f(t)$	$F(s)$
1	$1 \quad (\text{for } t > 0)$	$\dfrac{1}{s}$		6	$e^{-\sigma_1 t} \sin \omega_1 t$	$\dfrac{\omega_1}{(s^2 + \sigma_1^2) + \omega_1^2}$
2	$1 \quad (\text{for } t > a)$	$\dfrac{1}{s}\,e^{-a t}$		7	t	$\dfrac{1}{s^2}$
3	$e^{-\sigma_1 t}$	$\dfrac{1}{\sigma_1 + s}$		8	t^n	$\dfrac{n!}{s^{n+1}}$
4	$\sin \omega_1 t$	$\dfrac{\omega_1}{s^2 + \omega_1^2}$		9	$t\,e^{-\sigma_1 t}$	$\dfrac{1}{(\sigma_1 + s)^2}$
5	$\cos \omega_1 t$	$\dfrac{s}{s^2 + \omega_1^2}$		10	$t^n\,e^{-\sigma_1 t}$	$\dfrac{n!}{(\sigma_1 + s)^{n+1}}$

1.6 Important Properties of the Laplace Transform

It is useful to know some of the most important properties of the $\mathcal{L}$ transform:

1.6.1 Linearity (1)

$$\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\} \tag{1.6.1}$$

Example:

$$\mathcal{L}\{t + \sin\omega_1 t\} = \mathcal{L}\{t\} + \mathcal{L}\{\sin\omega_1 t\} = \frac{1}{s^2} + \frac{\omega_1}{s^2 + \omega_1^2} \tag{1.6.2}$$

1.6.2 Linearity (2)

$$\mathcal{L}\{K\, f(t)\} = K\,\mathcal{L}\{f(t)\} \tag{1.6.3}$$

where K is a real constant.
Example:

$$\mathcal{L}\{4\,\mathrm{e}^{-\sigma_1 t}\sin\omega_1 t\} = 4\,\mathcal{L}\{\mathrm{e}^{-\sigma_1 t}\sin\omega_1 t\} = 4\,\frac{\omega_1}{(s+\sigma_1)^2 + \omega_1^2} \tag{1.6.4}$$

1.6.3 Real Differentiation

$$\mathcal{L}\left\{\frac{d f(t)}{dt}\right\} = s\,F(s) - f(0^+) \tag{1.6.5}$$

The transform of a derivative of the function $f(t)$ is obtained if we multiply $F(s)$ by s and subtract the value of $f(t)$ if $0 \leftarrow t$ from the **right** side, denoted by the + sign at $f(0^+)$ (the direction is important because the values $f(0^-)$ and $f(0^+)$ can be different). We will prove this statement by deriving it from the definition of the $\mathcal{L}$ transform:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,\mathrm{e}^{-st}\,dt \tag{1.6.6}$$

We will integrate by parts by making $f(t) = u$ and $\mathrm{e}^{-st}\,dt = dv$. The result is:

$$\int_0^\infty f(t)\,\mathrm{e}^{-st}\,dt = f(t)\,\frac{1}{-s}\,\mathrm{e}^{-st}\Big|_{t=0}^{t=\infty} + \frac{1}{s}\int_0^\infty \left(\frac{d\,f(t)}{dt}\right)\mathrm{e}^{-st}\,dt$$

$$= \frac{f(0^+)}{s} + \frac{1}{s}\int_0^\infty \left(\frac{d\,f(t)}{dt}\right)\mathrm{e}^{-st}\,dt$$

$$= \frac{f(0^+)}{s} + \frac{1}{s}\,\mathcal{L}\left\{\frac{d f(t)}{dt}\right\} \tag{1.6.7}$$

By rearranging, we prove the statement expressed in Eq. 1.6.6:

$$s\int_0^\infty f(t)\,\mathrm{e}^{-st}\,dt - f(0^+) = s\,F(s) - f(0^+)$$

$$= \int_0^\infty \left[\frac{d\,f(t)}{dt}\right]\mathrm{e}^{-st}\,dt = \mathcal{L}\left\{\frac{d\,f(t)}{dt}\right\} \tag{1.6.8}$$

Example:

$$\mathcal{L}\left\{\frac{d\,(\mathrm{e}^{-\sigma_1 t})}{dt}\right\} = s\,F(s) - f(0^+) = s\,\frac{1}{s+\sigma_1} - 1 = \frac{-\sigma_1}{s+\sigma_1} \tag{1.6.9}$$

We may also check the result by first differentiating the function $\mathrm{e}^{-\sigma_1 t}$:

$$\frac{d\,(\mathrm{e}^{-\sigma_1 t})}{dt} = -\sigma_1\,\mathrm{e}^{-\sigma_1 t} \tag{1.6.10}$$

and then applying the $\mathcal{L}$ transform:

$$\mathcal{L}\{-\sigma_1\,\mathrm{e}^{-\sigma_1 t}\} = -\sigma_1\,\mathcal{L}\{\mathrm{e}^{-\sigma_1 t}\} = \frac{-\sigma_1}{s+\sigma_1} \tag{1.6.11}$$

The result is the same.

By now the advantage of the $\mathcal{L}$ transform against differential equations should have become obvious. In the s domain the derivative of the function $f(t)$ corresponds to $F(s)$ multiplied by s and subtracting the value $f(0^+)$. The reason that t must approach zero from the right (+) side is our prescribing $f(t)$ to be zero for $t < 0$. In other words, we have a unilateral transform.

The higher derivatives are obtained by repeating the above procedure. If for the first derivative we have obtained:

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0^+) \tag{1.6.12}$$

then the $\mathcal{L}$ transform of the second derivative is:

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\left(\mathcal{L}\{f'(t)\} - f'(0^+)\right) \\ &= s\left(s\,F(s) - f(0^+)\right) - f'(0^+) = s^2\,F(\mathrm{s}) - s\,f(0^+) - f'(0^+)\end{aligned} \tag{1.6.13}$$

By a similar procedure the $\mathcal{L}$ transform of the third derivative is:

$$\begin{aligned}\mathcal{L}\{f'''(t)\} &= s\left(s^2\,F(s) - s\,f(0^+) - f'(0^+)\right) - f''(0^+) = \\ &= s^3\,F(s) - s^2\,f(0^+) - s\,f'(0^+) - f''(0^+)\end{aligned} \tag{1.6.14}$$

Thus the $\mathcal{L}$ transform of the n^{th} derivative is simply:

$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} &= s^n\,F(s) - s^{n-1}f(0^+) - s^{n-2}f'(0^+) - s^{n-3}f''(0^+) - \cdots \\ &\cdots - s^2\,f^{(n-3)}(0^+) - s\,f^{(n-2)}(0^+) - f^{(n-1)}(0^+)\end{aligned} \tag{1.6.15}$$

1.6.4 Real Integration

We intend to prove that:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s} \tag{1.6.16}$$

We will derive the proof from the basic definition of the $\mathcal{L}$ transform:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \int_0^\infty \left[\int_0^t f(\tau)\,d\tau\right] \mathrm{e}^{-st}\,dt \tag{1.6.17}$$

For the integration by parts we assign:

$$u = \int_0^t f(\tau)\,d\tau \qquad du = f(\tau)\,d\tau \qquad v = \frac{1}{-s}\,\mathrm{e}^{-st} \qquad dv = \mathrm{e}^{-st}\,dt \tag{1.6.18}$$

By considering all this we may write the integral:

$$\int_0^\infty \int_0^t f(\tau)\,d\tau\,\mathrm{e}^{-st}\,dt = \left[\int_0^t \frac{1}{-s}\,\mathrm{e}^{-st}\,f(\tau)\,d\tau\right]_{t=0}^{t=\infty} - \int_0^\infty \frac{1}{-s}\,\mathrm{e}^{-st}\,f(t)\,dt \tag{1.6.19}$$

The term between both limits is zero for $t = 0$ because $\int_0^0 \cdots = 0$, and for $t = \infty$ as well because the exponential function $\mathrm{e}^{-\infty} = 0$. Thus only the last integral remains, from which we can factor out the term $1/s$. The result is:

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\int_0^\infty f(t)\,\mathrm{e}^{-st}\,dt = \frac{F(s)}{s} \tag{1.6.20}$$

and thus the statement expressed by Eq. 1.6.16 is proved.

Example:

$$f(t) = \mathrm{e}^{-\sigma_1 t}\sin\omega_1 t \tag{1.6.21}$$

We have already calculated the transform of this function (Eq. 1.5.10) and it is:

$$F(s) = \frac{\omega_1}{(s+\sigma_1)^2 + \omega_1^2}$$

Let us now calculate the integral of this function according to Eq. 1.6.20 by introducing a dummy variable τ:

$$\mathcal{L}\left\{\int_0^t \mathrm{e}^{-\sigma_1\tau}\sin\omega_1\tau\,d\tau\right\} = \frac{F(s)}{s} = \frac{\omega_1}{s\left[(s+\sigma_1)^2 + \omega_1^2\right]} \tag{1.6.22}$$

This expression describes the step response of a network, having a complex conjugate pole pair. We meet such functions very often in the analysis of inductive peaking circuits or in calculating the step response of an amplifier with negative feedback.

We may obtain the transform of multiple integrals by repeating the procedure expressed by Eq. 1.6.16.

By doing so we obtain for the

single integral: $$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

double integral: $$\mathcal{L}\left\{\int_0^t\int_0^{\tau_1} f(\tau)\,d\tau\right\} = \frac{F(s)}{s^2}$$

triple integral: $$\mathcal{L}\left\{\int_0^t\int_0^{\tau_1}\int_0^{\tau_2} f(\tau)\,d\tau\right\} = \frac{F(s)}{s^3}$$

n^{th} integral: $$\mathcal{L}\left\{\underbrace{\int_0^t\cdots\int_0^{\tau_{n-1}}}_{n\text{ integrals}} f(\tau)\,d\tau\right\} = \frac{F(s)}{s^n} \tag{1.6.23}$$

The $\mathcal{L}$ transform of the integral of the function $f(t)$ gives the complex function $F(s)/s$. The function $F(s)$ must be divided by s as many times as we integrate.

Here again we see a great advantage of the $\mathcal{L}$ transform, for **we can replace the integration in the time domain** (often a rather demanding procedure) **by a simple division by s in the (complex) frequency domain.**

1.6.5 Change of Scale

We have the function:

$$\mathcal{L}\{f(a\,t)\} = \int_0^\infty f(a\,t)\,\mathrm{e}^{-st}\,dt \tag{1.6.24}$$

We introduce a new variable $v = a\,t$, and for this $dv = a\,dt$ and also $t = v/a$. Thus we obtain:

$$\mathcal{L}\{f(a\,t)\} = \frac{1}{a}\int_{v=0}^\infty f(v)\,\mathrm{e}^{-\frac{st}{a}}\,dv = \frac{1}{a}\,F\left(\frac{s}{a}\right) \tag{1.6.25}$$

Example: we have the function:

$$f(t) = t\,\mathrm{e}^{-3\,t} \tag{1.6.26}$$

We have already calculated the $\mathcal{L}$ transform of a similar function by Eq. 1.5.15. For the function above the result is:

$$F(s) = \frac{1}{(s+3)^2} \tag{1.6.27}$$

Now let us change the scale tenfold. The new function is:

$$g(t) = f(10\,t) = 10\,t\,\mathrm{e}^{-30\,t} \tag{1.6.28}$$

According to Eq. 1.6.25 it follows that:

$$\mathcal{L}\{g(t)\} = \frac{1}{10}\,F\left(\frac{s}{10}\right) = \frac{1}{10\left(\dfrac{s}{10}+3\right)^2} = \frac{10}{(s+30)^2} \tag{1.6.29}$$

1.6.6 Impulse $\delta(t)$

In Fig. 1.6.1a we have a square pulse A_1 with amplitude 1 and duration $t = 1$. The surface S_1 under the pulse, equal to the time integral of this pulse, is *amplitude* $\times$ *time* and thus equal to 1. It is obvious that we may obtain the same time integral if the duration of the pulse is halved and its amplitude doubled (A_2). The pulse A_4 has a four times higher amplitude and its duration is only $t = 0.25$ and still has the same time-integral.

If we keep narrowing the pulse and adjusting the amplitude accordingly to keep the value of the time integral 1, we eventually arrive at a situation where the duration of the pulse becomes infinitely small, $t = \varepsilon \to 0$, and its amplitude infinitely large, $A = (1/\varepsilon) \to \infty$, as shown in Fig. 1.6.1b.

This impulse is denoted $\delta(t)$ and it is called the *Dirac*[2] *function*. Mathematically we express this function as:

$$\delta(t) = f_\varepsilon(t) = \left.\begin{cases} 1/\varepsilon & \text{when} \quad 0 \le t \le \varepsilon \\ 0 & \text{when} \qquad t > \varepsilon \end{cases}\right|_{\varepsilon \to 0} \tag{1.6.30}$$

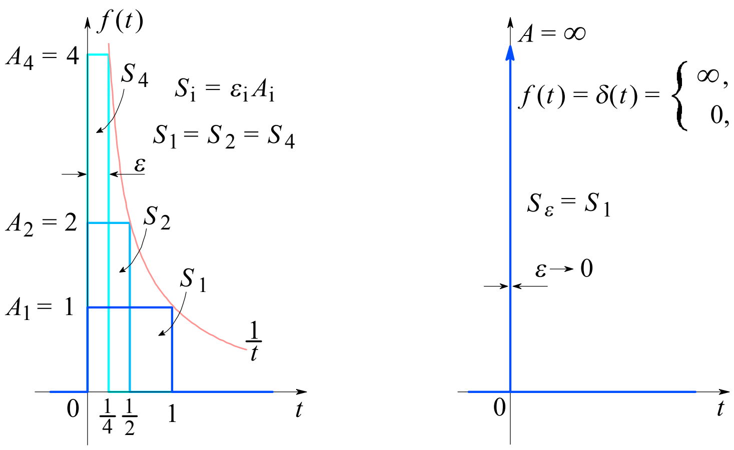

Fig. 1.6.1: The Dirac function as the limiting case of narrowing the pulse width, while keeping the time integral constant: **a)** If the pulse length is decreased, its amplitude must increase accordingly. **b)** When the pulse length $\varepsilon \to 0$ the amplitude is $(1/\varepsilon) \to \infty$.

Let us calculate the $\mathcal{L}$ transform of this function:

$$\mathcal{L}\{f_\varepsilon(t)\} = \int_0^\infty f_\varepsilon(t)\,\mathrm{e}^{-st}\,dt = \int_0^\varepsilon \frac{1}{\varepsilon}\,\mathrm{e}^{-st}\,dt + \int_\varepsilon^\infty (0)\,\mathrm{e}^{-st}\,dt$$

$$= -\frac{1}{s\,\varepsilon}\,\mathrm{e}^{-st}\Big|_{t=0}^{t=\varepsilon} = \frac{1-\mathrm{e}^{-s\,\varepsilon}}{s\,\varepsilon} \tag{1.6.31}$$

---

[2] *Paul Dirac*, 1902–1984, English physicist, Nobel Prize winner in 1933 (together with Erwin Schrödinger).

Now we express the function $\mathrm{e}^{-s\,\varepsilon}$ in this result by the following series:

$$\frac{1-\mathrm{e}^{-s\,\varepsilon}}{s\,\varepsilon}=\frac{1-[1-s\,\varepsilon+(s\,\varepsilon)^2/2!-(s\,\varepsilon)^3/3!+\cdots]}{s\,\varepsilon}=1-\frac{s\,\varepsilon}{2!}+\frac{(s\,\varepsilon)^2}{3!}-\cdots$$

and by letting $\varepsilon\to 0$ we obtain:

$$\mathcal{L}\{\delta(t)\}=\lim_{\varepsilon\to 0}\frac{1-\mathrm{e}^{-s\,\varepsilon}}{s\,\varepsilon}=\lim_{\varepsilon\to 0}\left[1-\frac{s\,\varepsilon}{2!}+\frac{(s\,\varepsilon)^2}{3!}-\cdots\right]=1 \tag{1.6.32}$$

Therefore the magnitude of the spectrum envelope of this function is **one** and it is independent of frequency. This means that the Dirac impulse $\delta(t)$ contains **an infinite number of frequency components**, the amplitude of each component being $A=1$.

1.6.7 Initial and Final Value Theorems

The **initial value theorem** is expressed as:

$$\lim_{0\leftarrow t} f(t)=\lim_{s\to\infty} s\,F(s) \tag{1.6.33}$$

We have written the notation $0\leftarrow t$ in order to emphasize that t approaches zero from the right of the coordinate system. From real differentiation we know that:

$$\mathcal{L}\left\{\frac{d\,f(t)}{dt}\right\}=\mathcal{L}\{f'(t)\}=\int_0^\infty f'(t)\,\mathrm{e}^{-st}\,dt=s\,F(s)-f(0^+) \tag{1.6.34}$$

The limit of this integral when $s\to\infty$ is zero:

$$\lim_{s\to\infty}\int_0^\infty f'(t)\,\mathrm{e}^{-st}\,dt=0 \tag{1.6.35}$$

If we assume that $f(t)$ is continuous at $t=0$ we may write the limit of the right hand side of Eq. 1.6.33:

$$0=\lim_{s\to\infty} s\,F(s)-f(0^+) \tag{1.6.36}$$

or, in a form more useful for practical calculations:

$$\boxed{\lim_{s\to\infty} s\,F(s)=f(0^+)+\lim_{0\leftarrow t} f(t)=\lim_{0\leftarrow t} f(t)} \tag{1.6.37}$$

Even if $f(t)$ is not continuous at $t=0$, this relation is still valid, although the proof is slightly more difficult [Ref. 1.10]. The expression $f(0^+)$ is introduced because we are dealing with a unilateral transform, in which it is assumed that $f(t)=0$ for $t<0$, so to calculate the actual initial value we must approach it from the positive side of the time axis.

For the functions which we will discuss in the rest of the book we can, in a similar way, prove the **final value theorem**, which is stated as:

$$\lim_{t\to\infty} f(t)=\lim_{s\to 0} s\,F(s) \tag{1.6.38}$$

(note that for some functions, such as $\sin\omega_1 t$ or $\cos\omega_1 t$ or the square wave, this limit does not exist, since the value oscillates with the time integral of the function!).

