Appendix 2.3
General Solutions for the Step Response of Third- and Fourth-Order Systems
(with some unpleasant surprises!)

There is no such thing as instant experience!
(Oppenheimer's Law)

For every experience you pay with part of your life.
For inexperience you pay with your whole life.
(Yi-Ching)

Are you experienced?
(Jimi Hendrix)

Contents:

List of Figures:

A2.3.1 Third-order system with one pole real and one complex-conjugate pole-pair

A normalized general form of a third-order all pole s domain function:

$$F(s) = \frac{(-1)^3 s_1 s_2 s_3}{(s-s_1)(s-s_2)(s-s_3)} \tag{A2.3.1.1}$$

The pole components are:

$$\begin{aligned} s_1 &= \sigma_1 + j\omega_1 \\ s_2 &= \sigma_1 - j\omega_1 \\ s_3 &= \sigma_3 \end{aligned} \tag{A2.3.1.2}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s} F(s) = \frac{-s_1 s_2 s_3}{s(s-s_1)(s-s_2)(s-s_3)} \tag{A2.3.1.3}$$

The Inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{3} \mathrm{res}_i\,[\,G(s)\,]$$

Residue 0:

$$\mathrm{res}_0 = \lim_{s\to 0} s\, \frac{-s_1 s_2 s_3}{s(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st} \tag{A2.3.1.4}$$

$$= \lim_{s\to 0} \frac{-s_1 s_2 s_3}{(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-s_1 s_2 s_3}{(0-s_1)(0-s_2)(0-s_3)}\, \mathrm{e}^{0t} = 1$$

Residue 1:

$$\mathrm{res}_1 = \lim_{s\to s_1} (s-s_1)\, \frac{-s_1 s_2 s_3}{s(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st} \tag{A2.3.1.5}$$

$$= \lim_{s\to s_1} \frac{-s_1 s_2 s_3}{s(s-s_2)(s-s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-s_1 s_2 s_3}{s_1(s_1-s_2)(s_1-s_3)}\, \mathrm{e}^{s_1 t}$$

$$= \frac{-s_2 s_3}{(s_1-s_2)(s_1-s_3)}\, \mathrm{e}^{s_1 t}$$

Residue 2:

$$\mathrm{res}_2 = \lim_{s \to s_2} (s - s_2) \frac{-\, s_1\, s_2\, s_3}{s\,(s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.1.6}$$

$$= \lim_{s \to s_2} \frac{-\, s_1\, s_2\, s_3}{s\,(s - s_1)(s - s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-\, s_1\, s_2\, s_3}{s_2\,(s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_2 t}$$

$$= \frac{-\, s_1\, s_3}{(s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_1 t}$$

Residue 3:

$$\mathrm{res}_3 = \lim_{s \to s_3} (s - s_3) \frac{-\, s_1\, s_2\, s_3}{s\,(s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.1.7}$$

$$= \lim_{s \to s_3} \frac{-\, s_1\, s_2\, s_3}{s\,(s - s_1)(s - s_2)}\, \mathrm{e}^{st}$$

$$= \frac{-\, s_1\, s_2\, s_3}{s_3\,(s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

$$= \frac{-\, s_1\, s_2}{(s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

If we insert the pole components the residues are:

$$\mathrm{res}_0 = 1 \tag{A2.3.1.8}$$

$$\mathrm{res}_1 = \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3}{(\sigma_1 + j\,\omega_1 - \sigma_1 + j\,\omega_1)(\sigma_1 + j\,\omega_1 - \sigma_3)}\, \mathrm{e}^{(\sigma_1 + j\,\omega_1)t} \tag{A2.3.1.9}$$

$$= \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3}{2j\,\omega_1(\sigma_1 - \sigma_3 + j\,\omega_1)}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\,\omega_1 t}$$

$$= \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3\,(\sigma_1 - \sigma_3 - j\,\omega_1)}{2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\,\omega_1 t}$$

$$= \frac{-\,\sigma_3\,[\sigma_1(\sigma_1 - \sigma_3) - j\,\omega_1(\sigma_1 - \sigma_3) - j\,\omega_1\sigma_1 -\, \omega_1^2]}{2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\,\omega_1 t}$$

$$= \frac{-\,\sigma_3\,[\sigma_1(\sigma_1 - \sigma_3) -\, \omega_1^2 - j\,\omega_1(2\,\sigma_1 - \sigma_3)]}{2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\,\omega_1 t}$$

$$\mathrm{res}_2 = \frac{-\,(\sigma_1 + j\,\omega_1)\,\sigma_3}{(\sigma_1 - j\,\omega_1 - \sigma_1 - j\,\omega_1)(\sigma_1 - j\,\omega_1 - \sigma_3)}\;\mathrm{e}^{(\sigma_1 - j\,\omega_1)t} \tag{A2.3.1.10}$$

$$= \frac{-\,(\sigma_1 + j\,\omega_1)\,\sigma_3}{-\,2j\,\omega_1(\sigma_1 - \sigma_3 - j\,\omega_1)}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$= \frac{-\,(\sigma_1 + j\,\omega_1)\,\sigma_3\,(\sigma_1 - \sigma_3 + j\,\omega_1)}{-\,2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$= \frac{-\,\sigma_3\,[\sigma_1(\sigma_1 - \sigma_3) + j\,\omega_1(\sigma_1 - \sigma_3) + j\,\omega_1\sigma_1 -\,\omega_1^2]}{-\,2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$= \frac{-\,\sigma_3\,[\sigma_1(\sigma_1 - \sigma_3) -\,\omega_1^2 + j\,\omega_1(2\,\sigma_1 - \sigma_3)]}{-\,2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$\mathrm{res}_3 = \frac{-\,(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)}{(\sigma_3 - \sigma_1 - j\,\omega_1)(\sigma_3 - \sigma_1 + j\,\omega_1)}\;\mathrm{e}^{\sigma_3 t} \tag{A2.3.1.11}$$

$$= \frac{-\,(\sigma_1^2 +\,\omega_1^2)}{(\sigma_3 - \sigma_1)^2 +\,\omega_1^2}\;\mathrm{e}^{\sigma_3 t}$$

We shall simplify the expressions of residues by using the following substitutions:

$$A = \sigma_1(\sigma_1 - \sigma_3) -\,\omega_1^2 \tag{A2.3.1.12}$$

$$B = 2\,\sigma_1 - \sigma_3 \tag{A2.3.1.13}$$

$$C = (\sigma_1 - \sigma_3)^2 + \omega_1^2 \tag{A2.3.1.14}$$

For the computation of $g(t)$ we could simply sum all residues in the complex exponential form, but for the human mind it is always easier to interpret the equations in terms of expressions which represent some part of the system behavior. The complex conjugate pole pair determines the system resonant frequency and we would like to see its influence on the step response.

$$\mathrm{res}_1 + \mathrm{res}_2 = \frac{-\,\sigma_3\,(A - j\,\omega_1 B)}{2j\,\omega_1 C}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t} + \frac{-\,\sigma_3\,(A + j\,\omega_1 B)}{-\,2j\,\omega_1 C}\;\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t} \tag{A2.3.1.15}$$

$$= \frac{-\,\sigma_3}{C}\;\mathrm{e}^{\sigma_1 t}\left[\frac{A - j\,\omega_1 B}{2j\,\omega_1}\;\mathrm{e}^{j\,\omega_1 t} + \frac{A + j\,\omega_1 B}{-\,2j\,\omega_1}\;\mathrm{e}^{-j\,\omega_1 t}\right]$$

$$= \frac{-\,\sigma_3}{C}\;\mathrm{e}^{\sigma_1 t}\left[\frac{A}{2j\,\omega_1}\;\mathrm{e}^{j\,\omega_1 t} - \frac{A}{2j\,\omega_1}\;\mathrm{e}^{-j\,\omega_1 t} - \frac{j\,\omega_1 B}{2j\,\omega_1}\;\mathrm{e}^{j\,\omega_1 t} - \frac{j\,\omega_1 B}{2j\,\omega_1}\;\mathrm{e}^{-j\,\omega_1 t}\right]$$

$$= \frac{-\,\sigma_3}{C}\;\mathrm{e}^{\sigma_1 t}\left[\frac{A}{\omega_1}\;\frac{\mathrm{e}^{j\,\omega_1 t} -\,\mathrm{e}^{-j\,\omega_1 t}}{2j} - B\,\frac{\mathrm{e}^{j\,\omega_1 t} +\,\mathrm{e}^{-j\,\omega_1 t}}{2}\right]$$

$$= \frac{-\sigma_3}{\omega_1 C}\, \mathrm{e}^{\sigma_1 t}\left[A\,\frac{\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}}{2j} - \omega_1 B\,\frac{\mathrm{e}^{j\omega_1 t} + \mathrm{e}^{-j\omega_1 t}}{2}\right]$$

The two complex exponentials can be transformed into a sine and cosine:

$$\mathrm{res}_1 + \mathrm{res}_2 = -\frac{\sigma_3}{\omega_1 C}\, \mathrm{e}^{\sigma_1 t}\left[A\sin(\omega_1 t) - \omega_1 B\cos(\omega_1 t)\right] \tag{A2.3.1.16}$$

The trigonometric sum in brackets can be further interpreted as a single sine function, shifted in phase and with modified amplitude, in accordance with the following trigonometric transformation:

$$P\sin(\omega_1 t) + Q\cos(\omega_1 t) = \sqrt{P^2 + Q^2}\,\sin\!\left(\omega_1 t + \arctan\frac{Q}{P}\right) \tag{A2.3.1.17}$$

So:

$$\mathrm{res}_1 + \mathrm{res}_2 = -\frac{\sigma_3}{\omega_1 C}\, \mathrm{e}^{\sigma_1 t}\sqrt{A^2 + \omega_1^2 B^2}\,\sin\!\left(\omega_1 t + \arctan\frac{-\omega_1 B}{A}\right) \tag{A2.3.1.18}$$

Let us replace the arctangent expression with the angle θ:

$$\theta = \arctan\frac{-\omega_1 B}{A} = \arctan\frac{-\omega_1(2\sigma_1 - \sigma_3)}{\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2} \tag{A2.3.1.19}$$

Finally, we can write the step response function:

$$\boxed{g(t) = 1 - \frac{\sigma_3}{\omega_1 C}\sqrt{A^2 + \omega_1^2 B^2}\, \mathrm{e}^{\sigma_1 t}\sin(\omega_1 t + \theta) - \frac{\sigma_1^2 + \omega_1^2}{C}\, \mathrm{e}^{\sigma_3 t}} \tag{A2.3.1.20}$$

Important note on θ:

If we insert the normalized pole values (of, say, a 3rd-order Butterworth) in Eq. A2.3.1.20 and check the initial and final values of $g(t)$ for $t = 0$ and $t = \infty$, we obtain correctly $g(0) = 0$ and $g(\infty) = +1$.

However, if we plot $g(t)$ from $t = 0$ to $t = 5$ with $\Delta t = 0.05$, the response will be **wrong**, (see Fig. A2.3.1), since we know that for any function having the number of poles greater than the number of zeros by at least 2 the derivative $dg(t)/dt$ must also be zero for $t \rightarrow 0^+$.

Where is the error?

By trial and error one might discover that the plot would be correct if the sign of the second term in $g(t)$ is positive. But by rechecking the calculation of $\mathrm{res}_1 + \mathrm{res}_2$ we can verify that the sign should be negative. So the error could be in the sign of θ. But if we check its value we can see that $\theta = 0$ (or at least very close to zero, depending on the precision of the pole values), therefore the sign of θ does not make any difference. **Now, what??**

The problem is hidden in the tangent function, which can not distinguish between the angles of 0 and π radians (apart from our inability to use a true value of π, but instead only a finite number of its decimal places). For the same reason the function $\arctan(\tan\pi)$ always returns 0 and never π.

However, whilst (for $t = 0$) $\sin(\omega_1 0 + 0) = \sin(\omega_1 0 + \pi) = 0$, for $t < 1/\omega_1$ the value of $\sin(\omega_1 t + 0)$ is positive and the value of $\sin(\omega_1 t + \pi)$ is negative!

In short, it must be (**but not always!!!, see** Fig. A2.3.3 **in Sec. A2.3.5**):

$$\boxed{\theta \Rightarrow \theta + \pi} \tag{A2.3.1.21}$$

This can be verified if we plot $g(t)$ by using Eq. A2.3.1.16 in place of the middle term of Eq. A2.3.1.20 — the sine-cosine combination retains the correct phase information! Likewise the correct result is obtained if we simply sum the residues in their complex exponential form!

