3. Weibull Distributions and Their Applications

Weibull models are used to describe various types of observed failures of components and phenomena. They are widely used in reliability and survival analysis. In addition to the traditional two-parameter and three-parameter Weibull distributions in the reliability or statistics literature, many other Weibull-related distributions are available. The purpose of this chapter is to give a brief introduction to those models, with the emphasis on models that have the potential for further applications. After introducing the traditional Weibull distribution, some historical development and basic properties are presented. We also discuss estimation problems and hypothesis-testing issues, with the emphasis on graphical methods. Many extensions and generalizations of the basic Weibull distributions are then summarized. Various applications in the reliability context and some Weibull analysis software are also provided.

The Weibull distribution is one of the best-known lifetime distributions. It adequately describes observed failures of many different types of components and phenomena. Over the last three decades, numerous articles have been written on this distribution. *Hallinan* [3.1] gives an insightful review by presenting a number of historical facts, the many forms of this distribution as used by practitioners, and possible confusions and errors that arise due to this non-uniqueness. *Johnson* et al. [3.2] devote a comprehensive chapter to a systematic study of this distribution. More recently, *Murthy* et al. [3.3] presented a monograph that contains nearly every facet relating to the Weibull distribution and its extensions.

In Sect. 3.1, we first define the three-parameter Weibull distribution and then look at its historical development and relations to other distributions. Section 3.2 studies the properties of the Weibull distribution, in particular those relevant to reliability. A brief discussion on simulation of Weibull variates is also included. We consider estimation problems and hypothesis testing in Sect. 3.3. In particular, we emphasize the graphical methods as a tool for selection and parameter estimation of a Weibull model. We devote Sect. 3.4 to Weibull-derived models, which includes many extensions and generalizations. Finally, in Sect. 3.5, we outline various applications, especially those in the reliability context. Because of the vast literature, we are unable to refer to all the source authors and we apologize in advance for any omissions in this regard.

Symbols and Abbreviations.

T	Random variable
$F(t)$	CDF, cumulative distribution function
$f(t)$	PDF, probability density function
$h(t)$	Failure rate function (hazard rate)
α, β, τ	Parameters
WPP	Weibull probability plot

3.1 Three-Parameter Weibull Distribution

According to *Hallinan* [3.1] the Weibull distribution has appeared in five different forms. The two common forms of the distribution function are as follows:

$$F(t) = 1 - \exp\left[-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right], \quad t \geq \tau \tag{3.1}$$

and

$$F(t) = 1 - \exp\left[\lambda(t-\tau)^{\beta}\right], \quad t \geq \tau\,. \tag{3.2}$$

The parameters of the distribution are given by the set $\boldsymbol{\theta} = \{\alpha, \beta, \tau\}$ with $\alpha > 0$, $\beta > 0$ and $\tau \geq 0$; where α is a scale parameter, β is the shape parameter that determines the appearance or shape of the distribution and τ is the location parameter. The parameter λ combines both scale and shape features as $\lambda = \alpha^{-\beta}$.

Although one should use $F(t, \boldsymbol{\theta})$ instead of $F(t)$, where $\boldsymbol{\theta} = (\alpha, \beta, \tau)$ denotes the vector of parameters, for notational convenience we suppress the parameter and use $F(t)$ to denote $F(t, \boldsymbol{\theta})$ in this chapter. Also, we do not intend to give an exhaustive review. Rather, we confine our discussion to aspects that are of relevance to the context of reliability theory.

For $\tau = 0$, (3.1) and (3.2) become the two-parameter Weibull distribution with

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\alpha}\right)^{\beta}\right], t \geq 0 \tag{3.3}$$

and

$$F(t) = 1 - \exp\left(-\lambda t^{\beta}\right), \quad t \geq \tau\,. \tag{3.4}$$

Murthy et al. [3.3] refer to this as the *standard* Weibull model, *Johnson* et al. [3.2] refer to a standard Weibull when $\alpha = 1$ (or $\lambda = 1$) in (3.3), (3.4).

3.1.1 Historical Development

The Weibull distribution is named after its originator, the Swedish physicist Waloddi Weibull, who in 1939 used it to model the distribution of the breaking strength of materials [3.4] and in 1951 for a wide range of other applications [3.5]. The distribution has been widely studied since its inception. It has been known that Weibull may not be the first to propose this distribution. The name Fréchet distribution is also sometimes used due to the fact that it was *Fréchet* [3.6] who first identified this distribution to be an extremal distribution (later shown to be one of the three possible solutions by *Fisher* and *Tippett* [3.7]). According to *Hallinan* [3.4], it was Weibull who suggested a scale parameter and a location parameter that made the distribution meaningful and useful.

3.1.2 Relations to Other Distributions

The Weibull distribution includes the exponential ($\beta = 1$) and the Rayleigh distribution ($\beta = 2$) as special cases. If X denotes the Weibull variable, then $-X$ has a type 3 extreme-value distribution [3.8, Chapt. 22]. A simple log transformation will transform the Weibull distribution into the Gumbel distribution (type 1 extreme-value distribution).

The Burr XII distribution, is given by

$$F(t) = 1 - \left[1 + \left(\frac{t}{\alpha}\right)\right]^{-k}, t \geq 0\,; k\,, \alpha > 0\,. \tag{3.5}$$

Let $k = \alpha$; as $k \to \infty$, then the Burr distribution approaches the Weibull (see, for example, [3.9]).

3.2 Properties

3.2.1 Basic Properties

Density Function

The probability density functions (PDF) (Fig. 3.1) of (3.1) and (3.2) are

$$f(t) = \beta\alpha^{-\beta}(t-\tau)^{\beta-1}\exp\left[-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right], \quad t \geq \tau \tag{3.6}$$

and

$$f(t) = \beta\lambda(t-\tau)^{\beta-1}\exp\left[-\lambda(t-\tau)^{\beta}\right], \quad t \geq \tau\,. \tag{3.7}$$

Mode

It follows from (3.6) that the mode is at $t = \alpha((\beta-1)/\beta)^{1/\beta} + \tau$ for $\beta > 1$ and at τ for $0 < \beta \leq 1$.

Median

It follows from (3.3) that the median of the distribution is at $\alpha(\log 2)^{1/\beta} + \tau$.

