11. Statistical Methods for Product and Process Improvement

The first part of this chapter describes a process model and the importance of product and process improvement in industry. Six Sigma methodology is introduced as one of most successful integrated statistical tool.

Then the second section describes the basic ideas for Six Sigma methodology and the (D)MAIC(T) process for better understanding of this integrated process improvement methodology.

In the third section, "Product Specification Optimization", optimization models are developed to determine optimal specifications that minimize the total cost to both the producer and the consumer, based on the present technology and the existing process capability. The total cost consists of expected quality loss due to the variability to the consumer, and the scrap or rework cost and inspection or measurement cost to the producer. We set up the specifications and use them as a counter measure for the inspection or product disposition, only if it reduces the total cost compared with the expected quality loss without inspection. Several models are presented for various process distributions and quality loss functions.

The fourth part, "Process Optimization", demonstrates that the process can be improved during the design phase by reducing the bias or variance of the system output, that is, by changing the mean and variance of the quality characteristic of the output. Statistical methods for process optimization, such as experimental design, response surface methods, and Chebyshev's orthogonal polynomials are reviewed. Then the integrated optimization models are developed to minimize the total cost to the system of producers and customers by determining the means and variances of the controllable factors. Finally, a short summary is given to conclude this chapter.

Improving manufacturing or service processes is very important for a business to stay competitive in today's marketplace. Companies have been forced to improve their business processes because customers are always demanding better products and services. During the last 20 years, industrial organizations have become more and more interested in process improvement. Statistical methods contribute much to this activity, including design of experiments, regression analysis, response surface methodology, and their integration with optimization methods.

A process is a collection of activities that takes one or more kinds of inputs and creates a set of outputs that are of value to the customer. Everyone may be involved in various processes in their daily life, for example, ordering books from an Internet retailer, checking out in a grocery store, remodeling a home, or developing new products. A process can be graphed as shown in Fig. 11.1. The purpose of this model is to define the supplier, process inputs, the process, associated outputs, and the customer. The loops for the feedback information for continuous improvement are also shown.

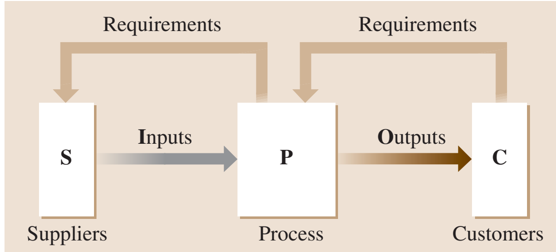


Fig. 11.1 Process model

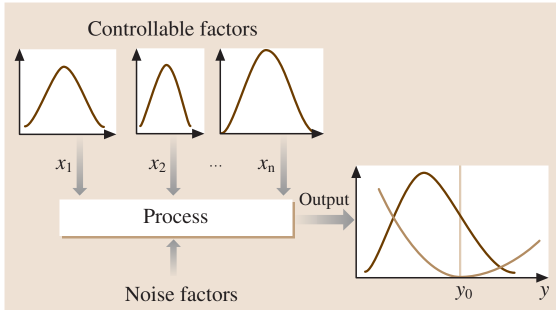


Fig. 11.2 General process with one output variable

As mentioned above, a process consists of many input variables and one or multiple output variables. The input variables include both controllable and uncontrollable or noise factors. For instance, for an electric circuit designed to obtain a target output voltage, the designer can specify the nominal values of resistors or capacitor, but he cannot control the variability of resistors or capacitors at any point in time or over the life cycle of the product. A typical process with one output variable is given in Fig. 11.2, where $X_1, X_2, \ldots, X_n$ are controllable variables and y is the realization of the random output variable Y.

Many companies have implemented continuous process improvement with Six Sigma methodology, such as Motorola [11.1] and GE [11.2]. Six Sigma is a customer-focused, data-driven, and robust methodology that is well rooted in mathematics and statistics. A typical process for Six Sigma quality improvement has six phases: define, measure, analyze, improve, control, and technology transfer, denoted by (D)MAIC(T). The section "Six Sigma Methodology and the (D)MAIC(T) Process" introduces the basic ideas behind Six Sigma methodology and the (D)MAIC(T) process for a better understanding of this integrated process-improvement methodology.

In the section "Product Specification Optimization," we create optimization models to develop specifications that minimize the total cost to both the producer and the consumer, based on present technology and existing process capabilities. The total cost consists of expected quality loss due to the variability to the consumer and the scrap or rework cost and inspection or measurement cost to the producer. We set up the specifications and use them as a countermeasure for inspection or product disposition only if it reduces the total cost compared with the expected quality loss without inspection. Several models are presented for various process distributions and quality-loss functions.

In the section "Process Optimization," we assume that the process can be improved during the design phase by reducing the bias or variance of the system output, that is, by changing the mean and variance of the quality characteristic of the output. Statistical methods for process optimization, such as experimental design, response surface methods, and Chebyshev's orthogonal polynomials, are reviewed. Then the integrated optimization models are developed to minimize the total cost to the system of producers and customers by determining the means and variances of the controllable factors.

11.1 Six Sigma Methodology and the (D)MAIC(T) Process

The traditional evaluation of quality is based on average measures of a process/product. But customers judge the quality of process/product not only on the average, but also by the variance in each transaction or use of the product. Customers value consistent, predictable processes that deliver best-in-class levels of quality. This is what Six Sigma process strives to produce. Six Sigma methodology focuses first on reducing process variation and thus on improving the process capability.

The typical definition of a process capability index, C_{pk}, is $C_{pk} = \min((\text{USL} - \hat{\mu})/(3\hat{\sigma}), (\hat{\mu} - \text{LSL})/(3\hat{\sigma}))$, where USL is the upper specification limit, LSL is the lower specification limit, $\hat{\mu}$ is the point estimator of the mean, and $\hat{\sigma}$ is the point estimator of the standard deviation. If the process is centered at the middle of the specifications, which is also interpreted as the target value, i.e., $\hat{\mu} = (\text{USL} + \text{LSL})/(2) = y_0$, then the Six Sigma process means that $C_{pk} = 2$. In the literature, it is typically mentioned that the Six Sigma process results in 3.4 defects per million opportunities (DPMO). For this statement, we assume that the process shifts by 1.5σ over time from the target (which is assumed to be the middle point of the specifications). It implies that the realized C_{pk} is 1.5 for the Six Sigma process over time. Thus, it is obvious that 6σ requirements or C_{pk} of 1.5 is not the goal; the ideal objective is to continuously improve the process based on some economic or other higher-level objectives for the system.

At the strategic level, the goal of Six Sigma is to align an organization to its marketplace and deliver real improvement to the bottom line. At the operational level, Six Sigma strives to move product or process characteristics within the specifications required by customers, shrink process variation to the six sigma level, and reduce the cause of defects that negatively affect quality [11.3].

Six Sigma continuous improvement is a rigorous, data-driven, decision-making approach to analyzing the root causes of problems and improve the process capability to the six sigma level. It utilizes a systematic six-phase, problem-solving process called (D)MAIC(T): define, measure, analyze, improve, control, and technology transfer. Traditionally, a four-step process, MAIC, is often referred to as a general process for Six Sigma process improvement in the literature. We extend it to the six-step process, (D)MAIC(T). We want to emphasize the importance of the define (D) phase as the first phase for the problem definition and project selection, and we want to highlight technology transfer (T) as the never-ending phase for continuous applications of Six Sigma technology to other parts of the organization. The process of (D)MAIC(T) stays on track by establishing deliverables for each phase, by creating engineering models over time to reduce process variation, and by continuously improving the predictability of system performance. Each of the six phases in the (D)MAIC(T) process is critical to achieving success.

11.1.1 Define: What Problem Needs to Be Solved?

It is important to define the scope, expectations, resources, and timelines for the selected project. The definition phase for the Six Sigma approach identifies the specific scope of the project, defines the customer and critical-to-quality (CTQ) issues from the viewpoint of the customer, and develops the core processes.

11.1.2 Measure: What Is the Current Capability of the Process?

Design for Six Sigma is a data-driven approach that requires quantifying and benchmarking the process using actual data. In this phase, the performance or process capability of the process for the CTQ characteristics are evaluated.

11.1.3 Analyze: What Are the Root Causes of Process Variability?

Once the project is understood and the baseline performance documented, it is time to do an analysis of the process. In this phase, the Six Sigma approach applies statistical tools to determine the root causes of problems. The objective is to understand the process at a level sufficient to be able to formulate options for improvement. We should be able to compare the various options with each other to determine the most promising alternatives. In general, during the process of analysis, we analyze the data collected and use process maps to determine root causes of defects and prioritize opportunities for improvement.

11.1.4 Improve: Improving the Process Capability

During the improvement phase of the Six Sigma approach, ideas and solutions are incorporated to initialize the change. Based on the root causes discovered and validated for the existing opportunity, the target process is improved by designing creative solutions to fix and prevent problems. Some experiments and trials may be implemented in order to find the best solution. If a mathematical model is developed, then optimization methods are utilized to determine the optimum solution.

11.1.5 Control: What Controls Can Be Put in Place to Sustain the Improvement?

The key to the overall success of the Six Sigma methodology is its sustainability, which seeks to make everything incrementally better on a continuous basis. The sum of all these incremental improvements can be quite large. Without continuous sustenance, over time things will get worse until finally it is time for another attempt at improvement. As part of the Six Sigma approach, performance-tracking mechanisms and measurements are put in place to assure that the gains made in the project are not lost over time and the process remains on the new course.

11.1.6 Technology Transfer: Where Else Can These Improvements Be Applied?

Ideas and knowledge developed in one part of an organization can be transferred to other parts of the organization. In addition, the methods and solutions developed for one product or process can be applied to other similar products or processes. Numbering by infinity, we keep on transferring technology, which is a never-ending phase for achieving Six Sigma quality. With technology transfer, the Six Sigma approach starts to create phenomenal returns.