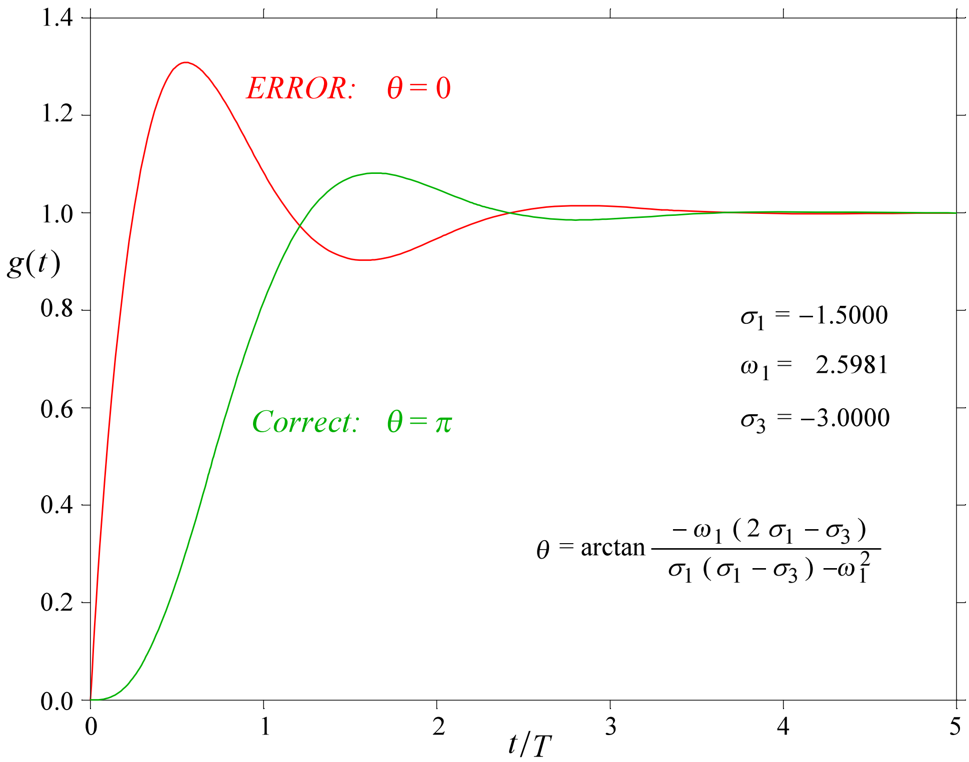

Fig. A2.3.1: To obtain the correct step response plot we must assign the value π to the initial phase angle θ (and not zero, as suggested by the arctangent operation).

This is a nice illustration of why some mathematicians refer to trigonometric transformations like Eq. A2.3.1.17 as 'malversations'!

Whilst it is very convenient to think of circuit action in terms of sine waves, for numerical calculation we shall rather stick to the complex exponential form.

Interpretation of the resulting expression

In terms of circuit action the general third-order expression for $g(t)$ can be interpreted as follows:

a) The value of res_0 (which in the case of normalized poles and normalized gain is equal to 1) represents the (normalized) final value to which the system will stabilize when $t \rightarrow \infty$ (or, since frequency is the inverse of time, $s = 0$).

b) The system reaction owed to the complex conjugate pole pair is represented by the second term: the input step excites the system into a sinusoidal oscillation at its resonant frequency, ω_1, shifted initially in phase by θ. The amplitude of this oscillation is normalized by a real correction factor, the value of which is determined by the impedance values at resonance. The oscillation is damped by the real and negative factor σ_1 in the exponent, which denotes energy dissipation in resistive circuit elements and thus forces the oscillations to decrease with time.

c) The final term represents the system reaction owed to the real pole $s_3 = \sigma_3$. Since the value of σ_3 is also negative this is also a decaying exponential function, again corrected in amplitude by a real factor, so that the sum of all terms at the time $t = 0$ is zero.

d) There remains to be cleared up the apparently mysterious initial phase shift θ. Any circuit with a complex conjugate pole pair has a capacitance and an inductance (in 'active' circuits the inductive behavior can be simulated by a capacitance with regenerative feedback). When coupled together they form a resonant circuit in which the voltage and current become derivatives of each other, i.e., a sine–cosine relationship. This makes sense of relations such as Eq. A2.3.1.16. The transient excitation is a signal which changes in a very short time interval, too short for the system to follow, so all the energy goes into the excitation of the system's resonance and its consequent relaxation. At resonance there is only one frequency (for each complex conjugate pole pair) by which the voltage and current vary, so there is some sense in interpreting the phase as the ratio of the relative voltage and current amplitudes, which in turn are set by the impedance ratios.

A2.3.2 Third-order system with all poles real, one pole pair coincident

A normalized general form of a third-order double pole s domain function is:

$$F(s) = \frac{(-1)^3\, s_1^2\, s_3}{(s-s_1)^2 (s-s_3)} \tag{A2.3.2.1}$$

The pole components (all poles real, one coincident pair) are:

$$\begin{aligned} s_1 &= s_2 = \sigma_1 \\ s_3 &= \sigma_3 \end{aligned} \tag{A2.3.2.2}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s}\, F(s) = \frac{-\, s_1^2\, s_3}{s\,(s-s_1)^2 (s-s_3)} \tag{A2.3.2.3}$$

The inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{3} \operatorname{res}_i [\,G(s)\,]$$

Residue 0:

$$\begin{aligned} \operatorname{res}_0 &= \lim_{s\to 0} s\, \frac{-\, s_1^2\, s_3}{s\,(s-s_1)^2 (s-s_3)}\, \mathrm{e}^{st} \\ &= \lim_{s\to 0} \frac{-\, s_1^2\, s_3}{(s-s_1)^2 (s-s_3)}\, \mathrm{e}^{st} \\ &= \frac{-\, s_1^2\, s_3}{(0-s_1)^2 (0-s_3)}\, \mathrm{e}^{0t} \\ &= \frac{-\, s_1^2\, s_3}{s_1^2 (-\, s_3)}\, 1 = 1 \end{aligned} \tag{A2.3.2.4}$$

Owing to the double pole at s_1, the calculation of res_1 is done by limiting the derivative:

$$\begin{aligned} \operatorname{res}_1 &= \lim_{s\to s_1} \frac{d}{ds} \left[(s-s_1)^2\, \frac{-\, s_1^2\, s_3}{s\,(s-s_1)^2 (s-s_3)}\, \mathrm{e}^{st} \right] \\ &= \lim_{s\to s_1} \frac{d}{ds} \left[\frac{-\, s_1^2\, s_3}{s\,(s-s_3)}\, \mathrm{e}^{st} \right] \\ &= \lim_{s\to s_1} \left[s_1^2\, s_3\, \frac{s_3(st-1) - s(st-2)}{s^2\,(s_3-s)^2}\, \mathrm{e}^{st} \right] \\ &= \lim_{s\to s_1} \left[s_1^2\, s_3\, \frac{-\, s^2 t + 2s + s_3 s_1 t - s_3}{s^2\,(s_3-s)^2}\, \mathrm{e}^{st} \right] \end{aligned} \tag{A2.3.2.5}$$

$$= s_1^2\, s_3\, \frac{-\, s_1^2 t + 2 s_1 + s_3 s_1 t - s_3}{s_1^2\,(s_3 - s_1)^2}\, \mathrm{e}^{s_1 t}$$

$$= s_3\, \frac{-\, s_1^2 t + s_1(s_3 t + 2) - s_3}{(s_3 - s_1)^2}\, \mathrm{e}^{s_1 t}$$

Residue 2 is calculated in the usual way:

$$\mathrm{res}_2 = \lim_{s \to s_3}\, (s - s_3)\, \frac{-\, s_1^2\, s_3}{s\,(s - s_1)^2 (s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.2.6}$$

$$= \lim_{s \to s_3}\, \frac{-\, s_1^2\, s_3}{s\,(s - s_1)^2}\, \mathrm{e}^{st}$$

$$= \frac{-\, s_1^2\, s_3}{s_3\,(s_3 - s_1)^2}\, \mathrm{e}^{s_3 t}$$

$$= \frac{-\, s_1^2}{(s_3 - s_1)^2}\, \mathrm{e}^{s_3 t}$$

We insert the pole components:

$$\mathrm{res}_0 = 1 \tag{A2.3.2.7}$$

$$\mathrm{res}_1 = \sigma_3\, \frac{-\, \sigma_1^2 t + \sigma_1(\sigma_3 t + 2) - \sigma_3}{(\sigma_3 - \sigma_1)^2}\, \mathrm{e}^{\sigma_1 t} \tag{A2.3.2.8}$$

$$\mathrm{res}_2 = \frac{-\, \sigma_1^2}{(\sigma_3 - \sigma_1)^2}\, \mathrm{e}^{\sigma_3 t} \tag{A2.3.2.9}$$

So the step response is:

$$g(t) = 1 - \sigma_3\, \frac{\sigma_1^2 t - \sigma_1(\sigma_3 t + 2) + \sigma_3}{(\sigma_3 - \sigma_1)^2}\, \mathrm{e}^{\sigma_1 t} - \frac{\sigma_1^2}{(\sigma_3 - \sigma_1)^2}\, \mathrm{e}^{\sigma_3 t} \tag{A2.3.2.10}$$

With a little rearrangement of the second term we can simplify the linear function of t to finally obtain:

$$\boxed{g(t) = 1 - \frac{\sigma_3^2}{(\sigma_3 - \sigma_1)^2} \left[\frac{\sigma_1}{\sigma_3} (\sigma_1 - \sigma_3)\, t + 2 \frac{\sigma_1}{\sigma_3} + 1 \right] \mathrm{e}^{\sigma_1 t} - \frac{\sigma_1^2}{(\sigma_3 - \sigma_1)^2}\, \mathrm{e}^{\sigma_3 t}} \tag{A2.3.2.11}$$

Since all three poles are real the system has no characteristic resonant frequency but only two decaying exponentials. Again, the value of the first residue ($\mathrm{res}_0 = 1$) is the normalized final value to which the system will settle. Also, since at $t = 0$ both exponentials are equal to 1, the sum of the factors that multiply them must be equal to -1 in order to make $g(0) = 0$. The linear time function within the second term slows down the response for small values of t.

A2.3.3 Third-order system with one real pole, one complex conjugate pole pair and one real zero

A normalized general form of a 3-pole and 1-zero s domain function is:

$$F(s) = \frac{(-1)^3\, s_1\, s_2\, s_3}{(s-s_1)(s-s_2)(s-s_3)} \cdot \frac{(s-s_\mathrm{z})}{-s_\mathrm{z}} \tag{A2.3.3.1}$$

The pole and zero components are:

$$\begin{aligned} s_1 &= \sigma_1 + j\,\omega_1 \\ s_2 &= \sigma_1 - j\,\omega_1 \\ s_3 &= \sigma_3 \\ s_\mathrm{z} &= \sigma_\mathrm{z} \end{aligned} \tag{A2.3.3.2}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s}\,F(s) = \frac{1}{s} \cdot \frac{s_1\, s_2\, s_3}{(s-s_1)(s-s_2)(s-s_3)} \cdot \frac{(s-s_\mathrm{z})}{s_\mathrm{z}} \tag{A2.3.3.3}$$

The inverse Laplace transform of $G(s)$:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{3} \mathrm{res}_i\,[\,G(s)\,]$$

Residue 0:

$$\mathrm{res}_0 = \lim_{s\to 0} s\,\frac{(s-s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\,(s-s_1)(s-s_2)(s-s_3)}\,\mathrm{e}^{st} \tag{A2.3.3.4}$$

$$= \lim_{s\to 0} \frac{(s-s_\mathrm{z})\, s_1\, s_2\, s_3}{s_\mathrm{z}\,(s-s_1)(s-s_2)(s-s_3)}\,\mathrm{e}^{st}$$

$$= \frac{(0-s_\mathrm{z})\, s_1\, s_2\, s_3}{s_\mathrm{z}\,(0-s_1)(0-s_2)(0-s_3)}\,\mathrm{e}^{0t} = 1$$

Residue 1:

$$\mathrm{res}_1 = \lim_{s\to s_1} (s-s_1)\,\frac{(s-s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\,(s-s_1)(s-s_2)(s-s_3)}\,\mathrm{e}^{st} \tag{A2.3.3.5}$$

$$= \lim_{s\to s_1} \frac{(s-s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\,(s-s_2)(s-s_3)}\,\mathrm{e}^{st}$$

$$= \frac{(s_1-s_\mathrm{z})\, s_1\, s_2\, s_3}{s_1\, s_\mathrm{z}\,(s_1-s_2)(s_1-s_3)}\,\mathrm{e}^{s_1 t}$$

$$= \frac{(s_1-s_\mathrm{z})\, s_2\, s_3}{s_\mathrm{z}\,(s_1-s_2)(s_1-s_3)}\,\mathrm{e}^{s_1 t}$$

Residue 2:

$$\mathrm{res}_2 = \lim_{s \to s_2} (s - s_2)\, \frac{(s - s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\, (s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.3.6}$$

$$= \lim_{s \to s_2} \frac{(s - s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\, (s - s_1)(s - s_3)}\, \mathrm{e}^{st}$$

$$= \frac{(s_2 - s_\mathrm{z})\, s_1\, s_2\, s_3}{s_2\, s_\mathrm{z}\, (s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_2 t}$$

$$= \frac{(s_2 - s_\mathrm{z})\, s_1\, s_3}{s_\mathrm{z}\, (s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_2 t}$$

