2. Statistical Reliability with Applications

This chapter reviews fundamental ideas in reliability theory and inference. The first part of the chapter accounts for lifetime distributions that are used in engineering reliability analysis, including general properties of reliability distributions that pertain to lifetime for manufactured products. Certain distributions are formulated on the basis of simple physical properties, and other are more or less empirical. The first part of the chapter ends with a description of graphical and analytical methods to find appropriate lifetime distributions for a set of failure data.

The second part of the chapter describes statistical methods for analyzing reliability data, including maximum likelihood estimation and likelihood ratio testing. Degradation data are more prevalent in experiments in which failure is rare and test time is limited. Special regression techniques for degradation data can be used to draw inference on the underlying lifetime distribution, even if failures are rarely observed.

The last part of the chapter discusses reliability for systems. Along with the components that comprise the system, reliability analysis must take account of the system configuration and (stochastic) component dependencies. System reliability is illustrated with an analysis of logistics systems (e.g., moving goods in a system of product sources and retail outlets). Robust reliability design can be used to construct a supply chain that runs with maximum efficiency or minimum cost.

2.1 Introduction and Literature Review

In every day use, words like reliability and quality have meanings that vary depending on the context. In engineering, *reliability* is defined as the ability of an item to perform its function, usually measured in terms of probability as a function of time. *Quality* denotes how the item conforms to its specifications, so reliability is a measure of the item's quality over time.

Since the time of *Birnbaum* and *Sanders* [2.1], when system reliability emerged as its own discipline, research has centered on the operation of simple systems with identical parts working independently of each other. Today's systems do not fit this mold; system representation must include multifaceted components with several component states that can vacillate between perfect operation and terminal failure. Not only do components interact within systems, but many systems are dynamic in that the system configuration can be expected to change during its operation, perhaps due to component failures or external stresses. Computer software, for example, changes its failure structure during the course of design, testing and implementation.

Statistical methods for reliability analysis grew from this concept of system examination, and system reliability is often gauged through component lifetime testing. This chapter reviews the current framework for statistical reliability and considers some modern needs from experimenters in engineering and the physical sciences.

Statistical analysis of reliability data in engineering applications cannot be summarized comprehensively in a single book chapter such as this. The following books (listed fully in the reference section) serve as an excellent basis for a serious treatment of the subject:

1. Statistical Theory of Reliability and Life Testing by *Barlow* and *Proschan* [2.2]
2. Practical Methods for Reliability Data Analysis by *Ansell* and *Phillips* [2.3]
3. Reliability: Probabilistic Models and Statistical Methods by *Leemis* [2.4]
4. Applied Reliability by *Tobias* and *Trindade* [2.5]
5. Engineering Reliability by *Barlow* [2.6]
6. Reliability for Technology, Engineering and Management by *Kales* [2.7]
7. Statistical Methods for Reliability Data by *Meeker* and *Escobar* [2.8]
8. Reliability Modeling, Prediction, and Optimization by *Blischke* and *Murthy* [2.9]
9. Statistical Methods for the Reliability of Repairable Systems by *Rigdon* and *Basu* [2.10] and
10. Modern Reliability Analysis: A Bayesian Perspective by *Johnson* et al. [2.11]

Some of the books in this list focus on reliability theory, and others focus exclusively on reliability engineering. From the more inclusive books, [2.8] provides a complete, high-level guide to reliability inference tools for an engineer, and most examples have an engineering basis (usually in manufacturing). For reliability problems closely associated with materials testing, *Bogdanoff* and *Kozin* [2.12] connects the physics of degradation to reliability models. *Sobczyk* and *Spencer* [2.13] also relate fatigue to reliability through probability modeling. For reliability prediction in software performance, *Lyu* [2.14] provides a comprehensive guide of engineering procedures for software reliability testing, while a more theoretical alternative by *Singpurwalla* and *Wilson* [2.15] emphasizes probability modeling for software reliability, including hierarchical Bayesian methods. Closely related to reliability modeling in engineering systems, *Bedford* and *Cooke* [2.16] goes over methods of probabilistic risk assessment, which is an integral part of reliability modeling for large and complex systems.

Other texts emphasize reliability assessment in a particular engineering field of interest. For statistical reliability in geotechnical engineering, *Baecher* and *Christian* [2.17] is recommended as it details statistical problems with soil variability, autocorrelation (i. e., Kriging), and load/resistance factors. *Ohring* [2.18] provides a comprehensive guide to reliability assessment for electrical engineering and electronics manufacturing, including reliability pertaining to degradation of contacts (e.g., crack growth in solder), optical-fiber reliability, semiconductor degradation and mass-transport-induced failure. For civil engineering, *Melchers'* [2.19] reliability text has a focus on reliability of structural systems and loads, time-dependent reliability and resistance modeling.

2.2 Lifetime Distributions in Reliability

While engineering research has contributed a great deal of the current methods for reliability life testing, an equally great amount exists in the biological sciences, especially relating to epidemiology and biostatistics. Life testing is a crucial component to both fields, but the bio-related sciences tend to focus on mean lifetimes and numerous risk factors. Engineering methods, on the other hand, are more likely to focus on upper (or lower) percentiles of the lifetime distribution as well as the stochastic dependencies between working components. Another crucial difference between the two research areas is that engineering models are more likely to be based on principles of physics that lead to well-known distributions such as Weibull, log–normal, extreme value and so on.

The failure-time distribution is the most widely used probability tool for modeling product reliability in science and industry. If $f(x)$ represents the probability density function for the product's failure time, then the its reliability is $R(x) = \int_x^\infty f(u)\,\mathrm{d}u$, and $R(t) = 1 - F(t)$ where F is the cumulative distribution function (CDF) corresponding to f. A *quantile* is the CDF's inverse; The p-th quantile of F is the lifetime value t_p such that $F(t_p) = p$. To understand the quality of a manufactured product through these lifetime probability functions, it is often useful to consider the notion of *aging*. For example, the (conditional) reliability of a product that has been working t units of time

is

$$R(x|t) = \frac{R(t+x)}{R(t)}, \text{ if } R(t) > 0\,. \tag{2.1}$$

The rate of change of $R(x|t)$ is an important metric for judging a product's quality, and the conditional *failure rate* function $h(t)$ is defined as

$$h(t) = \lim_{x\to\infty} x^{-1}\frac{R(t)-R(t+x)}{R(t)} = \frac{f(t)}{R(t)}\,. \tag{2.2}$$

The *cumulative failure rate* (sometimes called the *hazard function*) is $H(t) = \int_0^t h(u)\,\mathrm{d}u$, and has many practical uses in reliability theory because of its monotonicity and the fact that $H(t) = -\log R(t)$.