Moments

Let T denote the random variable from the three-parameter Weibull distribution given by (3.1). Then the transformed variable $T' = (T-\tau)/\alpha$, is the standard form (in the sense of *Johnson* et al. [3.2]) with the density function given by

$$f(t) = \beta t^{\beta-1} \exp\left(-t^{\beta}\right) \ , x > 0 \ , \beta > 0 \ . \qquad (3.8)$$

The moments of T (about zero) are easily obtained from the moments of T' which are given below:

$$\mu_r' = E(T'^r) = \Gamma\left(\frac{r}{\beta}+1\right) \ , \qquad (3.9)$$

from which we get

$$E(T') = \Gamma\left(\frac{1}{\beta}+1\right) \ , \qquad (3.10)$$

$$\mathrm{Var}(T) = \Gamma\left(\frac{2}{\beta}+1\right) - \left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2 \ . \qquad (3.11)$$

Skewness and Kurtosis

The distribution is positively skewed for small values of β. The skewness index $\sqrt{\beta_1}$ decreases and equals zero for $\beta = 3.6$ (approximately). Thus, for values of β in the vicinity of 3.6, the Weibull distribution is similar in shape to a normal distribution. The coefficient of kurtosis β_2 also decreases with β and then increases, β_2 has a minimum value of about 2.71 when $\beta = 3.35$ (approximately).

Order Statistics

Let $T_1, T_2, \cdots, T_n$ denote n independent and identically distributed three-parameter Weibull random variables with density function given in (3.6) and cumulative distribution function (CDF) in (3.1). Further, let $T_{(1)} \le T_{(2)} \le \cdots \le T_{(n)}$ denote the order statistics from a sample of n observations. The probability density function of $T_{(1)}$, is given by

$$f_{(1)}(t) = n[1-F(t)]^{n-1} f(x) = \frac{n\beta}{\alpha}\left(\frac{t-\tau}{\alpha}\right)^{\beta-1} \mathrm{e}^{-n[(t-\tau)/\alpha]^{\beta}} , t \ge \tau \ . \qquad (3.12)$$

It is obvious from (3.12) that $T_{(1)}$ is also distributed as a Weibull random variable, except that α is replaced by $\alpha n^{-1/\beta}$.

The density function of $T_{(r)} (1 \le r \le n)$ is given by

$$f_{(r)}(t) = \frac{n!}{(r-1)!(n-r)!}\left(1-\mathrm{e}^{-[(t-\tau)/\alpha]^{\beta}}\right)^{r-1} \mathrm{e}^{-[(t-\tau)/\alpha]^{\beta}(n-r+1)} \beta\alpha^{-\beta}(t-\tau)^{\beta-1}, \ t \ge 0 \ . \qquad (3.13)$$

It can be shown that

$$E\left[(T_{(r)})^k\right] = \sum_{i=0}^{k} \tau^i \alpha^{k-i} \omega_{(r)}^{k-i} \ , \qquad (3.14)$$

where

$$\omega_{(r)}^k = \frac{n!}{(r-1)!(n-r)!} \Gamma\left(1+\frac{k}{\beta}\right) \times \sum_{i=0}^{r-1} \frac{(-1)^r \binom{r-1}{i}}{(n-r+i+1)^{1+(k/\beta)}} \ .$$

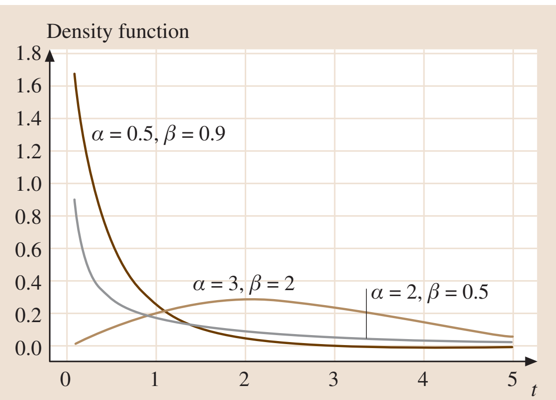

Fig. 3.1 Two-parameter Weibull density plots

3.2.2 Properties Related to Reliability

In this section, we consider only the first form of the Weibull distribution.

The survival function of the Weibull distribution is

$$\bar{F}(t) = 1 - F(t) = \exp\left[-\left(\frac{t-\tau}{\alpha}\right)^{\beta}\right], \ t \ge \tau \ . \qquad (3.15)$$

Note that at $t = \tau + \alpha$, $\bar{F}(\tau+\alpha) = 1 - \mathrm{e}^{-1} \approx 0.3679$.

Failure Rate

The failure rate function (also known as the hazard rate) for the three-parameter Weibull is

$$h(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\beta}{\alpha}\left(\frac{t-\tau}{\alpha}\right)^{\beta-1} \ . \qquad (3.16)$$

For the two-parameter case, it is given by

$$h(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1} \ . \qquad (3.17)$$

It is obvious that $h(t)$ is a decreasing function when $\beta < 1$, constant when $\beta = 1$ (the exponential case), and an increasing function when $\beta > 1$. Because of the behaviour of the failure rate function, the Weibull distribution often becomes suitable when the conditions for *strict randomness* of the exponential distribution are not satisfied, with the shape parameter β having a value depending upon the fundamental nature being considered.

In some way, having a failure rate function of monotonic shape has limitations in reliability applications. For this reason, several generalized or modified Weibull distributions have been proposed (see Sect. 3.4 for more details). Figure 3.2 shows plots of the failure rate functions for some selected parameters values.

Mean Residual Life

The mean residual life (MRL) of a lifetime random variable T is defined as

$$\mu(t) = E(T - t | T > t) = \frac{\int_t^\infty \bar{F}(x)\,\mathrm{d}x}{\bar{F}(t)} . \tag{3.18}$$

For the Weibull distribution, the MRL is complicated except for the two special cases $\beta = 1$ (exponential) and $\beta = 2$ (Rayleigh distribution). Assuming the location parameter $\tau = 0$, the MRL of the Rayleigh distribution with scale parameter $\alpha = \sqrt{2}\sigma$ is

$$\mu(t) = \sqrt{2\pi}\sigma\,[1 - \Phi(t/\sigma)]\,\mathrm{e}^{\frac{1}{2\sigma^2}t^2} , \quad t > 0 , \tag{3.19}$$

where $\Phi(t/\sigma)$ denotes the distribution function of the standard normal variable.

Relative Ageing of Two Two-Parameter Weibull Distributions

Suppose we have two Weibull random variables X and Y with distribution functions $F(x)$ and $G(y)$, respectively, given by

$$F(x) = 1 - \exp\left[-(x/\alpha_2)^{\beta_2}\right] ,$$
$$G(y) = 1 - \exp\left[-(y/\alpha_1)^{\beta_1}\right] . \tag{3.20}$$

We say that X ages faster than Y if the ratio of the failure rate of X over the failure rate of Y is an increasing function of t. This ratio is given by

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} \times \frac{\alpha_1^{\beta_2-1}}{\alpha_2^{\beta_1-1}} t^{\beta_2-\beta_1} \tag{3.21}$$

which is an increasing function of t if $\beta_2 > \beta_1$.