Residue 3:

$$\mathrm{res}_3 = \lim_{s \to s_3} (s - s_3)\, \frac{(s - s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\, (s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.3.7}$$

$$= \lim_{s \to s_3} \frac{(s - s_\mathrm{z})\, s_1\, s_2\, s_3}{s\, s_\mathrm{z}\, (s - s_1)(s - s_2)}\, \mathrm{e}^{st}$$

$$= \frac{(s_3 - s_\mathrm{z})\, s_1\, s_2\, s_3}{s_3\, s_\mathrm{z}\, (s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

$$= \frac{(s_3 - s_\mathrm{z})\, s_1\, s_2}{s_\mathrm{z}\, (s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

We now insert the real and imaginary components of the poles and the zero into the last three residues:

$$\mathrm{res}_1 = \frac{(\sigma_1 + j\,\omega_1 - \sigma_\mathrm{z})\,(\sigma_1 - j\,\omega_1)\,\sigma_3}{\sigma_\mathrm{z}\,(\sigma_1 + j\,\omega_1 - \sigma_1 + j\,\omega_1)(\sigma_1 + j\,\omega_1 - \sigma_3)}\, \mathrm{e}^{(\sigma_1 + j\,\omega_1)t} \tag{A2.3.3.8}$$

$$= \frac{\sigma_3\,[\sigma_1^2 + \omega_1^2 - \sigma_\mathrm{z}\,(\sigma_1 - j\,\omega_1)]}{\sigma_\mathrm{z}\, 2j\,\omega_1(\sigma_1 - \sigma_3 + j\,\omega_1)}\, \mathrm{e}^{\sigma_1 t}\mathrm{e}^{j\,\omega_1 t}$$

$$= \frac{\sigma_3\,[\sigma_1^2 + \omega_1^2 - \sigma_\mathrm{z}\,(\sigma_1 - j\,\omega_1)](\sigma_1 - \sigma_3 - j\,\omega_1)}{\sigma_\mathrm{z}\,(\sigma_1 - \sigma_3 + j\,\omega_1)(\sigma_1 - \sigma_3 - j\,\omega_1)}\, \mathrm{e}^{\sigma_1 t}\, \frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1}$$

$$= \frac{\sigma_3\,[\sigma_1^2 + \omega_1^2 - \sigma_\mathrm{z}\,(\sigma_1 - j\,\omega_1)](\sigma_1 - \sigma_3 - j\,\omega_1)}{\sigma_\mathrm{z}\,\left[(\sigma_1 - \sigma_3)^2 + \omega_1^2\right]}\, \mathrm{e}^{\sigma_1 t}\, \frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1}$$

The common real factor, which will be used also in res_2, is:

$$K_1 = \frac{\sigma_3}{\sigma_\mathrm{z}\,\left[(\sigma_1 - \sigma_3)^2 + \omega_1^2\right]} \tag{A2.3.3.9}$$

The remaining factor in the numerator is separated into its real and imaginary part:

$$
\begin{aligned}
&[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1-j\,\omega_1)](\sigma_1-\sigma_3-j\,\omega_1)\\
&\quad=[(\sigma_1^2+\omega_1^2)-\sigma_z\,\sigma_1](\sigma_1-\sigma_3)+j\,\omega_1\sigma_z\,(\sigma_1-\sigma_3)\\
&\qquad\quad-j\,\omega_1[(\sigma_1^2+\omega_1^2)-\sigma_z\,\sigma_1]+\omega_1^2\sigma_z\\
&\quad=(\sigma_1-\sigma_3)[(\sigma_1^2+\omega_1^2)-\sigma_z\,\sigma_1]+\omega_1^2\sigma_z\\
&\qquad\quad-j\,\omega_1[(\sigma_1^2+\omega_1^2)-\sigma_z\,(2\,\sigma_1-\sigma_3)]
\end{aligned}
$$

With the following substitutions:

$$
\begin{aligned}
A&=(\sigma_1-\sigma_3)[\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1]+\omega_1^2\sigma_z\\
B&=[\sigma_1^2+\omega_1^2-\sigma_z(2\,\sigma_1-\sigma_3)]
\end{aligned}
\tag{A2.3.3.10}
$$

we can write:

$$
\mathrm{res}_1=K_1\,\mathrm{e}^{\sigma_1 t}\,(A-j\,\omega_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1}
\tag{A2.3.3.11}
$$

Equally, for res_2:

$$
\mathrm{res}_2=\frac{(\sigma_1-j\,\omega_1-\sigma_z)\,(\sigma_1+j\,\omega_1)\,\sigma_3}{\sigma_z\,(\sigma_1-j\,\omega_1-\sigma_1-j\,\omega_1)(\sigma_1-j\,\omega_1-\sigma_3)}\;\mathrm{e}^{(\sigma_1-j\,\omega_1)t}
\tag{A2.3.3.12}
$$

$$
=\frac{\sigma_3\,[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1+j\,\omega_1)]}{\sigma_z\,(\,-2j\,\omega_1)(\sigma_1-\sigma_3-j\,\omega_1)}\;\mathrm{e}^{\sigma_1 t}\mathrm{e}^{-j\,\omega_1 t}
$$

$$
=\frac{\sigma_3\,[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1+j\,\omega_1)](\sigma_1-\sigma_3+j\,\omega_1)}{\sigma_z\,(\sigma_1-\sigma_3-j\,\omega_1)(\sigma_1-\sigma_3+j\,\omega_1)}\;\mathrm{e}^{\sigma_1 t}\;\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1}
$$

$$
=\frac{\sigma_3\,[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1+j\,\omega_1)](\sigma_1-\sigma_3+j\,\omega_1)}{\sigma_z\,\left[(\sigma_1-\sigma_3)^2+\omega_1^2\right]}\;\mathrm{e}^{\sigma_1 t}\;\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1}
$$

We extract the common real factor K_1 as before and rearrange the remaining factor into its real and imaginary part:

$$
\begin{aligned}
&[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1+j\,\omega_1)](\sigma_1-\sigma_3+j\,\omega_1)\\
&\quad=[\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1](\sigma_1-\sigma_3)-\sigma_z\,j\,\omega_1(\sigma_1-\sigma_3)\\
&\qquad\quad+j\,\omega_1[\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1]+\omega_1^2\sigma_z\\
&\quad=[\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1](\sigma_1-\sigma_3)+\omega_1^2\sigma_z\\
&\qquad\quad+j\,\omega_1[\sigma_1^2+\omega_1^2-\sigma_z\,(2\,\sigma_1-\sigma_3)]
\end{aligned}
$$

By using the same substitutions A and B as before, we can write:

$$\mathrm{res}_2 = K_1\,\mathrm{e}^{\sigma_1 t}\,(A + j\,\omega_1 B)\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1} \tag{A2.3.3.13}$$

For res_3 we have:

$$\begin{aligned}\mathrm{res}_3 &= \frac{(\sigma_3 - \sigma_\mathrm{z})\,(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)}{\sigma_\mathrm{z}\,(\sigma_3 - \sigma_1 - j\,\omega_1)(\sigma_3 - \sigma_1 + j\,\omega_1)}\,\mathrm{e}^{\sigma_3 t} \\ &= \frac{(\sigma_3 - \sigma_\mathrm{z})\,(\sigma_1^2 + \omega_1^2)}{\sigma_\mathrm{z}\,[(\sigma_3 - \sigma_1)^2 + \omega_1^2]}\,\mathrm{e}^{\sigma_3 t} \\ &= K_3\,\mathrm{e}^{\sigma_3 t}\end{aligned} \tag{A2.3.3.14}$$

The step response is the sum of the residues:

$$g(t) = 1 + K_1\,\mathrm{e}^{\sigma_1 t}\,(A - j\,\omega_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1} + K_1\,\mathrm{e}^{\sigma_1 t}\,(A + j\,\omega_1 B)\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1} + K_3\,\mathrm{e}^{\sigma_3 t} \tag{A2.3.3.15}$$

Again, we transform the two complex conjugate residues into a sine–cosine pair:

$$\begin{aligned}&(A - j\,\omega_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1} + (A + j\,\omega_1 B)\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1} = \\ &= \frac{A}{\omega_1}\,\frac{\mathrm{e}^{j\omega_1 t} - \mathrm{e}^{-j\omega_1 t}}{2j} - B\,\frac{\mathrm{e}^{j\omega_1 t} + \mathrm{e}^{-j\omega_1 t}}{2} \\ &= \frac{A}{\omega_1}\sin\omega_1 t - B\cos\omega_1 t \\ &= \frac{1}{\omega_1}\,[\,A\sin\omega_1 t - \omega_1 B\cos\omega_1 t\,]\end{aligned}$$

This sine–cosine pair can be further transformed into a sine function with an appropriate phase and amplitude correction:

$$A\sin\omega_1 t - \omega_1 B\cos\omega_1 t = \sqrt{A^2 + \omega_1^2 B^2}\;\sin\!\left(\omega_1 t + \arctan\frac{-\,\omega_1 B}{A}\right)$$

Finally, we write the step response as:

$$\boxed{g(t) = 1 + \frac{1}{\omega_1} K_1\,\mathrm{e}^{\sigma_1 t}\,\sqrt{A^2 + \omega_1^2 B^2}\;\sin(\omega_1 t + \theta) + K_3\,\mathrm{e}^{\sigma_3 t}} \tag{A2.3.3.16}$$

where:

$$\theta = \arctan\frac{-\,\omega_1 B}{A} \tag{A2.3.3.17}$$

However, as has already been shown in the first example, here too we must increase the phase angle by π:

$$\theta \Rightarrow \theta + \pi \tag{A2.3.3.18}$$

It is interesting to note that the zero $s_z = \sigma_z$ is a free factor in the denominator of both K_1 (Eq. A2.3.3.9) and K_3 (Eq. A2.3.3.14). It will therefore decrease the group delay, and, if brought too close to the complex plane origin, it would also increase the overshoot.

A2.3.4 Third-order system with one real pole, one complex conjugate pole pair, and one complex conjugate zero pair

A normalized general form of a 3-pole and 2-zero s domain function is:

$$F(s) = \frac{(-1)^3\, s_1\, s_2\, s_3}{(s-s_1)(s-s_2)(s-s_3)} \cdot \frac{(s-s_4)(s-s_5)}{(-1)^2 s_4 s_5} \tag{A2.3.4.1}$$

The pole and zero components are:

$$\begin{aligned} s_{1,2} &= \sigma_1 \pm j\,\omega_1 \\ s_3 &= \sigma_3 \\ s_{4,5} &= \sigma_\mathrm{z} \pm j\,\omega_\mathrm{z} \end{aligned} \tag{A2.3.4.2}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s}\,F(s) = \frac{1}{s} \cdot \frac{s_1\, s_2\, s_3}{(s-s_1)(s-s_2)(s-s_3)} \cdot \frac{(s-s_\mathrm{z})}{s_\mathrm{z}} \tag{A2.3.4.3}$$

The inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{3} \mathrm{res}_i\,[\,G(s)\,]$$

Residue 0:

$$\mathrm{res}_0 = \lim_{s\to 0} s\, \frac{-\,s_1\, s_2\, s_3 (s-s_4)(s-s_5)}{s\, s_4\, s_5\,(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st} \tag{A2.3.4.4}$$

$$= \lim_{s\to 0} \frac{-\,s_1\, s_2\, s_3 (s-s_4)(s-s_5)}{s_4\, s_5\,(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-\,s_1\, s_2\, s_3 (0-s_4)(0-s_5)}{s_4\, s_5\,(0-s_1)(0-s_2)(0-s_3)}\, \mathrm{e}^{0t} = 1$$

Residue 1:

$$\mathrm{res}_1 = \lim_{s\to s_1} (s-s_1)\, \frac{-\,s_1\, s_2\, s_3 (s-s_4)(s-s_5)}{s\, s_4\, s_5\,(s-s_1)(s-s_2)(s-s_3)}\, \mathrm{e}^{st} \tag{A2.3.4.5}$$

$$= \lim_{s\to s_1} \frac{-\,s_1\, s_2\, s_3 (s-s_4)(s-s_5)}{s\, s_4\, s_5\,(s-s_2)(s-s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-\,s_1\, s_2\, s_3 (s_1-s_4)(s_1-s_5)}{s_1 s_4\, s_5\,(s_1-s_2)(s_1-s_3)}\, \mathrm{e}^{s_1 t}$$

$$= \frac{-\,s_2\, s_3 (s_1-s_4)(s_1-s_5)}{s_4\, s_5\,(s_1-s_2)(s_1-s_3)}\, \mathrm{e}^{s_1 t}$$

Residue 2:

$$\mathrm{res}_2 = \lim_{s\to s_2} (s - s_2)\, \frac{-\, s_1\, s_2\, s_3 (s - s_4)(s - s_5)}{s\, s_4\, s_5\, (s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.4.6}$$

$$= \lim_{s\to s_2} \frac{-\, s_1\, s_2\, s_3 (s - s_4)(s - s_5)}{s\, s_4\, s_5\, (s - s_1)(s - s_3)}\, \mathrm{e}^{st}$$

$$= \frac{-\, s_1\, s_2\, s_3 (s_2 - s_4)(s_2 - s_5)}{s_2 s_4\, s_5\, (s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_2 t}$$

$$= \frac{-\, s_1\, s_3 (s_2 - s_4)(s_2 - s_5)}{s_4\, s_5\, (s_2 - s_1)(s_2 - s_3)}\, \mathrm{e}^{s_2 t}$$

