23. Statistical Approaches to Planning of Accelerated Reliability Testing

This chapter presents a few statistical methods for designing test plans in which products are tested under harsher environment with more severe stresses than usual operating conditions. Following a short introduction, three different types of testing conditions are dealt with in Sects. 23.2, 23.3, and 23.4; namely, life testing under constant stress, life testing in which stresses are increased in steps, and accelerated testing by monitoring degradation data. Brief literature surveys of the work done in these areas precede presentations of methodologies in each of these sections.

In Sect. 23.2, we present the conventional framework for designing accelerated test plans using asymptotic variance of maximum likelihood estimators (MLE) derived from the Fisher information matrix. We then give two possible extensions from the framework for accelerated life testing under three different constant stress levels; one based on a nonlinear programming (NLP) formulation so that experimenters can specify the desired number of failures, and one based on an enlarged solution space so that the design of the test plan can be more flexible in view of the many possible limitations in practice. These ideas are illustrated using numerical examples and followed by a comparison across different test plans.

We then present planning of accelerated life testing (ALT) in which stresses are increased in steps and held constant for some time before the next increment. The design strategy is based on a target acceleration factor which specifies the desired time compression needed to complete the test compared to testing under use conditions.

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Using a scheme similar to backward induction in dynamic programming, an algorithm for planning multiple-step step-stress ALT is presented.

In Sect. 23.4, we consider planning problems for accelerated degradation test (ADT) in which degradation data, instead of lifetime data, are used to predict a product's reliability. We give a unifying framework for dealing with both constant-stress and step-stress ADT. An NLP model which minimizes cost with precision constraint is formulated so that the tradeoff between getting more data and the cost of conducting the test can be quantified.

Technology and market forces have created generation after generation of highly reliable products. For most products, it is no longer feasible to test products/components in the usual manner at their design conditions as the time needed to obtain sufficient failure information so as to understand the products' behaviors and to quantify their reliability is prohibitively long. In applications where safety and mission success are critical, accelerated reliability testing (ART) is commonly deployed to quantify products' reliability. The basic idea of ART is to achieve time compression so that failure information can be precipitated within a reasonable test duration. To achieve the requisite time compression, products are tested either under a harsher environment or under more intensive usage than the usual use condition. For many products which are always on, only the former is feasible and will be dealt with in this chapter.

One of the important issues in ART is how to plan a test by determining the testing time, stress levels, sample sizes for different stress levels, etc., so as to achieve cost saving and/or efficiency. Good planning does not only lead to shorter test time or fewer test specimens or both; but more importantly; a good test plan will result in a more precise estimate for the reliability measure of interest, which could be critical, in some cases, in meeting a design or customer specification.

In this chapter, we present test plans for three types of acceleration tests, namely, constant-stress accelerated life test, step-stress accelerated life test and accelerated degradation test, in the following three sections, respectively. A quick review of the literature in each of these areas will first be presented at the beginning of the respective sections. The statistical approach for planning these experiments will then follow. At the end of each section, a numerical example will be given to illustrate the application of the planning methodology.

23.1 Planning Constant-Stress Accelerated Life Tests

The most commonly adopted ART is the constant-stress accelerated life test (CSALT) which comprises multiple sub-samples tested at different but fixed stress levels, at which time-censored failure times are recorded. For ease of administration, commonly used CSALT test plans consist of equally spaced test stresses, each with the same number of test specimens. Such standard plans are highly inefficient for estimating product reliability at the design stress as fewer failures are expected at lower stress levels unless the test duration is much longer than other higher stress levels. Pioneering work by *Chernoff* [23.1] and *Meeker* and *Nelson* [23.2] proposed statistically optimal plans for constant-stress ALT which involved only two stress levels. As a result, these plans, though statistically optimal, cannot be used to validate the assumed stress–life relationship. To remedy this problem, *Meeker* and *Hahn* [23.3] proposed the use of a 4:2:1 allocation ratio for low-, middle- and high-stress levels and gave the optimal low-stress level by assuming that the middle stress is the average of the high- and low-stress levels. An alternative approach is to set equally spaced stress levels with equal allocations as in *Nelson* and *Kielspinski* [23.4] and *Nelson* and *Meeker* [23.5]. The main reason for using a predetermined allocation and middle stress is that the statistically optimal plan only involves two stress levels once the stress–life relation is given. *Nelson* [23.6] provides a comprehensive treatment of these plans and thus they will not be presented in this chapter.

Another approach adopted by *Yang* [23.7] and *Yang* and *Jin* [23.8] to overcome this problem is to add test constraints involving expected minimum number of failures to be observed at the mid-stress levels. The resulting plan is sensitive to the value of the constraint for expected minimum number of failures as it indirectly determines the mid-stress levels and corresponding sample allocation. *Tang* [23.9] imposed constraints that limit the probability of having probability plots with best-fit lines crossing at the lower tails. In this way, the stress levels and their corresponding sample allocations are best suited to infer whether the shape parameters are indeed different at different stress levels. Motivated by the fact that the mid-stress level is meant for validating the assumed stress–life model, *Tang* et al. [23.10] considered test plans in which the mid-stress level will have the least influence on the slope of the stress–life relationship plot. Although these plans are more robust, variances of the estimates at design stress for these compromised plans may be much higher than those under the two-stress-level optimum plans. To address this, *Tang* and *Yang* [23.11] proposed a graphical approach for planning multiple constant-stress-level ALT so that the uncertainty involved for some estimate of interest is not worse than that of a statistical optimum plan by a margin determined by the experimenter before the test. Recently, *Tang* and *Xu* [23.12] presented a general framework for planning ALT. We shall summarize these plans in this Section and provide a simple comparison.

There are many other papers in statistical planning of CSALT that are not presented here. Most of the work has been summarized in *Nelson* [23.6] and *Meeker* and *Escobar* [23.13]. A comprehensive bibliography is given in *Nelson* [23.14]. For example, *Meeter* and *Meekers* [23.15] presented optimal plans involving a nonconstant shape parameter, in the case of Weibull distribution, and *Tang* et al. [23.16] gave optimal plans involving a failure-free life represented by a location parameter.

Notations and Assumptions

S_d, S_i, S_h	Stress (design, i, high) levels, $i = 1, 2, \ldots, h-1$
ξ_i	$\frac{S_i - S_h}{S_d - S_h}$: normalized stress levels, $i = 1, 2, \ldots, h$, so that $\xi_h = 0$, $\xi_d = 1$
π_i	Proportion of units tested at ξ_i
n	Total sample size
n_i	Actual number of test units needed at ξ_i
h	Total number of stress levels; also denotes the high stress level
τ_i	Censoring time at low S_i, $i = 1, 2, \ldots, h$
$\mathbf{F}$	Fisher information matrix for the plan with a unit of n independent observations
σ	Scale parameter
μ	Location parameter of the smallest extreme value distribution
μ_D, μ_H	μ (design, highest test stress)
p_i	Probability that units tested at ξ_i fail before τ_i
m	A maximum bound
γ_i	Expected number of failure at stress ξ_i
Avar[log(t)]	Asymptotic variance of natural logarithm of the estimate of a fixed percentile of interest t
$\hat{t}_{0.43}$	The estimate of the 43rd percentile of the time to failure; which is also the maximum likelihood estimation (MLE) of the mean life of the smallest extreme value distribution at design stress

23.1.1 The Common Framework

For ease of discussion and comparison, the common assumptions and framework of these plans are:

1. The lifetime follows a Weibull distribution of which the natural logarithm of life, $y = \ln(t)$, has a smallest extreme-value distribution with a reliability function given by
$$R(y) = \exp\left\{-\exp\left[{}^{(y-\mu)}\big/\sigma\right]\right\} \quad -\infty < y < +\infty\,, \tag{23.1}$$
where μ is the location parameter and σ is the scale parameter.
2. The scale parameter does not depend on the stress level and;
3. The location parameter is a linear function of the transformed stress:
$$\mu(\xi) = \alpha + \beta\,\xi\,, \tag{23.2}$$
where, α, β are unknown parameters to be estimated from test data.
4. For planning purpose, the initial guess values for σ, p_D and p_H are given.
5. Test units allocated to ξ_i are tested simultaneously until τ_i.
6. The highest test stress is specified.
7. The test stresses are above the design stress.
8. The measure of statistical precision is given by the asymptotic variance of the MLE of the mean life of the smallest extreme value distribution, which is the 43rd percentile ($\hat{t}_{0.43}$), at design stress.

