Appendix 7.1

Transfer Function Analysis of the MFB-3 circuits

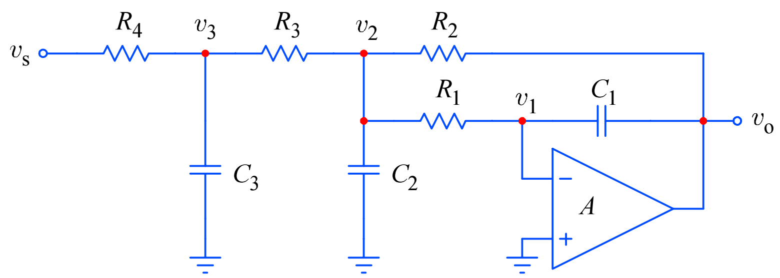

Fig. A7.1.1: The ‘Multiple Feedback’ 3-pole (MFB-3) low pass filter configuration

We start the transfer function analysis by considering the node voltages as shown in Fig. A7.1.1. If we assume an ideal opamp approximation, while there is enough feedback the inverting amplifier tries to set $v_1 \approx 0$. So by summing the currents at v_1 we have:

$$\frac{v_2 - v_1}{R_1} = \frac{v_1 - v_o}{\dfrac{1}{s\,C_1}}$$

$$\Rightarrow \qquad v_2 = -v_o\, s\, C_1 R_1 \tag{A7.1.1}$$

and by summing the currents at v_2 we obtain:

$$\frac{v_3 - v_2}{R_3} = \frac{v_2 - v_o}{R_2} + v_2\, s\, C_2 + \frac{v_2}{R_1} \tag{A7.1.2}$$

$$\Rightarrow \qquad v_3 = v_2\left(1 + \frac{R_3}{R_2} + \frac{R_3}{R_1} + s\, C_2 R_3\right) - v_o \frac{R_3}{R_2} \tag{A7.1.3}$$

We can substitute v_2 from Eq. A7.1.1:

$$v_3 = -v_o\left[s\, C_1 R_1\left(1 + \frac{R_3}{R_2} + \frac{R_3}{R_1}\right) + s^2 C_1 C_2 R_1 R_3 + \frac{R_3}{R_2}\right] \tag{A7.1.4}$$

Next, by summing the currents at v_3:

$$\frac{v_s - v_3}{R_4} = \frac{v_3 - v_2}{R_3} + v_3\, s\, C_3 \tag{A7.1.5}$$

which we can rewrite as:

$$v_s = v_3\left(1 + \frac{R_4}{R_3} + s\, C_3 R_4\right) - v_2 \frac{R_4}{R_3} \tag{A7.1.6}$$

Again, by substituting v_2 from Eq. A7.1.1 and v_3 from Eq. A7.1.4:

$$v_{\mathrm{s}} = -v_{\mathrm{o}}\left\{\left[s\,C_1 R_1\left(1+\frac{R_3}{R_2}+\frac{R_3}{R_1}\right)+s^2 C_1 C_2 R_1 R_3+\frac{R_3}{R_2}\right]\left(1+\frac{R_4}{R_3}+s\,C_3 R_4\right)-s\,C_1 R_1\frac{R_4}{R_3}\right\} \tag{A7.1.7}$$

With only v_{s} and v_{o} left we can write the transfer function (the non-canonic form):

$$\frac{v_{\mathrm{o}}}{v_{\mathrm{s}}} = \frac{-1}{\left[s\,C_1 R_1\left(1+\frac{R_3}{R_2}+\frac{R_3}{R_1}\right)+s^2\,C_1 C_2 R_1 R_3+\frac{R_3}{R_2}\right]\left(1+\frac{R_4}{R_3}+s\,C_3 R_4\right)-s\,C_1 R_1\frac{R_4}{R_3}} \tag{A7.1.8}$$