We repeat the statement from Eq. 1.6.34:

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)\,\mathrm{e}^{-st}\,dt = s\,F(s) - f(0^+) \tag{1.6.39}$$

Now let $s \to 0$ (using q as an intermediate dummy variable):

$$\lim_{s\to 0} \int_0^\infty f'(t)\mathrm{e}^{-st}\,dt = \int_0^\infty f'(t)\,dt = \lim_{q\to\infty} \int_0^q f'(t)\,dt$$

$$= \lim_{q\to\infty} [f(q) - f(0^+)] = \lim_{t\to\infty} f(t) - f(0^+) \tag{1.6.40}$$

Although the lower limit of the integral is a (simple) zero we have nevertheless written 0^+ in the result, to emphasize the unilateral transform. The limit of the right hand side of Eq. 1.6.39, when $s \to 0$ is:

$$\lim_{s\to 0} s\,F(s) - f(0^+) \tag{1.6.41}$$

By comparing the results of Eq. 1.6.34, 1.6.39 and 1.6.41 we may write:

$$\lim_{t\to\infty} f(t) - f(0^+) = \lim_{s\to 0} s\,F(s) - f(0^+) \tag{1.6.42}$$

or as stated initially:

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)} \tag{1.6.38, again}$$

The Eq. 1.6.37 and 1.6.38 are extremely useful for checking the results of complicated calculations by the direct or the inverse Laplace transform, as we will encounter in the following parts of the book. Should the check by these two equations fail, then we have obviously made a mistake somewhere.

However, this is a necessary, but not a sufficient condition: if the check was passed we are not guaranteed that other ‘sneaky’ mistakes will not exist, which may become obvious when we plot the resulting function.

1.6.8 Convolution

We need a process by which we can calculate the response of two systems connected so that the output of the first one is the input of the second one and their individual responses are known. We have two functions [Ref. 1.19]:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \qquad \text{and} \qquad g(t) = \mathcal{L}^{-1}\{G(s)\} \tag{1.6.43}$$

and we are looking for the inverse transform of the product:

$$y(t) = \mathcal{L}^{-1}\{F(s)\cdot G(s)\} \tag{1.6.44}$$

The product of functions is equal to the product of their Laplace transforms:

$$F(s)\cdot G(s) = \int_0^\infty f(\tau)\,\mathrm{e}^{-s\tau}\,d\tau \cdot \int_0^\infty g(v)\,\mathrm{e}^{-sv}\,dv \tag{1.6.45}$$

In order to distinguish better between $f(t)$ and $g(t)$, we assign the letter u for the argument of f and v for the argument of g; thus, $f(t) \to f(u)$ and $g(t) \to g(v)$. Since both variables are now well separated we may write the above integral also in the form:

$$F(s) \cdot G(s) = \int_0^\infty \left[\int_0^\infty f(u)\, g(v)\, \mathrm{e}^{-s(u+v)}\, dv \right] du \tag{1.6.46}$$

Let us integrate the expression inside the brackets to the variable v. To do so we introduce a new variable τ:

$$\tau = u + v \qquad \text{so} \qquad v = \tau - u \qquad \text{and} \qquad dv = d\tau \tag{1.6.47}$$

We consider the variable τ in the inner integral to be a (variable) parameter. From the above expressions it follows that $v = 0$ if $\tau = u$. By considering all this we may transform Eq. 1.6.46 into:

$$F(s) \cdot G(s) = \int_0^\infty \left[\int_\tau^\infty f(u)\, g(\tau - u)\, \mathrm{e}^{-s(u+\tau-u)}\, d\tau \right] du \tag{1.6.48}$$

We may also change the sequence of integration. Thus we may choose a fixed t_1 and first integrate from $\tau = 0$ to $\tau = t_1$. In the second integration we integrate from $u = 0$ to $u = \infty$. Then the above expression obtains the form:

$$F(s) \cdot G(s) = \int_0^\infty \left[\int_0^{t_1} f(u)\, g(\tau - u)\, du \right] \mathrm{e}^{-s\tau}\, d\tau \tag{1.6.49}$$

Now we can return from u back to the usual time variable t:

$$F(s) \cdot G(s) = \int_0^\infty \underbrace{\left[\int_0^{t_1} f(t)\, g(\tau - t)\, dt \right]}_{y(t)} \mathrm{e}^{-s\tau}\, d\tau \tag{1.6.50}$$

The expression inside the brackets is the function $y(t)$ which we are looking for, whilst the outer integral is the usual Laplace transform. Thus we define the convolution process, denoted by $g(t) * f(t)$, as:

$$\boxed{y(t) = \mathcal{L}^{-1}\{F(s) \cdot G(s)\} = g(t) * f(t) = \int_0^{t_1} f(t)\, g(\tau - t)\, dt} \tag{1.6.51}$$

The operator symbolized by the asterisk $(*)$ means 'convolved with'. *Convolutio* is the Latin word for folding. The German name for convolution is *die Faltung* and this also means folding. Obviously:

$$g(\tau - t) = g(-t)\Big|_{\tau=0}$$

and

$$g(\tau - t) = g(0)\Big|_{\tau=t}$$

This means that the function is ‘folded’ in time around the ordinate, from the right to the left side of the coordinate system. At the end of this part, after we master the network analysis in Laplace space, we will make an example (Fig. 1.15.1) in which this ‘folding’ and the convolution process will be explicitly shown, step by step.

In general we convolve whichever of the two functions is simpler. We may do so because the convolution is commutative:

$$g(t) * f(t) = f(t) * g(t) \tag{1.6.52}$$

The main properties of the Laplace transform are listed in Table 1.6.1.

Table 1.6.1: **The main properties of Laplace transform**

Property	$f(t)$	$F(s)$
Real Differentiation	$\dfrac{d\,f(t)}{dt}$	$s\,F(s) - f(0^+)$
Real Integration	$\displaystyle\int_0^{t_1} f(t)\,dt$	$\dfrac{F(s)}{s}$
Time-Scale Change	$f(at)$	$\dfrac{1}{a}\,F\left(\dfrac{s}{a}\right)$
Impulse function	$\delta(t)$	1
Initial Value	$\lim\limits_{0 \leftarrow t} f(t)$	$\lim\limits_{s \to \infty} s\,F(s)$
Final Value	$\lim\limits_{t \to \infty} f(t)$	$\lim\limits_{s \to 0} s\,F(s)$
Convolution	$\displaystyle\int_0^{t_1} f(t)\,g(\tau - t)\,dt$	$F(s) \cdot G(s)$

(blank page)

1.7 Application of the $\mathcal{L}$ transform in Network Analysis

1.7.1 Inductance

As we have discussed the fundamentals of $\mathcal{L}$ transform, we will now apply it in the network analysis. From basic electrical engineering we know that the instantaneous voltage $v(t)$ across an inductance L, through which a current $i(t)$ flows, as in Fig. 1.7.1a, is:

$$v(t) = \frac{di}{dt} L \tag{1.7.1}$$

By assuming time $t > 0$ and $i(0^+) = 0$, then, according to Eq. 1.6.5, the $\mathcal{L}$ transform of the above equation is the voltage across the inductance in the s domain:

$$V(s) = s\,L\,I(s) \tag{1.7.2}$$

where $I(s)$ is the current in the s domain. The inductive reactance is then:

$$\boxed{\frac{V(s)}{I(s)} = s\,L} \tag{1.7.3}$$

Here $s = \sigma + j\omega$ and thus it is complex; it can lie anywhere in the s plane. In the **special case** when $\sigma = 0$, and considering only the positive $j\omega$ axis, s degenerates into $j\omega$. Then the inductive reactance becomes the familiar $j\omega L$, as is known from the usual 'phasor' analysis of networks.

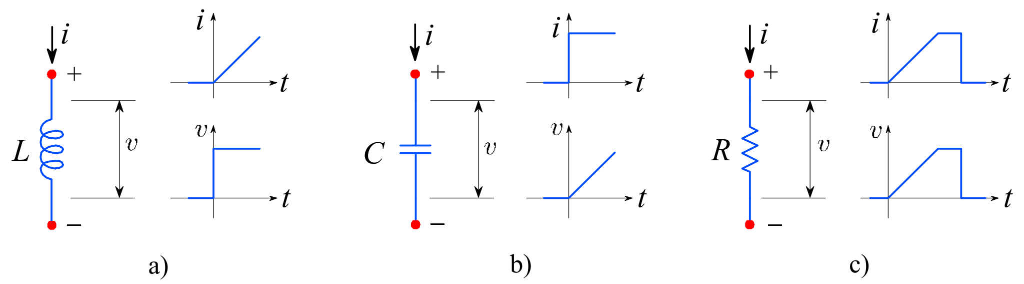

Fig. 1.7.1: The instantaneous voltage v as a function of the instantaneous current i: **a)** on an inductance L; **b)** on a capacitance C; **c)** on a resistance R.

1.7.2 Capacitance

From basic electrical engineering we also know that the instantaneous voltage $v(t)$ across a capacitance through which a current $i(t)$ flows during a time $t \geq 0$ is:

$$v(t) = \frac{q(t)}{C} = \frac{1}{C}\int_0^t i\,dt \tag{1.7.4}$$

as shown in Fig. 1.7.1b. Here $q(t)$ is the instantaneous charge on the capacitor C. By applying Eq. 1.6.20 we may calculate the voltage on the capacitor in the s domain:

$$V(s) = \frac{1}{s\,C}\,I(s) \tag{1.7.5}$$

The capacitive reactance in the s domain is:

$$\boxed{\frac{V(s)}{I(s)} = \frac{1}{s\,C}} \tag{1.7.6}$$

Here, too, s degenerates to $j\omega$ if $\sigma = 0$. In this case the capacitive reactance becomes simply $1/j\,\omega\,C$.

1.7.3 Resistance

For a resistor (Fig. 1.7.1c) the instantaneous voltage is simply:

$$v(t) = R\,i(t) \tag{1.7.7}$$

and, as there are no time-derivatives the same holds in the s domain, with the corresponding values $V(s)$ and $I(s)$:

$$V(s) = R\,I(s) \tag{1.7.8}$$

yielding:

$$\boxed{\frac{V(s)}{I(s)} = R} \tag{1.7.9}$$

1.7.4 Resistor and capacitor in parallel

By applying the Eq. 1.7.3, 1.7.6 and 1.7.9 we may transform a differential equation in the t domain into an algebraic equation in the s domain. Thus we may express an impedance $Z(s)$ or an admittance $Y(s)$ of more complicated networks by simple algebraic equations. Let us express a parallel combination of a resistor and a capacitor in s domain as shown in the upper part of Fig. 1.7.2. The impedance is:

$$Z(s) = \frac{1}{s\,C + \dfrac{1}{R}} = R\,\frac{-\left(-\dfrac{1}{RC}\right)}{s - \left(-\dfrac{1}{RC}\right)} = R\,\frac{-s_1}{s - s_1} = R\,G_1(s) \tag{1.7.10}$$

where the (real) pole is at $s_1 = \sigma_1 = -1/RC$ and G_1 represents the frequency dependence. The pole of a function is that particular value of the argument for which the function denominator is equal to zero and, consequently, the function value goes to infinity.

Now let us apply a current step, $I(s) = 1\,\mathrm{V}/R$, to our network expressed in the s domain as $1/(sR)$, according to Eq. 1.5.1. We introduced the factor $1/R$ in order to get a voltage of 1 V on our RC combination when $t \to \infty$. The corresponding function is then:

$$\begin{aligned} F(s) = V(s) = I(s)\cdot Z(s) &= \frac{1}{sR}\cdot R\,G_1(s) = \frac{1}{s}\,G_1(s) \\ &= \frac{1}{RC}\cdot\frac{1}{s}\cdot\frac{1}{s - \left(-\dfrac{1}{RC}\right)} \end{aligned} \tag{1.7.11}$$

From $1/s$ a second pole at $s = 0$ is introduced, as is drawn in Fig. 1.7.2b and c. To obtain the time domain function of the voltage across our impedance, we apply Eq. 1.5.3 and 1.6.20. First we discuss only the function $G_1(s)$. According to Eq. 1.5.3:

$$\mathcal{L}\{\mathrm{e}^{\sigma_1 t}\} = \frac{1}{s - \sigma_1} \tag{1.7.12}$$

or inversely:

$$\mathcal{L}^{-1}\left\{\frac{1}{s - \sigma_1}\right\} = \mathrm{e}^{\sigma_1 t} \tag{1.7.13}$$

By comparing Eq. 1.7.11 with Eq. 1.7.13 we see that $\sigma_1 = -1/RC$:

$$\mathcal{L}^{-1}\left\{\frac{1}{s - (-1/RC)}\right\} = \mathrm{e}^{-t/RC} \tag{1.7.14}$$

From Eq. 1.6.20 we concluded that the division in the s domain corresponds to the real integration in the t domain. By considering this together with Eq. 1.5.1, we obtain:

$$v_\mathrm{o}(t) = f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1/RC}{s\,[s - (-1/RC)]}\right\} = \frac{1}{RC}\int_0^t \mathrm{e}^{-t/RC}\,dt$$

$$= \frac{1}{RC}\left[-RC\left(\mathrm{e}^{-t/RC}\right)\Big|_0^t\right] = -\mathrm{e}^{-t/RC} - (-1) = 1 - \mathrm{e}^{-t/RC} \tag{1.7.15}$$

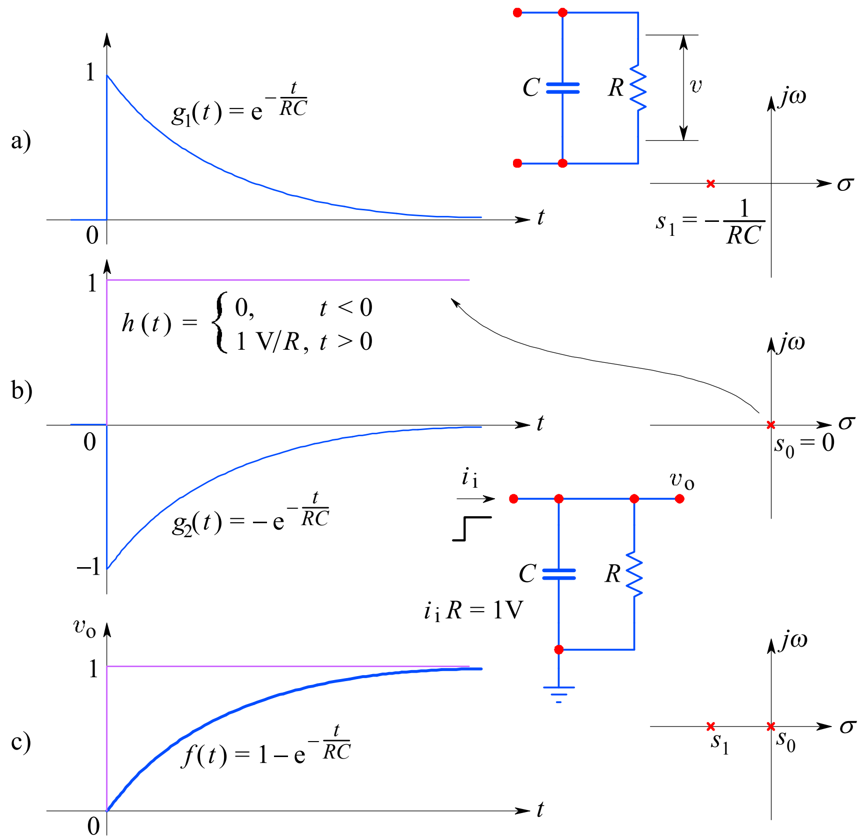

Fig. 1.7.2: The course of mathematical operations for a parallel RC network excited by a unit step current i_i. The t domain functions are on the left, the s domain functions are on the right. **a)** The self-discharge network function is equal to the impulse function $g_1(t)$. **b)** The unit step in t domain, $h(t)$, is represented by a pole at the origin (s_0) in the s domain. The function $g_2(t)$ is the reaction of the network to the unit step excitation. **c)** The output voltage is the sum of both functions, $v_o = f(t) = h(t) + g_2(t)$.

From this simple example we obtain the idea of how to use tables of $\mathcal{L}$ transforms to obtain the response in the t domain, which should otherwise be calculated by differential equations. In addition to this we may state a very important conclusion for the s-domain:

$$\boxed{\Big(\textit{output function}\Big) = \Big(\textit{excitation function}\Big) \times \Big(\textit{network function}\Big)}$$

In our case it was:

excitation function $\qquad H(s) = \dfrac{1}{sR}$

network function $\qquad R\,G_1(s) = R \cdot \dfrac{\dfrac{1}{RC}}{s + \dfrac{1}{RC}}$ $\qquad$ (also named '*impulse response*')

output function $\qquad F(s) = \dfrac{1}{s} \cdot \dfrac{\dfrac{1}{RC}}{s + \dfrac{1}{RC}}$

However, in general, especially for more complicated networks, the calculation of the corresponding function in the t domain is not as easy as shown above. Of course, one may always apply the formula for the inverse Laplace transform (Eq. 1.4.4):

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, \mathrm{e}^{st}\, ds$$

but it would not be fair to leave the reader to grind through this integral of his $F(s)$ with the best of his/her knowledge. In essence the above expression is a *contour integral*. **Knowledge of contour integration is a necessary prerequisite for calculating the inverse Laplace transform.** We will discuss this in the following section. After studying it the reader will realize that the calculation of the step-response in the t domain by contour integration is — although a little more difficult than in the above example of the simple RC circuit — still a relatively simple procedure.

1.8 Complex Line Integrals

In order to learn how to calculate contour integrals the first step is the calculation of complex line integrals. Both require a knowledge of the basics of complex variable theory, also called *the theory of analytic functions*. We will discuss only that part of this theory which is relevant to the inverse Laplace transform of rational functions (which are important in the calculation of amplifier step and impulse response). The reader who would like to know more about the complex variable theory, should study at least one of the books listed at the end of Part 1 [Ref. 1.4, 1.9, 1.10, 1.11, 1.12, 1.13, 1.14, 1.15], of which [Ref. 1.10 and 1.13] (in English), [Ref. 1.4] (in German), and [Ref. 1.11] (in Slovenian) are especially recommended.

The definition of an analytical function is:

In a certain domain (which we are interested in) a function $f(z)$ is analytical if it is:

1) continuous;
2) single valued (at each argument value); and
3) has a derivative at any selected point z, independently of from which side we approach that point.

From the calculus we know that a definite integral of a function of a **real** variable, such as $y = f(x)$, is equal to the area A between the function (curve) and the real axis x and between both limits x_1 to x_2. An example is shown in Fig. 1.8.1, where the integral of a simple function $1/x$, integrated from x_1 to x_2 is displayed.

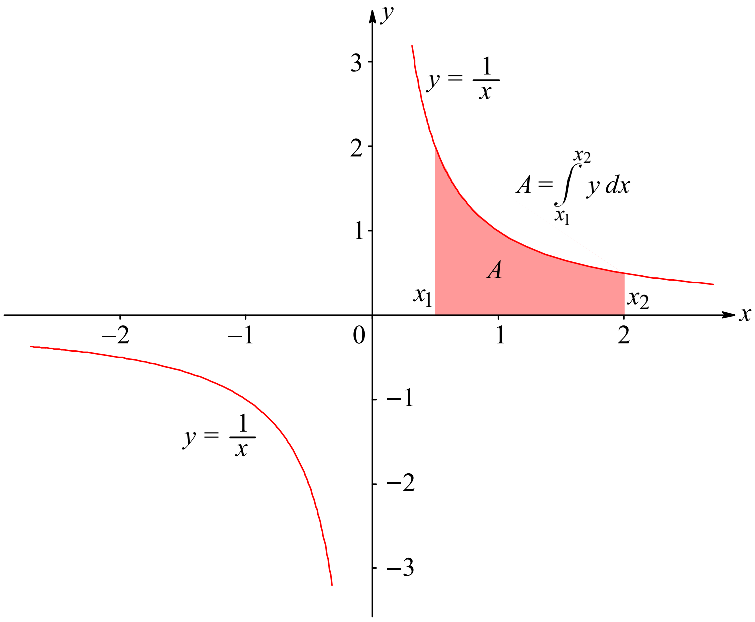

Fig. 1.8.1: The integral of a real function $y = 1/x$ between the limits x_1 and x_2 corresponds to the area A.