There are many optimization problems in the six phases of this methodology. In the following sections, several statistical methods and optimization models are reviewed or developed to improve the quality of product or process to the six sigma level, utilizing the tools of probabilistic design, robust design, design of experiments, multivariable optimization, and simulation techniques. The goal is to investigate and explore the engineering, mathematical, and statistical bases of (D)MAIC(T) process.

11.2 Product Specification Optimization

For any process, strategic decisions have to be made in terms of the disposition of the output of the process, which may be some form of inspection or other countermeasures such as scrapping or reworking the output product. We may do zero inspection, 100% inspection, or use sampling inspection. Some of the problems with acceptance sampling were articulated by *Deming* [11.4], who pointed out that this procedure, while minimizing the inspection cost, does not minimize the total cost to the producer. *Orsini* [11.5] in her doctoral thesis explained how this results in a process of suboptimization.

Deming's inspection criterion indicates that inspection should be performed either 100% or not at all, depending on the total cost to the producer, which includes the cost of inspection, k_1, and the detrimental cost of letting a nonconforming item go further down into production, k_2. The criterion involves k_1, k_2, and p, the proportion of incoming nonconforming items. The break-even point is given by $k_1/k_2 = p$. If $k_1/k_2 < p$, then 100% inspection is called for; if $k_1/k_2 > p$, then no inspection is done under the assumption that the process is in a state of statistical control. The practicality and usefulness of Deming's criterion for a manufacturing company was illustrated by *Papadakis* [11.6], who formulated models to decide if we should do either 100% inspection or zero inspection based on the total cost to the producer.

Deming [11.4] also concludes that k_1 and k_2 are not the only costs to consider. As manufacturers try hard to meet or exceed customer expectations, the cost to the customers should be considered when planning for the inspection strategy. To meet the requirements of the current competitive global markets, we consider the cost to both consumers and producers, thus the total cost to the whole system in the general inspection model. If we decide to do 100% inspection, we should also know what specification limits are for the purpose of inspection, so that we can make decisions about the disposition of the output. The work done by Deming and others does not explicitly consider the specification limits for inspection and how to determine them.

In the following discussion, several economic models are proposed that not only explain when to do 100% inspection but also develop the specifications for the inspection. A general optimization model is developed to minimize the total cost to the system, including both the producer and the customer, utilizing the quality loss function based on some of the contribution of *Taguchi*'s work [11.7, 8]. In particular, the optimization models with the symmetric and asymmetric quadratic quality loss function are presented to determine the optimal process mean and specification limits for inspection.

11.2.1 Quality Loss Function

The traditional concept of conformance to specifications is a binary evaluation system (Fig. 11.3). Units that meet the specification limits are labeled "good" or "conforming," and units out of specification limits are "bad" or "nonconforming." In the traditional quality concept, quality evaluation systems focus only on the nonconforming units and cost of quality is defined as cost of nonconformance. We can easily recognize the simplicity of this binary (go/no go) evaluation system, as the quality may not differ very much between a "good" item that is just within specifications and a "bad" item that is just outside specifications.

A better evaluation system should measure the quality of all the items, both within and outside specifications. As shown in Fig. 11.4, the concept of quality loss function provides a quantitative evaluation of loss caused by functional variation. We describe the derivation of the quadratic quality loss function in what follows.

Let $L_1(y)$ be a measure of losses, disutility, failure rate, or degradation associated with the quality characteristic y. $L_1(y)$ is a differentiable function in the neighborhood of the target y_0. Using Taylor's series expansion, we have

$$L_1(y) = L_1(y_0) + L_1'(y_0)(y - y_0) + L_1''(y_0)\frac{(y-y_0)^2}{2!} + \cdots .$$

The minimum quality loss should be obtained at y_0, and hence $L_1'(y_0) = 0$. Since $L_1(y_0)$ is a constant quality loss at y_0, we define the deviation loss of y from y_0 as

$$L(y) = L_1(y) - L_1(y_0) = L_1''(y_0)\frac{(y-y_0)^2}{2!} + \cdots .$$

By ignoring the higher-order terms, $L(y)$ can be approximated using a quadratic function:

$$L(y) \approx k(y - y_0)^2 ,$$

where

$$k = \frac{L_1''(y_0)}{2} .$$

If the actual quality loss function $L(y)$ is known, we should use it instead of the approximated loss function.

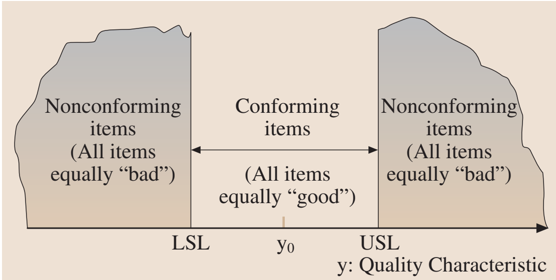


Fig. 11.3 Conformance to specifications concept of quality

Let $f(y)$ be the probability density function (pdf) of the random variable Y; then the expected loss for any given $L(y)$ is

$$\mathcal{L} = E[L(y)] = \int_{\text{all } y} L(y) f(y)\,\mathrm{d}y .$$

From this equation we can see that the expected loss depends heavily on the distribution of Y. To reduce the expected quality loss, we need to improve the distribution of Y, not just reduce the number of items outside specification limits. It is quite different from the traditional evaluation policy, which only measures the cost incurred by nonconforming quality characteristics. In the following sections, different quality loss functions are discussed for different types of quality characteristics.

"The Smaller the Better" Quality Characteristics

The objective is to reduce the value of the quality characteristic. Usually the smallest possible value for such characteristics is zero, and thus $y_0 = 0$ is the "ideal" or target value, as shown in Fig. 11.5. Some examples are wear, degradation, deterioration, shrinkage, noise

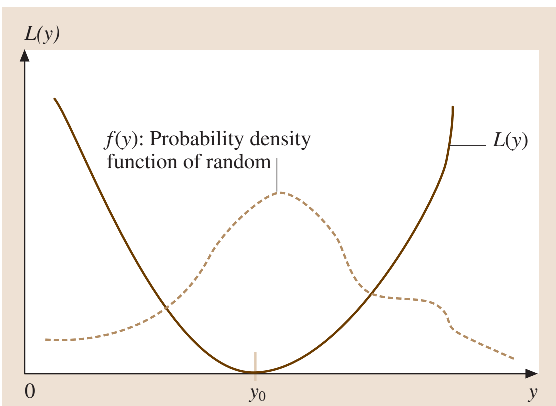


Fig. 11.4 Quality loss function $L(y)$

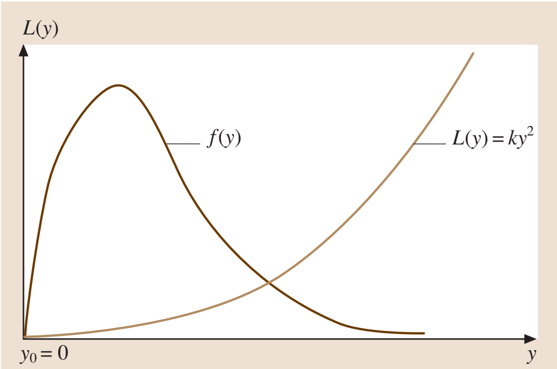


Fig. 11.5 "The smaller the better" quality characteristics

level, harmful effects, level of pollutants, etc. For such characteristics, engineers generally have an upper specification limit (USL). A good approximation of $L(y)$ is $L(y) = ky^2$, $y \geq 0$.

The expected quality loss is calculated as

$$\begin{aligned}\mathcal{L} &= E[L(y)] \\ &= \int\limits_{\text{all } y} L(y) f(y) \mathrm{d}y \\ &= \int\limits_{\text{all } y} ky^2 f(y) \mathrm{d}y \\ &= \int\limits_{\text{all } y} k\left[(y-\mu)^2 + 2(y-\mu)\mu + \mu^2\right] f(y) \mathrm{d}y \\ &= k\left(\sigma^2 + \mu^2\right) .\end{aligned}$$

To reduce the loss, we must reduce the mean μ and the variance σ^2 simultaneously.

"The Larger the Better" Quality Characteristics

For such quality characteristics, we want to increase their value as much as possible (within a given frame of reference), as shown in Fig. 11.6. Some examples are strength, life of a system (a measure of reliability), fuel efficiency, etc. An ideal value may be infinity, though impossible to achieve. For such characteristics, engineers generally have a lower specification limit (LSL). A good approximation of $L(y)$ is

$$L(y) = \frac{k}{y^2}, \quad y \geq 0 .$$

The expected quality loss is given by

$$\mathcal{L} = E[L(y)] = \int\limits_{\text{all } y} L(y) f(y) \mathrm{d}y = \int\limits_{\text{all } y} \frac{k}{y^2} f(y) \mathrm{d}y .$$

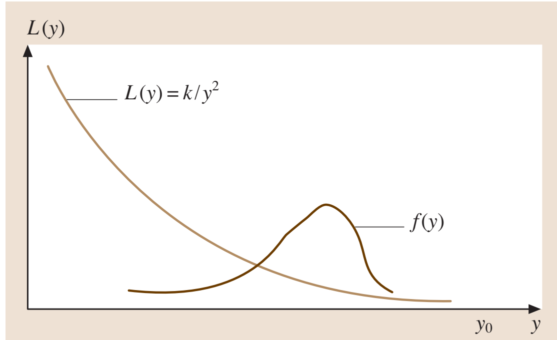


Fig. 11.6 "The larger the better" quality characteristics

Using Taylor's series expansion for $1/y^2$ around μ, we have

$$\begin{aligned}\frac{1}{y^2} &= \mu^{-2} \\ &\quad + \left(-2y^{-3}\big|_{\mu}\right)(y-\mu) \\ &\quad + 6y^{-4}\big|_{\mu} \frac{(y-\mu)^2}{2!} \\ &\quad + \cdots .\end{aligned}$$

By ignoring higher-order terms, we have

$$\frac{1}{y^2} \approx \frac{1}{\mu^2} + \frac{2}{\mu^3}(y-\mu) + \frac{3}{\mu^4}(y-\mu)^2 .$$

Finally, we have

$$\begin{aligned}E[L(y)] &\approx k \int\limits_{\text{all } y} \left[\frac{1}{\mu^2} + \frac{2}{\mu^3}(y-\mu) \right. \\ &\qquad \left. + \frac{3}{\mu^4}(y-\mu)^2\right] f(y) \mathrm{d}y \\ &\approx k\left(\frac{1}{\mu^2} + \frac{3\sigma^2}{\mu^4}\right) .\end{aligned}$$

To reduce quality losses for the "larger the better" quality characteristics, we must increase the mean μ and reduce the variance σ^2 of Y simultaneously.

