Appendix 1.1: Simple Poles, Complex Spaces

Mathematics is just an opinion.
(Stephen Wolfram, Wolfram Research)

It is always useful to make a graphical representation of the mathematical expression of the problem which we are trying to solve. The plot will often give us not only a better understanding, but also a kind of 'feeling' for a solution or a range of possible solutions.

However, plotting complex functions of complex variables is, well, complex! We have a two-dimensional argument domain (real and imaginary abscissae), and consequently a two-dimensional function domain (real and imaginary ordinates), and each axis should be orthogonal to every other one, so we need a four-dimensional (4D) plotting space. As if this were not enough, our plotting media have only two dimensions, so we can actually plot only a 2D projection of a 3D object, at best.

We have encountered this problem in Part 1, Fig. 1.13.1, where we wanted to see the complex line integral of a simple single-pole function. Traditionally this has been treated in literature in a way which, although helping to visualize the solution, is, mildly speaking, incorrect: by plotting a 3D magnitude (absolute value) of the function over the complex plane, and using the line in the complex plane and its projection on the magnitude surface to border the area (result of integration). What is incorrect in such a plot is that it shows us the integral of the absolute value of the complex function, or $\int |F(z)|\, dz$. Instead, the area between the integration path in the complex plane and its complex function line (the line lying on the complex function surface) is, generally, a complex-valued integral of the complex function. Let us see if we can find a way of plotting it more realistically.

We can first try to simplify the task by making separate plots for the real and imaginary value of the function over the complex plane. Let us take the simplest possible function, that of a simple single-pole at the origin:

$$F(z) = \frac{1}{z} \qquad \text{where} \qquad z = x + j\,y \tag{A1.1}$$

which for $y = 0$ is equal to the real function $1/x$. But for $x = 0$ and $y \neq 0$ this function is rotated around the pole by 90° and inverted in sign (both owing to j) because of the denominator rationalization:

$$F(z)\Big|_{x=0} = F(j\,y) = \frac{1}{j\,y} = \frac{j}{j^2\,y} = \frac{j}{-1\cdot y} = -j\,\frac{1}{y} \tag{A1.2}$$

For both $x \neq 0$ and $y \neq 0$ the function is twisted owing to its phase angle φ:

$$F(z) = \frac{1}{|z|\,\mathrm{e}^{j\varphi}} \qquad \text{where} \qquad |z| = \sqrt{x^2 + y^2} \qquad \text{and} \qquad \varphi = \arctan\frac{y}{x} \tag{A1.3}$$

Fig. A1.1 shows the real value of the function, $\Re\{F(z)\}$ whilst Fig. A1.2 shows the imaginary value of the function, $j\Im\{F(z)\}$.

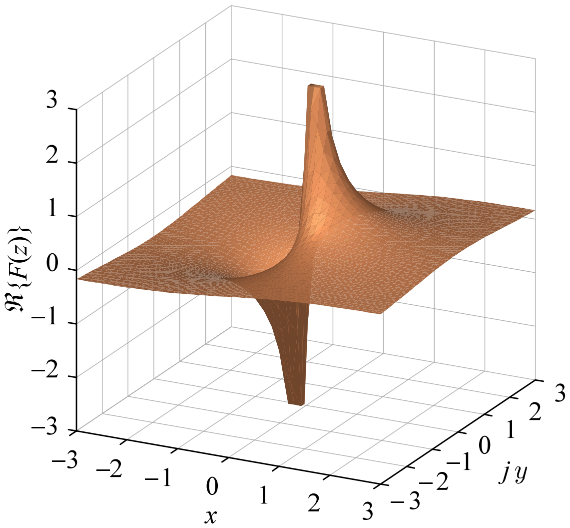

Fig. A1.1: The real part of the function $F(z) = 1/z$ for $z = x + jy$ in the range $-3 < x < 3$ and $-j3 < y < j3$. Note how $\Re\{F(z)\}$ changes sign along the x direction.