For test conducted at three stress levels, the above variance is given by:

$$\begin{aligned} y_{\text{var}} &= \text{Avar}\left[\hat{t}_{0.43}(1)\right] \\ &= [1\,,1\,,-0.577\,22]\mathbf{F}^{-1}[1\,,1\,,-0.577\,22]'\,, \end{aligned} \tag{23.3}$$

where $\mathbf{F}$ is the Fisher information matrix for the plan given below

$$\mathbf{F} = \frac{n}{\sigma^2}\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}\,,$$

$$M_{11} = \sum_{i=1}^{3} \pi_i A(\psi_i)\,,$$

$$M_{22} = \sum_{i=1}^{3} \pi_i \xi_i^2 A(\psi_i)\,,$$

$$M_{33} = \sum_{i=1}^{3} \pi_i C(\psi_i)\,,$$

$$M_{12} = M_{21} = \sum_{i=1}^{3} \pi_i \xi_i A(\psi_i)\,,$$

$$M_{13} = M_{31} = \sum_{i=1}^{3} \pi_i B(\psi_i)\,,$$

$$M_{32} = M_{23} = \sum_{i=1}^{3} \pi_i \xi_i B(\psi_i)\,,$$

$$A(\psi_i) = 1 - \exp\left\{-\exp\left[(\psi_i - \theta)\big/\sigma\right]\right\}\,,$$

Part C | 23.1

$$B(\psi_i) = \int_0^{\exp(\psi_i)} \ln(w) w \exp(-w)\,\mathrm{d}w + \left(\exp\left\{-\exp\left[(\psi_i - \theta)/\sigma\right]\right\}\right) \times \psi_i \exp(\psi_i)\,,$$

$$C(\psi_i) = 1 - \exp\left\{-\exp\left[(\psi_i - \theta)/\sigma\right]\right\} + \int_0^{\exp(\psi_i)} \ln^2(w) w \exp(-w)\,\mathrm{d}w + \left(\exp\left\{-\exp\left[(\psi_i - \theta)/\sigma\right]\right\}\right) \times \psi_i^2 \exp(\psi_i)\,,$$

$$\psi_i = [\tau_i - (\alpha + \beta S_H)]/\sigma[\beta(S_D - S_H)]/\sigma\,.$$

Note that, usually, the standardized asymptotic variance, $\text{Svar} = \text{Avar}^*(n/\sigma^2)$ is used in optimization.

23.1.2 Yang's Approach

Yang [23.7] formulated a constrained nonlinear programme to obtain an optimal test plan. He allowed for varying censoring time at different stress levels and aimed to minimize the total test duration while achieving statistical efficiency and robustness. The objective function to be minimized consists of a weighted sum of the standardized asymptotic variance and the product of the test duration at the lowest stress level and the sum of test duration; i. e.

$$\begin{aligned}
&\text{Minimize: } \omega\,\text{AVar}\left[\log(t_{0.43})\right] + (1-\omega)\tau_1 \sum_i \tau_i\,,\\
&\qquad 0 \le \omega \le 1\,,\\
&\text{Subject to: } \tau_i > \tau_j \quad \text{for } i < j\,,\\
&\qquad \sum_i \pi_i = 1\,;\\
&\qquad \xi_i > xi_j > \xi_h = 0 \quad \text{for } i < j\,,\\
&\qquad n\pi_i p_i \ge \gamma_i \quad \text{for all } i\,.
\end{aligned} \tag{23.4}$$

Yang [23.7] presented the solution for four stress levels, i. e. $h = 4$, and $\gamma_i = 10$ for all i. He transformed the above into an unconstrained optimization using a penalty-function approach to obtain nearly optimal plans. A similar approach is given in *Yang* and *Jin* [23.8] with 3 stress levels, known as the three-level best-compromised test plan, where the middle stress is the average of the low and high stresses.

23.1.3 Flexible Near-Optimal Plans

In the above plans, the inclusion of additional stress levels will give rise to less-precise estimates compared to the statistically optimal two-level CSALT. For better management of risk in planning ALT, it would be useful to know the extent by which the variance of the percentile of interest is inflated. For practical implementation, test plans should also be sufficiently flexible, in that some range of stress levels and the corresponding allocations are provided instead of stipulating specific stress levels and allocations. In the following, we present flexible near-optimal plans proposed by *Tang* and *Yang* [23.11] that provides the flexibility while limiting the loss in precision. For simplicity, we assume that the censoring time at all stress levels are identical, i. e. $\tau_i = T$ for all i.

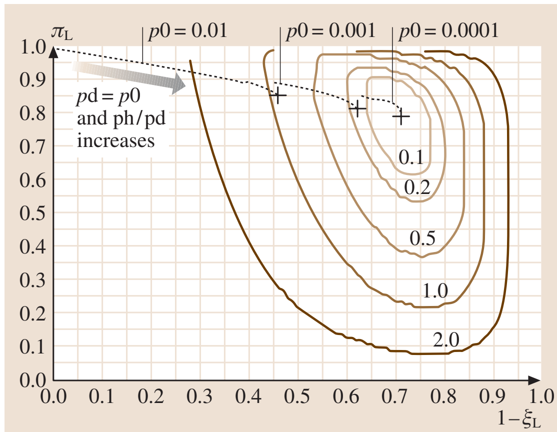

Fig. 23.1 Solution space of a two-stress-level CSALT

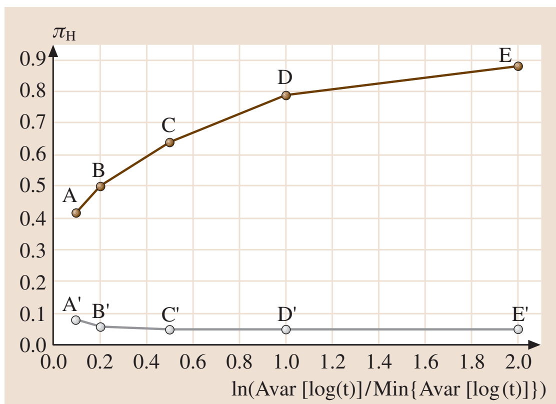

Fig. 23.2 Feasible region of π_H for different values of m in (23.5)

Consider a solution of two-stress-level CSALT depicted in Fig. 23.1. The statistically optimal solutions specify the low stress level (ξ_L) and the corresponding allocation (π_L), which are marked by "+" for different p_d. A possible approach in enlarging the solution space is to consider combinations of (ξ_L, π_L) such that the following ratio is restricted to a maximum bound tolerable, say m:

$$\ln\left(\frac{\text{Avar}\left[\log(t)\right]}{\text{Min}\left\{\text{Avar}\left[\log(t)\right]\right\}}\right) \le m\,. \tag{23.5}$$

Figure 23.1 depicts the contours that enclosed (ξ_L, π_L) for m ranging from 0.1–2.