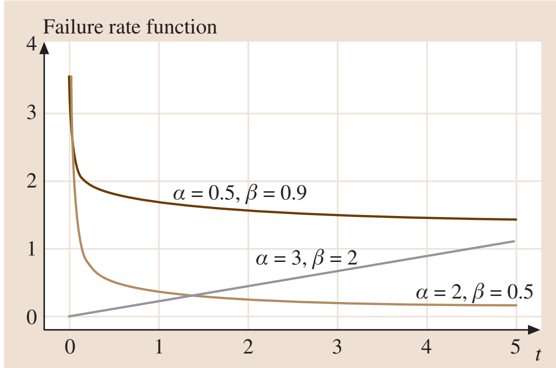

Fig. 3.2 Two-parameter Weibull failure rate functions

Suppose $E(X) = E(Y)$, i.e., $\alpha_2\Gamma(1+1/\beta_2) = \alpha_1\Gamma(1+1/\beta_1)$. *Lai* and *Xie* [3.10] show that $\mathrm{Var}(X) \le \mathrm{Var}(Y)$.

Effectiveness of Parallel Redundancy – The Weibull Case

Denote the lifetime of one component by T and the lifetime of a parallel system of two such independent components by T_p. The effectiveness of parallel redundancy of the component is defined by [3.11]

$$e_\mathrm{p} = [E(T_\mathrm{p}) - E(T)]/E(T) . \tag{3.22}$$

Suppose that T is a two-parameter Weibull distribution with shape parameter β and scale parameter α. *Xie* and *Lai* [3.11] show that

$$e_\mathrm{p} = 1 - 2^{-1/\beta} , \tag{3.23}$$

which decreases as β increases. Thus, a parallel redundancy is more effective for $\beta < 1$.

3.2.3 Simulation

The two-parameter Weibull distribution with parameters β and λ has the probability density function

$$f(x) = \lambda\beta x^{\beta-1}\,\mathrm{e}^{-\lambda x^\beta} . \tag{3.24}$$

The simple inverse-probability integral-transform method applied to the standard Weibull distribution ($\lambda = 1$) is quite efficient. The formula is simply

$$x = (-\log u)^{1/\beta} ,$$

where u is a uniform number between 0 and 1. Of course, an acceptance/rejection method could also be used to avoid the evaluation of the logarithm in the distribution function. The standard Weibull variate is then scaled by $\lambda^{-(1/\beta)}$ to obtain a two-parameter Weibull observation with density (3.24).

3.3 Modeling Failure Data

Failure data can be classified into two types: complete and incomplete (censored). For complete data, the actual realized values are known for each observation in the data set, whereas for censored data, the actual realized values are not known for some or all of the observations. Further, there are at least two types of censoring, details of which may be found *Lawless* [3.12] and *Nelson* [3.13].

In their introduction, *Murthy* et al. [3.3] point out that the two different approaches to building mathematical models are as follows.

1. *Theory-based modeling*: here, the modeling is based on the established theories for component failures. This kind of model is also called a *physics based model* or *white-box model*.
2. *Empirical modeling*: here, the data available forms the basis for the model building. This kind of model is also called a *data-dependent model* or *black-box model*.

In empirical modeling, the type of mathematical formulations needed for modeling is dictated by a preliminary analysis of data available. If the analysis indicates that there is a high degree of variability, then one needs to use models that can capture this variability. This requires probabilistic and stochastic models to model a given data set.

The process of black box modeling involves the following steps:

- Step 1 – Model selection,
- Step 2 – Parameter estimation,
- Step 3 – Model validation.

To select a model out of many possible models, one requires a good understanding of their different properties. This often is a trial-and-error process. *Liao* and *Shimokawa* [3.14] consider goodness-of-fit testing as a key procedure for selecting the statistical distribution that best fits the observed data. To execute the remaining two steps, we need various tools and techniques.

A large number of Weibull-related models have been derived in the literature. In Sect. 3.4, various such models are grouped into several categories. Selecting an appropriate model from the family of Weibull-related distributions can often be based on some of the probability plots to be discussed below. Moreover, they can provide crude estimates of model parameters. There are many different statistical tests for validating a model and we postpone this discussion until the next section.

3.3.1 Probability Plots

Weibull Probability Plot

Weibull probability plots (WPP) can be constructed in several ways [3.15]. In the early 1970s a special paper was developed for plotting the data in the form of $F(t)$ versus t on a graph paper with a log–log scale on the vertical axis and a log scale on the horizontal axis. A WPP plotting of data involves computing the empirical distribution function, which can be estimated in different ways; the two standard methods are:

- $\hat{F}(t_i) = i/(n+1)$, the "mean rank" estimator, and
- $\hat{F}(t_i) = (i-0.5)/n$, the "median rank" estimator.

Here, the data consists of successive failure times t_i, $t_1 < t_2 < \ldots t_n$. For censored data (right-censored or interval), the approach to obtain the empirical distribution functions needs to be modified; see, for example, *Nelson* [3.13].

These days, most computer reliability software packages contain programs to produce these plots automatically given a data set. A well-known statistical package MINITAB provides a Weibull probability plot under the Graph menu ≫ Probability Plot.

We may use an ordinary graph paper or spreadsheet software with unit scale for plotting. Taking logarithms twice of both sides of each of the CDF in (3.3) yields

$$\log[-\log \bar{F}(t)] = \beta \log(t-\tau) - \beta \log \alpha \,. \qquad (3.25)$$

Let $y = \log\left[-\log \bar{F}(t)\right]$ and $x = \log(t-\tau)$. Then we have

$$y = \beta x - \beta \log \alpha \,. \qquad (3.26)$$

The plot is now on a linear scale.

Weibull Hazard Plot

The hazard plot is analogous to the probability plot, the principal difference being that the observations are plotted against the cumulated hazard (failure) rate rather than the cumulated probability value. Moreover, this is designed for censored data.

Let $H(t)$ denote the cumulative hazard rate, then $\bar{F}(t) = \exp[-H(t)]$, so

$$H(t) = -\log \bar{F}(t) = \left(\frac{t-\tau}{\alpha}\right)^{\beta}, \tag{3.27}$$

$$H(t)^{1/\beta}/\alpha = (t-\tau). \tag{3.28}$$

Let $y = \log(t-\tau)$ and $x = \log H(t)$, then we have

$$y = \log\alpha + \frac{1}{\beta}x. \tag{3.29}$$

Rank the n survival times (including the censoreddata) in ascending order and let K denote the reverse ranking order of the survival time, i. e., $K = n$ for the smallest survival time and $K = 1$ for the largest survival time. The hazard is estimated from $100/K$ (a missing value symbol is entered at a censored failure time). The cumulative hazard is obtained by cumulating the hazards. The Weibull hazard plot is simply the plot arising from (3.29). See *Nelson* [3.16] for further details.