The failure rate clearly communicates how the product ages during different spans of its lifetime. Many manufactured products have an increasing failure rate, but the rate of increase is rarely stable throughout the product's lifetime. If $r(t)$ remains constant, it is easy to show the lifetime distribution is exponential $\left[f(x) = \theta\exp(-\theta x), x > 0\right]$ and the product exhibits no aging characteristics. Many electronic components and other manufactured items have brief initial period when failure rate is relatively high and decrease toward a steady state, where it stays until aging causes the rate to increase. This is called a bath-tub failure rate. The period in which early failures occur (called *infant mortality*) is called the *burn-in* period, and is often used by manufacturers to age products and filter out defectives (early failures) before being making it available to the consumer.

2.2.1 Alternative Properties to Describe Reliability

The failure rate function, reliability function, cumulative hazard function, and probability density describe different aspects of a lifetime distribution. The expected lifetime, or *mean time to failure* (MTTF) is an important measure for repairable systems. Several alternatives for characterizing properties of the lifetime distribution include:

- *Mean residual life* $= L(t) = E_X(X - t|X \geq t)$ is the expected residual life of a component that has already lasted t units of time. If $L(t)$ is less than the expected lifetime μ, the product is exhibiting aging by the time t.
- *Reversed hazard rate* $= \nu(t) = f(x)/F(x)$ provides a different aspect of reliability: the conditional failure frequency at the time just before t given that the product failed in $(0, t]$ (see Chapt. 1 of [2.20], for example).
- *Percentile residual life* $= Q_\alpha = F^{-1}\left[1-(1-\alpha)\times R(t)\right] - t$ is the α quantile of the residual life (the conditional lifetime distribution given that the product has lasted t units of time). The median residual life, where $\alpha = 1/2$ compares closely to $L(t)$.
- *Mill's ratio* $= R(x)/f(x) = 1/h(x)$, used in economics, is not an ordinary way to characterize reliability, but it is worth noting because of its close connection to failure rate.

2.2.2 Conventional Reliability Lifetime Distributions

So far, only one distribution (exponential) has been mentioned. Rather than presenting a formal review of commonly used reliability distributions, a summary of commonly applied lifetime distributions is presented in Table 2.1, including the exponential, gamma, Weibull, log–normal, logistic, Pareto and extreme value. In the table, $\Gamma(t) = \int_0^\infty x^{t-1}\,\mathrm{e}^{-x}\,\mathrm{d}x$ is the ordinary gamma function, and $IG(t, x)$ represents the corresponding incomplete Gamma function.

For manufacturing centers and research laboratories that conduct lifetime tests on products, lifetime data is an essential element of reliability analysis. However, a great deal of reliability analysis is based on field data, or reliability information sampled from day-to-day usage of the product. In many of these instances, lifetime data is a luxury not afforded to the reliability inference. Instead, historical event data and inspection counts are logged for the data analysis. Consequently, several discrete distributions (e.g., Poisson, binomial, geometric) are important in reliability applications. Chapter 4 has a more detailed discussion of these and other statistical distributions applied in engineering problems.

2.2.3 From Physics to Failure Distributions

Many of the distributions in Table 2.1 are derived based on physical principles. For example, *Weibull* [2.21] derived the distribution that takes his name to represent the breaking strength of materials based on the idea that some components are comparable to a chain that is no stronger than its weakest link. From this premise, the distribution can be derived from properties of minimums, in contrast to the extreme value distribution, which can be derived through the properties of maximums (see [2.22], for example). In a short time after its introduction, the Weibull distribution was successfully applied to numerous modeling problems in engineering and has become the hallmark distribution in applied re-

Table 2.1 Common lifetime distributions used in reliability data analysis

Distribution	$f(t),\quad t>0$	$h(t)$	μ	σ^2	Parameter space
Exponential	$\theta\,\mathrm{e}^{-\theta t}$	θ	$1/\theta$	$1/\theta^2$	$\theta>0$
Weibull	$\lambda\kappa t^{\kappa-1}\,\mathrm{e}^{-\lambda t^{\kappa}}$	$\lambda\kappa t^{\kappa-1}$	$\lambda^{-1/\kappa}\,\Gamma\left(1+\frac{1}{\kappa}\right)$	$\lambda^{-2/\kappa}\left[\Gamma\left(1+\frac{2}{\kappa}\right)-\Gamma^2\left(1+\frac{1}{\kappa}\right)\right]$	$\kappa>0,\lambda>0$
Gamma	$\lambda^r\,\Gamma^{-1}(r)t^{r-1}\,\mathrm{e}^{-\lambda t}$	$\dfrac{\lambda^r t^{r-1}\,\mathrm{e}^{-\lambda t}}{\Gamma(r)\,[1-IG(r,\lambda t)]}$	r/λ	r/λ^2	$r>0,\lambda>0$
Log–normal	$\dfrac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{\frac{-(\log t-\mu)^2}{2\sigma^2}}$	$f(t)/R(t)$	$\mathrm{e}^{\mu+\sigma^2/2}$	$\mathrm{e}^{2\mu+2\sigma^2}-\mathrm{e}^{2\mu+\sigma^2}$	$-\infty<\mu<\infty,\sigma>0$
Logistic	$\dfrac{\mathrm{e}^{-(t-\lambda)/\beta}}{\beta\left(1+\mathrm{e}^{-(t-\lambda)/\beta}\right)^2}$	$\left[\beta\left(1+\mathrm{e}^{-(t-\lambda)/\beta}\right)\right]^{-1}$	λ	$(\beta\pi)^2/3$	$-\infty<\lambda<\infty,\beta>0$
Pareto	$\dfrac{m\theta^m}{t^{m+1}}$	$\dfrac{m}{t}$	$\dfrac{m\theta}{m-1}$	$\dfrac{m\theta^2}{(m-1)^2(m-2)}$	$t>\theta,m>0$
Extreme value	$\dfrac{\exp[-(t-a)/b]}{b\exp[-\exp(-(t-a)/b)]}$	$\dfrac{\exp[-(t-a)/b]}{b\exp[-\exp(-(t-a)/b)]-1}$	$a-b\Gamma'(1)$	$(b\pi)^2/6$	$-\infty<a<\infty,b>0$