"Nominal the Best" Quality Characteristics

For such quality characteristics, we have an ideal or nominal value, as shown in Fig. 11.7. The performance of the product deteriorates as we move from each side of the nominal value. Some examples are dimensional characteristics, voltage, viscosity of a fluid, shift pressure, clearance, and so on. For such characteristics, engineers generally have both LSL and USL. An approximation of quality loss function for "nominal the best" quality characteristics is $L(y) = k(y - y_0)^2$.

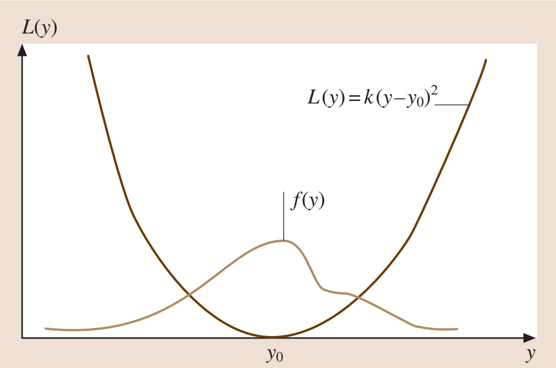


Fig. 11.7 "Nominal the best" quality characteristics

The expected quality loss is calculated as

$$\begin{aligned}\mathcal{L} = E[L(y)] &= \int_{\text{all } y} L(y)f(y)\,\mathrm{d}y \\ &= \int_{\text{all } y} k(y-y_0)^2 f(y)\,\mathrm{d}y \\ &= k\left[\sigma^2 + (\mu - y_0)^2\right] .\end{aligned}$$

Given the constant k, we must reduce bias $|\mu - y_0|$ and variance σ^2 to reduce the losses.

11.2.2 General Product Specification Optimization Model

Quality loss relates to cost or "loss" in dollars, not just to the manufacturer at the time of production, but also to the next consumer. The intangible losses (customer dissatisfaction, loss of customer loyalty, and eventual loss of market share), along with other tangible losses (rework, defects, down time, etc.), make up some of the components of the quality loss. Quality loss function is a way to measure losses due to variability from the target values and transform them to economic values. The greater the deviation from target, the greater the economic loss.

Variability means some kind of waste, but it is impossible to have zero variability. The common response has been to set not only a target level for performance but also a range of tolerance about that target, or specification limits, which represents "acceptable" performance. Thus if a quality characteristic falls anywhere within the specifications, it is regarded as acceptable, while if it falls outside that specifications, it is not acceptable. If the inspection has to be done to decide what is acceptable, we must know the specification limits. We consider the specifications not just from the viewpoint of the customer or the producer but from the viewpoint of the whole system. The issue is not only to decide when to do inspection, but also to decide what specifications will be applied for the inspection.

Suppose a process has been improved to its optimal capability using the present technology; then we consider the following two questions:

Question 1: Should we perform 100% inspection or zero inspection before shipping the output to the next or downstream customers?

Question 2: If 100% inspection is to be performed, how do we determine the optimal specification limits that minimize the total cost to the system, which includes both producers and consumers?

To answer the above two questions, the decision maker has to choose between the following two decisions:

Decision 1: No inspection is done, and thus we ship the whole distribution of the output to the next customer. One economic interpretation of cost to the downstream customers is the expected quality loss.

Decision 2: Do 100% inspection. It is clear that we will do the inspection and truncate the tails of the distribution only if it reduces total cost to both the producer and the consumer. If we have some arbitrary specification limits, we may very well increase the total cost by doing inspection. When we truncate the distribution by using certain specification limits, some additional costs will be incurred, such as the measurement or inspection cost (to evaluate if units meet the specifications), the rework cost, and the scrap cost. The general optimization model is

$$\text{Minimize ETC} = \text{EQL} + \text{ESC} + \text{IC} ,$$

where

ETC = Expected total cost per produced unit
EQL = Expected quality loss per unit
ESC = Expected scrap cost per unit
IC = Inspection cost per unit

and where the specification limits are the decision variables in the optimization model. Based on this general optimization model, models have been formulated under the following assumptions:

- The nature of the quality characteristics:
  - "The smaller the better"
  - "The larger the better"
  - "Target the best"

- The nature of the underlying distributions of the output:
  - Normal distribution
  - Lognormal distribution
  - Weibull distribution
- The relationship between the process mean and the target value:
  - The process mean is centered at the target: $\mu = y_0$
  - The process mean is not centered at the target: $\mu \neq y_0$
- The shape of the quality loss function:
  - Symmetric
  - Asymmetric
- The number of quality characteristics:
  - Single quality characteristic
  - Multiple quality characteristics

Kapur [11.9], *Kapur* and *Wang* [11.10], *Kapur* and *Cho* [11.11], and *Kapur* and *Cho* [11.12] have developed several models for various quality characteristics and illustrated the models with several numerical problems. *Kapur* and *Wang* [11.10] and *Kapur* [11.13] considered the normal distribution for the "target the best" single quality characteristic to develop the specification limits based on the symmetric quality loss function and also used the lognormal distribution to develop the model for the "smaller the better" single quality characteristic. For the "smaller or larger the better" single quality characteristic, *Kapur* and *Cho* [11.11] used the Weibull distribution to approximate the underlying skewed distribution of the process, because a Weibull distribution can model various shapes of the distribution by changing the shape parameter β. *Kapur* and *Cho* [11.12] proposed an optimization model for multiple quality characteristics with the multivariate normal distribution based on the multivariate quality loss function.

In the next two subsections, two optimization models are described to determine the optimal specification limits. The first model is developed for a normal distributed quality characteristic with a symmetric quality loss function, published by *Kapur* and *Wang* [11.10] and *Kapur* [11.13]. The second model is formulated for a normal distributed quality characteristic with an asymmetric quality loss function, proposed by *Kapur* and *Feng* [11.14].

11.2.3 Optimization Model with Symmetric Loss Function

We summarize the basic assumptions presented in *Kapur* and *Wang* [11.10] and *Kapur* [11.13] as below:

- The single quality characteristic is "target the best," and the target is y_0.
- The process follows a normal distribution: $Y \propto N(\mu, \sigma^2)$.
- The process mean is centered at the target: $\mu = y_0$.
- The quality loss function is symmetric about the target y_0 and given as $L(y) = k(y - y_0)^2$.

Based on these assumptions, the expected quality loss without inspection is calculated as:

$$\begin{aligned}\mathcal{L} = E[L(Y)] &= \int_{-\infty}^{\infty} k(y-y_0)^2 f(y)\,\mathrm{d}y \\ &= k\left\{[E(Y)-y_0]^2 + Var(Y)\right\} \\ &= k\left[\sigma^2 + (y_0-\mu)^2\right] .\end{aligned}$$

After setting the process mean at the target, $\mu = y_0$, the expected loss only has the variance term, which is $\mathcal{L} = k\sigma^2$.

If we do 100% inspection, we will truncate the tails of the distribution at specification limits, which should be symmetric about the target:

$$\begin{aligned}\mathrm{LSL} &= \mu - n\sigma ,\\ \mathrm{USL} &= \mu + n\sigma .\end{aligned}$$

In order to optimize the model, we need to determine the variance of the truncated normal distribution (the distribution of the units shipped to the customer), which is $\mathrm{V}(Y_\mathrm{T})$. Let $f_\mathrm{T}(y_\mathrm{T})$ be the probability density function for the truncated random variable Y_T; then we have

$$f_\mathrm{T}(y_\mathrm{T}) = \frac{1}{q} f(y_\mathrm{T}) = \frac{1}{q\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{(y_\mathrm{T}-\mu)^2}{2\sigma^2}} ,$$

where

$$\begin{aligned}q &= 2\Phi(n) - 1 \\ &= \text{fraction of units shipped to customers} \\ &\quad \text{or area under normal distribution within} \\ &\quad \text{specification limits}\end{aligned}$$

and

$$\mu - n\sigma \leq y_\mathrm{T} \leq \mu + n\sigma ,$$

where $\phi(\cdot)$ is the pdf for the standard normal variable and $\Phi(\cdot)$ is the cdf for the standard normal variable. From the probability density function (pdf) we can derive the mean and variance of the truncated normal distribution as

$$E(Y_T) = \mu ,$$
$$V(Y_T) = \sigma^2 \left[1 - \frac{2n}{2\Phi(n) - 1}\phi(n)\right] .$$

It is clear that the quantity of $V(Y_T)$ is less than σ^2, which means that we reduce the variance of units shipped to the customer (Y_T). Then the expected quality loss, $\mathcal{L}_T$, for the truncated distribution is

$$\mathcal{L}_T = k\left\{[E(Y_T) - y_0]^2 + V(Y_T)\right\} = kV(Y_T) .$$

Then the expected quality loss per unit EQL is $q\mathcal{L}_T$, because the fraction of units shipped to customers is q. Given k, SC, and IC, we have the optimization model with only one decision variable n as

$$\text{Minimize ETC} = q\mathcal{L}_T + (1-q)\text{SC} + \text{IC} ,$$
$$\text{subject to } \mathcal{L}_T = k\sigma^2\left[1 - \frac{2n}{2\Phi(n)-1}\phi(n)\right] ,$$
$$q = 2\Phi(n) - 1 ,$$
$$n \geq 0 .$$

The above objective function is unimodal and differentiable, and hence the optimal solution can be found by differentiating the objective function with respect to n and setting it equal to zero. Thus we solve $(\partial \text{ETC}/\partial n) = 0$, and the solution is $n^* = \sqrt{\text{SC}/(k\sigma^2)}$.