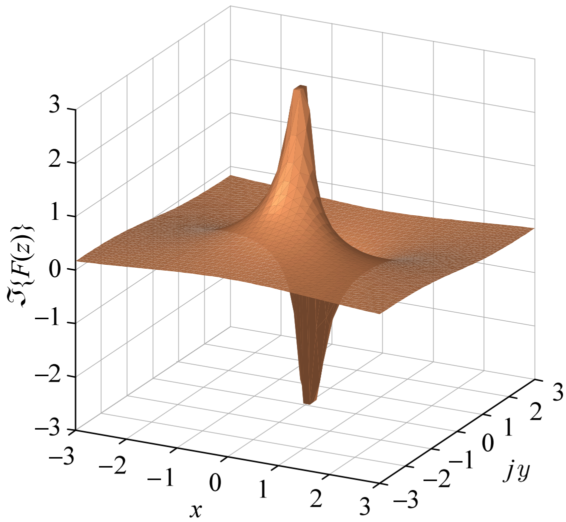

Fig. A1.2: The imaginary part of the function $F(z) = 1/z$ for $z = x + jy$ in the range $-3 < x < 3$ and $-j3 < y < j3$. Note how $\Im\{F(z)\}$ changes sign along the jy direction.

Each of these figures tells us only half of the story. Also bear in mind that the vertical axes of these two figures, $\Re\{F(z)\}$ and $\Im\{F(z)\}$ should also be mutually orthogonal.

The magnitude of a complex function (see Fig. A1.3) can be calculated as the square root of the product of the function with its own complex conjugate:

$$|F(z)| = \sqrt{\Big(\Re\{F(z)\} + j\Im\{F(z)\}\Big)\Big(\Re\{F(z)\} - j\Im\{F(z)\}\Big)}$$

$$= \sqrt{\Big(\Re\{F(z)\}\Big)^2 + \Big(\Im\{F(z)\}\Big)^2} \tag{A1.4}$$

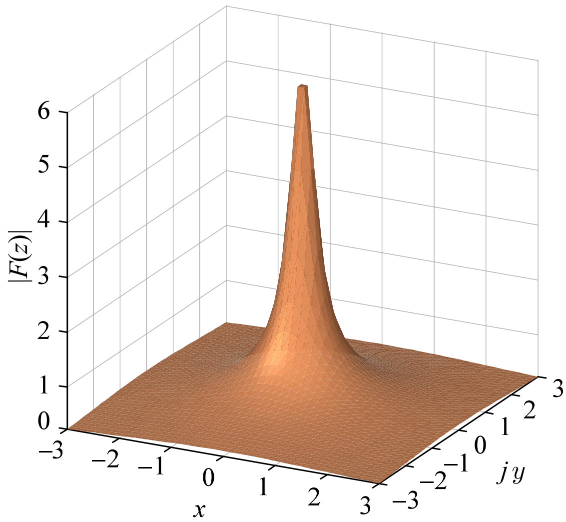

Fig. A1.3: Magnitude (absolute value) of $F(z)$. The shape of the surface is the same as would be obtained if the positive side of the real function $1/x$ is rotated around the vertical axis passing through the pole. Obviously any phase information is lost.

In Part 1, Sec. 1.8 we wanted to show how the value of the complex line integral depends only on the starting and end point of the integration path and is independent of the actual path taken. If we imagine a line on the surface of Fig. A1.3 and its projection in the z domain, the integral value would be equal to the area bordered by these two lines. But from Fig. A1.3 it is clear that a path closer to the pole would yield a larger area then a path more distant from the pole. Therefore the magnitude is not very useful for explaining the properties of the complex line integral. It is useful, nevertheless, for explaining other features of complex functions (see, for example, Jordan's Lemma in Sec. 1.13, Eq. 1.13.10, or the relationship between the frequency response and the location of the poles in Fig. 1.14.2).

Now that we have seen some of the awkward aspects of complex function graphing, let us try a different approach. But first we must review some basic concepts of complex numbers. Historically, the first glimpse of the existence of complex numbers came from realizing that there are some real functions, even very simple ones, which for a range of real argument values do not have real solutions. A typical example is the square root function, which is not defined for negative numbers (at least in real space). Mathematicians have agreed to represent $\sqrt{-N^2}$ as $\pm N\sqrt{-1}$, or the root of the positive number, multiplied by the root of the negative (imaginary) unit: jN. This opened the door into a new dimension, literally.

