5. Two-Dimensional Failure Modeling

Two-Dimensi

For many products (for example, automobiles), failures depend on age and usage and, in this case, failures are random points in a two-dimensional plane with the two axes representing age and usage. In contrast to the one-dimensional case (where failures are random points along the time axis) the modeling of two-dimensional failures has received very little attention. In this chapter we discuss various issues (such as modeling process, parameter estimation, model analysis) for the two-dimensional case and compare it with the one-dimensional case.

All products are unreliable in the sense that they degrade and fail with age and/or usage, and ultimately fail. Reliability of a product conveys the concept of dependability and the absence of failures. Reliability theory deals with various aspects of product reliability and encompasses various reliability issues. These include the following:

1. Reliability science to understand the degradation leading to failures.
2. Reliability engineering to design and manufacture reliable products.
3. Reliability management to manage the activities during the design and manufacture of products and the operation of unreliable products.
4. Reliability modeling to build models to obtain solutions to a variety of reliability-related problems in predicting, estimating, and optimizing the performance of unreliable products, the impact of unreliability and actions to mitigate the impact.

The modeling of failures is an important element of reliability modeling. In one-dimensional failure modeling, failures are random points along a one-dimensional axis representing age or usage. For many products (for example, automobiles), failures depend on age and usage and, in this case, failures are random points in a two-dimensional plane with the two axes representing age and usage. Models play an important role in decision-making. Many different types of models are used and these can be found in [5.1–4]. One-dimensional modeling has received considerable attention and so there is a vast literature covering this area. In contrast, two-dimensional failure modeling has received relatively little attention. In this chapter we discuss various issues relating to two-dimensional failure modeling.

The outline of the chapter is as follows. In Sect. 5.1 we discuss various issues relating to the modeling of failures. Section 5.2 deals with the black-box approach (or empirical modeling) where the modeling is based on failure and operational data. One-dimensional modeling based on the black-box approach is reviewed in Sect. 5.3, where we discuss the different issues and illustrate through an example involving a real case. This sets the background for the two-dimensional modeling discussed in Sect. 5.4. Two different approaches have been proposed for two-dimensional failure modeling. We discuss these two approaches and indicate topics for future research. In Sect. 5.5 we discuss a new approach and this is illustrated using the case discussed earlier. Finally, we conclude with some comments and remarks in Sect. 5.6.

5.1 Modeling Failures

In this section we briefly discuss various issues of importance in the modeling of failures.

5.1.1 Product Failures

Products can vary from simple (such as an electric kettle) to complex (such as an aeroplane). A product can be viewed as a system consisting of several parts and can be decomposed into a hierarchy of levels with the system at the top level and components at the lowest level and several levels (such as sub-system, assembly, sub-assembly and so on) in between. The failure of a product is due to the failure of one or more of its components.

The occurrence of failure depends on several factors. These include decisions made during the design and manufacture of the product, usage intensity and operating environment, and the maintenance actions (corrective and preventive) carried out during the operating life.

The modeling of failures can be done at any level ranging from system to component level.

5.1.2 Approaches to Modeling

The approach to modeling depends on the kind of information available and the goal of the modeling. There are two basic approaches to modeling failures, as indicated below.

1. Black-box approach: here the modeling is based solely on failure and censored data for similar items. This approach is used when there is very little understanding of the different mechanisms that lead to product failure or when the unit is too complex. This approach is also known as *data-based or empirical modeling*.
2. White-box approach: here the failure modeling at the component level is based on the different mechanisms that lead to failure. At the system level, the failure is done in terms of the failures of the different components. This approach is also known as *physics-based modeling*.

It should be noted that most statistical modeling incorporates features from both approaches and that the collaborative nature of such modeling is critical. The engineer or scientist needs to bring their expertise to develop the appropriate models in the white-box approach, whereas the statistician needs to be able to determine the appropriate data analysis for the given data set in its context.

5.1.3 First and Subsequent Failures

One needs to differentiate between the first failure and subsequent failures. The subsequent failures depend on the type of actions used to rectify the failures. In the case of a nonrepairable item (component, system or something in between), the failed item needs to be replaced by either a new or used item to make the product functional. In the case of a repairable item, the product can be made operational through the repair of the failed item. Three types of repair are indicated below:

1. Minimal repair, which restores the item to the condition just before failure;
2. Perfect repair (which makes the item as good as new); and
3. Imperfect repair that results in the item being better than it was prior to failure but not as good as new.

5.2 Black-Box Modeling Process

In the black-box approach to modeling failures, the data are the starting point that forms the basis for the model building. In this section we briefly discuss the different data types and then outline the modeling process.

5.2.1 Data Types

The data can be either item failure times or counts of item failures over an interval. In the former case, the data are continuous-valued and in the latter case they are integer-valued.

Lifetime data can be complete, censored or truncated. In the case of complete data, the data relate to the age at failure. With censored data, the lifetimes are only known to exceed some value(s). This could result from the item not having failed during the period of observation and hence still being operational for a certain length of time afterwards (right-censored data). When the data are the failures of an item over different disjoint time intervals we have grouped data. When failures of different components are pooled together, we have pooled data. In both cases, the data can be considered as categorical or, if they are in the form of counts, they are discrete-valued.

5.2.2 Modeling Process

The modeling process involves the following four steps.

Step 1: Exploratory analysis of data
Step 2: Model selection
Step 3: Parameter estimation
Step 4: Model validation

These are discussed further in Sects. 5.3–5.5.

5.3 One-Dimensional Black-Box Failure Modeling

In this section we give a brief review of one-dimensional failure modeling. The item under consideration can be the product or some component of the product.

5.3.1 Modeling First Failure

Let T denote the time to first failure. It is modeled by a failure distribution function, $F(t\,;\theta)$, which characterizes the probability $P\{T \le t\}$ and is defined as

$$F(t\,;\theta) = P(T \le t)\,,\ t \ge 0 \tag{5.1}$$

θ denotes the parameters (or parameter set) of the distribution function. If $F(t\,;\theta)$ is a differentiable function, then the failure density function, $f(t\,;\theta)$, is given by

$$f(t) = \mathrm{d}F(t;\theta)/\,\mathrm{d}t\,. \tag{5.2}$$

The survivor function, $\bar{F}(t)$, is given by

$$\bar{F}(t\,;\theta) = 1 - F(t\,;\theta) = P(T > t)\,. \tag{5.3}$$

The hazard function, $h(t\,;\theta)$, is given by

$$h(t\,;\theta) = f(t\,;\theta)/\bar{F}(t\,;\theta)\,. \tag{5.4}$$

The cumulative hazard function, $H(t\,;\theta)$, is given by

$$H(t\,;\theta) = \int_0^t h(u\,;\theta)\,\mathrm{d}u = -\log[1 - F(t\,;\theta)]\,. \tag{5.5}$$

Many different distributions have been used for modeling lifetimes. The shapes of the density and hazard functions depend on the form of the distribution and the parameter values.