The corresponding mathematical expression is:

$$A = \int_{x_1}^{x_2} \frac{1}{x}\,dx = F(x_2) - F(x_1) = \ln x_2 - \ln x_1 = \ln \frac{x_2}{x_1} \tag{1.8.1}$$

The area above the x axis is counted as positive and the area below the x axis (if any) as negative. The area A in Fig. 1.8.1 represents the difference of the integral values at the upper limit, $F(x_2)$ and the lower limit, $F(x_1)$. As shown in Fig. 1.8.1 the integration path was from x_1 along the x axis up to x_2.

For a comparison let us now calculate a similar integral, but with a **complex** variable $z = x + jy$:

$$W = \int_{z_1}^{z_2} \frac{1}{z}\, dz = F(z_2) - F(z_1) = \ln z_2 - \ln z_1 = \ln \frac{z_2}{z_1} \tag{1.8.2}$$

So far we can not see any difference between Eq. 1.8.1 and 1.8.2 (a close investigation of the result would show that it may be multi-valued in the case the path from z_1 to z_2 circles the pole one or more times; but we will not discuss such cases). The whole integration procedure is the same in both cases. The difference in the result of the second equation becomes apparent when we express the complex variable z in the exponential form:

$$z_1 = |z_1|\, \mathrm{e}^{j\theta_1} \qquad \text{and} \qquad z_2 = |z_2|\, \mathrm{e}^{j\theta_2} \tag{1.8.3}$$

then:

$$\ln \frac{z_2}{z_1} = \ln \frac{|z_2|\, \mathrm{e}^{j\theta_2}}{|z_1|\, \mathrm{e}^{j\theta_1}} = \ln \frac{|z_2|}{|z_1|}\, \mathrm{e}^{j(\theta_2 - \theta_1)} = \ln \frac{|z_2|}{|z_1|} + j(\theta_2 - \theta_1) = u + jv \tag{1.8.4}$$

where:

$$u = \ln \frac{|z_2|}{|z_1|} \qquad \text{and} \qquad v = \theta_2 - \theta_1 \tag{1.8.5}$$

and also:

$$|z_\mathrm{i}| = \sqrt{x_\mathrm{i}^2 + y_\mathrm{i}^2} \qquad \text{and} \qquad \theta_\mathrm{i} = \arctan \frac{y_\mathrm{i}}{x_\mathrm{i}} \tag{1.8.6}$$

Obviously the result of Eq. 1.8.2, as shown in Eq. 1.8.4, is complex. It can not be plotted as simply as the integral of Fig. 1.8.1, since for displaying the complex function of a complex argument we would need a 4D graph, whilst the present state of technology allows us to plot only a 2D projection of a 3D graph, at best.

We can, however, restrict the z argument's domain, as in Fig. 1.8.2, by making its real part a constant, say, $x = c$ and then make plots of $F(c + jy) = 1/(c + jy)$ for some selected value of c. In Fig. 1.8.2 we have chosen $c = 0$ and $c = 0.5$, whilst the imaginary part was varied from $-j3$ to $+j3$.

In this way we have plotted two graphs, labeled A and B. The graph A belongs to $c = 0$ and lies in the $\Im\{s\} \times \Im\{F(s)\}$ plane; it looks just like the one in Fig. 1.8.1, but changed in sign, owing to the following rationalization of the function's denominator:

$$\frac{1}{jy} = \frac{j}{j^2 y} = \frac{j}{-1 \cdot y} = -\frac{j}{y}$$

The graph B belongs to $c = 0.5$ and is a 3D curve, twisting in accordance with the phase angle of the function. To aid the 3D view the three projections of B have also been plotted.

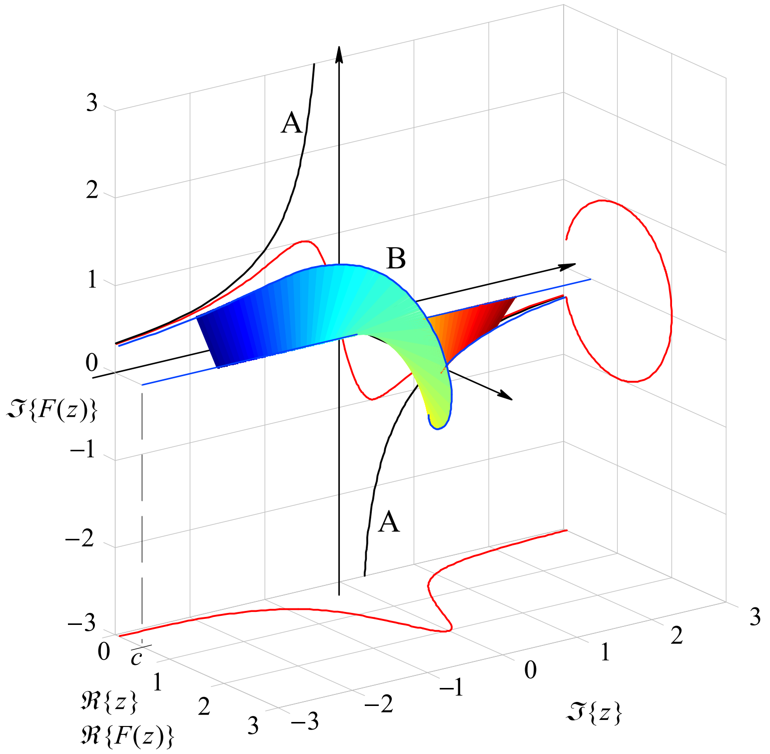

Fig. 1.8.2: By reducing the complex domain $x + jy$ to $c + jy$, where c is a constant, we can plot the complex function $F(c + jy)$ in a 3D graph. Here we have $c = 0$ (graph A) and $c = 0.5$ (graph B). Also shown are the three projections of B. The twisted surface is the integral of $F(c + jy)$ for $c = 0.5$ and y in the range $-j2 < y < j2$. See Appendix 1 (disk) for details.

Let us determine a few characteristic points of the graph B:

a) for the first point on the left we have $c = 0.5$ and $y = -3$, thus:

$$F(c + jy) = \frac{1}{0.5 - j3} = \frac{0.5 + j3}{(0.5 - j3)(0.5 + j3)}$$

$$= \frac{0.5 + j3}{0.5^2 + 3^2} = \frac{0.5 + j3}{0.25 + 9} = \frac{0.5}{9.25} + j\frac{3}{9.25} = 0.0541 + j0.3243$$

b) next, let us have $c = 0.5$ and $y = -0.5$, thus:

$$F(c + jy) = \frac{1}{0.5 - j0.5} = \frac{0.5 + j0.5}{(0.5 - j0.5)(0.5 + j0.5)}$$

$$= \frac{0.5 + j0.5}{0.5^2 + 0.5^2} = \frac{0.5 + j0.5}{0.25 + 0.25} = \frac{0.5}{0.5} + j\frac{0.5}{0.5} = 1 + j$$

(here both the real and the imaginary part are 1 — this is the top point of the curve).

c) an obvious choice is $c = 0.5$ and $y = 0$, thus:

$$F(c + jy) = \frac{1}{0.5 - j0} = \frac{1}{0.5} = 2$$

(here the real part is 2, the imaginary part is 0 and this is the rightmost point on the curve; also, it is its only real value point).

For positive imaginary values, $F(z)$ is the complex conjugate of the values above.

Now that we have some idea of how $F(z)$ looks, let us return to our integral problem. If the integration path is parallel to the imaginary axis, $-j2 < y < +j2$, and displaced by $x = \Re\{z\} = c = 0.5$, the result of integration would be the surface indicated in Fig. 1.8.2. But for an arbitrary path, with x not constant, we should make many such plots as above and then trace the integration path to appropriate curves. The area bounded by the integration path and its trace on those curves would be the result we seek.

For a detailed treatment of complex function plotting see Appendix 1 (on the disk).

Returning to the result of Eq. 1.8.4 we may draw an interesting conclusion:

The complex line integral depends only on the initial value z_1 and the final value z_2, which represent both limits of the integral.

The result of the integration is independent of the actual path beneath these limits, providing that the path lies on the same side of the pole.

All the significant differences between an integral of a real function and the line integral of a complex function are listed in Table 1.8.1.

The x axis is the argument's domain for a real integral, whilst for a complex integral it is the whole z plane. Do not confuse the z plane (the complex plane, $x + jy$, with the diagram's z axis (vertical axis), which here is $F(z) = F(x + jy)$. We recommend the readers to ponder over Fig. 1.8.2 and try to acquire a clear idea of the differences between both types of integral, since this is necessary for the understanding of the discussion which follows.

Table 1.8.1 Differences between real and complex integration

	real variable	complex variable
integral	$\displaystyle\int_{x_1}^{x_2} \frac{1}{x}\, dx$	$\displaystyle\int_{z_1}^{z_2} \frac{1}{z}\, dz$
independent variable	x	$z = x + j\,y$
dependent variable	$y = \dfrac{1}{x}$	$w = \dfrac{1}{z} = u + j\,v = \dfrac{x}{x^2+y^2} - j\,\dfrac{y}{x^2+y^2}$
integration path	from x_1 to x_2 along the x axis	from $z_1 = x_1 + j\,y_1$ to $z_2 = x_2 + j\,y_2$ anywhere in the z plane*
result	$\ln \dfrac{x_2}{x_1}$ **(real)**	$\ln \dfrac{z_2}{z_1} = \ln \dfrac{\lvert z_2\rvert}{\lvert z_1\rvert} + j\,(\theta_2 - \theta_1)$ **(complex)**

* except through the pole, where $z = 0$

To understand the theory better let us give a few examples:

1.8.1 Example 1

We have a function $f(z) = 3z$ which we shall integrate from $2j$ to $1+j$:

$$\int_{2j}^{1+j} 3z\,dz = \frac{3\,z^2}{2}\bigg|_{2j}^{1+j} = \frac{3}{2}\left[(1+j)^2 - (2\,j)^2\right] = 6 + 3\,j$$

1.8.2 Example 2

The integration limits are the same as in the previous example, whilst the function is different, $f(z) = 1 + z^2$:

$$\int_{2j}^{1+j} \left(1+z^2\right) dz = z\bigg|_{2j}^{1+j} + \frac{z^3}{3}\bigg|_{2j}^{1+j} = \frac{1}{3} + \frac{7}{3}\,j$$

1.8.3 Example 3

The same function as in Example 1, except that both limits are interchanged:

$$\int_{1+j}^{2j} 3\,z\,dz = \frac{3\,z^2}{2}\bigg|_{1+j}^{2j} = \frac{3}{2}\left[(2\,j)^2 - (1+j)^2\right] = -6 - 3\,j$$

We see that although the function under the integral is complex, the same rules apply for integration as for a function of a real variable. The last example shows us that if the limits of the integral of a complex function are exchanged the result of the integration changes the sign.

As already mentioned, the result of the integration of a complex function is independent of the actual path of integration between the limits z_1 and z_2 (see Fig. 1.8.3), provided that no pole lies between the extreme paths L_1 and L_2. Thus for all the paths shown the result of integration is the same. This means that the function in the area between L_1 and L_2 is *analytic*. When at least one pole of the function lies between L_1 and L_2, the integral along the path L_1 is in general no more equal to the integral along the path L_2. In Fig. 1.8.4 we show such a case, in which the function is non-analytic (or *non-regular*) inside a small area between z_1 and z_2 (in the remaining area the function is analytic).

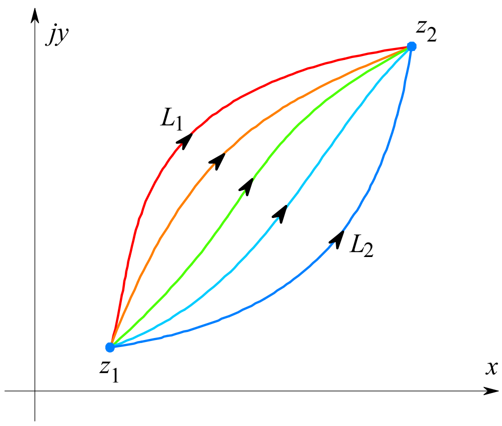

Fig. 1.8.3: A line integral from z_1 to z_2 along the line L_1, L_2, or any other line lying between these two yields the same result because between L_1 and L_2 the function has no pole.

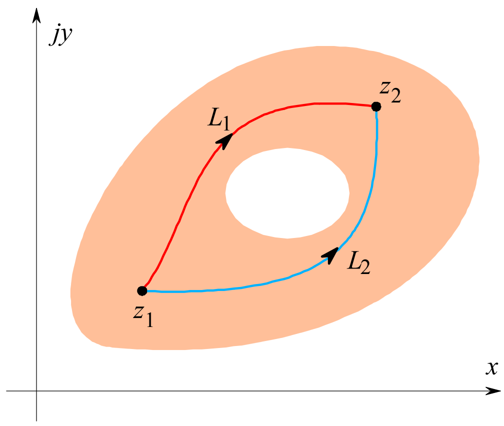

Fig. 1.8.4: Here the function has a non-analytic domain area between L_1 and L_2. Now the integral along the path L_1 is **not** equal to the integral along the path L_2.

Let us prove the above statement by two simple examples.

1.8.4 Example 4

We will again take the function $f(z) = 1/z$ and integrate along a part of a circle with the radius of 1, from $-j$ to $+1$ as is drawn in Fig. 1.8.5 (the pole z_0 lies at $z = 0$). We first calculate the integral:

$$\int_{-j}^{1} \frac{1}{z}\, dz \quad \text{along the path } L_1$$

On the circle with radius $|z| = 1$ it is:

$$z = (1) \cdot \mathrm{e}^{j\theta} \quad \text{and} \quad dz = j\,\mathrm{e}^{j\theta} d\theta$$

When we integrate along L_1 the angle θ goes from $-\pi/2$ to 0. Thus it is:

$$\int_{-\pi/2}^{0} \frac{j\,\mathrm{e}^{j\theta}}{\mathrm{e}^{j\theta}}\, d\theta = j\,\theta \Big|_{-\pi/2}^{0} = j\,\frac{\pi}{2}$$

1.8.5 Example 5

Here everything is the same as in the previous example, except that we will integrate along the path L_2 of Fig. 1.8.5. In this case the angle θ goes from $3\pi/2$ to 0:

$$\int_{3\pi/2}^{0} \frac{j\,\mathrm{e}^{j\theta}}{\mathrm{e}^{j\theta}}\, d\theta = j\,\theta \Big|_{3\pi/2}^{0} = -j\,\frac{3\pi}{2}$$

In Example 4, the integration path goes counterclockwise (which in mathematics is the positive sense) and we obtain a positive result. But in Example 5, in which the integration path goes clockwise, the result is negative, and, moreover, it has a different value, because the integration path lies on the other side of the pole, even if the limits of integration remain the same as in Example 4.

1.8.6 Example 6

We would like to see whether there is any difference in the result of Example 4 if we choose not to integrate along the circle, but instead along a straight line from $-j$ to $+1$ (L_3 in Fig. 1.8.6):

$$\int_{-j}^{1} \frac{1}{z}\, dz = \ln 1 - \ln(-j) = -\ln \mathrm{e}^{-j\pi/2} = j\,\frac{\pi}{2}$$

because $\ln 1 = 0$ and $-j = \mathrm{e}^{-j\pi/2}$. The result is the same as in Example 4.

In general if we consider Fig. 1.8.7, the integral along the path L_a or L_b, or any path in between, is always equal to $j\,\pi/2$. Similarly, the integral along the path L_c or L_d, or any path in between, is equal to $-j\,3\pi/2$.

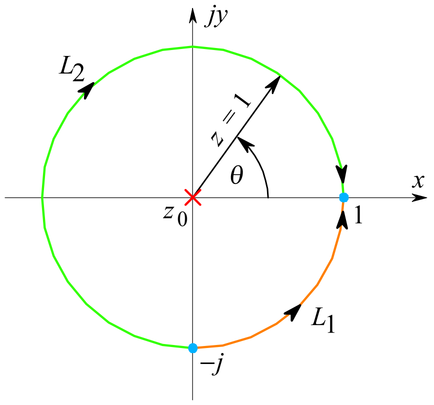

Fig. 1.8.5: The integral along the path L_1 is not equal to the integral along the path L_2 because the function has a pole which lies between both paths

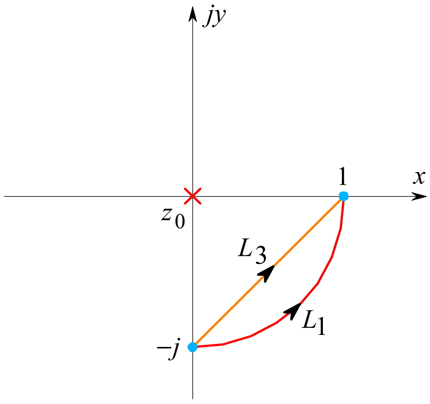

Fig. 1.8.6: The integral along the straight path L_3 is the same as the integral along the circular path L_1

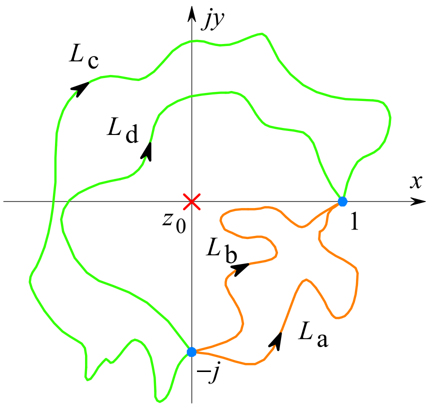

Fig. 1.8.7: The integrals along the paths L_a and L_b are equal to $j\pi/2$. However, those along L_c and L_d are equal to $-j3\pi/2$.

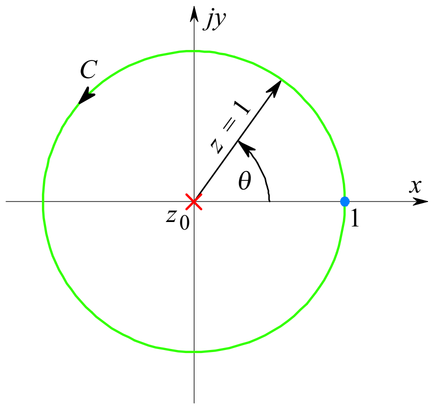

Fig. 1.8.8: The integral along the circular path C around the pole is $2\pi j$. See Sec.1.9.