This principle of enlarging the solution space can be generalized to 3SCSALT. In the following, we give a step-by-step description on how a contour plot of the solution space for 3SCSALT is constructed.

1. For different combinations of p_d and p_h, the statistically optimal 2SCSALT plan with the optimal value of π_H can be determined. By allowing the value of m in (23.5) to vary, a range of π_H as a function of m can easily be determined. The results are depicted in Fig. 23.2 for m= 0.1, 0.2, 0.5, 1.0, 2.0.
2. For each value of π_H, the solution space of ξ_M and ξ_L can be obtained using the same criterion as in (23.5). An example for $\pi_H = 0.15$ is given in Fig. 23.3 for m= 0.1, 0.2, 0.5, 1.0, 2.0. Interestingly, the solution space of ξ_M and ξ_L is enclosed in an approximately right-angled triangle sharing the common slope of 1 as $\xi_L > \xi_M$.
3. As one would usually like to ensure that ξ_L and ξ_M are sufficiently far apart, the preferred solution will be at the vertex of the right angle. Tracing the vertices for different values of m, the various combinations of ξ_L and ξ_M form a straight line, as depicted in Fig. 23.2.
4. Repeated applications of this procedure for different values of π_H result in a plot depicting the solution space of ξ_L and ξ_M of 3SCSALT, as shown in Fig. 23.4. In Fig. 23.4, we use the boundary values of π_H marked by A, B, C, D, E, A', B', C', D' and E' in Fig. 23.2, and some intermediate values of π_H (0.13, 0.18, 0.25, 0.33) to generate the corresponding lines that give the preferred solutions of ξ_L and ξ_M for different values of m.
5. The contours of various setting of m (0.1, 0.2, 0.5, 1, 2) are superimposed on these lines so that ξ_L and ξ_M can be read off by interpolation between the lines of π_H for a given m value.

To determine π_L and π_M, since the main purpose of having a middle stress is to validate the stress life model, one may prefer minimum allocation to the middle stress such that there are sufficient failures to detect nonlinearity, if it exists. In this case, π_M is given by

$$\pi_M = \frac{\gamma_m}{n F_M(T)}\,, \tag{23.6}$$

where γ_M is the minimum number of failures expected under the middle stress level, and $F_M(T)$ is the prob-

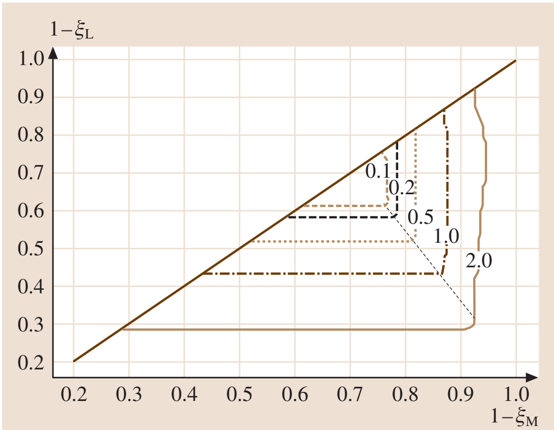

Fig. 23.3 Solution space for the middle and low stress levels for $\pi_H = 0.15$

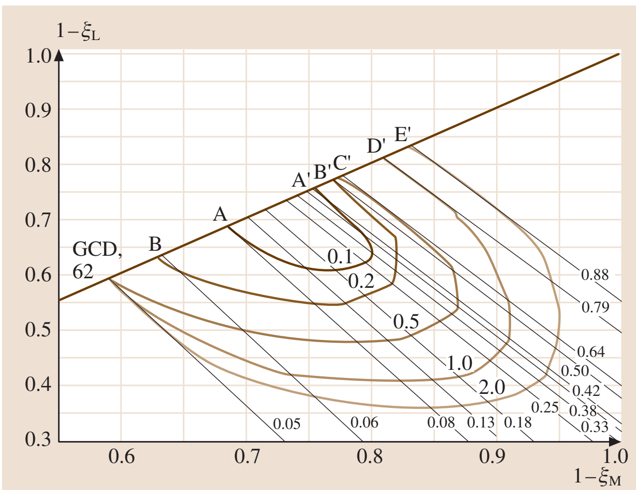

Fig. 23.4 Contour plots of the solution space for 3SCSALT

ability of failure by the end of the test at the middle stress.

Alternatively, as suggested by *Tang* [23.9], a good planning strategy is to set the centroid of the lower and middle stress levels, weighted by their respective allocation, in a near-optimal 3SCSALT plan equal to the optimal low stress in the statistical optimal plan, i. e.

$$\frac{\xi_{3M}\pi_{3M}+\xi_{3L}\pi_{3L}}{\pi_{3M}+\pi_{3L}}=\xi_{2L}^{*}$$

$$\Rightarrow \xi_{3M}\pi_{3M}+\xi_{3L}\pi_{3L}$$

$$=\xi_{2L}^{*}\left(1-\pi_{H}^{*}\right)\ , \tag{23.7}$$

where the subscript numbers 3 and 2 represent the number of stresses under which the test plans are designed, superscript "*" means optimal values; note that $\pi_{3H}=\pi_{H}^{*}$.

In summary, the steps in obtaining a 3SCSALT test plan are:

- For the given inputs, solve for the optimal π_H^*.
- For a given value of m, obtain ξ_L and ξ_M from Fig. 23.4 by interpolation between the lines of π_H to that corresponds to the optimal π_H.
- Compute π_M from (23.6) or (23.7) noting that $\pi_L = 1-\pi_M-\pi_H$.

23.1.4 Numerical Example

In this section, we give an example to show the procedure for planning a three-stress-level CSALT given that $p_d = 0.0001$, $p_h = 0.9$ and $n = 300$, $T = 300$, $\sigma = 1$, and $m = 0.1$. These give rise to $\mu_H = 4.8698$ and $\mu_D = 14.9148$.

1. Solve for the statistically optimal plan: $\xi_{2L^*} = 0.29$ and $\pi_H = 0.21$. (see Fig. 23.1)
2. From Fig. 23.4, with $\pi_H = 0.21$ and $m = 0.1$, $\xi_L = 0.38$ and $\xi_M = 0.22$.
3. From (23.6), since $\mu(\xi_M) = 7.0795$ and $F(300) = 0.223$, assuming that $\gamma_M = 21$, we have $\pi_M = 0.314$ and $\pi_L = 0.477$.
4. Alternatively, from (23.7), we have $0.22(0.79-\pi_L)+0.38\pi_L = 0.29\times 0.79$ which gives $\Rightarrow \pi_L = 0.35\ ,\ \pi_M = 0.44$.

Since the low stress levels are determined with $m < 0.1$, the resulting asymptotic variances should be less than 1.1 times the best achievable variance. Note that the sample allocation ratio is approximately 5:3:2, which is quite different from the 4:2:1 recommended by *Meeker* and *Hahn* [23.3]. Despite the lower allocation to the low stress level, the expected number of failures is about 8–10 at the lower stress in these plans, which is sufficient to make some meaningful statistical inference.

