46. Statistical Models on Maintenance

This chapter discusses a variety of approaches to performing maintenance. The first section describes the importance of preparing for maintenance correctly, by collecting data on unit lifetimes and estimating the reliability of the units statistically using quantities such as their mean lifetimes, failure rates and failure distributions.

Suppose that the time that the unit has been operational is known (or even just the calendar time since it was first used), and its failure distribution has been estimated statistically. The second section of the chapter shows that the time to failure is approximately given by the reciprocal of the failure rate, and the time before preventive maintenance is required is simply given by the pth percentile point of the failure distribution. Standard replacement policies, such as age replacement, in which a unit undergoes maintenance before it reaches a certain age, and periodic replacement, where the unit undergoes maintenance periodically, are also presented.

Suppose that the failure of a unit can only be recorded at discrete times (so the unit completes a specific number of cycles before failure). In the third section, the age replacement and periodic replacement models from the previous section are converted into discrete models. Three replacement policies in which the unit undergoes maintenance after a specific number of failures, episodes of preventive maintenance or repairs, are also presented. The optimum number of units for a parallel redundant system is derived for when each unit fails according to a failure distribution and fails upon some shock with a certain probability.

Suppose that the unit fails when the total amount of damage caused by shocks has exceeded a certain failure level. The fourth section describes the replacement policy in where the unit undergoes maintenance before failure for a cumulative damage model. The optimum damage level at which the unit should be replaced when it undergoes minimal repair upon failure is also derived analytically.

The last part introduces the repair limit policy, where the unit is replaced instead of being repaired if the repair time is estimated to be more than a certain time limit, as well as the inspection with human error policy, where units are checked periodically and failed units are only detected and replaced upon inspection. Finally, the maintenance of a phased array radar is analyzed as an example of the practical use of maintenance models. Two maintenance models are considered in this case, and policies that minimize the expected cost rates are obtained analytically and numerically.

Although system maintenance is important, performing it without understanding the operational parameters of the system first would probably do more harm than good. Therefore, the first step of maintenance is preparation: we have to collect data on the components used in the system, in order to be able to statistically estimate quantities such as mean lifetimes, failure rates, failure distributions, and so on. Secondly, designers, engineers and managers engaged in maintenance work need to be taught standard maintenance techniques from reliability theory, and how to apply them to real systems. After any necessary training has been performed, and data has been collecting about the system, we construct a maintenance plan, which depends upon the system environment and the resources (monetary and manpower) available. After much trial and error, we establish an maintenance scheme that is optimized for the system in question. This approach allows us to minimize or even eliminate any need for hasty maintenance after a severe system failure.

This chapter summarizes standard maintenance policies that can be applied (statistically and stochastically) to practical models. These policies are largely based upon the author's work [46.1]. If we monitor the age of a unit (in either the calendar time, the operating time or the usage time), it is best to perform maintenance at certain times. On the other hand, if we monitor the number of cycles or system uses, or the amount of total damage, wear or stress incurred by a unit, it is best to perform maintenance after a certain number of cycles or amount of wear, respectively. Moreover, it might be necessary to adopt combinations of these approaches.

In this chapter, we begin (in Sect. 46.1) by estimating the "mean time to failure" using the reciprocal of the failure rate. Then we introduce standard replacement policies, such as "age replacement" and "periodic replacement". Section 46.2 considers discrete versions of the general replacement models derived in Sect. 46.1 (in other words, where the maintenance can only be performed at certain times, which is a more realistic scenario). It also presents three models where replacement occurs after a certain number of events, including the number of system failures, the number of times it has undergone preventive maintenance, or the number of repairs that have been made to the system. Further, we discuss how many units should be allowed to fail in a parallel redundant system before the system is replaced. Section 46.3 introduces maintenance models based on cumulative damage; in other words, those where maintenance occurs after a certain amount of wear, stress, fatigue, corrosion, erosion and garbage. Finally, Sect. 46.4 presents another two useful maintenance models. In the first, known as the "repair limit policy", a failed unit is repaired unless the repair time is too long, in which case it is replaced. Second, we consider the "inspection model", with two types of human error. Finally, we give a practical example – the maintenance of a phased array radar – in which failed elements are either replaced at planned times or when they exceed a managerial number.

46.1 Time-Dependent Maintenance

If the failure of a unit during operation would cause serious damage to the whole system, it is important to know the total amount of time the unit has been in operation and to determine when to replace units or perform maintenance before the failure occurs. This is called *age replacement* or *preventive maintenance* (PM). The optimum age replacement policy (which minimizes the expected cost) and the optimum PM policy (which maximizes the availability are derived in *Barlow* and *Proschan* [46.2] and *Nakagawa* [46.1].