(blank page)

1.9 Contour Integrals

Let us take again our familiar function $f(z) = 1/z$ and calculate the integral along the full circle C (Fig. 1.8.8), where $|z| = 1$. We use the same notation for z and dz as we did in Example 4 and start the integration at $\theta = 0$, going counterclockwise (positive by definition):

$$\int_0^{2\pi} \frac{dz}{z} = j\int_0^{2\pi} \frac{e^{j\theta}\,d\theta}{e^{j\theta}} = j\int_0^{2\pi} d\theta = 2\pi j = \oint_C \frac{dz}{z} \tag{1.9.1}$$

The resulting integral along the circle C is called *the contour integral*; the arrow in the symbol indicates the direction of encircling the pole (at $z = 0$).

Now let us move the pole from the origin to the point $a = x_a + j\,y_a$. The corresponding function is then $f(z) = 1/(z - a)$. The first attempt would be to integrate along the contour C as shown in Fig. 1.9.1. Inside this contour the domain of the function is analytic, except for the point a. Unfortunately C is a random contour and can not be expressed in a convenient mathematical way. Since a is the only pole inside the contour C, we may select another, simpler integration path. As we have already mastered the integration around a circular path, we select a circle C_c with the radius ε that lies inside the contour C. From Fig. 1.9.1 it is evident that:

$$\varepsilon = |z - a| \quad \text{or} \quad z - a = \varepsilon\, e^{j\theta} \tag{1.9.2}$$

Thus:

$$z = \varepsilon\, e^{j\theta} + a \tag{1.9.3}$$

where the angle θ can have any value in the range $0 \dots 2\pi$. Furthermore it follows that:

$$dz = j\,\varepsilon\, e^{j\theta} d\theta \tag{1.9.4}$$

The contour integral around the pole a is then:

$$\oint_{C_c} \frac{dz}{z - a} = \int_0^{2\pi} \frac{j\,\varepsilon\, e^{j\theta} d\theta}{\varepsilon\, e^{j\theta}} = j\int_0^{2\pi} d\theta = 2\,\pi\,j \tag{1.9.5}$$

The result is the same as we have obtained for the function $f(z) = 1/z$, in which the pole was at the origin of the z plane.

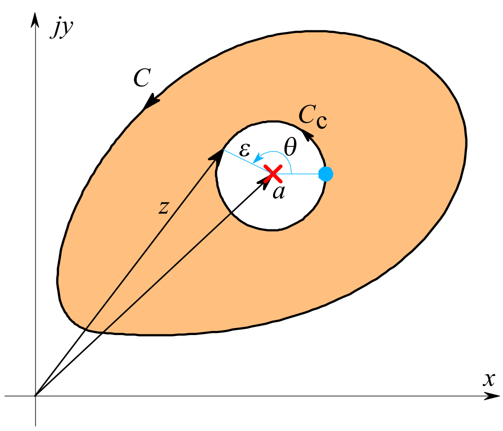

Fig. 1.9.1: Contour integral around the pole at a.

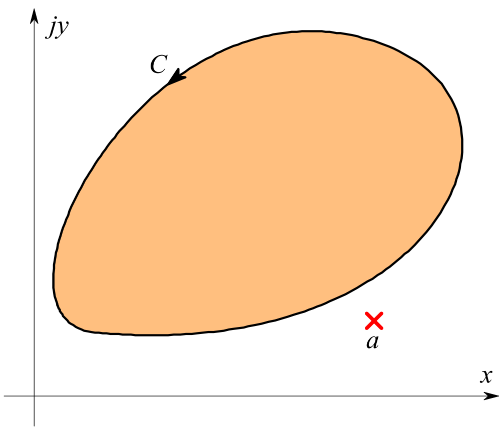

Fig. 1.9.2: The integral around the contour C is zero because a, the only pole of the function, lies outside the contour.

We look again at Fig. 1.8.3, where the integral around the path L_1 is equal to the integral around the path L_2 because there is no pole between L_1 and L_2. It would be interesting to make the integral from z_1 to z_2 along the path L_2 and then back again from z_2 to z_1 along the path L_1, making a closed loop (contour) integral:

$$\underbrace{\int_{z_1}^{z_2} f(z)\,dz}_{\text{along } L_2} + \underbrace{\int_{z_2}^{z_1} f(z)\,dz}_{\text{along } L_1} = \underbrace{\int f(z)\,dz}_{\text{along } L_2 + L_1} = 0 \tag{1.9.6}$$

Since both integrals have the same magnitude, by exchanging the limits of the second integral, thus making it negative, their sum is zero. This statement affords us the conclusion that the integral around the contour C in Fig. 1.9.2, which encircles an area where the function is analytic, is zero (the only pole a in the vicinity lies outside the contour of integration). This is expressed as:

$$\oint_C f(z)\,dz = 0 \tag{1.9.7}$$

The expressions in Eq. 1.9.6 and 1.9.8 were derived by the French mathematician *Augustine Louis Cauchy* (1788–1857). In all the calculations so far we have integrated in a counterclockwise sense, having the integration field, including the pole, always on the **left** side. In the case of a clockwise direction, let us again take Eq. 1.9.1 and integrate clockwise from 2π to 0:

$$\int_{2\pi}^{0} \frac{dz}{z} = j\int_{2\pi}^{0} \frac{\mathrm{e}^{j\theta}\,d\theta}{\mathrm{e}^{j\theta}} = j\int_{2\pi}^{0} d\theta = -2\pi j = \oint_C \frac{dz}{z} \tag{1.9.8}$$

Note that the sign of the result changes if we change the direction of encircling. So we may write in general:

$$\oint_C f(z)\,dz = -\oint_C f(z)\,dz \tag{1.9.9}$$

1.10 Cauchy's Way of Expressing Analytic Functions

Let us take a function $f(z)$ which is analytic inside a contour C. There are no regulations for the nature of $f(z)$ outside the contour, where $f(z)$ may also have poles. So this function is analytic also at the point a (inside C) where its value is $f(a)$.

Now we form another function:

$$g(z) = \frac{f(z)}{z-a} \tag{1.10.1}$$

This function is also analytic inside the contour C, except at the point a, where it has a pole, as shown in Fig. 1.10.1. Let us take the integral around the closed contour C:

$$\oint_C \frac{f(z)}{z-a}\,dz \tag{1.10.2}$$

which is similar to the integral in Eq. 1.9.5, except that here we have $f(z)$ in the numerator. Because at the point a the function under the integral is not analytic, the path of integration must avoid this point. Therefore we go around it along a circle of the radius ε, which can be made as small as required (but not zero).

For the path of integration we shall use the required contour C **and** the circle C_c. To make the closed contour the complete integration path will start at point 1 and go **counterclockwise** around the contour C to come back to the point 1; then from the point 1 to the point 2 along the dotted line; then **clockwise** around the circle C_c back to the point 2; and finally from the point 2 back to the point 1 along the dotted line. In this way, the contour of integration is closed. The integral from the point 1 to 2 and back again is zero. Thus there remain only the integrals around the contour C and around the circle C_c.

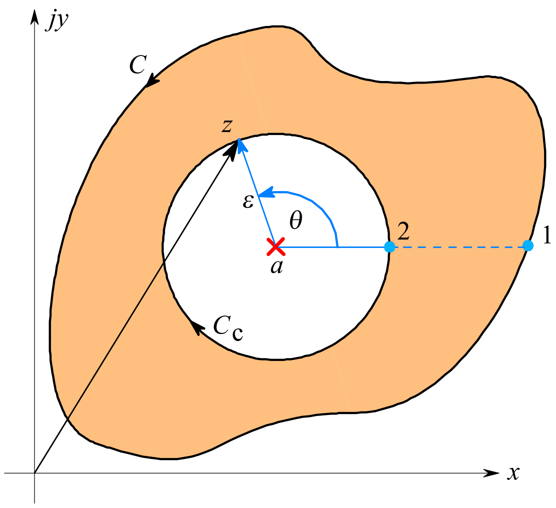

Fig. 1.10.1: Cauchy's method of expressing analytic functions (see text).

Since around the complete integration path the domain on the left hand side of the contour was always analytical, the resulting integral must be zero. Thus:

$$\oint\limits_{C-C_c} \frac{f(z)}{z-a}\,dz = \oint\limits_{C} \frac{f(z)}{z-a}\,dz + \oint\limits_{C_c} \frac{f(z)}{z-a}\,dz = 0 \tag{1.10.3}$$

and so it follows that:

$$\oint\limits_{C} \frac{f(z)}{z-a}\,dz = \oint\limits_{C_c} \frac{f(z)}{z-a}\,dz \tag{1.10.4}$$

Here we have changed the second integral sign by reversing the sense of encircling.

Similarly as in Eq. 1.9.2 and 1.9.4 we write:

$$z-a = \varepsilon\,\mathrm{e}^{j\theta} \qquad dz = j\,\varepsilon\,\mathrm{e}^{j\theta}d\theta \qquad \text{and} \qquad \frac{dz}{z-a} = j\,d\theta$$

Nothing would change if in Eq. 1.10.4 we write:

$$f(z) = f(a) + [f(z) - f(a)] \tag{1.10.5}$$

thus obtaining:

$$\oint\limits_{C} \frac{f(z)}{z-a}\,dz = j\,f(a)\int_0^{2\pi} d\theta + j\int_0^{2\pi} [f(z)-f(a)]\,d\theta \tag{1.10.6}$$

The integration must go from 0 to 2π in order to encircle the point a in the required direction. The value of the first integral on the right is:

$$j\,f(a)\int_0^{2\pi} d\theta = 2\pi j\,f(a) \tag{1.10.7}$$

and we will prove that the second integral is zero. Its magnitude is:

$$M < 2\pi \max\{|f(z) - f(a)|\} \tag{1.10.8}$$

The function $f(z)$ is continuous everywhere inside the field bordered by C and C_c; therefore the point z can be as close to the point a as desired. Consequently $|f(z) - f(a)|$ may also be as small as desired. The radius of the circle C_c inside the contour C is $\varepsilon = |z-a|$, and in Eq. 1.9.5 we have already observed that the value of the integral is independent of ε. If we take the limit $\varepsilon \to 0$ we obtain:

$$\lim_{\varepsilon\to 0} \int_0^{2\pi} [f(z) - f(a)]\,d\theta = 0 \tag{1.10.9}$$

Thus:

$$2\pi j\,f(a) = \oint\limits_{C} \frac{f(z)\,dz}{z-a} \tag{1.10.10}$$

and:

$$\boxed{f(a) = \frac{1}{2\pi j}\oint\limits_{C} \frac{f(z)\,dz}{z-a}} \tag{1.10.11}$$

where the point a may be any point inside the contour C.

Eq. 1.10.11 is of essential importance for the inverse Laplace transform; we name it *Cauchy's expression for an analytic function*. By means of this integral it is possible to calculate the value of an analytic function at any desired point (say, a) if all the values on the contour surrounding this point are known. Thus if:

$$g(z) = \frac{f(z)}{z - a}$$

then the value $f(a)$ is called the *residue* of the function $g(z)$ for the pole a.

To make the term 'residue' clear let us make a practical example. Suppose $g(z)$ is a rational function of two polynomials:

$$g(z) = \frac{P(z)}{Q(z)} = \frac{z^m + b_{m-1} z^{m-1} + b_{m-2}\, z^{m-2} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + a_{n-2}\, z^{n-2} + \cdots + a_1 z + a_0} \tag{1.10.12}$$

where b_{i} and a_{i} are real constants and $n > m$. Eq. 1.10.12 represents a general form of a frequency response of an amplifier, where z can be replaced by the usual $s = \sigma + j\omega$ and b_0/a_0 is the DC amplification (at frequency $s = 0$). Instead of the sums, the polynomials $P(z)$ and $Q(z)$, and thus $g(z)$, may also be expressed in the product form:

$$g(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_2) \cdots (z - p_n)} \tag{1.10.13}$$

In this equation, $z_1,\ z_2, \dots,\ z_m$ are the roots of the polynomial $P(z)$, so they are also the zeros of $g(z)$. Similarly, $p_1,\ p_2, \dots,\ p_n$ are the roots of the polynomial $Q(z)$ and therefore also the poles of $g(z)$. Both statements are valid if $p_{\mathrm{i}} \neq z_{\mathrm{i}}$ for any i that can be applied to Eq. 1.10.13 (if $z - z_1$ were equal to, say, $z - p_3$, there would be no pole at p_3, because this pole would be canceled by the zero z_1). Now we factor out the term with one pole, i.e., $1/(z - p_2)$ and write:

$$g(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)(z - p_3) \cdots (z - p_n)} \cdot \frac{1}{(z - p_2)} = f(z) \frac{1}{(z - p_2)} \tag{1.10.14}$$

where:

$$f(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_m)}{(z - p_1)\,(z - p_3) \cdots (z - p_n)} \tag{1.10.15}$$

If we focus only on $f(z)$ and let $z \to p_2$, we obtain the residue of the function $g(z)$ for the pole p_2 and this residue is equal to $f(p_2)$. Since we have taken the second pole we have appended the index '2' to the residue. By performing the suggested operation we obtain:

$$\mathrm{res}_2\, g(z) = \lim_{z \to p_2} (z - p_2)\, g(z) = f(p_2) \tag{1.10.16}$$

The word 'residue' is of Latin origin and means the *remainder*. However, since a remainder may also appear when we divide a polynomial by another, we shall keep using the expression 'residue' in order to avoid any confusion. Also in our further practical calculations we will simply write, say, res_2 instead of the complete expression $\mathrm{res}_2\, F(s)$.

The reader could obtain a rough idea of a residue by the following similarity: suppose we have a big circus tent, the canvas of which is supported by, say, four poles. If one of the poles is removed, the canvas sags. The height of the canvas above the ground where we have removed the pole is something similar to a residue for that pole. However, in this comparison two important facts are different: first, in the complex variable theory our 'canvas' as well as 'ground' are complex and, second, the poles are infinitely high (actually $+\infty$ on one side of the pole and $-\infty$ on the other; see Appendix 1 on the disk).

In the following examples we shall see that the calculation of residues is a relatively simple matter.

From now on we shall replace the variable z by our familiar complex variable $s = \sigma + j\omega$. Also, in order to distinguish more easily the functions of complex frequency from functions of time, we shall write the former with capitals, like $F(s)$ or $G(s)$ and the later with small letters, like $f(t)$ or $g(t)$.

To prove that the calculation of residues is indeed a simple task let us calculate two examples.

1.10.1 Example 1

Let us take a function: $$F(s) = \frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)}$$

We need to calculate the three residues of $F(s)$ for the poles at $s=-4$, $s=-5$ and $s=-6$:

$$\mathrm{res}_1 = \lim_{s\to-4}(s+4)\,\frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)} = \frac{(-4+2)(-4+3)}{(-4+5)(-4+6)} = 1$$

and in a similar way:

$$\mathrm{res}_2 = \lim_{s\to-5}(s+5)\,\frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)} = -6$$

$$\mathrm{res}_3 = \lim_{s\to-6}(s+6)\,\frac{(s+2)(s+3)}{(s+4)(s+5)(s+6)} = 6$$

An interesting fact here is that since all the poles are real, all the residues are real as well; in other words, a real pole causes the residue of that pole to be real.

1.10.2 Example 2

Our function is:

$$F(s) = \frac{(s+2)\,\mathrm{e}^{st}}{3\,s^2+9\,s+9}$$

Here we must consider that the variable of the function $F(s)$ is only s and not t. First we tackle the denominator to find both roots, which are the poles of our function:

$$3\,s^2+9\,s+9 = 3\,(s^2+3\,s+3) = 0$$

Thus:

$$s_{1,2} = \sigma_1 \pm j\,\omega_1 = -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - 3} \; = -\frac{3}{2} \pm j\,\frac{\sqrt{3}}{2}$$

and by expressing the function $F(s)$ with both poles we have:

$$F(s) = \frac{(s+2)\,\mathrm{e}^{st}}{3\,(s-s_1)(s-s_2)}$$

We shall carry out a general calculation of the two residues and then introduce the numerical values for σ_1 and ω_1.

$$\mathrm{res}_1 = \lim_{s \to s_1} (s-s_1)\,\frac{(s+2)\,\mathrm{e}^{st}}{3\,(s-s_1)(s-s_2)} = \frac{(s_1+2)\,\mathrm{e}^{st}}{3\,(s_1-s_2)}$$

$$= \frac{(\sigma + j\,\omega + 2)\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}}{3\,(\sigma_1 + j\,\omega_1 - \sigma_1 + j\,\omega_1)} = \frac{(\sigma + j\,\omega + 2)\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}}{6\,j\,\omega_1}$$

We now set $\sigma_1 = -3/2$ and $\omega_1 = \sqrt{3}\,/2$ to obtain the numerical value of the residue:

$$\mathrm{res}_1 = \frac{\left(-3/2 + j\,\sqrt{3}\,/2 + 2\right)\,\mathrm{e}^{-3\,t/2}\,\mathrm{e}^{j\sqrt{3}\,t/2}}{6\,j\,\sqrt{3}}$$

$$= \frac{1 + j\,\sqrt{3}}{12\,j\,\sqrt{3}}\,\mathrm{e}^{-3\,t/2}\,\mathrm{e}^{j\sqrt{3}\,t/2} = \frac{\sqrt{3}\, - j}{12\,\sqrt{3}}\,\mathrm{e}^{-3\,t/2}\,\mathrm{e}^{j\sqrt{3}\,t/2}$$

In a similar way we calculate the second residue:

$$\mathrm{res}_2 = \lim_{s \to s_2} (s-s_2)\,\frac{(s+2)\,\mathrm{e}^{st}}{3\,(s-s_1)(s-s_2)} = \frac{(s_2+2)\,\mathrm{e}^{st}}{3\,(s_2-s_1)}$$

$$= \frac{\sqrt{3}\, + j}{12\,\sqrt{3}}\,\mathrm{e}^{-3\,t/2}\,\mathrm{e}^{-j\sqrt{3}\,t/2}$$

Since both poles are complex conjugate, both residues are complex conjugate as well. In rational functions, which will appear in the later sections, all the poles will be either real, or complex conjugate, or both. Therefore **the sum of all residues of these functions (that is, the time function) will always be real.**

(blank page)

1.11 Residues of Functions with Multiple Poles, the Laurent Series

When a function contains multiple poles it is not possible to calculate the residues in the way shown in the previous section. As an example let us take the function:

$$G(s) = \frac{F(s)}{(s-a)^n} \tag{1.11.1}$$

To calculate the residue we first expand $F(s)$ into a *Taylor* series [Ref. 1.4, 1.11]:

$$F(s) = (s-a)^n\, G(s) \tag{1.11.2}$$

$$= \frac{F(a)}{0!} + \frac{F(a)'(s-a)}{1!} + \frac{F(a)''(s-a)^2}{2!} + \cdots + \frac{F(a)^{(n-1)}(s-a)^{n-1}}{(n-1)!} + \cdots$$

Now we divide all the fractions in this equation by $(s-a)^n$ (considering that $0! = 1$ by definition):

$$G(s) = \frac{F(s)}{(s-a)^{\,n}} \tag{1.11.3}$$

$$= \frac{F(a)}{(s-a)^n} + \frac{F'(a)}{(s-a)^{n-1}} + \frac{F''(a)}{2!(s-a)^{n-2}} + \cdots + \frac{F^{(n-1)}(a)}{(n-1)!\,(s-a)} + \cdots$$

The values $F(a)$, $F'(a)$, $F''(a)/2!$, $\ldots$, $F^{(n-1)}(a)/(n-1)!$, $\ldots$ are constants and we write them as A_{-n}, $A_{-(n-1)}$, $A_{-(n-2)}$, $\ldots$, A_{-1}, A_0, A_1, A_2, $\ldots$.