Residue 3:

$$\mathrm{res}_3 = \lim_{s\to s_3} (s - s_3)\, \frac{-\, s_1\, s_2\, s_3 (s - s_4)(s - s_5)}{s\, s_4\, s_5\, (s - s_1)(s - s_2)(s - s_3)}\, \mathrm{e}^{st} \tag{A2.3.4.7}$$

$$= \lim_{s\to s_3} \frac{-\, s_1\, s_2\, s_3\, (s - s_4)(s - s_5)}{s\, s_4\, s_5\, (s - s_1)(s - s_2)}\, \mathrm{e}^{st}$$

$$= \frac{-\, s_1\, s_2\, s_3\, (s_3 - s_4)(s_3 - s_5)}{s_3 s_4\, s_5\, (s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

$$= \frac{-\, s_1\, s_2\, (s_3 - s_4)(s_3 - s_5)}{s_4\, s_5\, (s_3 - s_1)(s_3 - s_2)}\, \mathrm{e}^{s_3 t}$$

We insert the real and imaginary components of the poles ad zeros:

$$\mathrm{res}_1 = \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3(\sigma_1 + j\,\omega_1 - \sigma_\mathrm{z} - j\,\omega_\mathrm{z})(\sigma_1 + j\,\omega_1 - \sigma_\mathrm{z} + j\,\omega_\mathrm{z})}{(\sigma_\mathrm{z} + j\,\omega_\mathrm{z})(\sigma_\mathrm{z} - j\,\omega_\mathrm{z})(\sigma_1 + j\,\omega_1 - \sigma_1 + j\,\omega_1)(\sigma_1 + j\,\omega_1 - \sigma_3)}\, \mathrm{e}^{(\sigma_1 + j\,\omega_1)t}$$

$$= \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3[\sigma_1^2 - \omega_1^2 - 2\,\sigma_\mathrm{z}(\sigma_1 + j\,\omega_1) + \sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2]}{(\sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2)2\,j\,\omega_1(\sigma_1 - \sigma_3 + j\,\omega_1)}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\,\omega_1 t}$$

$$= \frac{-\,(\sigma_1 - j\,\omega_1)\,\sigma_3[\sigma_1^2 - \omega_1^2 - 2\,\sigma_\mathrm{z}(\sigma_1 + j\,\omega_1) + \sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2](\sigma_1 - \sigma_3 - j\,\omega_1)}{(\sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2)[(\sigma_1 - \sigma_3)^2 \ + \omega_1^2]}\, \mathrm{e}^{\sigma_1 t}\, \frac{\mathrm{e}^{j\,\omega_1 t}}{2\,j\,\omega_1}$$

Now we can extract the common real factor:

$$K_1 = \frac{\sigma_3}{(\sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2)[(\sigma_1 - \sigma_3)^2 \ + \omega_1^2]} \tag{A2.3.4.8}$$

and multiply and regroup the remaining factors into real and imaginary parts:

$$(\sigma_1 - j\,\omega_1)\big[\sigma_1^2 - \omega_1^2 - 2\,\sigma_\mathrm{z}(\sigma_1 + j\,\omega_1) + \sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2\big](\sigma_1 - \sigma_3 - j\,\omega_1)$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - j\,\omega_1(2\,\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 - 2\,\sigma_\mathrm{z}(\sigma_1 + j\,\omega_1) + \sigma_\mathrm{z}^2 + \omega_\mathrm{z}^2]$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2 - j\,\omega_1(2\,\sigma_1 - \sigma_3)][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z(\sigma_1 + j\,\omega_1)]$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\,\sigma_1]$$

$$- j\,\omega_1(2\,\sigma_1 - \sigma_3)[\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\sigma_1]$$

$$- 2\,\sigma_z j\,\omega_1[\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2]$$

$$- 2\,\sigma_z\,\omega_1^2\,(2\,\sigma_1 - \sigma_3)$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\,\sigma_1] - 2\,\sigma_z\,\omega_1^2\,(2\,\sigma_1 - \sigma_3)$$

$$- j\,\omega_1\{(2\,\sigma_1 - \sigma_3)[\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\sigma_1] + 2\,\sigma_z[\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2]\}$$

We substitute:

$$A = [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\,\sigma_1] - 2\,\sigma_z\,\omega_1^2\,(2\,\sigma_1 - \sigma_3) \tag{A2.3.4.9}$$

$$B = (2\,\sigma_1 - \sigma_3)[\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\sigma_1] + 2\,\sigma_z[\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2] \tag{A2.3.4.10}$$

With this, res_1 can be written as:

$$\mathrm{res}_1 = -K_1\,\mathrm{e}^{\sigma_1 t}\,(A - j\,\omega_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2\,j\,\omega_1} \tag{A2.3.4.11}$$

Repeat the procedure for res_2:

$$\mathrm{res}_2 = \frac{-(\sigma_1 + j\,\omega_1)\,\sigma_3(\sigma_1 - j\,\omega_1 - \sigma_z - j\,\omega_z)(\sigma_1 - j\,\omega_1 - \sigma_z + j\,\omega_z)}{(\sigma_z + j\,\omega_z)(\sigma_z - j\,\omega_z)(\sigma_1 - j\,\omega_1 - \sigma_1 - j\,\omega_1)(\sigma_1 - j\,\omega_1 - \sigma_3)}\,\mathrm{e}^{(\sigma_1 - j\,\omega_1)t}$$

$$= \frac{-(\sigma_1 + j\,\omega_1)\,\sigma_3[\sigma_1^2 - \omega_1^2 - 2\,\sigma_z(\sigma_1 - j\,\omega_1) + \sigma_z^2 + \omega_z^2]}{(\sigma_z^2 + \omega_z^2)(-2\,j\,\omega_1)(\sigma_1 - \sigma_3 - j\,\omega_1)}\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$= \frac{-(\sigma_1 + j\,\omega_1)\,\sigma_3[\sigma_1^2 - \omega_1^2 - 2\,\sigma_z(\sigma_1 - j\,\omega_1) + \sigma_z^2 + \omega_z^2](\sigma_1 - \sigma_3 + j\,\omega_1)}{(\sigma_z^2 + \omega_z^2)[(\sigma_1 - \sigma_3)^2 + \omega_1^2]}\,\mathrm{e}^{\sigma_1 t}\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-2\,j\,\omega_1}$$

We extract K_1 and reorder the remaining numerator factors:

$$(\sigma_1 + j\,\omega_1)\big[\sigma_1^2 - \omega_1^2 - 2\,\sigma_z(\sigma_1 - j\,\omega_1) + \sigma_z^2 + \omega_z^2\big](\sigma_1 - \sigma_3 + j\,\omega_1)$$

$$= [\sigma_1(\sigma_1 - \sigma_3) + j\,\omega_1(2\,\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 - 2\,\sigma_z(\sigma_1 - j\,\omega_1) + \sigma_z^2 + \omega_z^2]$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2 + j\,\omega_1(2\,\sigma_1 - \sigma_3)][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z(\sigma_1 - j\,\omega_1)]$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\,\sigma_1]$$

$$+ j\,\omega_1(2\,\sigma_1 - \sigma_3)[\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\sigma_1]$$

$$+ 2\,\sigma_z j\,\omega_1[\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2]$$

$$- 2\,\sigma_z\,\omega_1^2\,(2\,\sigma_1 - \sigma_3)$$

$$= [\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2][\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\,\sigma_1] - 2\,\sigma_z\,\omega_1^2\,(2\,\sigma_1 - \sigma_3)$$

$$+ j\,\omega_1\{(2\,\sigma_1 - \sigma_3)[\sigma_1^2 - \omega_1^2 + \sigma_z^2 + \omega_z^2 - 2\,\sigma_z\sigma_1] + 2\,\sigma_z[\sigma_1(\sigma_1 - \sigma_3) - \omega_1^2]\}$$

Using the same substitutions as for res_1, we have:

$$\mathrm{res}_2 = \; - K_1\,\mathrm{e}^{\sigma_1 t}\,(A + j\,\omega_1 B)\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2\,j\,\omega_1} \tag{A2.3.4.12}$$

For res_3 we have:

$$\mathrm{res}_3 = \frac{-\,(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)(\sigma_3 - \sigma_z - j\,\omega_z)(\sigma_3 - \sigma_z + j\,\omega_z)}{(\sigma_z + j\,\omega_z)(\sigma_z - j\,\omega_z)(\sigma_3 - \sigma_1 - j\,\omega_1)(\sigma_3 - \sigma_1 + j\,\omega_1)}\,\mathrm{e}^{\sigma_3 t}$$

$$= \frac{-\,(\sigma_1^2 + \omega_1^2)[(\sigma_3 - \sigma_z)^2 + \omega_z^2]}{(\sigma_z^2 + \omega_z^2)[(\sigma_3 - \sigma_1)^2 + \omega_1^2]}\,\mathrm{e}^{\sigma_3 t}$$

$$= \; - K_3\,\mathrm{e}^{\sigma_3 t} \tag{A2.3.4.13}$$

The sum of the residues is the step response sought:

$$g(t) = 1 - K_1\,\mathrm{e}^{\sigma_1 t}\,(A - j\,\omega_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2\,j\,\omega_1} - K_1\,\mathrm{e}^{\sigma_1 t}\,(A + j\,\omega_1 B)\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2\,j\,\omega_1} - K_3\,\mathrm{e}^{\sigma_3 t} \tag{A2.3.4.14}$$

which can be rewritten as:

$$g(t) = 1 - K_1\,\mathrm{e}^{\sigma_1 t}\left(\frac{1}{\omega_1}\,A\,\frac{\mathrm{e}^{j\,\omega_1 t} - \mathrm{e}^{-j\,\omega_1 t}}{2\,j} - B\,\frac{\mathrm{e}^{j\,\omega_1 t} + \mathrm{e}^{-j\,\omega_1 t}}{2}\right) - K_3\,\mathrm{e}^{\sigma_3 t}$$

and also as:

$$g(t) = 1 - \frac{1}{\omega_1}\,K_1\,\mathrm{e}^{\sigma_1 t}\,(\,A\sin\omega_1 t - \omega_1 B\cos\omega_1 t) - K_3\,\mathrm{e}^{\sigma_3 t} \tag{A2.3.4.15}$$

and also as:

$$\boxed{g(t) = 1 - \frac{1}{\omega_1}\,K_1\,\mathrm{e}^{\sigma_1 t}\sqrt{A^2 + \omega_1^2 B^2}\;\sin(\omega_1 t + \theta) - K_3\,\mathrm{e}^{\sigma_3 t}} \tag{A2.3.4.16}$$

where the phase angle θ is:

$$\theta = \arctan\left(\frac{-\,\omega_1 B}{A}\right) + \pi \qquad \left(\begin{array}{c}\text{Increase by } \pi \text{ where appropriate. When in}\\ \text{in doubt, use either A2.3.4.14, or A2.3.4.15}\end{array}\right)$$

A2.3.5 Fourth-order system with two complex conjugate pole pairs

A four-pole normalized general function is:

$$F(s) = \frac{(-1)^4\, s_1\, s_2\, s_3\, s_4}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \tag{A2.3.5.1}$$

In general we have two complex conjugate pole pairs, with the following components:

$$\begin{aligned} s_{1,2} &= \sigma_1 \pm j\,\omega_1 \\ s_{3,4} &= \sigma_3 \pm j\,\omega_3 \end{aligned} \tag{A2.3.5.2}$$

The step response in the s domain is a fifth-order function:

$$G(s) = \frac{1}{s}\,F(s) = \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \tag{A2.3.5.3}$$

The inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{4} \operatorname{res}_i\,[\,G(s)\,]$$

Residue 0:

$$\begin{aligned} \operatorname{res}_0 &= \lim_{s\to 0} s\,\frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\,\mathrm{e}^{st} \\ &= \lim_{s\to 0} \frac{s_1\, s_2\, s_3\, s_4}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\,\mathrm{e}^{st} \\ &= \frac{s_1\, s_2\, s_3\, s_4}{(0-s_1)(0-s_2)(0-s_3)(0-s_4)}\,\mathrm{e}^{0t} = 1 \end{aligned} \tag{A2.3.5.4}$$

Residue 1:

$$\begin{aligned} \operatorname{res}_1 &= \lim_{s\to s_1} (s-s_1)\,\frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\,\mathrm{e}^{st} \\ &= \lim_{s\to s_1} \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_2)(s-s_3)(s-s_4)}\,\mathrm{e}^{st} \\ &= \frac{s_1\, s_2\, s_3\, s_4}{s_1\,(s_1-s_2)(s_1-s_3)(s_1-s_4)}\,\mathrm{e}^{s_1 t} \\ &= \frac{s_2\, s_3\, s_4}{(s_1-s_2)(s_1-s_3)(s_1-s_4)}\,\mathrm{e}^{s_1 t} \end{aligned} \tag{A2.3.5.5}$$

