20. Failure Rates in Heterogeneous Populations

Failure Rates

Most of the papers on failure rate modeling deal with homogeneous populations. Mixtures of distributions present an effective tool for modeling heterogeneity. In this chapter we consider nonasymptotic and asymptotic properties of mixture failure rates in different settings.

After a short introduction, in the first section of this chapter we show (under rather general assumptions) that the mixture failure rate is 'bent-down' compared with the corresponding unconditional expectation of the baseline failure rate, which has been proved in the literature for some specific cases. This property is due to an effect where 'the weakest populations die out first', explicitly proved mathematically in this section. This should be taken into account when analyzing failure data for heterogeneous populations in practice. We also consider the problem of mixture failure rate ordering for the ordered mixing distributions. Two types of stochastic ordering are analyzed: ordering in the likelihood ratio sense and ordering the variances when the means are equal. Mixing distributions with equal expectations and different variances can lead to corresponding ordering for mixture failure rates in $[0,\infty)$ in some specific cases. For a general mixing distribution, however, this ordering is only guaranteed for sufficiently small t.

In the second section, the concept of proportional hazards (PH) in a homogeneous population is generalized to a heterogeneous case. For each subpopulation, the PH model is assumed to exist. It is shown that this proportionality is violated for observed (mixture) failure rates. The corresponding bounds for a mixture failure rate are obtained in this case. The change point in the environment is discussed. Shocks – changing the mixing distribution – are also considered. It is shown that shocks with the stochastic properties described also bend down the initial mixture failure rate.

inally, the third section is devoted to new results on the asymptotic behavior of mixture failure rates. The suggested lifetime model generalizes all three conventional survival models (proportional hazards, additive hazards and accelerated life) and makes it possible to derive explicit asymptotic results. Some of the results obtained can be generalized to a wider class of lifetime distributions, but it appears that the class considered is 'optimal' in terms of the trade-off between the complexity of a model and the tractability (or applicability) of the results. It is shown that the mixture failure rate asymptotic behavior depends only on the behavior of a mixing distribution near to zero, and not on the whole mixing distribution.

Although most studies that model failure rates deal with homogeneous cases, homogeneous populations are rare in real life. Neglecting the existence of heterogeneity can lead to substantial errors during stochastic analysis, reliability, survival and risk analysis, and in other disciplines.

Mixtures of distributions usually present an effective approach to modeling heterogeneity. There may be a physical origin for such mixing in practice. This may happen, for instance, if different (heterogeneous) types of devices that all perform the same function, and are not distinguishable during operation, are mixed together. This occurs in real life when we have 'identical' items that originate from different brands. A similar situation arises when data from different distributions are pooled to enlarge the sample size.

It is well-known that mixtures of decreasing failure rate (DFR) distributions are always also DFR [20.1]. On the other hand, mixtures of increasing failure rate distributions (IFR) can decrease, at least over some intervals of time, which means that the IFR class of distributions is not closed under the operation of mixing [20.2]. As IFR distributions are usually used to model lifetimes governed by aging processes, this means that the operation of mixing can change the pattern of aging dramatically; for example from positive aging (IFR) to negative aging (DFR). It should be noted, however, that the change in the aging pattern usually occurs at sufficiently large item age, and so asymptotic methods are clearly important in this type of analysis. These facts and other implications of heterogeneity should be taken into account in applications.

One specific natural approach to this modeling exploits a notion of a non-negative random unobserved parameter (the frailty) Z, introduced by *Vaupel* et al. [20.3] for a gamma-distributed Z. This, in fact, can be interpreted as a subjective approach and leads to a consideration of a random failure rate $\lambda(t, Z)$. Some interesting applications of the frailty concept in survival analysis were studied by *Aalen* [20.4]. Since the failure rate is a conditional characteristic, the 'ordinary' expectation $E[\lambda(t, Z)]$ with respect to Z does not define a mixture failure rate $\lambda_{\mathrm{m}}(t)$, and a proper conditioning should be performed [20.5]. It is worth mentioning that a random failure rate is a specific case of a hazard rate process [*Kebir* [20.6] and *Yashin* and *Manton* [20.7]]. A convincing 'experiment' that shows a deceleration in the observed failure rate is performed by nature. It is well-known that human mortality follows the *Gompertz* [20.8] lifetime distribution with an exponentially increasing mortality rate. Assume that heterogeneity can be described by the proportional gamma frailty model:

$$\lambda(t, Z) = Z\alpha \exp(\beta t),$$

where α and β are positive constants. Due to its computational simplicity, the gamma frailty model is practically the only one that has been used in applications so far. It can be shown (see, e.g., *Finkelstein* and *Esaulova* [20.9]) that the mixture failure rate $\lambda_{\mathrm{m}}(t)$ in this case is monotonic in $[0, \infty)$ and asymptotically tends to a constant as $t \to \infty$. However, $\lambda_{\mathrm{m}}(t)$ monotonically increases for real values of the parameters of this model. This fact explains the recently observed deceleration in human mortality for the oldest humans (human mortality plateau, as in *Thatcher* [20.10]). A similar result has been experimentally obtained for a large cohort of medflies by *Carey* et al. [20.11]. On the other hand, in engineering applications a mixing operation can result in a failure rate that increases for $[0, t_{\mathrm{m}})$, $t_{\mathrm{m}} > 0$ and decreases asymptotically to 0 for (t_{m}, ∞), which has been experimentally observed by *Finkelstein* [20.12] for example for a heterogeneous sample of miniature light bulbs (Example 20.1). This fact is easily explained theoretically using the gamma frailty model with a baseline failure rate that increases as a power function (Weibull law) [20.9, 13].

When considering heterogeneous populations in different environments, the problem of ordering mixture failure rates for stochastically ordered random mixing variables arises. This topic has not been addressed in the literature before. In Sect. 20.1 we show that the natural type of ordering for the mixing models under consideration is ordering by likelihood ratio [20.14, 15]. This correlates with the general considerations of *Block* et al. [20.16] with respect to burn-in of heterogeneous populations. Specifically, when two frailties are ordered in this way, the corresponding mixture failure rates are naturally ordered as functions of time in $[0, \infty)$. Some specific results for the case of frailties with equal means and different variances are also obtained.

In Sect. 20.2 we discuss a 'combination' of a frailty and a proportional hazards (PH) model. The case of a step-stress change-point in the proportional hazards framework is considered and the corresponding bounds for the mixture failure rate are also obtained. Another example deals with a special type of shock, which performs a burn-in for heterogeneous populations.

Section 20.3 is devoted to the important topic of the asymptotic behavior of mixture failure rates. In *Block* et al. [20.17], it was proved that, if the failure rate of each subpopulation converges to a constant and this convergence is uniform, then the mixture failure rate converges to the failure rate of the strongest subpopulation: in other words, the weakest subpopulations die out first. (For convenience, from now on we shall use, where appropriate, the term "population" instead of "subpopulation") This result is a generalization of the case where populations have constant failure rates, as considered by *Clarotti* and *Spizzichino* [20.18], and it also represents a further development of the work by *Block* et al. in [20.16] (see also [20.19,20]). In *Block* and *Joe* [20.21] the following asymptotic result, which addresses the issue of ultimate monotonicity, was obtained. Let z_0 be a realization of a frailty Z, which corresponds to the strongest population. If $\lambda(t, z)/\lambda(t, z_0)$ uniformly decreases as $t \to \infty$, then $\lambda_{\mathrm{m}}(t)/\lambda(t, z_0)$ also decreases. If, in addition, $\lim_{t\to\infty} \lambda(t, z_0)$ exists, then this quotient decreases to 1. Although the lifetime model obtained from these findings may be rather general, the analytical restrictions, such as uniform convergence, are rather stringent. Besides, the strongest population cannot always be identified.

We suggest a class of distributions that generalizes the proportional hazards, the additive hazards and the accelerated life models and we prove a simple asymptotic result for the mixture failure rate for this class of lifetime distributions. It turns out that the asymptotic behavior of mixture failure rates depends only on the behavior of the mixing distribution in the neighborhood of the left end point of its support, and not on the whole mixing distribution.

Notation

The following notation is used in this chapter

T	lifetime random variable,
$F(t)$	cumulative distribution function of T,
Z	unobserved random variable (frailty),
$F(t, z)$	cumulative distribution function indexed by parameter z,
$\Pi(z)$	distribution function of Z,
$\Pi(z\|t)$	conditional distribution function of Z,
$\pi(z)$	probability density function of Z,
$\pi_k(z)$	probability density function of kZ,
$\pi(z\|t)$	conditional probability density of Z,
$\lambda(t, z)$	failure rate indexed by parameter z,
$\Lambda(t, z)$	cumulative failure rate indexed by parameter z,
$\lambda_{\mathrm{m}}(t)$	mixture failure rate,
$\lambda_{\mathrm{P}}(t)$	unconditional expectation in the family of failure rates,
$\lambda_{\mathrm{m}k}(t)$	mixture failure rate for the PH model,
$\tilde{\lambda}_{\mathrm{m}k}(t)$	notation for $k\lambda_{\mathrm{m}}(t)$,
$\lambda_{\mathrm{ms}}(t)$	mixture failure rate after a shock,
$g(z)$	function decreasing in z
$\varepsilon(t)$	baseline stress,
$\varepsilon_{\mathrm{s}}(t)$	more severe stress,
$A(s)$	function defining the general survival model,
$\phi(t)$	scale function in the general survival model,
$\Psi(t)$	additive part of the general survival model.

