45. Statistical Maintenance Modeling for Complex Systems

The first part of this chapter provides a brief introduction to statistical maintenance modeling subject to multiple failure processes. It includes a description of general probabilistic degradation processes.

The second part discusses detailed reliability modeling for degraded systems subject to competing failure processes without maintenance actions. A generalized multi-state degraded-system reliability model with multiple competing failure processes including degradation processes and random shocks is presented. The operating condition of the multi-state system is characterized by a finite number of states. A methodology to generate the system states when multi-failure processes exist is also discussed. The model can be used not only to determine the reliability of the degraded systems in the context of multi-state functions but also to obtain the probabilities of being in a given state of the system.

The third part describes the inspection–maintenance issues and reliability modeling for degraded repairable systems with competing failure processes. A generalized condition-based maintenance model for inspected degraded systems is discussed. An average long-run maintenance cost rate function is derived based on an expression for degradation paths and cumulative shock damage, which are measurable. An inspection sequence is determined based on the minimal maintenance cost rate. Upon inspection, a decision will be made on whether to perform preventive maintenance or not. The optimum preventive maintenance thresholds for degradation processes and inspection sequences are also determined based on a modified Nelder–Mead downhill simplex method.

Finally, the last part is given over to the conclusions and a discussion of future perspectives for degraded-system maintenance modeling.

Technology advances mean that most new products are, on one hand, more reliable, but on the other hand, very difficult to maintain during the product life cycle. Designers have been challenged to find new, effective approaches to evaluate reliability in a timely fashion and to maintain such systems in an optimum way. This chapter presents reliability and maintenance models for degraded systems subject to competing failure processes. The accuracy of reliability estimation through a degradation model cannot be ensured unless the unit-to-unit initial variation and within-unit degradation-rate variation are considered. This chapter also discusses a generalized random-coefficient degradation process and randomized logistic degradation process to model the degradation.

Literature Review

As degradation occurs, system performance changes from perfect functioning to complete failure; the binary assumption used to analyze, model and compute system reliability is relaxed. Using degradation measures to assess reliability has seen some important findings in the literature. *Tomsky* [45.1] investigated two regression models for detecting degradation reliability. *Nelson* [45.2] briefly surveyed the application of accelerated degradation. *Lu* [45.3] introduced a nonlinear mixed-effects model and estimated model parameters in a two-stage way. Recently, multi-state reliability has received considerable attention. *Levitin* [45.4] extended the reliability importance measures for multi-state systems with different measures of performance. When the multi-state nature of a system is addressed, a better understanding of the system reliability behavior is obtained. Our third new development is to build a methodology based on the formulation of degradation in terms of a finite discrete state.

It is well known that the effectiveness of a system depends on both the quality of its design and manufacturing process as well as the proper inspection–maintenance actions to prevent it from failing. Inspection–maintenance issues are considered in the second part of this chapter.

Maintenance has evolved from a simple model that deals with machinery breakdowns, to time-based preventive maintenance, to today's condition-based maintenance. It is of great importance to avoid the failure of a system during its actual operating; especially, when such failures are dangerous and costly. Time-based and condition-based maintenance are the two major approaches for maintenance. Condition-based maintenance is often profitable since it can be used to avoid failure occurrence at the lowest cost and to improve the availability and reliability of complex systems. This chapter examines the problem of developing a mathematical maintenance cost model to determine both the optimal inspection interval time and preventive maintenance threshold of degraded systems with competing failure processes subject to a condition-based maintenance policy.

Pham et al. [45.5] presented a Markov model for predicting the reliability of k-out-of-n systems in which components are subject to multi-stage degradation as well as catastrophic failures. Due to the aging effect, the failure rate of the component will increase. They considered the state-dependent transition rates for the degradation process.

Pham et al. [45.6] derived models for predicting the availability and mean lifetime of multistage degraded systems with partial repairs.

In some production systems failures are not possible to detect but can only be determined by inspection [45.7]. Several authors [45.8–22] have proposed various inspection policies and models for systems with a degradation process. *Grall* et al. [45.8] studied a system subject to a random deterioration process. They developed a model based on a stationary process to determine both the preventive maintenance threshold and inspection dates that minimized the average long-run cost rate. *Chelbi* and *Ait-Kadi* [45.10] addressed optimal inspection strategies for deteriorating equipment subject to preventive and corrective maintenance.

Klutke and *Yang* [45.11] studied the availability of maintained systems subject to both the effects of the degradation and random shocks. They considered the degradation process as a deterministic function of time t and that shocks occurred according to a Poisson process in which the shock magnitudes are independent and identically distributed (iid) random variables. *Pham* and *Xie* [45.13] developed a generalized surveillance model consisting of dual, mutually dependent stochastic processes for surveillance systems. Their model can be used to better understand both the inspection process, the repair unit itself, and to provide information that can be used to assist inspectors in scheduling and prioritizing their future inspections.

The choice of the inspection schedule and preventive maintenance threshold(s) obviously has an important influence on the economic performance of the maintenance policy. The inspection dates and the preventive maintenance threshold(s) are two main decision variables. However, in industrial applications of condition-based maintenance, the preventive maintenance threshold is usually decided based upon the recommendation made by the maintenance people and the inspection schedule often appears to be set by little more than a rule of thumb. Because of the lack of appropriate modeling support, the preventive maintenance threshold is likely to be set conservatively and the inspection schedule may be performed more than is perhaps necessary. The need for a maintenance model with cost consideration is obvious in this case.

The chapter is organized as follows. The basic concepts and a review of maintenance, as well as a brief description of probabilistic processes for the modeling of degradation and random shocks, is discussed in Sect. 45.1. A general reliability model for degraded nonrepairable systems subject to multiple competing processes is discussed in Sect. 45.2. The inspection–maintenance modeling issues and detection policies for degraded repairable systems considering multiple competing processes are described in Sect. 45.3. Several numerical examples are given in Sects. 45.2 and 45.3. Finally in Sect. 45.4, several future research perspectives and conclusions are briefly discussed.

Acronyms

CM	Corrective maintenance
PM	Preventive maintenance
rv	Random variable

Notation

C_c	Cost per CM action
C_p	Cost per PM action
C_m	Loss per unit idle time
C_i	Cost per inspection
$Y(t)$	Degradation process
$Y_1(t)$	Degradation process 1
$Y_2(t)$	Degradation process 2
$D(t)$	Cumulative shock damage value up to time t
G_1	Critical threshold value for degradation process 1
G_2	Critical threshold value for degradation process 2
S	Critical value for shock damage
L_1	PM critical threshold value for degradation process 1
L_2	PM critical threshold value for degradation process 2
$C(t)$	Cumulative maintenance cost up to time t
$E[C_1]$	Average total maintenance cost during a cycle
$E[W_1]$	Mean cycle length
$E[N_I]$	Mean number of inspections during a cycle
$E[\xi]$	Mean idle time during a cycle
$\{I_i\}_{i\in N}$	Inspection sequence
$\{U_i\}_{i\in N}$	Inter-inspection sequence
$\{W_i\}_{i\in N}$	Renewal times
T	Time to failure
P_{i+1}	Probability that there are a total of $(i+1)$ inspections in a renewal cycle
P_p	Probability that a renewal cycle ends by a PM action
P_c	Probability that a renewal cycle ends by a CM action
$EC(L_1, L_2, I_1)$	Expected long-run cost rate function
Ω_U	$\{M, \dots, 1, 0, F\}$ a system state space where state M is a perfect (good) state; state 0 is a degraded failure state; states $M-1 \dots l$ are intermediate degradation states; state F is the catastrophic failure state
Ω	$\{M, \dots, 1, 0\}$, a system degradation state space without catastrophic failure
Ω_i	$\{M_i, \dots, 1_i, 0_i\}$, a state space corresponding to the degradation processes of the i-th state, where 0_i is a degraded failure state due to the i-th degradation process, and M_i is a good state of the degradation process i, $i=1,2$
R	$\Omega_1 \times \Omega_2$, the Cartesian product of Ω_1 and Ω_2
R_i	i-th equivalence class, $i = 0, 1, \dots, M$

45.1 General Probabilistic Processes Description

We consider three random processes. The first two are used to model degradation, while the third is a compound Poisson process used for modeling random shocks:

1. $Y(t) = A + Bg(t)$ is called a random-coefficient degradation path, where $A > 0$and $B > 0$ are independent random variables and $g(t)$ is an increasing time-dependent function. The random variable A measures the initial value of degradation due to a different manufacturer, the manufacturing quality control of new items, variable deterioration during storage until the item is put into service, and so forth [45.14]. Therefore, the initial degradation value A is a random variable. The variable B is the degradation rate ($B > 0$) and represents the variations among the population; $g(t)$ is an increasing function.
2. $Y(t) = \frac{W\,\mathrm{e}^{Bt}}{A+\mathrm{e}^{Bt}}$ is called a randomized logistic degradation path function where A and B are independent non-negative random variables, and W is a constant. The random variable A represents the initial threshold level of degradation and B describes the rate at which degradation accumulates. It should be noted that $Y(t) = \frac{W\,\mathrm{e}^{Bt}}{A+\mathrm{e}^{Bt}}$ is an S-shaped curve and describes the degradation process well. It matches the

path of the cumulative degradation of many systems in practice. The S-shaped curve reflects an initial run-in period of low usage, followed by a period of steady rate of usage, and finally ending with an increasing rate of use due to the aging of the system. We establish the relationship between the two random variables A and B via some rearrangements as follows:

$$W\frac{\mathrm{e}^{Bt}}{A+\mathrm{e}^{Bt}} < H \Rightarrow B < \frac{1}{t}\ln\frac{u_1 A}{1-u_1}\,, \qquad (45.1)$$

where $u_1 = \frac{H}{W}$.

3. Let $D(t) = \sum_{i=0}^{N(t)} X_i$ represent a sequence of random shocks in which each shock causes independent damage X_i to the whole system where the X_i are iid with a probability distribution function (pdf) of $f_X(x)$, and a cumulative distribution function (cdf) of $F_X(x)$; $\{N(t), t \geq 0\}$ is a Poisson process with parameter $\lambda > 0$ that is independent of the sequence $\{X_i\}$; $F_X^{(k)}(x)$ denotes the k-th convolution.
   The stochastic process $D(t) = \sum_{i=0}^{N(t)} X_i$ is called a compound Poisson process where $N(t)$ is the number of shocks that have occurred up to time t, X_i is the damage caused by the i-th shock and $D(t)$ is the cumulative damage up to time t.

45.2 Nonrepairable Degraded Systems Reliability Modeling

This section addresses reliability models for nonrepairable degraded systems. First, we discuss a model for systems subject to two competing processes. Then, we present a generalized situation where systems are subjected to three competing processes.

45.2.1 Degraded Systems Subject to Two Competing Processes

Model description

The modeling assumptions are as follows:

1. Each system has a state space $\Omega_{\mathrm{U}} = \{M, \ldots, 1, 0, F\}$.
2. The system fails either due to degradation $(Y(t) > G)$ or catastrophic failure $\left[D(t) = \sum_{i=1}^{N_2(t)} X_{\mathrm{i}} > S\right]$. The system may either go from state i to the next degraded state $i-1$ or directly to the catastrophic failure state F, $i = M, \ldots, 1$.
3. No repair or maintenance is performed on the system.
4. Since $Y(t)$ describes the total damage up to time t, it is natural to assume that it is nondecreasing.
5. The two processes $Y(t)$ and $D(t)$ are independent.
6. At time $t = 0$, the system is in state M.

We consider a degradable system suited at a random environment where degradation and random shocks can contribute to an effect of the life of a system. In this section, we discuss the case where systems are subject to two failure processes, called a continuous and increasing degradation process $Y(t)$, and the a andom shock process $D(t)$. Whichever process occurs first causes the system to failure.

Figure 45.1 illustrates the system flow diagram of the two competing failure processes. In Fig. 45.1, we use either of the random processes described in Sect. 45.1 to represent a degradation process where random shocks are represented by a stationary and independent increment process. Then, we discuss a method to formulate these two processes from a multi-state standing point. That is, suppose that the operating conditions of the system at any point in time could be classified into one of a finite number of the states, say $\Omega_{\mathrm{U}} = \{M, \ldots, 1, 0, F\}$. We view the degradation process in terms of a finite number of states. For example, when the value of the degradation process $Y(t)$ falls into a predefined interval then its corresponding state will be determined. Let us define as follows:

$[0, W_M], \ldots, (W_2, W_1]$ are the intervals associated with the degradation process where $W_M < W_{M-1} < \ldots < W_2 < W_1$. A one-to-one relationship between the element of $\Omega = \{M, \ldots, 1, 0\}$ and its corresponding interval is set up as follows:

$$\begin{aligned}
\text{when } Y(t) &\in [0, W_M] \Rightarrow \text{in state } M\,,\\
\text{when } Y(t) &\in \left(W_M, W_{M-1}\right] \Rightarrow \text{in state } M-1\,,\\
&\vdots\\
\text{when } Y(t) &\in \left(W_i, W_{i-1}\right] \Rightarrow \text{in state } i\,,\\
\text{when } Y(t) &\in \left(W_2, W_1\right] \Rightarrow \text{in state } 1\,,\\
\text{when } Y(t) &> W_1 \Rightarrow \text{in state } 0\,.
\end{aligned}$$

Reliability Evaluation

The most general situation is to allow each degradation process to be described by a number of different discrete

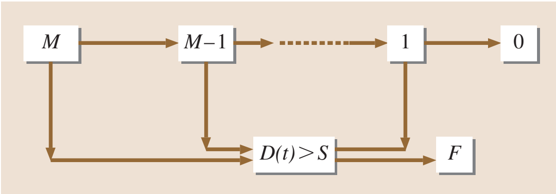

Fig. 45.1 The flow diagram of a system subjected to two competing failure processes

states. We now define the probability in each state. Let $P_i(t)$ be the probability that the value of $Y(t)$ will fall within a predefined interval corresponding to state i with $D(t) \le S$. From state i, the system will make a direct transition to state $i-1$ due to gradual degradation, or to state F due to a random shock.

The reliability function is defined as:

$$\begin{aligned} R_M(t) &= P\,(\text{state} \ge 1) \\ &= \sum_{i=1}^{M} P_i(t) \\ &= P[Y(t) \le G,\, D(t) \le S]\,, \end{aligned} \qquad (45.2)$$

where $P_i(t)$ is the probability of being in state i.

Suppose a system fails if the degradation process crosses some threshold, say G; or the shock damage process crosses some threshold, say S; T is defined as:

$$T = \inf\,[t > 0 : Y(t) > G \text{ or } D(t) > S] \qquad (45.3)$$

The mean time to failure is expressed as:

$$\begin{aligned} E[T] &= \int_0^\infty P[T > t]\,\mathrm{d}t \\ &= \int_0^\infty P[Y(t) \le G,\, D(t) \le S]\,\mathrm{d}t \\ &= \int_0^\infty P[Y(t) \le G] \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t}}{j!} F_X^{(j)}(S)\,\mathrm{d}t \end{aligned}$$

or, equivalently, that

$$E[T] = \sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \int_0^\infty P[Y(t) \le G](\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t}\,\mathrm{d}t\,. \qquad (45.4)$$

The specific expression for $E[T]$ depends on the probability function $P[Y(t) \le G]$. Sometimes, it is hard to find a closed-form solution. In this case, one can use a numerical method to solve the problem in (45.4).