3.3.2 Estimation and Hypothesis Testing

Parameter estimation is the second step of our modeling process. We consider both graphical and statistical methods for the three-parameter Weibull distribution.

Graphical Methods

Graphical plots are designed not just to assess if the data follows a Weibull population, they are also useful for estimating Weibull parameters (at least for initial estimates). These can be obtained from the smooth fit to a WPP plot of data and involve exploiting properties such as asymptotes, intersection points, slope, or points of inflection.

Estimation of Location Parameter. The determination of a suitable location parameter τ is not a simple task. If a data set graphs as a straight line on a WPP, then the data set is indeed adequately described by a Weibull distribution. However, an incorrect selection of τ will yield a curved plot of data that is in fact Weibull-distributed.

If the graph of a set of data appears concave upward, then the plot can be straightened by decreasing the value of τ. Conversely, if the data are concave downward, then the plot can be straightened by increasing the value of τ. The value of τ cannot be chosen larger than the smallest failure-time value, say $T_{(1)}$, in the sample. It is generally suggested to set the initial value of $\tau = T_{(1)}$ and then adjust this estimate to achieve a straight line. We also note that it is common for τ to be 0.

Estimation of the Shape Parameter. It follows from (3.26) that the shape parameter can be estimated from the slope of the WPP. However, care needs need to be given because possible confusion may arise due to several possible forms of the Weibull distribution.

Estimation of the Scale Parameter. To estimate the scale parameter, we first observe from the standard Weibull that, when t is at the 63.2 percentile, $F(t) = 1 - e^{-1}$, i. e., $\bar{F}(t) = e^{-1}$. Or equivalently, for $t = \tau + \alpha$, $\bar{F}(\tau+\alpha) = e^{-1}$ (so $\tau + \alpha$ is the 63.2 percentile). It now follows from (3.26) that $\log(t-\tau) = \log\alpha$ so that $\hat{\alpha} = t - \hat{\tau}$, where t is the 63.2 percentile. Hence the scale parameter is simply estimated from the x-intercept of the plot (3.26): $\hat{\alpha} = e^{(x\text{-intercept})}$.

Under the probability plot option, MINITAB not only gives a WPP, but also provides an estimate for the scale and shape parameters α and β.

Figure 3.3 illustrates how a WPP plot appears in a MINITAB graph with $\hat{\alpha} = 9.062$ and $\hat{\beta} = 1.661$.

Statistical Methods of Estimation

There are several statistical methods for estimating the model parameters. These include the method of moment, the method of percentile, the method of maximum likelihood, and the Bayesian method. The estimates can be

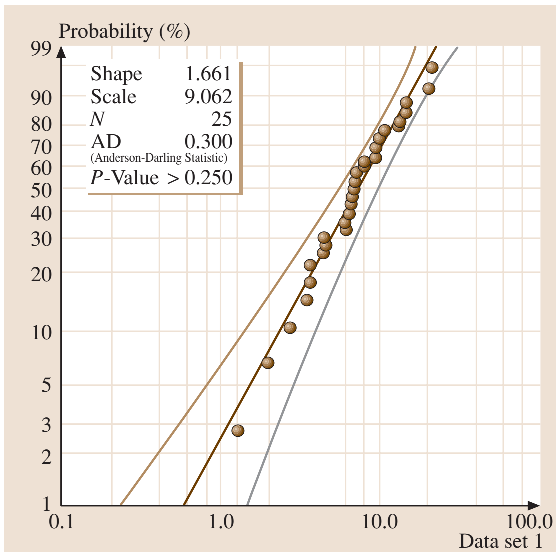

Fig. 3.3 A typical Weibull probability plot

either point estimates or interval estimates. The details of these methods can be found, for example, in *Kalbfleisch* and *Prentice* [3.17].

We note that these methods may not be appropriate for small data sets. Hence, the estimates obtained from the WPP plot may be taken as the final estimates when the data size is small.

Moment Estimation. By equating the first three moments of the Weibull distribution to the first three sample moments and solving, it is possible to find the moment estimators of α, β and τ. Now, the first moment ratio $\sqrt{\beta_1}$ depends only on the shape parameter β, and hence once $\sqrt{\beta_1}$ has been estimated from the sample coefficient of skewness, $\hat{\beta}$ can be determined numerically. Using $\hat{\beta}$, $\hat{\alpha}$ can be determined from the standard deviation, and lastly $\hat{\tau}$ is determined from the sample mean. If τ is known, then β can be estimated from the ratio (standard deviation/(mean-τ).

For modified moment estimation, see *Cohen* et al. [3.18].

Maximum-Likelihood Method. Maximum-likelihood methods for the three-parameter Weibull distribution have been reviewed by *Zanakis* and *Kyparisis* [3.19]; see also *Johnson* et al. [3.2, Chapt. 21] for other details.

The most common situation is when the location parameter τ is known. Without loss of generality, we may assume that $\tau = 0$ so that the model becomes the two-parameter Weibull distribution.

The maximum-likelihood estimators, $\hat{\beta}$ and $\hat{\alpha}$ of β and α, respectively, satisfy the equations

$$\hat{\alpha} = \left(\frac{1}{n} \sum_{i=1}^{n} X_i^{\hat{\beta}} \right)^{1/\hat{\beta}} \tag{3.30}$$

and

$$\hat{\beta} = \left[\left(\sum_{i=1}^{n} X_i^{\hat{\beta}} \log X_i \right) \left(\sum_{i=1}^{n} X_i^{\hat{\beta}} \right)^{-1} - \frac{1}{n} \sum_{i=1}^{n} \log X_i \right]^{-1} , \tag{3.31}$$

where X_i, $i = 1, 2, \ldots n$ are n independent observations from the two-parameter Weibull distribution. If $\tau \neq 0$, then each X_i is replaced by $X_i - \tau$ in the above equations. The value $\hat{\beta}$ needs to be solved from (3.31) and then (3.30) is used to obtain $\hat{\alpha}$.