20.1 Mixture Failure Rates and Mixing Distributions

20.1.1 Definitions

Let $T \geq 0$ be a lifetime random variable with the cumulative distribution function (Cdf) $F(t)[\bar{F}(t) \equiv 1 - F(t)]$. Assume that $F(t)$ is indexed by a random variable Z in the following sense

$$P(T \leq t|Z = z) \equiv P(T \leq t|z) = F(t, z)$$

and that the probability density function (pdf) $f(t, z)$ exists. Then the corresponding failure rate $\lambda(t, z)$ is $f(t, z)/\bar{F}(t, z)$. Let Z be interpreted as a non-negative random variable with support $[a, b]$, $a \geq 0$, $b \leq \infty$ and pdf $\pi(z)$. Thus, a mixture Cdf is defined by

$$F_{\mathrm{m}}(t) = \int_a^b F(t, z)\pi(z)\,\mathrm{d}z\,.$$

As the failure rate is a conditional characteristic, the mixture failure rate $\lambda_{\mathrm{m}}(t)$ should be defined in the following way (see, e.g., *Finkelstein* and *Esaulova* [20.9]):

$$\lambda_{\mathrm{m}}(t) = \frac{\int_a^b f(t, z)\pi(z)\,\mathrm{d}z}{\int_a^b \bar{F}(t, z)\pi(z)\,\mathrm{d}z} = \int_a^b \lambda(t, z)\pi(z|t)\,\mathrm{d}z\,, \tag{20.1}$$

where the conditional pdf (on the condition that $T > t$) is:

$$\pi(z|t) \equiv \pi(z|T > t) = \pi(z)\frac{\bar{F}(t, z)}{\int_a^b \bar{F}(t, z)\pi(z)\,\mathrm{d}z}\,. \tag{20.2}$$

Therefore, this pdf defines a conditional random variable $Z|t$, $Z|0 \equiv Z$ with the same support. On the other hand, consider the following *unconditional characteristic*

$$\lambda_P(t) = \int_a^b \lambda(t,z)\pi(z)\,dz\,, \tag{20.3}$$

which, in fact, defines an expected value (as a function of t) for a specific stochastic process $\lambda(t, Z)$. It follows from definitions (20.1) and (20.3) that $\lambda_m(0) = \lambda_P(0)$. The function $\lambda_P(t)$ is a supplementary one, but as a trend function of a stochastic process, it captures the monotonic pattern of the family $\lambda(t, z)$. Therefore, under certain conditions, $\lambda_P(t)$ has a similar shape to $\lambda(t, z)$: if, e.g., $\lambda(t, z)$, $z \in [a, b]$ increases with t, then $\lambda_P(t)$ increases as well. For some specific cases (see later) $\lambda_P(t)$ also characterizes the shape of the baseline failure rate. On the other hand, the mixture failure rate $\lambda_m(t)$ can have a different pattern: it can ultimately decrease, for instance, or it can preserve the property that it increases with t, as in *Lynch* [20.2]. There is even the possibility of a few oscillations. However, despite all of the patterns that are possible, it will be proved that the mixture failure rate is majorized by $\lambda_P(t)$:

$$\lambda_m(t) < \lambda_P(t),\ \ t > 0 \tag{20.4}$$

and under some additional assumptions, that

$$[\lambda_P(t) - \lambda_m(t)] \uparrow,\ \ t \geq 0. \tag{20.5}$$

Definition 20.1
[20.22]. Relation (20.4) defines a weak bending-down property for the mixture failure rate, whereas relation (20.5) is the definition of a strong bending-down property.

20.1.2 Multiplicative Model

Consider the following specific multiplicative model

$$\lambda(t,z) = z\,\lambda(t), \tag{20.6}$$

where $\lambda(t)$ is a baseline failure rate. This setting defines the widely used frailty (multiplicative) model. On the other hand, it can be also viewed as a proportional hazards (PH) model. Applying definition (20.1) gives:

$$\lambda_m(t) = \int_a^b \lambda(t,z)\pi(z|t)\,d\theta = \lambda(t)E[Z|t]. \tag{20.7}$$

The conditional expectation $E[Z|t](E[Z|0] \equiv E[Z])$ plays a crucial role in defining the shape of the mixture failure rate $\lambda_m(t)$ in this model. The following result was proved in *Finkelstein* and *Esaulova* [20.9]:

$$E'_t[Z|t] = -\lambda(t)Var(Z|t) < 0,$$

which means that the conditional expectation of Z is a decreasing function of $t \in [0, \infty)$. On the other hand, (20.3) becomes

$$\lambda_P(t) = \int_a^b \lambda(t,z)\pi(z)\,dz = \lambda(t)E[Z|0]. \tag{20.8}$$

Therefore

$$\lambda_P(t) - \lambda_m(t) = \lambda(t)(E[Z|0] - E[Z|t]) > 0$$

and relation (20.4) holds, whereas under the additional sufficient condition that $\lambda(t)$ is increasing, the strong bending-down property (20.5) occurs.

20.1.3 Comparison with Unconditional Characteristics

The main additional assumption that will be needed for the following result is that the family of failure rates $\lambda(t, z)$, $z \in [a, b]$ should be ordered in z.

Theorem 20.1
Let the failure rate $\lambda(t, z)$ in the mixing model (20.1) be differentiable with respect to both arguments and be ordered as

$$\lambda(t,z_1) < \lambda(t,z_2),\ z_1 < z_2, \forall z_1, z_2 \in [a,b], t \geq 0. \tag{20.9}$$

Assume that the conditional and unconditional expectations in relations (20.1) and (20.3), respectively, are finite for $\forall t \in [0, \infty)$.

Then:

a) The mixture failure rate $\lambda_m(t)$ bends down with time, weakly at least.

b) If, additionally, $\frac{\partial\lambda(t,z)}{\partial z}$ increases with t, then $\lambda_m(t)$ strongly bends down with time.

Proof: It is clear that ordering (20.9) is equivalent to the condition that $\lambda(t, z)$ increases with z for each $t \geq 0$. In accordance with (20.1) and (20.3), and integrating by parts [20.5]:

$$\begin{aligned}\Delta\lambda(t) &\equiv \int_a^b \lambda(t,z)[\pi(z)-\pi(z|t)]\,\mathrm{d}z\\ &= \lambda(t,z)[\Pi(z)-\Pi(z|t)]|_a^b\\ &\quad - \int_a^b \lambda'_z(t,z)[\Pi(z)-\Pi(z|t)]\,\mathrm{d}z\\ &= \int_a^b -\lambda'_z(t,z)[\Pi(z)-\Pi(z|t)]\,\mathrm{d}z > 0,\ \ t>0\,,\end{aligned} \quad (20.10)$$

where

$$\Pi(z) = P(Z \le z); \quad \Pi(z|t) = P(Z \le z|T>t)$$

and the term $\lambda(t,z)[\Pi(z)-\Pi(z|t)]|_a^b$ vanishes for $b=\infty$ as well. Inequality (20.10), and therefore the first part of the theorem, follows from $\lambda'_z(t,z)>0$ and the following inequality:

$$\Pi(z) - \Pi(z|t) < 0,\ \forall t>0,\ z\in(a,b). \quad (20.11)$$

Inequality (20.11) can be interpreted as: "the weakest populations die out first". This interpretation is widely used in various specific cases, especially in demographic literature [20.3]. To obtain (20.11), it is sufficient to prove that

$$\Pi(z|t) = \frac{\int_a^z \bar{F}(t,u)\pi(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi(u)\,\mathrm{d}u}$$

increases with t, which can be easily done by considering the corresponding derivative [20.22].

The derivative $\Pi'_t(z|t)>0$ if

$$\frac{\int_a^z \bar{F}'_t(t,u)\pi(u)\,\mathrm{d}u}{\int_a^z \bar{F}(t,u)\pi(u)\,\mathrm{d}u} > \frac{\int_a^b \bar{F}'_t(t,u)\pi(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi(u)\,\mathrm{d}u}.$$

As $\bar{F}'_t(t,z) = -\lambda(t,z)\bar{F}(t,z)$, it is sufficient to show that

$$B(t,z) \equiv \frac{\int_a^z \lambda(t,u)\bar{F}(t,u)\pi(u)\,\mathrm{d}u}{\int_a^z \bar{F}(t,u)\pi(u)\,\mathrm{d}u}$$

increases with z. Inequality $B'_z(t,z)>0$ is equivalent to

$$\lambda(t,z)\int_a^z \bar{F}(t,u)\pi(u)\,\mathrm{d}u > \int_0^z \lambda(t,u)\bar{F}(t,u)\pi(u)\,\mathrm{d}u.$$

Thus, due to the additional assumption in Theorem 20.1b), the integrand at the end of (20.10) does increase and therefore $\Delta\lambda(t)$ does as well, which immediately leads to the strong bending-down property (20.5). ■

The following example shows the strong bending-down property of the mixture failure rate in practice.

Example 20.1: Technical devices have parameters that are also usually quite heterogeneous and should exhibit a similar deceleration in the failure rate or may even bend down practically to 0. In order to support this statement and to show that the effect of heterogeneity is significantly underestimated by most reliability practitioners, the following experiment was conducted at the Max Planck Institute for Demographic Research [20.12]. We recorded the failure times for a population of 750 miniature lamps and constructed an empirical failure rate function (in relative units) for a time interval of 250 h, which is shown in Fig. 20.1.