The probability density function of the time to failure, $f_{\mathrm{T}}(t)$ is as follows:

$$\begin{aligned} f_{\mathrm{T}}(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} R(t) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} \{P\,[Y(t) \le G]\; P\,[D(t) \le S]\} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} \left\{ P[Y(t) \le G] \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t}}{j!} F_X^{(j)}(S) \right\} \\ &= -\sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \frac{\mathrm{d}}{\mathrm{d}t} \left\{ P[Y(t) \le G]\,(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t} \right\} \end{aligned}$$

Let $F_G(t) = P[Y(t) \le G]$, then $f_G(t) = \frac{\mathrm{d}}{\mathrm{d}t} F_G(t)$.

$$\begin{aligned} f_{\mathrm{T}}(t) = -\sum_{j=1}^{\infty} \frac{F_X^{(j)}(S)}{j!} \Big[& f_G(t)(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t} \\ & + F_G(t)\, j\lambda_2 (\lambda_2 t)^{j-1}\,\mathrm{e}^{-\lambda_2 t} \\ & - \lambda_2 F_G(t)(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t} \Big]\,. \end{aligned} \qquad (45.5)$$

Reliability models
Model 1:

$$\begin{cases} Y(t) = A + Bg(t) \\ D(t) = \sum_{i=0}^{N_2(t)} X_i \end{cases}$$

$$\begin{cases} \text{case 1}: A \sim \text{normal},\ B \sim \text{normal} \\ \text{case 2}: A \sim U[0, a],\ B \sim \operatorname{Exp}(b) \end{cases}$$

Model 2:

$$\begin{cases} Y(t) = W \frac{\mathrm{e}^{Bt}}{A + \mathrm{e}^{Bt}}, \text{ where } A \sim U[0, a],\ B \sim \operatorname{Exp}(b) \\ D(t) = \sum_{i=0}^{N_2(t)} X_i \end{cases}.$$

The two reliability models for the system are depicted in Fig. 45.1. In the following, we will take model 2 as an example to illustrate the results in this section. One can also easily apply it for the model 1.

Assume that the degradation process is described by the function $Y(t) = W \frac{\mathrm{e}^{Bt}}{A + \mathrm{e}^{Bt}}$, where the two random variables A and B are independent, and A follows a uniform distribution with parameter interval [0, a] and B follows an exponential distribution with parameter $\beta > 0$. In short, $A \sim U[0, a]$, $a > 0$ and $B \sim \operatorname{Exp}(\beta)$, $\beta > 0$.

The probability that the system is in state M is as follows:

$$\begin{aligned}
P_M(t) &= P\left[Y(t) = W\frac{e^{Bt}}{A+e^{Bt}} \leq W_M\,,\right.\\
&\qquad \left. D(t) = \sum_{i=0}^{N_2(t)} X_i \leq S\right]\\
&= \left[\int_{\forall A} P\left(B < \frac{1}{t}\ln\frac{u_1 A}{1-u_1}|A=x\right) f_A(x)\,dx\right]\\
&\quad \times P\left[D(t) = \sum_{i=0}^{N_2(t)} X_i \leq S\right]\\
&= \left[1-\frac{1}{a}\left(\frac{1-u_1}{u_1}\right)^{\frac{\beta}{t}}\left(\frac{t}{t-\beta}\right)\left(a^{1-\frac{\beta}{t}}-1\right)\right]\\
&\quad \times e^{-\lambda_2 t}\sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\,. \qquad (45.6)
\end{aligned}$$

The probability that the system is in state i is calculated as follows:

$$\begin{aligned}
P_i(t) &= P\left[W_{i+1} < W\frac{e^{Bt}}{A+e^{Bt}} \leq W_i\,,\right.\\
&\qquad \left. D(t) = \sum_{i=0}^{N_2(t)} X_i \leq S\right]\\
&= \left[\int_0^a P\left(\frac{1}{t}\ln\frac{u_{i-1}A}{1-u_{i-1}} < B\right.\right.\\
&\qquad \left.\left.\leq \frac{1}{t}\ln\frac{u_i A}{1-u_i}|A=x\right) f_A(x)\,dx\right]\\
&\quad \times e^{-\lambda_2 t}\sum_{j=1}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\\
&= \left\{\frac{1}{a}\left(\frac{t}{t-\beta}\right)\left(a^{1-\frac{\beta}{t}}\right)\left[\left(\frac{1-u_i}{u_i}\right)^{\frac{\beta}{t}}\right.\right.\\
&\qquad \left.\left. -\left(\frac{1-u_{i-1}}{u_{i-1}}\right)^{\frac{\beta}{t}}\right]\right\}\\
&\quad \times e^{-\lambda_2 t}\sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\,, \qquad (45.7)
\end{aligned}$$

where $\mu_i = \frac{W_i}{W}$, $i = M-1, .., 1$.

Similarly, the probability that the system is in state 0 is as follows:

$$\begin{aligned}
P_0(t) &= P\left[Y(t) = W\frac{e^{Bt}}{A+e^{Bt}} > G\,,\right.\\
&\qquad \left. D(t) = \sum_{i=0}^{N_2(t)} X_i \leq S\right]\\
&= \left[\frac{1}{a}\left(\frac{1-u_M}{u_M}\right)^{\frac{\beta}{t}}\left(\frac{t}{t-\beta}\right)\left(a^{1-\frac{\beta}{t}}\right)\right]e^{-\lambda_2 t}\\
&\quad \times\sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\,. \qquad (45.8)
\end{aligned}$$

The probability for a catastrophic failure state F is given by:

$$\begin{aligned}
P_F(t) &= P\left[Y(t) = W\frac{e^{Bt}}{A+e^{Bt}} \leq G\,,\right.\\
&\qquad \left. D(t) = \sum_{i=0}^{N_2(t)} X_i > S\right]\\
&= \left[1-\frac{1}{a}\left(\frac{1-u_1}{u_1}\right)^{\frac{\beta}{t}}\left(\frac{t}{t-\beta}\right)\left(a^{1-\frac{\beta}{t}}\right)\right]\\
&\quad \times\left[1-e^{-\lambda_2 t}\sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\right]. \qquad (45.9)
\end{aligned}$$

The reliability $R_M(t)$ is expressed as:

$$\begin{aligned}
R_M(t) &= \sum_{k=1}^{M} P_k(t)\\
&= \left[1-\frac{1}{a}\left(\frac{1-u_M}{u_M a}\right)^{\frac{\beta}{t}}\left(\frac{t}{t-\beta}\right)\left(a^{1-\frac{\beta}{t}}\right)\right]\\
&\quad \times\left[e^{-\lambda_2 t}\sum_{j=0}^{\infty}\frac{(\lambda_2 t)^j}{j!}F_X^{(j)}(S)\right]. \qquad (45.10)
\end{aligned}$$

A Numerical Example

Assume that the degradation is modeled as the function $Y(t) = W\frac{e^{Bt}}{A+e^{Bt}}$ where $A \sim U[0,5]$ and $B \sim \mathrm{Exp}(10)$. The critical values for the degradation and the shock damage are $G = 500$ and $S = 200$, respectively. The random shocks are measured by the function $D(t) = \sum_{i=1}^{N_2(t)} X_i$, where $X_i \sim \mathrm{Exp}(0.3)$ and X_is are

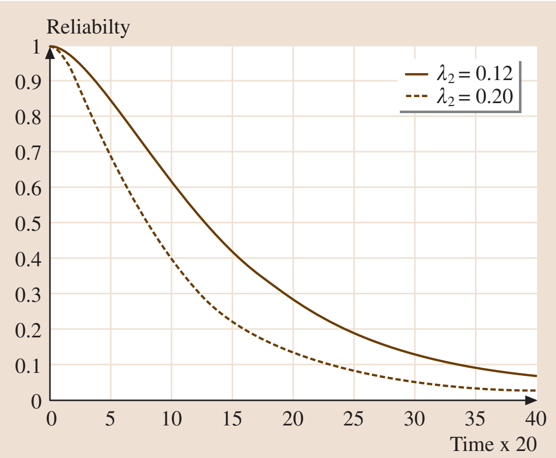

Fig. 45.2 Reliability versus time

iid. Figure 45.2 shows the reliability of the system as a function of time, where the solid line represents $N_2(t)$ with $\lambda_2 = 0.12$ and the dotted line represents $N_2(t)$ with $\lambda_2 = 0.20$.

45.2.2 Systems Subject to Three Competing Processes

System Description

In some applications, the systems are subjected to a variety of governing failure processes. In this section, we consider three independent competing failure processes in which two of them are degradation processes (called degradation process 1, which is measured by the function $Y_1(t)$, and degradation process 2, which is measured by $Y_2(t)$) and the third is a random shock process $D(t)$ [45.15]. Whichever process occurs first causes the system to fail.

Initially, the system is considered to be in its good state (i. e., M_1 and M_2). As time progresses, it can either go to the first degraded state [i. e., $(M-1)_1$ or $(M-1)_2$] upon degradation or can go to a failed state (state F), if subject to random shocks. When a system reaches the first degraded state, it can either stay in that state until the mission time, or it can go to the second degradation state [i. e., $(M-2)_1$ or $(M-2)_2$] upon degradation, or it can go to a failed state (F state) upon random shocks.

The same process will be continued for all stages of degradation except the last degradation, either stage 0_1 or stage 0_2. If the system reaches the last degradation state, it cannot perform its functions satisfactorily and must be treated as a failure (state 0).

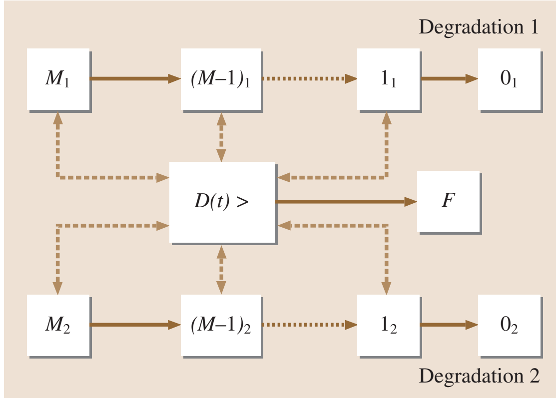

Fig. 45.3 The flow diagram of a system subjected to multiple failure processes [45.15]

Figure 45.3 shows the system flow diagram of the multiple competing transition processes. In Fig. 45.3, the above represents the degradation process 1; the bottom represents the degradation process 2; F represents a catastrophic failure state due to random shocks.

Assumptions.

1. The system consists of $(M+2)$ states where state 0 and state F are both complete failure states. State i is a degradation state, $1 < i < M$.
2. No repair or maintenance is performed on the system.
3. We assume that $Y_i(t)$, $i = 1, 2$ is a nonnegative nondecreasing function at time t, since degradation is an irreversible accumulation of damage.
4. $Y_i(t)$, $i = 1, 2$ and $D(t)$ are statistically independent. The independence assumption implies that the state of one process will have no effect on the state of the others.
5. At time $t = 0$, the system is in state M.
6. The system can fail either due to any of the degradation process when $Y_i(t) > G_i$, $i = 1, 2$ or due to random shocks (in which case it goes to a catastrophic failure state F), i. e. $D(t) = \sum_{i=1}^{N(t)} X_i > S$.
7. The critical threshold value G_i depends upon a function of the states of the degraded systems.

Methodology

In this section, we consider that the degradation paths are modeled by some continuous probabilistic functions. Since the operating condition of the systems is characterized by a finite number of states, let us call the system state space Ω_U. First, we need the discrete continuous processes. In Step 1 below, we discuss a procedure for forcing two degradation processes to become discrete in order to obtain Ω_1 and Ω_2, which correspond to degradation process 1 and 2, respectively. After we have obtained the degradation process spaces Ω_1 and Ω_2, we present a methodology for how to establish a relationship between the system state space Ω_U and the degradation and random shock state spaces $\{\Omega_1, \Omega_2, F\}$ in Step 2.

Step 1: formulate the degradation processes in terms of discrete state sets. The two-degradation-process case is considered here. The most general situation is to allow each degradation process to be described by a number of different discrete states. The state space denoted by $\Omega_1 = \{M_1, \ldots, 1_1, 0_1\}$ corresponds to degradation process 1 with $M_1 + 1$ states. Similarly, the state space denoted by $\Omega_2 = \{M_2, \ldots, 1_2, 0_2\}$ is associated with degradation process 2, having $M_2 + 1$ states. M_1 and M_2 may or may not be the same, and $M_\mathrm{i} < \infty$, $i = 1, 2$.

We view the degradation process from the perspective of a finite number of states. For example, when the value of degradation process 1 $Y_1(t)$ falls into a predefined interval, then its corresponding state will be determined. Let us define as follows:

$[0, W_M], \ldots, (W_2, W_1]$ are the intervals on the degradation 1 curve (Fig. 45.4a) corresponding to state M_1, 0_1, where $W_M < W_{M-1} < \cdots < W_1$ and $[0, A_\mathrm{M}], \ldots, (A_2, A_1]$ are intervals associated with the curve for degradation process 2 (Fig. 45.4b) corresponding to state M_2, 0_2, where $A_M < A_{M-1} < \ldots < A_1$.