We may now express the function $G(s)$ as:

$$G(s) = \frac{A_{-n}}{(s-a)^n} + \frac{A_{-(n-1)}}{(s-a)^{n-1}} + \frac{A_{-(n-2)}}{(s-a)^{n-2}} + \cdots + \frac{A_{-1}}{(s-a)}$$

$$+ A_0 + A_1(s-a) + A_2(s-a)^2 + \cdots \tag{1.11.4}$$

The sum of all fractions from the above function we call the *principal part* and the rest is the *analytic part* (also known as the *regular part*).

Eq. 1.11.4 is named the *Laurent series*, after the French mathematician *Pierre-Alphonse Laurent*, 1813–1854, who in 1843 described “a series with negative powers”.

A general expression for the Laurent series is:

$$\boxed{F(s) = \sum_{n=-m}^{+\infty} A_n\,(s-a)^n} \tag{1.11.5}$$

where m and n are integers.

Let us calculate the contour integral of the above function:

$$\oint_C F(s)\,ds = \oint_C \sum_{n=-m}^{+\infty} A_n\,(s-a)^n\,ds \tag{1.11.6}$$

We shall integrate each part of the series separately.

Again, $A_{-n},\ \dots,\ A_{-2},\ A_{-1},\ A_0,\ A_1,\ A_2,\ \dots$ are constants and they may be put in front of the integral sign. In general we need to know the solution of the integral:

$$\oint_C (s-a)^n\,ds \tag{1.11.7}$$

where n is an integer either positive or negative. As in Eq. 1.9.2 we write again:

$$s-a=\varepsilon\,\mathrm{e}^{j\theta} \qquad \Rightarrow \qquad ds = j\,\varepsilon\,\mathrm{e}^{j\theta}d\theta \qquad \Rightarrow \qquad (s-a)^n = \varepsilon^n\,\mathrm{e}^{j\,n\,\theta}$$

where the radius ε is considered to be constant. Thus:

$$\oint_C (s-a)^n\,ds = \int_0^{2\pi} \left(\varepsilon^n\,\mathrm{e}^{j\,n\,\theta}\right) j\,\varepsilon\,\mathrm{e}^{j\theta}d\theta = j\,\varepsilon^{n+1}\int_0^{2\pi} \mathrm{e}^{j(n+1)\theta}\,d\theta \tag{1.11.8}$$

If $n \neq -1$, the result of this integration is:

$$\frac{j\,\varepsilon^{n+1}}{j\,(n+1)}\,\mathrm{e}^{j(n+1)\theta}\bigg|_{\theta=0}^{\theta=2\pi} = \frac{\varepsilon^{n+1}}{(n+1)}\left[\mathrm{e}^{j(n+1)2\pi}-1\right]=0 \tag{1.11.9}$$

because $\mathrm{e}^{j\,k\,2\pi}=1$, for any positive or negative integer k, including 0. For $n=-1$, we derive from Eq. 1.11.8:

$$\oint_C \frac{ds}{s-a} = \int_0^{2\pi} \frac{j\,\varepsilon\,\mathrm{e}^{j\theta}\,d\theta}{\varepsilon\,\mathrm{e}^{j\theta}} = 2\pi j \tag{1.11.10}$$

In order that the result corresponds to the Laurent series we must add the constant factor with $n=-1$ and this is A_{-1}. Eq. 1.11.8 to 1.11.10 prove that the contour integration for the complete Laurent series $G(s)$ yields only:

$$\boxed{\oint_C G(s)\,ds = A_{-1}2\pi j} \tag{1.11.11}$$

Thus from the whole series (Eq. 1.11.4) only the part with A_{-1} remained after the integration. If we return to Eq. 1.11.3 we conclude that:

$$A_{-1} = \frac{1}{2\pi j}\oint_C G(s)\,ds = \operatorname{res}\frac{F(s)}{(s-a)^n} = \frac{F^{(n-1)}(a)}{(n-1)!}$$

$$= \lim_{s\to a}\frac{1}{(n-1)!}\left[\frac{d^{(n-1)}}{ds^{(n-1)}}\,(s-a)^n\,G(s)\right] \tag{1.11.12}$$

is the residue of the function $G(s)=F(s)/(s-a)^n$ for the pole a. The following examples will show how we calculate the residues for multiple poles in practice.

1.11.1 Example 1

We take a function:

$$G(s) = \frac{F(s)}{(s-a)^3}$$

Our task is to calculate the general expression for the residue of the triple pole $(n = 3)$ at $s = a$. According to Eq. 1.11.12 it is:

$$\begin{aligned} \mathrm{res} &= \lim_{s \to a} \frac{1}{(3-1)!} \left[\frac{d^{(3-1)}}{ds^{(3-1)}} (s-a)^3 \, G(s) \right] \\ &= \frac{1}{2} \left[\frac{d^2}{ds^2} (s-a)^3 \, G(s) \right]_{s \to a} \end{aligned}$$

1.11.2 Example 2

Here we shall calculate with numerical values.

We intend to find the residues for the double pole at $s = -2$ and for the single pole at $s = -3$ of the function:

$$F(s) = \frac{5}{(s+2)^2 \, (s+3)}$$

Solution:

$$\begin{aligned} \mathrm{res}_1 &= \lim_{s \to -2} \frac{1}{(2-1)!} \left[\frac{d^{(2-1)}}{ds^{(2-1)}} (s+2)^2 \, \frac{5}{(s+2)^2 \, (s+3)} \right] \\ &= \left[\frac{d}{ds} \, \frac{5}{(s+3)} \right]_{s \to -2} = \left[\frac{-5}{(s+3)^2} \right]_{s \to -2} = -5 \end{aligned}$$

$$\mathrm{res}_2 = \lim_{s \to -3} \left[(s+3) \, \frac{5}{(s+2)^2 \, (s+3)} \right] = \left[\frac{5}{(s+2)^2} \right]_{s \to -3} = 5$$

It is important to remember the required order of the operations: first we multiply by the expression containing the multiple pole and then find the derivative. To do it the opposite way is wrong! Finally, we insert the numerical value for the pole.

(blank page)

1.12 Complex Integration Around Many Poles: the Cauchy–Goursat Theorem

So far we have calculated a contour integral around one pole (simple or multiple). Now we will integrate around more poles, either single or multiple.

Cheese is a regular part of French meals. So we may imagine that the great mathematician Cauchy observed a slice of Emmentaler cheese like that in Fig. 1.12.1 (the characteristics of this cheese is big holes) on his plate and reflected in the following way:

Suppose all that is cheese is an analytic (regular) domain R of a function $F(s)$. In the holes are the poles $s_1, \dots, s_5$. We are not interested in the domain outside the cheese. How could we 'mathematically' encircle the cheese around the crust and around the rims of all the holes, so that the cheese is always on the left side of the contour?

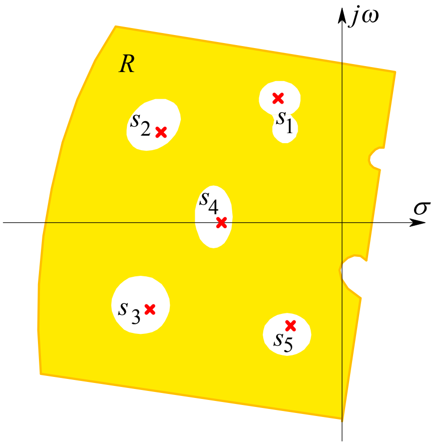

Fig. 1.12.1: The cheese represents a regular (analytic) domain R of a function which has one simple pole in each hole.

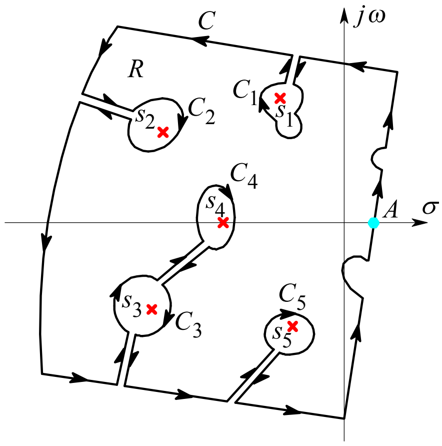

Fig. 1.12.2: Encircling the poles by contours $C, C_1, \dots, C_5$, so that the regular domain of the function is always on the left side.

Impossible? No! If we take a knife and make a cut from the crust towards each hole without removing any cheese, we provide the necessary path for the suggested contour, as shown in Fig. 1.12.2.

Now we calculate a contour integral starting from the point A in the suggested (counterclockwise) direction until we come to the cut towards the first pole, s_1. We follow the cut towards contour C_1, follow it around the pole and then go along the cut again, back to the crust. We continue around the crust up to the cut of the next pole and so on, until we arrive back to point A and close the contour. Since we have not removed any cheese in making the cuts, the paths from the crust to the corresponding hole and back again cancel out in this integration path. As we have proved by Eq. 1.9.5:

$$\int_a^b F(s)\,ds + \int_b^a F(s)\,ds = 0$$

Therefore, only the contour C around the crust and the small contours $C_1, \dots, C_5$ around the rims of the holes containing the poles are what we must consider in the integration around the contour in Fig. 1.12.2. The contour C was taken **counterclockwise**, whilst the contours $C_1, \dots, C_5$ were taken **clockwise**.

We write down the complete contour integral:

$$\oint_C F(s)\,ds + \oint_{C_1} F(s)\,ds + \cdots + \oint_{C_5} F(s)\,ds = 0 \tag{1.12.1}$$

The result of integration is zero because along this circuitous contour of integration we have had the regular domain always on the left side. By changing the sense of encircling of the contours $C_1, \dots, C_5$ we may write Eq. 1.12.1 also in the form:

$$\oint_C F(s)\,ds = \oint_{C_1} F(s)\,ds + \cdots + \oint_{C_5} F(s)\,ds \tag{1.12.2}$$

When we changed the sense of encircling, we changed the sign of the integrals; this allows us to put them on the right hand side with a positive sign. Now all the integrals have positive (counterclockwise) encircling. Therefore the integral encircling all the poles is equal to the sum of the integrals encircling each particular pole.

By observing this equation we realize that the right hand side is the sum of residues for all the five poles, multiplied by $2\pi j$. Thus for the general n-pole case the Eq. 1.12.2 may also be written as:

$$\boxed{\oint_C F(s)\,ds = 2\pi j\,[\,\mathrm{res}_1 + \cdots + \mathrm{res}_n] = 2\pi j \sum_{i=1}^{n} \mathrm{res}_i} \tag{1.12.3}$$

Eq. 1.12.2 and 1.12.3 are called the *Cauchy–Goursat theorem*; they are of essential importance for the inverse Laplace transform.

1.13 Equality of the Integrals $\oint_C F(s)\,e^{st}\,ds$ and $\int_{c-j\infty}^{c+j\infty} F(s)\,e^{st}\,ds$

The reader is invited to examine Fig. 1.13.1, where the function $|F(s)| = |1/s|$ was plotted. The function has one simple pole at the origin of the complex plane. The resulting surface has been cut between $-j$ and 1 to expose an arbitrarily chosen integration path L between $s_1 = x_1 + j\,y_1 = 0 - j\,0.5$ and $s_2 = x_2 + j\,y_2 = 0.5 + j\,0$ (see the integration path in the plot of the s domain in Fig. 1.13.2).

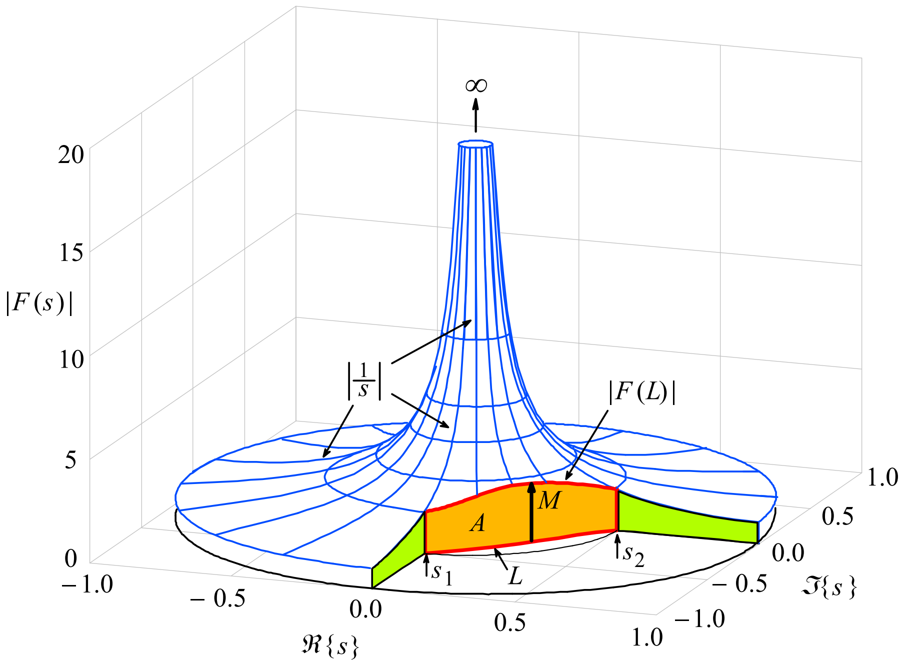

Fig. 1.13.1: The complex function magnitude, $|F(s)| = |1/s|$. The resulting surface has been cut between $-j$ and $+1$ to expose an arbitrarily chosen integration path L, starting at $s_1 = 0 - j\,0.5$ and ending at $s_2 = 0.5 + j\,0$. On the path of integration the function $|F(s)|$ has a maximum value M.

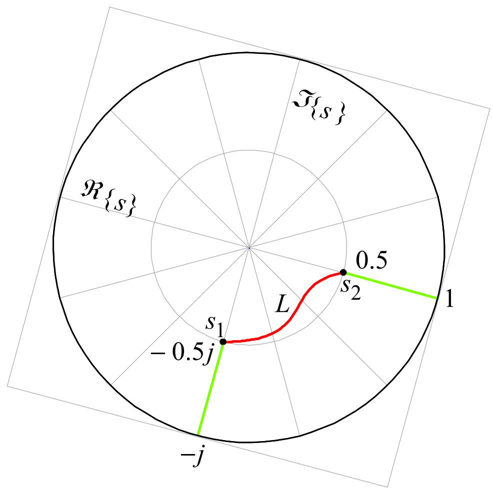

Fig. 1.13.2: The complex domain of Fig. 1.13.1 shows the arbitrarily chosen integration path L, which starts at $s_1 = 0 - j\,0.5$ and ends at $s_2 = 0.5 + j\,0$.

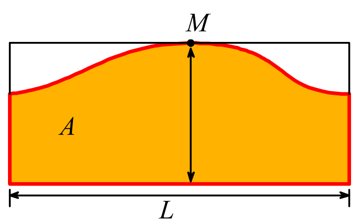

Fig. 1.13.3: The area A of Fig. 1.13.1 has been laid flat in order to show that it must be smaller than or at best equal to the area of the rectangle $M \times L$.

Let us take a closer look at the area A between s_1, s_2, $|F(s_1)|$ and $|F(s_2)|$, shown in Fig 1.13.3. The area A corresponds to the integral of $F(s)$ from s_1 to s_2 and it can be shown that it is always smaller than, or at best equal to, the rectangle $M \times L$:

$$\underbrace{\left|\int_{s_1}^{s_2} F(s)\,ds\right| = \left|\int_{s_1}^{s_2} \frac{ds}{s}\right| \le \int_{s_1}^{s_2} \frac{|ds|}{|s|}}_{\text{along the path } L} \le \int_{s_1}^{s_2} M\,|ds| = M L \tag{1.13.1}$$

Here M is the greatest value of $|F(s)|$ for this particular path of integration L, as shown in Fig. 1.13.3, in which the resulting 3D area between s_1, s_2, $|F(s_1)|$ and $|F(s_2)|$ was stretched flat. So:

$$\left|\int_{s_1}^{s_2} F(s)\,ds\right| \le \int_{s_1}^{s_2} |F(s)\,ds| \le M L \tag{1.13.2}$$

Eq. 1.13.2 is an essential tool in the proof of the inverse $\mathcal{L}$ transform via the integral around the closed contour.

Let us now move to network analysis, where we have to deal with rational functions of the complex variable $s = \sigma + j\,\omega$. These functions have a general form:

$$F(s) = \frac{s^m + b_{m-1}\,s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}\,s^{n-1} + \cdots + a_1 s + a_0} \tag{1.13.3}$$

where $m < n$ and both are positive and real. Since we can also express $s = R\,\mathrm{e}^{j\theta}$ (as can be derived from Fig. 1.13.4), we may write Eq. 1.13.3 also in the form:

$$F(s) = \frac{R^m\,\mathrm{e}^{j\,m\,\theta} + b_{m-1}\,R^{m-1}\,\mathrm{e}^{j(m-1)\theta} + \cdots + b_1 R\,\mathrm{e}^{j\,\theta} + b_0}{R^n\,\mathrm{e}^{j\,n\,\theta} + a_{n-1}\,R^{n-1}\,\mathrm{e}^{j(n-1)\theta} + \cdots + a_1 R\,\mathrm{e}^{j\,\theta} + a_0} \tag{1.13.4}$$

According to Eq. 1.13.2 and 1.13.4 we have:

$$\big|F(s)\big| = \left|\frac{R^m\,\mathrm{e}^{j\,m\,\theta} + \cdots + b_0}{R^n\,\mathrm{e}^{j\,n\,\theta} + \cdots + a_0}\right| \le \frac{K}{R^{n-m}} = M \tag{1.13.5}$$

where K is a real constant and M is the maximum value of $\big|F(s)\big|$ within the integration interval, according to Fig. 1.13.1 and 1.13.3 (in [Ref. 1.10, p. 212] the interested reader can find the complete derivation of the constant K).

$$s_1 = \sigma_1 + j\,\omega_1 = R\,\mathrm{e}^{j\theta}$$
$$\sigma_1 = R\cos\theta \qquad \omega_1 = R\sin\theta$$
$$R = \sqrt{(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)}$$
$$= \sqrt{\sigma_1^2 + \omega_1^2}$$
$$\theta = \arctan\frac{\omega_1}{\sigma_1}$$

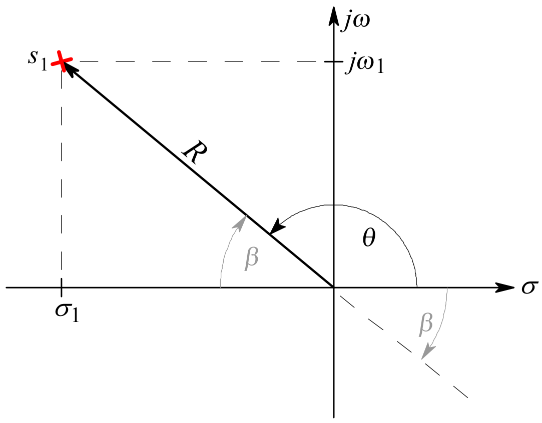

Fig. 1.13.4: Cartesian and polar representations of a complex number (note: $\tan\theta$ is equal for the counterclockwise defined θ from the positive real axis and for the clockwise defined $\beta = \theta - \pi$).