Let us now consider an example for a normal process with $\mu = 10$, $\sigma = 0.50$, $y_0 = 10$, $k = 5$, IC = \$0.10, and SC = \$2.00.

Decision 1: If we do not conduct any inspection, the total expected quality loss per unit is calculated as $\text{TC} = \mathcal{L} = k\sigma^2 = 5 \times 0.50^2 = \$1.25$.

Decision 2: Let us determine the specification limits that will minimize the total expected cost by using the following optimization model:

$$\begin{aligned}\text{Minimize ETC} &= q\mathcal{L}_T + (1-q)\text{SC} + \text{IC} \\ &= 5 \times 0.5^2\,[2\Phi(n) - 1 - 2n\phi(n)] \\ &\quad + [2 - 2\Phi(n)] \times 2.00 + 0.10 \\ \text{subject to } & n \geq 0 .\end{aligned}$$

The optimal solution is given by $n^* = \sqrt{\text{SC}/(k\sigma^2)} = 1.26$, and $\text{ETC}^* = \$0.94 < \$1.25$. Thus, the optimal strategy is to have LSL = 9.37 and USL = 10.63, and do 100% inspection to screen the nonconforming units. The above model presents a way to develop optimum specification limits by minimizing the total cost. Also, Fig. 11.8 gives the relationship between the expected total cost ETC and n, where we can easily observe that the minimum is when $n = 1.26$.

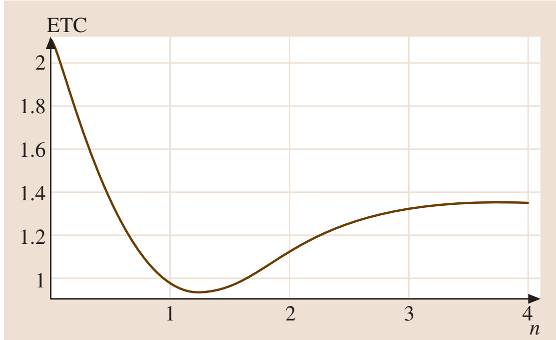


Fig. 11.8 Expected total cost vs. n

In addition to the above model for the "target the best" quality characteristic, *Kapur* and *Wang* [11.10] used the lognormal distribution to develop a model for the "smaller the better" quality characteristic. For the "smaller or larger the better" quality characteristic, *Kapur* and *Cho* [11.11] used the Weibull distribution to approximate the underlying skewed distribution of the process because a Weibull distribution can model various shapes of the distribution by changing the shape parameter β.

11.2.4 Optimization Model with Asymmetric Loss Function

The following assumptions are presented to formulate this optimization model [11.14]:

- The single quality characteristic is "target the best," and the target is y_0.
- The process follows a normal distribution: $Y \approx N(\mu, \sigma^2)$, and the probability density function of Y is $f(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$.
- The mean of the process can be easily adjusted, but the variance is given based on the present technology or the inherent capability of the process.
- The process mean may not be centered at the target: $\mu \neq y_0$, which is a possible consequence of an asymmetric loss function.
- The quality loss function is asymmetric about the target y_0, which means that the performance of the product deteriorates in the different ways as the

quality characteristic deviates to either side of the target value. An asymmetric quality loss function is given as:

$$\begin{cases} k_1(y-y_0)^2, & y \le y_0 , \\ k_2(y-y_0)^2, & y > y_0 . \end{cases}$$

Based on these assumptions, if we ship the whole distribution of the output to the next customer as for Decision 1, the total cost is just the expected quality loss to the customer. We can prove that the expected quality loss without truncating the distribution is:

$$\begin{aligned} \mathrm{ETC}_1 = \mathcal{L} = & \int_{-\infty}^{y_0} k_1(y-y_0)^2 f(y)\,\mathrm{d}y \\ & + \int_{y_0}^{\infty} k_2(y-y_0)^2 f(y)\,\mathrm{d}y \\ = & (k_1-k_2)\sigma(y_0-\mu)\phi\left(\frac{y_0-\mu}{\sigma}\right) \\ & + \left[\sigma^2+(y_0-\mu)^2\right] \\ & \times \left[(k_1-k_2)\Phi\left(\frac{y_0-\mu}{\sigma}\right)+k_2\right] , \end{aligned}$$

where $\phi(\cdot)$ is the pdf for the standard normal variable and $\Phi(\cdot)$ is the cdf for the standard normal variable. Given k_1, k_2, and y_0 and the standard deviation σ, the total cost or the expected quality loss to the customer in this case should be minimized by finding the optimal process mean μ^*. The optimization model for Decision 1 is:

$$\begin{aligned} \text{Minimize } \mathrm{ETC}_1 = & (k_1-k_2)\sigma(y_0-\mu)\phi\left(\frac{y_0-\mu}{\sigma}\right) \\ & + \left[\sigma^2+(y_0-\mu)^2\right] \\ & \times \left[(k_1-k_2)\Phi\left(\frac{y_0-\mu}{\sigma}\right)+k_2\right] \\ & \text{subject to } \mu \in \mathbb{R} . \end{aligned}$$

Given k_1, k_2, y_0, and σ, ETC_1 or $\mathcal{L}$ is a convex differential function of μ, because the second derivative $\frac{\mathrm{d}^2\mathcal{L}}{\mathrm{d}\mu^2} > 0$. We know that a convex differential function obtains its global minimum at $\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\mu} = 0$, which is given by

$$\begin{aligned} \frac{\mathrm{d}\mathcal{L}}{\mathrm{d}\mu} = & 2(k_2-k_1)\left[\sigma^2 f(y_0)+(\mu-y_0)\int_{y_0}^{\mu} f(y)\,\mathrm{d}y\right] \\ & + (k_1+k_2)(\mu-y_0) \\ = & 0 . \end{aligned} \tag{11.1}$$

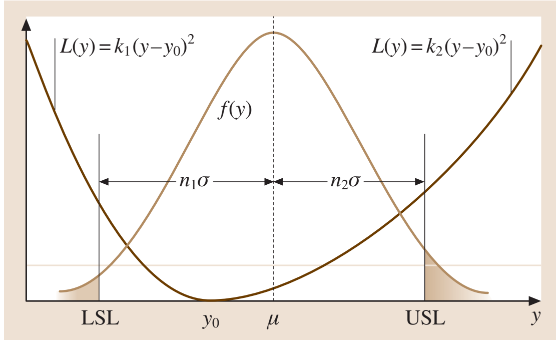


Fig. 11.9 Optimization model with asymmetric loss function

Thus, the optimal value of the process mean μ^* is obtained by solving the above equation of μ. Since the root of (11.1) cannot be found explicitly, we can use Newton's method to search the numerical solution by Mathematica.

If we do the 100% inspection as for Decision 2, we should truncate the tails of the distribution at asymmetric specification limits as shown in Fig. 11.9, where

$$\begin{aligned} \mathrm{LSL} &= \mu - n_1\sigma , \\ \mathrm{USL} &= \mu + n_2\sigma . \end{aligned}$$

Let $f_{\mathrm{T}}(y_{\mathrm{T}})$ be the probability density function for the truncated random variable Y_{T}; then we have

$$\begin{aligned} & f_{\mathrm{T}}(y_{\mathrm{T}}) = \frac{1}{q} f(y_{\mathrm{T}}) = \frac{1}{q\sigma\sqrt{2\pi}}\mathrm{e}^{-\frac{(y_{\mathrm{T}}-\mu)^2}{2\sigma^2}} , \\ & \text{where} \quad q = \Phi(n_1)+\Phi(n_2)-1 , \\ & \text{and} \quad \mu - n_1\sigma \le y_{\mathrm{T}} \le \mu + n_2\sigma . \end{aligned}$$

Using the above information, we can prove that the expected quality loss for the truncation distribution is:

$$\begin{aligned} \mathcal{L}_{\mathrm{T}} = & \frac{1}{q}\{k_1\sigma\,[2(\mu-y_0)-n_1\sigma]\,\phi(n_1) \\ & + k_2\sigma\,[2(y_0-\mu)-n_2\sigma]\,\phi(n_2)\} \\ & + \frac{1}{q}\left\{\sigma(y_0-\mu)(k_1-k_2)\phi\left(\frac{y_0-\mu}{\sigma}\right)\right. \\ & \left. + (k_1-k_2)\left[\sigma^2+(y_0-\mu)^2\right]\Phi\left(\frac{y_0-\mu}{\sigma}\right)\right\} \\ & + \frac{1}{q}\left\{\left[\sigma^2+(y_0-\mu)^2\right]\right. \\ & \left. \times [k_1\Phi(n_1)+k_2\Phi(n_2)-k_1]\right\} . \end{aligned}$$

Then the expected quality loss per unit EQL is $q\mathcal{L}_{\mathrm{T}}$, because the fraction of units shipped to customers is q. If k_1, k_2, y_0, ESC, and IC are given, we can minimize ETC_2 to find the optimal value of n_1, n_2, and the process mean value μ. The optimization model for Decision 2 is

$$\begin{aligned}
\text{Min ETC}_2 &= q\mathcal{L}_{\mathrm{T}} + (1-q)\mathrm{SC} + \mathrm{IC}\\
&= \Bigg\{ k_1\sigma[2(\mu-y_0) - n_1\sigma]\phi(n_1)\\
&\quad + k_2\sigma[2(y_0-\mu) - n_2\sigma]\phi(n_2)\\
&\quad + \sigma(y_0-\mu)(k_1-k_2)\phi\left(\frac{y_0-\mu}{\sigma}\right)\\
&\quad + (k_1-k_2)\left[\sigma^2 + (y_0-\mu)^2\right]\\
&\quad \times \Phi\left(\frac{y_0-\mu}{\sigma}\right)\\
&\quad + \left[\sigma^2 + (y_0-\mu)^2\right]\\
&\quad \times [k_1\Phi(n_1) + k_2\Phi(n_2) - k_1]\Bigg\}\\
&\quad + [2 - \Phi(n_1) - \Phi(n_2)]\,\mathrm{SC} + \mathrm{IC}\,.
\end{aligned}$$

To choose from the alternative decisions, we should optimize the model for Decision 1 with zero inspection first and have the minimum expected total cost ETC_1^*. Then we optimize the model for Decision 2 with 100% inspection and have the optimal expected total cost ETC_2^*. If $\mathrm{ETC}_1^* < \mathrm{ETC}_2^*$, we should adjust the process mean to the optimal mean value given by the solutions and then ship all the output to the next or downstream customers without any inspection because the total cost to the system will be minimized in this way. Otherwise, we should take Decision 2, adjust the process mean, and do 100% inspection at the optimal specification limits given by the solutions of the optimization model.