liability. A primary reason for its suitability to lifetime analysis is its flexible failure rate; unlike other distributions listed in Table 2.1, the Weibull failure rate is simple to model, easy to demonstrate and it can be either increasing or decreasing. A mixture of two Weibull distributions can be used to portray a bath-tub failure rate (as long as only one of the shape parameters is less than one). *Mudholkar* et al. [2.23] introduce a new shape parameter to a generalized Weibull distribution that allows bath-tub-shaped failure rates as well as a broader class of monotone failure rates.

For materials exposed to constant stress cycles with a given stress range, lifetime is measured in number of cycles until failure (N). The Whöler curve (or S–N curve) relates stress level (S) to N as $NS^b=k$, where b and k are material parameters (see [2.13] for examples). By taking logarithms of the S–N equation, we can express cycles to failure as a linear function: $Y=\log N=\log k-b\log S$. If N is log–normally distributed, then Y is normally distributed and regular regression models can be applied for predicting cycles to failure (at a given stress level). In many settings, the log–normal distribution is applied as the failure time distribution when the corresponding degradation process based on rates that combine multiplicatively. Despite having a concave-shaped (or upside-down bath-tub shape) failure rate, the log–normal is especially useful in modeling fatigue crack growth in metals and composites.

Birnbaum and *Saunders* [2.1] modeled the damage to a test item after n cycles as $B_n=\zeta_1+\cdots+\zeta_n$, where ζ_i represents the damage amassed in the i-th cycle. If failure is determined by B_n exceeding a fixed damage threshold value B^*, and if the ζ_i are identically and independently distributed,

$$P(N\le n)=P(B_n>B^*)\approx\Phi\left(\frac{B^*-n\mu}{\sigma\sqrt{n}}\right)\,, \tag{2.3}$$

where Φ is the standard normal CDF. This results because B_n will be approximately normal if n is large enough. The reliability function for the test unit is

$$R(t)\approx\Phi\left(\frac{B^*-n\mu}{\sigma\sqrt{n}}\right) \tag{2.4}$$

which is called the *Birnbaum–Saunders* distribution. It follows that

$$W=\frac{\mu\sqrt{N}}{\sigma}-\frac{B^*}{\sigma\sqrt{N}} \tag{2.5}$$

has a normal distribution, which leads to accessible implementation in lifetime modeling (see [2.24] or [2.12] for more properties).

2.2.4 Lifetime Distributions from Degradation Modeling

These examples show how the product's lifetime distribution can be implied by knowledge of how it degrades in time. In general, degradation measurements have great potential to improve lifetime data analysis, but they also introduce new problems to the statistical inference. Lifetime models have been researched and refined for

many manufactured products that are put on test. On the other hand, degradation models tend to be empirical (e.g., nonparametric) or based on simple physical properties of the test item and its environment (e.g., the Paris crack law, Arrhenius rule, power law) which often lead to obscure lifetime models. *Meeker* and *Escobar* [2.8] provide a comprehensive guide to degradation modeling, and show that many valid degradation models will not yield lifetime distributions with closed-form solutions. Given the improving computational tools available to researchers, this is no deterrent to using degradation analysis; users of the S-plus programming software can access advanced tools for degradation analysis from SPLIDA (S-plus functions for life data analysis) developed by *Meeker* [2.25].

In a setting where the lifetime distribution is known, but the degradation distribution is unknown, degradation information does not necessarily complement the available lifetime data. For example, the lifetime data may be distributed as Weibull, but conventional degradation models will contradict the Weibull assumption (actually, the rarely used *reciprocal Weibull* distribution for degradation with a fixed failure threshold leads to Weibull lifetimes).

In selecting a degradation model based on longitudinal measurements of degradation, monotonic models are typically chosen under the assumption that degradation is a one-way process. In some cases, such as the measured luminosity of light displays (vacuum fluorescent displays, plasma display devices), the degradation is not necessarily monotonic because, during the first phase of product life, impurities inside the light display's vacuum are slowly burned off and luminosity increases. After achieving a peak level, usually before 100 hours of use, the light slowly degrades in a generally monotonic fashion. See *Bae* and *Kvam* [2.26, 27] for details on the modeling of non-monotonic degradation data. Degradation data analysis is summarized in Sect. 2.3.3.

2.2.5 Censoring

For most products tested in regular-use conditions (as opposed to especially harsh conditions), the allotted test time is usually too short to allow the experimenter to witness failure times for the entire set that is on test. When the item is necessarily taken off test after a certain amount of test time, its lifetime is *right censored*. This is also called type I censoring. Type II censoring corresponds to tests that are stopped after a certain number of failures (say k out of n, $1 \le k \le n$) occur.

Inspection data are lifetimes only observed at fixed times of inspection. If the inspection reveals a failed test item, it must be *left censored* at that fixed time. Items that are still working at the time of the last inspection are necessarily right censored. This is sometimes called *interval censoring*.

Censoring is a common hindrance in engineering applications. Lifetime data that are eclipsed by censoring cause serious problems in the data analysis, but it must be kept in mind that each observation, censored or not, contributes information and increases precision in the statistical inference, overall.