Mathematically, the relationship between the degradation process states $\Omega_1 = \{M_1, \ldots, 1_1, 0_1\}$,

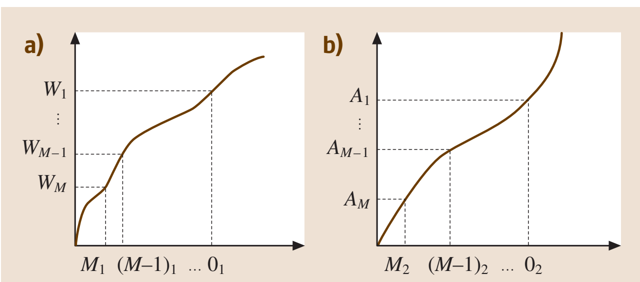

Fig. 45.4a,b The degradation process functions in multi-state terms for: **(a)** degradation 1, **(b)** degradation 2

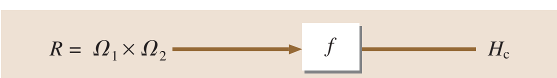

Fig. 45.5 A mapping function

$\Omega_2 = \{M_2, \ldots, 1_2, 0_2\}$, and their corresponding degradation intervals are given as follows:

Degradation process 1

$$
\begin{aligned}
0 < Y_1(t) &\le W_M, \text{ state } M_1 \\
W_M < Y_1(t) &\le W_{\mathrm{M}-1}, \text{ state } (M-1)_1 \\
&\vdots \\
W_2 < Y_1(t) &\le W_1, \text{ state } 1_1 \\
G_1 = W_1 &< Y_1(t), \text{ state } 0_1
\end{aligned}
$$

Degradation process 2

$$
\begin{aligned}
0 < Y_2(t) &\le A_M, \text{ state } M_2 \\
A_M < Y_2(t) &\le A_{\mathrm{M}-1}, \text{ state } (M-1)_2 \\
&\vdots \\
A_2 < Y_2(t) &\le A_1, \text{ state } 1_2 \\
G_2 = A_1 &< Y_2(t), \text{ state } 0_2
\end{aligned}
$$

Step 2: generate the system state space. The system state space is defined as $\Omega_U = \{M, \ldots, 1, 0, F\}$, and consists of $M + 2$ states. In this step, we discuss a methodology to develop a function to generate a relationship between the system state space Ω_U and the degradation state spaces $\{\Omega_1, \Omega_2, F\}$. For example, at a given time t, suppose that degradation process 1 is at state $i_1 \in \Omega_1$, and degradation process 2 is at state $j_2 \in \Omega_2$; what is the system state? This question is addressed as follows.

Let us assume that at the current time the system is not in a catastrophic failure state. So state F can be ignored for the time being. Therefore, we can simply look at ways to define a function that has a relationship between Ω and $\{\Omega_1, \Omega_2\}$ instead of Ω_U and $\{\Omega_1, \Omega_2, F\}$.

The operation is described by a mapping function f, which can be written as

$$f : R = \Omega_1 \times \Omega_2 \to \Omega = \{M, .., 1, 0\}$$

where $R = \Omega_1 \times \Omega_2 = \{(i_1, j_2) | i_1 \in \Omega_1, j_2 \in \Omega_2\}$ is a *Cartesian* product as the input space domain, as shown in Fig. 45.5. The matrix H_c given below is an output space consisting of $M + 1$ elements corresponding to each input-space domain through the function f.

$$H_c = \begin{array}{c c} & \begin{array}{cccc} 0_1 & 1_1 & \cdots & M_1 \end{array} \\ \begin{array}{c} 0_2 \\ 1_2 \\ \vdots \\ M_2 \end{array} & \begin{pmatrix} \times & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & M \end{pmatrix} \end{array} .$$

The top row of this matrix H_c represents the state from degradation process 1. The leftmost column represents the state from degradation process 2. The elements of H_c represent $f(i_1, j_2) = k$ where $i_1 \in \Omega_1$, $j_2 \in \Omega_2$ and $k \in \Omega$. Notice that, in the matrix H_c, all the elements in the first row and first column are zero except that denoted by $\times$ because the system will go to a degraded failure state (state 0) when either of the degradations reaches state 0_i, $i = 1, 2$. Besides, some elements in the matrix H_c are also zeros since we define that, when degradation 1 is in some low state $l_1 (0_1 < l_1 < M_1)$ and degradation 2 is also in some low state $l_2 (0_2 < l_2 < M_2)$, we consider it a degradation failure. It is also observed that $f(M_1, M_2) = M$, because initially the system is in a brand-new state (perfect state M).

As we mentioned above, the first element in H_c is marked by $\times$, which means it does not exist. The reason is presented as follows. We define the time to failure as

$$T = \inf\,[t : Y_1(t) > G_1, Y_2(t) > G_2 \text{ or } D(t) > S] \,. \tag{45.11}$$

It should be noted that all three processes are competing against each other for the life of a system. However, only one of the three processes (whichever occurs first when its corresponding critical threshold value is exceeded) causing the system to fail. Hence, the following events will not happen:

$$P\,[Y_1(t) > G_1, Y_2(t) > G_2, D(t) \le S] = 0 \,,$$
$$P\,[Y_1(t) > G_1, Y_2(t) > G_2, D(t) > S] = 0 \,,$$
$$P\,[Y_1(t) > G_1, Y_2(t) < G_2, D(t) > S] = 0 \,,$$

and

$$P\,[Y_1(t) < G_1, Y_2(t) > G_2, D(t) > S] = 0 \,.$$

Because $f(0_1, 0_2) = P[Y_1(t) > G_1, Y_2(t) > G_2, D(t) \le S]$, so the combination of $f(0_1, 0_2)$ does not exist.

The function $f : R = \Omega_1 \times \Omega_2 \to \Omega = \{M, .., 1, 0\}$ is defined with following requirements:

1. $f(0_1, b) = f(a, 0_2) = 0$, where $b \in \Omega_2, a \in \Omega_1$
   $f(M_1, M_2) = M$

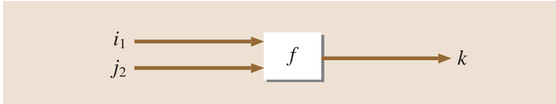

Fig. 45.6 A representation of a system state-generating box

2. f is monotonic and nondecreasing in each argument.

   For instance,

   $f(a_1, b_2) \ge f(l_1, b_2)$ if $a_1 \ge l_1$,
   $f(a_1, b_2) \ge f(a_1, l_2)$ if $b_2 \ge l_2$.

Figure 45.6 demonstrates the system's state-generating box. There are two inputs i_1 and j_2 and an output k. The inside mapping mechanism is performed by the function f. At time t, suppose that degradation 1 is at state i_1 and degradation 2 is at state j_2; i_1 and j_2 are inputs. Via matrix H_c, the system state k is then generated as output.

In the matrix H_c different state-combination inputs can generate the same results for the system state. To explain this, we need the following definition of the equivalence class.

Definition 45.1
The i-th equivalence class, R_i, is defined as follows:

$$R_i = [(k_1, j_2) \text{ where } k_1 \in \Omega_1, j_2 \in \Omega_2 | f(k_1, j_2) = i] \,,$$
$$i = 0, 1, \ldots, M \,, \tag{45.12}$$

R_i represents all possible state combinations that generate the system state i; $R_0, \ldots, R_M$ are disjointed sets that partition R into $(M+1)$ equivalence classes, so that

$$R = \bigcup_{i=0}^{M} R_i \,.$$

45.2.3 Reliability Evaluation

In this section, the probability density functions and the system mean time to failure are derived based on the state probabilities given in Sect. 45.2.1. Now, we derive the probability of being in each state. Initially, the system is in a brand-new state; i.e., in state $M = f(R_M)$. The probability for state M is given by

$$P_t(M) = P_t[f(R_M)] \,. \tag{45.13}$$

As defined previously, R_i represents all possible state combinations generating the system state i. The probability of being in state i is the union of all the elements in R_i

$$P_t(i) = P[f(R_i)]\,. \quad (45.14)$$

The probability for a catastrophic failure state F is given by

$$P_t(F) = P[Y_1(t) \le G_1, Y_2(t) \le G_2, D(t) > S]\,. \quad (45.15)$$

The reliability $R(t)$ can be calculated as follows:

$$\begin{aligned} R(t) &= P(\text{system state } \ge 1) \\ &= \bigcup_{i=1}^{M} P[f(R_i)] \\ &= \sum_{i=1}^{M} P_t(i)\,, \end{aligned} \quad (45.16)$$

where $P_t(i)$ is the probability of being in state i.

The mean time to failure is expressed as [45.15]:

$$\begin{aligned} E[T] &= \int_0^\infty P(T > t)\,\mathrm{d}t \\ &= \int_0^\infty P[Y_1(t) \le G_1] P[Y_2(t) \le G_2] \\ &\quad \times \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j\, \mathrm{e}^{-\lambda_2 t}}{j!} F_X^{(j)}(S) \end{aligned}$$

or, equivalently, that

$$\begin{aligned} E[T] &= \sum_{j=0}^{\infty} \frac{F_X^{(j)}(S)}{j!} \int_0^\infty P[Y_1(t) \le G_1] \\ &\quad \times P[Y_2(t) \le G_2](\lambda_2 t)^j\, \mathrm{e}^{-\lambda_2 t}\,\mathrm{d}t\,. \end{aligned} \quad (45.17)$$

The result in (45.17) obviously would depend on the expression $P[Y_1(t) \le G_1]P[Y_2(t) \le G_2]$. The probability density function of time to failure, $f_{\mathrm{T}}(t)$ is therefore as follows:

$$\begin{aligned} f_{\mathrm{T}}(t) &= -\frac{\mathrm{d}}{\mathrm{d}t}[P(T > t)] \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{ P[Y_1(t \le G_1)] P[Y_2(t) \le G_2] \\ &\quad \times \sum_{j=0}^{\infty} \frac{(\lambda_2 t)^j\, \mathrm{e}^{-\lambda_2 t}}{j!} F_X^{(j)}(S) \Bigg\}\,. \end{aligned} \quad (45.18)$$

45.2.4 Numerical Examples

This example aims to illustrate the results discussed in the previous sections. Consider a system subjected to two degradation processes and random shocks.

Assume that degradation process 1 is described by the function $Y_1(t) = A + Bg(t)$, where the random variables A and B are independent and both follow normal distributions, with mean 90 and variance 2.5, and mean 78 and variance 6, respectively. In short, $A \sim N(90, 2.5)$ and $B \sim N(78, 6)$. The degradation function is assumed to be $g(t) = t^3$. Also $G_1 = 2500$, $W_3 = 1500$, $W_2 = 2000$, and $W_1 = 2500$.

Assume that degradation process 2 is described by $Y_2(t) = W \frac{\mathrm{e}^{BBt}}{AA + \mathrm{e}^{BBt}}$, where the random variables AA and BB are independent and follow uniform distributions with parameter interval [0,100] and an exponential distribution with parameter 0.1, respectively. In other words, $AA \sim U[0, 100]$ and $BB \sim \mathrm{Exp}(0.01)$. Also $G_2 = 5000$, $A_2 = 2600$, $A_1 = 5000$, and $W = 7000$. Assume that the random shock is represented by $D(t) = \sum_{i=0}^{N(t)} X_i$ with critical value $S = 200$, where $X_i \sim \mathrm{Exp}(0.1)$ and the X_i are iid.

Assume that the states associated with degradation process 1 and degradation 2 are, respectively, $\Omega_1 = \{3_1, 2_1, 1_1, 0_1\}$ and $\Omega_2 = \{2_2, 1_2, 0_2\}$. We define the system state space as $\Omega_{\mathrm{U}} = \{3, 2, 1, 0, F\}$ and the matrix H_{c} is given as

$$H_{\mathrm{c}} = \begin{array}{c c} & \begin{array}{cccc} 0_1 & 1_1 & 2_1 & 3_1 \end{array} \\ \begin{array}{c} 0_2 \\ 1_2 \\ 2_2 \end{array} & \begin{pmatrix} \times & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \end{array}\,.$$

Then we obtain

$$\begin{aligned} R = \{&(0_1, 1_2), (0_1, 2_2), (1_1, 0_2), (2_1, 0_2), (3_1, 0_2), \\ &(1_1, 1_2), (2_1, 1_2), (3_1, 1_2), (1_1, 2_2), (2_1, 2_2), \\ &(3_1, 2_2)\} \end{aligned}$$

The equivalence classes can be listed as follows:

$$\begin{aligned} R_0 &= \{(0_1, 1_2), (0_1, 2_2), (1_1, 0_2), (2_1, 0_2), (3_1, 0_2), \\ &\qquad (1_1, 1_2)\}\,, \\ R_1 &= \{(1_1, 2_2)\}\,, \\ R_2 &= \{(2_1, 1_2), (2_1, 2_2)\}\,, \\ R_3 &= \{(3_1, 1_2), (3_1, 2_2)\}\,, \\ R &= \sum_{i=0}^{3} R_i\,. \end{aligned}$$

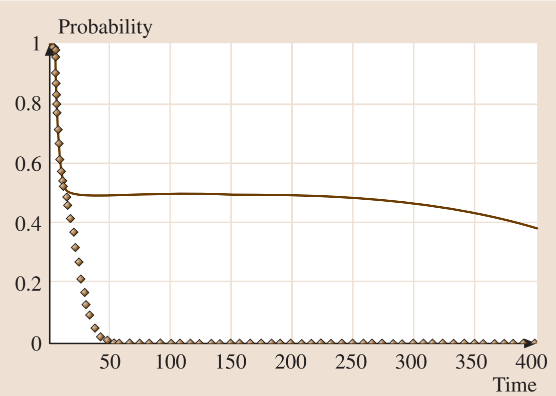

Fig. 45.7 Probability plot for state 3 versus time

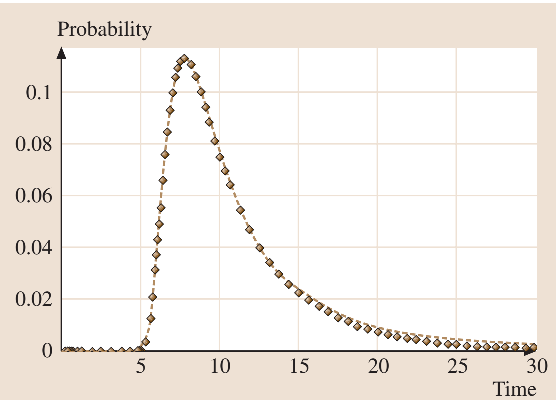

Fig. 45.8 Probability plot for state 2 versus time

According to this expression for H_c, the probability of the system being in state 3 is the sum of the probability $f(3_1, 2_2)$ and of the probability $f(3_1, 1_2)$. That sum is calculated as

$$\begin{aligned} P_t(3) &= P_t\left[f(R_3)\right] \\ &= \Phi\left(\frac{1500-(90+78t)}{\sqrt{2.5+6t^6}}\right)\left[1-\frac{1}{100}\,(0.4)^{\frac{0.01}{t}}\right. \\ &\quad \times\left.\left(\frac{t}{t-0.01}\right)\left(0.01^{1-\frac{0.01}{t}}\right)\right] \\ &\quad \times \mathrm{e}^{-\lambda_2 t}\sum_{j=0}^{\infty}\left(\frac{\lambda_2 t}{j!}\right)F_X^{(j)}(200)\,. \end{aligned} \tag{45.19}$$

Figure 45.7 shows the probability for the system to be in state 3 as a function of time t, where the solid line represents the compound Poisson process $D(t)=\sum_{i=0}^{N(t)} X_i$ with rate $\lambda=0.04$ and the dotted line represents the compound Poisson process with rate $\lambda=0.8$. In Fig. 45.7 we observe that, as t progresses to 50, the probability that the system is in state 3 quickly approaches 0 when the rate is given as $\lambda=0.8$, and is stable with $\lambda=0.04$.