23.2 Planning Step-Stress ALT (SSALT)

The simplest form of SSALT is a partially ALT considered by *Degroot* and *Goel* [23.23] in which products are first tested under use condition for a period of time before the stress is increased and maintained at a level throughout the test. This is a special case of a simple SSALT where only two stress levels are used. Much work has been done in this area and literature appeared before 1989 has been covered by *Nelson* [23.6], which has a chapter devoted to step-stress and progressive-stress ALT assuming exponential failure time. A survey

Table 23.1 A summary of the characteristics of literature on optimal design of SSALT

Paper	Problem addressed	Input	Output	Lifetime distribution
Bai et al. [23.17]	Planning two-step SSALT	p_d, p_h	Optimal hold time	Exponential
Bai, *Kim* [23.18]	Planning two-step SSALT	p_d, p_h, shape parameter	Optimal hold time	Weibull
Khamis, *Higgins* [23.19]	Planning three-step SSALT without censoring	All parameters of stress–life relation	Optimal hold time	Exponential
Khamis [23.20]	Planning m-step SSALT without censoring	All parameters of stress–life relation	Optimal hold time	Exponential
Yeo, *Tang* [23.21]	Planning m-step SSALT	p_h and a target acceleration factor	Optimal hold time and lower stress	Exponential
Park, *Yum* [23.22]	Planning two-step SSALT with ramp rate	p_d, p_h, ramp rate	Optimal hold time	Exponential

of the subsequent work from 1989 has been given in *Tang* [23.24]. A summary of the work relating to optimal design of SSALT is presented in Table 23.1. The term optimal usually refers to minimizing the asymptotic variance of the log(MTTF), where MTTF is the mean time to failure, or that of a percentile at use condition. As we can see from Table 23.1, with the exception of *Bai* and *Kim* [23.18], all this work deals with exponential failure time. This is due to simplicity as well as practicality, as it is hard to know the shape parameter of the Weibull distribution in advance.

23.2.1 Planning a Simple SSALT

We first consider a two-level SSALT in which n test units are initially placed on S_1. The stress is changed to S_2 at $\tau_1 = \tau$, after which the test is continued until all units fail or until a predetermined censoring time T. For simplicity, we assume that, at each stress level, the life distribution of the test units is exponential with mean θ_i, and that the linear cumulative exposure model (LCEM) of *Nelson* [23.6] applies.

The typical design problem for a two-step SSALT is to determine the optimal hold time, with a given low stress level. In the following, we shall give the optimal plan that includes both optimal low stress and hold time as in *Yeo* and *Tang* [23.21].

The Likelihood Function

Under exponential failure time and LCEM assumptions, the likelihood function under simple step-stress is

$$\begin{aligned}L(\theta_1,\ \theta_2) = &\prod_{j=1}^{n_1}\left[\frac{1}{\theta_1}\exp\left(-\frac{t_{1,j}}{\theta_1}\right)\right]\\ &\times\prod_{j=1}^{n_2}\left[\frac{1}{\theta_2}\exp\left(-\frac{t_{2,j}-\tau}{\theta_2}-\frac{\tau}{\theta_1}\right)\right]\\ &\times\prod_{j=1}^{n_c}\exp\left(-\frac{\tau}{\theta_1}-\frac{T-\tau}{\theta_2}\right),\end{aligned} \quad (23.8)$$

where the notations are defined as follows:

- n_i number of failed units at stress level S_i, $i = 1, 2, \ldots, h$,
- n_c number of censored units at S_h (at end of test),
- $t_{i,j}$ failure time j of test units at stress level S_i, $i = 1, 2, \ldots, h$,
- θ_i mean life at stress S_i, $i = 1, 2, \ldots, h$,
- τ_i hold time at low stress levels S_i, $i = 1, 2, \ldots, h-1$,
- T censoring time.

MLE and Asymptotic Variance

The MLE of $\log(\theta_0)$ can be obtained by differentiating the log-likelihood function in (23.8):

$$\log\left(\theta_0\hat{}\right) = \frac{\log\left(\frac{U_1}{n_1}\right) - (1-\xi_1)\log\left(\frac{U_2}{n_2}\right)}{\xi_1}, \quad (23.9)$$

where

$$U_1 = \sum_{j=1}^{n_1} t_{1,j} + (n-n_1)\cdot\tau\ ;$$

and

$$U_2 = \sum_{j=1}^{n_2}\left(t_{2,j}-\tau\right) + (n-n_c)\cdot(T-\tau)\ .$$

From the Fisher information matrix, the asymptotic variance of the MLE of the log(mean life) at the design stress is:

$$\begin{aligned}V(\xi_1,\ \tau) = &\frac{\left(\frac{1}{\xi_1}\right)^2}{1-\exp\left(-\frac{\tau}{\theta_1}\right)}\\ &+\frac{\left(\frac{1-\xi_1}{\xi_1}\right)^2}{\exp\left(-\frac{\tau}{\theta_1}\right)\left[1-\exp\left(-\frac{1-\tau}{\theta_2}\right)\right]}\ .\end{aligned} \quad (23.10)$$

To obtain the optimal test plan, we need to express (23.10) in terms of ξ_1,τ, and other input variables. To do this, we need to assume a stress–life relation. For illustration, suppose the mean life of a test unit is a log-linear function of stress:

$$\log(\theta_i) = \alpha + \beta S_i\ , \quad (23.11)$$

where α, β $(\beta < 0)$ are unknown parameters. (This is a common choice for the life–stress relationship because it includes both the power-law and the Arrhenius relation as special cases.) From the log-linear relation of the mean in (23.11), we have

$$\frac{\theta_2}{\theta_1} = \left(\frac{\theta_2}{\theta_0}\right)^{\xi_1} = \exp\left(\beta\xi_1\right)\ .$$

And since $p_h = 1-\exp\left(-\frac{1}{\theta_2}\right)$, it follows that $V(\xi,\ \tau)$ is given by:

$$\begin{aligned}V(\xi_1,\ \tau) = &\frac{\left(\frac{1}{\xi_1}\right)^2}{1-(1-p_h)^{\omega}}\\ &+\frac{\left(\frac{1-\xi_1}{\xi_1}\right)^2}{(1-p_h)^{\omega}\left[1-(1-p_h)^{1-\tau}\right]}\ ,\end{aligned} \quad (23.12)$$

Part C | 23.2

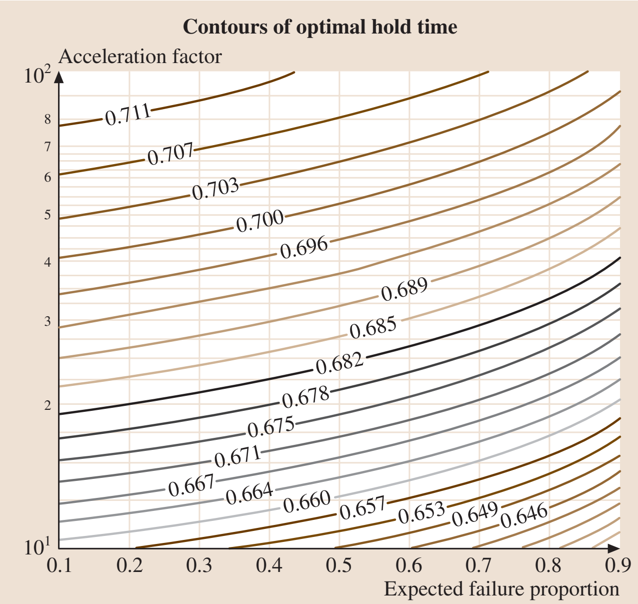

Fig. 23.5 Contours of optimal hold time at low stress for a two-step SSALT. For a given (p, ϕ), the optimal hold time can be read off by interpolation between the contours