2.2.6 Probability Plotting

Probability plotting is a practical tool for checking the adequacy of a fitted lifetime distribution to a given set of data. The rationale is to transform the observed data according to a given distribution so a linear relationship exists if the distribution was specified correctly. In the past, probability plotting paper was employed to construct the transformation, but researchers can find plotting options on many computer packages that feature data analysis (e.g., SAS, S-Plus, Matlab, Minitab, SPSS) making the special plotting paper nearly obsolete. Despite the applicability of this technique, few engineering texts feature in-depth discussion on probability plotting and statistics texts tend to focus on theory more than implementation. *Rigdon* and *Basu* [2.10] provide a thorough discussion of basic probability plotting, and *Atkinson* [2.28] provides a substantial discussion of the subject in the context of regression diagnostics. Advanced plotting techniques even allow for censored observations (see *Waller* and *Turnbull* [2.29], for example).

To illustrate how the plot works, we first linearize the CDF of the distribution in question. For example, if we consider the two-parameter Weibull distribution, the *quantile function* is

$$t_p = \left(\frac{-\log p}{\lambda}\right)^{1/\kappa}, \tag{2.6}$$

which implies that the plot of $\log t$ has a linear relationship with the log–log function of $p = F(t)$. Hence, Weibull probability plots are graphed on log–log probability paper. Figure 2.1 shows a Weibull plot (using Minitab) for the fatigue life of 67 alloy specimens that failed before $n = 300\,000$ cycles. This data set is from *Meeker* and *Escobar* [2.8] and the plot also includes 95% confidence bands that identify the uncertainty associated with the plot. In this case the curvature (especially

2.3 Analysis of Reliability Data

Once the lifetime distribution of a test item is determined, the data can be used to estimate important properties of the distribution, including mean, standard deviation, failure rate, reliability (at a fixed time t) and upper or lower quantiles that pertain to early or late failure times.

There are two fundamental methods for approaching the analysis of lifetime data: Bayesian methods and, for the lack of an optimal term, non-Bayesian methods. Although Bayesian methods are accepted widely across many fields of engineering and physical science, non-Bayesian statistics, mostly frequentist and likelihood methods, are still an industry standard. This chapter will not detail how methods of statistical inference are derived in various frameworks of statistical ideology. Accelerated life testing, an important tool for designing reliability experiments, is discussed in detail in Chapt. 22 and is only mentioned in this chapter. Instead, a summary of important procedures is outlined for statistical estimation, confidence intervals and hypothesis tests.

2.3.1 Maximum Likelihood

Parametric likelihood methods examine a family of probability distributions and choose the parameter combination that best fits the data. A likelihood function is generally defined by the observed probability model; if the lifetime data $X_1, \ldots, X_n$ are independently and identically (i.i.d.) distributed with density function $f_X(x;\theta)$, the likelihood function is

$$L(\theta) = \prod_{i=1}^{n} f_X(x_i;\theta) \tag{2.7}$$

and the *maximum likelihood estimator* (MLE) is the value of θ that maximizes $L(\theta)$. Single-parameter distributions such as the exponential generate easily solved MLEs, but distributions with two or more parameters are not often straightforward. Samples that are not IID lead to complicated likelihood functions and numerical methods are usually employed to solve for MLEs. If an observation x represents a right-censoring time, for example, then $P(\text{censor}) = R(x)$ and this information contributes the term $R(x)$ to the likelihood instead of $f(x)$. *Leemis* [2.4] provides a thorough introduction to likelihood theory for reliability inference.

For most parametric distributions of interest, the MLE $(\hat{\theta})$ has helpful limit properties. As the sample size $n \to \infty$, $\sqrt{n}(\hat{\theta}-\theta) \to N\left[0, i(\theta)^{-1}\right]$, where

$$i(\theta) = \mathrm{E}\left[\left(\frac{\partial}{\partial\theta}\log f\right)^2\right] = -\mathrm{E}\left(\frac{\partial^2}{\partial\theta^2}\log f\right) \tag{2.8}$$

is the estimator's *Fisher information*. For other parameters of interest, say $\psi(\theta)$, we can construct approximate confidence intervals based on an estimated variance using the Fisher information:

$$\hat{\sigma}^2\left[\psi(\hat{\theta})\right] \approx \frac{1}{\psi(\hat{\theta})^2 i(\hat{\theta})}\,. \tag{2.9}$$

This allows the analyst to make direct inference for the component reliability $\left[\psi(\theta;t) = R_\theta(t)\text{, for example}\right]$.

Example. MLE for failure rate with exponential data $(X_1, \ldots, X_n)$: the likelihood is based on $f(x) = \theta\exp(-\theta x)$ and is easier to maximize in its natural-log form

$$\log L(\theta) = \log\left(\prod_{i=1}^{n}\theta\,\mathrm{e}^{-\theta x_i}\right) = n\log\theta - \theta\sum_{i=1}^{n} x_i.$$

The maximum occurs at $\hat{\theta} = 1/\bar{x}$, and the Fisher information $i(\theta) = n/\theta^2$, so an approximate $(1-\alpha)$ confidence interval is

$$\frac{1}{\bar{x}} \pm z_{\frac{\alpha}{2}}\, i(\hat{\theta})^{-1/2} = \frac{1}{\bar{x}} \pm z_{\frac{\alpha}{2}}\frac{\hat{\theta}}{\sqrt{n}} = \frac{1}{\bar{x}} \pm z_{\frac{\alpha}{2}}\left(\bar{x}\sqrt{n}\right)^{-1}. \tag{2.10}$$

In this case, the approximation above is surpassed by an exact interval that can be constructed from the statistic $2\theta(X_1+\cdots+X_n)$ which has a chi-squared distribution with $2n$ degrees of freedom. The confidence statement $P\left[\chi^2_{2n}(1-\alpha/2) \le (X_1+\cdots+X_n) \le \chi^2_{2n}(\alpha/2)\right] = 1-\alpha$, where $\chi^2_{2n}(\alpha)$ represents the α quantile of the chi-squared distribution with $2n$ degrees of freedom, leads to a $1-\alpha$ confidence interval for θ of

$$\left(\frac{\chi^2_{2n}(1-\alpha/2)}{2n\bar{x}}, \frac{\chi^2_{2n}(\alpha/2)}{2n\bar{x}}\right). \tag{2.11}$$