By multiplying the various parts of the denominator we obtain a 3rd-order function of s:

$$\frac{v_{\mathrm{o}}}{v_{\mathrm{s}}} = -\frac{1}{K_3\,s^3+K_2\,s^2+K_1\,s+K_0} \tag{A7.1.9}$$

where the coefficients at their relative power of s are :

$$K_3 = C_1 C_2 C_3 R_1 R_3 R_4 \tag{A7.1.10}$$

$$K_2 = C_1 C_2 R_1 R_3\left(1+\frac{R_4}{R_3}\right)+C_1 C_3 R_1 R_4\left(1+\frac{R_3}{R_2}+\frac{R_3}{R_1}\right) \tag{A7.1.11}$$

$$K_1 = C_1 R_1\left(1+\frac{R_3}{R_2}+\frac{R_3}{R_1}+\frac{R_4}{R_2}+\frac{R_4}{R_1}\right)+C_3 R_4\frac{R_3}{R_2} \tag{A7.1.12}$$

$$K_0 = \frac{R_3}{R_2}\left(1+\frac{R_4}{R_3}\right) \tag{A7.1.13}$$

To put the transfer function into its canonical form, we must divide it by the coefficient of the highest power of s, which is K_3, so we obtain a new set of coefficients:

$$Q_2 = \frac{K_2}{K_3} = \frac{1}{C_3 R_4}\left(1+\frac{R_4}{R_3}\right)+\frac{1}{C_2 R_3}\left(1+\frac{R_3}{R_2}+\frac{R_3}{R_1}\right) \tag{A7.1.14}$$

$$Q_1 = \frac{K_1}{K_3} = \frac{1+(R_3+R_4)\left(\frac{1}{R_2}+\frac{1}{R_1}\right)}{C_2 C_3 R_3 R_4}+\frac{\frac{R_3}{R_2}}{C_1 C_2 R_1 R_3} \tag{A7.1.15}$$

$$Q_0 = \frac{K_0}{K_3} = \frac{R_3}{R_2}\left(1+\frac{R_4}{R_3}\right)\frac{1}{C_1 C_2 C_3 R_1 R_3 R_4} \tag{A7.1.16}$$

Now the transfer function looks like :

$$\frac{v_{\mathrm{o}}}{v_{\mathrm{s}}} = -\frac{\frac{1}{C_1 C_2 C_3 R_1 R_3 R_4}}{s^3+s^2\,Q_2+s\,Q_1+Q_0} \tag{A7.1.17}$$

In order to obtain the correctly normalized system gain we must make the numerator equal to the coefficient at the lowest power of s (that is at s^0), which is Q_0. Besides the system time constants, Q_0 also contains a scaling factor:

$$\frac{R_3}{R_2}\left(1+\frac{R_4}{R_3}\right) = \frac{R_3+R_4}{R_2} = -\frac{1}{A_0} \tag{A7.1.18}$$

and from the schematic diagram we see that A_0 is the system's DC gain. Note that because the circuit inverts the signal, A_0 must include the negative sign. Therefore we can write the general form:

$$\frac{v_\mathrm{o}}{v_\mathrm{s}} = A_0\,\frac{Q_0}{s^3+s^2\,Q_2+s\,Q_1+Q_0} \tag{A7.1.19}$$

From the gain factor and the form of the coefficients it is clear that we can optimize the component values if we set:

$$R_1 = R_3 = R_4 = R \tag{A7.1.20}$$

Finally we can write the complete transfer function explicitly:

$$\frac{v_\mathrm{o}}{v_\mathrm{s}} = \frac{-\dfrac{R_2}{2R}\cdot\dfrac{\dfrac{2R}{R_2}}{R^3\,C_1C_2C_3}}{s^3+s^2\left(\dfrac{2}{C_3R}+\dfrac{2+\dfrac{R}{R_2}}{C_2R}\right)+s\left(\dfrac{3+\dfrac{2R}{R_2}}{C_2C_3R^2}+\dfrac{\dfrac{R}{R_2}}{C_1C_2R^2}\right)+\dfrac{\dfrac{2R}{R_2}}{R^3\,C_1C_2C_3}} \tag{A7.1.21}$$

So the DC gain is:

$$A_0 = -\frac{R_2}{2R} \tag{A7.1.22}$$

and the cut off frequency is:

$$\omega_\mathrm{h3} = \sqrt[3]{\frac{2R}{R_2}\cdot\frac{1}{R^3\,C_1C_2C_3}} \tag{A7.1.23}$$