For example, we need to make decisions in terms of the disposition of the output of a process that has the following parameters: the output of the process has a target value $y_0 = 10$; the quality loss function is asymmetrical about the target with $k_1 = 10$ and $k_2 = 5$, based on the input from the customer; the distribution of the process follows a normal distribution with $\sigma = 1.0$; the inspection cost per unit is $\mathrm{IC} = \$0.10$, and the scrap cost per unit is $\mathrm{SC} = \$4.00$. Should we do 100% inspection or zero inspection? If 100% inspection is to be done, what specification limits should be used?

First, we minimize the optimization model for Decision 1:

$$\begin{aligned}
\text{Min ETC}_1 &= (k_1-k_2)\sigma(y_0-\mu)\phi\left(\frac{y_0-\mu}{\sigma}\right)\\
&\quad + \left[\sigma^2 + (y_0-\mu)^2\right]\\
&\quad \times \left[(k_1-k_2)\Phi\left(\frac{y_0-\mu}{\sigma}\right) + k_2\right]\\
&= 5(10-\mu)\phi\,(10-\mu)\\
&\quad + \left[1 + (10-\mu)^2\right]\\
&\quad \times [5\Phi\,(10-\mu) + 5]
\end{aligned}$$
subject to $\mu \geq 0$.

Using Mathematica to solve the equation with the given set of parameters, we have the optimal solution $\mu^* = 10.28$, and $\mathrm{ETC}_1^* = \$6.96$. Also, Genetic Algorithm by *Houck* et al. [11.15] gives us the same optimal solution.

Then, we optimize the model for Decision 2 given by

$$\begin{aligned}
\text{Min ETC}_2 &= \Big\{(20\mu - 10n_1 - 200)\phi(n_1)\\
&\quad + (100 - 10\mu - 5n_2)\phi(n_2)\\
&\quad + (50 - 5\mu)\phi\,(10-\mu)\\
&\quad + \left[5 + 5(10-\mu)^2\right]\Phi\,(10-\mu)\\
&\quad + \left[1 + (10-\mu)^2\right]\\
&\quad \times [10\Phi(n_1) + 5\Phi(n_2) - 10]\Big\}\\
&\quad + 4\,[2 - \Phi(n_1) - \Phi(n_2)] + 0.1
\end{aligned}$$
subject to $n_1 \geq 0,\; n_2 \geq 0$.

This can be minimized using Genetic Algorithm provided by *Houck* et al. [11.15], which gives us $n_1^* = 0.72,\; n_2^* = 0.82,\; \mu^* = 10.08$, and $\mathrm{TC}^* = \$2.57 < \$6.96$.

Since $\mathrm{ETC}_1^* > \mathrm{ETC}_2^*$, we should adjust the process mean to 10.08 given by the optimal solution from Decision 2 and do 100% inspection with respect to $\mathrm{LSL} = 9.36$ and $\mathrm{USL} = 10.90$ to screen the nonconforming units. In this way, the expected total cost to the whole system will result in a reduction of \$4.39, or 63% decrease in ETC. This example presents a way to determine the optimal process mean value and specification limits by minimizing the total cost to both producer and consumer.

11.3 Process Optimization

In the previous section, it is assumed that it is difficult to improve the process because of the constraint of the current technology, cost, or capability. To improve the performance of the output, we screen or inspect the product before shipping to the customer by setting up optimal specification limits on the distribution of the output. Thus the focus is on inspection of the product. To further optimize the performance of the system, it is supposed that the process can be improved during the design phase, which is also called offline quality engineering. Then the process should be designed and optimized with any effort to meet the requirements of customers economically. During offline quality engineering, three design phases need to be taken [11.7]:

- ***System design***: The process is selected from knowledge of the pertinent technology. After system design, it is often the case that the exact functional relationship between the output variables and input variables cannot be expressed analytically. One needs to explore the functional relationship of the system empirically. Design of experiments is an important tool to derive this system transfer function. Orthogonal polynomial expansion also provides an effective means of evaluating the influences of input variables on the output response.
- ***Parameter design***: The optimal settings of input variables are determined to optimize the output variable by reducing the influence of noise factors. This phase of design makes effective use of experimental design and response surface methods.
- ***Tolerance design***: The tolerances or variances of the input variables are set to minimize the variance of output response by directly removing the variation causes. It is usually true that a narrower tolerance corresponds to higher cost. Thus cost and loss due to variability should be carefully evaluated to determine the variances of input variables. Experimental design and response surface methods can be used in this phase.

In the following sections, the statistical methods involved in the three design phases are reviewed, including experimental design method, orthogonal polynomial expansion, and response surface method. Since the ultimate goal is to minimize the total cost to both producers and consumers, or the whole system, some integrated optimization models are developed from the system point of view.

11.3.1 Design of Experiments

Introduction to Design of Experiments

Experiments are typically operations on natural entities and processes to discover their structure, functioning, or relationships. They are an important part of the scientific method, which entails observation, hypothesis, and sequential experimentation. In fact, experimental design methods provide us the tools to test the hypothesis, and thus to learn how systems or processes work.

In general, experiments are designed to study the performance of processes or systems. The process or system model can be illustrated by Fig. 11.2 as given in the introduction of this chapter. The process consists of many input variables and one or multiple output variables. The input variables include both controllable factors and uncontrollable or noise factors.

Experimental design methods have broad applications in many disciplines such as agriculture, biological and physical sciences, and design and analysis of engineering systems. They can be used to improve the performance of existing processes or systems and also to develop new ones. The applications of experimental design techniques can be found in:

- Improving process yields
- Reducing variability including both bias from target value and variance
- Evaluating the raw material or component alternatives
- Selecting of component-level settings to make the output variables robust
- Reducing the total cost to the organization and/or the customer

Procedures of Experimental Design

To use statistical methods in designing and analyzing an experiment, it is necessary for experimenters to have a clear outline of procedures as given below.

Problem Statement or Definition. A clear statement of the problem contributes substantially to better understanding the background, scope, and objective of the problem. It is usually helpful to list the specific problems that are to be solved by the experiment. Also, the physical, technological, and economic constraints should be stated to define the problem.

Selection of Response Variable. After the statement of the problem, the response variable y should be selected. Usually, the response variable is a key performance measurement of the process, or the critical-to-quality (CTQ) characteristic. It is important to have precise measures of the response variable. If at all possible, it should be a quantitative (variable) quality characteristic, which would make data analysis easier and meaningful.

Choice of Factors, Levels, and Ranges. Cause and effect diagrams should be developed by a team or panels of experts in the area. The team should represent all points of view and should also include people necessary for implementation. A brainstorming approach can be used to develop theories for the construction of cause and effect diagrams.

From the cause and effect diagrams a list of factors that affect the response variables is developed, including both qualitative and quantitative variables. Then the factors are decomposed into control factors and noise factors. Control factors are factors that are economical to control. Noise factors are uncontrollable or uneconomical to control. Three types of noise factors are outer noise, inner noise, and production noise.

The list of factors is generally very large, and the group may have to prioritize the list. The number of factors to include in the study depends on the priorities, difficulty of experimentation, and budget. The final list should include as many control factors as possible and some noise factors that tend to give high or low values of the response variable.

Once the factors have been selected, the experimenter must choose the number of levels and the range for each factor. It also depends on resource and cost considerations. Usually, factors that are expected to have a linear effect can be assigned two levels, while factors that may have a nonlinear effect should have three or more levels. The range over which the factors are varied should also be chosen carefully.

Selection of Experimental Design. The selection of experimental design depends on the number of factors, the number of levels for each factor, and the number of replicates that provides the data to estimate the experimental error variance. Also, the determination of randomization restrictions is involved, such as blocking or not. Randomization justifies the statistical inference methods of estimation and tests of hypotheses. In selecting the design, it is important to keep the experimental objectives in mind. Several books review and discuss the types of experimental designs and how to choose an appropriate experimental design for a wide variety of problems [11.16–18].

Conduction of the Experiment. Before performing the experiment, it is vital to make plans for special training if required, design data sheets, and schedule for experimentation etc. In the case of product design experimentation, sometimes the data can be collected through the use of simulation programs rather than experiments with actual hardware. Then the computer simulation models need to be developed before conducting the experiment.

When running the experiment in the laboratory or a full-scale environment, the experimenter should monitor the process on the right track, collect all the raw data, and record unexpected events.

Analysis and Interpretation of the Data. Statistical methods are involved in data analysis and interpretation to obtain objective conclusions from the experiment. There are many software packages designed to assist in data analysis, such as SAS, S-Plus, etc. The statistical data analysis can provide us with the following information:

- Which factors and interactions have significant influences on the response variable?
- What are the rankings of relative importance of main effects and interactions?
- What are the optimal factor level settings so that the response is near the target value? (parameter design)
- What are the optimal factor level settings so that the effects of the noise factors are minimized? (robust design)
- What are the best factor level settings so that the variability of the response is reduced?
- What is the functional relationship between the controllable factors and response, or what is the empirical mathematical model relating the output to the input factors?

Statistical methods lend objectivity to the decision-making process and attach a level of confidence to a statement. Usually, statistical techniques will lead to solid conclusions with engineering knowledge and common sense.