Because $R_2=\{(2_1, 1_2), (2_1, 2_2)\}$, the probability of being in state 2 is given by

$$\begin{aligned} P_\mathrm{t}(2) &= P_\mathrm{t}[f(2_1, 1_2)]+P_\mathrm{t}[f(2_1, 2_2)] \\ &= (UV)\,\mathrm{e}^{-\lambda_2 t}\sum_{j=0}^{\infty}\left(\frac{\lambda_2 t}{j!}\right)F_X^{(j)}(200)\,, \end{aligned} \tag{45.20}$$

where

$$\begin{aligned} U &= \Phi\left(\frac{2000-(90+78t)}{\sqrt{2.5+6t^6}}\right) \\ &\quad -\Phi\left(\frac{1500-(90+78t)}{\sqrt{2.5+6t^6}}\right), \end{aligned}$$

and

$$\begin{aligned} V &= 1-\frac{1}{100}\left(\frac{t}{t-0.01}\right)(0.4)^{\frac{0.01}{t}} \\ &\quad \times\left(\frac{t}{t-0.01}\right)(0.01)^{1-\frac{0.01}{t}}\,. \end{aligned}$$

Figure 45.8 shows the probability of being in state 2 as a function of time t, where the solid line represents the compound Poisson process $D(t)=\sum_{i=0}^{N(t)} X_i$ with rate $\lambda=0.04$, and the dotted line represents the compound Poisson process with rate $\lambda=0.8$. In Fig. 45.8, we observe that, before the time t progresses to 5, the probability of being in state 2 stays close to zero for both rates $\lambda=0.8$ and $\lambda=0.04$. It should be noted that the two curves are almost the same for the different values of the rate $\lambda=0.8$ and $\lambda=0.04$.

Similarly, the probability of being in state 1 is calculated as:

$$\begin{aligned} P_t(1) &= P_\mathrm{t}[f(1_1, 2_2)] \\ &= E_1 E_2\,\mathrm{e}^{-\lambda_2 t}\sum_{j=0}^{\infty}\left(\frac{\lambda_2 t}{j!}\right)F_X^{(j)}(200)\,, \end{aligned}$$

where $$\begin{aligned} E_1 &= \Phi\left(\frac{2500-(90+78t)}{\sqrt{2.5+6t^6}}\right) \\ &\quad -\Phi\left(\frac{2000-(90+78t)}{\sqrt{2.5+6t^6}}\right), \\ E_2 &= 1-\frac{1}{100}\left(\frac{t}{t-0.01}\right)\left(\frac{22}{13}\right)^{\frac{0.01}{t}}(0.01)^{1-\frac{0.01}{t}}\,. \end{aligned} \tag{45.21}$$

Part E | 45.2

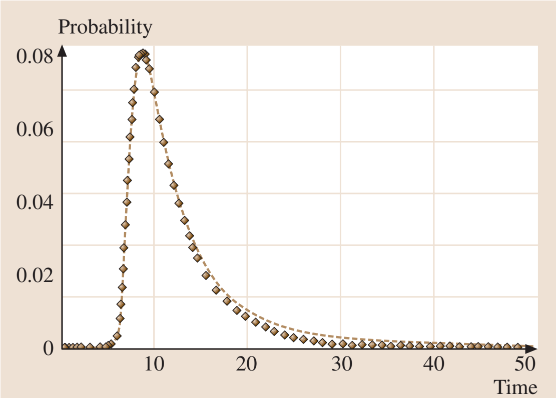

Fig. 45.9 Probability plot for state 1 versus time

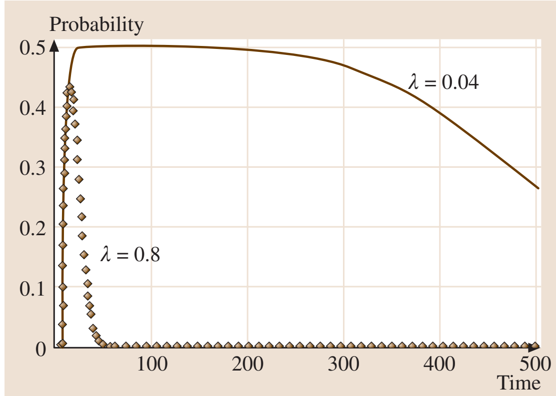

Fig. 45.10 Probability plot for state 0 versus time

Figure 45.9 shows the probability of being in state 1 versus time t, where the solid line represents the compound Poisson process $D(t) = \sum_{i=0}^{N(t)} X_i$ with rate $\lambda = 0.04$, and the dotted line represents the compound Poisson process with rate $\lambda = 0.8$. In Fig. 45.9, we observe that, before the time t progresses to 15, the probability of being in state 1 for both rates $\lambda = 0.8$ and $\lambda = 0.04$ are about the same.

We can also easily obtain the probability of being in state 0 as follows:

$$\begin{aligned} P_t(0) &= P\big[f(0_1,1_2) + f(0_1,2_2) + f(1_1,0_2) \\ &\quad + f(2_1,0_2) + f(3_1,0_2) + f(1_1,1_2)\big] \\ &= (X_1Y_1 + X_2Y_2 + X_3Y_3)\mathrm{e}^{-\lambda_2 t} \\ &\quad \times \sum_{j=0}^{\infty} \left(\frac{\lambda_2 t}{j!}\right) F_X^{(j)}(200)\,, \end{aligned}$$

where $X_1 = 1 - \Phi\left(\dfrac{2500-(90+78t)}{\sqrt{2.5+6t^6}}\right)$,

$$\begin{aligned} X_2 &= \Phi\left(\frac{2500-(90+78t)}{\sqrt{25+6t^2}}\right)\,, \\ Y_1 &= 1 - \frac{1}{100}(0.4)^{\frac{0.01}{t}}\left(\frac{t}{t-0.01}\right) \\ &\quad \times \left(0.01^{1-\frac{0.01}{t}}\right)\,, \\ Y_2 &= \frac{1}{100}(0.4)^{\frac{0.01}{t}}\left(\frac{t}{t-0.01}\right) \\ &\quad \times \left(0.01^{1-\frac{0.01}{t}}\right)\,, \\ X_3 &= \Phi\left(\frac{2500-(90+78t)}{\sqrt{2.5+6t^6}}\right) \\ &\quad - \Phi\left(\frac{2000-(90+78t)}{\sqrt{2.5+6t^6}}\right)\,, \end{aligned}$$

$$\begin{aligned} Y_3 &= 1 - \frac{1}{100}\left[\left(\frac{22}{13}\right)^{\frac{0.01}{t}} + (0.4)^{\frac{0.01}{t}}\right] \\ &\quad \times \left(\frac{t}{t-0.01}\right)\left(0.01^{1-\frac{0.01}{t}}\right)\,. \end{aligned} \tag{45.22}$$

Figure 45.10 shows the probability that the system is in state 0 versus the time t, where the solid line represents the compound Poisson process $D(t) = \sum_{i=0}^{N(t)} X_i$ with rate $\lambda = 0.04$, and the dotted line represents the compound Poisson process with rate $\lambda = 0.8$. In Fig. 45.10, we observe that the probability of being in state 0 is close to zero when $t > 100$ for the rate $\lambda = 0.8$.

The probability of being in state F is calculated as:

$$\begin{aligned} P_t(F) &= P\left[Y_1(t) \le G_1, Y_2(t) \le G_2, D(t) > S\right] \\ &= KL\left[1 - \mathrm{e}^{-\lambda_2 t}\sum_{j=0}^{\infty}\left(\frac{\lambda_2 t}{j!}\right)F_X^{(j)}(200)\right]\,, \end{aligned}$$

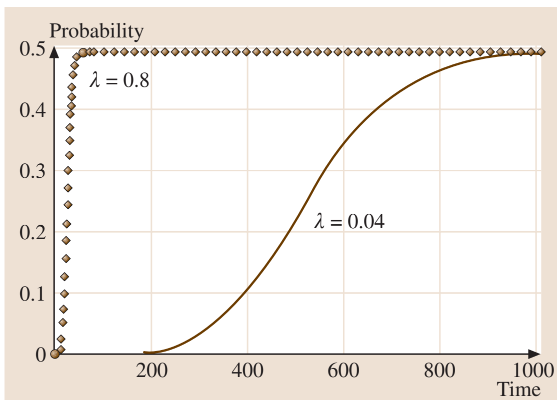

Fig. 45.11 Probability plot for state F versus time

Part E | 45.2

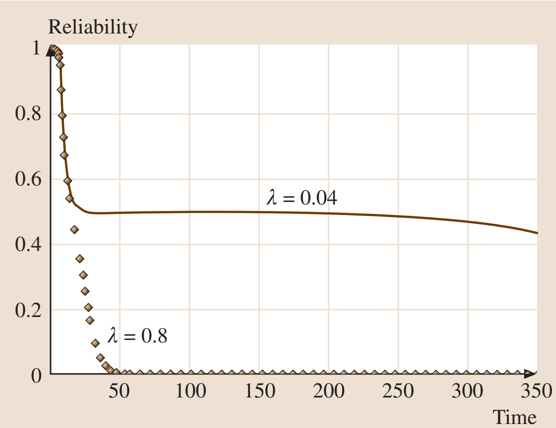

Fig. 45.12 Reliability versus time

where
$$K = \Phi\left(\frac{2500-(90+78t)}{\sqrt{2.5+6t^6}}\right),$$
$$L = 1-\frac{1}{100}(0.4)^{\frac{0.01}{t}}\left(\frac{t}{t-0.01}\right)\left(0.01^{1-\frac{0.01}{t}}\right). \quad (45.23)$$

Figure 45.11 shows the probability of being in state F as a function of time t, where the solid line represents the compound Poisson process $D(t)=\sum_{i=0}^{N(t)} X_i$ with rate $\lambda = 0.04$, and the dotted line represents the compound Poisson process with rate $\lambda = 0.8$.

Finally, the system reliability $R(t)$ is given by

$$\begin{aligned} R(t) &= P(\text{system state} \geq 1) \\ &= \sum_{i=1}^{3} P_t(i) \\ &= X_3 Y_3 \mathrm{e}^{-\lambda_2 t} \sum_{j=0}^{\infty}\left(\frac{\lambda_2 t}{j!}\right) F_X^{(j)}(200), \end{aligned}$$

where
$$\begin{aligned} X_3 &= \Phi\left(\frac{2000-(90+78t)}{\sqrt{2.5+6t^6}}\right) \\ &\quad \times \left\{1-\frac{1}{100}\left[(0.4)^{\frac{0.01}{t}}+\left(\frac{22}{13}\right)^{\frac{0.01}{t}}\right]\right. \\ &\quad \left. \times\left(\frac{t}{t-0.01}\right)\left(0.01^{1-\frac{0.01}{t}}\right)\right\}, \\ Y_3 &= \left[\Phi\left(\frac{2500-(90+78t)}{\sqrt{2.5+6t^6}}\right)\right. \\ &\quad \left. -\Phi\left(\frac{2000-(90+78t)}{\sqrt{2.5+6t^6}}\right)\right] \\ &\quad \times\left[1-\frac{1}{100}\left(\frac{t}{t-0.01}\right)\right. \\ &\quad \left. \times\left(\frac{22}{13}\right)^{\frac{0.01}{t}} 0.01^{1-\frac{0.01}{t}}\right]. \end{aligned} \quad (45.24)$$

Figure 45.12 shows the system reliability versus time t, where the solid line represents the compound Poisson process with rate $\lambda = 0.04$, and the dotted line represents the compound Poisson process with rate $\lambda = 0.8$. As for the rate $\lambda = 0.8$ we observe that the system will probably fail after a time t of 50. It seems that the random shock process governs the behavior of the reliability function. Therefore, the dotted line quickly approaches the failure caused by the shock damage.

45.3 Repairable Degraded Systems Modeling

45.3.1 Inspection–Maintenance Model Subject to Two Competing Processes

Model description

Assumptions. The system starts in a new condition. The assumptions are as follows [45.22]:

1. The system is not continuously monitored, its state can be detected only by inspection, but system failure is self-announcing without inspection.
2. After a PM or CM action, the system will be restored back to an as-good-as-new state.
3. A CM action is more costly than a PM, and a PM costs much more than an inspection. This implies $C_c > C_p > C_i$.
4. The two processes $Y(t)$ and $D(t)$ are independent.
5. Repair time is not negligible.

Although continuous monitoring processes are feasible for some systems, the cost to monitor the process and the labor required would, however, not make it realistic in practice. Therefore, we need to improve the system performance by determining the periodic inspections with maintenance action that will minimize the average total system maintenance cost. Since system deterioration while running leads to system failure, it proves better to assume that the degradation paths are continuous and increasing functions.

Inspection–Maintenance Policy. It is proposed that the system is periodically inspected at times $\{I, 2I, \cdots, nI, \cdots\}$. We assume that the degradation $(\{Y(t)\}_{t\geq 0})$ and random shock processes $(\{D(t)\}_{t\geq 0})$ are independent. Let T denote the time to failure, defined as

$$T = \inf\left[t > 0 : Y(t) > G \text{ or } D(t) > S\right] ,$$

where G is the critical value for $\{Y(t)\}_{t\geq 0}$ and S is the threshold level for $\{D(t)\}_{t\geq 0}$.

The two threshold values L and G (where G is fixed) effectively divide the system state into three regions, as illustrated in Fig. 45.13. They are: the doing-nothing zone; the PM zone; and the CM zone. The maintenance action will be performed when either of the following situations occurs.

1. The current inspection reveals that the system condition falls into the PM zone, and this state was not found on previous inspection. At inspection time iI, the system falls into the PM zone, which means $\{Y[(i-1)I] \leq L,\ D[(i-1)I] \leq S\} \cap \{L < Y(iI) \leq G,\ D(iI) \leq S\}$. Then PM action is performed and will take a random time R_1.

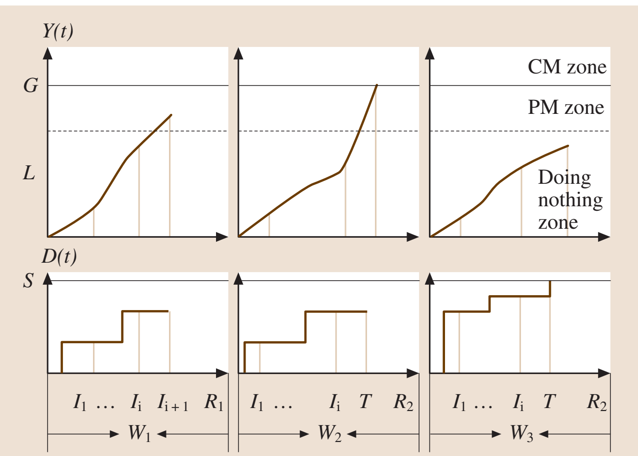

Fig. 45.13 The evolution of the system

2. When the system fails at T, a CM action is taken immediately and takes time R_2.

It is assumed that both PM and CM actions are considered to be perfect. Even though both PM and CM actions bring the system back to an as-good-as-new state, they are, physically, not necessarily the same, since a CM has to performed on a worse system. Hence, CM is likely to be more complex and expensive. Therefore, it is realistic to assume that the repair time is not negligible. This chapter considers that the PM action will take a random amount of time R_1 and that a CM action will take a random amount of time R_2. After a PM or a CM action is performed, the system is renewed. A new sequence of the inspection would start again, defined in the same way.