Suppose the location parameter τ is also unknown, then the maximum-likelihood estimates $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\tau}$ satisfy the following equations

$$\hat{\alpha} = \left[\frac{1}{n} \sum_{i=1}^{n} (X_i - \hat{\tau})^{\hat{\beta}} \right]^{1/\hat{\beta}} \tag{3.32}$$

$$\hat{\beta} = \left\{ \left[\sum_{i=1}^{n} (X_i - \hat{\tau})^{\hat{\beta}} \log(X_i - \hat{\tau}) \right] \times \left[\sum_{i=1}^{n} (X - \hat{\tau})^{\hat{\beta}} \right]^{-1} - \frac{1}{n} \sum_{i=1}^{n} \log(X_i - \hat{\tau}) \right\}^{-1} \tag{3.33}$$

and

$$(\hat{\beta} - 1) \sum_{i=1}^{n} (X_i - \hat{\tau})^{-1} = \hat{\beta} \hat{\alpha}^{-\hat{\beta}} \sum_{i=1}^{n} (X_i - \hat{\tau})^{\hat{\beta}-1} . \tag{3.34}$$

If the value $\hat{\tau}$ satisfying (3.32)–(3.34) is larger than $X_1 = \min X_i$, then it is the maximum-likelihood estimate of τ. Otherwise, we set that $\hat{\tau} = X_1$ and then (3.32) and (3.33) must be solved for $\hat{\alpha}$ and $\hat{\beta}$.

The method is suitable for large data sets because of its asymptotic properties. Modifications of these estimates can be found, for example, in *Johnson* et al. [3.2].

The reader may consult Chapt. 21 of *Johnson* et al. [3.2] and Chapt. 4 of *Murthy* et al. [3.3] for the following methods of estimation,

- Best linear unbiased estimation,
- Percentile estimator,
- Bayesian estimator,
- Interval estimation,
- Minimum quantile distance estimation.

For the two-parameter Weibull distribution (i. e., $\tau = 0$), methods of inference can be obtained via the type 1 extreme-value distribution.

3.3.3 Hypothesis Testing

Goodness-of-Fit Tests for the Weibull Distribution

There are other general goodness-of-fit tests based on the empirical distribution functions derived. For example, the chi-square test, Kolmogorov–Smirnov test, Cramer–von Mises test and Anderson–Darling test, and so on, could also be used to test the goodness of fit of the Weibull model.

Generally, a goodness-of-fit test for Weibull distribution can be described as: H_0: the population follows a Weibull model, versus H_1: the Weibull model is not suitable.

MINITAB under probability plot gives the Anderson–Darling statistic as well as the confidence bands.

Lawless [3.12] summarizes the goodness-of-fit tests for the Weibull or extreme-value distribution. Four tests (the likelihood ratio test as a sub-model of the gamma distribution, Mann–Scheuer–Fertig test, Tiku test and Cramer–von Mises test) are discussed in detail. *Dodson* [3.20] also includes and compares some goodness-of-fit tests for Weibull distributions in Chapt. 4 of his book. *Liao* and *Shimokawa* [3.14] construct a new goodness-of-fit test for the two-parameter Weibull and its power is compared with other traditional goodness-of-fit tests.

Hypothesis Testing

Hypothesis testing of Weibull parameters may be performed by the likelihood ratio, score and Wald tests. It is well known that all these tests are asymptotically optimal.

For testing Weibull versus the exponential distribution, we test $\alpha = 1$ (or $\lambda = 1$ of the second form) versus $\alpha \neq 0$.

3.4 Weibull-Derived Models

There are many extensions, generalizations and modifications to the Weibull distribution. They arise out of the need to model features of empirical data sets that cannot be adequately described by a three-parameter Weibull model: for example, the monotonic property of the Weibull, which is unable to capture the behaviour of a data set that has a bathtub-shape failure rate. *Xie* et al. [3.21] review several Weibull-related distributions that exhibit bathtub-shaped failure rates. Plots of mean residual life from several of these Weibull-derived models are given in *Lai* et al. [3.22]. For simplicity, we simply refer to these Weibull-related models as Weibull models.

3.4.1 Taxonomy for Weibull Models

According to *Murthy* et al. [3.3], the taxonomy for Weibull models involves seven types, each of which divided into several sub-types. These models can be grouped into three groups:

1. Univariate models (types I–V),
2. Multivariate models (type VI),
3. Stochastic process models (type VII).

In this chapter we confine our discussion to univariate and stochastic process models, while *Murthy* et al. [3.23] looks at bivariate models (a special case of multivariate models). In this chapter, only some selective models will be included. We refer our readers to the book by *Murthy* et al. [3.3] for more details.

3.4.2 Univariate Models

The starting point is the two-parameter Weibull model with distribution function $F(t)$. Let $G(t)$ denote the derived Weibull model. T is a random variable from $F(t)$ and Z is a random variable from $G(t)$.

Type I Models (Transformation of Weibull Variable)

Here Z and T are related by a transformation. The transformation can be either linear or nonlinear. These include the following:

Reflected Weibull Distribution.

$$G(t) = \exp\left[-\left(\frac{\tau - t}{\alpha}\right)^{\beta}\right], -\infty < t < \tau . \quad (3.35)$$

This is also known as a type 3 extreme-value distribution (see *Johnson* et al. [3.8], Chapt. 22).

Double Weibull Distribution.

$$g(t) = \beta(1/2)\,|t|^{(\beta-1)} \exp(-|t|^{\beta}), -\infty < t < \infty . \quad (3.36)$$

This is an obvious extension to include the negative real line as its support.

Log Weibull Distribution. The distribution is derived from the logarithmic transformation of the two-parameter Weibull distribution. The distribution function is

$$G(t) = 1 - \exp\left[-\exp\left(\frac{t-\tau}{\alpha}\right)\right], -\infty < t < \infty . \quad (3.37)$$

This is also known as a type 1 extreme-value distribution or Gumbel distribution. In fact, it is the most commonly referred to in discussions of extreme-value distributions (see [3.8], Chapt. 22). The density function is

$$g(t)=\frac{1}{\alpha}\exp\left(\frac{t-\tau}{\alpha}\right)\exp\left[-\exp\left(\frac{t-\tau}{\alpha}\right)\right],$$

and the failure rate function is quite simple and given by

$$h(t)=\frac{1}{\alpha}\exp\left(\frac{t-\tau}{\alpha}\right).$$

Inverse (or Reverse) Weibull Model. Let $Z=\alpha^2/T$, where T has a two-parameter Weibull distribution, then the distribution function of T is

$$G(t)=\exp[-(t/\alpha)^{-\beta}]\,,\ t\geq 0\,. \tag{3.38}$$

This is also known as a type 2 extreme-value distribution or Fréchet distribution (see [3.8], Chapt. 22). The failure rate function is as follows:

$$h(t)=\frac{\beta\alpha^{\beta}t^{-\beta-1}\,\mathrm{e}^{-(\alpha/t)^{\beta}}}{1-\mathrm{e}^{-(\alpha/t)^{\beta}}}\,.$$

Type II Models (Modification/Generalization of the Weibull Distribution)

Here $G(t)$ is related to $F(t)$ by some relationship.