Conclusions and Recommendations. After data analysis, the experimenter should draw some conclusions and recommend an action plan. Usually, a confirmation experiment is run to verify the reproducibility of the optimum recommendation. If the result is not confirmed

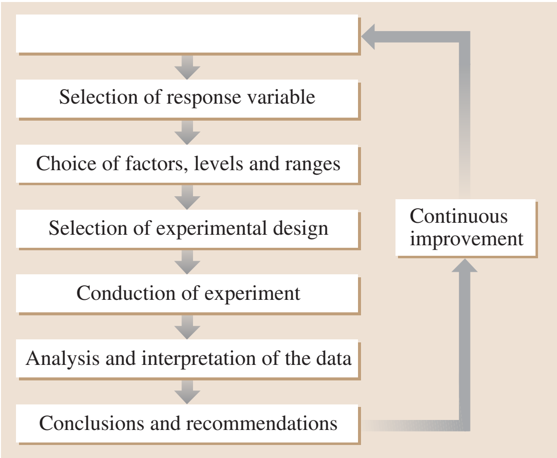


Fig. 11.10 Iterative procedures of experimental design

or is unsatisfactory, additional experimentation may be required.

Based on the results of the confirmation experiment and the previous analysis, the experimenter can develop sound conclusions and recommendations.

Continuous Improvement. The entire process is actually a learning process, where hypotheses about a problem are tentatively formulated, experiments are conducted to investigate these hypotheses, and new hypotheses are then formulated based on the experimental results. By continuous improvement, this iterative process moves us closer to the "truth" as we learn more about the system at each stage (Fig. 11.10). Statistical methods enter this process at two points: (1) selection of experimental design and (2) analysis and interpretation of the data [11.16].

11.3.2 Orthogonal Polynomials

Most research in engineering is concerned with the derivation of the unknown functional relationship between input variables and output response. In many cases, the model is often easily and elegantly constructed as a series of orthogonal polynomials [11.19–21]. Compared with other orthogonal functions, the orthogonal polynomials are particularly convenient for at least two reasons. First, polynomials are easier to work with than irrational or transcendental functions; second, the terms in orthogonal polynomials are statistically independent, which facilitates both their generation and processing. One of the other advantages of orthogonal polynomials is that users can simply develop their own system of functions in accordance with the particular problem. More often, a problem can be transformed to one of the standard families of polynomials, for which all significant relations have already been worked out.

Orthogonal polynomials can be used whether the values of controllable factors Xs are equally or unequally spaced [11.22]. However, the computation is relatively easy when the values of factor levels are in equal steps. For a system with only one equal-step input variable X, the general orthogonal polynomial model of the functional relationship between response variable Y and X is given as

$$y = \mu + \alpha_1 P_1(x) + \alpha_2 P_2(x) + \alpha_3 P_3(x) + \cdots + \alpha_n P_n(x) + \varepsilon \,, \tag{11.2}$$

where x is the value of factor level, y is the measured response [11.17], μ is the grand mean of all responses, and $P_k(x)$ is the kth-order orthogonal polynomial of factor X. The transformations for the powers of x into orthogonal polynomials $P_k(x)$ up to the cubic degree are given below:

$$\begin{aligned} P_1(x) &= \lambda_1 \left(\frac{x-\bar{x}}{d}\right) \,, \\ P_2(x) &= \lambda_2 \left[\left(\frac{x-\bar{x}}{d}\right)^2 - \left(\frac{t^2-1}{12}\right)\right] \,, \\ P_3(x) &= \lambda_3 \left[\left(\frac{x-\bar{x}}{d}\right)^3 - \left(\frac{x-\bar{x}}{d}\right)\left(\frac{3t^2-7}{20}\right)\right] \,, \end{aligned} \tag{11.3}$$

where $\bar{x}$ is the average value of factor levels, t is the number of levels of the factor, d is the distance between factor levels, and the constant λ_k makes $P_k(x)$ an integral value for each x.

Since t, d, $\bar{x}$, and x are known, $P_k(x)$ can be calculated for each x. For example, a four-level factor X ($t = 4$) can fit a third-degree equation in x. The orthogonal polynomials can be tabulated based on the calculation of (11.3) as below:

	$P_1(x)$	$P_2(x)$	$P_3(x)$
x_1	−3	1	−1
x_2	−1	−1	3
x_3	1	−1	−3
x_4	3	1	1

The values of the orthogonal polynomials $P_k(x)$ have been tabulated up to $t = 104$ [11.21].

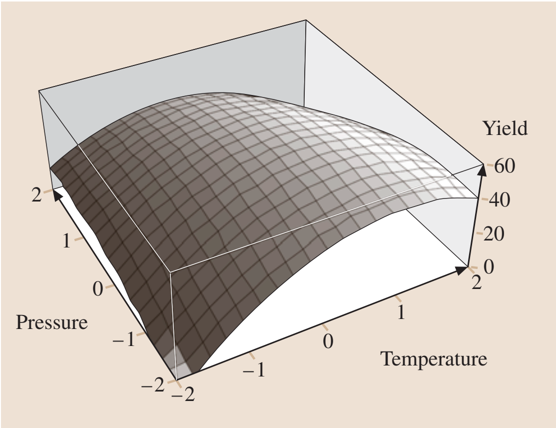
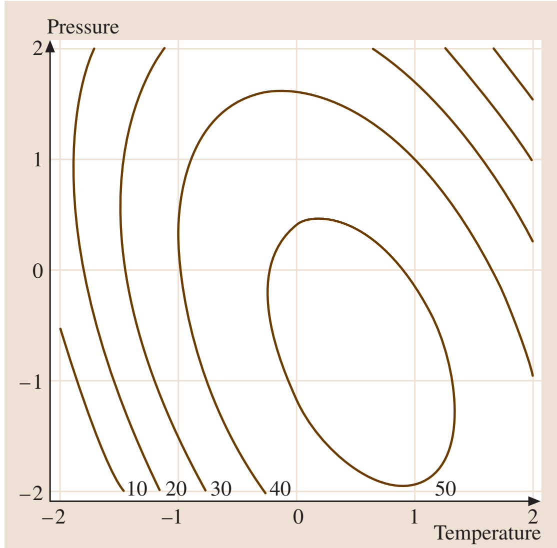


Fig. 11.11a,b Response surface (a) and contour plot (b) for a chemical process



Given the response y_i for the ith level of X, x_i, $i = 1, 2, \ldots, t$, the estimates of the α_k coefficients for the orthogonal polynomial (11.2) are calculated as

$$\alpha_k = \sum_{i=1}^{t} y_i P_k(x_i) \Big/ \sum_{i=1}^{t} P_k(x_i)^2$$

for $k = 1, 2, \ldots, n$. The estimated orthogonal polynomial equation is found by substituting the estimates of $\mu, \alpha_1, \alpha_2, \cdots, \alpha_n$ into (11.2).

It is desirable to find the degree of polynomials that adequately represents the functional relationship between the response variable and the input variables. One strategy to determine the polynomial equation is to test the significance of the terms in the sequence: linear, quadratic, cubic, and so forth. Beginning with the simplest polynomial, a more complex polynomial is constructed as the data require for adequate description. The sequence of hypotheses is $H_0: \alpha_1 = 0$, $H_0: \alpha_2 = 0$, $H_0: \alpha_3 = 0$, and so forth. These hypotheses about the orthogonal polynomials are each tested with the F test ($F = \mathrm{MSC/MSE}$) for the respective polynomial. The sum of square for each polynomial needs to be calculated for the F test, which is

$$SS_{P_k} = \left(\sum_{i=1}^{t} y_i P_k(x_i) \right)^2 \Big/ \sum_{i=1}^{t} P_k(x_i)^2$$

for $k = 1, 2, \ldots, n$. The system function relationship can be developed by including the statistically significant terms in the orthogonal polynomial model.

For the multiple equal-step input variables $X_1, X_2, \ldots, X_n$, the orthogonal polynomial equation is found in a similar manner as for the single input variable. *Kuel* [11.17] gives an example of water uptake by barley plants to illustrate procedures to formulate the functional relationship between the amount of water uptake and two controllable factors: salinity of medium and age of plant.

11.3.3 Response Surface Methodology

Response surface methodology (RSM) is a specialized experimental design technique for developing, improving, and optimizing products and processes. The method can be used in the analysis and improvement phases of the (D)MAIC(T) process. As a collection of statistical and mathematical methods, RSM consists of an experimental strategy for exploring the settings of input variables, empirical statistical modeling to develop an appropriate approximating relationship between the response and the input variables, and optimization methods for finding the levels or values of the input variables that produce desirable response values.

Figure 11.11 illustrates the graphical plot of response surface and the corresponding contour plot for a chemical process, which shows the relationship between the response variable yield and the two process variables: temperature and pressure. Thus, when the response surface is developed by the design of experiments and constructed graphically, optimization of the process becomes easy using the response surface.

The process model given in Fig. 11.2 is also very useful for RSM. Through the response surface methodology, it is desirable to make the process box "transparent" by obtaining the functional relationship between the output response and the input factors. In fact, successful use of RSM is critically dependent upon the development of a suitable response function. Usually, either a first-order or second-order model is appropriate in a relatively small region of the variable space.

In general, a first-order response model can be written as

$$Y = b_0 + b_1 X_1 + b_2 X_2 + \cdots + b_n X_n + \varepsilon \,.$$

For a system with nonlinear behavior, a second-order response model is used as given below:

$$Y = b_0 + \sum_i b_i X_i + \sum_i b_{ii} X_i^2 + \sum_i \sum_j d_{ij} X_i X_j + \varepsilon \,.$$

The method of least squares estimation is used to estimate the coefficients in the above polynomials. The second-order model is widely used in response surface methodology.

As an extended branch of experimental design, RSM has important applications in the design, development, and formulation of new products, as well as in the improvement of existing product designs. The applications of RSM can be found in many industrial settings where several variables influence the desired outcome (e.g., minimum fraction defective or maximum yield), including the semiconductor, electronic, automotive, chemical, and pharmaceutical industries.