Note: in the future we will omit θ for notational ease so that we have $h(t)$ instead of $h(t\,;\theta)$ and similarly for the other functions.

A commonly used model is the two-parameter Weibull distribution, which is given by

$$F(t\,;\theta) = 1 - \exp[-(t/\alpha)]^{\beta}\,,\ t \ge 0 \tag{5.6}$$

with $\theta = \{\alpha, \beta\}$. Here, α is the scale parameter and β is the shape parameter.

The Weibull models are a family of distributions derived from the two-parameter Weibull distribution. *Lai* et al. [5.5] discuss a few of these models and, for more details, see *Murthy* et al. [5.6]. Many other distributions have been used in modeling time to failure and these can be found in most books on reliability. See, for example, *Blischke* and *Murthy* [5.2], *Meeker* and *Escobar* [5.4], *Lawless* [5.3], *Nelson* [5.7] and *Kalbfleisch* and *Prentice* [5.8]. *Johnson* and *Kotz* [5.9, 10] give more details of other distributions that can be used for failure modeling.

5.3.2 Modeling Subsequent Failures

Minimal Repair

In minimal repair, the hazard function after repair is the same as that just before failure. In general, the repair time is small relative to the mean time between failures so that it can be ignored and the repairs treated as instantaneous. In this case, failures over time occur according to a nonhomogeneous Poisson point process with intensity function $\lambda(t) = h(t)$, the hazard function. Let $N(t)$

denote the number of failures over the interval $[0, t)$. Define

$$\Lambda(t) = \int_{o}^{t} \lambda(u)\,\mathrm{d}u \tag{5.7}$$

Then we have the following results:

$$P(N(t)=n) = \frac{\mathrm{e}^{-\Lambda(t)}[\Lambda(t)]^n}{n!}\,, n=0\,,1\,,2\,,\ldots \tag{5.8}$$

and

$$E[N(t)] = \Lambda(t) \tag{5.9}$$

For more details, see *Nakagawa* and *Kowada* [5.11] and *Murthy* [5.12].

Perfect Repair

This is identical to replacement by a new item. If the failures are independent, then the times between failures are independent, identically distributed random variables from $F(t)$ and the number of failures over $[0, t)$ is a renewal process with

$$P\{N(t)=n\} = F^{(n)}(t) - F^{(n+1)}(t)\,, \tag{5.10}$$

where $F^{(n)}(t)$ is the n-fold convolution of $F(t)$ with itself, and

$$E[N(t)] = M(t)\,, \tag{5.11}$$

where $M(t)$ is the renewal function associated with $F(x\,;\theta)$ and is given by

$$M(t) = F(t) + \int_{0}^{t} M(t-u)\,\mathrm{d}F(u)\,. \tag{5.12}$$

For more on renewal processes, see *Cox* [5.13], *Cox* and *Isham* [5.14]) and *Ross* [5.15].

Imperfect Repair

Many different imperfect repair models have been proposed. See *Pham* and *Wang* [5.16] for a review of these models. In these models, the intensity function $\lambda(t)$ is a function of $\Im_t$, the history of failures over $[0, t)$. Two models that have received considerable attention are: (i) reduction in failure intensity, and (ii) virtual age. See *Doyen* and *Gaudoin* [5.17] for more on these two models.

5.3.3 Exploratory Data Analysis

The first step in constructing a model is to explore the data through plots of the data. By so doing, information can be extracted to assist in model selection. The plots can be either nonparametric or parametric and the plotting is different for perfect repair and imperfect repair situations. The data comprises both the failure times and the censored times.

Perfect Repair

Plot of Hazard Function (Nonparametric). The procedure (for complete or censored data) is as follows:

Divide the time axis into cells with cell i defined by $[t_i, t_{i+1})$, $i \geq 0$, $t_0 = 0$ and $t_i = i\delta$, where δ is the cell width. Define the following quantities:

N_i^{f} : Number of items with failure times in cell $i\,, i \geq 0$;
N_i^{c} : Number of items with censoring times in cell $i\,, i \geq 0$;
$N_i^{\mathrm{f}|ri}$: Number of failures in cells i and beyond $\left(=\sum_{j=i}^{\infty} N_j^{\mathrm{f}}\right)$.

Similarly define $N_i^{\mathrm{c}|ri}$ for censored data.

The estimator of the hazard function is given by

$$\hat{h}_i = \frac{N_i^{\mathrm{f}}}{N_i^{\mathrm{f}|ri} + N_i^{\mathrm{c}|ri}}\,, i \geq 0 \tag{5.13}$$

Plot of Density Function (Nonparametric). The simplest form of nonparametric density estimator is the histogram. Assuming the data is complete, the procedure is to calculate the relative frequencies for each cell,

$$\hat{f}_i = \frac{N_i^{\mathrm{f}}}{\sum_{j=0}^{\infty} N_j^{\mathrm{f}}}\,, \tag{5.14}$$

and then plot these against the cell midpoints. As histograms can be very unreliable for exploring the shape of the data, especially if the data set is not large, it is desirable to use more sophisticated density-function estimators (*Silverman* [5.18]).

Weibull Probability Plots (Parametric). The Weibull probability plot (WPP) provides a systematic procedure to determine whether one of the Weibull-based models is suitable for modeling a given data set or not, and is more reliable than considering just a simple histogram. It is based on the Weibull transformations

$$y = \ln\{-\ln[1-F(t)]\} \quad \text{and} \quad x = \ln(t)\,. \tag{5.15}$$

The plot of y versus x gives a straight line if $F(t)$ is a two-parameter Weibull distribution.

Thus, if $F(t)$ is estimated for (complete) data from a Weibull distribution and the equivalent transformations and plot obtained, then a *rough* linear relationship should be evident. To estimate $F(t)$, we need an empirical estimate of $F(t_i)$ for each failure time t_i.

Assuming the t_i's are ordered, so that $t_1 \le t_2 \le \ldots \le t_n$, a simple choice (in the case of complete data) is to take the empirical distribution function

$$\hat{F}(t_i) = i/(n+1)\,. \tag{5.16}$$

We then plot $\hat{y}_i = \ln\left\{-\ln\left[1-\hat{F}(t_i)\right]\right\}$ versus $x_i = \ln(t_i)$ and assess visually whether a straight line could describe the points.

We illustrate by considering real data. The data refers to failure times and usage (defined through distance traveled between failures) for a component of an automobile engine over the warranty period given by three years and 60 000 miles. Here we only look at the failure times in the data set. Figure 5.1 shows a Weibull probability plot of the inter-failure times of a component that we shall call component C-1. This clearly shows a curved relationship and so a simple Weibull model would not be appropriate.

Note: the plotting of the data depends on the type of data. So, for example, the presence of censored observations would necessitate a change in the empirical failure estimates (see *Nelson* [5.7] for further details).