Extended Weibull Distribution [3.24].

$$G(t)=1-\frac{\nu\,\mathrm{e}^{-(\lambda t)^{\beta}}}{1-(1-\nu)\mathrm{e}^{-(\lambda t)^{\beta}}}\,. \tag{3.39}$$

The failure rate function is

$$h(t)=\frac{\nu\beta(\lambda t)^{\beta-1}}{\left[1-(1-\nu)\mathrm{e}^{-(\lambda t)^{\beta}}\right]}\,.$$

The failure rate function is increasing if $\nu\geq 1$, $\beta\geq 1$ and decreasing if $\nu\leq 1$, $\beta\leq 1$.

Modified Weibull Distribution [3.25].

$$G(t)=1-\exp(-\lambda t^{\beta}\,\mathrm{e}^{\nu t})\,,\ t\geq 0 \tag{3.40}$$

with

$$h(t)=\lambda(\beta+\nu t)t^{\beta-1}\exp(\nu t)\,. \tag{3.41}$$

For $\nu=0$, this reduces to a Weibull distribution. For $0<\beta<1$, $h(t)$ is initially decreasing and then increasing in t, implying that the failure rate function has a bathtub shape. When $\beta>1$, $h(t)$ is increasing in t.

Exponentiated Weibull Distribution [3.26].

$$G(t)=[F(t)]^{\nu}=\{1-\exp[-(t/\alpha)^{\beta}]\}^{\nu}\,,\ t\geq 0\,. \tag{3.42}$$

The density function is

$$g(t)=\frac{\beta\nu}{\alpha^{\beta}}t^{\beta-1}\,\mathrm{e}^{-(t/\alpha)^{\beta}}\left\{1-\mathrm{e}^{-(t/\alpha)^{\beta}}\right\}^{\nu-1}.$$

For an appropriate choice of the parameter set, it will give rise to a bathtub-shaped failure function. *Jiang* and *Murthy* [3.27] use a graphical approach to study this distribution. The failure rate function is given by

$$h(t)=\frac{\beta\nu}{\alpha^{\beta}}t^{\beta-1}\,\mathrm{e}^{-(t/\alpha)^{\beta}}\frac{\left(1-\mathrm{e}^{-(t/\alpha)^{\beta}}\right)^{\nu-1}}{\left[1-(1-\mathrm{e}^{-(t/\alpha)^{\beta}})^{\nu}\right]}\,.$$

Four-Parameter Weibull Distribution [3.28].

$$G(t)=1-\exp\left[-\lambda\left(\frac{t-a}{b-t}\right)^{\beta}\right],\quad 0\leq a\leq t\leq b<\infty\,. \tag{3.43}$$

Doubly Truncated Weibull Distribution.

$$G(t)=\frac{F(t)-F(a)}{F(b)-F(a)}\,,\quad 0<a\leq t\leq b<\infty\,. \tag{3.44}$$

Modified Weibull Extension [3.29].

$$G(t)=1-\exp\left[-\lambda\alpha\left(\mathrm{e}^{(t/\alpha)^{\beta}}-1\right)\right],\quad t\geq 0,\,\alpha,\,\beta,\,\lambda>0\,. \tag{3.45}$$

This is known as the generalized exponential power model originally studied by *Smith* and *Bain* [3.30]. The case of $\alpha=1$ is also studied in *Chen* [3.31]. The failure rate function is

$$h(t)=\lambda\beta(t/\alpha)^{\beta-1}\exp\left[(t/\alpha)^{\beta}\right]. \tag{3.46}$$

It approaches to a two-parameter Weibull distribution when $\lambda\to\infty$ with α in such a manner that $\alpha^{\beta-1}/\lambda$ is held constant.

Type III Models (Models Involving Two or More Distributions)

These are univariate models derived from two or more distributions with at least one being either the standard Weibull model or a distribution derived from it.

Part A | 3.4

n-Fold Mixture Model.

$$G(t) = \sum_{i=1}^{n} p_i F_i(t)\,,\ p_i \geq 0\,,\ \sum_{i=1}^{n} p_i = 1\,. \qquad (3.47)$$

Jiang and *Murthy* [3.32, 33] categorize the possible shapes of the failure rate function for a mixture of any two Weibull distributions in terms of five parameters. *Gurland* and *Sethurama* [3.34] also consider a mixture of the Weibull distribution with failure rate $\lambda\beta t^{\beta-1}$ and the exponential distribution with failure rate λ_1. For $\beta > 1$, the Weibull distribution is IFR (Increasing failure rate). They found that the resulting mixture distribution is ultimately DFR (Decreasing failure rate). In fact, the mixture has a failure rate with an upside-down bathtub shape.

n-Fold Competing Risk Model.

$$G(t) = 1 - \prod_{i=1}^{n}[1 - F_i(t)]\,. \qquad (3.48)$$

Jiang and *Murthy* [3.35] also give a parametric study of a competing risk model involving two Weibull distributions.

n-Fold Multiplicative Model (Complementary Risk Model).

$$G(t) = \prod_{i=1}^{n} F_i(t)\,. \qquad (3.49)$$

The multiplicative model involving two Weibull distribution is considered in *Jiang* and *Murthy* [3.36].

n-Fold Sectional Model.

$$G(t) = \begin{cases} k_1 F_1(t)\,, & 0 \leq t \leq t_1\,,\\ 1 - k_2 \bar{F}_2(t)\,, & t_1 < t \leq t_2\,,\\ \ldots & \\ 1 - k_n \bar{F}_n(t)\,, & t > t_{n-1}\,, \end{cases} \qquad (3.50)$$

where the sub-populations $F_i(t)$ are two- or three-parameter Weibull distributions and the t_is (called partition points) are an increasing sequence. *Jiang* and *Murthy* [3.37] and *Jiang* et al. [3.38] consider the case $n = 2$ in details.

Type IV Models (Weibull Models with Varying Parameters)

For models belonging to this group, the parameters of the model are either functions of the independent variable (t) or some other variables (such as the stress level s, etc.), or are random variables.

Arrhenius Weibull Model.

$$\alpha(S) = \exp(\gamma_0 + \gamma_1 S)\,. \qquad (3.51)$$

Power Weibull Model.

$$\alpha(S) = \frac{e^{\gamma_0}}{S^{\gamma_1}}\,. \qquad (3.52)$$

These types of models have been used extensively in accelerated life testing [3.39] in reliability theory. As a result, they are referred to as *accelerated failure models*.

Weibull Proportional Hazard Models.

$$h(t) = \psi(S) h_0(t)\,, \qquad (3.53)$$

where $h_0(t)$ is called the baseline hazard for a two-parameter Weibull distribution. The only restriction on the scalar function $\psi(S)$ is that it be positive. Many different forms for $\psi(S)$ have been proposed. One such is the following:

$$\psi(S) = \exp\left(b_0 + \sum_{i=1}^{k} b_i s_i\right)\,. \qquad (3.54)$$

For more on such models, see *Cox* and *Oakes* [3.39] and *Kalbfleisch* and *Prentice* [3.17].