For real functions we usually map the results on the real–real plane, of which one axis (the abscissa) represents the argument's values and the other (the ordinate) the appropriate function values. If the axes are to be independent they should be mutually orthogonal, so that moving in parallel to one axis does not change the value on the other (this is usually referred to as the Cartesian system of coordinates, after *Cartesius — René Descartes*, 1596–1650, even if in his *La géométrie* he never required that the two 'reference lines' should be perpendicular). If we want to achieve the same independence of axes for imaginary numbers the imaginary ordinate must be orthogonal to both the real ordinate and the real abscissa. So, our space is now 3D. Also all our axes should have a common origin point, and, for obvious reasons, the zero is the most convenient one.

We can now reinterpret the imaginary ordinate as having the same scale as the real ordinate, but is rotated by 90° (as in Fig. A1.4, in which the function $y = \sqrt{x}$ is shown) owing to the multiplication by the imaginary unit.

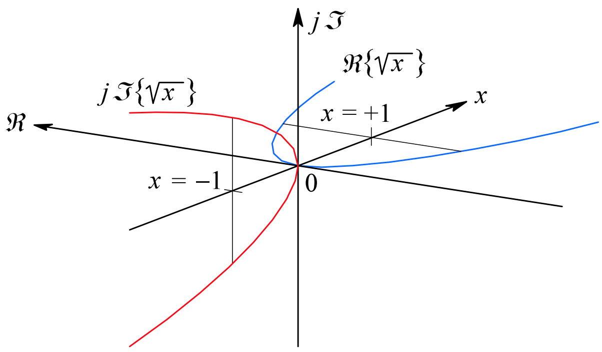

Fig. A1.4 : The complete representation of the square root function. The imaginary part is the same as the real, but rotated by $\pi/2$. The only common point is at the origin.

In general each new multiplication by j means an additional rotation by 90°. Now remember that in Laplace space we have a complex argument plane, therefore because of this 90° rotation the imaginary abscissa must also be orthogonal to the real abscissa, the real ordinate and the imaginary ordinate . No problem: we just add another dimension, orthogonal to the previous three, ending up in a 4D space (that is, no problem, apart from the fact of the human mind being conditioned to operate optimally in 1D, reasonably well in 2D, almost acceptably in 3D, but rather poorly in 4D).

Owing to the rotational definition of imaginary axes there is no way of forming any non-orthogonal relationship between the imaginary axis of the function and the imaginary axis of its argument domain.

But there is no such limitation for the two real axes. We can always think of the real function as a real to real transform, or simply a way of remapping the real axis. The same is true for the real part of the complex function. With this in mind we can return to the 3D space, with one real axis and two imaginary.

To understand the axis remapping let us examine Fig. A1.5; there we see that the function $1/x$ reflects the >1 part into the <1 part and vice versa; the origin goes to infinity and the infinity into the origin; the unity point is remapped into itself.

By using this remapping with complex arguments we are able to see the real part of the argument from the perspective of the function's real part. For example, a circle in the argument's complex plane, having a unit radius, is transformed by remapping into a double 'U' shape, as shown in Fig. A1.6. But then the $1/z$ function of the circle is transformed into a unit circle in the function complex plane.

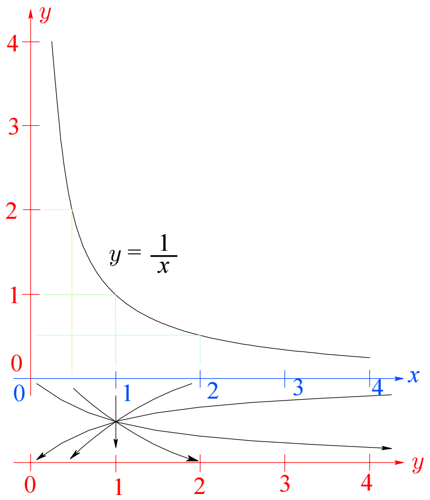

Fig. A1.5: The real to real remapping for the $1/x$ function. The axis is inverted about the point unity (because $1/1 = 1$).