Minimal Repair

Plot of Cumulative Intensity Function (Nonparametric). The procedure is as follows: With δ and the cells defined as before, define the following:

M: Number of items at the start;
N_i^{f}: Total number of failures over $[0, i\delta)$;
M_i^{c}: Number of items censored in cell i;
λ_i: Cumulative intensity function till cell i.

The estimator of the cumulative intensity is given by

$$\hat{\lambda}_0 = \frac{N_0^{\mathrm{f}}}{M} \quad \text{and}$$

$$\hat{\lambda}_i = \frac{N_i^{\mathrm{f}} - \sum\limits_{j=0}^{i-1} M_j^{\mathrm{c}}\hat{\lambda}_j}{\left[M - \sum\limits_{j=0}^{i-1} M_j^{\mathrm{c}}\right]}\,,\; i \ge 1\,. \tag{5.17}$$

Graphical Plot (Parametric). When the failure distribution is a two-parameter Weibull distribution, from (5.9) we see that a plot of $y = \ln\{E[N(t)]/t\}$ versus $x = \ln(t)$ is a straight line. *Duane* [5.19] proposed plotting $y = \ln[N(t)/t]$ versus $x = \ln(t)$ to determine if a Weibull distribution is a suitable model or not to model a given data set. For a critical discussion of this approach, see *Rigdon* and *Basu* [5.20].

5.3.4 Model Selection

We saw in Fig. 5.1 that a simple Weibull model was clearly not adequate to model the failures of component C-1. However, there are many extensions of the Weibull model that can fit a variety of shapes. *Murthy* et al. [5.6] give a taxonomic guide to such models and give steps for model selection. This particular curve is suited to modeling with a mixture of two Weibull components. Figure 5.2 shows the WPP plot of Fig. 5.1 with the transformed probability curve for this mixture. (Details about estimating this curve are given in Sect. 5.3.5.) This seems to fit the pattern quite well, although it misses

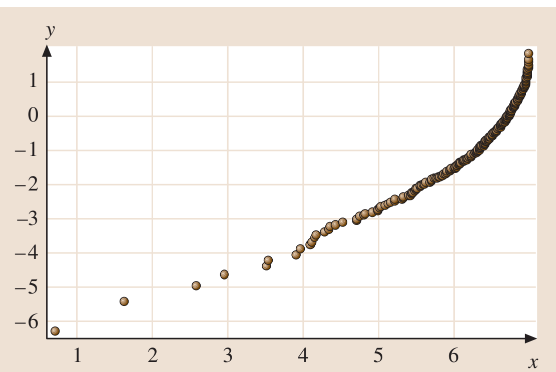

Fig. 5.1 WPP of days to failure of component C-1

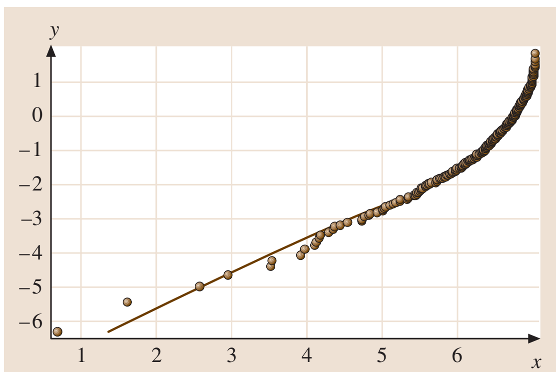

Fig. 5.2 WPP of component C-1 failures with Weibull mixture

the shape of the curve present in the few small failure times.

Figure 5.3 gives the empirical plot of the density function and the density function based on the mixture model. As can be seen, the model matches the data reasonably well. The plots illustrate the way in which the second Weibull component is being used. The nonparametric density estimate suggests that there is a small failure mode centered around 200 d. The second Weibull component, with a weight of 24.2%, captures these early failure times while the dominant component, with a weight of 75.8%, captures the bulk of the failures.

5.3.5 Parameter Estimation

The model parameters can be estimated either based on the graphical plots or by using statistical methods. Many different methods (method of moments, method of maximum likelihood, least squares, Bayesian and so on) have been proposed. The graphical methods yield crude estimates whereas the statistical methods are more refined and can be used to obtain confidence limits for the estimates. Most books on statistical reliability (some of which are mentioned in Sect. 5.3.1) deal with this topic in detail.

The parameters for the Weibull mixture model in Fig. 5.3 were estimated by minimizing the squared error between the points and the curve on the Weibull probability plot. The estimates are

$$\hat{p} = 0.242\,, \quad \hat{\beta}_1 = 1.07\,, \quad \hat{\beta}_2 = 4.32\,,$$
$$\hat{\eta}_1 = 381 \quad \text{and} \quad \hat{\eta}_2 = 839\,.$$

Similar estimates can be obtained without computer software using the graphical methods given by *Jiang* and *Murthy* [5.21].

Alternatively, we can use the standard statistical approach of maximum-likelihood estimation to get the parameter estimates. We find

$$\hat{p} = 0.303\,, \hat{\beta}_1 = 1.46\,, \hat{\beta}_2 = 5.38\,,$$
$$\hat{\eta}_1 = 383 \quad \text{and} \quad \hat{\eta}_2 = 870\,.$$

These values are less affected by the small failures times.

5.3.6 Model Validation

Validation of statistical models is highly dependent on the nature of the models being used. In many situations, it can simply involve an investigation of the shape of the data through plots such as quantile–quantile plots (for example, normal probability plots and WPP) and through tests for goodness of fit (general tests, such as the χ^2 goodness-of-fit test, or specific tests, such as the Anderson–Darling test of normality). Many introductory statistics texts cover these plots and tests (see, for example, *Vardeman* [5.22] and *D'Agostino* and *Stephen* [5.23]). In more complex situations, these approaches need to be used on residuals obtained after fitting a model involving explanatory variables. An alternative approach, which can be taken when the data set is large, is to take a random sample from the data set, fit the model(s) to this sub-sample and then evaluate (through plots and tests) how well the model fits the sub-sample consisting of the remaining data.