Maintenance Cost Modeling

In this section, an explicit expression for the average long-run maintenance cost per unit time is derived. The objectives of the model are to determine the optimal PM threshold L and the optimal inspection time I. From the basics of renewal reward theory, we have

$$\lim_{t\to\infty} \frac{C(t)}{t} = \frac{E[C_1]}{E[W_1]} .$$

We now model the average total maintenance cost per unit time on a single renewal cycle instead of $\lim_{t\to\infty} \frac{C(t)}{t}$; then we will analyze $E[C_1]$ and $E[W_1]$.

Expected maintenance cost analysis in a cycle. The expected total maintenance cost during a cycle $E[C_1]$ is expressed as [45.22]:

$$E[C_1] = C_\mathrm{i} E[N_I] + C_\mathrm{p} E[R_1] P_\mathrm{p} + C_\mathrm{c} E[R_2] P_\mathrm{c} . \tag{45.25}$$

During a renewal cycle, activities in terms of costs include: inspection cost, time to repair, and PM or CM actions. The renewal cycle will end by either a PM or a CM action. With a probability of P_p, the cycle will end with a PM action and it will take on average an amount of time $E[R_1]$ to complete a PM action, with a corresponding cost of $C_\mathrm{p} E[R_1] P_\mathrm{p}$. Similarly, if a cycle ends with a CM action with probability P_c, it will take on average an amount of time $E[R_2]$ to complete a CM action, with a corresponding cost of $C_\mathrm{c} E[R_2] P_\mathrm{c}$. In the following, we will perform the analysis of $E[C_1]$.

Calculate $E[N_I]$. Let $E[N_I]$ denote the expected number of inspections during a cycle. $E[N_I]$ can be obtained as:

$$E[N_I] = \sum_{i=1}^{\infty} (i) P(N_I = i) \,. \quad (45.26)$$

Obviously $\sum_{i=1}^{\infty} P(N_I = i) = 1$. There will be a total of i inspections during a cycle if the first PM trigger falls within the time interval $[(i-1)I, iI]$, or if the system condition is in the doing-nothing zone before time iI and the system fails during the interval $[iI, (i+1)I]$. In other words, the inspection will stop when the i-th inspection finds that a PM condition is satisfied while this situation was not revealed in the previous inspection, or the system fails during the interval $[iI < T \leq (i+1)I]$ while the system is in the doing-nothing zone before iI.

Let $P(N_I = i)$ denote the probability that there a total of i inspections occur in a renewal cycle. Then we have

$$\begin{aligned} P(N_I = i) = {} & P\{Y[(i-1)I] \leq L, D[(i-1)I] \leq S\} \\ & \times P[L < Y(iI) \leq G, D(iI) \leq S] \\ & + P[Y(iI) \leq L, D(iI) \leq S] \\ & \times P[iI < T \leq (i+1)I] \,. \end{aligned} \quad (45.27)$$

Hence,

$$\begin{aligned} E[N_I] = \sum_{i=1}^{\infty} & i\{P\{Y[(i-1)I] \leq L, D[(i-1)I] \leq S\} \\ & \times P[L < Y(iI) \leq G, D(iI) \leq S] \\ & + P[Y(iI) \leq L, D(iI) \leq S] \\ & \times P[iI < T \leq (i+1)I]\} \,. \end{aligned} \quad (45.28)$$

We now calculate the probabilities $P\{Y[(i-1)I] \leq L, D[(i-1)I] \leq S\}$ and $P[L < Y(iI) \leq G, D(iI) \leq S]$ with the following two different expressions for $Y(t)$.

A) Assume $Y(t) = A + Bg(t)$ where $A \sim N\left(\mu_{\mathrm{A}}, \sigma_{\mathrm{A}}^2\right)$, $B \sim N\left(\mu_{\mathrm{B}}, \sigma_{\mathrm{B}}^2\right)$, and A and B are independent. Given $g(t) = t$. $D(t) = \sum_{i=0}^{N(t)} X_{\mathrm{i}}$ where the X_{i} are iid and $N(t) \sim \mathit{Poisson}(\lambda)$. Then

$$\begin{aligned} & P\{Y[(i-1)I] \leq L, D[(i-1)I] \leq S\} \\ & = P[A + B(i-1)I \leq L] \\ & \quad \times P\left\{D[(i-1)I] = \sum_{i=0}^{N[(i-1)I]} X_{\mathrm{i}} \leq S\right\} \\ & = \Phi\left(\frac{L - (\mu_{\mathrm{A}} + \mu_{\mathrm{B}}(i-1)I)}{\sqrt{\sigma_A^2 + \sigma_{\mathrm{B}}^2((i-1)I)^2}}\right) \mathrm{e}^{-\lambda(i-1)I} \\ & \quad \times \sum_{j=0}^{\infty} \frac{(\lambda(i-1)I)^j}{j!} F_X^{(j)}(S) \end{aligned} \quad (45.29)$$

and

$$\begin{aligned} & P[L < Y(iI) \leq G, D(iI) \leq S] \\ & = \left[\Phi\left(\frac{G - (\mu_{\mathrm{A}} + \mu_{\mathrm{B}} iI)}{\sqrt{\sigma_{\mathrm{A}}^2 + \sigma_{\mathrm{B}}^2 (iI)^2}}\right)\right. \\ & \quad \left. - \Phi\left(\frac{L - (\mu_{\mathrm{A}} + \mu_{\mathrm{B}} iI)}{\sqrt{\sigma_A^2 + \sigma_{\mathrm{B}}^2 (iI)^2}}\right)\right] \mathrm{e}^{-\lambda iI} \\ & \quad \times \sum_{j=0}^{\infty} \frac{(\lambda iI)^j}{j!} F_X^{(j)}(S) \,. \end{aligned} \quad (45.30)$$

B) Assume $Y(t) = W \frac{\mathrm{e}^{Bt}}{A + \mathrm{e}^{Bt}}$, where W is a constant, $A \sim U[0, a]$, $a > 0$; $B \sim \mathrm{Exp}(\beta)$, $\beta > 0$, A and B are independent. $D(t) = \sum_{i=0}^{N(t)} X_{\mathrm{i}}$ where the X_{i} are iid and $N(t) \sim \mathit{Possion}(\lambda)$. Then

$$\begin{aligned} & P\{Y[(i-1)I] \leq L, D[(i-1)I] \leq S\} \\ & = \left[1 - \frac{1}{a}\left(\frac{1 - u_1}{u_1}\right)^{\frac{\beta}{I_{i-1}}} \left(\frac{(i-1)I}{(i-1)I - \beta}\right)\right. \\ & \quad \left. \times \left(a^{1 - \frac{\beta}{(i-1)I_1}} - 1\right)\right] \mathrm{e}^{-\lambda(i-1)I} \\ & \quad \times \sum_{j=0}^{\infty} \frac{[\lambda(i-1)I]^j}{j!} F_X^{(j)}(S) \,, \end{aligned} \quad (45.31)$$

where $u_1 = L/W$. Similarly,

$$\begin{aligned} & P[L < Y(iI) \leq G, D(iI) \leq S] \\ & = \left\{\frac{1}{a}\left(\frac{iI}{iI - \beta}\right)\right. \\ & \quad \left. \times \left(a^{1 - \frac{\beta}{iI}}\right)\left[\left(\frac{1 - u_3}{u_3}\right)^{\frac{\beta}{iI}} - \left(\frac{1 - u_2}{u_2}\right)^{\frac{\beta}{iI}}\right]\right\} \mathrm{e}^{-\lambda iI} \\ & \quad \times \sum_{j=0}^{\infty} \frac{(\lambda iI)^j}{j!} F_X^{(j)}(S) \,, \end{aligned} \quad (45.32)$$

where $u_2 = G/W$, $u_3 = L/W$.

Secondly, we discuss the calculation of $P[iI < T \leq (i+1)I]$. The definition of T is $T = \inf[t > 0 : Y(t) > G$ or $D(t) > S]$. According to the definition, we derive the expression:

$$\begin{aligned} & P[iI < T \leq (i+1)I] \\ & = P\{Y(iI) \leq L, Y[(i+1)I] > G\} \\ & \quad \times P\{D[(i+1)I] \leq S\} \\ & \quad + P\{Y[(i+1)I] \leq L\} \\ & \quad \times P\{D(iI) \leq S, D[(i+1)I] > S\} \,. \end{aligned} \quad (45.33)$$

In (45.33), since $Y(iI)$, $Y[(i+1)I]$ are not independent, we could obtain the joint pdf $f_{Y(iI),Y[(i+1)I]}(y_1, y_2)$ in order to compute $P\{Y(iI) \le L, Y[(i+1)I] > G\}$. We consider two different expressions for $Y(t)$. The details are as follows:
A) Assume $Y(t) = A + Bg(t)$ where $A > 0$ and $B > 0$ are two independent random variables, and $g(t)$ is an increasing function of time t. Assume that $A \sim f_A(a)$, $B \sim f_B(b)$. Let

$$\begin{cases} y_1 = a + bg(iI) \\ y_2 = a + bg[(i+1)I] \end{cases} .$$

After simultaneously solving the above equations in terms of y_1 and y_2, we obtain:

$$a = \frac{y_1 g[(i+1)I] - y_2 g(iI)}{g[(i+1)I] - g(iI)} = h_1(y_1, y_2) ,$$
$$b = \frac{y_2 - y_1}{g[(i+1)I] - g(iI)} = h_2(y_1, y_2) .$$

The *Jacobian* J is given by

$$J = \begin{vmatrix} \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} \end{vmatrix} = \left| \frac{1}{g(iI) - g[(i+1)I]} \right| .$$

Then the random vector $\{Y(iI), Y[(i+1)I]\}$ has a joint continuous pdf as follows

$$f_{Y(iI),Y[(i+1)I]}(y_1, y_2) = |J|\, f_A[h_1(y_1, y_2)] f_B[h_2(y_1, y_2)] . \qquad (45.34)$$

B) Assume $Y(t) = \frac{W e^{At}}{B + e^{At}}$ where $A > 0$ and $B > 0$ are independent. Assume $A \sim f_A(a)$, $B \sim f_B(b)$. Let

$$\begin{cases} y_1 = \frac{W e^{aiI}}{b + e^{aiI}} \\ y_2 = \frac{W e^{a(i+1)I}}{b + e^{a(i+1)I}} \end{cases} .$$

The solutions for a and b can be easily found from the above equations in terms of y_1 and y_2 as follows:

$$\begin{cases} a = \frac{\ln\left(\frac{y_2(y_1 - W)}{y_1(y_2 - W)}\right)}{I} = h_1(y_1, y_2) \\ b = -\frac{e^{\frac{\ln\left(\frac{y_2(y_1-W)}{y_1(y_2-W)}\right)(i+1)I}{I}}(y_2 - W)}{y_2} = h_2(y_1, y_2) \end{cases} .$$

It can be shown that the random vector $\{Y(iI), Y[(i+1)I]\}$ has a joint density function given by

$$f_{Y(iI),Y[(i+1)I]}(y_1, y_2) = |J|\, f_A[h_1(y_1, y_2)] f_B[h_2(y_1, y_2)] , \qquad (45.35)$$

where the Jacobian determinant J is given in Appendix A.

As for the term $P\{D(iI) \le S, D[(i+1)I] > S\}$ in (45.30), since $D(t) = \sum_{i=0}^{N(t)} X_i$ is a compound Poisson process, the compound Poisson process has a stationary independent increment property. Therefore, the random variables $D(iI)$ and $D[(i+1)I] - D(iI)$ are independent. Using the *Jacobian* transformation, the random vector $\{D(iI), D[(i+1)I] - D(iI)\}$ is distributed in the same way as vector $\{D(iI), D[(i+1)I]\}$. Note that $D(iI)$ and $D(I_{i+1})$ are independent, therefore,

$$P\{D(iI) \le S, D[(i+1)I] > S\} = P[D(iI) \le S] P\{D[(i+1)I] > S\} . \qquad (45.36)$$

Calculate P_p. Note that either a PM or CM action will end a renewal cycle. In other words, these two events are mutually exclusive at the renewal time point. As a consequence, $P_p + P_c = 1$. The probability P_p can be obtained as follows:

$$P_p = P(\text{PM ending a cycle}) = \sum_{i=1}^{\infty} P\{Y[(i-1)I] \le L, L < Y(iI) \le G\} \times P[D(iI) \le S] . \qquad (45.37)$$

Analysis of expected cycle length. Since the renewal cycle ends either by a PM action with probability P_p or a CM action with probability P_c, the mean cycle length $E[W_1]$ is calculated as follows:

$$\begin{aligned} E[W_1] &= \sum_{i=1}^{\infty} E[(iI + R_1) I_{\text{PM occurs in}[(i-1)I, iI]}] + E\,[(T + R_2) 1_{\text{CM occurs}}] \\ &= \left\{ \sum_{i=1}^{\infty} iI P\{Y[(i-1)I] \le L, D[(i-1)I] \le S\} P[L < Y(iI) \le G, D(iI) \le S] \right\} \\ &\quad + E[R_1] P_p + (E[T] + E[R_2]) P_c , \end{aligned} \qquad (45.38)$$

where $I_{\text{PM occurs in}[(i-1)I, iI]}$ and $I_{\text{CM occurs}}$ are the indicator functions.

The mean time to failure, $E[T]$ is given by [45.22]:

$$E[T]=\int_0^\infty P\{T>t\}\,\mathrm{d}t$$
$$=\int_0^\infty P[Y(t)\le G,\,D(t)\le S]\,\mathrm{d}t$$
$$=\int_0^\infty P[Y(t)\le G]\sum_{j=0}^\infty \frac{(\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t}}{j!}F_X^{(j)}(S)\,\mathrm{d}t$$

or, equivalently:

$$E[T]=\sum_{j=0}^\infty \frac{F_X^{(j)}(S)}{j!}\int_0^\infty P[Y(t)\le G](\lambda_2 t)^j\,\mathrm{e}^{-\lambda_2 t}\,\mathrm{d}t \quad (45.39)$$

The expression $E[T]$ depends on the probability $P[Y(t)\le G]$ and cannot always be easily be obtained in closed form.