Let us insert the pole components here:

$$\operatorname{res}_1 = \frac{(\sigma_1 - j\,\omega_1)(\sigma_3 + j\,\omega_3)(\sigma_3 - j\,\omega_3)\,\mathrm{e}^{(\sigma_1 + j\,\omega_1)t}}{(\sigma_1 + j\,\omega_1 - \sigma_1 + j\,\omega_1)(\sigma_1 + j\,\omega_1 - \sigma_3 - j\,\omega_3)(\sigma_1 + j\,\omega_1 - \sigma_3 + j\,\omega_3)} \tag{A2.3.5.6}$$

$$= \frac{(\sigma_1 - j\,\omega_1)(\sigma_3^2 + \omega_3^2)\,\mathrm{e}^{(\sigma_1 + j\,\omega_1)t}}{2j\,\omega_1[(\sigma_1 - \sigma_3) + j\,(\omega_1 - \omega_3)][(\sigma_1 - \sigma_3) + j\,(\omega_1 + \omega_3)]}$$

$$= \frac{(\sigma_1 - j\,\omega_1)(\sigma_3^2 + \omega_3^2)\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1\{(\sigma_1 - \sigma_3)^2 + j(\sigma_1 - \sigma_3)[(\omega_1 - \omega_3) + (\omega_1 + \omega_3)] - (\omega_1 - \omega_3)(\omega_1 + \omega_3)\}}$$

$$= \frac{(\sigma_1 - j\,\omega_1)(\sigma_3^2 + \omega_3^2)\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1[(\sigma_1 - \sigma_3)^2 + 2j\,\omega_1(\sigma_1 - \sigma_3) - (\omega_1^2 - \omega_3^2)]}$$

$$= \frac{(\sigma_3^2 + \omega_3^2)(\sigma_1 - j\,\omega_1)}{2j\,\omega_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2j\,\omega_1(\sigma_1 - \sigma_3)]}\,\mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}$$

We shall rationalize the denominator (but only the part in the brackets — the factor $2j$ in front will be used later with the $\mathrm{e}^{j\,\omega_1 t}$ term to transform it into a sine and cosine, as was done in the third-order example) by multiplying both the numerator and the denominator by the complex conjugate of the denominator:

$$= \frac{(\sigma_3^2 + \omega_3^2)(\sigma_1 - j\,\omega_1)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) - 2j\,\omega_1(\sigma_1 - \sigma_3)]}{[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)]^2 + 4\,\omega_1^2(\sigma_1 - \sigma_3)^2}\,\mathrm{e}^{\sigma_1 t}\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1}$$

Before regrouping the terms in the numerator into a real and imaginary part we extract the common real factor:

$$K_1 = \frac{(\sigma_3^2 + \omega_3^2)}{[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)]^2 + 4\,\omega_1^2(\sigma_1 - \sigma_3)^2} \tag{A2.3.5.7}$$

The remaining expression in the numerator must be first multiplied and then rearranged to separate the real and imaginary part:

$$(\sigma_1 - j\,\omega_1)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) - 2j\,\omega_1(\sigma_1 - \sigma_3)]$$

$$= \sigma_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)] - j\,\omega_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)]$$

$$- 2j\,\omega_1\sigma_1(\sigma_1 - \sigma_3) - 2\,\omega_1^2(\sigma_1 - \sigma_3)$$

$$= \sigma_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)] - 2\,\omega_1^2(\sigma_1 - \sigma_3)$$

$$- j\,\omega_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2\,\sigma_1(\sigma_1 - \sigma_3)]$$

In order to simplify the expression we again introduce some substitutions. These same substitutions will be used again for res_2, whilst slightly different substitutions will be used for res_3 and res_4.

$$A = (\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) \tag{A2.3.5.8}$$

$$B = 2\,(\sigma_1 - \sigma_3) \tag{A2.3.5.9}$$

So by taking into account the common factor K_1 from above we can write:

$$\mathrm{res}_1 = K_1 \left[\left(\sigma_1 A - \omega_1^2 B \right) - j\,\omega_1 (A + \sigma_1 B) \right] \mathrm{e}^{\sigma_1 t} \, \frac{\mathrm{e}^{j\,\omega_1 t}}{2 j\,\omega_1} \tag{A2.3.5.10}$$

Residue 2:

$$\mathrm{res}_2 = \lim_{s \to s_2} \, (s - s_2) \, \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \, \mathrm{e}^{st} \tag{A2.3.5.11}$$

$$= \lim_{s \to s_2} \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_3)(s-s_4)} \, \mathrm{e}^{st}$$

$$= \frac{s_1\, s_2\, s_3\, s_4}{s_2\,(s_2-s_1)(s_2-s_3)(s_2-s_4)} \, \mathrm{e}^{s_2 t}$$

$$= \frac{s_1\, s_3\, s_4}{(s_2-s_1)(s_2-s_3)(s_2-s_4)} \, \mathrm{e}^{s_2 t}$$

Enter the pole components:

$$\mathrm{res}_2 = \frac{(\sigma_1 + j\,\omega_1)(\sigma_3 + j\,\omega_3)(\sigma_3 - j\,\omega_3)\, \mathrm{e}^{(\sigma_1 - j\,\omega_1)t}}{(\sigma_1 - j\,\omega_1 - \sigma_1 - j\,\omega_1)(\sigma_1 - j\,\omega_1 - \sigma_3 - j\,\omega_3)(\sigma_1 - j\,\omega_1 - \sigma_3 + j\,\omega_3)} \tag{A2.3.5.12}$$

$$= \frac{(\sigma_1 + j\,\omega_1)(\sigma_3^2 + \omega_3^2)\, \mathrm{e}^{(\sigma_1 - j\,\omega_1)t}}{-\,2 j\,\omega_1 [(\sigma_1 - \sigma_3) - j\,(\omega_1 + \omega_3)][(\sigma_1 - \sigma_3) - j\,(\omega_1 - \omega_3)]}$$

$$= \frac{(\sigma_1 + j\,\omega_1)(\sigma_3^2 + \omega_3^2)\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{-j\,\omega_1 t}}{-\,2 j\,\omega_1 [(\sigma_1 - \sigma_3)^2 - 2 j\,\omega_1 (\sigma_1 - \sigma_3) - (\omega_1 - \omega_3)(\omega_1 + \omega_3)]}$$

$$= \frac{(\sigma_1 + j\,\omega_1)(\sigma_3^2 + \omega_3^2)\ \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{-j\,\omega_1 t}}{-\,2 j\,\omega_1 [(\sigma_1 - \sigma_3)^2 - 2 j\,\omega_1 (\sigma_1 - \sigma_3) - (\omega_1^2 - \omega_3^2)]}$$

$$= \frac{(\sigma_3^2 + \omega_3^2)(\sigma_1 + j\,\omega_1)}{-\,2 j\,\omega_1 [(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) - 2 j\,\omega_1 (\sigma_1 - \sigma_3)]} \, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{-j\,\omega_1 t}$$

Again, we shall associate the factor $-2j\,\omega_1$ with the imaginary exponential and then rationalize the expression in the brackets of the denominator by multiplying it (and, of course, the numerator as well) with its own complex conjugate:

$$= \frac{(\sigma_3^2 + \omega_3^2)(\sigma_1 + j\,\omega_1)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2 j\,\omega_1 (\sigma_1 - \sigma_3)]}{\left[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)\right]^2 + 4\,\omega_1^2 (\sigma_1 - \sigma_3)^2} \, \mathrm{e}^{\sigma_1 t} \, \frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2 j\,\omega_1}$$

We now extract the same common real factor K_1 as in res_1 (A2.3.73) and again regroup the remaining terms of the numerator into real and imaginary parts:

$$(\sigma_1 + j\,\omega_1)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2 j\,\omega_1 (\sigma_1 - \sigma_3)]$$

$$= \sigma_1 [(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)] - 2\,\omega_1^2 (\sigma_1 - \sigma_3)$$

$$+ 2 j\,\omega_1 \sigma_1 (\sigma_1 - \sigma_3) + j\,\omega_1 [(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)]$$

$$= \sigma_1[(\sigma_1-\sigma_3)^2-(\omega_1^2-\omega_3^2)]-2\,\omega_1^2(\sigma_1-\sigma_3)$$

$$+\,j\,\omega_1[(\sigma_1-\sigma_3)^2-(\omega_1^2-\omega_3^2)+2\sigma_1(\sigma_1-\sigma_3)]$$

By using the same substitutions K_1, A, and B as for res_1, it is obvious that:

$$\mathrm{res}_2 = K_1\,\left[\left(\sigma_1 A-\omega_1^2 B\right)+j\,\omega_1(A+\sigma_1 B)\right]\mathrm{e}^{\sigma_1 t}\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1} \qquad \text{(A2.3.5.13)}$$

Residue 3:

$$\mathrm{res}_3 = \lim_{s\to s_3}\,(s-s_3)\,\frac{s_1\,s_2\,s_3\,s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\,\mathrm{e}^{st} \qquad \text{(A2.3.5.14)}$$

$$= \lim_{s\to s_3}\,\frac{s_1\,s_2\,s_3\,s_4}{s\,(s-s_1)(s-s_2)(s-s_4)}\,\mathrm{e}^{st}$$

$$= \frac{s_1\,s_2\,s_3\,s_4}{s_3\,(s_3-s_1)(s_3-s_2)(s_3-s_4)}\,\mathrm{e}^{s_3 t}$$

$$= \frac{s_1\,s_2\,s_4}{(s_3-s_1)(s_3-s_2)(s_3-s_4)}\,\mathrm{e}^{s_3 t}$$

Insert the pole components:

$$\mathrm{res}_3 = \frac{(\sigma_1+j\,\omega_1)(\sigma_1-j\,\omega_1)(\sigma_3-j\,\omega_3)\;\mathrm{e}^{(\sigma_3+j\,\omega_3)t}}{(\sigma_3+j\,\omega_3-\sigma_1-j\,\omega_1)(\sigma_3+j\,\omega_3-\sigma_1+j\,\omega_1)(\sigma_3+j\,\omega_3-\sigma_3+j\,\omega_3)}$$

$$= \frac{(\sigma_3-j\,\omega_3)(\sigma_1^2+\omega_1^2)\;\mathrm{e}^{(\sigma_3+j\,\omega_3)t}}{2j\,\omega_3[(\sigma_3-\sigma_1)-j\,(\omega_1-\omega_3)][(\sigma_3-\sigma_1)+j\,(\omega_1+\omega_3)]} \qquad \text{(A2.3.5.15)}$$

By taking out the sign of the $(\sigma_3-\sigma_1)$ term and using $-(\sigma_1-\sigma_3)$ instead, we can use similar substitution factors as with the previous two residues:

$$= \frac{(\sigma_1^2+\omega_1^2)(\sigma_3-j\,\omega_3)\;\mathrm{e}^{\sigma_3 t}\;\mathrm{e}^{j\,\omega_3 t}}{2j\,\omega_3[\,-\,(\sigma_1-\sigma_3)-j\,(\omega_1-\omega_3)][\,-\,(\sigma_1-\sigma_3)+j\,(\omega_1+\omega_3)]}$$

$$= \frac{(\sigma_3-j\,\omega_3)(\sigma_1^2+\omega_1^2)\;\mathrm{e}^{\sigma_3 t}\;\mathrm{e}^{j\,\omega_3 t}}{2j\,\omega_3[(\sigma_1-\sigma_3)^2+(\omega_1^2-\omega_3^2)-2j\,\omega_3(\sigma_1-\sigma_3)]}$$

$$= \frac{(\sigma_1^2+\omega_1^2)(\sigma_3-j\,\omega_3)}{(\sigma_1-\sigma_3)^2+(\omega_1^2-\omega_3^2)-2j\,\omega_3(\sigma_1-\sigma_3)}\;\mathrm{e}^{\sigma_3 t}\;\frac{\mathrm{e}^{j\,\omega_3 t}}{2j\,\omega_3}$$

Rationalize the denominator:

$$= \frac{(\sigma_1^2+\omega_1^2)(\sigma_3-j\,\omega_3)[(\sigma_1-\sigma_3)^2+(\omega_1^2-\omega_3^2)+2j\,\omega_3(\sigma_1-\sigma_3)]}{\left[(\sigma_1-\sigma_3)^2+(\omega_1^2-\omega_3^2)\right]^2+4\,\omega_3^2(\sigma_1-\sigma_3)^2}\;\mathrm{e}^{\sigma_3 t}\;\frac{\mathrm{e}^{j\,\omega_3 t}}{2j\,\omega_3}$$

We now extract the common real factor K_3, which for res_3 and res_4 is slightly different from K_1:

$$K_3 = \frac{(\sigma_1^2 + \omega_1^2)}{\left[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)\right]^2 + 4\,\omega_3^2(\sigma_1 - \sigma_3)^2} \tag{A2.3.5.16}$$