Sequential Procedures of RSM

The applications of RSM are sequential in nature [11.23]. That is, at first we perform a screening experiment to reduce the list of candidate variables to a relatively few, so that subsequent experiments will be more efficient and require few tests. Once the important independent variables are identified, the next objective is to determine if the current levels or settings of the independent variables result in a value of the response that is near the optimum. If they are not consistent with optimum performance, a new set of adjustments to input variables should be determined to move the process toward the optimum. When the process is near the optimum, a model is needed to accurately approximate the true response function within a relatively small region around the optimum. Then, the model can be analyzed to identify the optimum conditions for the process. We can list the sequential procedures as follows [11.24]:

Step 0: *Screening experiment.* Usually the list of input variables is rather long, and it is desirable to start with a screening experiment to identify the subset of important variables. After the screening experiment, the subsequent experiments will be more efficient and require fewer runs or tests.

Step 1: *Determine if the optimal solution is located inside the current experimental region.* Once the important variables are identified through screening experiments, the experimenter's objective is to determine if the current settings of the input variables result in a value of response that is near optimum. If the current settings are not consistent with optimum performance, then go to step 2; otherwise, go to step 3.

Step 2: *Search the region that contains the optimal solution.* The experimenter must determine a set of adjustments to the process variables that will move the process toward the optimum. This phase of response surface methodology makes considerable use of the first-order model with two-level factorial experiment, and an optimization technique called the method of steepest ascent. Once the region containing the optimum solution is determined, go to step 3.

Step 3: *Establish an empirical model to approximate the true response function within a relatively small region around the optimum.* The experimenter should design and conduct a response surface experiment and then collect the experimental data to fit an empirical model. Because the true response surface usually exhibits curvature near the optimum, a nonlinear empirical model (often a second-order polynomial model) will be developed.

Step 4: *Identify the optimum solution for the process.* Optimization methods will be used to determine the optimum conditions. The techniques for the analysis of the second-order model are presented by *Myers* [11.23].

The sequential nature of response surface methodology allows the experimenter to learn about the process or system as the investigation proceeds. The investigation procedures involve several important topics/methods, including two-level factorial designs, method of steepest ascent, building an empirical model, analysis of second-order response surface, and response surface experimental designs, etc. For more detailed information, please refer to *Myers* [11.23] and *Yang* and *El-Haik* [11.24].

11.3.4 Integrated Optimization Models

The ultimate objective of Six Sigma strategy is to minimize the total cost to both producer and consumer, or the whole system. The cost to the consumer is related to the expected quality loss of the output variable, and it is caused by the deviation from the target value. The cost to the producer is associated with changing probability distributions of input variables. If the system transfer function and the variance transmission equation are available, and the cost functions for different grades of input factors are given, the general optimization model to reflect the optimization strategy is given in Fig. 11.12.

General Optimization Problem

We usually consider the first two moments of the probability distributions of input variables, and then the optimization models will focus on the mean and variance values. Therefore, the expected quality loss to the consumer consists of two parts: the bias of the process and the variance of the process. The strategy to reduce bias is to find adjustment factors that do not affect variance and thus are used to bring the mean closer to the target value. Design of experiments can be used to find these adjustment factors. It will incur certain costs to the producer. To reduce the variance of Y, the designer should reduce the variances of the input variables, which will also increase costs. The problem is to balance the reduced expected quality loss with the increased cost for the reduction of the bias and variances of the input variables. Typically, the variance control cost for the ith input variable X_i is denoted by $C_i(\sigma_i^2)$, and the mean control cost for the ith input variable X_i is denoted by $D_i(\mu_i)$. By focusing on the first two moments of the probability distributions of $X_1, X_2, \ldots, X_n$, the general optimization model is formulated as

$$\begin{aligned}\text{Minimize TC} = &\sum_{i=1}^{n} C_i\left(\sigma_i^2\right) + \sum_{i=1}^{n} D_i(\mu_i)\\ &+ k\left[\sigma_Y^2 + (\mu_Y - y_0)^2\right] ,\\ \text{subject to } &\mu_Y \approx m(\mu_1, \mu_2, \cdots, \mu_n) ,\\ &\sigma_Y^2 \approx h\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2\right) . \end{aligned} \tag{11.4}$$

In this objective function, the first two terms,

$$\sum_{i=1}^{n} C_i\left(\sigma_i^2\right) \text{ and } \sum_{i=1}^{n} D_i(\mu_i) ,$$

are the control costs on the variances and means of input variables, or the cost to the producer; the last term, $k\left[\sigma_Y^2 + (\mu_Y - y_0)^2\right]$, is the expected quality loss to the customer, where k is a constant in the quality loss function. The first constraint, $\mu_Y \approx m(\mu_1, \mu_2, \cdots, \mu_n)$, is the model for the mean of the system, which can be obtained through the system transfer function. The second constraint, $\sigma_Y^2 \approx h(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2)$, is the variance transmission equation. A future research problem is to solve this optimization problem in such a way as to consider together both the mean and the variance.

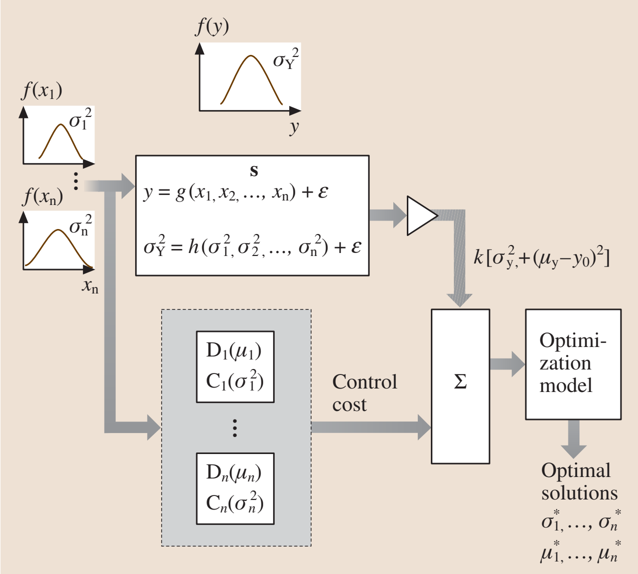


Fig. 11.12 General optimization model for system

Tolerance Design Problem

If we assume that the bias reduction has been accomplished, the general optimization problem given by (11.4) can be simplified as a tolerance design problem, which is given below:

$$\begin{aligned}\text{Minimize TC} = &\sum_{i=1}^{n} C_i\left(\sigma_i^2\right) + k\sigma_Y^2\\ \text{subject to } &\sigma_Y^2 \approx h\left(\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2\right) . \end{aligned} \tag{11.5}$$

The objective of the tolerance design is to determine the tolerances (which are related to variances) of the input variables to minimize the total cost, which consists of the expected quality loss due to variation $k\sigma_Y^2$ and the control cost on the tolerances of the input variables

$$\sum_{i=1}^{n} C_i\left(\sigma_i^2\right) .$$

Typically, $C_i(\sigma_i^2)$ is a nonincreasing function of each σ_i^2.

For this tolerance design problem, a RLC circuit example is given by *Chen* [11.25] to minimize the total cost to both the manufacturer and the consumer. Taguchi's method is used to construct the variance transmission equation as the constraint in Chen's example. *Bare* et al. [11.26] propose another optimization model to minimize the total variance control cost by finding the optimum standard deviations of input variables. Taylor's series expansion is used to develop the variance transmission equation in their model.

Case Study: Wheatstone Bridge Circuit Design

We use the Wheatstone bridge circuit design problem [11.7] as a case study to illustrate models described above [11.27]. The system transfer function is known for this example, and thus we will illustrate the development of variance transmission equation and optimization design models.

The Wheatstone bridge in Fig. 11.13 is used to determine an unknown resistance Y by adjusting a known resistance so that the measured current is zero. The resistor B is adjusted until the current X registered by the galvanometer is zero, at which point the resistance value B is read and Y is calculated from the formula $Y = BD/C$. Due to the measurement error, the current is not exactly zero, and it is assumed to be a positive or negative value of about 0.2 mA. In this case the resistance is given by the following system transfer function:

$$Y = \frac{BD}{C} - \frac{X}{C^2 E}[A(C+D)+D(B+C)] \times [B(C+D)+F(B+C)] \; .$$

The noise factors in the problem are variability of the bridge components, resistors A, C, D, F, and input voltage E. This is the case where control factors and noise factors are related to the same variables. Another noise factor is the error in reading the galvanometer X. Assuming that when the galvanometer is read as zero, there may actually be a current about 0.2 mA. Taguchi did the parameter design using L_{36} orthogonal arrays for the design of the experiment. When the parameter design cannot sufficiently reduce the effect of internal and external noises, it becomes necessary to develop the variance transmission equation and then control the variation of the major noise factors by reducing their tolerances, even though this increases the cost.

Let the nominal values or mean of control factors be the second level and the deviations due to the noise factors be the first and third level. The three levels of noise factors for the optimum combination based on parameter design are given in Table 11.1.