To exemplify model validation, 80% of the data was randomly taken and the mixed Weibull model above fitted. The fitted model was then compared using a WPP to the remaining 20% of the data. The upper top of Fig. 5.4 shows a Weibull plot of 80% of the failure data for component C-1, together with the Weibull mixture fit to the data. The remaining 20% of failure data are

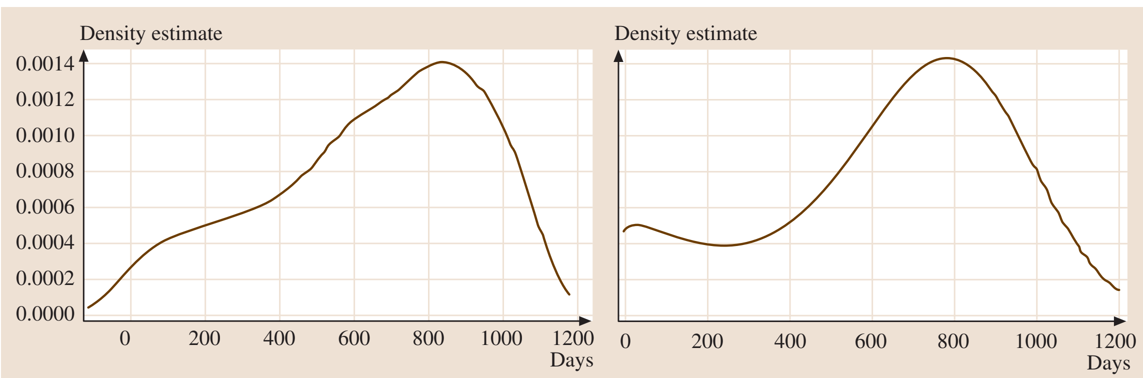

Fig. 5.3 Empirical density (*left*) and Weibull mixture density (*right*) for component C-1

plotted in the lower plot. The Weibull mixture curve with the same parameters as in the upper plot has been added here. Apart from the one short failure time, this curve seems to fit the test data quite well. This supports the use of the Weibull mixture for modeling the failures of this component.

5.4 Two-Dimensional Black-Box Failure Modeling

When failure depends on age and usage, one needs a two-dimensional failure model. Two different approaches (one-dimensional and two-dimensional) have been proposed and we discuss both of these in this section.

5.4.1 One-Dimensional Approach

Here, the two-dimensional problem is effectively reduced to a one-dimensional problem by treating usage as a random function of age.

Modeling First Failure

Let $X(t)$ denote the usage of the item at age t. In the one-dimensional approach, $X(t)$ is modeled as a linear function of t and so given by

$$X(t) = \Gamma t \tag{5.18}$$

where Γ, $0 \leq \Gamma < \infty$, represents the usage rate and is a nonnegative random variable with a distribution function $G(r)$ and density function $g(r)$.

The hazard function, conditional on $\Gamma = r$ is given by $h(t|r)$. Various forms of $h(t|r)$ have been proposed; one such is the following polynomial function:

$$h(t|r) = \theta_0 + \theta_1 r + \theta_2 t + \theta_3 X(t) + \theta_4 t^2 + \theta_5 t X(t)\,. \tag{5.19}$$

The conditional distribution function for the time to first failure is given by

$$F(t|r) = 1 - \exp\left[-\int_0^t h(u|r)\,\mathrm{d}u\right]\,. \tag{5.20}$$

On removing the conditioning, we have the distribution function for the time to first failure, given by

$$F(t) = \int_0^\infty \left\{1 - \exp\left[-\int_0^t h(u|r)\,\mathrm{d}u\right]\right\} g(r)\,\mathrm{d}r\,. \tag{5.21}$$

Modeling Subsequent Failures

The modeling of subsequent failures, conditional on $\Gamma = r$, follows along lines similar to that in Sect. 5.3.2. As a result, under minimal repair, the failures over time occur according to a nonhomogeneous Poisson process with intensity function $\lambda(t|r) = h(t|r)$ and, under perfect repair, the failures occur according to the renewal process associated with $F(t|r)$.

The bulk of the literature deals with a linear relationship between usage and age. See, for example, *Blischke* and *Murthy* [5.1], *Lawless* et al. [5.24] and *Gertsbakh* and *Kordonsky* [5.25]. *Iskandar* and *Blischke* [5.26] deal with motorcycle data. See *Lawless* et al. [5.24] and *Yang* and *Zaghati* [5.27] for automobile warranty data analysis.

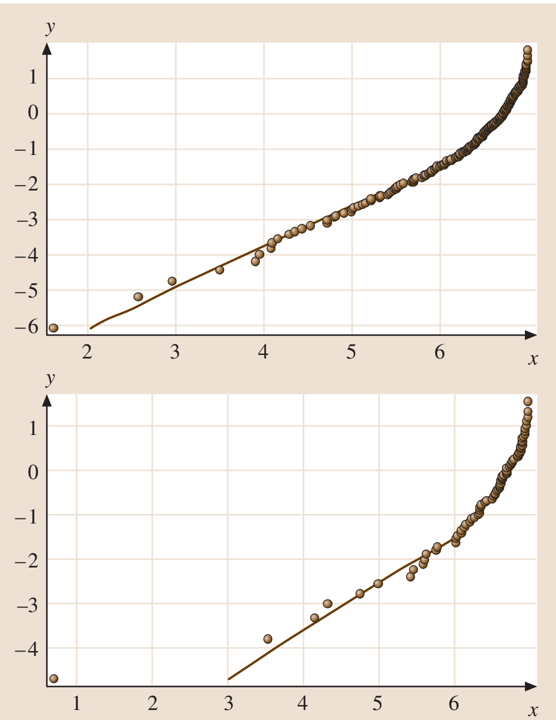

Fig. 5.4 Weibull plots of fitting data (*top*) and test data (*bottom*) for component C-1

5.4.2 Two-Dimensional Approach

Modeling First Failure

Let T and X denote the system's age and usage at its first failure. In the two-dimensional approach to modeling, (T, X) is treated as a nonnegative bivariate random variable and is modeled by a bivariate distribution function

$$F(t\,,x) = P(T \le t\,, X \le x)\,;\, t \ge 0\,, x \ge 0\,. \tag{5.22}$$

The survivor function is given by

$$\bar{F}(t\,,x) = Pr(T > t\,, X > x) = \int\limits_t^\infty \int\limits_x^\infty f(u, v)\,\mathrm{d}v\,\mathrm{d}u\,. \tag{5.23}$$

If $F(t, x)$ is differentiable, then the bivariate failure density function is given by

$$f(t\,,x) = \frac{\partial^2 F(t\,,x)}{\partial t \partial x}\,. \tag{5.24}$$

The hazard function is defined as

$$h(t\,,x) = f(t\,,x)/\bar{F}(t\,,x)\,, \tag{5.25}$$

with $h(t\,,x)\delta t \delta x$ defining the probability that the first system failure will occur in the rectangle $[t\,,t+\delta t)\times[x\,,x+\delta x)$ given that $T > t$ and $X > x$. Note, however, that this is not the same as the probability that the first system failure will occur in the rectangle $[t\,,t+\delta t)\times[x\,,x+\delta x)$ given that it has not occurred *before* time t and usage x, which is given by $(f(t\,,x)/[1-F(t\,,x)])\,\delta t\delta x$.

Bivariate Weibull Models

A variety of bivariate Weibull models have been proposed in the literature. We indicate the forms of the models, and interested readers can obtain more details from *Murthy* et al. [5.6].