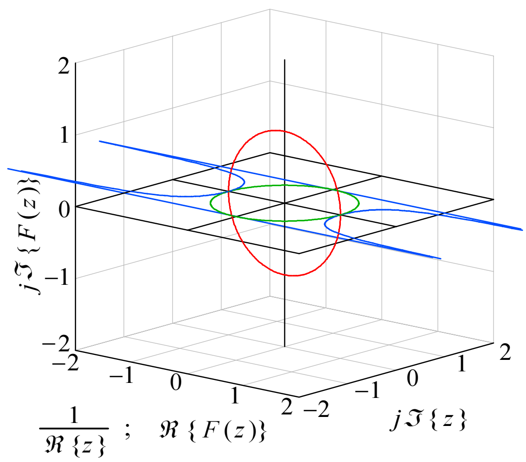

Fig. A1.6: A unit circle in the argument's complex plane (green) looks like a double 'U' shape (blue) from the function's complex plane (the argument's imaginary scale remains unchanged). In turn it is transformed into a circle (red) in the function space.

Let us write our function $F(z) = 1/z$ in another way:

$$F(z) = \frac{1}{z} = \frac{1}{x + jy} = \frac{x - jy}{(x + jy)(x - jy)} = \frac{x - jy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{jy}{x^2 + y^2} \tag{A1.5}$$

We now tabulate this function for some discrete values of x:

$$x = 0 \qquad \rightarrow \qquad F(0 + jy) = 0 - \frac{j}{y} \tag{A1.6}$$

$$x = 0.5 \qquad \rightarrow \qquad F(0.5 + jy) = \frac{0.5}{0.25 + y^2} - \frac{jy}{0.25 + y^2} \tag{A1.7}$$

$$x = 1 \qquad \rightarrow \qquad F(1 + jy) = \frac{1}{1 + y^2} - \frac{jy}{1 + y^2} \tag{A1.8}$$

$$x = 2 \qquad \rightarrow \qquad F(2 + jy) = \frac{2}{4 + y^2} - \frac{jy}{4 + y^2} \tag{A1.9}$$

$$x = \infty \qquad \rightarrow \qquad F(\infty + jy) = 0 - j0 = 0 \tag{A1.10}$$

In Eq. A1.6 we have taken into account that for $x \rightarrow 0$ then $\Re\{F(z)\} \rightarrow 0$ also, because x^2 approaches zero more quickly than x, so we are left with the ratio x/y^2, which is zero for $x = 0$. The remaining term is the $-j/y$, which we have already obtained at the beginning of our discussion.

From Eq. A1.7, A1.8, and A1.9 we find an interesting fact that $F(z)$ has **only one** real value for each x (when $y = 0$), and it is equal to $1/x$.

However, for $y \rightarrow \pm\infty$, $F(z) \rightarrow 0$; again, the positive part of the imaginary abscissa gives negative imaginary ordinate values.

In Eq. A1.10 we have taken into account that x^2 approaches infinity much faster than x. From this we conclude that the infinity is remapped into the origin.

Now let us try to plot this.

Since we want to display the shape of the function, the real axis will actually be $\Re\{F(z)\}$, the real ordinate, whilst the real part of the argument is inverted, $x \rightarrow 1/x$. Also, x is introduced parametrically, one value at a time.

The consequence of a parametric x is that $F(x + jy)$ is not presented as a surface, as in previous graphs, but instead we have one complex curve for each discrete value of x. But, of course, making a large number of line plots on the same graph, one line for each real argument, we can eventually approximate the complex function's surface.

By using this process, as in Fig. A1.7, we finally obtain a rough idea of the true function shape of $1/z$. We have plotted it for only a small number of real argument values, positive for the sake of simplicity, since for $x < 0$ the plot is a mirror image.

Because the function inverts the argument values we have used a geometrical progression for x in order to cover a wide enough range (from $0.1 < x < 100$, including also $x = 0$) to see the function tends towards zero as well as towards infinity.