If only the unit's calendar operational time (the period of time since the unit was first used; its "age") is known and its failure is not a relatively minor or inexpensive event, it is necessary to perform PM or replace it periodically. This section describes maintenance policies in which a unit undergoes maintenance according to its total operating time (the total amount of time that the unit has been operational) or the age of the unit.

Suppose that X is a random variable that represents this age or total operating time. We then denote the failure distribution by $F(t) \equiv \Pr(X \leq t)$ for $0 \leq t < \infty$. Let μ and $h(t)$, respectively, be the finite mean time to failure of X and the failure rate, so $\mu \equiv \int_0^\infty \overline{F}(t)\,\mathrm{d}t$ and $h(t) \equiv f(t)/\overline{F}(t)$ where f is the density of F and $\overline{F} \equiv 1 - F$. Further, $H(t) \equiv \int_0^t h(u)\,\mathrm{d}u$ is called the cumulative hazard function, and is given by the relation

$$\overline{F}(t) = \exp\left[-\int_0^t h(u)\,\mathrm{d}u\right] = \mathrm{e}^{-H(t)}\,. \qquad (46.1)$$

Thus, $F(t)$, $\overline{F}(t)$, $f(t)$, $h(t)$ and $H(t)$ determine one another. Throughout this paper, we use these notations.

46.1.1 Failure Distribution

The most important statistical parameter is to find the mean time to failure (MTTF) of the unit. This is obtained comparatively easily by collecting data on the unit lifetime. Given the estimated failure distribution $F(t)$ of a unit, then the MTTF is given by $\mu \equiv \int_0^\infty \overline{F}(t)\,\mathrm{d}t$.

Next, since the probability that a unit with age T will fail in an interval $(T, t+T)$ is $[F(t+T)-F(T)]/\overline{F}(T)$, its MTTF (which is called the mean residual life) is

$$\frac{1}{\overline{F}(T)}\int_0^\infty \overline{F}(t+T)\,\mathrm{d}t = \frac{1}{\overline{F}(T)}\int_T^\infty \overline{F}(t)\,\mathrm{d}t \qquad (46.2)$$

which decreases from μ to $1/h(\infty)$ when $F(t)$ is an IFR property [46.2]; in other words when $h(t)$ is increasing. Also, in this case

$$\frac{1}{\overline{F}(T)}\int_T^\infty \overline{F}(t)\,\mathrm{d}t \le \frac{1}{h(T)} \,. \qquad (46.3)$$

Thus, $1/h(T)$ is used as an upper estimator for the MTTF of a unit with age T.

When failures occur in a nonhomogeneous Poisson process, the expected number of failures during $(0\,,\,T]$ is given by $H(T)$ [46.3]. Thus, if T_k corresponds to the time that the expected number of failures is k $(k=1\,,2\,,\ldots)$, so $H(T_k)=k$, we have

$$\int_{T_{k-1}}^{T_k} h(t)\,\mathrm{d}t = 1 \quad \text{or} \quad \int_0^{T_k} h(t)\,\mathrm{d}t = k$$
$$(k=1\,,2\,,\ldots)\,. \qquad (46.4)$$

In particular, when $F(t)$ is IFR,

$$T_k \le \frac{1}{h(T_{k-1})} + T_{k-1} \qquad (46.5)$$

In other words, the time where the expected number of failures from T_{k-1} is 1 is less than $1/h(T_{k-1})$. Note that the equations hold in (46.3) and (46.5) when F is exponential.

From the above discussions, it is possible to estimate that the time to the next failure, if it fails at time t, is about $1/h(t)$.

The simplest way to prevent failure is to make sure that the probability of failure is less than $p(0<p<1)$; in other words to compute a pth percentile point that satisfies $F(T_p)=p$. Then, a unit will undergo replacement at time T_p. Of course, we may consider this replacement to be PM.

Another simple method of replacement is to balance the cost of replacement after failure against that of replacement before failure. A cost c_1 is incurred for each failed unit and c_2 $(<c_1)$ is incurred for each operational unit. Then, we have $c_1 F(T) = c_2 \overline{F}(T)$, so $F(T)=c_2/(c_1+c_2)$. We may compute a $p[=c_2/(c_1+c_2)]$th percentile point for the failure distribution $F(t)$.

46.1.2 Age Replacement

Suppose that the operating record of a unit and its failure distribution $F(t)$ are known, and that its failure rate increases with operating time. In this case, if a unit is replaced by a new one, it is called age replacement. The optimum policy for minimizing the expected cost rate is discussed. If a unit is preventively maintained and becomes as good as new, this is called perfect PM [46.3]. The optimum policy for maximizing the availability is discussed [46.2].