The results were very convincing: the failure rate initially increased (a tentative fit showed the Weibull law) and then it decreased to a very low level. This pattern for the observed failure rate is exactly the same as that predicted in *Finkelstein* and *Esaulova* [20.9] for the Weibull baseline Cdf.

We will now show now that the natural ordering for our mixing model is based on the likelihood ratio. Somewhat similar reasoning can be found in *Block*

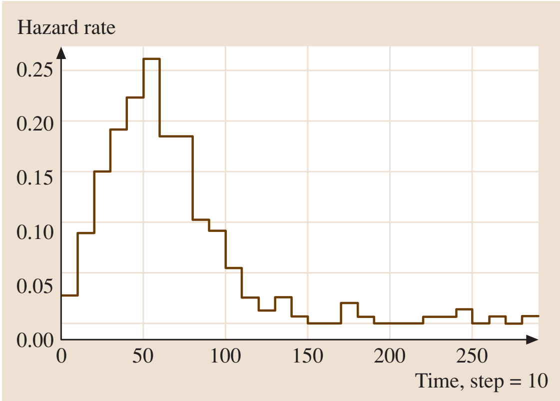

Fig. 20.1 Empirical hazard rate for a population of the 750 miniature lamps.

et al. [20.16] and *Shaked* and *Spizzichino* [20.23]. Let Z_1 and Z_2 be continuous non-negative random variables with the same support and with densities $\pi_1(z)$ and $\pi_2(z)$, respectively. Recall [20.14, 15] that Z_2 is smaller than Z_1 based on the likelihood ratio [20.24]:

$$Z_1 \geq_{\text{LR}} Z_2\,, \tag{20.12}$$

if $\pi_2(z)/\pi_1(z)$ is a decreasing function.

Definition 20.2
Let $Z(t)$, $t \in [0, \infty)$ be a family of random variables indexed by parameter t (time) with probability density functions $p(z, t)$. We say that $Z(t)$ decreases with t according to the likelihood ratio if

$$L(z, t_1, t_2) = \frac{p(z, t_2)}{p(z, t_1)}$$

decreases with z for all $t_2 > t_1$.

The following simple result states that our family of conditional mixing random variables $Z|t$, $t \in [0, \infty]$ decreases based on the likelihood ratio:

Theorem 20.2
Let the family of failure rates $\lambda(t, z)$ in the mixing model (20.1) be ordered as in the relation (20.9).

Then the family of random variables $Z|t \equiv Z|T > t$ decreases with $t \in [0, \infty)$ based on the likelihood ratio.

Proof: In accordance with (20.2):

$$\begin{aligned} L(z, t_1, t_2) &= \frac{\pi(z|t_2)}{\pi(z|t_1)} \\ &= \frac{\bar{F}(t_2, z) \int_a^b \bar{F}(t_1, z)\pi(z)\,\mathrm{d}z}{\bar{F}(t_1, z) \int_a^b \bar{F}(t_2, z)\pi(z)\,\mathrm{d}z}\,. \end{aligned} \tag{20.13}$$

Therefore, the monotonicity with z of $L(z, t_1, t_2)$ is defined by

$$\frac{\bar{F}(t_2, z)}{\bar{F}(t_1, z)} = \exp\left\{-\int_{t_1}^{t_2} \lambda(u, z)\,\mathrm{d}u\right\},$$

which, due to ordering given in (20.9), decreases with z for all $t_2 > t_1$. ■

20.1.4 Likelihood Ordering of Mixing Distributions

For the mixing model (20.1) and (20.2), consider two different mixing random variables Z_1 and Z_2 with probability density functions $\pi_1(z)$, $\pi_2(z)$ and cumulative distribution functions $\Pi_1(z)$, $\Pi_2(z)$, respectively. Assuming some type of stochastic ordering for Z_1 and Z_2, we intend to achieve simple ordering of the corresponding mixture failure rates. Using simple examples, it becomes apparent that the 'usual' stochastic ordering (stochastic dominance) is too weak to do this. It was shown in the previous section that likelihood ratio ordering is the natural one for the family of random variables $Z|t$ in our mixing model. Therefore, it seems reasonable to order Z_1 and Z_2 in this sense too.

Lemma 20.1
Let

$$\pi_2(z) = \frac{g(z)\pi_1(z)}{\int_a^b g(z)\pi_1(z)\,\mathrm{d}z}\,, \tag{20.14}$$

where $g(z)$ is a decreasing function.

Then Z_1 is stochastically larger than Z_2:

$$Z_1 \geq_{st} Z_2 \quad (\Pi_1(z) \leq \Pi_2(z),\ z \in [a, b]) \tag{20.15}$$

Proof:

$$\begin{aligned} \Pi_2(z) &= \frac{\int_a^z g(u)\pi_1(u)\,\mathrm{d}u}{\int_a^b g(u)\pi_1(u)\,\mathrm{d}u} \\ &= \frac{\int_a^z g(u)\pi_1(u)\,\mathrm{d}u}{\int_a^z g(u)\pi_1(u)\,\mathrm{d}u + \int_z^b g(u)\pi_1(u)\,\mathrm{d}u} \\ &= \frac{g*(a, z)\int_a^z \pi_1(u)\,\mathrm{d}u}{g*(a, z)\int_a^z \pi_1(u)\,\mathrm{d}u + g*(z, b)\int_a^z \pi_1(u)\,\mathrm{d}u} \\ &\geq \int_a^z \pi_1(u)\,\mathrm{d}u = \Pi_1(z), \end{aligned} \tag{20.16}$$

where $g*(a, z)$ and $g*(z, b)$ are the mean values of the function $g(z)$ in the corresponding integrals. As this function decreases, $g*(z, b) \leq g*(a, z)$. ■

Equation (20.14) with decreasing $g(z)$ means that $Z_1 \geq_{\mathrm{LR}} Z_2$, and it is well-known (see, e.g., [20.14]) that likelihood ratio ordering implies corresponding stochastic ordering. However, we need the previous reasoning to derive the following result.

Theorem 20.3

Let relation (20.14) (where $g(z)$ is a decreasing function) hold, which means that Z_1 is larger than Z_2 based on likelihood ratio ordering. Assume that the ordering from (20.9) holds.

Then for $\forall t \in [0, \infty)$:

$$\lambda_{\mathrm{m}1}(t) \equiv \frac{\int_a^b f(t,z)\pi_1(z)\,\mathrm{d}z}{\int_a^b \bar{F}(t,z)\pi_1(z)\,\mathrm{d}z} \geq \frac{\int_a^b f(t,z)\pi_2(z)\,\mathrm{d}z}{\int_a^b \bar{F}(t,z)\pi_2(z)\,\mathrm{d}z} \equiv \lambda_{\mathrm{m}2}(t). \tag{20.17}$$

Proof: Inequality (20.17) means that the mixture failure rate obtained for the stochastically larger (in the likelihood ratio ordering sense) mixing distribution is larger for $\forall t \in [0, \infty)$ than the one obtained for the stochastically smaller mixing distribution.

We shall first prove that

$$\Pi_1(z|t) = \frac{\int_a^z \bar{F}(t,u)\pi_1(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi_1(u)\,\mathrm{d}u} \leq \frac{\int_a^z \bar{F}(t,u)\pi_2(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi_2(u)\,\mathrm{d}u} \equiv \Pi_2(z|t). \tag{20.18}$$

Indeed:

$$\frac{\int_a^z \bar{F}(t,u)\pi_2(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi_2(u)\,\mathrm{d}u} = \frac{\int_a^z \bar{F}(t,u)\frac{g(u)\pi_1(u)}{\int_a^b g(u)\pi_1(u)\,\mathrm{d}u}\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\frac{g(u)\pi_1(u)}{\int_a^b g(u)\pi_1(u)\,\mathrm{d}u}\,\mathrm{d}u}$$
$$= \frac{\int_a^Z g(u)\bar{F}(t,u)\pi_1(u)\,\mathrm{d}u}{\int_a^b g(u)\bar{F}(t,u)\pi_1(u)\,\mathrm{d}u} \geq \frac{\int_a^Z \bar{F}(t,u)\pi_1(u)\,\mathrm{d}u}{\int_a^b \bar{F}(t,u)\pi_1(u)\,\mathrm{d}u},$$

where the last inequality follows using exactly the same argument, as in inequality (20.16) of Lemma 20.1. Similar to (20.10), and taking into account relation (20.18):

$$\begin{aligned}
\lambda_{\mathrm{m}1}(t)-\lambda_{\mathrm{m}2}(t) &= \int_a^b \lambda(t,z)[\pi_1(z|t)-\pi_2(z|t)]\,\mathrm{d}z \\
&= \lambda(t,z)[\Pi_1(z|t)-\Pi_2(z|t)]\big|_a^b \\
&\quad - \int_a^b \lambda_z'(t,z)[\Pi_1(z|t)-\Pi_2(z|t)]\,\mathrm{d}z \\
&= \int_a^b -\lambda_z'(t,z)[\Pi_1(z|t) \\
&\quad - \Pi_2(z|t)]\,\mathrm{d}z \geq 0, \quad t>0.
\end{aligned} \tag{20.19}$$