Let us draw the poles of Eq. 1.13.3 in the complex plane to calculate the integral around an inverted 'D' shaped contour, as shown in Fig. 1.13.5 (for convenience only three poles have been drawn there). Since Eq. 1.13.3 is assumed to describe a real passive system, all poles must lie either on the left side of the complex plane or at the origin. As we know, the integral around the closed contour embracing all the poles is equal to the sum of residues of the function $F(s)$:

$$\oint_L F(s)\,ds = \int_{\sigma_a - j\omega_1}^{\sigma_a + j\omega_1} F(s)\,ds + \int_\Gamma F(s)\,ds = 2\pi j \sum_{i=1}^{n} \mathrm{res}_i \tag{1.13.6}$$

The contour has two parts: the straight line from $\sigma_a - j\omega_1$ to $\sigma_a + j\omega_1$, where σ_a is a constant (which we will define more exactly later) and the arc $\Gamma = R\gamma$, where γ is the arc angle and R is its radius. According to Eq. 1.13.2, the line integral along the path L is:

$$\left| \oint_L F(s)\,ds \right| \le ML \tag{1.13.7}$$

where M is the maximum value of the integral (magnitude!) on the path L. In our case:

$$M = \frac{K}{R^{n-m}} \quad \text{and} \quad L = \gamma\, R = \Gamma \tag{1.13.8}$$

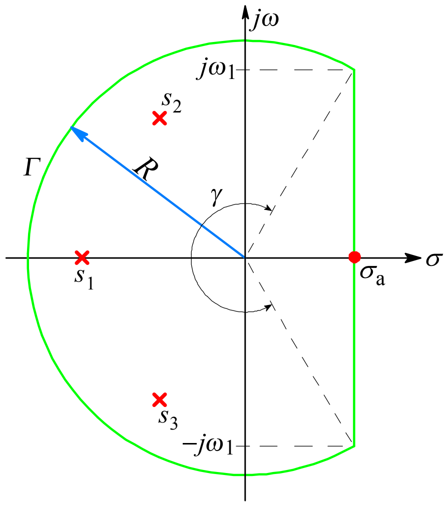

Fig. 1.13.5: The integral along the inverted ‘D’ shaped contour encircling the poles is equal to the sum of the residues of each pole. This contour is used to prove the inverse Laplace transform, where the integral along the arc vanishes if $R \to \infty$, provided that the number of poles exceeds the number of zeros by at least 2 (in this example no zeros are shown).

For a very large R we may write:

$$\left| \int_\Gamma F(s)\,ds \right| \le \frac{K}{R^{n-m}}\,\gamma\, R = \frac{K\gamma}{R^{n-m-1}} \tag{1.13.9}$$

If $R \to \infty$:

$$\lim_{R\to\infty} \frac{K\gamma}{R^{n-m-1}} = 0 \quad \textbf{only if} \quad n - m \ge 2 \tag{1.13.10}$$

and this procedure is called *Jordan’s lemma*.

If the condition of Eq. 1.13.10 holds, only the straight part of the contour counts because if $R \to \infty$ then also $\omega_1 \to \infty$, thus changing the limits of the integral along the straight path accordingly. If we make these changes to Eq. 1.13.6, it shrinks to:

$$\int_{\sigma_\mathrm{a}-j\infty}^{\sigma_\mathrm{a}+j\infty} F(s)\,ds = \oint_L F(s)\,ds = 2\pi j \sum_{i=1}^{n} \mathrm{res}_i \tag{1.13.11}$$

The function $F(s)$ may also contain the factor e^{st}, where $\Re(s) \geq \sigma_\mathrm{a}$ and $t \geq 0$. In this case the constant σ_a, which is called *the abscissa of absolute convergence* [Ref. 1.3, 1.5, 1.8], must be small enough to ensure the convergence of the integral. The factor e^{st} is always present in the inverse $\mathcal{L}$ transform. Let us write this factor down and let us divide Eq. 1.13.11 by the factor $2\pi j$. In this way the integral obtains the form:

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j} \int_{\sigma_\mathrm{a}-j\infty}^{\sigma_\mathrm{a}+j\infty} F(s)\,\mathrm{e}^{st}\,ds = \sum \mathrm{res}\left\{F(s)\,\mathrm{e}^{st}\right\}} \tag{1.13.12}$$

and this is the formula for the inverse $\mathcal{L}$ transform [Ref. 1.3, 1.5, 1.8]. The above integral is convergent for $t \geq 0$, which is the usual constraint in passive network analysis. This constraint will also apply to all derivations which follow.

In the condition written in Eq. 1.13.10 we see that the order of the denominator's polynomial must exceed the order of the numerator by at least **two**, otherwise we could not prove the inverse $\mathcal{L}$ transform by the method derived above. This means that the number of poles must exceed the number of zeros by at least two. However, in network theory we often deal with the input functions called *positive real functions* [Ref. 1.16]. The degree of the denominator in these functions may exceed the degree in the numerator by **one** only. To prove the inverse $\mathcal{L}$ transform for such a case, we must reach for another method. The proof is possible by using a rectangular contour [Ref. 1.5, 1.13, 1.17]:

When the degree of the denominator exceeds the degree of the numerator by one only, Eq. 1.13.5 is reduced to:

$$\left|F(s)\right| \leq \frac{K}{R} = M \tag{1.13.13}$$

so to prove the inverse Laplace transform we use:

$$F(s) = \frac{1}{s - s_\mathrm{p}} = \frac{1}{s - \sigma_\mathrm{p}} \tag{1.13.14}$$

This is a single-pole function, with the pole on the negative real axis (for our calculations it is not essential that the pole lies on the real axis, but in the theory of real passive networks, a single-pole **always** lies either on the negative σ axis or at the origin of the complex plane).

The pole and the rectangular contour with the sides Γ_1, Γ_2, Γ_3 and Γ_4 are shown in Fig. 1.13.6. We will integrate around this rectangular contour. At the same time we let both $\sigma_1 \to \infty$ and $\omega_1 \to \infty$. Next we will prove, considering these limits, that the line integrals along the sides Γ_2, Γ_3 and Γ_4 are all equal to zero.

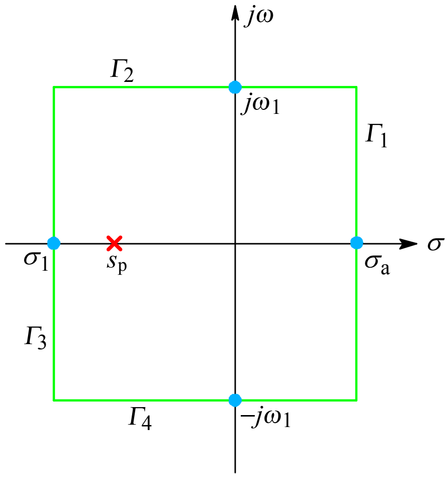

Fig. 1.13.6: By using a rectangular contour as shown it is possible to prove the inverse Laplace transform by means of the contour integral, even if the number of poles exceeds the number of zeros by only one. In this integral, encircling the single simple pole, we let $\sigma_1 \to \infty$ and $\omega_1 \to \infty$, so that the integrals along Γ_2, Γ_3 and Γ_4 vanish.

The proof must show that:

$$\oint_{\Gamma} F(s)\,\mathrm{e}^{st}\,ds = \lim_{\omega_1 \to \infty} \int_{\sigma_a - j\omega_1}^{\sigma_a + j\omega_1} F(s)\,\mathrm{e}^{st}\,ds = 2\pi j \sum \operatorname{res}\left\{F(s)\,\mathrm{e}^{st}\right\} \tag{1.13.15}$$

Here we will include the factor e^{st} (which always appears in the inverse $\mathcal{L}$ transform) at the very beginning, because it will help us in making the integral along Γ_3 convergent. Let us start with the integral along the side Γ_2, where ω_1 is constant:

$$\begin{aligned}\left|\int_{\Gamma_2} F(s)\,\mathrm{e}^{st}\,ds\right| &= \left|\int_{\sigma_a}^{-\sigma_1} F(\sigma + j\omega_1)\,\mathrm{e}^{(\sigma + j\omega_1)t}\,d\sigma\right| \\ &\le \int_{-\sigma_1}^{\sigma_a} \frac{K}{\sigma_1}\,\mathrm{e}^{\sigma t}\,d\sigma \\ &= \frac{K}{\sigma_1}\cdot\frac{1}{t}\left(\mathrm{e}^{\sigma_a t} - \mathrm{e}^{-\sigma_1 t}\right) \to 0\Big|_{\sigma_1 \to \infty}\end{aligned} \tag{1.13.16}$$

Since we are calculating the absolute value, we can exchange the limits of the last integral. The integral along Γ_4 is almost equal:

$$\begin{aligned}\left|\int_{\Gamma_4} F(s)\,\mathrm{e}^{st}\,ds\right| &= \left|\int_{-\sigma_1}^{\sigma_a} F(\sigma + j\omega_1)\,\mathrm{e}^{(\sigma + j\omega_1)t}\,d\sigma\right| \\ &\le \int_{-\sigma_1}^{\sigma_a} \frac{K}{\sigma_1}\,\mathrm{e}^{\sigma t}\,d\sigma \\ &= \frac{K}{\sigma_1}\cdot\frac{1}{t}\left(\mathrm{e}^{\sigma_a t} - \mathrm{e}^{-\sigma_1 t}\right) \to 0\Big|_{\sigma_1 \to \infty}\end{aligned} \tag{1.13.17}$$

In the integral along Γ_3, σ_1 is constant:

$$\left|\int_{\Gamma_3} F(s)\,\mathrm{e}^{st}\,ds\right| = \left|\int_{-j\omega_1}^{j\omega_1} F(-\sigma_1 + j\omega)\,\mathrm{e}^{(-\sigma_1+j\omega)t}\,d\omega\right|$$

$$\leq \int_{-\omega_1}^{\omega_1} \frac{K}{\sigma_1}\,\mathrm{e}^{-\sigma_1 t}\,d\omega$$

$$= \frac{K}{\sigma_1}\,\mathrm{e}^{-\sigma_1 t}(\omega_1 + \omega_1) \;\rightarrow 0\Big|_{\substack{\sigma_1\rightarrow\infty\\ \omega_1\rightarrow\infty}} \tag{1.13.18}$$

Since the integrals along Γ_2, Γ_3 and Γ_4 are all equal to zero if $\sigma_1 \rightarrow \infty$ and $\omega_1 \rightarrow \infty$, only the integral along Γ_1 remains, which, in the limit, is equal to the integral along the complete rectangular contour and, in turn, to the sum of the residues of the poles of $F(s)$:

$$\lim_{\omega_1\rightarrow\infty} \int_{\sigma_\mathrm{a}-j\omega_1}^{\sigma_\mathrm{a}+j\omega_1} F(s)\,\mathrm{e}^{st}\,ds = \int_{\sigma_\mathrm{a}-j\infty}^{\sigma_\mathrm{a}+j\infty} F(s)\,\mathrm{e}^{st}\,ds = \oint_{\Gamma} F(s)\,\mathrm{e}^{st}\,ds$$

$$= 2\pi j \sum \mathrm{res}\left\{F(s)\,\mathrm{e}^{st}\right\} \tag{1.13.19}$$

If this equation is divided by $2\pi j$, we again obtain the Eq. 1.13.12 which is the inverse Laplace transform of the function $F(s)$.

Although there was only a single pole in our $F(s)$ in Eq. 1.13.14 the result obtained is valid in the general case, when $F(s)$ has n poles and $n-1$ zeros.

Thus we have proved the $\mathcal{L}^{-1}$ transform by means of a contour integral for positive real functions. As in Eq. 1.13.12, here, too, the *abscissa of absolute convergence* σ_a must be chosen so that $\Re\{s\} \geq \sigma_\mathrm{a}$ and also $t \geq 0$ in order to ensure the convergence of the integral. However, we may also integrate along a straight path, where $\sigma < \sigma_\mathrm{a}$, provided that all the poles remain on the left side of the path.

From all the complicated equations above the reader must remember only one important fact, which we will use very frequently in the following sections: **By means of the $\mathcal{L}^{-1}$ transform of $F(s)$, the complex transfer function of a linear network, we obtain the real time function, $f(t)$, as the sum of the residues of all the poles of the complex frequency function $F(s)\,\mathrm{e}^{st}$.**

Let us put this in the symbolic form:

$$\boxed{f(t) = \mathcal{L}^{-1}\{F(s)\} = \sum \mathrm{res}\left\{F(s)\,\mathrm{e}^{st}\right\}} \tag{1.13.20}$$

1.14 Application of the Inverse Laplace Transform

In the following parts of the book we will very frequently need the inverse Laplace transform of two-pole and three-pole systems, in which the third pole at the origin, $1/s$, is the $\mathcal{L}$ transform of the unit step function. Therefore it would be useful to perform our first example of the $\mathcal{L}^{-1}$ transform calculation on such a network function.

Fig. 1.14.1 shows a typical two-pole network. Our task is to calculate the voltage on the resistor R as a function of time for $t > 0$. First we will apply an input current i_i in the form of an impulse $\delta(t)$, and next the input current will have a unit step form. Both results will be used in many cases in the following parts of the book. In the same way as for $F(s)$ and $f(t)$ we will label the voltages and currents with capitals (V, I) when they are the functions of frequency and with small letters (v, i) when they are functions of time.

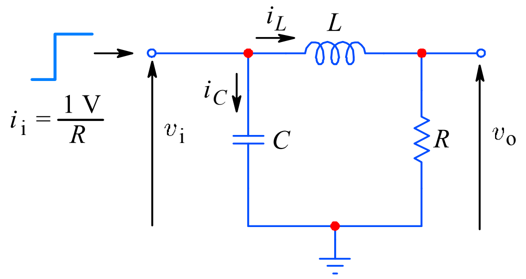

Fig. 1.14.1: A simple RLC circuit, driven by a current step, often found in electrical and electronics networks.

The input current is composed of two components, the current through the capacitor I_C, and through the inductor I_L (and the resistor R) and V_i is the input voltage:

$$I_\mathrm{i} = I_C + I_L = V_\mathrm{i}\, s\, C + \frac{V_\mathrm{i}}{s\, L + R} = V_\mathrm{i} \frac{s^2 L\, C + s\, R\, C + 1}{s\, L + R} \tag{1.14.1}$$

Correspondingly:

$$\boxed{\frac{V_\mathrm{i}}{I_\mathrm{i}} = \frac{s\, L + R}{s^2 L\, C + s\, R\, C + 1}} \tag{1.14.2}$$

This is a typical *input function* [Ref. 1.16], in this case it has the form of an (input) impedance, Z_i. The characteristics of an input function is that the number of poles exceeds the number of zeros by **one** only. The output voltage V_o is:

$$V_\mathrm{o} = V_\mathrm{i}\, \frac{R}{s\, L + R} \tag{1.14.3}$$

and so:

$$\frac{V_\mathrm{o}}{I_\mathrm{i}} = \frac{R}{s\, L + R} \cdot \frac{s\, L + R}{s^2 L\, C + s\, R\, C + 1} = \frac{R}{s^2 L\, C + s\, R\, C + 1} \tag{1.14.4}$$

The result is the *transfer function* of the network (from input to output, but is expressed as the output to input ratio). Since the dimension of Eq. 1.14.4 is (complex) Ohms it is also named the *transimpedance*. In general we will assume that the input current is $1\,\mathrm{V}/R$, in order to obtain a *normalized* transfer function:

$$\boxed{V_\mathrm{o} = \frac{1\,[\mathrm{V}]}{s^2 L\, C + s\, R\, C + 1} = G(s)} \tag{1.14.5}$$

In our later applications of the circuit in Fig. 1.14.1 the denominator of Eq. 1.14.5 must have **complex** roots (although, in general, the roots can also be real). Now let us calculate both roots of the denominator from its canonical form:

$$s^2 + s\,\frac{R}{L} + \frac{1}{L\,C} = 0 \tag{1.14.6}$$

with the roots:

$$s_{1,2} = \sigma_1 \pm j\,\omega_1 = -\frac{R}{2\,L} \pm \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}} \tag{1.14.7}$$

In special cases, some of which we shall analyze in the later parts of the book, the roots may also be double and real.

Expressing the transfer function, Eq. 1.14.5, by its roots, we obtain:

$$G(s) = \frac{1}{L\,C} \cdot \frac{1}{(s-s_1)(s-s_2)} \tag{1.14.8}$$

From the $\mathcal{L}^{-1}$ transform of this function we obtain the system's impulse response in the time domain, $g(t) = g\{\delta(t)\}$. The factor $1/LC$ is the system resonance, ω_1^2, which in a different network may take a different form (in the general normalized second-order case it is equal to the product of the two poles, $s_1 s_2$). Thus, we put $K = 1/LC$:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{K}{(s-s_1)(s-s_2)}\right\}$$

$$= \frac{K}{2\pi j} \oint_C \frac{\mathrm{e}^{st}}{(s-s_1)(s-s_2)}\,ds = K \sum \operatorname{res}\left\{\frac{\mathrm{e}^{st}}{(s-s_1)(s-s_2)}\right\} \tag{1.14.9}$$

The contour of integration in Eq. 1.14.9 must encircle both poles.

Since the network in Fig. 1.14.1 is passive, both poles s_1 and s_2 lie on the left side of the complex plane. As an example, in Fig. 1.14.2 we have drawn the magnitude of $G(s)$ for the special case of a 2$^{\text{nd}}$-order *Butterworth* network, for which the absolute values of the pole components are equal, $|\sigma| = |\omega| = 1/\sqrt{2} = 0.707$.

In this figure the Laplace transformed system transfer function, $\left|G(s)\right|$, is represented by the surface over the complex plane, peaking to infinity over the poles.