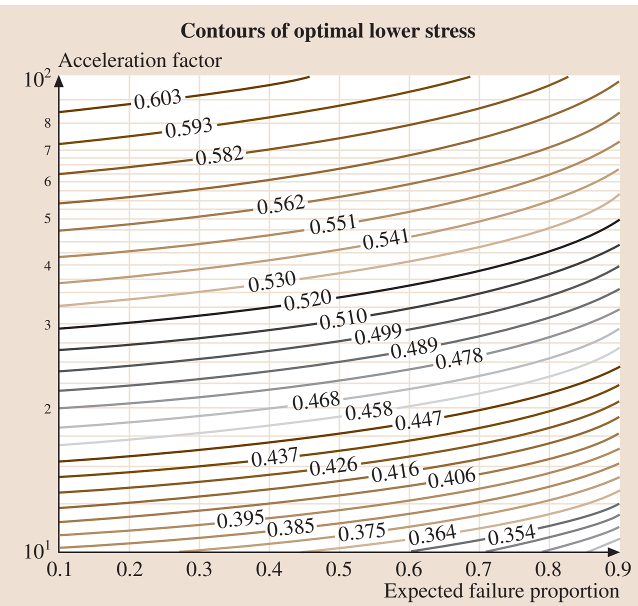

Fig. 23.6 Contours of optimal low stress level for a two-step SSALT. For a given (p, ϕ), the optimal low stress can be read off by interpolation between the contours

where

$$\omega \equiv \tau \left(\frac{\theta_2}{\theta_0}\right)^{\xi_1} = \tau \left[\exp(\beta \xi_1)\right] .$$

As β is unknown, another input variable is needed. We propose using the target acceleration factor (AF) as it is a measure of the amount of extrapolation and a time-compression factor. AF is easier to estimate compared to the commonly used probability of failure at design stress. Given the time constraint that determines the maximum test duration and some guess of the test duration if the test were conducted at use condition, the target AF, denoted by ϕ, is given by the ratio of the two, i. e.

$$\phi = \frac{\text{time to failure at design stress}}{\text{time to failure under test plan}} . \tag{23.13}$$

For exponential lifetime under the LCEM assumption, the equivalent operating time for the test duration T at the design stress given is by $\theta_0 \left(\frac{\tau}{\theta_1} + \frac{T-\tau}{\theta_2}\right)$. As a result, the AF is:

$$\phi = \frac{\tau \frac{\theta_0}{\theta_1} + (T-\tau) \frac{\theta_0}{\theta_2}}{T} . \tag{23.14}$$

From the log-linear stress–life relation in (23.11), without loss of generality, let $S_0 = 0$, $S_1 = x$, $S_2 = 1$, $T = 1$. Then, (23.14) becomes:

$$\phi = (1-\tau)\exp(-\beta) + \tau \exp(-\beta x) . \tag{23.15}$$

The optimal low stress and the corresponding hold time can be obtained by solving the following constrained nonlinear programme (NLP):

$$\begin{aligned} \text{min: } V(x, \tau) = & \frac{\left(\frac{1}{1-x}\right)^2}{1-(1-p_h)^{\omega}} \\ & + \frac{\left(\frac{x}{1-x}\right)^2}{(1-p_h)^{\omega}\left[1-(1-p_h)^{1-\tau}\right]} \end{aligned}$$

$$\text{subject to: } (1-\tau)\exp(-\beta) + \tau\exp(-\beta x) = \phi , \tag{23.16}$$

where $x = 1 - \xi_1$.

The results are given graphically in Figs. 23.5, 23.6, with (p_h, ϕ) on the x–y axis, for ϕ ranging from 10–100 and p ranging from 0.1–0.9. Figure 23.5 shows the contours of the optimal normalized hold time τ and Fig. 23.6 gives the contours of the optimal normalized low stress (x). Given a pair of (p_h, ϕ) the simultaneous optimal low stress and hold time can be read from the graphs.

Both sets of contours for the optimal hold time and the optimal low stress show an upward convex trend. The results can be interpreted as follows. For the same p, in situations where the time constraint calls for a higher AF, it is better to increase the low stress and extend the hold time at low stress. For a fixed AF, a longer test time (so that p increases) will result in a smaller optimal low stress and a shorter optimal hold time.

23.2.2 Planning Multiple-Step SSALT

To design optimal plans for multiple-step SSALT, we adopt a similar idea as in CSALT where the low stress level and its sample allocation are split into two portions. As all units are tested in a single step-stress pattern, the analogy of the sample allocation in a CSALT is the hold time in a SSALT. As a result, for a three-step SSALT, the hold time at the high stress level is kept at $(1-\tau)$ while the hold time at low stress is split into two for the additional intermediate stress level. In doing so, we need to ensure that after splitting the optimal low stress of a two-step SSALT into two stress levels, the AF achieved in the newly created three-step SSALT is identical to that the original two-step SSALT. Since the high stress and its hold time remain intact, the AF contributed by the first two steps of the three-step SSALT must be the same as that contributed by the low stress in the original two-step SSALT. In essence, given the target AF and p_h, we first solve the optimal design problem for a two-step SSALT. Then, a new target AF corresponding to the AF contributed by the low stress of the optimal two-step SSALT is used as input to solve for a new two-step SSALT; which, after combining with the earlier result, forms a three-step SSALT. To achieve the new target AF, the resulting middle stress will be slightly higher than the optimal low stress. The new two-step SSALT uses this middle stress as the high stress and the optimal τ as the test duration. The optimal hold time and low stress for a three-step SSALT can be solved using (23.16).

The above procedure can be generalized to m steps SSALT which has $m-1$ cascading stages of two-step SSALT, as it has the structure of a typical dynamic programme. The number of steps corresponds to the stage of a dynamic programme and the alternatives at each stage are the low stress level and test duration.

For example, in a three-step SSALT having two cascading stages of simple SSALT, the results of stage 1 of the simple SSALT gives the optimal low stress level and its hold time. At stage 2, this low stress level is split into a simple SSALT that maintains the overall target AF. As a result AF is one of the state variables which is additive under LCEM; i.e. the new target AF for stage 2 is the AF contributed by the low stress in stage 1. From (23.15), the AF contributed by the low stress is given by

$$\phi' = \tau \ \exp(-\beta x) . \tag{23.17}$$

To solve the stage 2 NLP, this target AF needs to be normalized by the hold time, τ, due to change to time scale; i.e. the normalized AF, ϕ_2 at stage 2 is given by:

$$\phi_2 = \phi'/\tau = \exp(-\beta x) . \tag{23.18}$$

Note that, to meet the above target AF, the middle stress level x_m should be higher than the optimal low stress in stage 1. As a result, p_2, the expected proportion of failure at x_m during τ will also increase. At the same time, for consistency, we need to ensure that β of the stress life model is identical to that obtained in stage 1. These variables are interdependent and can only be obtained iteratively. The algorithm [23.24] that iteratively solves for p_2, x_m, that results in the same β is summarized as follows.