2.3.2 Likelihood Ratio

Uncertainty bounds, especially for multidimensional parameters, are more directly computed using the likelihood ratio (LR) method. Here we consider θ to have p components. Confidence regions are constructed by

actual contours (in p-dimensions) of the likelihood function. Define the LR as

$$\Lambda(\theta, \hat{\theta}) = \frac{L(\theta)}{L(\hat{\theta})} , \tag{2.12}$$

where $\hat{\theta}$ is the MLE of L. If θ is the true value of the parameter, then

$$-2 \log \Lambda \sim \chi^2_p ,$$

where χ^2_p is the chi-squared distribution with p degrees of freedom. A $(1-\alpha)$ confidence region for θ is

$$\left\{ \theta : -2 \log \Lambda(\theta, \hat{\theta}) \le \chi^2_p(\alpha) \right\} , \tag{2.13}$$

where $\chi^2_p(\alpha)$ represents the $1-\alpha$ quantile of the χ^2_p distribution.

Example. Confidence region for Weibull parameters: In this case, the MLEs for $\theta = (\lambda, r)$ must be computed using numerical methods. Many statistical software packages compute such estimators along with confidence bounds. With $(\hat{\lambda}, \hat{r})$, $L(\hat{\lambda}, \hat{r})$ standardizes the likelihood ratio so that $0 \le \Lambda(\theta, \hat{\theta}) \le 1$ and Λ peaks at $(\lambda, r) = (\hat{\lambda}, \hat{r})$. Figure 2.2 shows 50%, 90% and 95% confidence regions for the Weibull parameters based on a simulated sample of $n = 100$.

Empirical likelihood provides a powerful method for providing confidence bounds on parameters of inference without necessarily making strong assumptions about the lifetime distribution of the product (i. e., it is *nonparametric*). This chapter cannot afford the space needed to provide the reader with an adequate description of its method and theory; *Owen* [2.30] provides a comprehensive study of empirical likelihood including its application to lifetime data.

2.3.3 Degradation Data

As an alternative to traditional life testing, degradation tests can be effective in assessing product reliability when measurements of degradation leading to failure are observable and quantifiable. *Meeker* and *Escobar* [2.8] provide the most comprehensive discussion on modeling and analyzing degradation data for manufactured items that have either a soft failure threshold (i. e., an arbitrary fixed point at which the device is considered to have failed) or items that degrade before reaching a failed state. In the electronics industry, product lifetimes are far too long to test in a laboratory; some products in the lab will tend to become obsolete long before they actually fail. In such cases, accelerated degradation testing (ADT) is used to hasten product failure. In the manufacture of electronic components, this is often accomplished by increasing voltage or temperature. See Chapt. 22 for a review of recent results in ALT.

If the degradation path is modeled as

$$y_i(t) = \eta_i(t) + \epsilon_i(t) , \tag{2.14}$$

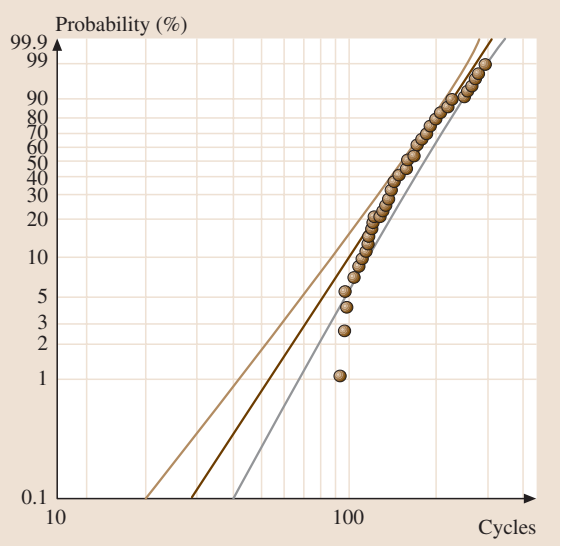

Fig. 2.1 Weibull probability plot for alloy T7987 fatigue life [2.8]

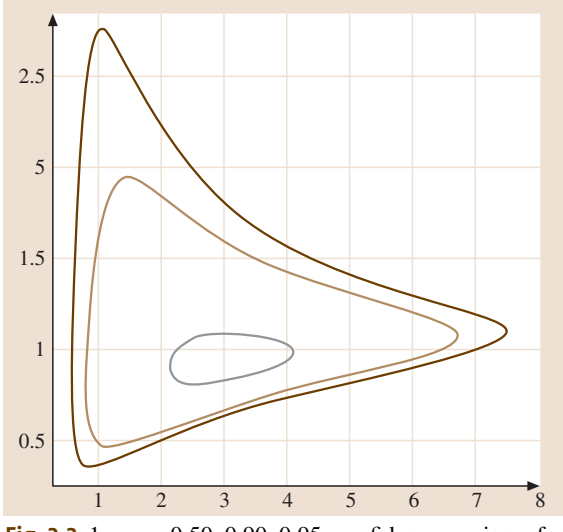

Fig. 2.2 $1-\alpha = 0.50, 0.90, 0.95$ confidence regions for Weibull parameters (λ, r) based on simulated data of size $n = 100$

where η_i is the path of the i-th tested unit ($i = 1, \ldots, n$) and ϵ_i represents an error term that has a distribution $H(\epsilon; \Sigma)$ with parameter Σ unknown. Failure would be declared once $y_i(t)$ passes a certain degradation threshold, say y^*. The lifetime distribution can be computed as (assuming degradation is an increasing function)

$$F(t) = P\left[y(t) > y^*\right] = P\left[\epsilon_i(t) > y^* - \eta_i(t)\right] . \tag{2.15}$$

If η is a deterministic function, the lifetime distribution is driven completely by the error term. This is not altogether realistic. In most cases, item-to-item variability exists and the function η contains *random coefficients*; that is, $\eta(t) = \eta(t, \lambda, \theta)$, where λ is a vector of unknown parameters (common to all units) and θ is a vector of random coefficients which have a distribution G (with further unknown parameters β) so that realizations of θ change from unit to unit. With an accumulated set of unknown parameters (λ, β, Σ), this makes for a difficult computation of the lifetime distribution. Numerical methods and simulations are typically employed to generate point estimates and confidence statements.