If we intersect the magnitude function by a vertical plane along the $j\omega$-axis, the surface edge (curve) at the intersection represents the frequency response $|G(j\,\omega)|$ of the function $|G(s)|$. The response is shown in a linear scale, and for the negative values of ω as well. In later sections we shall draw the frequency response graphs with a logarithmic scale for the frequency (positive only). Likewise, the magnitude will be logarithmic as well.

Eq. 1.14.9 has two residues:

$$\operatorname{res}_1 = \lim_{s\to s_1} (s-s_1)\,\frac{\mathrm{e}^{st}}{(s-s_1)(s-s_2)} = \frac{\mathrm{e}^{s_1 t}}{s_1 - s_2}$$

$$\operatorname{res}_2 = \lim_{s\to s_2} (s-s_2)\,\frac{\mathrm{e}^{st}}{(s-s_1)(s-s_2)} = \frac{\mathrm{e}^{s_2 t}}{s_2 - s_1} \tag{1.14.10}$$

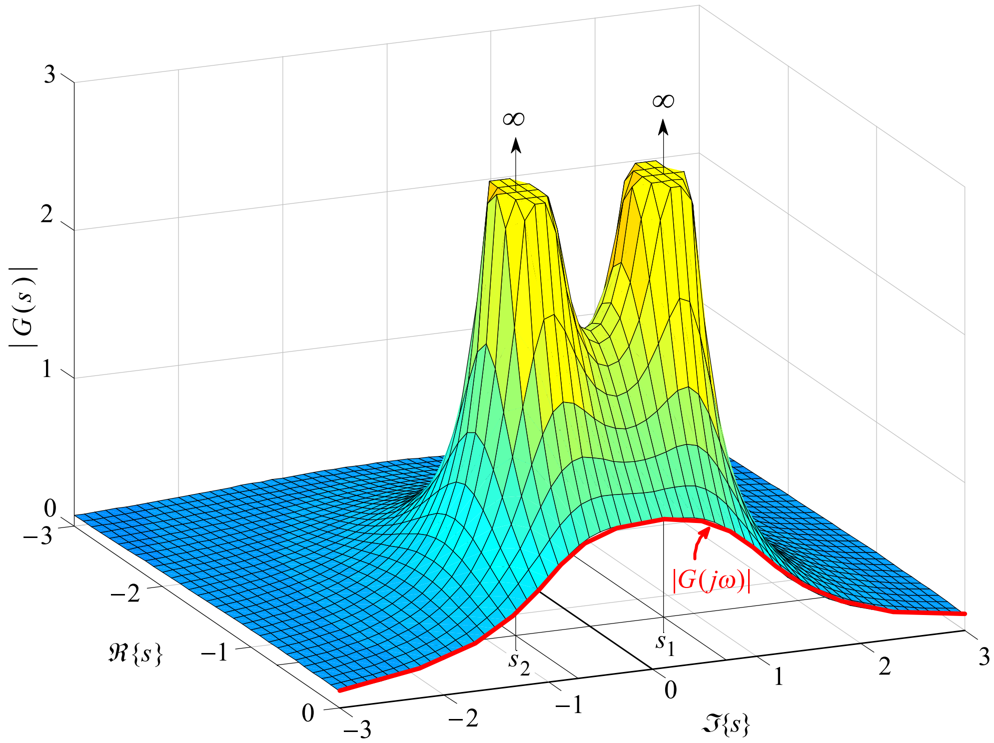

Fig. 1.14.2: The magnitude of the system transfer function, Eq. 1.14.8, for $s_{1,2} = (-1 \pm j)/\sqrt{2}$ and $K = 1$. For $\Re(s) = 0$, the surface $|G(s)|$ is reduced to the frequency response's magnitude curve, $|G(j\omega)|$. The height at $s_{1,2}$ is ∞, but was limited to 3 in order to see $|G(j\omega)|$ in detail.

The corresponding time function is the sum of both residues:

$$g(t) = \text{res}_1 + \text{res}_2 = \frac{K}{s_1 - s_2}\left(e^{s_1 t} - e^{s_2 t}\right) \tag{1.14.11}$$

Now we insert $\sigma_1 + j\omega_1$ for s_1 and $\sigma_1 - j\omega_1$ for s_2:

$$g(t) = \frac{K}{\sigma_1 + j\omega_1 - \sigma_1 + j\omega_1}\left[e^{(\sigma_1 + j\omega_1)t} - e^{(\sigma_1 - j\omega_1)t}\right] \tag{1.14.12}$$

We factor out $e^{\sigma_1 t}$ and rearrange the denominator to obtain:

$$g(t) = \frac{K}{2j\omega_1}\, e^{\sigma_1 t}\left[e^{j\omega_1 t} - e^{-j\omega_1 t}\right] = \frac{K}{\omega_1}\, e^{\sigma_1 t}\, \frac{e^{j\omega_1 t} - e^{-j\omega_1 t}}{2j} \tag{1.14.13}$$

Since:

$$\frac{e^{j\omega_1 t} - e^{-j\omega_1 t}}{2j} = \sin\omega_1 t \tag{1.14.14}$$

then:

$$g(t) = \frac{K}{\omega_1}\, e^{\sigma_1 t} \sin\omega_1 t \tag{1.14.15}$$

But K can also be expressed with σ_1 and ω_1:

$$K = \frac{1}{LC} = s_1 s_2 = \sigma_1^2 + \omega_1^2 \tag{1.14.16}$$

so:

$$g(t) = \frac{\sigma_1^2 + \omega_1^2}{\omega_1}\, e^{\sigma_1 t} \sin\omega_1 t = \frac{\sqrt{\sigma_1^2 + \omega_1^2}}{\sin\theta}\, e^{\sigma_1 t} \sin\omega_1 t \tag{1.14.17}$$

where θ is the angle between a pole and the positive σ axis, as in Fig. 1.13.4.

In our example, $\sigma_1^2 + \omega_1^2 = 1$ (Butterworth case), so Eq. 1.14.17 can be simplified:

$$g(t) = \frac{1}{\omega_1}\,\mathrm{e}^{\sigma_1 t}\sin\omega_1 t = \frac{1}{\sin\theta}\,\mathrm{e}^{\sigma_1 t}\sin\omega_1 t \tag{1.14.18}$$

Note that Eq. 1.14.13 and Eq. 1.14.17 are valid for any complex pole pair, not just for Butterworth poles. This completes the calculation of the impulse response.

The next case, in which we are interested more often, is the step response. In Example 1, Sec. 1.5, we have calculated that the unit step function in the time domain corresponds to $1/s$ in the frequency domain. To obtain the step response in the time domain, we need only to multiply the frequency response by $1/s$ and calculate the inverse $\mathcal{L}$ transform of the product. So by multiplying $G(s)$ by $1/s$ we obtain a new function:

$$F(s) = \frac{1}{s}\,G(s) = \frac{K}{s\,(s-s_1)(s-s_2)} \tag{1.14.19}$$

To calculate the step response in the time domain we use the $\mathcal{L}^{-1}$ transform:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\,\frac{K}{s\,(s-s_1)\,(s-s_2)} =$$

$$= \frac{K}{2\pi j}\oint_C \frac{\mathrm{e}^{st}}{s\,(s-s_1)(s-s_2)}\,ds = K\sum \operatorname{res}\left\{\frac{\mathrm{e}^{st}}{s\,(s-s_1)(s-s_2)}\right\} \tag{1.14.20}$$

The difference between Eq. 1.14.9 and Eq. 1.14.20 is that here we have an additional pole $s_0 = 0$, because of the factor $1/s$. Thus here we have three residues:

$$\begin{aligned}
\operatorname{res}_0 &= \lim_{s\to 0} s\,\frac{\mathrm{e}^{st}}{s\,(s-s_1)(s-s_2)} = \frac{1}{s_1\,s_2} \\
\operatorname{res}_1 &= \lim_{s\to s_1} (s-s_1)\,\frac{\mathrm{e}^{st}}{s\,(s-s_1)(s-s_2)} = \frac{\mathrm{e}^{s_1 t}}{s_1\,(s_1-s_2)} \\
\operatorname{res}_2 &= \lim_{s\to s_2} (s-s_2)\,\frac{\mathrm{e}^{st}}{s\,(s-s_1)(s-s_2)} = \frac{\mathrm{e}^{s_2 t}}{s_2\,(s_2-s_1)}
\end{aligned} \tag{1.14.21}$$

In the double-pole case (coincident pole pair, $s_1 = s_2$) the calculation is different (remember Eq. 1.11.12) and it will be shown in several examples in Part 2. The time domain function is the sum of all three residues ($K/s_1\,s_2$ is factored out):

$$f(t) = \frac{K}{s_1\,s_2}\left(1 + \frac{s_2}{s_1-s_2}\,\mathrm{e}^{s_1 t} + \frac{s_1}{s_2-s_1}\,\mathrm{e}^{s_2 t}\right) \tag{1.14.22}$$

By expressing $s_1 = \sigma_1 + j\omega_1$ and $s_2 = \sigma_1 - j\omega_1$ in each of the residues we obtain:

$$\frac{K}{s_1\,s_2} = \frac{K}{(\sigma_1 + j\omega_1)(\sigma_1 - j\omega_1)} = \frac{K}{\sigma_1^2+\omega_1^2} = 1 \qquad \text{(see Eq. 1.14.16)}$$

$$\begin{aligned}
\frac{s_2\,\mathrm{e}^{s_1 t}}{s_1 - s_2} &= \frac{(\sigma_1 - j\omega_1)\,\mathrm{e}^{(\sigma_1 + j\omega_1)t}}{\sigma_1 + j\omega_1 - \sigma_1 + j\omega_1} = \frac{\sigma_1 - j\omega_1}{2\,j\omega_1}\,\mathrm{e}^{(\sigma_1 + j\omega_1)t} \\
\frac{s_1\,\mathrm{e}^{s_2 t}}{s_2 - s_1} &= \frac{(\sigma_1 + j\omega_1)\,\mathrm{e}^{(\sigma_1 - j\omega_1)t}}{\sigma_1 - j\omega_1 - \sigma_1 - j\omega_1} = \frac{\sigma_1 + j\omega_1}{-2\,j\omega_1}\,\mathrm{e}^{(\sigma_1 - j\omega_1)t}
\end{aligned} \tag{1.14.23}$$

We put these results into Eq. 1.14.22 and obtain:

$$f(t) = 1 + \frac{\sigma_1 - j\omega_1}{2j\omega_1}\,\mathrm{e}^{(\sigma_1 + j\omega_1)t} + \frac{\sigma_1 + j\omega_1}{-2j\omega_1}\,\mathrm{e}^{(\sigma_1 - j\omega_1)t} \tag{1.14.24}$$

By factoring out $\mathrm{e}^{\sigma_1 t}$, and with a slight rearranging, we arrive at:

$$f(t) = 1 + \mathrm{e}^{\sigma_1 t}\left[\frac{\sigma_1}{\omega_1}\left(\frac{\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}}{2j}\right) - \left(\frac{\mathrm{e}^{j\omega_1 t} + \mathrm{e}^{-j\omega_1 t}}{2}\right)\right] \tag{1.14.25}$$

Since $(\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t})/2j = \sin\omega_1 t$ and $(\mathrm{e}^{j\omega_1 t} + \mathrm{e}^{-j\omega_1 t})/2 = \cos\omega_1 t$ we can simplify Eq. 1.14.25 into the form:

$$\boxed{f(t) = 1 + \mathrm{e}^{\sigma_1 t}\left(\frac{\sigma_1}{\omega_1}\sin\omega_1 t - \cos\omega_1 t\right)} \tag{1.14.26}$$

We could now numerically calculate the response, but we want to show two things:

1) how the formula relates to the physical circuit behavior;

2) explain an error, all too often ignored (even by experienced engineers!).

We can further simplify the sine–cosine term by using the vector sum of the two phasors (this relation can be found in any mathematics handbook):

$$A\sin\alpha + B\cos\alpha = \sqrt{A^2 + B^2}\,\sin(\alpha + \theta) \qquad \text{where} \qquad \theta = \arctan(B/A)$$

By putting $A = \sigma_1/\omega_1$ and $B = -1$ we arrive at:

$$f(t) = 1 + \sqrt{1 + \left(\frac{\sigma_1}{\omega_1}\right)^2}\;\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta) \tag{1.14.27}$$

where:

$$\theta = \arctan\left(\frac{-\omega_1}{\sigma_1}\right)$$

For the Butterworth case, the square root is equal to $\sqrt{2}$, but in the general case it is:

$$\sqrt{1 + \left(\frac{\sigma_1}{\omega_1}\right)^2} = \frac{\sqrt{\sigma_1^2 + \omega_1^2}}{\omega_1} = \left|\frac{1}{\sin\theta}\right| \tag{1.14.28}$$

Note that for any value of σ_1 and ω_1 their square can never be negative, which is reflected in the absolute value notation at the end; on the other hand, it is important to preserve the correct sign of the phase shifting term in $\sin(\omega_1 t + \theta)$. By putting Eq. 1.14.28 back into Eq. 1.14.27 we obtain a relatively simple expression:

$$\boxed{f(t) = 1 + \left|\frac{1}{\sin\theta}\right|\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta)} \tag{1.14.29}$$

If we now insert the numerical values for σ_1, ω_1 and θ and plot the function for t in the interval from 0 to 10, the resulting graph will be obviously **wrong!** What happened?

Let us check our result by applying the rule of initial and final value from Sec. 1.6. We will use Eq. 1.14.29 and Eq. 1.14.8, considering that $K = s_1 s_2$ (Eq. 1.14.16).

1. Check the initial value in the frequency-domain, $s \to \infty$:

$$f(0) = \lim_{s\to\infty} s\,F(s) = \lim_{s\to\infty} \frac{s_1 s_2}{(s-s_1)(s-s_2)} = 0 \tag{1.14.30}$$

which is correct. But in the time-domain at $t = 0$:

$$f(0) = 1 + \frac{1}{|\sin\theta|}\,\mathrm{e}^{\sigma_1 0}\sin(\omega_1 0 + \theta) = 2 \tag{1.14.31}$$

which is **wrong!**

2. Check the final value for $t \to \infty$:

$$f(\infty) = \lim_{t\to\infty}\left[1 + \frac{1}{|\sin\theta|}\,\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta)\right]$$

$$= s\,F(0) = \frac{s_1 s_2}{(0-s_1)(0-s_2)} = 1 \tag{1.14.32}$$

and at least this one is correct in both the time and frequency domain. Note that in both checks the pole at $s = 0$ is canceled by the multiplication of $F(s)$ by s.

Considering the error in the initial value in the time domain, many engineers wrongly assume that they have made a **sign** error and change the time domain equation to:

$$f(t) = 1 - \left|\frac{1}{\sin\theta}\right|\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta) \qquad \textbf{(wrong!)}$$

Although the step response plot will now be correct, a careful analysis shows that the negative sign is completely unjustified! Instead we should have used:

$$\theta = \pi + \arctan\left(\frac{-\omega_1}{\sigma_1}\right) \tag{1.14.33}$$

The reason for the added π lies in the *tangent* function, which repeats with a period of π radians (and not 2π, as the *sine* and *cosine* do). This results in a lost sign since the *arctangent* can not tell between angles in the first quadrant from those in third, and angles in the second quadrant from those in fourth. See Appendix 2.3 (on the disk) for more of such cases in 3rd- and 4th-order systems.

A graphical presentation of the step response solution, given by Eq. 1.14.29 and with the correct initial phase angle, Eq. 1.14.33, is displayed in Fig. 1.14.3.

The physical circuit behavior can be explained as follows:

The system resonance term, $\sin(\omega_1 t)$, is first shifted by θ, the characteristic angle of the pole, becoming $\sin(\omega_1 t + \theta)$ (the time shift is θ/ω_1). At resonance the voltage and current in reactive components are each others' derivatives (a sine–cosine relationship, see Eq. 1.14.26), the initial phase angle θ reflects their impedance ratios.

The amplitude of the shifted function is then corrected by the absolute value of the function at $t = 0$, which is $|1/\sin\theta|$. Thus the starting value is equal to -1, and in addition the slope is precisely identical to the initial slope of the exponential damping function, $e^{\sigma_1 t}$, so that their product has **zero** initial slope.

This product is the system reaction to the unit step excitation, $h(t)$, which sets the final value for $t \to \infty$ ($s_0 = 0$). By summing the residue at s_0 ($\text{res}_0 = 1$) with the reaction function gives the final result, the step response $f(t)$.

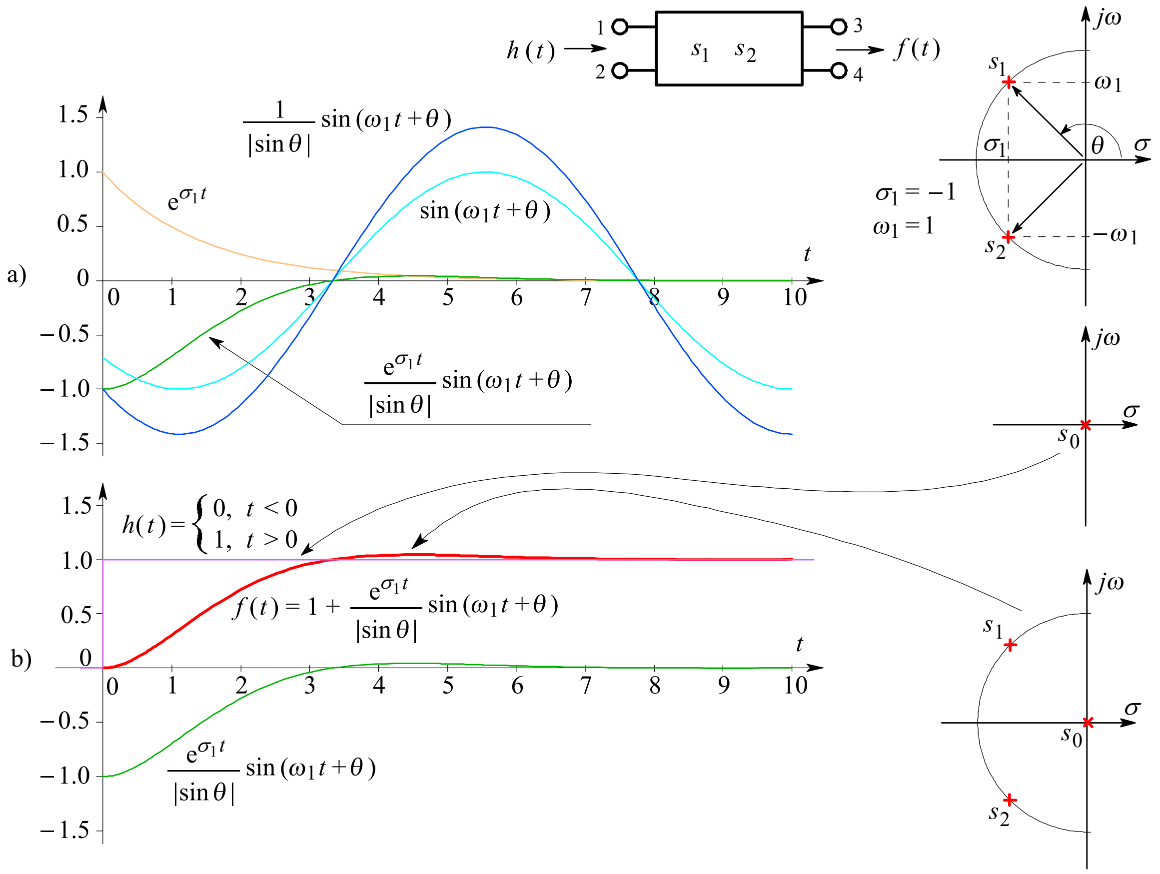

Fig. 1.14.3: Step by step graphic representation of the procedures used in the calculation of the step response of a system with two complex conjugate poles.