Suppose that a unit is replaced at failure or at a planned time T $(0<T\le\infty)$, whichever occurs first. Then the expected cost rate is [46.2]

$$C(T) = \frac{c_1 F(T) + c_2 \overline{F}(T)}{\int_0^T \overline{F}(t)\,\mathrm{d}t} \,, \qquad (46.6)$$

where c_1 is the cost of replacement at failure and c_2 is the cost of replacement at planned time T, with $c_2<c_1$. If $T=\infty$, then the policy corresponds to replacement upon failure, and the expected cost rate is $C(\infty)=c_1/\mu$.

We find an optimum time T^* which minimizes $C(T)$ in (46.6), provided the failure rate $h(t)$ is strictly increasing, with $h(\infty)\equiv \lim_{t\to\infty} h(t)$. Evidently, since $\lim_{T\to 0} C(T)=\infty$, a positive T^* $(0<T^*\le\infty)$ must exist. Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$h(T)\int_0^T \overline{F}(t)\,\mathrm{d}t - F(T) = \frac{c_2}{c_1-c_2} \,. \qquad (46.7)$$

It is easily to see that the left-hand side of (46.7) strictly increases from 0 to $h(\infty)\mu-1$ because $h(t)$ strictly increases. Thus, if $h(\infty) > c_1/[\mu(c_1-c_2)]$, then there is a finite and unique T^* $(0<T^*<\infty)$ which satisfies (46.7), and the expected cost rate is

$$C(T^*) = (c_1-c_2)h(T^*)\,. \qquad (46.8)$$

On the other hand, if $h(\infty)\le c_1/[\mu(c_1-c_2)]$ then $T^*=\infty$; in other words, the unit should be replaced at failure.

Next, the unit is repaired at failure or undergoes PM before failure at planned time T $(0<T\le\infty)$, whichever

occurs first. Then, the steady-state availability is

$$A(T) = \frac{\int_0^T \overline{F}(t)\,\mathrm{d}t}{\int_0^T \overline{F}(t)\,\mathrm{d}t + \beta_1 F(T) + \beta_2 \overline{F}(T)}\,, \tag{46.9}$$

where β_1 is the mean time of repair for a failed unit and β_2 is the mean time of PM for an operational unit at time T with $\beta_1 > \beta_2$. Thus, the policy that maximizes the availability is the same one that minimizes the expected cost $C(T)$ in (46.6), except that c_i is replaced by β_i $(i = 1, 2)$.

46.1.3 Periodic Replacement

If a system consists of many kinds of components and only its age is only known, it would be wise to make planned maintenances at periodic times $kT (k = 1, 2, \ldots)$ $(0 < T \leq \infty)$. We consider three periodic replacements in which a failed unit is replaced, undergoes minimal repair or remains failed.

A new unit begins to operate at time $t = 0$, and a failed unit is discovered instantly and replaced by a new one. Further, a unit is replaced at the periodic time kT, whatever its age. Let $M(t)$ be a renewal function of $F(t)$, so $M(t) \equiv \sum_{j=1}^{\infty} F^{(j)}(t)$, where $F^{(j)}(t)$ $(j = 1, 2, \ldots)$ is the j-fold Stieltjes convolution of $F(t)$ with itself and $F^{(0)}(t) = 1$ for $t \geq 0$. Then, the expected cost rate is [46.2],

$$C_1(T) = \frac{c_1 M(T) + c_2}{T}\,, \tag{46.10}$$

where c_1 is the cost of replacement at each failure.

We seek an optimum time T^* which minimizes the expected cost rate $C_1(T)$ in (46.10). Differentiating $C_1(T)$ with respect to T and setting it equal to zero implies

$$Tm(T) - \int_0^T m(t)\,\mathrm{d}t = \frac{c_2}{c_1}\,, \tag{46.11}$$

where $m(t)$ is the renewal density of $M(t)$, so $m(t) \equiv \mathrm{d}M(t)/\mathrm{d}t$. This equation is a necessary condition for a finite T^*, and in this case, the expected cost rate is

$$C_1(T^*) = c_1 m(T^*)\,. \tag{46.12}$$

Next, a unit is always replaced at times kT, but it is not replaced at failure, and hence it remains a failure for the time interval from its failure to its replacement. Then, the expected cost rate is [46.1]

$$C_2(T) = \frac{c_1 \int_0^T F(t)\,\mathrm{d}t + c_2}{T}\,, \tag{46.13}$$

where c_1 is the cost for the time elapsed between the failure and the replacement of the unit per unit of time.