■

The starting point of Theorem 20.3 was (20.14) with the crucial assumption of a decreasing $g(z)$ function. It should be noted, however, that this assumption can be rather formally and directly justified by considering the difference $\Delta\lambda(t) = \lambda_{\mathrm{m}1}(t) - \lambda_{\mathrm{m}2}(t)$ and using definitions (20.1) and (20.2). The corresponding numerator (the denominator is positive) is transformed into a double integral in the following way

$$\begin{aligned}
&\int_a^b \lambda(t,z)\bar{F}(t,z)\pi_1(z)\,\mathrm{d}z \int_a^b \bar{F}(t,z)\pi_2(z)\,\mathrm{d}z \\
&- \int_a^b \lambda(t,z)\bar{F}(t,z)\pi_2(z)\,\mathrm{d}z \int_a^b \bar{F}(t,z)\pi_1(z)\,\mathrm{d}z \\
&= \int_a^b \int_a^b \bar{F}(t,u)\bar{F}(t,s)[\lambda(t,u)\pi_1(u)\pi_2(s) \\
&\quad - \lambda(t,s)\pi_1(u)\pi_2(s)]\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_{\substack{a\\u>s}}^b \int_a^b \bar{F}(t,u)\bar{F}(t,s)\{\pi_1(u)\pi_2(s)[\lambda(t,u)-\lambda(t,s)] \\
&\quad + \pi_1(s)\pi_2(u)[\lambda(t,s)-\lambda(t,u)]\}\,\mathrm{d}u\,\mathrm{d}s \\
&= \int_{\substack{a\\u>s}}^b \int_a^b \bar{F}(t,u)\bar{F}(t,s)[\lambda(t,u)-\lambda(t,s)][\pi_1(u)\pi_2(s) \\
&\quad - \pi_1(s)\pi_2(u)]\,\mathrm{d}u\,\mathrm{d}s.
\end{aligned} \tag{20.20}$$

Therefore, the final double integral is positive if ordering (20.9) holds and $\pi_2(z)/\pi_1(z)$ is decreasing.

Part C | 20.1

20.1.5 Ordering Variances of Mixing Distributions

Let $\Pi_1(z)$ and $\Pi_2(z)$ be two mixing distributions with equal means. It follows from equation (20.7) that, for the multiplicative model considered in this section, $\lambda_{m1}(0) = \lambda_{m2}(0)$. Intuitive considerations and reasoning based on the principle: "the weakest populations die out first" suggest that, unlike (20.17), the mixture failure rates will be ordered as $\lambda_{m1}(t) < \lambda_{m2}(t)$ for all $t > 0$ if, e.g., the variance of Z_1 is larger than the variance of Z_2. We will show that this is true for a specific case and that for a general multiplicative model this ordering only holds for a sufficiently small time t. Therefore, it is necessary to formulate a stronger condition to apply when ordering the 'variabilities' of Z_1 and Z_2.

Example 20.2: For a meaningful specific example, consider the frailty model (20.6), where Z has a gamma distribution

$$\pi(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} \exp(-\beta z);\ \alpha > 0, \beta > 0.$$

Substituting this density into relation (20.1) gives

$$\lambda_m(t) = \frac{\lambda(t) \int_0^\infty \exp[-z\Lambda(t)] z\pi(z)\,dz}{\int_0^\infty \exp[-z\Lambda(t)]\pi(z)\,dz},$$

where $\Lambda(t) = \int_0^t \lambda(u)\,du$ is a cumulative baseline failure rate. Computing integrals results in

$$\lambda_m(t) = \frac{\alpha\lambda(t)}{\beta + \Lambda(t)}. \quad (20.21)$$

Equations (20.21) can now be written in terms of $E[Z]$ and $Var(Z)$:

$$\lambda_m(t) = \lambda(t) \frac{E^2[Z]}{E[Z] + Var(Z)\Lambda(t)}, \quad (20.22)$$

which, for the specific case $E[Z] = 1$, gives the result from *Vaupel* et al. [20.3], widely used in demography:

$$\lambda_m(t) = \frac{\lambda(t)}{1 + Var(Z)\Lambda(t)}.$$

Using equation (20.22), we can compare mixture failure rates of two populations with different Z_1 and Z_2 on the condition that $E[Z_2] = E[Z_1]$:

$$Var(Z_1) \geq Var(Z_2) \Rightarrow \lambda_{m1}(t) \leq \lambda_{m2}(t). \quad (20.23)$$

Intuitively it might be expected that this result would be valid for arbitrary mixing distributions in the multiplicative model. However, the mixture failure rate dynamics here can be much more complicated than this, even for this specific case, and this topic needs further attention in future research. A somewhat similar situation was observed in *Finkelstein* and *Esaulova* [20.9]: although the conditional variance $Var(Z|t)$ decreased with t for the multiplicative gamma frailty model, a counter example was constructed for the case of the uniform mixing distribution for [0, 1].

The following theorem shows that ordering variances is a sufficient and necessary condition for ordering mixture failure rates, but only for the initial time interval.

Theorem 20.4
Let Z_1 and Z_2 ($E[Z_2] = E[Z_1]$) be two mixing distributions in the multiplicative model (20.6), (20.7).

In this case, ordering the variances

$$Var(Z_1) > Var(Z_2) \quad (20.24)$$

is a sufficient and necessary condition for the ordering of mixture failure rates in the neighborhood of $t = 0$:

$$\lambda_{m1}(t) < \lambda_{m2}(t);\ t \in (0, \varepsilon), \quad (20.25)$$

where $\varepsilon > 0$ is sufficiently small.

Proof:
Sufficient condition:
From the results in Sect. 20.1.3:

$$\Delta\lambda(t) = \lambda_{m1}(t) - \lambda_{m2}(t) = \lambda(t)(E[Z_1|t] - E[Z_2|t]), \quad (20.26)$$

$$E'_t[Z_i|t] = -\lambda(t)Var(Z_i|t) < 0,\ i = 1, 2, t \geq 0, \quad (20.27)$$

where

$$E[Z_i|0] \equiv E[Z_i],\ Var(Z_i|t) \equiv Var(Z_i). \quad (20.28)$$

As the means of the mixing variables are equal, relation (20.26) for $t = 0$ reads $\Delta\lambda(0) = 0$, and therefore the time interval in (20.25) is opened. Thus, if the ordering in (20.24) holds, the ordering in (20.25) then follows immediately after, considering the derivative of

$$\frac{\lambda_{m1}(t)}{\lambda_{m2}(t)} = \frac{E[Z_1|t]}{E[Z_2|t]}$$

at $t = 0$ and taking into account relations (20.27) and notation (20.28).

Necessary condition:

Similar to (20.20), the numerator of the difference $\Delta\lambda(t)$ is

$$\lambda(t)\int_a^b\int_a^b\{\exp[-\Lambda(t)(u+s)]\}(u-s)\pi_1(u)\pi_2(s)\,\mathrm{d}u\,\mathrm{d}s,$$

where, as previously, $\Lambda(t)=\int_0^t\lambda(u)\,\mathrm{d}u$. After changing variables to $x=(u+s)/2$, $y=(u-s)/2$, the double integral is transformed to the iterated integral and denoted by $G(t)$:

$$G(t)\equiv\int_a^b\exp[-2\Lambda(t)x]\times\int_{-x}^{x}y\pi_1(x+y)\pi_2(x-y)\,\mathrm{d}y\,\mathrm{d}x. \tag{20.29}$$

Denote the internal integral in (20.29) by $g(x)$. Then:

$$G(t)=\int_a^b\{\exp[-2\Lambda(t)x]\}g(x)\,\mathrm{d}x.$$

On the other hand, reverting back to the initial variables of integration and taking into account that $\Lambda(0)=0$, we get

$$\begin{aligned}G(0)&=\int_a^b g(x)\,\mathrm{d}x=\int_a^b\int_a^b(u-s)\pi_1(u)\pi_2(s)\,\mathrm{d}u\,\mathrm{d}s\\&=\int_a^b u\pi_1(u)\,\mathrm{d}u-\int_a^b u\pi_2(u)\,\mathrm{d}u\\&=E[Z_1]-E[Z_2]=0.\end{aligned}$$

Assume, firstly, that $\lambda(0)\neq 0$. As $G(0)=0$, the function $G(t)$ is negative in the neighborhood of 0 if $G'(0)<0$:

$$G'(t)=-2\lambda(t)\int_a^b\{\exp[-2\Lambda(t)x]\}xg(x)\,\mathrm{d}x,$$

$$G'(0)<0\Rightarrow\int_a^b xg(x)\,\mathrm{d}x>0.$$

If $\Delta\lambda(t)<0$, $t\in(0,\varepsilon)$ [condition (20.25)], then $G(t)<0$, $t\in(0,\varepsilon)$, and taking into account that

$$\begin{aligned}\int_a^b xg(x)\,\mathrm{d}x&=\int_a^b\int_a^b\frac{u+s}{2}(u-s)\pi_1(s)\pi_2(s)\,\mathrm{d}u\,\mathrm{d}s\\&=\frac{1}{2}\int_a^b\int_a^b(u^2-s^2)\pi_1(u)\pi_2(s)\,\mathrm{d}u\,\mathrm{d}s\\&=\frac{1}{2}[Var(Z_1)-Var(Z_2)],\end{aligned}$$

we arrive at the ordering given in (20.24).

Similar considerations are valid for $\lambda(0)=0$. The function $G(t)$ is negative in this case in the neighborhood of 0 if $G''(0)<0$. As

$$G''(0)=-2\lambda'(0)\int_a^b xg(x)\,\mathrm{d}x$$

and $\lambda'(0)>0$ [since $\lambda(t)>0$, $t>0$ and $\lambda(0)=0$], the same reasoning used for the case $\lambda(0)\neq 0$, also holds here. ■

A trivial but important consequence of this theorem is as follows:

Corollary

Let mixture failure rate ordering (20.25) hold for $t\in(0,\infty)$. Then inequality (20.24) holds.