Optimization of the maintenance cost rate policy

We determine the optimal inspection time I and PM threshold L such that the long-run average maintenance cost rate $EC(L,I)$ is minimized. Mathematically, we wish to minimize the following objective function [45.22]:

$$\begin{aligned}EC(L,I)&\\ =&\frac{\sum_{i=1}^\infty iP_1P_2}{\left\{\sum_{i=1}^\infty I_iP_1P_2\right\}+E[R_1]P_\mathrm{p}+E[R_2]P_\mathrm{c}}\\ &+\frac{\sum_{i=1}^\infty iV_i\,\{P_3P_4+P_5P_6\}}{\left\{\sum_{i=1}^\infty I_iP_1P_2\right\}+E[R_1]P_\mathrm{p}+E[R_2]P_\mathrm{c}}\\ &+\frac{C_\mathrm{p}E[R_1]\sum_{i=1}^\infty P_1P_2}{\left\{\sum_{i=1}^\infty I_iP_1P_2\right\}+E[R_1]P_\mathrm{p}+E[R_2]P_\mathrm{c}}\\ &+\frac{C_\mathrm{c}E[R_2]\left\{1-\sum_{i=1}^\infty P_1P_2\right\}}{\left\{\sum_{i=1}^\infty I_iP_1P_2\right\}+E[R_1]P_\mathrm{p}+E[R_2]P_\mathrm{c}}\,,\end{aligned} \quad (45.40)$$

where $I_{i-1}=(i-1)I$, $I_i=iI$, $I_{i+1}=(i+1)I$ and $V_i=P[Y(iI)\le L,\,D(iI)\le S]$, P_1: $P[Y(I_{i-1})\le L,\,D(I_{i-1})\le S]$, P_2: $P[L<Y(I_i)\le G,\,D(I_i)\le S]$, P_3: $P[Y(I_i)\le L,\,Y(I_{i+1})>G]$, P_4: $P[D(I_{i+1})\le S]$, P_5: $P[Y(I_{i+1})\le L]$, P_6: $P[D(I_i)\le S,\,D(I_{i+1})>S]$

This complex objective function is a nonlinear optimization problem and it is hard to obtain closed-form optimal solutions for L and I. *Nelder* and *Mead* [45.23] introduced a downhill simplex method that does not require the calculation of derivatives. A simplex is the most elementary geometrical scheme that can be formed in n dimensions and has $(n+1)$ vertices. A brief summary of the steps of the method is: each iteration generates a new vertex for the simplex. If the new point is better than at least one of the existing vertices, it then replaces the worst vertex. The search direction is generated through reflection, expansion and contraction operations.

A step-by-step algorithm proposed by *Li* and *Pham* [45.21] based on the Nelder–Mead downhill simplex method is summarized as follows:

- Step 1: choose (n+1) distinct vertices as an initial set $\left\{Z^{(1)},\cdots,Z^{(n+1)}\right\}$. Then calculate the function value $f(Z)$ for $i=1,2,\ldots,(n+1)$, where $f(Z)=EC(I,L)$. Put the values $f(Z)$ in an increasing order where $f(Z^{(1)})=\min[EC(I,L)]$ and $f\left(Z^{(n+1)}\right)=\max[EC(I,L)]$. Set $k=0$.
- Step 2: compute the best-n centroid $X^{(k)}=\frac{1}{n}\sum_{i=1}^n Z^{(i)}$.
- Step 3: use the centroid $X^{(k)}$ in Step 2 to compute the away-from-worst move direction
$$\Delta X^{(k+1)}=X^{(k)}-Z^{(n+1)}\,.$$
- Step 4: set $\lambda=1$ and compute $f(X^{(k)}+\lambda\Delta X^{(k+1)})$. If $f(X^{(k)}+\lambda\Delta X^{(k+1)})\le f(Z^{(1)})$ then go to Step 5. Otherwise, if $f(X^{(k)}+\lambda\Delta X^{(k+1)})\ge f(Z^{(n)})$ then go to Step 6. Otherwise, fix $\lambda=1$ and go to Step 8.
- Step 5: Set $\lambda=2$ and compute $f(X^{(k)}+2\Delta X^{(k+1)})$. If $f(X^{(k)}+2\Delta X^{(k+1)})\le f(X^{(k)}+\Delta X^{(k+1)})$ then set $\lambda=2$. Otherwise, set $\lambda=1$. Then go to Step 8.
- Step 6: If $f(X^{(k)}+\lambda\Delta X^{(k+1)})\le f(Z^{(n+1)})$ then set $\lambda=1/2$. Compute $f(X^{(k)}+\frac{1}{2}\Delta X^{(k+1)})$. If $f(X^{(k)}+\frac{1}{2}\Delta X^{(k+1)})\le f(Z^{(n+1)})$ then set $\lambda=1/2$ and go to Step 8. Otherwise, set $\lambda=-1/2$ and, if $f(X^{(k)}-\frac{1}{2}\Delta X^{(k+1)})\le f(Z^{(n+1)})$, then set $\lambda=-1/2$ and go to Step 8. Otherwise, go to Step 7.
- Step 7: shrink the current solution set toward the best $Z^{(1)}$ by $Z^{(i)}=\frac{1}{2}(Z^{(1)}+Z^{(i)})$, $i=2,\cdots,n+1$. Compute the new $f(Z^{(2)}),\cdots,f(Z^{(n+1)})$, let $k=k+1$, and return to Step 2.
- Step 8: Replace the worst $Z^{(n+1)}$ by $X^{(k)}+\lambda\Delta X^{(k+1)}$. If $\sqrt{\frac{1}{n+1}\sum_{i=1}^{n+1}[f(Z^{(i)})-\overline{f}\,]^2}<0.5$, where $\overline{f}$ is an average value, then STOP. Otherwise, let $k=k+1$ and return to Step 2. (It should be noted that the criterion in Step 8 is not unique but will depend on how soon you would like the algorithm to stop when the function values at the vertices are close. Here we do this when the difference be-

tween the maximum and the minimum values of f is less than 0.5.)

A Numerical Example

Here we present an example to illustrate the results and the step-by-step application procedure.

Assume that the degradation process is described by $Y(t) = A + Bg(t)$, where A and B are independent and follow a uniform distribution with parameter interval [0,4] and an exponential distribution with parameter 0.3, i. e., $A \sim U(0,4)$ and $B \sim \mathrm{Exp}(-0.3t)$, respectively, and $g(t) = \sqrt{t}\,\mathrm{e}^{0.005t}$.

Assume that the random shock damage is described by $D(t) = \sum_{i=1}^{N(t)} X_i$, where X_i follows an exponential distribution, i. e., $X_i \sim \mathrm{Exp}(-0.04t)$ and $N(t) \sim Poisson(0.1)$. Also $G = 50$, $S = 100$, $C_i = 900$/inspection, $C_c = 5600$/CM, $C_p = 3000$/PM,$R_1 \sim \mathrm{Exp}(-0.1t)$, and $R_2 \sim \mathrm{Exp}(-0.04t)$. We now determine both the values of I and L so that the average total cost per unit time $EC(I, L)$ is minimized. Following are step-by-step procedure [45.22]:

- Step 1: since there are two decision variables I and L, we need $(n+1) = 3$ initial distinct vertices, which are $Z^{(1)} = (25, 20)$, $Z^{(2)} = (20, 18)$, and $Z^{(3)} = (15, 10)$. Set $k = 0$. We calculate the value of $f\left(Z^{(.)}\right)$ corresponding to each vertex and sort them in increasing order of $EC(I, L)$.
- Step 2: calculate the centroid: $X^{(0)} = \left(Z^{(1)} + Z^{(1)}\right)/2 = (22.5, 19)$.
- Step 3: generate the search direction: $\Delta X = X^{(0)} - Z^{(2)} = (7.5, 9)$.
- Step 4: set $\lambda = 1$, which will produce a new minimal $EC(30, 28) = 501.76$ that leads us to try an expansion with $\lambda = 2$, that is (37.5, 38).
- Step 5: set $\lambda = 2$. Similarly, calculate $f(Z)$, which leads to $EC(37.5, 38) = 440.7$. Go to Step 8. This result turns out to be a better solution, hence (15, 10) is replaced by (37.5, 38).

The iteration continues and stops at $k = 6$ (Table 45.1) since $\sqrt{\frac{1}{3}\sum_{i=1}^{3}\left[EC(Z^{(i)}) - \overline{EC(I, L)}\right]^2} < 0.5$, where $\overline{EC(I, L)}$ is the average value.

Table 45.1 illustrates the process of the Nelder–Mead algorithm. In Table 45.1, $Z^{(.)} = (I, L)$. From Table 45.1, we observe that a set of the optimal values is

$$I^* = 37.5,\ L^* = 38$$

and the corresponding cost value is $EC^*(I, L) = 440.7$.

Table 45.2 illustrates the various values of L on P_c for given $I = 37.5$. From Table 45.2, we observe that the probability P_c increases as L increases. In other words, a larger value for L will put the system at high risk of failure.

Figure 45.14 shows the relationship between L and P_c for different I values, such as $I = 35$, $I = 37.5$, and $I = 40$. From Fig. 45.14, we observe that P_c is an increasing function of L. This means a higher preventive-maintenance threshold is more likely to result in a failure.

Figure 45.15 depicts the effect of the first inspection time on P_p for various L values such as $L = 33$, $L = 35$, $L = 37$ and $L = 39$. Shorter inspection times will cause more-frequent inspection and, as

Table 45.1 Optimal values I and L

k	$Z^{(1)}$	$Z^{(2)}$	$Z^{(3)}$	Search result
0	(25,20) $EC(I, L) = 564.3$	(20,18) $EC(I, L) = 631.1$	(15,10) $EC(I, L) = 773.6$	(37.5, 38) $EC(I, L) = 440.7$
1	(37.5,38) $EC(I, L) = 440.7$	(25,20) $EC(I, L) = 564.3$	(20,18) $EC(I, L) = 631.1$	(42.5,40) $EC(I, L) = 481.2$
2	(37.5,38) $EC(I, L) = 440.7$	(42.5,40) $EC(I, L) = 481.2$	(25,20) $EC(I, L) = 564.3$	(32.5,29) $EC(I, L) = 482.2$
3	(37.5,38) $EC(I, L) = 440.7$	(42.5,40) $EC(I, L) = 481.2$	(32.5,29) $EC(I, L) = 482.2$	(32.5,33.5) $EC(I, L) = 448.9$
4	(37.5,38) $EC(I, L) = 440.7$	(32.5,33.5) $EC(I, L) = 448.9$	(42.5,40) $EC(I, L) = 481.2$	(38.75,37.125) $EC(I, L) = 441.0$
5	(37.5,38) $EC(I, L) = 440.7$	(38.75,37.125) $EC(I, L) = 441.0$	(32.5,33.5) $EC(I, L) = 448.9$	(35.3125,35.25) $EC(I, L) = 441.1$
6	(37.5,38) $EC(I^*, L^*) = 440.7$	(38.75,37.125) $EC(I, L) = 441.0$	(35.3125,35.25) $EC(I, L) = 441.4$	Stop

Table 45.2 The effect of L on P_c for $I = 37.5$

L	P_c
33	0.465
35	0.505
37	0.654
39	0.759

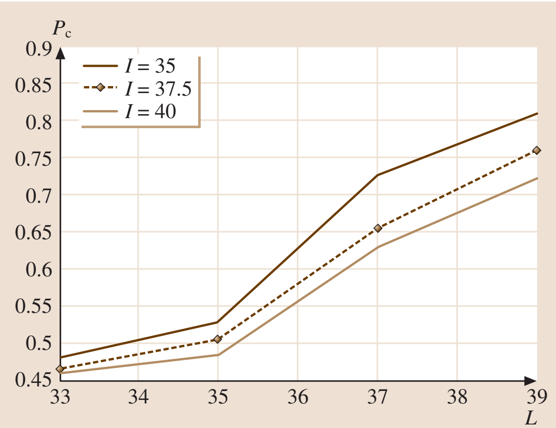

Fig. 45.14 P_c versus L

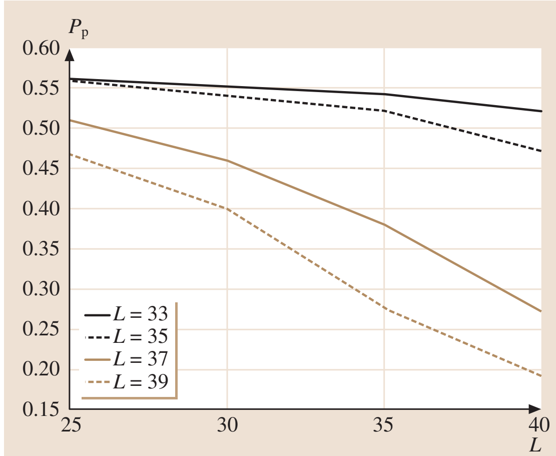

Fig. 45.15 The effect of the inspection sequence on P_p for given L

a result, will increase the probability of a PM. From Fig. 45.15, we also observe that, for smaller L values ($L = 33$ and $L = 35$), the curve decreases slightly as I increases; while, for larger values of L such as $L = 37$ and $L = 39$, the curve has a larger decrease as I increases. We also observe that the curve is more sensitive to the value of L, especially when L is large.

In summary, we observe that, on one hand, a lower value of L will result in frequent PM action and prevents full usage of the residual life of the systems. Frequent PM actions might reduce the chance of high deterioration and failures, but will also be costly. On the other hand, a higher L value will keep the system working in a higher-risk condition. Also, frequent inspections will reduce the probability of failure, while incurring additional cost.

45.3.2 Inspection–Maintenance Model for Degraded Systems with Three Competing Processes

General Inspection–Maintenance Description

This section considers systems with inspection-based maintenance subject to three failure processes that are competing for the life of such systems: two of these are degradation processes called degradation process i (measured by $Y_i(t)$ for $i = 1, 2$) and the third is a random shock process measured by the function $D(t)$ [45.21].

We assume that the three processes are independent and whichever process occurs first will cause the system to fail, where the failure of the system is defined as when $Y_1(t) > G_1$, $Y_2(t) > G_2$ or $D(t) > S$. The state of the system can only be revealed through inspection.

Assumptions.

1. System failure is only detected by inspection. Inspections are assumed to be instantaneous, perfect and nondestructive. Since the system is not continuously monitored, if the system fails it will remain failed until the next inspection, which causes a loss of C_m per unit time. In this case, a maintenance action is begun instantaneously at the inspection time.
2. After a maintenance action, either PM or CM, the system state will start as good as new.
3. A CM action will cost more than a PM action. Similarly, a PM action will cost much more than an inspection itself. This implies that $C_c > C_p > C_i$.
4. The three nondecreasing processes $Y_1(t)$, $Y_2(t)$, and $D(t)$ are independent.
5. No continuous monitoring is performed on the system.
6. The time for a CM or PM action is negligible.

We consider a system subject to three competing processes; two of them are continuous, gradual degradation processes with different characteristics, and the third is a random shock process. Applications of such systems can be found in the Space Shuttle computer complex due to critical mission phases such as boost, reentry and landing and in electric generator power systems due to the loss of commercial power systems. More related applications can be found in [45.13].

Although a continuous monitoring process is feasible for some systems, the cost of monitoring the process and the labor required would not make it realistic in practice. Therefore the criteria we consider here is to improve the system performance by performing periodic inspections, with a maintenance action if necessary, to minimize the total system maintenance cost.