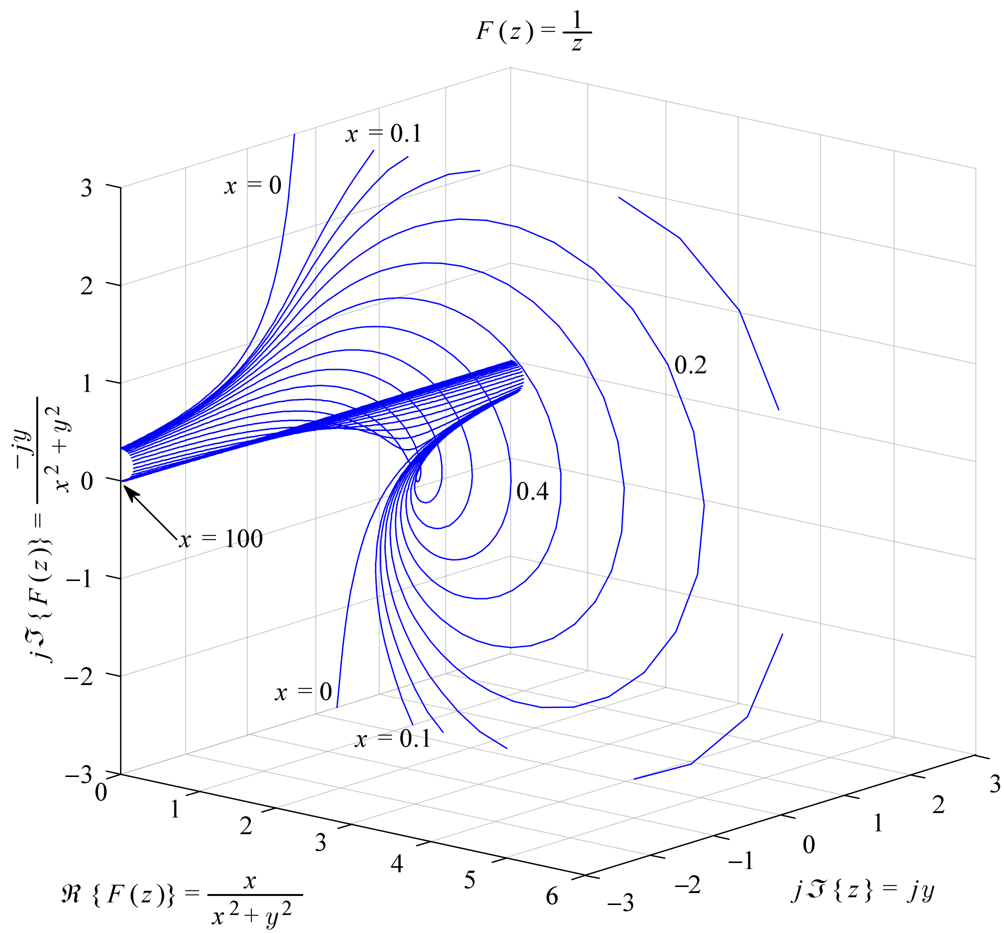

Fig. A1.7: The complex function $F(z) = 1/z$ in the complex space. The range of the argument's imaginary axis, $\Im\{z\}$, is $-3 < y < 3$, whilst $\Re\{z\}$ was parametrized in a geometrical progression, from $0.1 < x < 100$, including $x = 0$, in order to cover a wide enough range to obtain the function trends at extreme values of x. The real axis is $\Re\{F(z)\}$ so $F(z)$ crosses it at exactly $1/x$. The vertical axis is $\Im\{F(z)\}$. In this way a 3D view of $F(z)$ is made possible. For $x = 0$ $F(z)$ follows the same shape as the real function, but inverted by j, extending to infinities at the pole. But for $0 < x \leq 1$ $F(z)$ twists in the imaginary plane in accordance with its phase angle. For $x > 1$ the twisting becomes progressively smaller, approaching the imaginary axis along the whole jy range for $x = \infty$. For $x < 0$ (not shown here) $F(z)$ is a mirror image of this figure, reflected in the $j\Im\{F(z)\}$, $j\Im\{z\}$ plane.

Let us now plot the surface using the same axis assignments as in Fig. A1.7, but extending the real range to negative values as well in order to have a complete view. Owing to the finite density of plotted data we must limit the function values so that its absolute value remains within the axis range. This will enable us to see some detail.