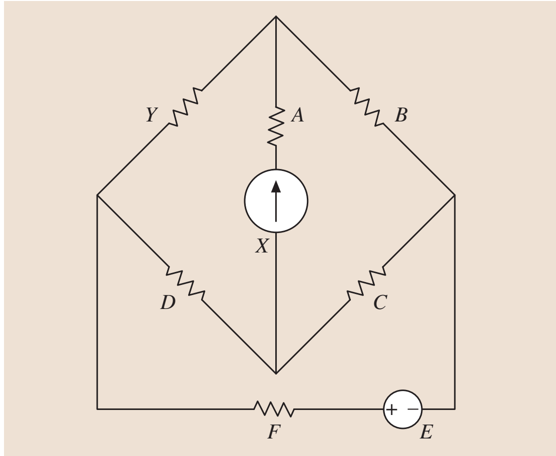


Fig. 11.13 Wheatstone bridge and parameter symbols

We use three methods to develop the variance transmission equation: Taylor series approximation, response surface method, and experimental design method. The results for various approaches are given in Table 11.2. RSM (L_{36}) and DOE (L_{36}) have the same L_{36} orthogonal array design layout for comparison purposes. Improved RSM and improved DOE use the complete design with $N = 3^7 = 2187$ design points for the unequal-mass three-level noise factors. For comparison purposes, we also perform the complete design with 2187 data points for the equal-mass three-level noise factors, which are denoted as RSM (2187) and DOE (2187) in Table 11.2. Without considering the different design layouts, it seems that the improved method gives better approximation of variance. We can see that the improved DOE's VTE does not differ much from the original one in its ability to approximate the variance of the response. Because the improved DOE method requires the complete evaluation at all combinations of levels, it is costly in terms of time and resources. If

Table 11.1 Noise factor levels for optimum combination

Factor	Level 1	Level 2	Level 3
$A(\Omega)$	19.94	20	20.06
$B(\Omega)$	9.97	10	10.03
$C(\Omega)$	49.85	50	50.15
$D(\Omega)$	9.97	10	10.03
$E(\mathrm{V})$	28.5	30	31.5
$F(\Omega)$	1.994	2	2.006
$X(\mathrm{A})$	−0.0002	0	0.0002

Table 11.2 Comparison of results from different methods

Methods	VTE	σ_Y^2
Linear Taylor	$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 276.002\,84\sigma_X^2 + O(\sigma^3)$	$7.939\,01 \times 10^{-5}$
Nonlinear Taylor	$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 276.002\,84\sigma_X^2 + 3.84 \times 10^{-6}\sigma_C^4 + O(\sigma^5)$	$7.939\,10 \times 10^{-5}$
RSM (L_{36})	$\sigma_Y^2 = 0.040\,04\sigma_B^2 + 0.001\,62\sigma_C^2 + 0.040\,36\sigma_D^2 + 300.373\,96\sigma_X^2 + 1.42 \times 10^{-8}$	$8.045\,28 \times 10^{-5}$
RSM (2187)	$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 299.598\,75\sigma_X^2 + 1.00 \times 10^{-8}$	$8.000\,03 \times 10^{-5}$
IPV RSM	$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 299.601\,30\sigma_X^2 + 1.43 \times 10^{-8}$	$8.000\,51 \times 10^{-5}$
DOE (L_{36})	$\sigma_Y^2 = 0.041\,18\sigma_B^2 + 0.001\,66\sigma_C^2 + 0.041\,50\sigma_D^2 + 308.939\,35\sigma_X^2 + 1.42 \times 10^{-8}$	$8.274\,94 \times 10^{-5}$
DOE (2187)	$\sigma_Y^2 = 0.040\,02\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,02\sigma_D^2 + 299.735\,65\sigma_X^2 + 1.00 \times 10^{-8}$	$8.004\,08 \times 10^{-5}$
IPV DOE	$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 299.768\,00\sigma_X^2 + 1.43 \times 10^{-8}$	$8.000\,95 \times 10^{-5}$
Monte Carlo	1 000 000 observations	$7.998\,60 \times 10^{-5}$

Note: The calculation of σ_Y^2 is for $\sigma_B = 0.024\,49$, $\sigma_C = 0.122\,47$, $\sigma_D = 0.024\,49$, $\sigma_X = 0.000\,16$; RSM (2187) is the response surface method applied on the same data set as Taguchi's VTE (2187); improved (IPV) RSM is the response surface method applied on the same data set as the improved (IPV) Taguchi VTE

the high cost of the complete design is a concern, the original DOE's equal-mass three-level method using orthogonal array is preferred. If the complete evaluation can be accomplished by simulation without much difficulty, the improved DOE method should be applied to ensure high accuracy. Thus, the variance transmission equation for this Wheatstone bridge circuit is determined as

$$\sigma_Y^2 = 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 + 0.040\,00\sigma_D^2 + 299.768\,00\sigma_X^2 + 1.43 \times 10^{-8} .$$

For such a problem, we can easily develop the mean model and use it with the above VTE to develop the general optimization model. It is well understood that the tolerances or variances on resistors, voltage, and current impact the cost of the design, i. e., tighter tolerances result in higher cost. Thus we can develop the variance control cost functions $C_i(\sigma_i^2)$ for each component. Similarly, the mean control cost functions $D_i(\mu_i)$ for any problem can be developed. For this problem, if the cost associated with changing the mean values is relatively small or insignificant, then we can just focus on the tolerance design problem given by (11.5), which is

$$\begin{aligned} \text{Minimize TC} &= C_B\left(\sigma_B^2\right) + C_C\left(\sigma_C^2\right) + C_D\left(\sigma_D^2\right) \\ &\quad + C_X\left(\sigma_X^2\right) + k\sigma_Y^2 , \\ \text{subject to } \sigma_Y^2 &= 0.040\,00\sigma_B^2 + 0.001\,60\sigma_C^2 \\ &\quad + 0.040\,00\sigma_D^2 + 299.768\,00\sigma_X^2 \\ &\quad + 1.43 \times 10^{-8} . \end{aligned}$$

Based on the complexity of the cost functions $C_i(\sigma_i^2)$ and $D_i(\mu_i)$ and the constraint, such optimization problems can be solved by many optimization methods including software available for global search algorithms such as genetic algorithm optimization toolbox (GAOT) for Matlab 5 (http://www.ie.ncsu.edu/mirage/GAToolBox/gaot/).

11.4 Summary

In this chapter, we first introduce the Six Sigma quality and design for Six Sigma process. By focusing on the analysis and improvement phases of the (D)MAIC(T) process, we discuss the statistical and optimization strategies for product and process optimization, respectively. Specifically, for product optimization, we review the quality loss function and various optimization models for specification limits development. For process optimization, we discuss design of experiments, orthogonal polynomials, response surface methodology, and integrated optimization models. Those statistical methods play very important roles in the activities for process and product improvement.

References

11.1 Motorola University: *Home of Six Sigma methodology and practice* ((Online) Motorola Inc. Available from: https://mu.motorola.com/, Accessed 27 May 2004)

11.2 General Electric Company: *What is Six Sigma: The Roadmap to Customer Impact* ((Online) General Electric Company. Available from: http://www.ge.com/sixsigma/, Accessed 27 May 2004)

11.3 F. W. Breyfogle: *Implementing Six Sigma: Smarter Solutions Using Statistical methods, 2nd edn.* (Wiley, New York 2003)

11.4 W. E. Deming: *Quality, Productivity, and Competitive Position* (MIT, Center for Advanced Engineering Study, Cambridge 1982)

11.5 J. Orsini: Simple rule to reduce total cost of inspection and correction of product in state of chaos, Ph.D. Dissertation, Graduate School of Business Administration, New York University (1982)

11.6 E. P. Papadakis: The Deming inspection criterion for choosing zero or 100 percent inspection, J. Qual. Technol. **17**, 121–127 (1985)

11.7 G. Taguchi: *Introduction to Quality Engineering* (Asia Productivity Organization, Tokyo 1986)

11.8 G. Taguchi: *System of Experimental Design, Volume I and II, Quality Resources* (American Supplier Institute, Deaborn, MI 1987)

11.9 K. C. Kapur: *Quality Loss Function and Inspection*, Proc. TMI Conf. Innovation in Quality (Engineering Society of Detroit, Detroit, 1987)

11.10 K. C. Kapur, D. J. Wang: *Economic Design of Specifications Based on Taguchi's Concept of Quality Loss Function*, Proc. Am. Soc. Mech. Eng. (ASME, Boston, 1987)

11.11 K. C. Kapur, B. Cho: Economic design and development of specifications, Qual. Eng. **6(3)**, 401–417 (1994)

11.12 K. C. Kapur, B. Cho: Economic design of the specification region for multiple quality characteristics, IIE Trans. **28**, 237–248 (1996)

11.13 K. C. Kapur: An approach for development of specifications for quality improvement, Qual. Eng. **1(1)**, 63–78

11.14 Q. Feng, K. C. Kapur: Economic development of specifications for 100% inspection based on asymmetric quality loss function, IIE Trans. Qual. Reliab. Eng. (2003) in press

11.15 C. R. Houck, J. A. Joines, M. G. Kay: A Genetic Algorithm for Function Optimization: A Matlab Implementation, NCSU-IE Technical Report, 95-09, 1995

11.16 C. R. Hicks, K. V. Turner: *Fundamental Concepts in the Design of Experiments, 5th edn.* (Oxford University Press, New York 1999)

11.17 R. O. Kuehl: *Statistical Principles of Research Design and Analysis* (Duxbury Press, Belmont, CA 1994)

11.18 D. C. Montgomery: *Design and Analysis of Experiments, 5th edn.* (Wiley, New York 2001)

11.19 F. S. Acton: *Analysis of Straight-Line Data* (Wiley, New York 1959)

11.20 P. Beckmann: *Orthogonal Polynomials for Engineers and Physicists* (Golem Press, Boulder, CO 1973)

11.21 F. A. Graybill: *An Introduction to Linear Statistical Models* (McGraw-Hill, New York 1961)

11.22 A. Grandage: Orthogonal coefficients for unequal intervals, query 130, Biometrics **14**, 287–289 (1958)

11.23 R. H. Myers, D. C. Montgomery: *Response Surface Methodology: Process and Product Optimization Using Designed Experiments* (Wiley, New York 2002)

11.24 K. Yang, B. El-Haik: *Design for Six Sigma: A Roadmap for Product Development* (McGraw-Hill, New York 2003)

11.25 G. Chen: Product and process design optimization by quality engineering, Ph.D. Dissertation, Wayne State University, Detroit (1990)

11.26 J. M. Bare, K. C. Kapur, Z. B. Zabinsky: Optimization methods for tolerance design using a first-order approximation for system variance, Eng. Design Autom. **2**, 203–214 (1996)

11.27 K. C. Kapur, Q. Feng: Integrated optimization models and strategies for the improvement of the six sigma process, Int. J. Six Sigma Comp. Adv. **1**(2) (2005)