Model 1 [*Marshall* and *Olkin* [5.28]]

$$\bar{F}(t\,,x) = \exp\left\{-\left[\lambda_1 t^{\beta_1} + \lambda_2 x^{\beta_2} + \lambda_{12}\max(t^{\beta_1}\,, x^{\beta_2})\right]\right\}\,. \tag{5.26}$$

Model 2 [*Lee* [5.29]]

$$\bar{F}(t\,,x) = \exp\left\{-\left[\lambda_1 c_1^\beta t^\beta + \lambda_2 c_2^\beta x^\beta + \lambda_{12}\max\left(c_1^\beta t^\beta\,, c_2^\beta x^\beta\right)\right]\right\}\,. \tag{5.27}$$

Model 3 [*Lee* [5.29]]

$$\bar{F}(t\,,x) = \exp\left[-\lambda_1 t^{\beta_1} - \lambda_2 x^{\beta_2} - \lambda_0 \max(t\,,x)^{\beta_0}\right]\,. \tag{5.28}$$

Model 4 [*Lu* and *Bhattacharyya* [5.30]]

$$\bar{F}(t\,,x) = \exp\left\{-\left[(t/\theta_1)^{\beta_1/\delta} + (x/\theta_2)^{\beta_2/\delta}\right]^\delta\right\}\,, \tag{5.29}$$

$$\bar{F}(t\,,x) = \left[1 + \left(\left\{\exp\left[(t/\theta_1)^{\beta_1}\right]-1\right\}^{1/\gamma} + \left\{\exp\left[(x/\theta_2)^{\beta_2}\right]-\right\}^{1/\gamma}\right)^\gamma\right]^{-1}\,, \tag{5.30}$$

$$\bar{F}(t\,,x) = \exp\left[-(t/\alpha_1)^{\beta_1} - (x/\alpha_2)^{\beta_2} - \delta h(t\,,x)\right]\,. \tag{5.31}$$

Different forms for the function of $h(t\,,x)$ yield a family of models. One form for $h(t\,,x)$ is the following:

$$h(t\,,x) = \left[(t/\alpha_1)^{\beta_1/m} + (x/\alpha_2)^{\beta_2/m}\right] \tag{5.32}$$

which results in

$$\bar{F}(t\,,x) = \exp\left\{-(t/\alpha_1)^{\beta_1} - (x/\alpha_2)^{\beta_2} - \delta\left[(t/\alpha_1)^{\beta_1/m} + (x/\alpha_2)^{\beta_2/m}\right]^m\right\}\,. \tag{5.33}$$

Two other variations are

$$\bar{F}(t\,,x) = \exp\Big(-(t/\alpha_1)^{\beta_1} - (x/\alpha_2)^{\beta_2} - \delta\left\{1-\exp\left[-(t/\alpha_1)^{\beta_1}\right]\right\} \times\left\{1-\exp\left[-(x/\alpha_2)^{\beta_2}\right]\right\}\Big)\,, \tag{5.34}$$

$$\bar{F}(t\,,x) = \left\{1 + \left[\left\{\exp\left[(t/\alpha_1)^{\beta_1}\right]-1\right\}^{1/\gamma} + \left\{\exp\left[(x/\alpha_2)^{\beta_2}\right]-1\right\}^{1/\gamma}\right]^\gamma\right\}^{-1}\,. \tag{5.35}$$

Model 5 (*Sarkar* [5.31])

$$\bar{F}(t\,,x) = \begin{cases} \exp\Big(\left[-(\lambda_1+\lambda_{12})t^{\beta_1}\right] \\ \times\Big\{1-\left[A\left(\lambda_2 t^{\beta 1}\right)\right]^{-\gamma} \\ \times\left[A(\lambda_2 x^{\beta_2})\right]^{1+\gamma}\Big\}\Big)\,, \\ \qquad t \ge x > 0\,; \\ \exp\Big(\left[-(\lambda_2+\lambda_{12})x^{\beta_2}\right] \\ \times\Big\{1-\left[A(\lambda_1 x^{\beta_2})\right]^{-\gamma} \\ \times\left[A(\lambda_1 t^{\beta_1})\right]^{1+\gamma}\Big\}\Big)\,, \\ \qquad x \ge t > 0\,; \end{cases} \tag{5.36}$$

where $\gamma = \lambda_{12}/(\lambda_1 + \lambda_2)$ and $A(z) = 1 - \mathrm{e}^{-z}$, $z > 0$.
Model 6 [*Lee* [5.29]]

$$\bar{F}(t\,,x) = \exp\left[-\left(\lambda_1 t^{\beta_1} + \lambda_2 x^{\beta_2}\right)^{\gamma}\right]\,. \tag{5.37}$$

Comment: many other non-Weibull models can also be used for modeling. For more on this see *Johnson* and *Kotz* [5.32] and *Hutchinson* and *Lai* [5.33]. *Kim* and *Rao* [5.34], *Murthy* et al. [5.35], *Singpurwalla* and *Wilson* [5.36], and *Yang* and *Nachlas* [5.37] deal with two-dimensional warranty analysis.

Modeling Subsequent Failures

Minimal Repair. Let the system's age and usage at the j-th failure be given by t_j and x_j, respectively. Under minimal repair, we have that

$$h\left(t_j^+\,,x_j\right) = h\left(t_j^-\,,x_j\right)\,, \tag{5.38}$$

as the hazard function after repair is the same as that just before failure. Note that there is no change in the usage when the failed system is undergoing minimal repair.

Let $\{N(t\,,x) : t \geq 0\,, x \geq 0\}$ denote the number of failures over the region $[0\,,t) \times [0\,,x)$. Unfortunately, as there is no complete ordering of points in two dimensions, there is no analogous result to that obtained for minimal repair in one dimension. In particular, the hazard rate does not provide an intensity rate at a point $(t\,,x)$ as the failure after the last failure prior to $(t\,,x)$ may be either prior to time t (though after usage x) or prior to usage x (though after time t), as well as possibly being after both time t and usage x. Hence, not only is it more difficult to obtain the distribution for $\{N(t,x) : t \geq 0, x \geq 0\}$, it is also more difficult to obtain even the mean function for this process.

Perfect Repair. In this case, we have a two-dimensional renewal process for system failures and the following results are from *Hunter* [5.38]:

$$p_n(t\,,x) = F^{(n)}(t\,,x) - F^{(n+1)}(t\,,x)\,,\ n \geq 0\,, \tag{5.39}$$

where $F^{(n)}(t\,,x)$ is the n-fold bivariate convolution of $F(t\,,x)$with itself. The expected number of failures over $[0\,,t) \times [0\,,x)$ is then given by the solution of the two-dimensional integral equation

$$M(t\,,x) = F(t\,,x) + \int_0^t \int_0^x M(t-u\,,x-v) \times f(u\,,v)\,\mathrm{d}v\,\mathrm{d}u\,. \tag{5.40}$$

Imperfect Repair. This has not been studied and hence is a topic for future research.