Type V Models (Discrete Weibull Models)

Here T can only assume non-negative integer values and this defines the support for $F(t)$.

Model 1 [3.40].

$$F(t) = \begin{cases} 1 - q^{t^\beta} & t = 0, 1, 2, 3 \cdots\,,\\ 0 & t < 0\,. \end{cases} \qquad (3.55)$$

Model 2 [3.41]. The cumulative hazard function is given by

$$H(t) = \begin{cases} c t^{\beta-1} & t = 1, 2, \cdots, m\\ 0 & t < 0\,, \end{cases} \qquad (3.56)$$

where m is given by

$$m = \begin{cases} \text{int}\left(c^{-[1/(\beta-1)]}\right) & \text{if} \quad \beta > 1\,,\\ \infty & \text{if} \quad \beta \leq 1\,, \end{cases} \qquad (3.57)$$

and int($\bullet$) represents the integer part of the quantity inside the brackets.

Model 3 [3.42].

$$F(t) = 1 - \exp\left[-\sum_{i=1}^{t} r(i)\right] = 1 - \exp\left[-\sum_{i=1}^{t} c i^{\beta-1}\right], \quad t = 0, 1, 2, \cdots . \tag{3.58}$$

3.4.3 Type VI Models (Stochastic Point Process Models)

These are stochastic point process models with links to the standard Weibull model.

Power Law Process (*Bassin* [3.43])

This is a point process model with the intensity function given by

$$\lambda(t) = \left(\frac{\beta t^{\beta-1}}{\alpha^{\beta}}\right) . \tag{3.59}$$

This model has been called by many different names: *power law process*; *Rasch–Weibull process*; *Weibull intensity function*); *Weibull–Poisson process* and *Weibull process*. We note that the inter-event times do not have Weibull distributions. For further discussion on the Weibull process, see for example, *Bain* and *Engelhart* [3.44, Chapt. 9].

Proportional Intensity Model (*Cox* and *Oaks* [3.45])

The intensity function is given by

$$\lambda(t;\, S) = \lambda_0(t)\psi(S)\,,\, t \ge 0\,, \tag{3.60}$$

where $\lambda_0(t)$ is of the form given above and $\psi(S)$ is a function of the explanatory variables S. The only restriction on $\psi(S)$ is that $\psi(S) > 0$.

Ordinary Renewal Process (*Yannaros* [3.46])

Here the point process is a renewal process with the time between events being independent and identically distributed with the distribution function given by the standard Weibull distribution.

Modified Renewal Process

Here the distribution function for the time to first event, $F_0(t)$, is different from that for the subsequent inter-event times, which are identical and independent random variables with distribution function $F(\cdot)$. Note that $F_0(t)$ and/or $F(t)$ are standard Weibull distributions. When $F_0(t) = F(t)$, this reduces to the ordinary renewal process.

3.5 Empirical Modeling of Data

We have seen a large number of Weibull-related models which we simply refer as Weibull models. They exhibit a wide range of shapes for the density and failure rate functions, which make them suitable for modeling complex failure-data sets.

Recall, we mention that empirical modeling usually involves three steps: model selection, estimation of model parameters and model validation. In the context of Weibull models, a selection procedure may be based on WPP plots. This is possible due to the availability of WPP or generalized WPP plots for all the Weibull models of types I–III. Of course, the shape of the density and failure rate functions will also be valuable in the selection step. An added advantage of the WPP plots is that they provide crude estimates of model parameters; these serve as a starting point for steps 2 and 3.

It has been suggested that an alternative method to estimate model parameters is through a least-squares fit. Basically speaking, this involves selecting the parameters the parameters to minimize a function given

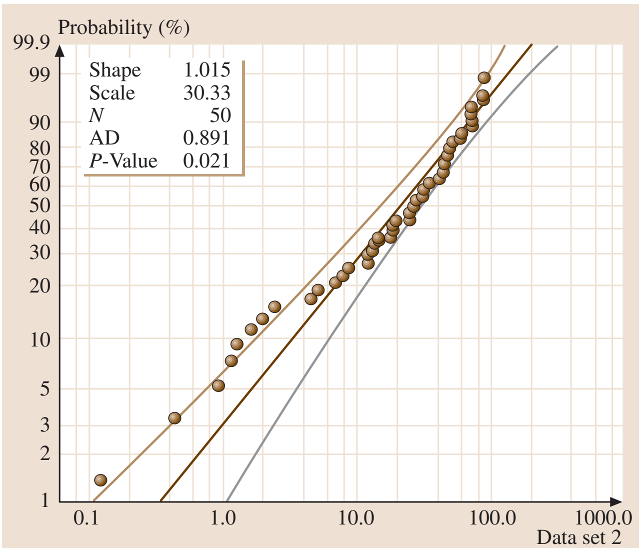

Fig. 3.4 Probability plot of data set from Table 3.1

Table 3.1 Data set of failure test (data set 2)

0.12	0.43	0.92	1.14	1.24	1.61	1.93	2.38	4.51	5.09
6.79	7.64	8.45	11.90	11.94	13.01	13.25	14.32	17.47	18.10
18.66	19.23	24.39	25.01	26.41	26.80	27.75	29.69	29.84	31.65
32.64	35.00	40.70	42.34	43.05	43.40	44.36	45.40	48.14	49.10
49.44	51.17	58.62	60.29	72.13	72.22	72.25	72.29	85.20	89.52

by

$$J(\boldsymbol{\theta}) = \sum_{i=1}^{n} [y(t_i;\boldsymbol{\theta}) - y_i]^2 , \tag{3.61}$$

where $y(t_i;\boldsymbol{\theta})$ uses the vector parameter $\boldsymbol{\theta}$ and y_i is the corresponding value obtained from the data. The optimization can be carried out using any standard optimization package. The least-squares method not only furnishes us with parameter estimates, but also helps to select a Weibull model. If one of the potential candidates has a value for $J(\boldsymbol{\theta})$ which is considerably smaller than that for the other models, then this can undoubtedly be accepted as the most appropriate model for modeling the given data set. If two or more Weibull models give rise to roughly the same value of $J(\boldsymbol{\theta})$, one would need to examine additional properties of the WPP plots to decide on the final model. Other approaches such as bootstrap and jackknife may be employed for the final selection. For more on this, see *Murthy* et al. [3.3, 47].

A couple of comments on step 3 of our empirical modeling may be in order. There are many statistical tests for validating a model. These generally require data that is different from the data used for model selection and parameter estimation. A smaller data set may pose a problem, as there will be no separate data left after model selection and parameter estimation. Various solutions have been proposed in the case of a small data set. We refer the readers to the book by *Meeker* and *Escobar* [3.48] for further details.