We have purposely presented the complete calculation of the step response for the RLC circuit in every detail for two reasons:

1) to show how the step response is calculated by means of the $\mathcal{L}^{-1}$ transform and the theory of residues; and

2) because we shall meet such functions very frequently in the following parts.

(blank page)

1.15 Convolution

In network analysis we often encounter a cascade of networks, so that the output of the preceding network is driving the input of the following one. The output of the later network is therefore a response to the response of the preceding network. We need a procedure to solve such problems. In the time domain this is done by the convolution integral [Ref. 1.2]. Fig. 1.15.1 displays the complete procedure of convolution.

In Fig. 1.15.1a there are two networks:

The network A has a Bessel pole pair with the following data:

$s_{1,2} = \sigma_1 \pm j\omega_1 = -1.500 \pm j0.866$;   the pole angle is $\theta_1 = \pm 150°$.

In addition, owing to the input unit step function, we have a third pole $s_0 = 0$.

The network B has a Butterworth pole pair with the following data:

$s_{3,4} = \sigma_2 \pm j\omega_2 = -0.7071 \pm j0.7071$;   the pole angle is $\theta_2 = \pm 135°$.

Bessel and Butterworth poles are discussed in detail in Part 4 and Part 6.

According to Eq. 1.14.29 the step response of the network A is:

$$f(t) = 1 + \frac{1}{|\sin\theta_1|}\,\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta_1)$$

and, according to Eq. 1.14.17, the impulse response of the network B is:

$$g(t) = \frac{1}{|\sin\theta_2|}\,\mathrm{e}^{\sigma_2 t}\sin\omega_2 t$$

Both functions are shown in Fig. 1.15.1a. We will convolve $g(t)$ because it is easier to do so. This convolving (folding) is done by time reversal about $t = 0$, obtaining $g(\tau - t)$. The reversion interval τ has to be chosen so that $g(t \geq \tau) = 0$ (or at least very close to zero), otherwise the convolution integral would not converge to the correct final value. The output function $y(t)$ is then the convolution integral:

$$y(t) = \int_0^{t_{\max}} \underbrace{\left[1 + \frac{1}{|\sin\theta_1|}\,\mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta_1)\right]}_{f(t)} \underbrace{\frac{1}{|\sin\theta_2|}\,\mathrm{e}^{\sigma_2(\tau - t)}\sin\omega_2(\tau - t)\,dt}_{g(\tau - t)} \qquad (1.15.1)$$

To solve this integral requires a formidable effort and the reader may be assured that we shall not attempt to solve it here, because — as we will see later — there is a more elegant method of doing so. We have written the complete integral merely to give the reader an example of the convolution based on the functions which we have already calculated. Nevertheless, it is a challenge for the reader who wants to do it by himself (for the construction of diagrams in Fig. 1.15.1, this integral has been solved!).

In Fig. 1.15.1b we first convolve the function $g(t)$ and introduce the time constant τ to obtain $g(\tau - t)$. Next, in Fig. 1.15.1c the function $g(\tau - t)$ is shifted right along the time axis to the position $t = 1$, obtaining $g(\tau - t + 1)$. The area A_1 under the product of the two signals is the value of the convolution integral for the interval $0 \leq t \leq 1$.

In a similar fashion, in Fig. 1.15.1d the function $g(\tau - t)$ is shifted to $t = 2$. Here the value of the convolution integral for the interval $0 \leq t \leq 2$ is equal to the area A_2.

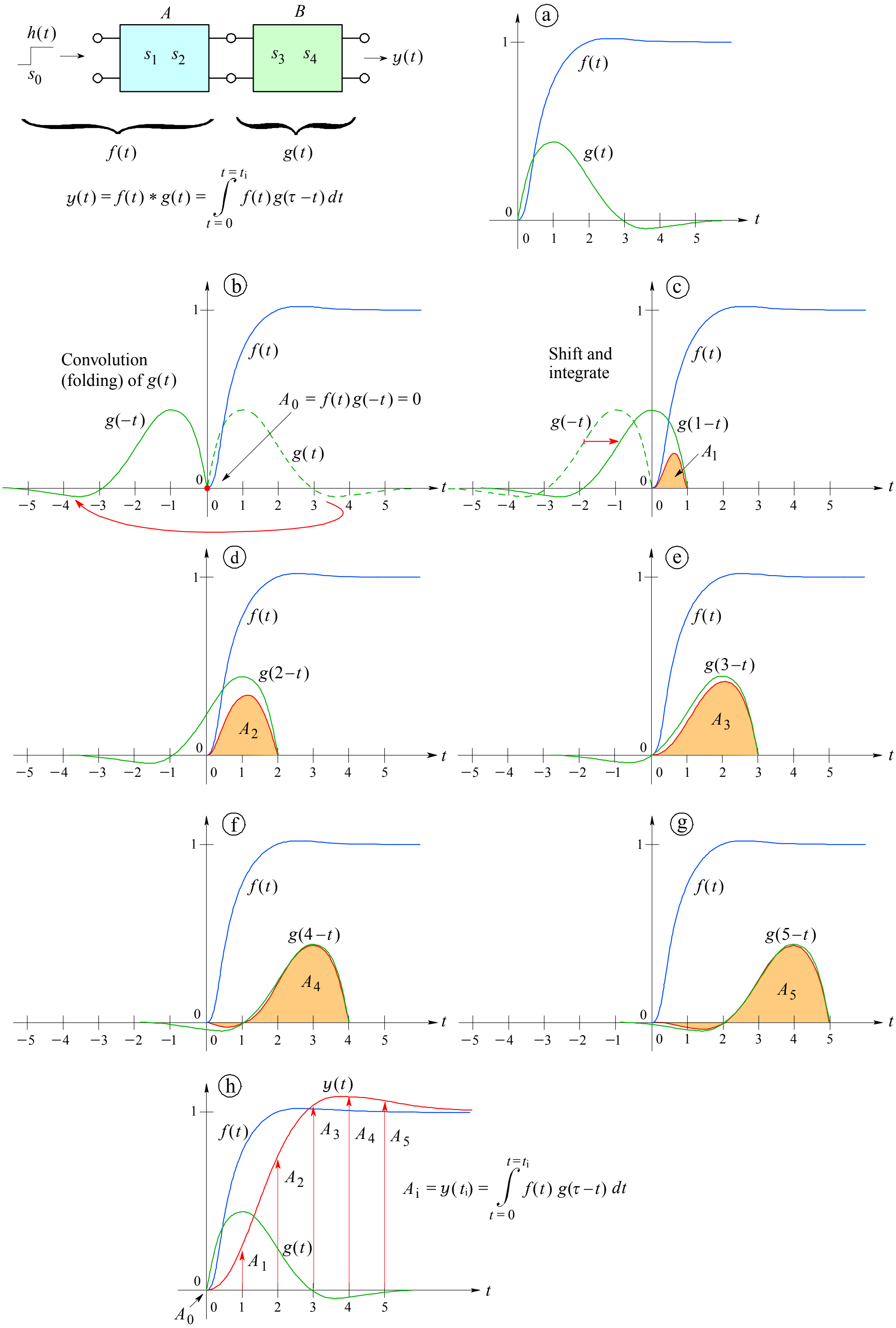

Fig. 1.15.1: Graphic presentation of the mathematical course of the convolution integral, $f(t) * g(t)$. See the text for the description. To see the movie click on this link: convmovie.avi (we apologize for the bad quality resulting from the poor conversion from vector graphics to bitmap format; readers who have access to Matlab are invited to run the 'convmovie.m' file).

In Fig. 1.15.1e, the function $g(\tau - t)$ is shifted to $t = 3$ to obtain the area A_3 and in Fig. 1.15.1f, the function $g(\tau - t)$ is shifted to $t = 4$, resulting in the area A_4, which is in part negative, owing to the shape of $g(\tau - t)$.

In Fig. 1.15.1g, $t = 5$ and the area A_5 is obtained. Since $f(t)$ has nearly reached its final value and $g(t)$ is almost zero for $t > 5$, any further shifting changes A_t only slightly.

Finally, in Fig. 1.15.1h the values of $A_1, \dots, A_5$ are inserted to point to the particular values of the output function $y(t)$. For comparison, the input of network B, $f(t)$, is also drawn. Although $f(t)$ has almost no overshoot, the Butterworth poles in the network B cause a large undershoot in $g(t)$, which results in an overshoot in the output signal $y(t)$.

Important note: In the last plot of Fig. 1.15.1 the system response $y(t)$ is plotted as if the network B had a unity gain. The impulse response of a unity gain system is characterized by the whole area under it being equal to 1; consequently, its peak amplitude would be very small compared to $f(t)$, so there would not be much to see. Therefore for $g(t)$ we have plotted its ideal impulse response. The normalization to a unity-gain is accomplished by dividing the ideal impulse-response by its own time integral (numerically, each instantaneous amplitude sample is divided by the sum of all samples). See Part 6 and Part 7 for more details.

From Eq. 1.6.51, it has become evident that convolution in the time domain corresponds to a simple frequency domain multiplication. This is also shown in Fig. 1.15.2. The upper half of the figure is the s domain whilst the bottom half is the t domain. Instead of making the convolution $g(t) * f(t)$ in the t domain, which is difficult, (see Eq. 1.15.1), we rather perform a simple multiplication $G(s) \cdot F(s)$ in the s domain. Then, by means of the $\mathcal{L}^{-1}$ transform we obtain the function $y(t)$ which we are looking for.

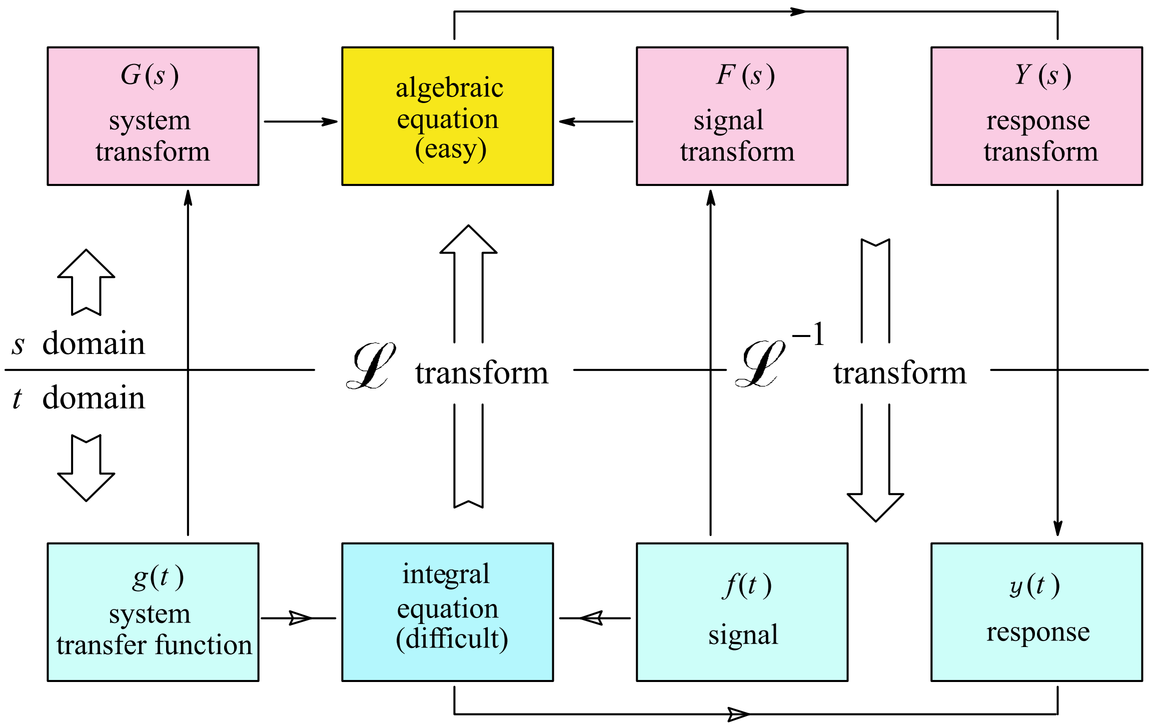

Fig. 1.15.2: Equivalence of the system response calculation in the time domain, $f(t)*g(t)$, and the frequency domain, $F(s) \cdot G(s)$. For analytical work the transform route is the easy way. For computer use the direct method is preferred.

By taking the transform route we need only to calculate the sum of all the residues (five in the case shown in Fig. 1.15.1), which is far less difficult than the calculation of the integral in Eq. 1.15.1.

The mathematical expression, which applies to this case, this is:

$$y(t) = \mathcal{L}^{-1}\{F(s)\cdot G(s)\}$$

$$= \sum \text{res} \underbrace{\left[\frac{s_1\, s_2}{s\,(s-s_1)(s-s_2)}\right]}_{F(s)} \underbrace{\left[\frac{s_3\, s_4}{(s-s_3)(s-s_4)}\right]}_{G(s)} \mathrm{e}^{st} \tag{1.15.2}$$

Here the numerators of both fractions have been normalized by introducing the products $s_1 s_2$ and $s_3 s_4$ respectively, to replace the constant K (according to Eq. 1.14.16) in the Eq. 1.14.19 and 1.14.8. A solution of the above equation can be found in Part 2, Sec. 2.6.

Fig. 1.15.2 also reveals another very important possibility. If the input signal $f(t)$ is known and a certain output signal $y(t)$ is desired, we can synthesize (not always!) the intermediate network $G(s)$ by taking the $\mathcal{L}$ transform of both time functions and calculating their quotient:

$$G(s) = \frac{Y(s)}{F(s)} \tag{1.15.3}$$

where $Y(s) = \mathcal{L}\{y(t)\}$ and $F(s) = \mathcal{L}\{f(t)\}$.

Eq. 1.15.1 has convinced us that the calculation of convolution is not an easy task, even for relatively simple functions. By using a PC computer, the convolution in time domain can be calculated **numerically**. Several good mathematical programs exist (we have been using Matlab™ [Ref. 1.18]), which simplify the convolution calculation to a matter of pure routine. This is explained in detail in Part 6 and Part 7.

Résumé of Part 1

So far we have discussed the Laplace transform and its inverse, only to the extent which the reader needs for understanding the rest of the book.

Since we shall calculate many practical examples of the $\mathcal{L}^{-1}$ transform in the following chapters, we have discussed extensively only the calculation of the time function of a simple two pole network with a complex conjugate pole pair, excited by the unit step function.

The readers who want to broaden their knowledge of the Laplace transform, can find enough material for further study in the references quoted.

(blank page)

References:

[1.1] *G.E. Valley & H. Wallman,* Vacuum Tube Amplifiers,
MIT Radiation Laboratory Series, Vol. 18, McGraw-Hill, New York, 1948.

[1.2] *R.B. Randall,* Frequency Analysis,
Brüel & Kjær, Nærum, 1987.

[1.3] *M.F. Gardner & J.L. Barnes*, Transients in Linear Systems Studied by Laplace Transform,
Twelfth Printing, John Wiley & Sons, New York, 1956.

[1.4] *O. Föllinger*, Laplace und Fourier Transformation,
AEG–Telefunken, Berlin, 1982.

[1.5] *G. Doetsch*, Introduction to the Theory and Application of the Laplace Transform,
Springer–Verlag, Berlin, 1970.

[1.6] *G. Doetsch*, Anleitung zum praktischen Gebrauch der Laplace Transformation und der
Z–Transformation, R. Oldenburg Verlag, Munich–Vienna, 1985.

[1.7] *T.F. Bogart, Jr.*, Laplace Transforms and Control Systems, Theory for Technology,
John Wiley & Sons, New York, 1982.

[1.8] *M. O'Flynn & E. Moriarthy*, Linear Systems, Time Domain and Transform Analysis,
John Wiley, New York, 1987.

[1.9] *G.A. Korn & T.M. Korn*, Mathematical Handbook for Scientists and Engineers,
McGraw-Hill, New York, 1961.

[1.10] *M.R. Spiegel*, Theory and Problems of Laplace Transforms,
Schaum's Outline Series, McGraw-Hill, New York, 1965.

[1.11] *J. Plemelj*, Teorija analitičnih funkcij,
Slovenska akademija znanosti in umetnosti, Ljubljana, 1953.

[1.12] *M.R. Spiegel*, Theory and Problems of Complex Variable, SI (Metric) Edition,
McGraw-Hill, New York, 1974.

[1.13] *I. Stewart & D. Tall*, Complex Variables,
Cambridge University Press, Cambridge, 1983.

[1.14] *R.W. Churchil & J.W. Brown*, Complex Variables and Applications, Fourth Edition,
International Student Edition, McGraw-Hill, Auckland, 1984.

[1.15] *W. Gellert, H. Küstner, M. Hellwich, & H. Kästner*, The VNR Concise Encyclopedia of
Mathematics, second edition, Van Nostrand Rheinhold, New York, 1992.

[1.16] *E. Van Valkenburg*, Introduction to Modern Network Synthesis,
John Wiley & Sons, New York, 1960.

[1.17] *P. Starič*, Proof of the Inverse Laplace Transform for Positive Real Functions,
Elektrotehniški vestnik, Ljubljana, 1991, pp. 23–27.

[1.18] *J.N. Little & C.B. Moler*, MATLAB User's Guide,
The MathWorks, Inc., South Natick, USA, 1990.

[1.19] *G.E. Hostetter*, Engineering Network Analysis,
Harper & Row, Publishers, New York, 1984.

[1.20] *J. Bednařík & J. Daněk*, Obrazové zesilovače pro televisí a měřicí techniku,
Statní nakladatelství technické literatury, Prague, 1957.

[1.21] *V. Bubeník*, Impulsová Technika,
Statní nakladatelství technické literatury, Prague, 1958.

[1.22] *D.E. Scott*, An Introduction to System Analysis, A System Approach,
McGraw-Hill, New York, 1987.

[1.23] *P. Kraniauskas*, Transforms in Signals and Systems,
Addison-Wesley, Wokingham, 1992.

[1.24] <http://en.wikipedia.org/wiki/Gibbs_phenomenon/>

(blank page)