20.2 Modeling the Impact of the Environment

20.2.1 Bounds in the Proportional Hazards Model

Consider the specific multiplicative frailty model (20.6) and (20.7). Formally combine this model with the proportional hazards (PH) model in a following way:

$$\lambda(t,z,k)=zk\lambda(t)\equiv z_k\lambda(t). \tag{20.30}$$

Therefore, the baseline $F(t)$ is indexed by the random variable $Z_k=kZ$ with the pdf $\pi_k(z)=\pi(z/k)$, whereas the corresponding conditional pdf $\pi_k(z|t)$ is given by the right hand side of (20.2), where $\pi(z)$ is substituted by $\pi_k(z)$. Equivalently, (20.30) can be interpreted as a frailty model with a mixing random variable Z and a baseline failure rate $k\lambda(t)$. These two simple and equivalent interpretations will help us in what follows. Without losing any generality, assume that $a = 0$ and $b = \infty$. Thus, similar to (20.6)–(20.9), the mixture failure rate in this case is

$$\lambda_{mk}(t) = k\lambda(t) \int_0^\infty z\pi_k(z|t)\,dz \equiv \lambda(t)E[Z_k|t]. \quad (20.31)$$

As $Z_k = kZ$, its density function is

$$\pi_k(z) = \frac{1}{k}\pi\left(\frac{z}{k}\right).$$

Theorem 20.5

Let the mixture failure rates for the multiplicative models (20.6) and (20.30) be given by relations (20.7) and (20.31), respectively, where $k > 1$.

Assume that the following quotient increases with z:

$$\frac{\pi_k(z)}{\pi(z)} = \frac{\pi\left(\frac{z}{k}\right)}{k\pi(z)} \uparrow \quad (20.32)$$

Then:

$$\lambda_{mk}(t) > \lambda_m(t); \ \forall t \in [0, \infty). \quad (20.33)$$

Proof: Although inequality (20.33) seems rather trivial at first sight, it is only valid for some specific cases of mixing (e.g., the multiplicative model). It is clear that (20.33) is always true for sufficiently small t, whereas with larger t the ordering can be different for general mixing models. Denote:

$$\Delta\lambda_m(t) = \lambda_{mk}(t) - \lambda_m(t).$$

Using definitions (20.1)–(20.2), it can be seen that, similar to the case for relation (20.20), the sign of this difference is defined by the sign of

$$\begin{aligned}
&\int_0^\infty z\bar{F}(t,z)\pi_k(z)\,dz \int_0^\infty \bar{F}(t,z)\pi(z)\,dz \\
&- \int_0^\infty z\bar{F}(t,z)\pi_k(z)\,dz \int_0^\infty \bar{F}(t,z)\pi(z)\,dz \\
&= \int_0^\infty \int_0^\infty \bar{F}(t,u)\bar{F}(t,s)[u\pi_k(u)\pi(s) \\
&\quad - s\pi_k(u)\pi(s)]\,du\,ds \\
&= \underset{u>s}{\int_0^\infty \int_0^\infty} \bar{F}(t,u)\bar{F}(t,s)[\pi_k(u)\pi(s)(u-s) \\
&\quad + \pi_k(s)\pi(u)(s-u)]\,du\,ds \\
&= \underset{u>s}{\int_0^\infty \int_0^\infty} \bar{F}(t,u)\bar{F}(t,s)(u-s)[\pi_k(u)\pi(s) \\
&\quad - \pi_k(s)\pi(u)]\,du\,ds. \qquad (20.34)
\end{aligned}$$

Therefore, the sufficient condition for inequality (20.33) is condition (20.32), which is, in fact, rather crude. It is easy to verify that this condition is satisfied, for example, for the gamma and the Weibull densities, which are often used for mixing. ■

Example 20.3: Consider the same setting as in Example 20.1. Condition (20.32) is satisfied for the gamma distribution. The mixture failure rate $\lambda_m(t)$ in this case is given by relation (20.22). A similar equation obviously exists for $\lambda_{mk}(t)$, and the corresponding comparison can be performed explicitly:

$$\begin{aligned}
\lambda_{mk}(t) &= \lambda(t)\frac{E^2[Z_k]}{E[Z_k] + Var(Z_k)\Lambda(t)} \\
&= \lambda(t)\frac{k^2E^2[Z]}{kE[Z] + k^2Var(Z)\Lambda(t)} > \lambda_m(t). \qquad (20.35)
\end{aligned}$$

Now we shall obtain an upper bound for $\lambda_{mk}(t)$.

Theorem 20.6

Let the mixture failure rates for the multiplicative models (20.6) and (20.30) be given by relations (20.7) and (20.31), respectively, where $k > 1$.

Then:

$$\lambda_{mk}(t) < k\lambda_m(t); \ \forall t \in (0, \infty). \quad (20.36)$$

Proof: Consider the difference $\lambda_{mk}(t) - k\lambda_m(t)$ similarly to (20.34), but in a slightly different way: $\lambda_{mk}(t)$ will be equivalently defined by the baseline failure rate $k\lambda(t)$ and the mixing variable Z (in (20.34) it was defined by the baseline $\lambda(t)$ and the mixing variable kZ). This means that:

$$\lambda_{mk}(t) - k\lambda_m(t) = k\lambda(t)(\hat{E}[Z|t] - E[Z|t]), \quad (20.37)$$

where the conditioning in $\hat{E}[Z|t]$ is different from the one in $E[Z|t]$ in the sense described. Denote:

$$\bar{F}_k(t,z) = \exp\left[-zk\Lambda(t)\right].$$

'Symmetrically' to (20.34), $\mathrm{sign}[\lambda_{\mathrm{m}k}(t) - k\lambda_{\mathrm{m}}(t)]$ is defined by

$$\mathrm{sign} \int_{0}^{\infty} \int_{0 \atop u>s}^{\infty} \pi(u)\pi(s)(u-s)[\bar{F}_k(t,u)\bar{F}(t,s) - \bar{F}(t,u)\bar{F}_k(t,s)]\,\mathrm{d}u\,\mathrm{d}s,$$

which is negative for all $t > 0$ since

$$\frac{\bar{F}_k(t,z)}{\bar{F}(t,z)} = \exp[-(k-1)z\Lambda(t)]$$

decreases with z. ■

It is worth noting that we do not need an additional condition for this bound as in the case of Theorem 20.5. Also, it is clear that $\lambda_{\mathrm{m}k}(0) = k\lambda_{\mathrm{m}}(0)$. As previously mentioned, model (20.30) defines a combination of a PH model and a frailty model. When $Z = 1$, it is an 'ordinary' PH model. In the presence of a random Z, as follows from (20.36), the observed failure rate $\lambda_{\mathrm{m}k}(t)$ cannot be obtained as $k\lambda_{\mathrm{m}}(t)$ due to the nature of the mixing.

Therefore, the PH model in each realization does not result in the PH model for the corresponding mixture failure rate.

Example 20.3 (continuation): We illustrate inequality (20.36):

$$\lambda_{\mathrm{m}k}(t) = \lambda(t)\frac{k^2E^2[Z]}{kE[Z] + k^2 Var(Z)\Lambda(t)} < \lambda(t)\frac{kE^2[Z]}{E[Z] + Var(Z)\Lambda(t)} = k\lambda_{\mathrm{m}}(t).$$

20.2.2 Change Point in the Environment

Assume that there are two possible environments (stresses), $\varepsilon(t)$ and $\varepsilon_s(t)$: the baseline and a more severe one, respectively. The baseline environment for our heterogeneous population corresponds to the observed failure rate $\lambda_{\mathrm{m}}(t)$ and the more severe one to $\lambda_{\mathrm{m}k}(t)$, $k > 1$. As we did previously, assume also that the PH model for each subpopulation (for each fixed z) holds. Consider a piece-wise constant step stress with a single change point at t_1:

$$\varepsilon(t_1) = \begin{cases} \varepsilon, & 0 \le t < t_1, \\ \varepsilon_k & t \ge t_1, \end{cases} \tag{20.38}$$

where the stresses ε and ε_k correspond to the failure rates $z\lambda(t)$ and $zk\lambda(t)$, respectively ($k > 1, z \ge 0$). In accordance with the 'memoryless property' of the PH model, the stress (20.38) results in the following failure rate for each subpopulation:

$$\lambda(t, t_1, z, k) = \begin{cases} z\lambda(t), & 0 \le t < t_1 \\ kz\lambda(t) & t \ge t_1 \end{cases} \tag{20.39}$$

Denote the resulting mixture failure rate in this case as:

$$\lambda_{\mathrm{m}}(t, t_1) = \begin{cases} \lambda_{\mathrm{m}}(t), & 0 \le t < t_1, \\ \tilde{\lambda}_{\mathrm{m}k}(t) & t \ge t_1, \end{cases} \tag{20.40}$$

where, similar to the previous section,

$$\tilde{\lambda}_{\mathrm{m}k}(t_1) = k\lambda_{\mathrm{m}}(t_1). \tag{20.41}$$

It is worth noting that relation (20.41) means that this model with a step stress is proportional for the mixture failure rates only at the switching point t_1. We want to prove the following inequality:

$$\lambda_{\mathrm{m}k}(t) < \tilde{\lambda}_{\mathrm{m}k}(t);\ \forall t \in [t_1, \infty). \tag{20.42}$$

In accordance with (20.40), consider two **initial** (for the interval $[0,\infty)$) mixing distributions: $Z_1 = Z|T_1 > t_1$, where T_1 is defined by the baseline failure rate $k\lambda(t)$ and $\tilde{Z}_1 = Z|\tilde{T}_1 > t_1$, where $\tilde{T}_1$ is defined by the baseline failure rate $\lambda(t)$. As follows from definition (20.2), the corresponding ratio

$$\frac{\tilde{\pi}(z,t_1)}{\pi(z,t_1)} = \exp[(k-1)z\Lambda(t_1)]$$

increases with z. Then inequality (20.42) follows immediately after taking the proof of Theorem 20.1 into account with obvious alterations caused by the change in the left end point of the interval from 0 to t_1.