Inspection–maintenance policy. The length of the inspection will be reduced as the system ages. In other words, the intervals between successive inspections become shorter as the system ages. A geometric sequence is applied in this study to develop the inter-inspection sequence. The inspection time is constructed as $I_n = \sum_{j=1}^{n} \alpha^{j-1} I_1$, where $0 < \alpha \leq 1$ and I_1 is the first inspection time. We define $U_n = I_n - I_{n-1} = \alpha^{n-1} I_1$ as the inter-inspection interval and $(U_i)_{i \in N}$ as a decreasing geometric sequence. According to the state detected at the inspection $I_n, n = 1, \cdots$, one of the following actions will happen [45.21]:

1. If both degradation values are below their PM thresholds and the shock damage value is less than its threshold, in other words $[Y_1(I_n) \leq L_1, Y_2(I_n) \leq L_2] \cap [D(I_n) \leq S]$, then the system is still in a good condition. In this case, we do nothing but determine the next inspection at $I_{n+1} = I_n + U_n$, where U_n is the inter-inspection time between the n-th and the $n+1$-th inspection interval.
2. If a degradation process falls into the PM zone $[L_i < Y_i(I_n) \leq G_i, i = 1, 2]$ and the other two processes are less than their corresponding critical thresholds, then the system calls for a PM action and it is instantaneously performed accordingly.
3. If any of the process values exceed their corresponding critical thresholds $[Y_i(t) > G_i, i = 1, 2$, or $D(t) > S]$, then the system calls for a CM action and it is instantaneously performed. In this case, the system has failed and a CM is performed on the system.

We assume that, after a maintenance action, i. e., PM or CM, the system will be restored to as good as new. A new sequence of inspection begins, defined in the same way, and the system maintenance follows the same decision rules outlined above. Figure 45.16 shows the evolution of the system, where $Y_1(t)$ and $Y_2(t)$ represent the degradation processes 1 and 2, respectively, and $D(t)$ represents a cumulative shock damage. $(W_i)_{i \in N}$ is a renewal sequence. Figure 45.17 shows the maintenance zone projected onto the $Y_1(t)$, $Y_2(t)$ planes; G_i and L_i are the CM and PM critical thresholds for $Y_1(t)$, and $Y_2(t)$ respectively.

Maintenance cost analysis

The expected total maintenance cost per cycle, $E[C_1]$, is given as:

$$E[C_1] = C_{\mathrm{i}} E[N_I] + C_{\mathrm{p}} P_{\mathrm{p}} + C_{\mathrm{c}} P_{\mathrm{c}} + C_{\mathrm{m}} E[\zeta] , \quad (45.41)$$

where C_{i} is the cost associated with each inspection, C_{p} is the cost associated with a PM action, and C_{c} is the CM action cost. Since failure is not self-announcing and it can occur at any given instant time T within the inspection time interval $[I_{\mathrm{i}}, I_{\mathrm{i}+1}]$, the system will remain idle during the interval $[T, I_{\mathrm{i}+1}]$. The cost coefficient C_{m}

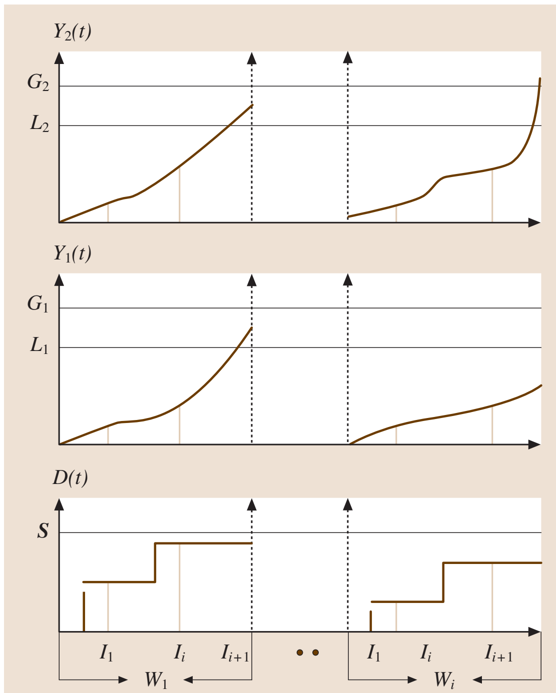

Fig. 45.16 The evolution of the system condition

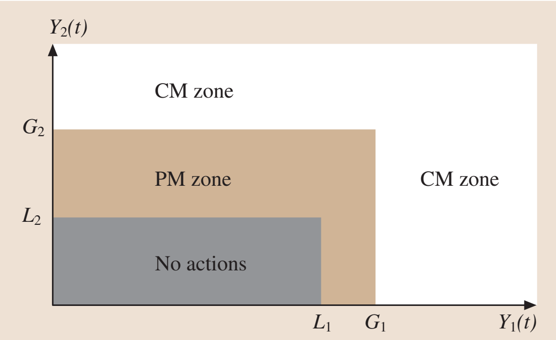

Fig. 45.17 Maintenance zone projected onto $Y_1(t)$, $Y_2(t)$

is defined as the penalty cost per unit time associated with such an event.

Calculation.

1. Let $P(N_I = i+1)$ be the probability that there are a total of $(i+1)$ inspections in the cycle. The expected number of inspections during a cycle, $E[N_I]$, is

$$E[N_I] = \sum_{i=0}^{\infty} (i+1) P_{i+1} \tag{45.42}$$

where $P_{i+1} = P(N_I = i+1)$. Note that

$$P_{i+1} = P(N_I = i+1) = \bigcup\nolimits_{j=1}^{17} P\left(E_j^{(i+1)}\right) ,$$

where $E_j^{(i+1)}(j = 1, \cdots, 17)$ denotes the renewal cycle that ends at the j-th possibile time I_{i+1}. The details of all $E_j^{(i+1)}$, where the $E_j^{(i+1)}$ are mutually disjoined events for $j = 1, \cdots, 18$ are listed in Appendix B.

There are a total of 18 system state combinations revealed at any given interval $(I_i, I_{i+1}]$ where there is only one state event, $E_{18}^{(i+1)}$ (Appendix B) representing that the system is in a good condition and that no maintenance action will be required. Any other remaining state events will trigger either a PM or a CM action at time I_{i+1}.

After some simplifications, we have

$$\begin{aligned} P_{i+1} = {} & P[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S] \\ & - P[Y_1(I_{i+1}) \le L_1, Y_2(I_{i+1}) \\ & \le L_2, D(I_{i+1}) \le S] \,. \end{aligned} \tag{45.43}$$

Therefore,

$$\begin{aligned} E[N_1] = {} & \sum_{i=0}^{\infty} (i+1) \{ P[Y_1(I_i) \le L_1, Y_2(I_i) \\ & \le L_2, D(I_i) \le S] \\ & - P[Y_1(I_{i+1}) \le L_1, Y_2(I_{i+1}) \\ & \le L_2, D(I_{i+1}) \le S] \} \,. \end{aligned}$$

2. There will be either a PM or CM action to end a renewal cycle. It is obvious that the two events (PM and CM) are mutually exclusive at the renewal time point: $P_\mathrm{p} + P_\mathrm{c} = 1$. We now calculate P_p as follows:

$$\begin{aligned} P_\mathrm{p} & = P(\text{the cycle ends due to a PM action}) \\ & = \sum_{i=0}^{\infty} \sum_{j=1}^{3} P\left[E_j^{(i+1)}\right] . \end{aligned}$$

After some simplifications, we obtain

$$\begin{aligned} P_\mathrm{p} = {} & \sum_{i=0}^{\infty} \{ P[Y_1(I_i) \le L_1, L_1 < Y_1(I_{i+1}) \le G_1] \\ & \times P[Y_2(I_i) \le L_2, Y_2(I_{i+1}) \le G_2] \\ & \times P[D(I_{i+1})] + P[Y_1(I_{i+1}) \le L_1] \\ & \times P[Y_2(I_i) \le L_2, L < Y_2(I_{i+1}) \le G_2] \\ & \times P[D(I_{i+1})] \} \end{aligned} \tag{45.44}$$

and $P_\mathrm{c} = 1 - P_\mathrm{p}$. We can obtain the joint probability density function $f_{Y(I_i),Y(I_{i+1})}(y_1, y_2)$ of $Y(I_i)$ and $Y(I_{i+1})$ by computing $P[Y_1(I_i) \le L_1, Y_1(I_{i+1}) \le G_1]$ and $P[Y_2(I_i) \le L_2, Y_2(I_{i+1}) \le G_2]$.

3. Let T denote the time to failure. That is $T = \inf[t : Y_1(t) > G_1, Y_2(t) > G_2 \text{ or } D(t) > S]$. If $I_i < T \le I_{i+1}$, the unit will be idle during the interval $[T, I_{i+1}]$. Let $E[\zeta]$ denote the average idle time between the failure occurrence epoch and its inspection during the cycle. Then $E[\xi]$ is calculated as follows:

$$\begin{aligned} E[\xi] & = \sum_{i=0}^{\infty} E\left[(I_{i+1} - T) 1_{I_i < T \le I_{i+1}}\right] \\ & = \sum_{j=0}^{\infty} R_j \int_{I_i}^{I_{i+1}} (I_{i+1} - t) \mathrm{d}F_T(t) \end{aligned} \tag{45.45}$$

where

$$
\begin{aligned}
R_j = \big\{ & P[Y_1(I_i) \le L_1, L_1 < Y_1(I_{i+1}) \le G_1] \\
& \times P[Y_2(I_i) \le L_2, L_1 < Y_1(I_{i+1})] \\
& + P[Y_2(I_i) \le L_2] \\
& \times P[Y_1(I_i) \le L_1, Y_1(I_{i+1}) > G_1] \\
& + P[Y_1(I_{i+1}) \le L_1] \\
& \times P[Y_2(I_i) \le L_2]\big\} P[D(I_i) \le S] \\
F(t) = {} & P[Y_1(t) > G_1, Y_2(t) \le G_2, D(I_i) \le S] \\
& + P[Y_1(t) \le G_1, Y_2(t) > G_2, D(I_i) \le S] \\
& + P[Y_1(t) \le G_1, Y_2(t) \le G_2, D(I_i) > S]
\end{aligned}
$$

and $1_{I_i < T \le I_{i+1}}$ is an indicator function.

Expected cycle length. The expected cycle length $E[W_1]$ is given as follows:

$$
\begin{aligned}
E[W_1] &= E[E[W_1|N_I]] \\
&= \sum_{i=0}^{\infty} E[W_1|N_I = i] P(N_i = i) \\
&= \sum_{i=0}^{\infty} I_{i+1} P_{i+1} \,,
\end{aligned}
\tag{45.46}
$$

where $P_{i+1}$is given in (45.43).

Therefore, the average long-run maintenance cost rate function $EC(L_1, L_2, I_1)$ is a function of the inspection times $\{I_1, \cdots, I_i, \cdots\}$ and the PM critical threshold values (L_1, L_2) through the functions P_p, P_c, $E[N_I]$, $E[\zeta]$ and $E[W_1]$. The average long-run maintenance cost rate is, in other words, $EC(L_1, L_2, I_1) = \frac{E[C_1]}{E[W_1]}$, and can be obtained by computing the two functions given in (45.41) and (45.46).

Optimization of the maintenance cost rate

The geometric inspection sequence $\{I_1, \cdots, I_i, \cdots\}$, where $I_n = \sum_{j=1}^{n} \alpha^{j-1} I_1$, depends on I_1 for given α. In this section, we develop a step-by-step algorithm based on the Nelder–Mead downhill simplex method to obtain the optimum decision variables (I_1, L_1, L_2) such that the long-run average maintenance cost rate $EC(L_1, L_2, I_1)$ is minimized. Mathematically, the optimization problem of the cost rate function can be formulated as follows [45.21]:

Optimization problem. Find I_1, L_1 and $L_2 (0 < L_1 \le G_1, 0 < L_2 \le G_2)$ such that

$$
\begin{aligned}
& EC(L_1, L_2, I_1) \\
&= \frac{C_i \sum_{i=0}^{\infty} (i+1) \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le S \right] - P\left[Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}}{\sum_{i=0}^{\infty} \left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le S \right] - P\left[Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}} \\
&+ \frac{C_p \sum_{i=0}^{\infty} \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le G_2 \right] P\left[Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le G_2 \right] P\left[D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}}{\sum_{i=0}^{\infty} \left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le S \right] - P\left[Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}} \\
&+ \frac{C_c \left(1 - \sum_{i=0}^{\infty} \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le G_2 \right] P\left[Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le G_2 \right] P\left[D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}\right)}{\sum_{i=0}^{\infty} \left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le S \right] - P\left[Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}} \\
&+ \frac{C_m \sum_{i=0}^{\infty} \left\{ (R_{1i} + R_{2i} + R_{3i}) P\left[D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\} \int_{\sum_{j=1}^{i} \alpha^{j-1} I_1}^{\sum_{j=1}^{i+1} \alpha^{j-1} I_1} \left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1 - t\right) \mathrm{d}F_T(t)}{\sum_{i=0}^{\infty} \left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \left\{ P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le S \right] - P\left[Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_1, Y_2\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le L_2, D\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le S \right] \right\}}
\end{aligned}
$$

is minimum, where

$$R_{1i} = P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, \right.$$
$$\left. L_1 < Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right) \le G_1\right]$$
$$\times P\left[Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2, \right.$$
$$\left. L_1 < Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right)\right],$$
$$R_{2i} = P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1, \right.$$
$$\left. G_1 < Y_1\left(\sum_{j=1}^{i+1} \alpha^{j-1} I_1\right)\right]$$
$$\times P\left[Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2\right],$$
$$R_{3i} = P\left[Y_1\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_1\right]$$
$$\times P\left[Y_2\left(\sum_{j=1}^{i} \alpha^{j-1} I_1\right) \le L_2\right].$$

This optimization function is a complex nonlinear function, the optimum solution of which is difficult to find. The Nelder–Mead downhill simplex method (discussed in Sect. 45.3.1) is the most popular direct-search method for obtaining the optimum solution of an unconstrained nonlinear function, and does not require the calculation of derivatives.

Numerical examples

This section illustrates the results in the Sect. 45.3.2. Assume that degradation process 1 is described as the function $Y_1(t) = \frac{W e^{B_1 t}}{A_1 + e^{B_1 t}}$, where the random variables A_1 and B_1 are independent and follow a uniform distribution with parameter interval [0,40], and exponential distribution with parameter 1, respectively. In short, $A_1 \sim U[0,40]$ and $B_1 \sim \mathrm{Exp}(1)$.

Similarly, assume that degradation process 2 is modeled as $Y_2(t) = A_2 + B_2 g(t)$ where $A_2 \sim U[0,2]$, $B_2 \sim \mathrm{Exp}(0.2)$ and $g(t) = \sqrt{t}\,e^{0.01t}$. Assume that the random shock is represented by the function $D(t) = \sum_{i=0}^{N_2(t)} X_i$, where $X_i \sim \mathrm{Exp}(0.04)$ and $N(t) \sim \mathit{Poisson}(0.1)$. Also $G_1 = 300$, $G_2 = 70$ and $S = 100$.