Example 3.1:
50 items are tested to failure. The failure times are automatically recorded and given in Table 3.1. The Weibull probability plot indicates that the two-parameter Weibull model is not appropriate.

Based on the plot, we observe that the plot shows a decreasing slope at the beginning, and an increasing slope at the end. This is an indication of a distribution with a bathtub-shaped failure-rate function. Some of the models of type II or type III can be used. One possibility is to fit the early part and last part of the plot with separate lines (Fig. 3.4). This will result in a two-fold competing risk model (3.48).

3.6 Applications

3.6.1 Applications in Reliability

Product reliability depends on the design, development and manufacturing decisions made prior to the launch (pre-launch stage) of the product and it in turn affects the failures when the product is put into operation after launch (post-launch stage).

The pre-launch stage involves several phases. In the feasibility phase, study is carried out using the specified target value for product reliability. During the design phase, product reliability is assessed in terms of part and component reliabilities. Product reliability increases as the design is improved. However, this improvement has an upper limit. If the target value is below this limit, then the design using available parts and components achieves the desired target value. If not, then a program to improve the reliability through test–fix–test cycles is carried out during the development phase. Here the prototype is tested until a failure occurs and the causes of the failure are analyzed. Based on this, design changes are made to overcome the identified failure causes. This process is continued until the reliability target is achieved. The reliability of the items produced during manufacturing tends to vary from the design target value due to variations resulting from the manufacturing process. Through proper process and quality control during the manufacturing phase, these variations are controlled.

In the post-launch stage, the reliability of an item decreases due to deterioration resulting from age and/or usage. This deterioration is affected by several factors, including the environment, operating conditions and maintenance. The rate of deterioration can be controlled through effective preventive maintenance actions. Poor reliability results in higher maintenance cost for the

Table 3.2 A sample of reliability applications

Author(s)	Topic
Weibull [3.5]	Yield strength of steel, fatigue life of steel
Keshevan et al. [3.49]	Fracture strength of glass
Sheikh et al. [3.50]	Pitting corrosion in pipes
Quereshi and *Sheikh* [3.51]	Adhesive wear in metals
Durham and *Padget* [3.52]	Failure of carbon-fiber composites
Almeida [3.53]	Failure of coatings
Fok et al. [3.54]	Failure of brittle materials
Newell et al. [3.55]	Failure of composite materials
Li et al. [3.56]	Concrete components

buyer. It also leads to higher warranty cost for the manufacturer resulting from the cost of rectifying all failures within the warranty period subsequent to the sale of the product.

Most products are composed of several components and the failure of the product is due to failure of its components. Weibull models (the two- and three-parameter models as well as a variety of types I–III Models) have been used to model the failures of many components and the literature is vast. A very small sample of the literature follows.

These models allow the determination of product reliability in terms of component reliabilities during the design phase.

During the development phase, it is often necessary to use accelerated testing to hasten the process leading to component failures. A variety of type IV models have been used for the design of experiments to carry out this testing. For more on such models, see *Nelson* [3.39] and *Meeker* and *Escobar* [3.48].

The improvement in reliability (also referred to as reliability growth) during the development phase has been modeled in many different ways. *Duane* [3.57] used a Weibull intensity model formulation to model the improvement in failure rate as a function of the development time. The breakthroughs leading to improvements can be viewed as random points along the time axis. *Crow* [3.58] modeled this by a Weibull power-law process (type VI Weibull model). For models that are modification of this model, see *Murthy* et al. [3.3].

In the manufacturing phase, the fraction of nonconforming items is small when the process is in-control while it increases significantly when the process goes out-of-control. The Weibull distribution has been used to model the in-control duration in the design of control charts to detect the change from in-control to out-of-control. For more on this, see *Rahim* [3.59], *Costa* and *Rahim* [3.60], and *Chen* and *Yang* [3.61]. *Nelson* [3.62] deals with control charts for items with Weibull failure distributions where conforming and nonconforming items differ in the scale parameter but have the same shape parameter. *Murthy* et al. [3.63] and *Djamaludin* et al. [3.64] deal with lot production and look at optimal lot size to control the occurrence of nonconforming items.

When the failure rate has a bathtub shape, it is prone to early failure. For products modeled by Weibull models exhibiting bathtub failure rates, burn-in is a technique that can be used to weed out such failures and improve product reliability before it is released for sale. For more on burn-in, see, e.g., *Kececioglu* and *Sun* [3.65].

Warranty cost analysis for products with a Weibull failure distribution has received a lot of attention. For more on this, see, e.g., *Blischke* and *Murthy* [3.66, 67] and *Murthy* and *Djamaludin* [3.68].

Preventive maintenance of products with a Weibull failure distribution has received considerable attention. In the age policy, an item is replaced preventively when it reaches some specified age and *Tadikmalla* [3.69] deals with this in the context of the Weibull failure distribution. In the block policy, items are replaced preventively at set clock times and *Blischke* and *Murthy* [3.70] deals with this in the context of the Weibull failure distribution.

3.6.2 Applications in Other Areas

Weibull distribution has been used as a model in diverse disciplines to study many different issues. There are

Table 3.3 A sample of other applications

Discipline	Topic	Author(s)
Geophysics	Wind-speed data analysis	*Al-Hasan* [3.71]
	Earthquake magnitude	*Huillet* and *Raynaud* [3.72]
	Volcanic occurrence data	*Bebbington* and *Lai* [3.73]
	Low-flow analysis	*Durrans* [3.74]
	Regional flood frequency	*Heo* et al. [3.75]
Food science	Sterility in thermal preservation method	*Mafart* et al. [3.76]
Social science	Unemployment duration	*Roed* and *Zhang* [3.77]
Environment	Environment radioactivity	*Dahm* et al. [3.78]
Nature	Ecological application	*Fleming* [3.79]
Medical science	Survival data	*Carroll* [3.80]

several thousand papers and we give a very small sample of this vast literature

3.6.3 Weibull Analysis Software

In any analysis of statistical data, computer software is required. In addition to standard statistical software such as MINITAB, SPSS, SAS, etc., or spreadsheet software such as Excel, some specialized Weibull analysis software are also available. Below is a list of some common ones.

- Weibull++ 6 by ReliaSoft Corp., Tucson, AZ; http://weibull.reliasoft.com/
- Relex Weibull by Relex Software Corp., Greensburg, PA; http://www.relexsoftware.com/products/weibull.asp
- WeibullPro by Isograph Inc., Newport Beach, CA; http://www.isograph-software.com/avsoverwbl.htm
- WinSMITH Weibull by Barringer & Associates, Inc., Humble, TX; http://www.barringer1.com/wins.htm

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