Comparison with 1-D Failure Modeling

For the first failure, in the one-dimensional failure modeling, we have

$$F(t) + \bar{F}(t) = 1\,, \tag{5.41}$$

and

$$\bar{F}(t) = \exp\left[-\int_0^t h(u)\,\mathrm{d}u\right]\,. \tag{5.42}$$

In two-dimensional failure modeling, however, we have

$$F(t\,,x) + \bar{F}(t\,,x) < 1\,, \tag{5.43}$$

since

$$F(t\,,x) + \bar{F}(t\,,x) + P\,(T \leq t\,,X > x) + P\,(T > t\,,X \leq x) = 1\,. \tag{5.44}$$

A Numerical Example

We confine our attention to a model proposed by *Lu* and *Bhattacharyya* [5.30], where the survivor function is given by (5.31) with $h(t\,,x)$ given by (5.32) with $\alpha_1\,,\alpha_2\,,\beta_1\,,\beta_2 > 0$, $\delta \geq 0$ and $0 < m \leq 1$. If $m = 1$ then the hazard function is given by

$$h(t\,,x) = (1+\delta)^2 \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} \frac{\beta_2}{\alpha_2}\left(\frac{t}{\alpha_2}\right)^{\beta_2 - 1}\,. \tag{5.45}$$

Let the model parameters be as follows:

$$\alpha_1 = 2\,, \alpha_2 = 3\,, \beta_1 = 1.5\,, \beta_2 = 2.0\,,$$
$$\delta = 0.5\,, m = 1$$

The units for age and usage are years and 10 000 km, respectively. The expected age and usage at first system failure are given by
$E(T_1) = \theta_1 \Gamma(1/\beta_1 + 1) = 1.81$ (years) and
$E(X_1) = \theta_2 \Gamma(1/\beta_2 + 1) = 2.66$ (10^3 km).

Figure 5.5 is a plot of the survivor function $\bar{F}(t\,,x)$ and Fig. 5.6 is a plot of the hazard function $h(t\,,x)$. Note that $h(t\,,x)$ increases as t (age) and x (usage) increase, since β_1 and β_2 are greater than 1.

Replacement. The expected number of system failures in the rectangle $[0\,,t) \times [0\,,x)$ under replacement is given by the renewal function $M(t\,,x)$ in (5.40). Figure 5.7 is

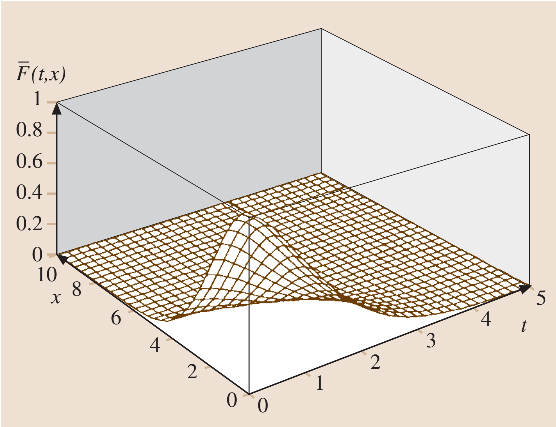

Fig. 5.5 Plot of the survivor function $\bar{F}(t, x)$

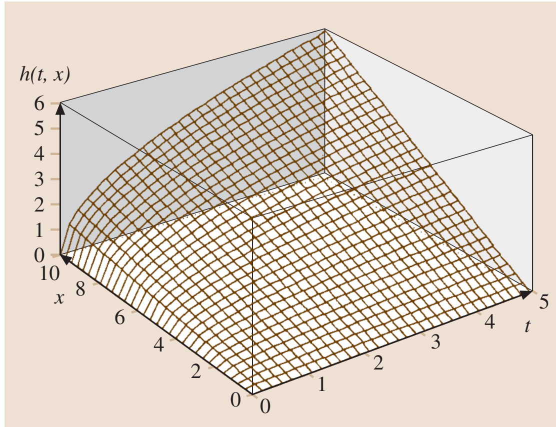

Fig. 5.6 Plot of the hazard function $h(t, x)$

a plot of $M(t, x)$, obtained using the two-dimensional renewal-equation solver from *Iskandar* [5.39].

5.4.3 Exploratory Data Analysis

In the one-dimensional case, the presence of censored observations causes difficulties in estimating the various functions (hazard rate, density function). When usage is taken into account, these difficulties are exacerbated, due to the information about usage being only observed at failure times. In particular, models which build conditional distributions for the failure times given usage (or usage rates) have to determine a strategy for assigning the censored failure times to some usage (or usage group).

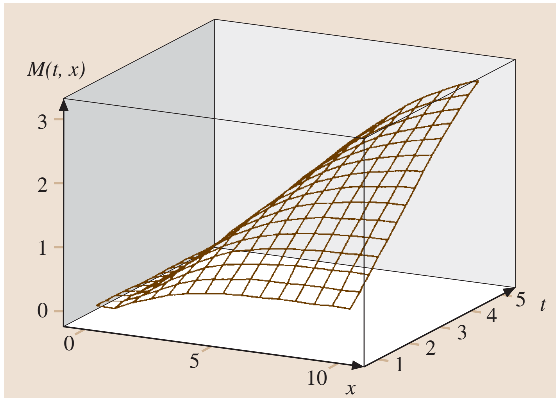

Fig. 5.7 Plot of the renewal function $M(t, x)$

1-D Approach

Perfect Repair. Firstly, we group the data into different groups based on the usage rate. Each group has a mean usage rate and the data is analyzed using the approach discussed in Sect. 5.3. This yields the model for the failure distribution *conditioned* on the usage rate. One then needs to determine whether the model structure is the same for different usages or not and whether the linear relationship [given by (5.16)] is valid or not. Next, exploratory plots of the usage rate need to be obtained to determine the kind of distribution appropriate to model the usage rate.

If the conditional failure distributions are two-parameter Weibull distributions then the WPP plots are straight lines. If the shape parameters do not vary with usage rate, then the straight lines are parallel to each other. One can view usage in a manner similar to stress level and use accelerated life-test models [*Nelson* [5.7]] to model the effect of usage on failure.

Imperfect Repair. The plotting (for a given usage rate) follows along the lines discussed in Sect. 5.3.3 and this allows one to determine the distribution appropriate to model the data. Once this is done, one again needs to examine exploratory plots of the usage rate to decide on the appropriate model.

2-D Approach

We confine our discussion to the case of perfect repair.

Plot of Hazard Function (Nonparametric Approach). We divide the region into rectangular cells. Cell (i, j) is given by $[i\delta_1, (i+1)\delta_1) \times [j\delta_2, (j+1)\delta_2)$, where δ_1 and δ_2 are the cells' width and height respectively.