1. Compute $p_2^{(0)}$, the expected proportion of failure at the low stress level of stage 1:
$$\begin{aligned} p_2^{(0)} &= 1-\exp\left(-\frac{\tau}{\theta_x}\right) \\ &= 1-\exp\left\{-\tau \log\left(\frac{1}{1-p}\right)\right. \\ &\qquad \left.\times \exp\left[\beta\left(1-x\right)\right]\right\} . \end{aligned} \tag{23.19}$$
2. Solve the constrained NLP in (23.16) to obtain $\left(\tau^*_{(1)}, x^*_{(1)}, \beta^*_{(1)}\right)$.
3. Compute the new middle stress $x_\mathrm{m}^{(1)}$
$$x_\mathrm{m}^{(1)} = \beta^*_{(1)} \Bigg/ \left[\log\left(\frac{\log\left(\frac{1}{1-p_2^{(0)}}\right)}{\tau\log\left(\frac{1}{1-p}\right)}\right) + \beta^*_{(1)}\right] , \tag{23.20}$$
4. Update p_2 using $x_\mathrm{m}^{(1)}$:
$$\begin{aligned} p_2^{(1)} &= 1-\exp\left\{-\tau \log\left(\frac{1}{1-p}\right)\right. \\ &\qquad \left.\times \exp\left[\beta\left(1-x_\mathrm{m}^{(1)}\right)\right]\right\} . \end{aligned} \tag{23.21}$$
5. Repeat steps 2 to 4, with $\left(p_2^{(1)}, \phi_2\right)$, $\left(p_2^{(2)}, \phi_2\right)$, $\left(p_2^{(3)}, \phi_2\right)$, ... until $\left|\beta^*_{(k)} - x_\mathrm{m}^{(k)}\beta\right| < \varepsilon$, for some prespecified $\varepsilon > 0$.

Suppose that $\left(\tau^*_{(k)}, x^*_{(k)}, \beta^*_{(k)}\right)$ are the solutions of the scheme. The new optimal low stress and hold time can be computed by combining the results from stages 1 and 2 to form the optimal plan for a three-step SSALT ($h = 3$) as follows:

$$x_1 = x_{\mathrm{m}}^{(k)} \cdot x^*_{(k)} , \quad x_2 = x_{\mathrm{m}}^{(k)} , \quad \tau_1 = \tau \cdot \tau^*_{(k)} , \quad \tau_2 = \tau . \tag{23.22}$$

In general, the above scheme can be carried out recursively to generate test plans for multiple-step SSALT.

23.2.3 Numerical Example

Suppose that a three-step SSALT test plan is needed to evaluate the breakdown time of insulating oil. The high and use stress levels are 50 kV and 20 kV, respectively, and the stress is in log(kV). The total test time is 20 h. It is estimated that the reliability is about 0.1 under 50 kV for 20 h ($p = 0.9$) and the target acceleration factor $\phi = 50$.

Solving (23.16) gives $\beta = 4.8823$, $\tau_2 = 0.6871$ and $x = 0.5203$. Using these as the inputs for the next stage, we have $\tau_1 = 0.4567$; $x_1 = 0.2865$ and $x_2 = 0.6952$. As a result, the voltages for conducting the SSALT are

$$\begin{aligned} S_1 &= \exp\left[S_0 + x_1(S_h - S_0)\right] \\ &= \exp\left\{\log(20) + 0.2865 \times \left[\log(50) - \log(20)\right]\right\} \\ &= 26.0\,\mathrm{kV} , \\ S_2 &= \exp\left[S_0 + x_2(S_h - S_0)\right] \\ &= \exp\left\{\log(20) + 0.6952 \times \left[\log(50) - \log(20)\right]\right\} \\ &= 37.8\,\mathrm{kV} . \end{aligned}$$

And the switching times for the lowest and the middle stress are

$$\begin{aligned} t_1 &= \tau_1 20 = 9.134\,\mathrm{h} = 548.0\,\mathrm{min} ; \\ t_2 &= \tau_2 20 = 13.74\,\mathrm{h} = 824.5\,\mathrm{min} . \end{aligned}$$

The three-step SSALT starts the test at 26 kV and holds for 548 min, then increase the stress to 37.8 kV and holds for another 276 min (=824.5–548), after which the stress is increased to 50 kV until the end of the test.

23.3 Planning Accelerated Degradation Tests (ADT)

For reliability testing of ultra-high-reliability products, ALT typically ends up with too few failures for meaningful statistical inferences. To address this issue, accelerated degradation tests (ADT), which eliminate the need to observe actual failures, were proposed. For successful application of ADT, it is imperative to identify a quantitative parameter (degradation measure) that is strongly correlated with product reliability and thus will degrade over time. The degradation path of this parameter is then synonymous to performance loss of the product. The time to failure is usually defined as the first passage time of the degradation measure exceeding a prespecified threshold.

Planning of ADT typically involve specifying the stress levels, sample size, sample allocations, inspection frequencies and number of inspections for a constant-stress ADT (CSADT). More samples and frequent inspections will result in more accurate statistical inferences; but at a higher testing cost. So there is a tradeoff between the attainable precision of the estimate and the total testing cost. *Park* and *Yum* [23.25] and *Yu* and *Chiao* [23.26] used precision constraints for optimal planning. *Boulanger* and *Escobar* [23.27] and *Yu* and *Tseng* [23.28] also derived cost functions according to their test procedures. *Wu* and *Chang* [23.29] and *Yang* and *Yang* [23.30] presented CSADT plans such that the asymptotic variance of a percentile of interest is minimized while the testing cost is kept at a prescribed level. *Tang* et al. [23.31] gave SSADT plans in which the testing cost is minimized while fulfilling a precision constraint. *Park* et al. [23.32] gave an SSADT plan with destructive inspections. *Yu* and *Tseng* [23.33] presented a CSDT plan in which the rate of degradation follows a reciprocal Weibull distribution.

23.3.1 Experimental Set Up and Model Assumptions

For simplicity, we consider an ADT, be it a CSADT or SSADT, with two stress levels. Some descriptions and assumptions are as follows:

1. The test stress X_k is normalized by $X_k = \frac{S_k - S_0}{S_2 - S_0}$, $k = 0, 1, 2$, in which the S_k are functions of the applied stresses. With such a transformation, $X_0 = 0 < X_1 < X_2 = 1$.
2. A total sample size is n of which n_k are assigned to the stress level X_k, so that the relationship between

n and n_k can be expressed by:

$$n = \begin{cases} \sum_{k=1}^{2} n_k & \text{for CSADT} \\ n_k & \text{for SSADT} \end{cases} . \tag{23.23}$$

3. The test duration at stress X_k is τ_k, and the stopping time of the whole test is T. The relationship between T and τ_k is:

$$T = \begin{cases} \text{maximum}(T_1, T_2) & \text{for CSADT} \\ \sum_{k=1}^{2} T_k & \text{for SSADT} \end{cases} . \tag{23.24}$$