We regroup the remaining numerator terms into real and imaginary part:

$$\begin{aligned}
&(\sigma_3 - j\,\omega_3)[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) + 2j\,\omega_3(\sigma_1 - \sigma_3)] \\
&\qquad = \sigma_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] + 2\,\omega_3^2(\sigma_1 - \sigma_3) \\
&\qquad\quad + 2j\,\omega_3\sigma_3(\sigma_1 - \sigma_3) - j\,\omega_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] \\
&\qquad = \sigma_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] + 2\,\omega_3^2(\sigma_1 - \sigma_3) \\
&\qquad\quad - j\,\omega_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) - 2\,\sigma_3(\sigma_1 - \sigma_3)]
\end{aligned}$$

From this, the following substitutions are used for both res_3 and res_4:

$$C = (\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) \tag{A2.3.5.17}$$

and again, as in Eq. A2.3.5.9:

$$B = 2\,(\sigma_1 - \sigma_3)$$

So:

$$\mathrm{res}_3 = K_3\left[\left(\sigma_3 C + \omega_3^2 B\right) - j\,\omega_3(C - \sigma_3 B)\right] \mathrm{e}^{\sigma_3 t}\,\frac{\mathrm{e}^{j\,\omega_3 t}}{2j\,\omega_3} \tag{A2.3.5.18}$$

Residue 4:

$$\mathrm{res}_4 = \lim_{s \to s_4} (s - s_4)\,\frac{s_1\,s_2\,s_3\,s_4}{s\,(s - s_1)(s - s_2)(s - s_3)(s - s_4)}\,\mathrm{e}^{st} \tag{A2.3.5.19}$$

$$\begin{aligned}
&= \lim_{s \to s_4} \frac{s_1\,s_2\,s_3\,s_4}{s\,(s - s_1)(s - s_2)(s - s_3)}\,\mathrm{e}^{st} \\
&= \frac{s_1\,s_2\,s_3\,s_4}{s_4\,(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)}\,\mathrm{e}^{s_4 t} \\
&= \frac{s_1\,s_2\,s_3}{(s_4 - s_1)(s_4 - s_2)(s_4 - s_3)}\,\mathrm{e}^{s_4 t}
\end{aligned}$$

Insert the pole components:

$$\begin{aligned}
\mathrm{res}_4 &= \frac{(\sigma_1 + j\,\omega_1)(\sigma_1 - j\,\omega_1)(\sigma_3 + j\,\omega_3)\;\mathrm{e}^{(\sigma_3 - j\,\omega_3)t}}{(\sigma_3 - j\,\omega_3 - \sigma_1 - j\,\omega_1)(\sigma_3 - j\,\omega_3 - \sigma_1 + j\,\omega_1)(\sigma_3 - j\,\omega_3 - \sigma_3 - j\,\omega_3)} \\
&= \frac{(\sigma_3 + j\,\omega_3)(\sigma_1^2 + \omega_1^2)\;\mathrm{e}^{(\sigma_3 - j\,\omega_3)t}}{-\,2j\,\omega_3[-(\sigma_1 - \sigma_3) - j\,(\omega_1 + \omega_3)][-(\sigma_1 - \sigma_3) + j\,(\omega_1 - \omega_3)]}
\end{aligned} \tag{A2.3.5.20}$$

$$= \frac{(\sigma_3 + j\omega_3)(\sigma_1^2 + \omega_1^2)\ \mathrm{e}^{\sigma_3 t}\ \mathrm{e}^{-j\omega_3 t}}{-\,2j\omega_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) + 2j\omega_3(\sigma_1 - \sigma_3)]}$$

$$= \frac{(\sigma_1^2 + \omega_1^2)(\sigma_3 + j\omega_3)}{(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) + 2j\omega_3(\sigma_1 - \sigma_3)}\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\omega_3 t}}{-\,2j\omega_3}$$

Rationalize the denominator:

$$= \frac{(\sigma_1^2 + \omega_1^2)(\sigma_3 + j\omega_3)[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) - 2j\omega_3(\sigma_1 - \sigma_3)]}{[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)]^2 + 4\,\omega_3^2(\sigma_1 - \sigma_3)^2}\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\omega_3 t}}{-\,2j\omega_3}$$

Let us again extract the same common real factor K_3 as for res_3 (A2.3.82) and regroup the remaining terms of the numerator into real and imaginary parts:

$$\begin{aligned}
(\sigma_3 + j\omega_3)[(\sigma_1 - \sigma_3)^2 &+ (\omega_1^2 - \omega_3^2) - 2j\omega_3(\sigma_1 - \sigma_3)] \\
&= \sigma_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] + 2\,\omega_3^2(\sigma_1 - \sigma_3) \\
&\quad - 2j\omega_3\sigma_3(\sigma_1 - \sigma_3) + j\omega_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] \\
&= \sigma_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2)] + 2\,\omega_3^2(\sigma_1 - \sigma_3) \\
&\quad + j\omega_3[(\sigma_1 - \sigma_3)^2 + (\omega_1^2 - \omega_3^2) - 2\,\sigma_3(\sigma_1 - \sigma_3)]
\end{aligned}$$

and, using the same substitutions K_2, C, and B, as for res_3, we obtain:

$$\mathrm{res}_4 = K_3\big[(\sigma_3 C + \omega_3^2 B) + j\omega_3(C - \sigma_3 B)\big]\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\omega_3 t}}{-\,2j\omega_3} \tag{A2.3.5.21}$$

Now we can sum all five residues and group the relative real and imaginary terms:

$$\begin{aligned}
\sum_{i=0}^{4} \mathrm{res}_i = 1 &+ K_1\,\big[(\sigma_1 A - \omega_1^2 B) - j\omega_1(A + \sigma_1 B)\big]\ \mathrm{e}^{\sigma_1 t}\ \frac{\mathrm{e}^{j\omega_1 t}}{2j\omega_1} \\
&+ K_1\,\big[(\sigma_1 A - \omega_1^2 B) + j\omega_1(A + \sigma_1 B)\big]\ \mathrm{e}^{\sigma_1 t}\ \frac{\mathrm{e}^{-j\omega_1 t}}{-\,2j\omega_1} \\
&+ K_3\,\big[(\sigma_3 C + \omega_3^2 B) - j\omega_3(C - \sigma_3 B)\big]\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{j\omega_3 t}}{2j\omega_3} \\
&+ K_3\big[(\sigma_3 C + \omega_3^2 B) + j\omega_3(C - \sigma_3 B)\big]\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\omega_3 t}}{-\,2j\omega_3}
\end{aligned} \tag{A2.3.5.22}$$

This we can write as:

$$\sum_{i=0}^{4} \mathrm{res}_i = 1 + \frac{K_1}{\omega_1}\left[\frac{\sigma_1 A - \omega_1^2 B}{2j} - \omega_1 \frac{A + \sigma_1 B}{2}\right] \mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{j\,\omega_1 t}$$

$$+ \frac{K_1}{\omega_1}\left[\frac{\sigma_1 A - \omega_1^2 B}{-\,2j} + \omega_1 \frac{A + \sigma_1 B}{-\,2}\right] \mathrm{e}^{\sigma_1 t}\,\mathrm{e}^{-j\,\omega_1 t}$$

$$+ \frac{K_3}{\omega_3}\left[\frac{\sigma_3 C + \omega_3^2 B}{2j} - \omega_3 \frac{C - \sigma_3 B}{2}\right] \mathrm{e}^{\sigma_3 t}\,\mathrm{e}^{j\,\omega_3 t}$$

$$+ \frac{K_3}{\omega_3}\left[\frac{\sigma_3 C + \omega_3^2 B}{-\,2j} + \omega_3 \frac{C - \sigma_3 B}{-\,2}\right] \mathrm{e}^{\sigma_3 t}\,\mathrm{e}^{-j\,\omega_3 t}$$

From the residue pairs $\mathrm{res}_{1,2}$ and $\mathrm{res}_{3,4}$ we extract the common factors and sum the exponentials with the imaginary exponent:

$$\sum_{i=0}^{4} \mathrm{res}_i$$

$$= 1 + \frac{K_1}{\omega_1}\,\mathrm{e}^{\sigma_1 t}\left[\left(\sigma_1 A - \omega_1^2 B\right)\frac{\mathrm{e}^{j\,\omega_1 t} - \mathrm{e}^{-j\,\omega_1 t}}{2j} - \omega_1 (A + \sigma_1 B)\,\frac{\mathrm{e}^{j\,\omega_1 t} + \mathrm{e}^{-j\,\omega_1 t}}{2}\right]$$

$$+ \frac{K_3}{\omega_3}\,\mathrm{e}^{\sigma_3 t}\left[\left(\sigma_3 C + \omega_3^2 B\right)\frac{\mathrm{e}^{j\,\omega_3 t} - \mathrm{e}^{-j\,\omega_3 t}}{2j} - \omega_3 (C - \sigma_3 B)\,\frac{\mathrm{e}^{j\,\omega_3 t} + \mathrm{e}^{-j\,\omega_3 t}}{2}\right]$$

Now we replace the complex exponential terms by their equivalent sine and cosine:

$$g(t) = 1 + \frac{K_1}{\omega_1}\,\mathrm{e}^{\sigma_1 t}\left[(\sigma_1 A - \omega_1^2 B)\sin(\omega_1 t) - \omega_1 (A + \sigma_1 B)\cos(\omega_1 t)\right]$$

$$+ \frac{K_3}{\omega_3}\,\mathrm{e}^{\sigma_3 t}\left[\left(\sigma_3 C + \omega_3^2 B\right)\sin(\omega_3 t) - \omega_3 (C - \sigma_3 B)\cos(\omega_3 t)\right] \tag{A2.3.5.23}$$

Finally, each sine–cosine pair can be transformed in a single sine function with the appropriate phase shift, as we did in the three-pole case (Eq. A2.3.17; but, as explained there, we must check if this operation will cause a wrong phase — see the next page!):

$$g(t) = 1 + \frac{K_1}{\omega_1}\,\mathrm{e}^{\sigma_1 t}\sqrt{\left(\sigma_1 A - \omega_1^2 B\right)^2 + \omega_1^2 (A + \sigma_1 B)^2}\;\sin(\omega_1 t + \theta_1)$$

$$+ \frac{K_3}{\omega_3}\,\mathrm{e}^{\sigma_3 t}\sqrt{\left(\sigma_3 C + \omega_3^2 B\right)^2 + \omega_3^2 (C - \sigma_3 B)^2}\;\sin(\omega_3 t + \theta_3) \tag{A2.3.5.24}$$

where:

$$\theta_1 = \arctan\frac{-\,\omega_1 (A + \sigma_1 B)}{\sigma_1 A - \omega_1^2 B} \qquad \text{and} \qquad \theta_3 = \arctan\frac{-\,\omega_3 (C - \sigma_3 B)}{\sigma_3 C + \omega_3^2 B} \tag{A2.3.5.25}$$

Phase error

The next two figures show the peculiar problem with the fourth-order system phase if it is calculated by Eq. A2.3.5.24 and A.2.3.5.25. In the case of an MFA system, Fig. A2.3.2, both angles should be increased by π radians in order to obtain the correct response. But for a MFED system, Fig. A2.3.3, only θ_3 needs to be corrected!

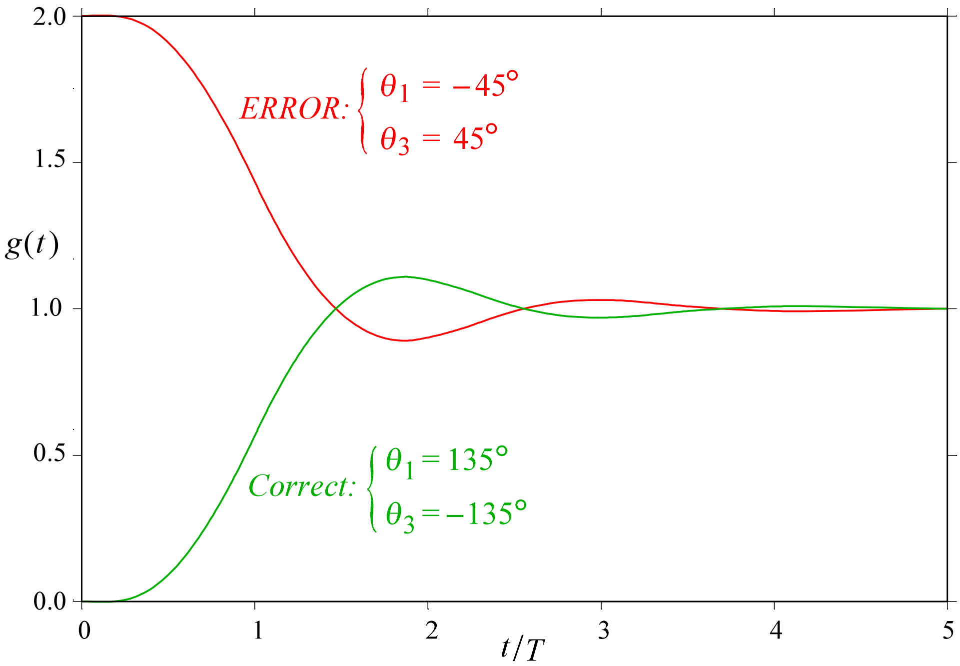

Fig. A2.3.2: A 4$^{\text{t}}$ -order MFA system step response is mirrored over the final value if the phase is not corrected.