Let us define:

N_{ij}^{f} : Number of items with failures times in cell $(i\,,\,j)\,,\,i \geq 0\,,\,j \geq 0\,;$

N_{ij}^{c} : Number of items with censoring times in cell $(i\,,\,j)\,,\,i \geq 0\,,\,j \geq 0\,;$

$N_{ij}^{\mathrm{f|sw}}$: Number of failures in cells to the southwest of

$$\mathrm{cell}(i\,,\,j)\left(=\sum_{i'=0}^{i-1}\sum_{j'=0}^{j-1} N_{i'j'}^{\mathrm{f}}\right)\;;$$

$N_{ij}^{\mathrm{f|ne}}$: Number of failures in cells to the northeast of

$$\mathrm{cell}(i-1\,,\,j-1)\left(=\sum_{i'=i}^{\infty}\sum_{j'=j}^{\infty} N_{i'j'}^{\mathrm{f}}\right)\;.$$

Similarly define $N_{ij}^{\mathrm{c|ne}}$ and $N_{ij}^{\mathrm{c|sw}}$ for censored data.

A nonparametric estimator of the hazard function is

$$\hat{h}_{ij} = \frac{N_{ij}^{\mathrm{f}}}{N_{ij}^{\mathrm{f|ne}} + N_{ij}^{\mathrm{c|ne}}}\,,\, i \geq 0\,,\, j \geq 0\,. \tag{5.46}$$

Plot of Renewal Function (Nonparametric Approach). A simple estimator of the renewal function in the case of complete data is given by the partial mean function over the cells; that is,

$$\hat{M}(t_i\,,\,x_i) = \frac{N_{ij}^{\mathrm{f|sw}}}{N}\,, \tag{5.47}$$

where N is the total number of observations. A contour plot of this versus t and x can then be obtained.

5.4.4 Model Selection

To determine if the estimate of the renewal function above corresponds to the renewal function for a given model, plots similar to quantile–quantile plots can be investigated. Firstly, the renewal function for the given model is estimated and then its values are plotted against the corresponding values of the nonparametric estimator above. If a (rough) linear relationship is present, then this would be indicative that the model is reasonable.

5.4.5 Parameter Estimation and Validation

Once an appropriate model for $h(t\,,\,x) = f(t\,,\,x)/\bar{F}(t\,,\,x)$ is determined, estimation of the parameters can be carried out using standard statistical procedures (least squares, maximum likelihood, and so on) in a similar fashion to the one-dimensional case, although we are unaware of any equivalent graphical methods which may be used. Similarly, model validation can be carried out as before. It should be noted that the procedure indicated in the previous section can also be used to validate the model, if not used to select it. In fact, a common approach when faced with a complex model may be to fit the model using an estimation procedure such as maximum-likelihood estimation (or generalized least squares using an empirical version of a functional such as the renewal function), then investigating the relationship between some other functional of the model and its empirical version. This area requires further investigation.

5.5 A New Approach to Two-Dimensional Modeling

One of the attractions of the one-dimensional approach taken in Sect. 5.4.1 is that it matches the manner in which the failures occur in practice; that is, the expectation is that the failure time is dependent on the amount of usage of the item—different usage leads to different distributions for the time until failure, with these distributions reflecting the ordering that higher usage leads to shorter time until failure. However, usage may vary over time for individual items and the one-dimensional approach does not allow for this aspect. The model described in this section overcomes this shortcoming by allowing usage to vary between failures.

5.5.1 Model Description

For convenience, consider a single item. Suppose that T_i is the time until the i-th failure and that X_i is the total usage at the i-th failure. Analogously to the one-dimensional approach, let $\Gamma_i = (X_i - X_{i-1})/(T_i - T_{i-1})$ be the usage rate between the $(i-1)$th and the i-th failures. Assuming that these usage rates are independent and come from a common distribution (an oversimplification but a useful starting point for developing models), the marginal distribution of the usage rates can be modeled, followed by the times until failure modeled for different usage rates after each failure in

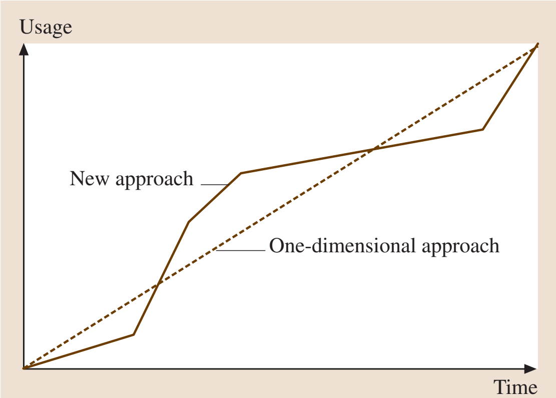

Fig. 5.8 Plot of usage versus time

a similar manner to the one-dimensional approach. This approach combines the two approaches discussed earlier. Figure 5.8 shows the plots of usage versus time for the proposed model and the model based on the one-dimensional approach.

Note that a more general approach is to model the usage as a cumulative stochastic process. This approach has received some attention. See *Lawless* et al. [5.24] and *Finkelstein* [5.40] and the references cited therein for further details.

5.5.2 An Application

We illustrate by considering the failure and usage data for component C-1 over the warranty period.

Modeling the Usage Rates

Before investigating the relationship between usage rate and time to failure, it is worthwhile investigating the days to failure and usage at failure for claims made within the warranty period. This is shown in Fig. 5.9. Only three of the failures were a second failure on the component; all of the others are the first failure since manufacture. There are three considerations to take into account when interpreting Fig. 5.9.

Firstly, the censoring by both time and usage ensures that only the initial part of the bivariate distribution of usage at failure and time to failure can be explored and the relationship between usage at failure and time to failure is distorted. Secondly, the proportions of components according to usage rate vary considerably, with very few components having high or low usage rates (as would be expected). Lastly, there are a greater number of short failure times than might be expected, suggesting that many early failures may not be related to usage and are more likely to be the result of quality-related problems during production. What seems like a linear trend (around the line *Usage* = 50× *Days*) is not valid in light of the above discussion.