The result is shown in Fig. A1.8. Now we have a clue of why the integration along a closed path (which does not encircle the pole) is always zero : such a path will cover an equal positive and negative area between the surface and the complex argument domain. Closer to the pole the height is larger, but the path length is shorter and vice versa.

It is now also clear why the integral along the path which encircles the pole once is always equal to $2\pi j$: owing to the shape of the surface any circle in the argument's domain is transformed into a circle in the function domain (as in Fig. A1.6), crossing the real axis in two points only, while all the remaining points of the circle are purely imaginary.

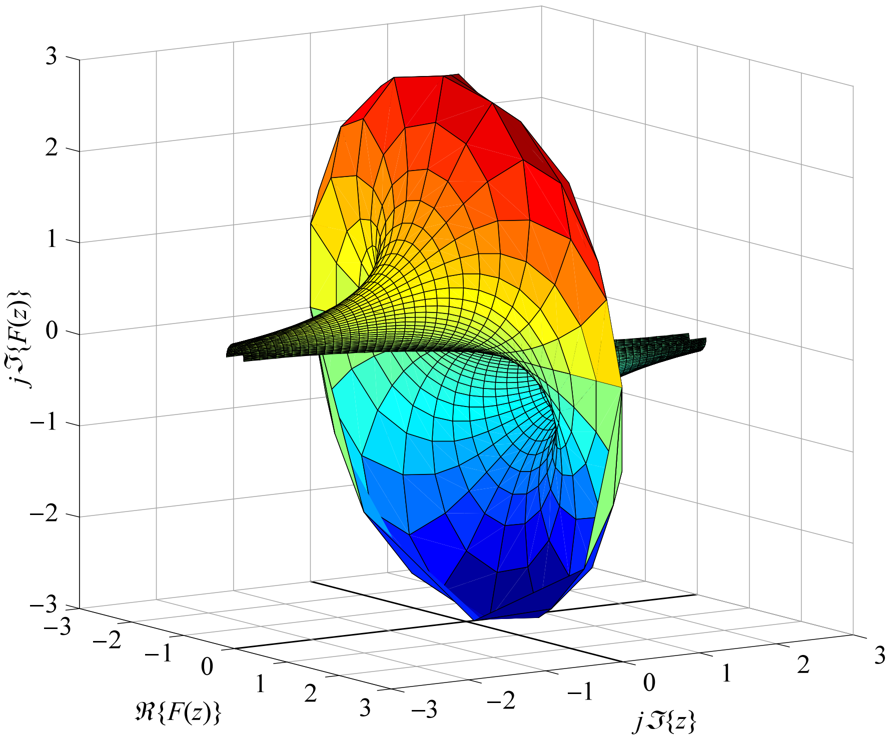

Fig. A1.8: As in Fig. A1.7, but using the surface plot. The shape of the surface looks like an old phonograph horn (but hyperbolically shaped, instead of exponentially), cut in half and with the upper side pulled out through the horn's mouth. The lines should be curved smoothly, but owing to the finite density of the argument domain grid, they quickly become straight lines. Note also that both 'throats' of the horn should form a full circle, the one in front lying all above zero and the rear one below zero; however, we have limited the $1/\Re\{z\}$ range to $-3 < x < 3$, in order to show the most interesting details and also to limit the graphics' size in bytes.

Finally, it is clear why the result of integration along a curve depends only on the starting and end points and not on the actual path taken: the area representing the integration result is also twisted in 3D. We shall explore this in a little more detail in Fig. A1.9 .

Let us have a straight line integration path L in the argument complex plane, starting at $z_1 = 0 - j\,0.5$ and ending at $z_2 = 0.5 + j\,0$, and let us make the plots of $F(z)$ in increments of $\Delta\,x = 0.05$.