Inequality (20.42) was graphically illustrated in *Vaupel* and *Yashin* ([20.25] Fig.10) for a specific case of a discrete mixture of two subpopulations and the Gompertz baseline failure rate. The demographic meaning of this was the following: suppose we decrease the mortality rates of the subpopulations during early life ($[0, t_1)$). Then the observed mortality rate for $[t_1, \infty)$ is *larger* than the observed mortality rate for the initial mixture without changes. In other words, early success results in more failure later on [20.25].

20.2.3 Shocks in Heterogeneous Populations

Now consider the general mixing model (20.1)–(20.2) and assume that an instantaneous shock occurs at time $t = t_1$. This shock affects the whole population: with corresponding complementary probabilities it either kills an individual or 'leaves him unchanged'. Without losing any generality, let $t_1 = 0$; otherwise a new initial mixing variable $Z|t_1$ needs to be defined and the corresponding procedure can be easily adjusted to this case. It is natural to suppose that the frailest individuals or populations (those with the largest failure rates) are more susceptible to being killed.

This setting can be defined probabilistically in the following way. Let $\pi_1(z)$ denote the frailty distribution of a random variable Z_1 after the shock and let $\lambda_{ms}(t)$ be the corresponding observed (mixture) failure rate after the shock. Assume that

$$\pi_1(z) = \frac{g(z)\pi(z)}{\int_a^b g(z)\pi(z)\,\mathrm{d}z}, \tag{20.43}$$

where $g(z)$ is a decreasing function and therefore $\pi_1(z)/\pi(z)$ also decreases. This means that the shock performs a kind of burn-in operation [20.16] and the random variables Z and Z_1 are ordered based on the likelihood ratio [20.14, 15]:

$$Z \geq_{\mathrm{LR}} Z_1 \tag{20.44}$$

We are now able to formulate the following result.

Theorem 20.7

Let relation (20.43), which defines the mixing density after a shock at $t = 0$ (and where $g(z)$ is a decreasing function), hold.

Also assume that the ordering given by (20.9) holds Then:

$$\lambda_{ms}(t) < \lambda_m(t);\ \forall t \in [0, \infty). \tag{20.45}$$

Proof: Inequality (20.9) is a natural ordering of the family of failure rates $\lambda(t, z)$, $z \in [0, \infty)$ and it trivially holds for the specific model (20.6). Performing all of the steps we used when obtaining relation (20.34), we finally obtain:

$$\begin{aligned}
&\operatorname{sign}[\lambda_{ms}(t) - \lambda_m(t)] \\
&= sign \underset{u>s}{\int_a^b \int_a^b} \bar{F}(t, u)\bar{F}(t, s)[\lambda(t, u) \\
&- \lambda(t, s)][\pi_1(u)\pi(s) - \pi_1(s)\pi(u)]\,\mathrm{d}u\,\mathrm{d}s,
\end{aligned}$$

which is negative due to definition (20.43) and the assumptions of this theorem. ■

At $t = 0$, for instance:

$$\lambda_m(0) - \lambda_{ms}(0) = \int_0^\infty \lambda(0, z)[\pi(z) - \pi_1(z)]\,\mathrm{d}z.$$

In accordance with inequality (20.45), the curve $\lambda_{ms}(t)$ lies beneath the curve $\lambda_m(t)$ for $t \geq 0$. This fact seems intuitively evident, but, in fact, it is only valid due to the rather stringent conditions of this theorem. It can be shown, for instance, that replacing condition (20.44) with a weaker one of stochastic dominance, $Z_{\mathrm{st}} \geq Z_1$, will not guarantee the ordering given in (20.45) for all t.

This result can be generalized to a sequence of shocks of the type described that occur at times $\{t_i\}$, $i = 1, 2, \ldots$

20.3 Asymptotic Behaviors of Mixture Failure Rates

20.3.1 Survival Model

The asymptotic behaviors of mixture failure rates have been studied by *Block* et al. [20.16], *Gurland* and *Sethuraman* [20.20], *Lynn* and *Singpurwalla* [20.19] and *Block* et al. [20.17], to name but a few. In *Finkelstein* and *Esaulova* [20.9] we considered the properties of $\lambda_m(t)$ as $t \to \infty$ for the multiplicative model (20.6). As $\lambda_m(t) = \lambda(t)E[Z|t]$, this product was analyzed for increasing $\lambda(t)$ and conditions implying convergence to 0 were derived, taking into account that the conditional expectation $E[Z|t]$, defined in (20.7), decreases with t. The approach taken in this section is different: we study a new lifetime model and derive explicit asymptotic formulae for mixture failure rates that generalize various specific results obtained for proportional hazards and additive hazards models. This approach also allows us to deal with the accelerated life model (ALM), which has not been studied in the literature.

We now define a class of distributions $F(t, z)$ and study the asymptotic behavior of the corresponding mixture failure rate $\lambda_m(t)$. To begin with it is more convenient to define this in terms of the cumulative failure rate $\Lambda(t, z)$, rather than in terms of $\lambda(t, z)$:

$$\Lambda(t, z) = A[z\phi(t)] + \psi(t). \tag{20.46}$$

General Assumptions for the Model (20.46):

The natural properties of the cumulative failure rate of the absolutely continuous distribution $F(t, z)$ (for $\forall z \in [0, \infty)$) imply that the functions $A(s)$, $\phi(t)$ and $\psi(t)$ are differentiable, the right hand side of (20.46) does not decrease with t and it tends to infinity as $t \to \infty$ and $A[z\phi(0)] + \psi(0) = 0$. Therefore, these properties will be assumed throughout this section, although some of them will not be needed for formal proofs.

An important additional simplifying assumption is that

$$A(s),\, s \in [0, \infty);\;\; \phi(t),\, t \in [0, \infty) \tag{20.47}$$

are increasing functions of their arguments and $A(0) = 0$, although some generalizations (e.g., only for ultimately increasing functions) are easily performed. Therefore, we will view $1 - \exp[-A(z\phi(t))]$, $z \neq 0$ in this chapter as a lifetime Cdf.

It should be noted that model (20.46) can be also easily generalized to the form $\Lambda(t, z) = A[g(z)\phi(t)] + \psi(t) + \eta(z)$ for some properly defined functions $g(z)$ and $\eta(z)$. However, we cannot generalize any further (at least, at this stage), and the multiplicative form of the arguments in $A[g(z)\phi(t)]$ is important to our method of deriving asymptotic relations. It is also clear that the additive term $\psi(t)$, although important in applications, provides only a slight generalization for further analysis of $\lambda_m(t)$, as (20.46) can be interpreted in terms of two components in series (or, equivalently, as two competing risks).

The failure rate, which corresponds to the cumulative failure rate $\Lambda(t, z)$, is

$$\lambda(t, z) = z\phi'(t)A'[z\phi(t)] + \psi'(t), \tag{20.48}$$

where, by $A'[z\phi(t)]$, we in fact mean $\mathrm{d}A[z\phi(t)]/\mathrm{d}[z\phi(t)]$.

Now we can explain why we start with the cumulative failure rate and not with the failure rate itself, which is common in lifetime modeling. The reason is that one can easily suggest intuitive interpretations of (20.46), whereas it is certainly not as simple to interpret the failure rate structure in the form (20.48) without stating that it just follows from the structure of the cumulative failure rate.

Relation (20.46) defines a rather broad class of survival models which can be used, for example, to model the impact of the environment on survival characteristics. The proportional hazards, additive hazards and accelerated life models, widely used in reliability, survival analysis and risk analysis, are the obvious specific cases of our relations (20.46) or (20.48):

PH (multiplicative) model:

Let

$$A(u) \equiv u,\;\; \phi(t) = \Lambda(t),\;\; \psi(t) \equiv 0.$$

Then

$$\lambda(t, z) = z\lambda(t), \qquad \Lambda(t, z) = z\Lambda(t). \tag{20.49}$$

ALM:

Let

$$A(u) \equiv \Lambda(u),\;\; \phi(t) = t,\;\; \psi(t) \equiv 0.$$

Then

$$\Lambda(t, z) = \int_0^{tz} \lambda(u)\,\mathrm{d}u, \qquad \lambda(t, z) = z\lambda(tz). \tag{20.50}$$

AH *model*:

Let

$$A(u) \equiv u,\;\; \phi(t) = t,\;\; \psi(t) \text{ is increasing, } \psi(0) = 0.$$

Then

$$\lambda(t, z) = z + \psi'(t), \qquad \Lambda(t, z) = zt + \psi(t). \tag{20.51}$$

The functions $\lambda(t)$ and $\psi'(t)$ act as baseline failure rates in equations (20.49), (20.50) and (20.51), respectively. Note that, in all of these models, the functions $\phi(t)$ and $A(s)$ increase monotonically.