Figure 5.10 looks at the conditional distribution of days to failures against the usage rate (miles/day) averaged over the time before the claim. Again, care needs to be taken in interpreting this plot. In the left of this plot, censoring due to time and the low number of components having low usage rate has distorted the distribution of failure times for each usage rate. From a usage rate of around 60 km/d, the key feature of the plot is the censoring due to reaching the usage limit. Thus, although it would be expected that the failure-time distribution would be concentrated around a decreasing

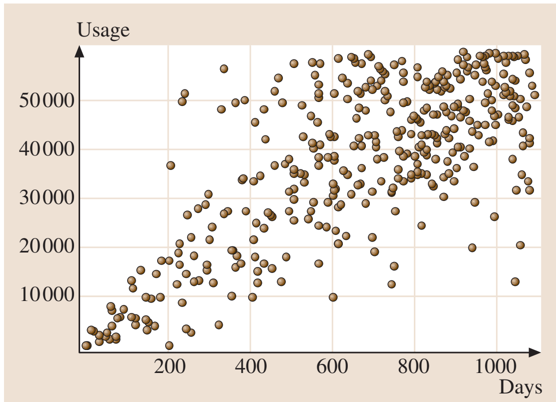

Fig. 5.9 Plot of days to failure and usage at failure for claims within warranty for component C-1

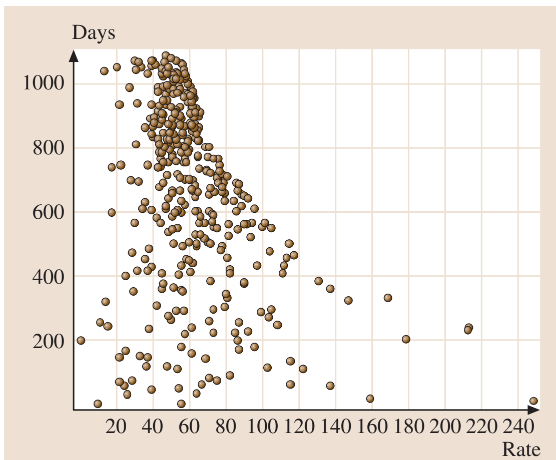

Fig. 5.10 Plot of days to failure against usage rate for component C-1

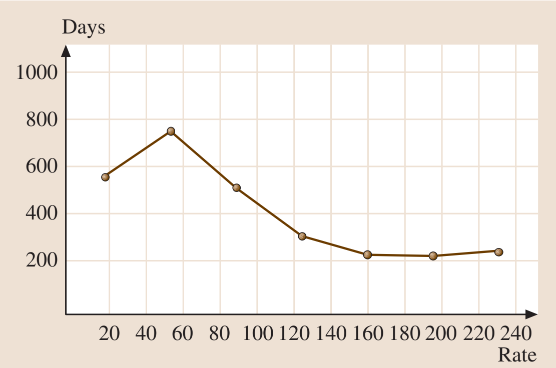

Fig. 5.11 Mean time to failure for usage-rate bands for component C-1

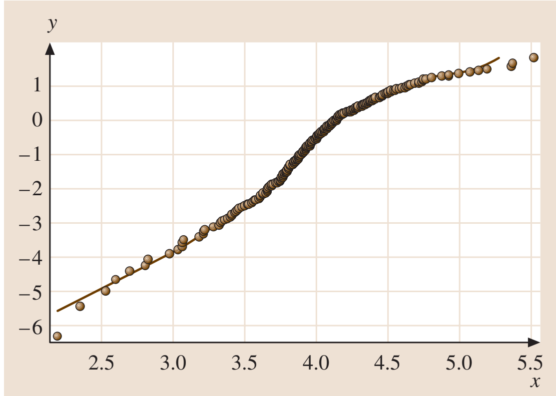

Fig. 5.12 WPP of usage rates for component C-1 with Weibull mixture

mean as usage rate increases, this is exaggerated by the censoring.

Figure 5.11 shows a plot of the mean time to failure for a split of failures into usage-rate bands. In this plot, we see the effects of the censoring, as indicated above. Thus, the mean time to failure actually increases in the low-usage-rate regime. For usage rate greater than 60 km/d, the mean time to failure decreases as the usage rate increases, as is expected (although, as discussed above, this is exaggerated by the censoring due to reaching the usage limit).

Figure 5.12 is a WPP plot of the usage rate. The plot indicates that a Weibull mixture involving two subpopulations is appropriate to model the usage rate. The parameter estimates of the fitted curve in Fig. 5.12 are

$$\hat{p} = 0.647\,,\ \hat{\beta}_1 = 5.50\,,\ \hat{\beta}_2 = 1.99\,,$$
$$\hat{\eta}_1 = 57.7\,,\ \hat{\eta}_2 = 75.7\,.$$

These give mean usage rates to failure of 53.3 km/d and 75.7 km/d, respectively.

Figure 5.13 gives the empirical plot of the density function and the density function based on the mixture model. As can be seen, the model matches the data reasonably well. The model estimates that around 65% of the failures come from a subpopulation with a mean usage rate of 53.3 km/d, giving a dominant peak in the observed density. The other subpopulation, with a higher mean usage rate of 75.7 km/d, accounts for the extra failures occurring for usage rates between around 50 and 100 km/d.

For different usage rates (one for each band in Fig. 5.11) one can obtain the conditional failure distribution $F(t|r)$ in a manner similar to that in Sect. 5.3. It is important to note that this ignores the censored data and as such would yield a model that gives conservative estimates for the conditional mean time to failure. Combining this with the distribution function for the usage rate yields the two-dimensional failure model that can be used to find solutions to decision problems.

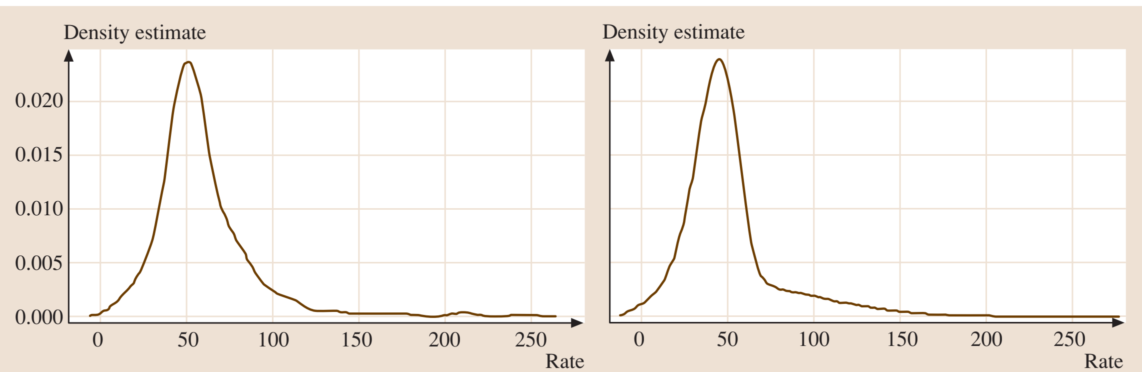

Fig. 5.13 Empirical density (*left*) and Weibull mixture density (*right*) for component C-1 usage rates

5.6 Conclusions

In this chapter we have looked at two-dimensional failure modeling. We have discussed the two approaches that have been proposed and suggested a new approach. However, there are several issues that need further study. We list these and hope that it will trigger more research in the future.

1. Different empirical plotting of two-dimensional failure data.
2. Study of models based on the two-dimensional approach and how this can be used in conjunction with the empirical plots to help in model selection.
3. Further study of the models based on the new approach discussed in Sect. 5.6.
4. Most failure data available for modeling is the data collected for products sold with two-dimensional warranties. In this case, the warranty can cease well before the time limit due to the usage limit being exceeded. This implies censored data with uncertainty in the censoring. This topic has received very little attention and raises several challenging statistical problems.

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