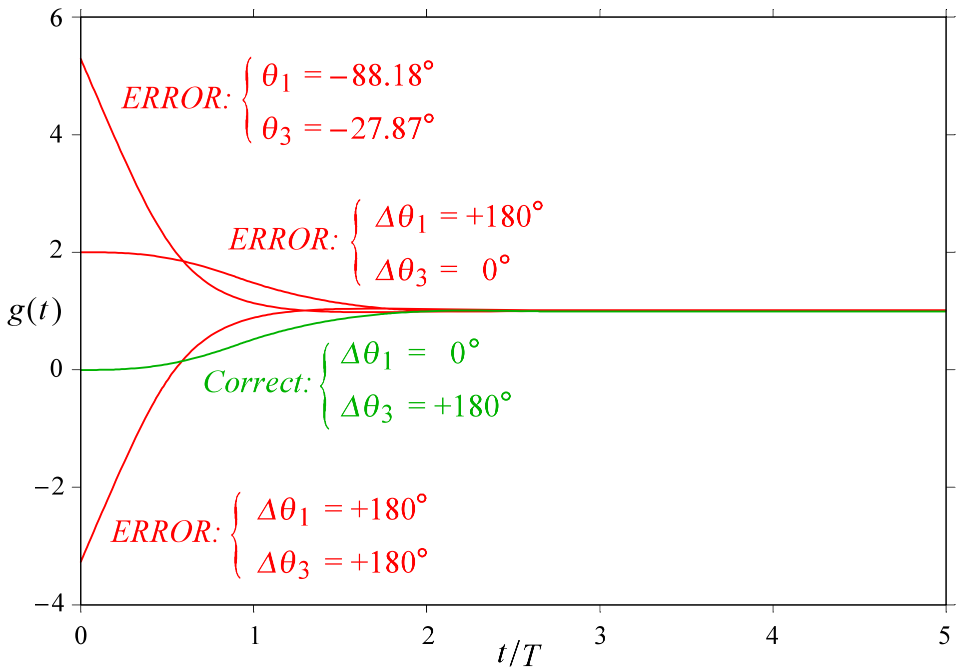

Fig. A2.3.3: Strangely, the 4$^{\text{t}}$ -order MFED system step response is correct if only the second phase angle is corrected. All the other combinations are wrong!

These figures clearly demonstrate the unpredictable nature of the sine–cosine to amplitude–phase transformation owed to the half-circle period of the arctangent function. Whilst some insight into the circuit's behavior can be gained by observing the initial amplitude of each resonance mode, it is advisable not to rely on this form of equation unless you are absolutely certain of the result.

To play it safe, the sum of the residues in their complex exponential form is recommended for the numerical computation.

Interpretation

As for all step response functions, the first residue represents the normalized final value. The fourth-order system has two distinct sinusoidal resonance modes, ω_1 and ω_3, each owed to its own complex conjugate pole pair, each with a different initial phase, θ_1 and θ_3, and with a different amplitude correction factor. Also, each mode has its own damping function, $e^{\sigma_1 t}$ and $e^{\sigma_3 t}$.

A2.3.6 Fourth-order system with two complex conjugate pole pairs and one real zero

A four-pole one-zero normalized general function is:

$$F(s) = \frac{(-1)^4\, s_1\, s_2\, s_3\, s_4}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z} \tag{A2.3.6.1}$$

In general, the two complex conjugate pole pairs and the zero are:

$$\begin{aligned} s_{1,2} &= \sigma_1 \pm j\,\omega_1 \\ s_{3,4} &= \sigma_3 \pm j\,\omega_3 \end{aligned} \qquad \text{and} \qquad s_z = \sigma_z \tag{A2.3.6.2}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s}\, F(s) = \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z} \tag{A2.3.6.3}$$

The inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{4} \mathrm{res}_i\,[\,G(s)\,]$$

Since only the poles have residues the step response calculation is similar to that in the previous section, with the difference that each residue is multiplied by the normalized difference between the appropriate pole and the zero, as shown below.

Residue 0:

$$\mathrm{res}_0 = \lim_{s\to 0}\, s\, \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z}\, \mathrm{e}^{st} \tag{A2.3.6.4}$$

$$= \lim_{s\to 0} \frac{s_1\, s_2\, s_3\, s_4}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z}\, \mathrm{e}^{st}$$

$$= \frac{s_1\, s_2\, s_3\, s_4}{(0-s_1)(0-s_2)(0-s_3)(0-s_4)} \cdot \frac{0-s_z}{-s_z}\, \mathrm{e}^{0t} = 1$$

Residue 1:

$$\mathrm{res}_1 = \lim_{s\to s_1}\, (s-s_1)\, \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z}\, \mathrm{e}^{st} \tag{A2.3.6.5}$$

$$= \lim_{s\to s_1} \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z}\, \mathrm{e}^{st}$$

$$= \frac{s_1\, s_2\, s_3\, s_4}{s_1\,(s_1-s_2)(s_1-s_3)(s_1-s_4)} \cdot \frac{s_1-s_z}{-s_z}\, \mathrm{e}^{s_1 t}$$

$$= \frac{s_2\, s_3\, s_4}{(s_1-s_2)(s_1-s_3)(s_1-s_4)} \cdot \frac{s_1-s_z}{-s_z}\, \mathrm{e}^{s_1 t}$$

Let us insert the pole and zero components here:

$$\mathrm{res}_1 = \frac{(\sigma_1 - j\omega_1)(\sigma_3 + j\omega_3)(\sigma_3 - j\omega_3)(\sigma_1 + j\omega_1 - \sigma_z)\,\mathrm{e}^{(\sigma_1 + j\omega_1)t}}{-\sigma_z(\sigma_1 + j\omega_1 - \sigma_1 + j\omega_1)(\sigma_1 + j\omega_1 - \sigma_3 - j\omega_3)(\sigma_1 + j\omega_1 - \sigma_3 + j\omega_3)} \tag{A2.3.6.6}$$

We can now follow the same path as in Sec. A2.3.5 for res_1 up to the rationalization of the denominator, but the ending part will be slightly different. By using the term K_1 from Eq. A2.3.5.7, the common factor which we can extract becomes:

$$\frac{K_1}{-\sigma_z} = \frac{(\sigma_3^2 + \omega_3^2)}{-\sigma_z\Big\{\big[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)\big]^2 + 4\,\omega_1^2(\sigma_1 - \sigma_3)^2\Big\}} \tag{A2.3.6.7}$$

and the remaining part of the numerator of Eq. A2.3.6.6 is rearranged as follows:

$$\begin{aligned}
&(\sigma_1 + j\omega_1 - \sigma_z)(\sigma_1 - j\omega_1)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) - 2j\omega_1(\sigma_1 - \sigma_3)] \\
&\qquad = (\sigma_1 - \sigma_z)\{\sigma_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)] - 2\,\omega_1^2(\sigma_1 - \sigma_3)\} \\
&\qquad\quad + j\omega_1\{\sigma_1[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2)] - 2\,\omega_1^2(\sigma_1 - \sigma_3)\} \\
&\qquad\quad - j\omega_1(\sigma_1 - \sigma_z)[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2\,\sigma_1(\sigma_1 - \sigma_3)] \\
&\qquad\quad + \omega_1^2[(\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) + 2\,\sigma_1(\sigma_1 - \sigma_3)]
\end{aligned}$$

To follow the further development more easily, let us use the same substitutions A and B, as in Eq. A2.3.5.8 and A.2.3.5.9:

$$A = (\sigma_1 - \sigma_3)^2 - (\omega_1^2 - \omega_3^2) \qquad \text{and} \qquad B = 2\,(\sigma_1 - \sigma_3)$$

and, by separating the real and imaginary parts, we can rewrite the remaining numerator terms as:

$$\begin{aligned}
&= (\sigma_1 - \sigma_z)[\sigma_1 A - \omega_1^2 B] + \omega_1^2(A + \sigma_1 B) \\
&\quad + j\omega_1\big\{\,[\sigma_1 A - \omega_1^2 B] - (\sigma_1 - \sigma_z)(A + \sigma_1 B)\big\}
\end{aligned}$$

We shall use two new substitutions for the above real and imaginary terms:

$$M = (\sigma_1 - \sigma_z)[\sigma_1 A - \omega_1^2 B] + \omega_1^2(A + \sigma_1 B) \tag{A2.3.6.8}$$

$$N = [\sigma_1 A - \omega_1^2 B] - (\sigma_1 - \sigma_z)(A + \sigma_1 B) \tag{A2.3.6.9}$$

So, we can write:

$$\mathrm{res}_1 = \frac{K_1}{-\sigma_z}(M + j\omega_1 N)\;\mathrm{e}^{\sigma_1 t}\;\frac{\mathrm{e}^{j\omega_1 t}}{2j\omega_1} \tag{A2.3.6.10}$$

By a similar procedure we arrive at the following relations for the remaining residues:

$$\mathrm{res}_2 = \frac{K_1}{-\,\sigma_\mathrm{z}}(M - j\,\omega_1 N)\ \mathrm{e}^{\sigma_1 t}\ \frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\omega_1} \tag{A2.3.6.11}$$

For res_3 and res_4 we must use a slightly different substitution, reflecting the difference between A, Eq. A2.3.5.8, and C, Eq. A2.3.5.17:

$$P = (\sigma_3 - \sigma_\mathrm{z})[\sigma_3 C + \omega_3^2 B] + \omega_3^2(C - \sigma_3 B) \tag{A2.3.6.12}$$

$$Q = [\sigma_3 C + \omega_3^2 B] - (\sigma_3 - \sigma_\mathrm{z})(C - \sigma_3 B) \tag{A2.3.6.13}$$

So:

$$\mathrm{res}_3 = \frac{K_3}{-\,\sigma_\mathrm{z}}(P + j\,\omega_3 Q)\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{j\,\omega_3 t}}{2j\omega_3} \tag{A2.3.6.14}$$

$$\mathrm{res}_4 = \frac{K_3}{-\,\sigma_\mathrm{z}}(P - j\,\omega_3 Q)\ \mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\,\omega_3 t}}{-\,2j\omega_3} \tag{A2.3.6.15}$$

where K_3 is the same as in Eq. A2.3.5.16.

Finally, we sum all the residues:

$$\begin{aligned} g(t) = \sum_{i=0}^{4} \mathrm{res}_i \ &= 1 - \frac{K_1}{\sigma_\mathrm{z}}(M + j\,\omega_1 N)\,\mathrm{e}^{\sigma_1 t}\ \frac{\mathrm{e}^{j\,\omega_1 t}}{2j\omega_1} \\ &\quad - \frac{K_1}{\sigma_\mathrm{z}}(M - j\,\omega_1 N)\,\mathrm{e}^{\sigma_1 t}\ \frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\omega_1} \\ &\quad - \frac{K_3}{\sigma_\mathrm{z}}(P + j\,\omega_3 Q)\,\mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{j\,\omega_3 t}}{2j\omega_3} \\ &\quad - \frac{K_3}{\sigma_\mathrm{z}}(P - j\,\omega_3 Q)\,\mathrm{e}^{\sigma_3 t}\ \frac{\mathrm{e}^{-j\,\omega_3 t}}{-\,2j\omega_3} \end{aligned} \tag{A2.3.6.16}$$

which can be rewritten as:

$$\begin{aligned} g(t) = 1 &- \frac{K_1}{\sigma_\mathrm{z}}\,\mathrm{e}^{\sigma_1 t}\left[\frac{M}{\omega_1}\cdot\frac{\mathrm{e}^{j\,\omega_1 t} - \mathrm{e}^{-j\,\omega_1 t}}{2j} + N\,\frac{\mathrm{e}^{j\,\omega_1 t} + \mathrm{e}^{-j\,\omega_1 t}}{2}\right] \\ &- \frac{K_3}{\sigma_\mathrm{z}}\,\mathrm{e}^{\sigma_3 t}\left[\frac{P}{\omega_3}\cdot\frac{\mathrm{e}^{j\,\omega_3 t} - \mathrm{e}^{-j\,\omega_3 t}}{2j} + Q\,\frac{\mathrm{e}^{j\,\omega_3 t} + \mathrm{e}^{-j\,\omega_3 t}}{2}\right] \end{aligned} \tag{A2.3.6.17}$$

and, by replacing the complex exponentials with their equivalent sine-cosine forms:

$$\begin{aligned} g(t) = 1 &- \frac{K_1}{\sigma_\mathrm{z}}\,\mathrm{e}^{\sigma_1 t}\left[\frac{M}{\omega_1}\sin(\omega_1 t) + N\cos(\omega_1 t)\right] \\ &- \frac{K_3}{\sigma_\mathrm{z}}\,\mathrm{e}^{\sigma_3 t}\left[\frac{P}{\omega_3}\sin(\omega_3 t) + Q\cos(\omega_3 t)\right] \end{aligned} \tag{A2.3.6.18}$$

No, we shall not attempt to examine the possibility of running into yet another set of phase errors by any further trigonometric transformation!