Assume that the cost parameters are as follows: $C_c = 560$ units/CM, $C_p = 400$ units/PM, $C_i = 100$ units/inspection, $C_m = 500$ units/unit time and $\alpha = 0.97$.

The inspection sequence $\{I_1, \cdots, I_n, \cdots\}$ is constructed with $I_n = \sum_{j=1}^{n} \alpha^{j-1} I_1$. We want to determine the values of I_1 and (L_1, L_2) so that the average long-run maintenance cost rate per unit time is minimized.

Following are step-by-steps using our proposed algorithm in Sect. 45.3.1:

- Step 1: there are three decision variables, say L_1, L_2, and I_1, so we need four distinct vertices as an initial set of values: $Z^{(1)} = (270, 56, 76)$, $Z^{(1)} = (280, 60, 72)$, $Z^{(2)} = (290, 52, 66)$ and $Z^{(3)} = (300, 50, 57)$. Set $k = 0$.

We now calculate the function value $f(Z)$ corresponding to each vertex and put them in increasing order of the objective value $EC(L_1, L_2, I_1)$ from smallest to highest.

- Step 2: compute the centroid: $X^{(0)} = \frac{1}{3}\left(Z^{(1)} + Z^{(2)} + Z^{(3)}\right) = (280, 56, 71.3)$.
- Step 3: search for the away-from-worst direction: $\Delta X = X^{(0)} - Z^{(4)} = (-20, 6, 14.3)$.
- Step 4: set $\lambda = 1$, which will generate a new minimal $EC(260, 60, 85.6) = 291.9$ that leads to an expansion with $\lambda = 2$ that is (240, 60, 99.9).
- Step 5: set $\lambda = 2$. Similarly, compute $f(Z)$, which leads to 247.9. Go to Step 8

This result turned out to be a better solution, hence (300, 50, 57) is replaced by (240, 60, 99.9).

The iteration continues and stops at $k = 4$ (see Table 45.3) since

$$\sqrt{\frac{1}{4}\sum_{i=1}^{4}\left[EC(Z^{(i)}) - \overline{EC(L_1, L_2, I_1)}\right]^2} = 0.449 < 0.5\,,$$

where $\overline{EC(L_1, L_2, I_1)}$ is the average value.

From Table 45.3, we obtain the optimal solution for (L_1, L_2, I_1) as: $\left(L_1^* = 172, L_2^* = 60, I_1^* = 144\right)$ and the corresponding average long-run maintenance cost rate is $EC\left(L_1^*, L_2^*, I_1^*\right) = 245.9$. Figure 45.18 depicts the average long-run maintenance cost-rate curve $EC(L_1, L_2, I_1)$ as a function of the inspection time interval I_1 for $L_1 = 172$ and $L_2 = 60$.

Table 45.4 presents a sensitivity analysis in terms of the probability that the cycle will end due to a PM action, P_p, for various values of (L_1, L_2) for $\alpha = 0.97$

Table 45.3 Nelder–Mead algorithm results

k	$Z^{(1)} = (L_1, L_2, I_1)$	$Z^{(2)}$	$Z^{(3)}$	$Z^{(4)}$	Search result
0	(270,56,76) $\frac{E[C_1]}{E[W_1]} = 300.7$	(280,60,72) $\frac{E[C_1]}{E[W_1]} = 332.2$	(290,52,66) $\frac{E[C_1]}{E[W_1]} = 360.4$	(300,50,57) $\frac{E[C_1]}{E[W_1]} = 388.2$	$\lambda = 2$ $\frac{E[C_1]}{E[W_1]} = 247.9$
1	(240,60,99.9) $\frac{E[C_1]}{E[W_1]} = 247.9$	(270,56,76) $\frac{E[C_1]}{E[W_1]} = 300.7$	(280,60,72) $\frac{E[C_1]}{E[W_1]} = 332.2$	(290,52,66) $\frac{E[C_1]}{E[W_1]} = 360.4$	$\lambda = 1$ $\frac{E[C_1]}{E[W_1]} = 248.0$
2	(236,60,99.2) $\frac{E[C_1]}{E[W_1]} = 247.9$	(240,60,99.9) $\frac{E[C_1]}{E[W_1]} = 248.0$	(270,56,76) $\frac{E[C_1]}{E[W_1]} = 300.7$	(280,60,72) $\frac{E[C_1]}{E[W_1]} = 332.2$	$\lambda = 2$ $\frac{E[C_1]}{E[W_1]} = 246.7$
3	(187,56,131) $\frac{E[C_1]}{E[W_1]} = 246.7$	(236,60,99.2) $\frac{E[C_1]}{E[W_1]} = 247.9$	(240,60,99.9) $\frac{E[C_1]}{E[W_1]} = 248.0$	(270,56,76) $\frac{E[C_1]}{E[W_1]} = 300.7$	$\lambda = 1$ $\frac{E[C_1]}{E[W_1]} = 245.9$
4	(172,60,144) $\frac{E[C_1]}{E[W_1]} = 245.9$	(187,56,131) $\frac{E[C_1]}{E[W_1]} = 246.7$	(236,60,99.2) $\frac{E[C_1]}{E[W_1]} = 247.9$	(240,60,99.9) $\frac{E[C_1]}{E[W_1]} = 248.0$	Stop

Table 45.4 The effect of (L_1, L_2) on P_p for a given inspection sequence

L_1	L_2	P_p
200	60	0.5910
190	58	0.5928
180	56	0.5936
170	54	0.5948
160	52	0.5950
150	50	0.5968

Table 45.5 The effect of the inspection sequence on P_p for fixed PM values

I_1	P_p
110	0.642
120	0.610
130	0.578
140	0.510
150	0.480
160	0.430

and $I_1 = 144$. From Table 45.4, we observe that the probability P_p slightly increases as both L_1 and L_2 decrease. This in fact shows that one would perform more PMs than CMs when L_1 and L_2 both become smaller.

Similarly, Table 45.5 presents the probability that the cycle will end due to a PM action, P_p, for various values of I_1 given $L_1 = 172$, $L_2 = 60$ and $\alpha = 0.97$. From Table 45.5, we observe that the probability P_p decreases as I_1 increases. In other words, the maintenance cycle will be more likely to end due to corrective rather than preventive maintenance if one delays in-

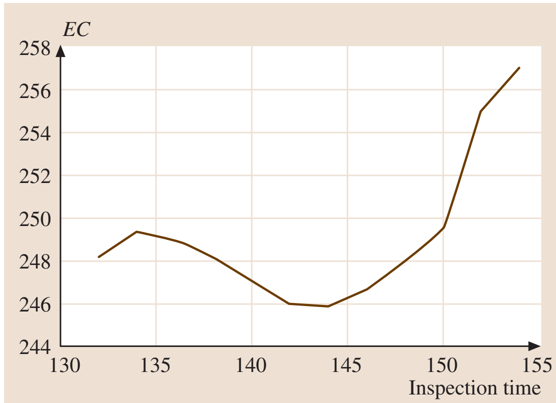

Fig. 45.18 The average maintenance cost $EC(L_1, L_2, I_1)$ versus I_1

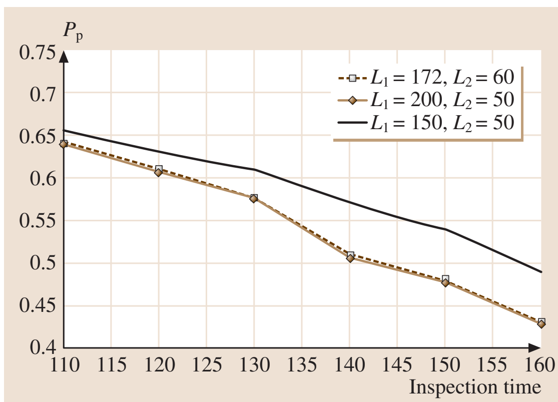

Fig. 45.19 The probability P_p versus I_1 for various pairs (L_1, L_2)

spection. This result can help maintenance managers or inspectors to allocate resources as well as time allocations.

Figure 45.19 shows the results for the probability that the cycle ends due to PM versus the inspection interval time I_1 for given values of the threshold PM levels (L_1, L_2). It is interesting to observe from Fig. 45.19 that the results are about the same for the two combinations of $(L_1 = 172,\ L_2 = 60)$ and $(L_1 = 200,\ L_2 = 50)$.

45.4 Conclusions and Perspectives

In this chapter, we present reliability and maintenance models for degraded systems with multiple competing failure processes. For mathematical modeling, it is always necessary to make some assumptions in order to make the model applicable in practice. For reliability and maintenance modeling, assumptions have often played an important role in determining the structure and complexity of the models.

The results of the maintenance models in this chapter can be used to help practitioners and inspectors as well as marketing managers to allocate resources and for the promotion strategies for new products. It would be of interest for future research to implement these results by collecting data and observing product system degradations in practice. Other research problems worth exploring in the future are as follows [45.22].

1. The objective function discussed in this chapter is to minimize the expected long-run maintenance cost. In practice, costs associated with inspections, preventive maintenance, corrective maintenance, and downtime are sometimes difficult to obtain, even when used in practice. For some critical systems, the overriding goal is to ensure that the system should be available when needed; availability is, therefore, of primary interest, and cost is secondary.
   To achieve a high level of availability for a specified inspection rate, it is worth to determine the optimum number of inspections with respect to imperfect repairs, such as minimal and opportunistic schemes, that maximizes the system availability. The time required for imperfect repairs and for replacement policies are random.
2. This chapter assumed that at any time there is unlimited supply of systems available for replacement. In reality, this assumption might not be true due to budget allocation and other constraints. In this case, a random lead time for delivering the new system when needed should be considered. It is essential and practical to analyze the effect of this random lead time on availability. When incorporating random lead time, the expected downtime will increase; therefore, system availability will decrease.

45.5 Appendix A

Jacobian determinant

Below is a 2×2 *Jacobian* determinant

$$J = \frac{y_1(y_2-W)\left(\frac{y_2}{y_1(y_2-W)} - \frac{y_2(y_1-W)}{y_1^2(y_2-W)}\right)\left[-d(y_1,y_2) - d_1(y_1,y_2) + d_2(y_1,y_2)\right]}{y_2(y_1-W)(I_{i+1}-I_i)} + d_3(y_1,y_2)\ \text{,where}$$

$$d(y_1,y_2) = \frac{\left[\left(\frac{y_1-W}{y_1(y_2-W)} - \frac{y_2(y_1-W)}{y_1(y_2-W)^2}\right) y_1(y_2-W)^2 I_{i+1}\,\mathrm{e}^{\left(\frac{\ln\left(\frac{y_2(y_1-W)}{y_1(y_2-W)}\right) I_{i+1}}{I_{i+1}-I_i}\right)}\right]}{y_2^2(y_1-W)(I_{i+1}-I_i)},\ y_1 \neq W,\ y_2 \neq W\ ,$$

$$d_1(y_1, y_2) = \frac{e^{\left(\frac{\ln\left(\frac{y_2(y_1-W)}{y_1(y_2-W)}\right) I_{i+1}}{I_{i+1}-I_i}\right)}}{y_2}, \quad y_2 \neq W,$$

$$d_2(y_1, y_2) = \frac{e^{\left(\frac{\ln\left(\frac{y_2(y_1-W)}{y_1(y_2-W)}\right) I_{i+1}}{I_{i+1}-I_i}\right)} (y_2 - W)}{y_2^2}, \quad y_2 \neq W\,,$$

$$d_3(y_1, y_2) = \frac{d_{31}(y_1, y_2) d_{32}(y_1, y_2)}{y_2^3 (y_1 - W)^2 (I_{i+1} - I_i)^2}, \quad y_1 \neq W\,,$$

$$d_{31}(y_1, y_2) = \left(\frac{y_1 - W}{y_1(y_2 - W)} - \frac{y_2(y_1 - W)}{y_1(y_2 - W)^2}\right) y_1^2 (y_2 - W)^3, \quad y_2 \neq W\,,$$

$$d_{32}(y_1, y_2) = \left(\frac{y_2}{y_1(y_2 - W)} - \frac{y_2(y_1 - W)}{y_1^2(y_2 - W)}\right) I_{i+1}\, e^{\left(\frac{\ln\left(\frac{y_2(y_1-W)}{y_1(y_2-W)}\right) I_{i+1}}{I_{i+1}-I_i}\right)}, \quad y_2 \neq W\,.$$

45.6 Appendix B

A list of all 18 events:

$$E_1^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) \le S\right]$$

$$E_2^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, Y_2(I_{i+1}) \le L_2, D(I_{i+1}) \le S\right]$$

$$E_3^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) \le L_1, L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) \le S\right]$$

$$E_4^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, Y_2(I_{i+1}) > G_2, D(I_{i+1}) \le S\right]$$

$$E_5^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) > S\right]$$

$$E_6^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, Y_2(I_{i+1}) > G_2, D(I_{i+1}) > S\right]$$

$$E_7^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, L_2 > Y_2(I_{i+1}), D(I_{i+1}) \le S\right]$$

$$E_8^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) \le S\right]$$

$$E_9^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, L_2 > Y_2(I_{i+1}), D(I_{i+1}) > S\right]$$

$$E_{10}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) > S\right]$$

$$E_{11}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, Y_2(I_{i+1}) > G_2, D(I_{i+1}) \le S\right]$$

$$E_{12}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[Y_1(I_{i+1}) > G_1, Y_2(I_{i+1}) > G_2, D(I_{i+1}) > S\right]$$

$$E_{13}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap \left[L_1 < Y_1(I_{i+1}) \le G_1, L_2 > Y_2(I_{i+1}), D(I_{i+1}) > S\right]$$

$$E_{14}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right]$$

$$\cap\left[L_1 > Y_1(I_{i+1}), L_2 > Y_2(I_{i+1}), D(I_{i+1}) \le S\right]$$

$$E_{15}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap\left[L_1 > Y_1(I_{i+1}), L_2 > Y_2(I_{i+1}), D(I_{i+1}) > S\right]$$

$$E_{16}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap\left[L_1 > Y_1(I_{i+1}), L_2 < Y_2(I_{i+1}) \le G_2, D(I_{i+1}) > S\right]$$

$$E_{17}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap\left[L_1 > Y_1(I_{i+1}), Y_2(I_{i+1}) > G_2, D(I_{i+1}) \le S\right]$$

$$E_{18}^{(i+1)} = \left[Y_1(I_i) \le L_1, Y_2(I_i) \le L_2, D(I_i) \le S\right] \cap\left[Y_1(I_{i+1}) \le L_1, Y_2(I_{i+1}) \le L_2, D(I_{i+1}) \le S\right]$$

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