Least squares or maximum likelihood can be used to estimate the unknown parameters in the degradation model. To estimate $F(t_0)$, one can simulate M degradation curves (choosing M to be large) from the estimated regression by generating M random coefficients $\theta_1, \ldots, \theta_M$ from the estimated distribution $G(\theta; \hat{\beta})$. Next compute the estimated degradation curve for y_i based on the model with θ_i and $\hat{\lambda}$: $y_i(t) = \eta_i(t; \hat{\lambda}, \theta_i)$. Then $\hat{F}(t_0)$ is the proportion of the M generated curves that have reached the failure threshold y^* by time t_0.

Meeker and Escobar use *bootstrap confidence intervals* for measuring the uncertainty in the lifetime distribution estimate. Their method follows the general algorithm for nonparametric bootstrap confidence intervals described in *Efron* and *Tibshirani* [2.31]. There are numerous bootstrap sampling methods for various uncertainty problems posed by complex models. This algorithm uses a nonparametric bootstrap sampling procedure which *resamples* n of the sample degradation curves *with replacement* (i. e., so some curves may not be represented in the sample while others may be represented multiple times). This resampled set will be termed the bootstrap sample in the following procedure for constructing confidence intervals.

1. Compute estimates of the parameters β, λ, Σ.
2. Use simulation (as above) to construct $\hat{F}(t_0)$.
3. Generate $N \geq 1000$ bootstrap samples, and for each one, compute estimates $\hat{F}^{(1)}(t_0), \ldots, \hat{F}^{(N)}(t_0)$. This is done as before except now the M simulated degradation paths are constructed with an error term generated from $H(\eta; \hat{\Sigma})$ to reflect variability in any single degradation path.
4. With the collection of bootstrap estimates from step 3, compute a $1 - \alpha$ confidence interval for $F(t_0)$ as $\left[\hat{F}^l(t_0), \hat{F}^u(t_0)\right]$, where the indexes $1 \leq l \leq u \leq N$ are calculated as $l/N = \Phi[2\Phi^{-1/2}(p_0) + \Phi^{-1/2} \times (\alpha/2)]$ and $u/N = \Phi[2\Phi^{-1/2}(p_0) + \Phi^{-1/2} \times (1 - \alpha/2)]$, and p_0 is the proportion of bootstrap estimates of $F(t_0)$ less than $\hat{F}(t_0)$.

Procedures based on realistic degradation models can obviously grow to be computationally cumbersome, but for important applications the increase in statistical efficiency can be dramatic. In the past, these computations have impeded degradation analysis from being a feature of reliability problem solving. Such analyses are easier to implement now, and the reliability analyst need not be coerced into using an overly simplistic model—for instance, a linear model that does not allow for random coefficients.

2.4 System Reliability

A *system* is an arrangement of components that work together for a common goal. So far, the discussion has fixated on the lifetime analysis of a single component, so this represents an extension of single-component reliability study. At the simplest level, a system contains n components of an identical type that are assumed to function independently. The mapping of component outcomes to system outcomes is through the system's structure function. The *reliability function* describes the system reliability as a function of component reliability.

A *series* system is such that the failure of any of the n components in the working group causes the system to fail. If the probability that a single component fails in its mission is p, the probability the system fails is $1 - P(\text{system succeeds}) = 1 - P(\text{all } n \text{ components succeed}) = 1 - (1 - p)^n$. More generally, in terms of component reliabili-

ties $(p_1, \ldots, p_n)$, the system reliability function Ψ is

$$\Psi(p_1, \ldots, p_n) = \prod_{i=1}^{n} (1 - p_i) \,. \tag{2.16}$$

A *parallel* system is just the opposite; it fails only after every one of its n working components fail. The system failure probability is then

$$\Psi(p_1, \ldots, p_n) = 1 - \prod_{i=1}^{n} p_i \,. \tag{2.17}$$

The parallel system and series system are special cases of a k-out-of-n system, which is a system that works as long as at least k out of its n components work. Assuming $p_i = p, i = 1, \ldots, n$, the reliability of a k-out-of-n systems is

$$\Psi(p) = \sum_{i=k}^{n} \binom{n}{i} (1-p)^i \, p^{n-1} \,. \tag{2.18}$$

Of course, most component arrangements are much more complex that a series or parallel system. With just three components, there are five unique ways of arranging the components in a coherent way (that is, so that each component success contributes positively to the system reliability). Figure 2.3 shows the *system structure* of those five arrangements in terms of a *logic diagram* including a series system (1), a 2-out-of-3 system (3), and a parallel system (5). Note that the 2-out-of-3 system cannot be diagrammed with only three components, so each component is represented twice in the logic diagram. Figure 2.4 displays the corresponding reliabilities, as a function of the component reliability $0 \le p \le 1$ of those five systems. Fundamental properties of coherent systems are discussed in [2.2] and [2.4].

2.4.1 Estimating System and Component Reliability

In many complex systems, the reliability of the system can be computed through the reliability of the components along with the system's structure function. If the exact reliability is too difficult to compute explicitly, reliability bounds might be achievable based on *minimum cut sets* (MCS) and *minimum path sets* (MPS). An MPS is the collection of the smallest component sets that are required to work in order to keep the system working. An MCS is the collection of the smallest component sets that are required to fail in order for the system to fail. Table 2.2 shows the minimum cuts sets and path sets for the three-component systems from Fig. 2.3.

In most industrial systems, components have different roles and varying reliabilities, and often the component reliability depends on the working status of other components. System reliability can be simplified through fault-tree analyses (see Chapt. 7 of [2.16], for example), but uncertainty bounds for system reliability are typically determined through simulation.