4. For each unit i, let $D_{i,1}, D_{i,2}, \ldots D_{i,j}$ be the recorded degradation values, which are the differences between the initial and current value of the degradation measurement at the preset time points $t_{i,1}, t_{i,2}, \ldots t_{i,j}, t_{i,0} = 0 < t_{i,1} < t_{i,2} < \ldots < t_{i,j} < \ldots t_{i,L} = T$; each item is measured L_1 times at X_1 and L_2 times at X_2. The total number of inspections is $L = L_1 + L_2$. Given the number of inspections, all samples are inspected simultaneously at equal interval Δt so that the stress-changing time is $T_1 = L_1 \Delta t$ and the experiment stopping time is $T = L\Delta t = (L_1 + L_2)\,\Delta t$.
5. The degradation is governed by a stochastic process $[D_k(t), t \geq 0]$ with drift $\eta_k > 0$ and diffusion $\sigma_k^2 > 0$ at X_k. We assume that the degradation increments follow a normal distribution, i. e. $\Delta D_{i,j} \sim N\left(\eta_k \Delta t_{i,j}, \sigma^2 \Delta t_{i,j}\right)$ with probability distribution function (PDF)

$$f\left(\Delta D_{i,j}\right) = \frac{1}{\sqrt{2\pi}\sqrt{\Delta t_{i,j}\sigma^2}} \exp \times \left(-\frac{\left(\Delta D_{i,j} - \Delta t_{i,j}\eta_k\right)^2}{2\Delta t_{i,j}\sigma^2}\right) \tag{23.25}$$

in which the drift is stress-dependent, and is given by:

$$\eta_k = a + bX_k \tag{23.26}$$

and the diffusion remains constant for all stresses:

$$\sigma_k^2 = \sigma^2 , \tag{23.27}$$

where a, b and σ^2 are unknown parameters that need to be pre-estimated either from engineering handbooks or other ways before experiment planning.
6. Only degradation increments are measured throughout the test. This assumption is mild since the products in ADT are always highly reliable and normally no physical failures occur.

The above model is applicable to stress-drift relations that can be linearized as in (23.26). For example, for degradation induced by humidity, (23.26) may be the result after taking logarithm of the rate of reaction versus the logarithm of the relative humidity. In the case of the Arrhenius model, the reaction rate is an exponential function of the stress factor ($= 1/$absolute temperature); taking logarithms on both sides of the equation results in a linear function between the log(Drift), which is η_k, versus the stress factor ($= 1/$absolute temperature).

23.3.2 Formulation of Optimal SSADT Plans

Here, we follow *Tang* et al. [23.31], in which an optimal SSADT plan is obtained such that the total test expense is minimized while the probability that the estimated mean lifetime at use stress locates within a predescribed range of its true value should not be less than a precision level p. For simplicity, the decision variables are the sample size and the number of inspections at each stress level, which also determine the test duration for a given inspection interval. The context of discussion can be generalized to that of a CSADT plan by noting that

$$\pi_k = \begin{cases} \frac{n_k}{n} & \text{for CSADT} , \\ \frac{T_k}{T} & \text{for SSADT} . \end{cases} \tag{23.28}$$

$$q_k = \begin{cases} \frac{T_k}{T} & \text{for CSADT} , \\ \frac{n_k}{n} & \text{for SSADT} . \end{cases} \tag{23.29}$$

For CSADT, $q_1 = 1, 0 < q_2 \leq 1$, while $q_k = 1$ for SSADT. With this definition, the proportion of sample allocation in CSADT is analogous to the holding time in SSADT. This is consistent with the analogy between CSALT and SSALT.

Precision Constraint in SSADT Planning

Suppose that the mean lifetime at use condition, $\mu(X_0)$, is of interest in our planning. To obtain an estimate close to its true value with a certain level of confidence, we impose a precision constraint by limiting the sampling risk in estimating $\mu(X_0)$ with its MLE, i. e. $\widehat{\mu}(X_0)$, to be reasonably small. Mathematically, this can be expressed as follows:

$$\Pr\left[\frac{\mu\left(X_0\right)}{c} \leq \widehat{\mu}\left(X_0\right) \leq c\mu\left(X_0\right)\right] \geq p , \tag{23.30}$$

where $c > 1$ and p are given constants. The asymptotic variance of $\widehat{\mu}(X_0)$ is needed for further explanation of (23.30). From (23.25), the log-likelihood for an individual degradation increment $D_{i,j}$ is

$$\ln L_{i,j} = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln\left(\Delta t_{i,j}\right) - \ln\sigma - \frac{U_{i,j}^2}{2}\,, \quad (23.31)$$

where

$$U_{i,j} = \frac{\left(\Delta D_{i,j} - \Delta t_{i,j}\eta_k\right)}{\sqrt{\Delta t_{i,j}}\sigma}\,, \begin{cases} k=1 & \text{if } j \le L_1\,, \\ k=2 & \text{otherwise}\,. \end{cases}$$

Hence, the log-likelihood function for all degradation increments of n items is given by

$$\ln L = \sum_{i=1}^{n}\sum_{j=1}^{L}\ln L_{i,j}\,. \quad (23.32)$$

Given the degradation critical value D_c, $\mu(X_0)$ is given by the ratio of this threshold value over drift at use condition:

$$\mu(X_0) = D_c/\eta_0 = D_c/a\,. \quad (23.33)$$

Let $\left\{\widehat{a}, \widehat{b}, \widehat{\sigma}\right\}$ be the MLE of $\{a, b, \sigma\}$, then, by the invariant property, the MLE of $\mu(X_0)$ is given by:

$$\widehat{\mu}(X_0) = D_c/\widehat{\eta_0} = D_c/\widehat{a} \quad (23.34)$$

Then the asymptotic variance of $\widehat{\mu}(X_0)$ can be obtained by:

$$\text{Avar}\left[\widehat{\mu}(X_0)\right] = \widehat{h}'\mathbf{F}^{-1}\widehat{h}\,, \quad (23.35)$$

where $h = \left(\frac{\partial\widehat{\mu}(X_0)}{\partial a}, \frac{\partial\widehat{\mu}(X_0)}{\partial b}, \frac{\partial\widehat{\mu}(X_0)}{\partial \sigma}\right)'$, and $\boldsymbol{F}$ is a Fisher information matrix displayed as follows, in which the caret (^) indicates that the derivative is evaluated at $\{a, b, \sigma\} = \left\{\widehat{a}, \widehat{b}, \widehat{\sigma}\right\}$. We make use of $E\left(U_{i,j}\right) = 0$ and $\text{Var}\left(U_{i,j}\right) = 1$

$$\boldsymbol{F} = \begin{pmatrix} E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial a^2}\right) & E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial a \partial b}\right) & E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial a \partial \sigma}\right) \\ & E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial b^2}\right) & E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial b \partial \sigma}\right) \\ \text{symmetrical} & & E\left(-\frac{\partial^2 \ln \widehat{L}}{\partial \sigma^2}\right) \end{pmatrix}$$

$$= \frac{n}{\widehat{\sigma}}^2 \begin{pmatrix} L\Delta t & \Delta t \sum_{k=1}^{2} X_k L_k & 0 \\ & \Delta t \sum_{k=1}^{2} X_k^2 L_k & 0 \\ \text{symmetrical} & & 2L \end{pmatrix}.$$

Thus we have

$$\text{Avar}\left[\widehat{\mu}(X_0)\right] = \frac{\widehat{\sigma}^2}{n}\frac{D_c^2}{\widehat{a}^4}\left(\frac{\sum_{k=1}^{2} X_k^2 L_k}{L\Delta t \sum_{k=1}^{2} X_k^2 L_k - \Delta t\left(\sum_{k=1}^{2} X_k L_k\right)^2}\right). \quad (23.36)$$

