Appendix 2.1

General Solutions for 1st-, 2nd-, 3rd- and 4th-order polynomials

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First-order polynomial: $a\,x + b = 0$

Canonical form: $x + \dfrac{b}{a} = 0$

Solution: $x = -\dfrac{b}{a}$

Second-order polynomial: $a\,x^2 + b\,x + c = 0$

Canonical form: $x^2 + \dfrac{b}{a}\,x + \dfrac{c}{a} = 0$

Solutions: $x_{1,2} = \dfrac{-b \mp \sqrt{b^2 - 4\,a\,c}}{2\,a}$

Third-order polynomial, canonical form:

$$x^3 + a\,x^2 + b\,x + c = 0$$

Solutions:
By substituting:

$$K = \sqrt{a^2 - 3b}$$

$$M = 4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2$$

$$N = 2a^3 - 9ab + 27c$$

the real solution is:

$$x_1 = -\frac{a}{3} - \frac{2}{3}\,K\,sin\,\frac{atan\,\dfrac{j\,N}{3\sqrt{3M}}}{3}$$

and the two complex conjugate solutions are:

$$x_2 = -\frac{a}{3} + K\,sin\,\frac{atan\,\dfrac{j\,N}{3\sqrt{3M}}}{3} - \frac{\sqrt{3}}{3}\,K\,cos\,\frac{atan\,\dfrac{j\,N}{3\sqrt{3M}}}{3}$$

$$x_3 = -\frac{a}{3} + K\,sin\,\frac{atan\,\dfrac{j\,N}{3\sqrt{3M}}}{3} + \frac{\sqrt{3}}{3}\,K\,cos\,\frac{atan\,\dfrac{j\,N}{3\sqrt{3M}}}{3}$$

Note: If preferred, here is a purely algebraic (non-trigonometric) result, obtained by using the new symbolic calculus capability of Matlab (Version 5.3 for Students). The command lines needed are simply:

```
syms x a b c      % define x, a, b and c as symbols
r = solve( x^3 + a*x^2 + b*x + c ) ;
```

The real solution is:

```
r(1) =
   1/6*
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)-
   6*(1/3*b-1/9*a^2)/
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)-
   1/3*a
```

The two complex-conjugate solutions are:

```
r(2) =
   -1/12*
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)+
   3*(1/3*b-1/9*a^2)/
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)-
   1/3*a+1/2*i*3^(1/2)*
   (1/6*
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)+
   6*(1/3*b-1/9*a^2)/
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)
   )
```

```
r(3) =
   -1/12*
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)+
   3*(1/3*b-1/9*a^2)/
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)-
   1/3*a-1/2*i*3^(1/2)*
   (1/6*
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)+
   6*(1/3*b-1/9*a^2)/
      (36*a*b-108*c-8*a^3+12*
         (12*b^3-3*b^2*a^2-54*b*c*a+81*c^2+12*c*a^3)^(1/2)
      )^(1/3)
   )
```

Fourth-order equation, canonical form:

$$x^4 + a\,x^3 + b\,x^2 + c\,x + d = 0$$

Solutions:
The roots are identical to the roots of two lower order equations:

$$x^2 + (a + A)\frac{x}{2} + \left(y + \frac{ay - c}{A}\right) = 0$$

where:

$$A = \pm\sqrt{8y + a^2 - 4b}$$

and y is any real root of the third-order equation:

$$8y^3 - 4by + (2ac - 8d)y + d(4b - a^2) - c^2 = 0$$

As has been proven by the French mathematician *Evariste Galois* (1811-1832), the solutions of polynomials of order 5 or higher can not be expressed analytically as rational functions of the polynomial coefficients. In such cases, the roots can be found by numerical computation methods (users of Matlab can try the ROOTS routine, which calculates the roots from polynomial coefficients by numerical methods; see also the POLY routine, which finds the coefficients from the roots).