In laboratory tests, component reliabilities are determined and the system reliability is computed as a function of the statistical inference of component lifetimes. In field studies, the tables are turned. Component manufacturers seeking reliability data outside laboratory

Table 2.2 Minimum cut sets and path sets for the systems in Fig. 2.3

System	Minimum path sets	Minimum cut sets
1	{A,B,C}	{A}, {B}, {C}
2	{A,B}, {C}	{A,C}, {B,C}
3	{A,B}, {A,C}, {B,C}	{A,B}, {A,C}, {B,C}
4	{A,B}, {A,C}	{A}, {B,C}
5	{A}, {B}, {C}	{A,B,C}

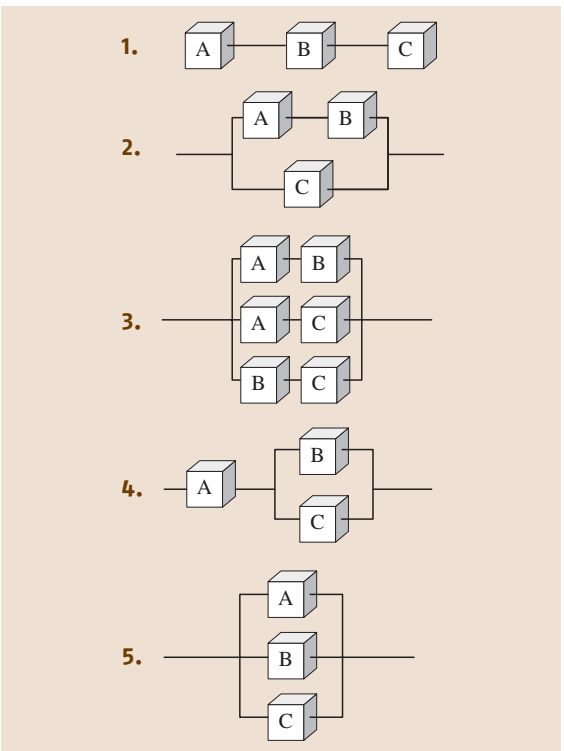

Fig. 2.3 Five unique systems of three components: (1) is series, (3) is 2-out-of-3 and (5) is parallel

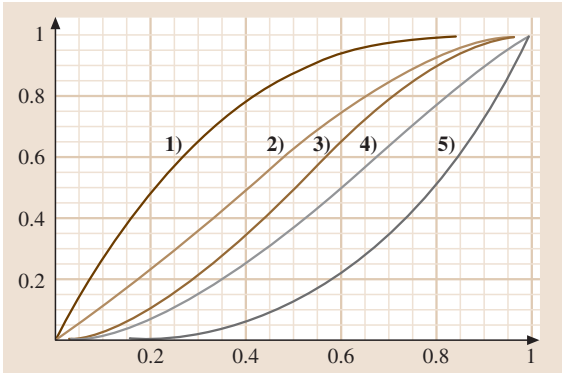

Fig. 2.4 System reliabilities of five system configurations in Fig. 2.3 from the series system (1) to the parallel system (5)

tests look to component lifetime data within a working system. For a k-out-of-n system, for example, the system lifetime represents an order statistic of the underlying distribution function. That is, if the ordered lifetimes form a set of independent and identically distributed components ($X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$), then $X_{n-k+1:n}$ represents the k-out-of-n system lifetime. The density function for $X_{r:n}$ is

$$f_{r:n}(t) = r\binom{n}{r} F(t)^{r-1}\,[1-F(t)]^{n-r}\,f(t), t>0\,. \tag{2.19}$$

Kvam and *Samaniego* [2.32] derived the nonparametric maximum likelihood estimator for $F(t)$ based on a sample of k-out-of-n system data, and showed that the MLE $\hat{F}(t)$ is consistent. If the i-th system ($i = 1, \ldots, m$) observed is a k_i-out-of-n_i system, the likelihood can be represented as

$$L(F) = \prod_{i=1}^{m} f_{k_i:n_i}(t_i) \tag{2.20}$$

and numerical methods are employed to find $\hat{F}$. *Huang* [2.33] investigated the asymptotic properties of this MLE, and *Chen* [2.34] provides an ad hoc estimator that examines the effects of censoring.

Compared to individual component tests, observed system lifetimes can be either advantageous or disadvantageous. With an equal number of k-out-of-n systems at each $1 \le k \le n$, *Takahasi* and *Wakimoto* [2.35] showed that the estimate of MTTF is superior to that of an equal number of individual component tests. With an unbalanced set of system lifetimes, no such guarantee can be made. If only series systems are observed, *Kvam* and *Samaniego* [2.36] show how the uncertainty in $\hat{F}(t)$ is relatively small in the lower quantiles of F (where system failures are observed) but explodes in the upper quantiles.

2.4.2 Stochastic Dependence Between System Components

Almost all basic reliability theory is based on systems with independently operating components. For realistic modeling of complex systems, this assumption is often impractical; system components typically operate at a level related to the quality and operational state of the other system components.

External events that cause the simultaneous failure of component groups is a serious consideration in reliability analysis of power systems. This can be a crucial point in systems that rely on built-in component redundancy to achieve high target system reliability. Shock models, such as those introduced by *Marshall* and *Olkin* [2.37], can be employed to demonstrate how multiple component failures can occur. An extra failure process is added to the otherwise independent component failure processes, representing the simultaneous failure of one or more components, thus making the component lifetimes positively dependent. This is the basis for most dependent failure models in probabilistic risk assessment, including *common cause failure* models used in the nuclear industry (alpha-factor model, beta-factor model, binomial failure rate model). See Chapt. 8 of *Bedford* and *Cooke* [2.16] for discussion about how these models are used in risk assessment.

In dynamic systems, where system configurations and component reliability can change after an external event or a failure of one or more of the system components, the shock model approach cannot be applied effectively. In some applications, a load-share model applies.

Early applications of the load-share system models were investigated by *Daniels* [2.38] for studying the reliability of composite materials in the textile industry. Yarns and cables fail after the last fiber (or wire) in the bundle breaks, thus a bundle of fibers can be considered a parallel system subject to a constant tensile load. An individual fiber fails in time with an individual rate that depends on how the unbroken fibers within the bundle share the load of this stress. Depending on the physical properties of the fiber composite, this load sharing has different meanings in the failure model. Yarn bundles or untwisted cables tend to spread the stress load uniformly after individual failures which defines an *equal*

load-share rule, implying the existence of a constant system load that is distributed equally among the working components.