If we split our integration path into N sections with same $\Delta\,x$, then, because L is at 45° with both axes, $\Delta\,y = \Delta\,x$. For each section L_k of our integration path we also calculate $F(L_k) = F(z_1 + k\,\Delta L) = F(z_1 + k\,\Delta x + j\,k\,\Delta y)$, where $k = 1, 2, \dots, N$. If we connect each point L_k with its appropriate $F(\,L_k\,)$ we end up with an approximation of the complex line integration result.

$$A = \sum_{k=0}^{N} F(L_k)\,\Delta\,L \tag{A1.11}$$

To obtain the true value we should have made our Δx and Δy very small, ideally $\rightarrow 0$, but then there would be an infinite number of segments to sum, exactly what the analytical integration requires:

$$A = \int_{z_1}^{z_2} F(z)\, dz \tag{A1.12}$$

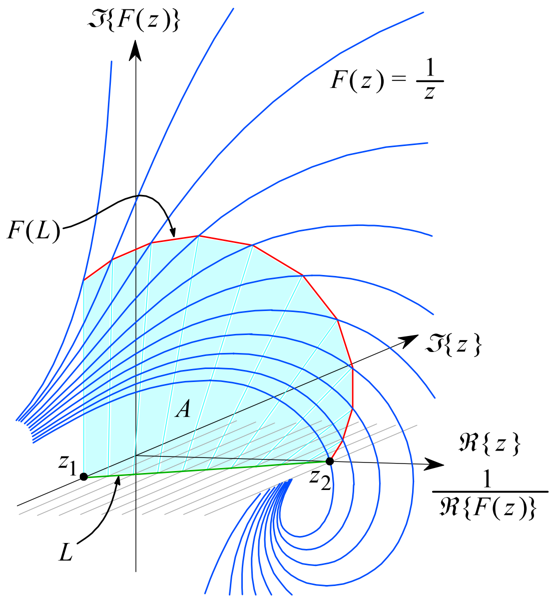

Fig. A1.9: The path L in the z plane (from z_1 to z_2) along which the integration of the complex function, shown as $F(L)$, is performed. The result is the area A.

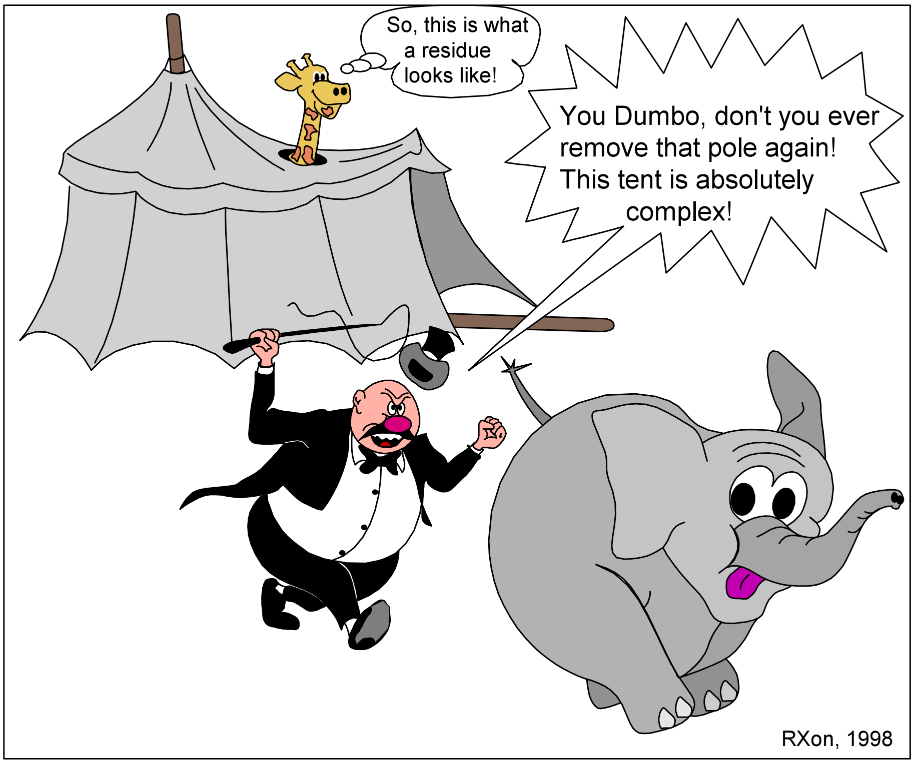

Fig. A1.10: Do not remove a pole unless you know exactly what you are doing!