The asymptotic behaviors of the mixture failure rates for the PH and AH models have been studied for some specific mixing distributions in, for example, *Gurland* and *Sethuraman* [20.20] and *Finkelstein* and *Esaulova* [20.9]. On the other hand, as far as we know, the mixture failure rate for the ALM has only been considered at a descriptive level in *Anderson* and *Louis* [20.26].

20.3.2 Main Result

The next theorem derives an asymptotic formula for the mixture failure rate $\lambda_m(t)$ under rather mild assumptions. We use an approach related to the ideology of generalized convolutions, for example *Laplace* and *Fourier* transforms and (especially) Mellin convolutions [20.27].

Theorem 20.8
Let the cumulative failure rate $\Lambda(t, z)$ be given by the model (20.46) and the mixing pdf $\pi(z)$ be defined as

$$\pi(z) = z^{\alpha}\pi_1(z), \tag{20.52}$$

where $\alpha > -1$ and $\pi_1(z)$, $\pi_1(0) \neq 0$ is a function that is bounded in $[0, \infty)$ and continuous at $z = 0$.

Assume also that $\phi(t)$ increases to infinity:

$$\phi(t) \to \infty \quad \text{as} \quad t \to \infty \tag{20.53}$$

and that

$$\begin{aligned} &\exp[-A(s)]s^{\alpha+1} \to 0 \quad \text{as} \quad t \to \infty, \\ &\int_0^{\infty} \exp[-A(s)]s^{\alpha}\,\mathrm{d}s < \infty. \end{aligned} \tag{20.54}$$

Then

$$\lambda_{\mathrm{m}}(t) - \psi'(t) \sim (\alpha + 1)\frac{\phi'(t)}{\phi(t)}. \tag{20.55}$$

By relation (20.55) we (as usual) mean asymptotic equivalence, and we write $a(t) \sim b(t)$ as $t \to \infty$, if $\lim_{t\to\infty}[a(t)/b(t)] = 1$.
Proof: Firstly, we need a simple lemma for the Dirac sequence of functions. ■

Lemma 20.2
Let $g(z)$, $h(z)$ be non-negative functions in $[0, \infty)$ that satisfy the following conditions:

$$\int_0^{\infty} g(z)\,\mathrm{d}z < \infty, \tag{20.56}$$

and $h(z)$ is bounded and continuous at $z = 0$.

Then, as $t \to \infty$:

$$t\int_0^{\infty} g(tz)h(z)\,\mathrm{d}z \to h(0)\int_0^{\infty} g(z)\,\mathrm{d}z. \tag{20.57}$$

Proof: Substituting $u = tz$,

$$t\int_0^{\infty} g(tz)h(z)\,\mathrm{d}z = \int_0^{\infty} g(u)h(u/t)\,\mathrm{d}u.$$

The function $h(u)$ is bounded and $h(u/t) \to 0$ as $t \to \infty$, thus convergence (20.57) holds by the dominated convergence theorem. ■

We are now able to prove Theorem 20.8. The proof is straightforward, as we use definition (20.1) and Lemma 20.2.

The survival function for the model (20.46) is

$$\bar{F}(t, z) = \exp\{-[A(z\phi(t)] - \psi(t)\}.$$

Taking into account that $\phi(t) \to \infty$ as $t \to \infty$, and applying Lemma 20.2 to the function $g(u) = \exp[-A(u)]u^{\alpha}$:

$$\begin{aligned} &\int_0^{\infty} \bar{F}(t, z)\pi(z)\,\mathrm{d}z = \int_0^{\infty} \exp\{-[A(z\phi(t))] \\ &- \psi(t)\}z^{\alpha}\pi_1(z)\,\mathrm{d}z \\ &\sim \frac{\exp[-\psi(t)]\pi_1(0)}{\phi(t)^{\alpha+1}}\int_0^{\infty} \exp[-A(s)]s^{\alpha}\,\mathrm{d}s, \end{aligned} \tag{20.58}$$

where the integral is finite due to the condition given in (20.54).

The corresponding probability density function is:

$$\begin{aligned} f(t, z) =&\{A'[z\phi(t)]z\phi'(t) \\ &+ \psi'(t)\}\exp\{-A[z\phi(t)] - \psi(t)\} \\ =&A'[z\phi(t)]z\phi'(t)\exp\{-A[z\phi(t)] \\ &- \psi(t)\} + \psi'(t)\bar{F}(t, z). \end{aligned}$$

Similarly, applying Lemma 20.2 gives:

$$\begin{aligned} &\int_0^{\infty} f(t, z)\pi(z)\,\mathrm{d}z - \psi'(t)\int_0^{\infty} \bar{F}(t, z)\pi(z)\,\mathrm{d}z \\ &= \phi'(t)\exp[-\psi(t)]\int_0^{\infty} A'[z\phi(t)] \\ &\exp\{-A[z\phi(t)]\}z^{\alpha+1}\pi_1(z)\,\mathrm{d}z \\ &\sim \frac{\phi'(t)\exp[-\psi(t)]\pi_1(0)}{\phi(t)^{\alpha+2}}\int_0^{\infty} A'(s) \\ &\exp[-A(s)]s^{\alpha+1}\,\mathrm{d}s \end{aligned} \tag{20.59}$$

Integrating by parts and using condition (20.54):

$$\begin{aligned} &\int_0^{\infty} A'(s)\exp[-A(s)]s^{\alpha+1}\,\mathrm{d}s \\ &= (\alpha + 1)\int_0^{\infty} \exp[-A(s)]s^{\alpha}\,\mathrm{d}s. \end{aligned} \tag{20.60}$$

Combining relations (20.58)–(20.60), finally:

$$\frac{\int_0^\infty f(t,z)\pi(z)\,\mathrm{d}z}{\int_0^\infty \bar{F}(t,z)\pi(z)\,\mathrm{d}z} - \psi'(t) \sim (\alpha+1)\frac{\phi'(t)}{\phi(t)}.$$

It is easy to see that assumption (20.52) holds for the main lifetime distributions, such as Weibull, gamma, log-normal etc. Assumption (20.53) states a natural condition for the function $\phi(t)$, which can often be viewed as a scale transformation. Conditions (20.54) mean that the Cdf $1-\exp[-A(s)]$ should not be 'too heavy-tailed' (as e.g. the Pareto distribution $1-s^{-\beta}$, for $s \geq 1$, $\beta-\alpha > 1$) and are equivalent to the condition that a moment of order $\alpha+1$ exists for this Cdf. The examples shown in the next subsection will clearly illustrate that these conditions are not stringent at all and can be easily met in most practical situations.

A crucial feature of this result is that the asymptotic behavior of the mixture failure rate depends only [omitting an obvious additive term $\psi(t)$] on the behavior of the mixing distribution near to zero and on the derivative of the logarithm of the scale function $\phi(t) : [\log\phi(t)]' = \phi'(t)/\phi(t)$. When $\pi(0) \neq 0$ and $\pi(z)$ is bounded in $[0,\infty)$, the result does not depend on the mixing distribution at all, as $\alpha = 0$!

20.3.3 Specific Models

Multiplicative (PH) Model

In the conventional notation, the baseline failure rate is usually denoted by $\lambda_0(t)$ [or $\lambda_\mathrm{b}(t)$]. Therefore, model (20.6) reads:

$$\lambda(t,z) = z\lambda_0(t), \quad \Lambda_0(t) = \int_0^t \lambda_0(u)\,\mathrm{d}u \tag{20.61}$$

and the mixture failure rate is given by

$$\lambda_\mathrm{m}(t) = \frac{\int_0^\infty z\lambda_0(t)\exp[-z\Lambda_0(t)]\pi(z)\,\mathrm{d}z}{\int_0^\infty \exp[-z\Lambda_0(t)]\pi(z)\,\mathrm{d}z}. \tag{20.62}$$

As $A(u) \equiv u$, $\phi(t) = \Lambda_0(t)$, $\psi(t) \equiv 0$ in this specific case, Theorem 20.8 simplifies to:

Corollary 20.1

Assume that the mixing pdf $\pi(z)$, $z \in [0,\infty)$ can be written as

$$\pi(z) = z^\alpha \pi_1(z), \tag{20.63}$$

where $\alpha \geq -1$ and $\pi_1(z)$ is bounded in $[0,\infty)$, continuous at $z=0$ and $\pi_1(0) \neq 0$.

Then the mixture failure rate for the multiplicative model (20.61) has the following asymptotic behavior:

$$\lambda_\mathrm{m}(t) \sim \frac{(\alpha+1)\lambda_0(t)}{\int_0^t \lambda_0(u)\,\mathrm{d}u}. \tag{20.64}$$

The mixture failure rate given by (20.62) can be obtained explicitly when the Laplace transform of the mixing pdf $\tilde{\pi}(t)$ is easily computed. As the cumulative failure rate increases monotonically with t, the mixture survival function is written in terms of the Laplace transform as:

$$\int_0^\infty \exp[-z\Lambda_0(t)]\pi(z)dz = \tilde{\pi}[\Lambda_0(t)].$$

Therefore, (20.62) becomes:

$$\lambda_\mathrm{m}(t) = -\frac{(\tilde{\pi}[\Lambda_0(t)])'}{\tilde{\pi}[\Lambda_0(t)]} = -(\log\tilde{\pi}[\Lambda_0(t)])'$$

and the corresponding inverse problem can also be solved; in other words, given the mixture failure rate and the mixing distribution, obtain the baseline failure rate [20.28].