As expected, a load-sharing structure within a system can increase reliability (if the load distribution saves the system from failing automatically) but reliability inference is hampered even by the simplest model. *Kvam* and *Peña* [2.39] show how the efficiency of the load-share system, as a function of component dependence, varies between that of a series system (equivalent to sharing an infinite load) and a parallel system (equivalent to sharing zero load).

2.4.3 Logistics Systems

Numerous studies have examined fundamental problems in network reliability [2.40], system performance degradation and workload rerouting for telecommunication, power and transportation networks [2.41,42]. In comparison, the literature on modeling logistics system reliability or performance degradation is scarce. Logistics systems that transport goods, energy (e.g., electricity and gas), water, sewage, money or information from origins to destinations are critical to every nation's economic prosperity. Unlike the hub in the typical Internet or telecommunication or network, where the messages are not mixed together, logistics distribution centers (DCs) tend to mix products from various sources for risk-pooling purposes [2.35]. Past studies [2.43] of road-network reliability mainly addressed connectivity and travel-time reliability. These developments have limited use in providing a first-cut analysis for system-level planning that involves robust logistics network design to meet reliability requirements or supply-chain cost and delivery-time evaluation for contract decisions [2.44].

Consider a logistics network consisting of many suppliers providing goods to several DCs, which support store operations to meet customer demands. The reliability of such a network can be evaluated in terms of the probability of delivering goods to stores in a prespecified time limit t_0. Traveling time in transport routes contains uncertainty, as does the processing time for products shipped through DCs. Random traveling time is a function of routing distances, road and traffic conditions and possible delays from seaport or security checkpoint inspections. Traveling distance depends on the configuration of logistics networks. Some retail chains use single-layer DCs, but others use multiple-layer DCs similar to airline hubs (e.g., regional DCs and global DCs) in aggregating various types of goods. Vehicle routing procedures typically involve trucks that carry similar products to several stores in an assigned region. Different products are consolidated in shipment for risk-pooling purposes and to more easily control delivery-time and store-docking operations.

When one DC cannot meet the demands from its regional stores (due to demand increase or the DC's limited capability), other DCs provide backup support to maintain the overall network's service reliability. Focusing on the operations between DCs and stores, *Ni* et al. [2.45] defined the following network reliability as a weighted sum of the individual reliabilities from each DC's operations:

$$r^*_{\text{system},k} = \left[\sum_{i=1, i\neq k}^{M} d_i P\left(T^*_{m,i} < t_0\right) + \sum_{i=1, i\neq k}^{M} p_i d_k P\left(T^*_{m,k,i} < t_0\right) \right] \Big/ \sum_{i=1}^{M} d_i \,, \tag{2.21}$$

where d_i is the demand aggregated at the i-th DC, $T^*_{m,i}$ is the motion time defined as the sum of traveling time from DC_i to its assigned stores (including material processing time at DC_i), p_i is the proportion of products rerouted from DC_k through DC_i due to the limited capability in DC_k and $T^*_{m,k,i}$ is the modified motion time including the rerouted traveling time.

For modeling the aggregated demand d_i and calculating routing distance, *Ni* et al. [2.45] proposed a multiscale approximation model to quantify demand patterns at spatially located clustered stores. Then, they evaluated product rerouting strategies for maintaining system service reliability, defined in (2.19). Based on the store locations of a major retail chain, several examples show the importance of designing a robust logistics network to limit service reliability degradation when a *contingency* (e.g., multiple DC failure) occurs in the network. Future work includes:

1. Modeling the low-probability but high-impact contingency in the DCs [2.45] and routes for calculating their relative importance to network reliability
2. Examining the tradeoff between the cost of adding more DCs and the improvement of service reliability
3. Resolving the *domino effect* when the added workload to DCs after a local DC failure causes further DC failures due to faulty predetermined rules of rerouting to maintain system reliability (e.g., the 2003 electricity blackout in the Northeastern region of the US).

2.4.4 Robust Reliability Design in the Supply Chain

Past studies of robust parameter design [2.46] focused on product quality issues and assumed that all the controllable variables are under single ownership. Recent outsourcing trends in automobile and electronic manufacturing processes motivate the presentation in this section. In an automobile manufacturing enterprize system, various parts suppliers have control of variables determining quality and reliability. Most of the automobile supply-chain systems assemble these parts into a subsystem and then move these systems to other locations owned by different partners for the final system-level assembly and testing. Every segment of the assembly operation controls a subset of variables leading to different levels of system reliability. Because the warranty policy is addressed to all of the part manufacturing and assembly processes in making the final product, it is important to extend the robust parameter design concept to the supply-chain-oriented manufacturing processes.

Supply-chain partners have their own operation objectives (e.g., maximize the profit of manufacturing parts to supply several automobile companies). Some of the objectives are aligned to manufacturing a specific type of product, but there are many potential situations with conflicting objectives. When there is no single ownership of all controllable variables in the manufacturing processes, negotiation is needed to resolve potential conflicts. Game theory [2.47] is commonly used in supply-chain contract decisions. Following the framework of *Chen* and *Lewis* [2.48], we can decompose the set of controllable variables into a few subsets owned by distinct partners and formulate the objectives of these partners. The supply-chain manager can define the product quality and reliability measures and build models to link them to the controllable and uncontrollable variables that are seen in robust parameter design.

Different negotiation situations (e.g., the final product assembly company has more bargaining power than other partners) will lead to distinct levels selected for the controllable variables (see *Charoensiriwath* and *Lu* [2.49] for examples in negotiations). As a result, the reliability of the final product can vary. Designing a supply-chain system that leads to the most reliable products (with minimum cost) presents an acute challenge, and warranty policies can be designed correspondingly. Because parts and subsystems are made by various partners, warranty responsibilities for certain parts are distributed among partners under the negotiated supply-chain contracts.

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