A2.3.7 Fourth-order system with two poles real, one complex conjugate pole pair and one real zero

A four-pole one-zero normalized general function is:

$$F(s) = \frac{(-1)^4\, s_1\, s_2\, s_3\, s_4}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z} \qquad \text{(A2.3.7.1)}$$

In the case of the two real poles, one complex conjugate pole pair, and one real zero, the components are:

$$\begin{aligned} s_{1,2} &= \sigma_1 \pm j\omega_1 \\ s_3 &= \sigma_3 \\ s_4 &= \sigma_4 \\ s_z &= \sigma_z \end{aligned} \qquad \text{(A2.3.7.2)}$$

The step response in the s domain is:

$$G(s) = \frac{1}{s}\, F(s) = \frac{s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z} \qquad \text{(A2.3.7.3)}$$

The inverse Laplace transform of $G(s)$ is:

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \sum_{i=0}^{4} \text{res}_i\,[\,G(s)\,]$$

The step response calculation is similar to that in the previous section, with the difference that, instead of the second complex conjugate pole pair, there are two simple real poles.

As before, $\text{res}_0 = 1$. The calculation of the remaining four residues follows.

$$\text{res}_1 = \lim_{s \to s_1} \frac{(s-s_1)\, s_1\, s_2\, s_3\, s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \cdot \frac{s-s_z}{-s_z}\, \mathrm{e}^{st} \qquad \text{(A2.3.7.4)}$$

$$= \frac{s_1\, s_2\, s_3\, s_4}{s_1\,(s_1-s_2)(s_1-s_3)(s_1-s_4)} \cdot \frac{s_1-s_z}{-s_z}\, \mathrm{e}^{s_1 t}$$

$$= \frac{(s_1-s_z)\, s_2\, s_3\, s_4}{-s_z\,(s_1-s_2)(s_1-s_3)(s_1-s_4)}\, \mathrm{e}^{s_1 t}$$

$$= \frac{(\sigma_1 + j\omega_1 - \sigma_z)(\sigma_1 - j\omega_1)\sigma_3\,\sigma_4}{-\sigma_z\,(\sigma_1 + j\omega_1 - \sigma_1 + j\omega_1)(\sigma_1 + j\omega_1 - \sigma_3)(\sigma_1 + j\omega_1 - \sigma_4)}\, \mathrm{e}^{(\sigma_1 + j\omega_1)t}$$

$$= \frac{[\sigma_1^2 + \omega_1^2 - \sigma_z\,(\sigma_1 - j\omega_1)]\sigma_3\,\sigma_4}{-\sigma_z\, 2j\omega_1(\sigma_1 - \sigma_3 + j\omega_1)(\sigma_1 - \sigma_4 + j\omega_1)}\, \mathrm{e}^{\sigma_1 t}\, \mathrm{e}^{j\omega_1 t}$$

$$= \frac{[\sigma_1^2 + \omega_1^2 - \sigma_z\,(\sigma_1 - j\omega_1)]\sigma_3\,\sigma_4}{-\sigma_z\,[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4) - \omega_1^2 + j\omega_1(2\sigma_1 - \sigma_3 - \sigma_4)]}\, \mathrm{e}^{\sigma_1 t}\, \frac{\mathrm{e}^{j\omega_1 t}}{2j\omega_1}$$

We rationalize the denominator by multiplying it with its own complex conjugate; to keep the ratio unchanged we also multiply the numerator:

$$\frac{\sigma_3\,\sigma_4[\sigma_1^2+\omega_1^2-\sigma_z\,(\sigma_1-j\,\omega_1)][(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2-j\,\omega_1(2\,\sigma_1-\sigma_3-\sigma_4)]}{-\,\sigma_z\left\{[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]^2+\omega_1^2(2\,\sigma_1-\sigma_3-\sigma_4)^2\right\}}$$

From this we extract the common real factor (the zero will be entered in the final expression):

$$K_1=\frac{\sigma_3\,\sigma_4}{\left\{[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]^2+\omega_1^2(2\,\sigma_1-\sigma_3-\sigma_4)^2\right\}} \tag{A2.3.7.5}$$

The remaining two brackets in the numerator should be multiplied and separated into a real and imaginary part:

$$\begin{aligned}
&[\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1+j\,\omega_1\sigma_z][(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2-j\,\omega_1(2\,\sigma_1-\sigma_3-\sigma_4)]\\
&\quad=(\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1)[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]\\
&\qquad+j\,\omega_1\sigma_z\,[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]\\
&\qquad-j\,\omega_1(2\,\sigma_1-\sigma_3-\sigma_4)(\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1)+\,\omega_1^2\sigma_z\,(2\,\sigma_1-\sigma_3-\sigma_4)\\
&\quad=(\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1)[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]+\,\omega_1^2\sigma_z\,(2\,\sigma_1-\sigma_3-\sigma_4)\\
&\qquad+j\,\omega_1\{\sigma_z\,[(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2]-(2\,\sigma_1-\sigma_3-\sigma_4)(\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1)\}
\end{aligned}$$

To simplify the expressions we shall use the following substitutions:

$$\begin{aligned}
A&=(\sigma_1-\sigma_3)(\sigma_1-\sigma_4)-\omega_1^2\\
B&=2\,\sigma_1-\sigma_3-\sigma_4\\
C&=\sigma_1^2+\omega_1^2-\sigma_z\,\sigma_1
\end{aligned} \tag{A2.3.7.6}$$

With these substitutions we write:

$$\mathrm{res}_1=\frac{K_1}{-\,\sigma_z}\Big[\big(CA+\,\omega_1^2\sigma_z\,B\big)+j\,\omega_1\Big(\sigma_z\,A-CB\Big)\Big]\,\mathrm{e}^{\sigma_1 t}\,\frac{\mathrm{e}^{j\,\omega_1 t}}{2j\,\omega_1} \tag{A2.3.7.7}$$

and res_2 is the complex conjugate of res_1:

$$\mathrm{res}_2=\frac{K_1}{-\,\sigma_z}\Big[\big(CA+\,\omega_1^2\sigma_z\,B\big)-j\,\omega_1\Big(\sigma_z\,A-CB\Big)\Big]\,\mathrm{e}^{\sigma_1 t}\,\frac{\mathrm{e}^{-j\,\omega_1 t}}{-\,2j\,\omega_1} \tag{A2.3.7.8}$$

Before going on to residues of real poles, let us rewrite the sum $\mathrm{res}_1+\mathrm{res}_2$ in the complex exponential form, as well as in the sine–cosine form, as usually:

$\mathrm{res}_1 + \mathrm{res}_2$

$$= \frac{K_1\,\mathrm{e}^{\sigma_1 t}}{-\,\sigma_z\,\omega_1}\left[\left(CA + \,\omega_1^2\sigma_z\,B\right)\frac{\mathrm{e}^{j\omega_1 t} -\,\mathrm{e}^{-j\omega_1 t}}{2j} + \omega_1\left(\sigma_z\,A - CB\right)\frac{\mathrm{e}^{j\omega_1 t} +\,\mathrm{e}^{-j\omega_1 t}}{2}\right]$$

$$= \frac{K_1}{-\,\sigma_z\,\omega_1}\,\mathrm{e}^{\sigma_1 t}\left[\left(CA + \,\omega_1^2\sigma_z\,B\right)\sin(\omega_1 t) + \omega_1\left(\sigma_z\,A - CB\right)\cos(\omega_1 t)\right] \tag{A2.3.7.9}$$

The calculation of the two residues of real poles is less demanding:

$$\mathrm{res}_3 = \lim_{s\to s_3}\frac{(s-s_3)\,s_1\,s_2\,s_3\,s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\cdot\frac{s-s_z}{-\,s_z}\,\mathrm{e}^{st} \tag{A2.3.7.10}$$

$$= \frac{s_1\,s_2\,s_4}{(s_3-s_1)(s_3-s_2)(s_3-s_4)}\cdot\frac{s_3-s_z}{-\,s_z}\,\mathrm{e}^{s_3 t}$$

$$= \frac{(\sigma_1+j\,\omega_1)(\sigma_1-j\,\omega_1)\,\sigma_4}{(\sigma_3-\sigma_1-j\,\omega_1)(\sigma_3-\sigma_1+j\,\omega_1)(\sigma_3-\sigma_4)}\cdot\frac{\sigma_3-\sigma_z}{-\,\sigma_z}\,\mathrm{e}^{\sigma_3 t}$$

$$= \frac{(\sigma_1^2+\omega_1^2)\,\sigma_4}{\left[(\sigma_3-\sigma_1)^2+\omega_1^2\right](\sigma_3-\sigma_4)}\cdot\frac{\sigma_3-\sigma_z}{-\,\sigma_z}\,\mathrm{e}^{\sigma_3 t}$$

$$\mathrm{res}_4 = \lim_{s\to s_4}\frac{(s-s_4)\,s_1\,s_2\,s_3\,s_4}{s\,(s-s_1)(s-s_2)(s-s_3)(s-s_4)}\cdot\frac{s-s_z}{-\,s_z}\,\mathrm{e}^{st} \tag{A2.3.7.11}$$

$$= \frac{s_1\,s_2\,s_3}{(s_4-s_1)(s_4-s_2)(s_4-s_3)}\cdot\frac{s_4-s_z}{-\,s_z}\,\mathrm{e}^{s_4 t}$$

$$= \frac{(\sigma_1+j\,\omega_1)(\sigma_1-j\,\omega_1)\,\sigma_3}{(\sigma_4-\sigma_1-j\,\omega_1)(\sigma_4-\sigma_1+j\,\omega_1)(\sigma_4-\sigma_3)}\cdot\frac{\sigma_4-\sigma_z}{-\,\sigma_z}\,\mathrm{e}^{\sigma_4 t}$$

$$= \frac{(\sigma_1^2+\omega_1^2)\,\sigma_3}{\left[(\sigma_4-\sigma_1)^2+\omega_1^2\right](\sigma_4-\sigma_3)}\cdot\frac{\sigma_4-\sigma_z}{-\,\sigma_z}\,\mathrm{e}^{\sigma_4 t}$$

We extract a common real factor from res_3 and res_4:

$$K_3 = \frac{(\sigma_1^2+\omega_1^2)}{\left[(\sigma_3-\sigma_1)^2+\omega_1^2\right](\sigma_3-\sigma_4)} \tag{A2.3.7.12}$$

and we write:

$$\mathrm{res}_3 = -\,\frac{K_3}{\sigma_z}\,\sigma_4(\sigma_3-\sigma_z)\,\mathrm{e}^{\sigma_3 t} \tag{A2.3.7.13}$$

$$\mathrm{res}_3 = \frac{K_3}{\sigma_z}\,\sigma_3(\sigma_4-\sigma_z)\,\mathrm{e}^{\sigma_4 t} \tag{A2.3.7.14}$$

Finally, we can write the system step response:

$$g(t) = 1 - \frac{K_1}{\sigma_z\,\omega_1}\,\mathrm{e}^{\sigma_1 t}\Big[\big(CA + \omega_1^2\sigma_z\,B\big)\sin(\omega_1 t) + \omega_1\Big(\sigma_z\,A - CB\Big)\cos(\omega_1 t)\Big]$$

$$- \frac{K_3}{\sigma_z}\,\sigma_4(\sigma_3 - \sigma_z)\,\mathrm{e}^{\sigma_3 t} + \frac{K_3}{\sigma_z}\,\sigma_3(\sigma_4 - \sigma_z)\,\mathrm{e}^{\sigma_4 t} \qquad \text{(A2.3.7.15)}$$

Interpretation:

The value of the first residue ($\mathrm{res}_0 = 1$) is the final value to which the system will settle when the time variable becomes many times greater than the largest time constant of the system.

The transient part of the response is governed by the system resonance, which in turn is determined by the imaginary part of the complex conjugate pole pair, ω_1. The initial phase of the system resonance is given by the ratio of the coefficients of the cosine and sine terms. Overall, the system's resonance amplitude is set by the real factor K_1 and the response is exponentially damped by the real part σ_1 of the complex conjugate pole pair.

The transient is additionally influenced by the two exponential decays owed to both real poles.

It is interesting to note that the zero appears as a free factor in the denominator of all four residues. This indicates that if the zero is placed too close to the complex plane's origin the system overshoot will increase.

Concluding Remarks:

The attentive reader has probably noted that the two residues of each complex conjugate pole pair also form a complex conjugate pair themselves. Since for the time domain response we need the sum of all residues, by summing each complex conjugate pair of residues will yield a double real part of the pair (the imaginary parts cancel). Mathematically this can be expressed as:

$$s_{1,2} = \sigma_1 \pm j\omega_1 \qquad \Rightarrow \qquad \mathrm{res}_{1,2} = A \pm jB$$

$$\mathrm{res}_1 + \mathrm{res}_2 = 2\,\Re\{\mathrm{res}_1\} = 2\,A$$

This means that we can spare ourselves a great deal of work if we calculate only one residue for each pole pair, take only its real part and double its value. Any real pole will also have a single real residue and it is simply added to the rest (no doubling here!).

Therefore, for a n^{th}-order system we only need to find the real part of $n/2$ residues if n is even, and $(n-1)/2 + 1$ residues if n is odd.