Because the MLE is asymptotically normal and consistent, for large n, approximately we have

$$\widehat{\mu}(X_0) \sim N\left\{\left[\mu(X_0)\right], \text{Avar}\left[\widehat{\mu}(X_0)\right]\right\}, \quad (23.37)$$

which can be rewritten as

$$\frac{\widehat{\mu}(X_0)}{\mu(X_0)} \sim N\left(1, \frac{\widehat{\sigma}^2}{n\widehat{a}^2}Q\right). \quad (23.38)$$

From (23.30), we have

$$\Pr\left(\frac{1}{c} \le \frac{\widehat{\mu}(X_0)}{\mu(X_0)} \le c\right) \ge p\,. \quad (23.39)$$

This translates into the precision constraint

$$\Phi\left(\frac{(c-1)\sqrt{n}}{\frac{\widehat{\sigma}}{\widehat{a}}\sqrt{Q}}\right) - \Phi\left(\frac{\left(\frac{1}{c}-1\right)\sqrt{n}}{\frac{\widehat{\sigma}}{\widehat{a}}\sqrt{Q}}\right) \ge p\,, c > 1\,, \quad (23.40)$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution and

$$Q = \frac{\sum_{k=1}^{2} X_k^2 L_k}{L \times \Delta t \times \sum_{k=1}^{2} X_k^2 L_k - \Delta t \times \left(\sum_{k=1}^{2} X_k L_k\right)^2}\,. \quad (23.41)$$

Cost Function in SSADT Planning

Typical cost components for testing consists of:

1. Operating cost, which mainly comprises the operator's salary and can be expressed as $\Delta t(C_{o1}L_1 + C_{o2}L_2)$, where C_{ok} is the salary of the operator per unit of time at X_k.
2. Measurement cost, which includes the cost of using measuring equipments and the expense of testing materials. Because depletion of equipments at higher stress is more severe than that at lower stress, measurement cost can be generated as $n(C_{m1}L_1 + C_{m2}L_2)$, where C_{mk} is the cost per measurement per device at X_k.

3. Sample cost, which is related to the number of samples, and can be formulated as $C_\mathrm{d}n$, where C_d is the price of an individual device.

   So, the total cost (TC) of testing is:

$$\begin{aligned}&TC\,(n\,,L_1\,,L_2|X_1\,,\Delta t)\\&=\Delta t\,(C_{\mathrm{o}1}L_1+C_{\mathrm{o}2}L_2)\\&\quad+n\,(C_{\mathrm{m}1}L_1+C_{\mathrm{m}2}L_2)+C_\mathrm{d}n\,,\\&C_{\mathrm{o}k}>0\,,C_{\mathrm{m}k}>0\,,C_\mathrm{d}>0\,.\end{aligned}\tag{23.42}$$

In some experiments, the lower test stress can be fixed because of practical limitations. For example, the temperature of a test oven can only be adjusted within a small range or even fixed at some particular values. Given the lower stress X_1, the two-step-stress ADT planning problem is to determine the sample size n, number of inspections L_1 and L_2. The problem is formulated as:

$$\begin{aligned}\text{Min: }&\mathrm{TC}\,(n\,,L_1\,,L_2|X_1\,,\Delta t)\\&=\Delta t\,(C_{\mathrm{o}1}L_1+C_{\mathrm{o}2}L_2)\\&\quad+n\,(C_{\mathrm{m}1}L_1+C_{\mathrm{m}2}L_2)+C_\mathrm{d}n\,,\\&\quad C_{\mathrm{o}k}>0\,,C_{\mathrm{m}k}>0\,,C_\mathrm{d}>0\,;\\\text{s.t.: }&\Phi\left(\frac{(c-1)\cdot\sqrt{n}}{\frac{\widehat{\sigma}}{\widehat{a}}\cdot\sqrt{Q}}\right)-\Phi\left(\frac{(\frac{1}{c}-1)\cdot\sqrt{n}}{\frac{\widehat{\sigma}}{\widehat{a}}}\cdot\sqrt{Q}\right)\\&\quad\geq p\,,\quad c>1\,.\end{aligned}\tag{23.43}$$

Due to the simplicity of the objective function and the integer restriction on the decision variables, the solution can be obtained by complete enumerations or using search methods given in *Yu* and *Tseng* [23.28].

23.3.3 Numerical Example

In this example, the operating temperature of a light-emitting diode (LED) in use condition is 25 °C. Historical experience indicates that the highest temperature that will not affect the failure mechanism is 65 °C. The lower test stress is set at 45 °C, which can be normalized by

$$\begin{aligned}X_1&=\frac{S_1-S_0}{S_2-S_0}\\&=\frac{1/(45+273)-1/(25+273)}{1/(65+273)-1/(25+273)}\\&=0.53\,.\end{aligned}$$

This normalization is consistent with the Arrhenius model in which stress takes the reciprocal of temperature.

To set the inspection time interval, we refer to a similar CSADT plan conducted at 25 °C in *Yu* and *Chiao* [23.26], which suggested an optimal inspection time interval of 240 h. Here, in view of adopting a higher stress, Δt should be shorter as the degradation rate is higher. Here, $\Delta t=120$ h to capture more degradation information. The operation and measurement coefficients are set at $C_{\mathrm{o}1}=0.3$, $C_{\mathrm{o}2}=0.4$, $C_{\mathrm{m}1}=4$ and $C_{\mathrm{m}2}=4.5$.

Here the c and p values represent the dependence on sampling risk. Smaller sampling risk implies smaller c and relatively larger p and vise versa. As an illustration, we present the case of $c=2$, $p=0.9$ by setting $\widehat{\sigma}=10^{-4}$ (which is comparable with the value used in *Yu* and *Chiao* [23.26]).

Substitute this information into (23.42) and (23.43), we have:

$$\begin{aligned}\text{Min:}\quad&\mathrm{TC}\,(n\,,L_1\,,L_2)\\&=120\cdot(0.3\cdot L_1+0.4L_2)\\&\quad+n\cdot(4L_1+4.5L_2)+86n\\\text{s.t.:}\quad&\Phi\left(\frac{\sqrt{n}}{100\sqrt{Q}}\right)-\Phi\left(\frac{-\frac{1}{2}\sqrt{n}}{100\sqrt{Q}}\right)\geq 0.9\,,\end{aligned}$$

where $Q=0.53^2L_1+L_2/120\big[(L_1+L_2)(0.53^2L_1+L_2)-(0.53L_1+L_2)^2\big]$.

This plan puts 16 samples at 45 °C for 3240 h, after which the temperature is increased to 65 °C and held for 720 h before the end of the test. Measurements are taken at 120 h interval.

23.4 Conclusions

In this chapter, literature surveys and statistical approaches for planning three types of accelerated reliability testing, namely, constant-stress accelerated life tests, step-stress accelerated life tests and step-stress accelerated degradation tests, are presented. We only focus on literature concerning the above three problems since the 1990s. A more comprehensive survey can be obtained from *Nelson* [23.14]. The general approach taken in solving for the optimal plan is to derive the asymptotic variance (or its approximation) of a percentile of interest at use condition and minimize it subject to a set of constraints. The constraints either help to define the logical solution space or help to narrow the solution space for ease of finding the solution. For more general considerations, the approach presented in *Tang* and *Xu* [23.12] can be adopted to generalize the current models so that other objectives and constraints can be incorporated.

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