Example 20.4: As for examples 20.1 and 20.3, consider a frailty model (20.61) where Z has a gamma distribution, which, for notational convenience, is written in a slightly different form:

$$\pi(z) = \left(\frac{z}{b}\right)^{c-1}\exp\left\{-\frac{z}{b}\right\}\frac{1}{b\Gamma(c)}, \tag{20.65}$$

where $b, c > 0$.

The expected value Z is bc and the variance is b^2c. The Laplace transform of $\pi(z)$ is $\tilde{\pi}(t) = c(tb+1)^{-c}$ and therefore the mixture failure rate is given by the following expression, which is the same as (20.22):

$$\lambda_\mathrm{m}(t) = \frac{bc\lambda_0(t)}{1+b\int_0^t \lambda_0(u)\,\mathrm{d}u}. \tag{20.66}$$

Obviously, the asymptotic behavior of $\lambda_m(t)$ can be analyzed explicitly. Consider two specific cases.

If the baseline distribution is Weibull with $\lambda_0(t) = \lambda t^\beta$, $\beta \geq 0$, then the mixture failure rate (20.66) is

$$\lambda_m(t) = \frac{(\beta+1)\lambda bct^\beta}{(\beta+1)+\lambda bt^{\beta+1}}, \tag{20.67}$$

which converges to 0 as $t \to \infty$ because it is $\sim (\beta+1)ct^{-1}$, exactly as prescribed by formula (20.64) of Corollary 20.1 ($c = \alpha + 1$).

If the baseline distribution is Gompertz with $\lambda_0(t) = \mu \exp(\beta t)$, then a simple transformations gives

$$\lambda_m(t) = \frac{\beta c \exp(\beta t)}{\exp(\beta t) + \left(\frac{\beta}{\mu b} - 1\right)}. \tag{20.68}$$

If $b = \beta/\mu$, then $\lambda_m(t) \equiv \beta c$; if $b > \beta/\mu$, then $\lambda_m(t)$ increases to β/μ; if $b < \beta/\mu$, it decreases to β/μ.

It is reasonable to compare the asymptotic behaviors of (20.67) and (20.68) for the same mixing distribution (20.65). For the Weibull Cdf, the mixture failure rate converges to 0. This means that, within the framework of the multiplicative model, where the family of failure rates is ordered in z, we can still speak of convergence to the failure rate of the strongest population, defining the $z = 0$ case as a 'generalized' (or formal) strongest failure rate: $\lambda(t, 0) = 0$. However, the failure rate for a Gompertz Cdf does not converge to 0 – it converges to a constant, thus violating the principle of converging to the failure rate of the strongest population, even when formulated in a 'generalized' form! The reason for this is the sharp increase in the function $\phi(t)$, which is proportional to $\exp(\beta t)$ in the latter case.

Accelerated Life Model

In the conventional notation, this model is written as:

$$\lambda(t, z) = z\lambda_0(tz),$$
$$\Lambda_0(tz) = \int_0^{tz} \lambda_0(u)\,du. \tag{20.69}$$

Although the ALM also has a very simple definition, the presence of the mixing parameter z in the arguments make analysis of the mixture failure rate more complex than in the multiplicative case. Therefore, as mentioned previously, this model is practically unstudied. The mixture failure rate in this specific case is

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda_0(tz)\exp[-\Lambda_0(tz)]\pi(z)\,dz}{\int_0^\infty \exp[-\Lambda_0(tz)]\pi(z)\,dz}. \tag{20.70}$$

The asymptotic behavior of $\lambda_m(t)$ can be described as a specific case of Theorem 20.8 with $A(s) = \Lambda_0(s)$, $\phi(t) = t$ and $\psi(t) \equiv 0$:

Corollary 20.2

Assume that the mixing pdf $\pi(z)$, $z \in [0, \infty)$ can be defined as $\pi(z) = z^\alpha \pi_1(z)$, where $\alpha > -1$, $\pi_1(z)$ is continuous at $z = 0$ and bounded in $[0, \infty)$, $\pi_1(0) \neq 0$.

Let the baseline distribution with cumulative rate $\Lambda_0(t)$ have a moment of order $\alpha + 1$. Then

$$\lambda_m(t) \sim \frac{\alpha+1}{t} \tag{20.71}$$

as $t \to \infty$.

The conditions of Corollary 20.2 are not that strong and are relatively natural. Most widely used lifetime distributions have all of the moments. The Pareto distribution will be discussed in the next example.

As already stated, the conditions on the mixing distribution hold for the gamma and the Weibull distributions, which are commonly used as mixing distributions.

Relation (20.71) is really surprising, as it does not depend on the baseline distribution, which seems strange, at least at first sight. It is also dramatically different to the multiplicative case (20.64). It follows from Example 20.4 that both asymptotic results coincide in the case of the Weibull baseline distribution; this is obvious, as the ALM can only be reparameterized to end up with a PH model and vice versa for the Weibull distribution.

The following example shows other possible asymptotic behaviors for $\lambda_m(t)$ when one of the conditions of Corollary 20.2 does not hold.

Example 20.5: Consider the gamma mixing distribution $\pi(z) = z^\alpha \exp(-x)/\Gamma(\alpha+1)$. Let the baseline distribution be the Pareto distribution with density $f_0(t) = \beta/t^{\beta+1}$ $t \geq 1, \beta > 0$.

For $\beta > \alpha + 1$ the conditions of Corollary 20.2 hold and relation (20.71) occurs. Let $\beta \leq \alpha + 1$, which means that the baseline distribution doesn't have the $(\alpha+1)$th moment. Therefore, one of the conditions of Corollary 20.2 does not hold. In this case:

$$\lambda_m(t) \sim \frac{\beta}{t}$$

as $t \to \infty$, which can be shown by direct integration:

$$\int_0^\infty z f_0(tz)\pi(z)\,dz$$

$$= \int_{1/t}^\infty \frac{\beta z}{\Gamma(\alpha+1)t^{\beta+1}z^{\beta+1}} \exp(-z) z^\alpha \,dz$$

$$= \frac{\beta}{\Gamma(\alpha+1)t^{\beta+1}} \int_{1/t}^\infty z^{\alpha-\beta} \exp(-z)\,dz$$

$$\sim \frac{\Gamma(\alpha-\beta+1)\beta}{\Gamma(\alpha+1)t^{\beta+1}}$$

and

$$\int_0^\infty \bar{F}_0(tz)\pi(z)\,dz = \int_0^{1/t} \frac{\exp(-z)z^\alpha}{\Gamma(\alpha+1)}\,dz + \int_{1/t}^\infty \frac{\exp(-z)z^\alpha}{t^\beta z^\beta \Gamma(\alpha+1)}\,dz .$$

As $t \to \infty$, the first integral on the right-hand side is equivalent to

$$\int_0^{1/t} \frac{z^\alpha}{\Gamma(\alpha+1)}\,dz = \frac{1}{t^{\alpha+1}\Gamma(\alpha+2)}$$

and the second integral is equivalent to $\Gamma(\alpha-\beta+1)/\Gamma(\alpha+1)t^\beta$, which decreases more slowly for $\beta \le \alpha$; therefore, the sum of the two integrals is $\Gamma(\alpha-\beta+1)/\Gamma(\alpha+1)t^\beta$. Eventually:

$$\lambda_m(t) \sim \frac{\Gamma(\alpha-\beta+1)\beta}{\Gamma(\alpha+1)t^{\beta+1}} \cdot \frac{\Gamma(\alpha+1)t^\beta}{\Gamma(\alpha-\beta+1)} = \frac{\beta}{t} .$$

If $\beta = \alpha+1$, then

$$\int_0^\infty z f_0(tz)\pi(z)\,dz$$

$$= \frac{\alpha+1}{\Gamma(\alpha+1)t^{\alpha+2}} \int_{1/t}^\infty z^{-1} \exp(-z)\,dz$$

and since

$$\int_0^{1/t} z^\alpha \,dz = o(t^{-\alpha-1}) \int_{1/t}^\infty z^{-1} \exp(-z)\,dz ,$$

we obtain

$$\int_0^\infty \bar{F}_0(tz)\pi(z)\,dz$$

$$= \frac{1}{\Gamma(\alpha+1)t^{\alpha+1}} \int_{1/t}^\infty z^{-1} \exp(-z)\,dz .$$

Therefore

$$\lambda_m(t) \sim \frac{\alpha+1}{t} = \frac{\beta}{t} .$$

and both cases can be combined into one relation

$$\lambda_m(t) \sim \frac{\min(\beta, \alpha+1)}{t} .$$

It can be shown that the same asymptotic relation holds not only for the gamma distribution, but also for any other mixing distribution $\pi(z)$ of the form $\pi(z) = z^\alpha \pi_1(z)$. If $\beta > \alpha+1$, the function $\pi_1(z)$ should be bounded and $\pi_1(0) \neq 0$.

Due to its simplicity, the asymptotic behavior of $\lambda_m(t)$ in the additive hazards model (20.51) does not warrant special attention. As $A(s) = s$ and $\phi(t) = t$, conditions (20.53) and (20.54) of Theorem 20.8 hold and the asymptotic result in (20.55) simplifies to:

$$\lambda_m(t) - \psi'(t) \sim \frac{\alpha+1}{t} .$$

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