Differentiating $C_2(T)$ with respect to T and setting it equal to zero,

$$TF(T) - \int_0^T F(t)\,\mathrm{d}t = \frac{c_2}{c_1}\,. \tag{46.14}$$

Thus, if $\mu > c_2/c_1$ then an optimum and unique T^* exists which satisfies (46.14), and the expected cost rate is

$$C_2(T^*) = c_1 F(T^*)\,. \tag{46.15}$$

Finally, a unit undergoes only minimal repair at failure; in other words its failure rate remains undisturbed by minimal repair. Let $H(t)$ be a cumulative hazard function. Then, the expected cost rate is [46.2]

$$C_3(T) = \frac{c_1 H(T) + c_2}{T}\,, \tag{46.16}$$

where c_1 is the cost of minimal repair at failure. Differentiating $C_3(T)$ with respect to T and setting it equal to zero,

$$Th(T) - \int_0^T h(t)\,\mathrm{d}t = \frac{c_2}{c_1}\,. \tag{46.17}$$

When the failure rate $h(t)$ is strictly increasing, the left-hand side of (46.17) is also strictly increasing. Thus, if a solution T^* exists to (46.17), it is unique and the expected cost rate is

$$C_3(T^*) = c_1 h(T^*)\,. \tag{46.18}$$

46.2 Number-Dependent Maintenance

Most maintenance models are based on the continuous time failure distributions shown in Sect. 46.1. However, the time to failure of a unit might be discrete; in other words it can be measured by the number of cycles of some kind (such as the number of rotations) before failure. Units such as switching devices, railroad tracks, ball bearings and airplane tires fall into this category. We would often choose to do this if the unit is used very frequently. In another case, the exact instant of failure of a unit is not recorded; instead the day or even year in which it occurred is noted.

This section summarizes maintenance models where the maintenance depends on the number of cycles completed by a property of the unit [46.4,5]. First, we convert the standard replacement models from the previous section into discrete time models. Second, we consider the case where a unit is maintained preventively by monitoring the number of occurrences of failure, preventive maintenance or repair. Thirdly, we consider a parallel redundant system where the system is replaced preventively if a specified number of units have failed.

46.2.1 Replacement Policies

Often, an operating unit cannot be replaced at the optimum time for some reason, such as a shortage of spare units, a lack of money or workers, and the inconvenience of having the system out of operation when the unit is replaced. Indeed, some units can only be replaced at idle times: a weekend, month-end or year-end.

This section converts the standard age and periodic replacement models in Sect. 46.1.2 and Sect. 46.1.3 into discrete ones. Units are only replaced at times kT where T $(0 < T < \infty)$ has been specified previously. The other notations we use here are the same ones as those used in Sect. 46.1.

Age Replacement

Only the total operating time of the unit is measured. It is assumed that replacement can occur at times kT $(k = 1, 2, \dots)$: replacement is only allowed in certain time periods kT. A unit is replaced at time NT or at failure, whichever occurs first, and failures are detected immediately when they occur.

From (46.6), the expected cost rate is given by

$$C(N) = \frac{c_1 F(NT) + c_2 \overline{F}(NT)}{\int_0^{NT} \overline{F}(t)\,\mathrm{d}t} \quad (N = 1, 2, \dots)\,. \tag{46.19}$$

We want to find an optimum number N^* that minimizes $C(N)$ when the failure rate $h(t)$ is strictly increasing. Forming the inequality $C(N+1) - C(N) \geq 0$, we have

$$\frac{F[(N+1)T] - F(NT)}{\int_{NT}^{(N+1)T} \overline{F}(t)\,\mathrm{d}t} \int_0^{NT} \overline{F}(t)\,\mathrm{d}t - F(NT) \geq \frac{c_2}{c_1 - c_2} \quad (N = 1, 2, \dots)\,. \tag{46.20}$$

It is easy to see that the left-hand side of (46.20) strictly increases to $h(\infty)\mu - 1$ because $h(t)$ strictly increases. Thus, if $h(\infty) > c_1/[\mu(c_1 - c_2)]$ then a finite and unique minimum N^* $(1 \leq N^* < \infty)$ exists that satisfies (46.20).

On the other hand, if $h(\infty) \leq c_1/[\mu(c_1 - c_2)]$ then $N^* = \infty$; in other words the unit should be replaced at failure.

Table 46.1 shows T_p, T^* and N^* values for a given c_1/c_2 ratio, where $T = 1$, $F(T_p) = c_2/(c_1 + c_2)$, and $F(t) = 1 - \exp(-t/100)^2$. This indicates that T^* and N^* have similar values for a given c_1/c_2. T_p is less than T^* at small c_1/c_2, but T_p becomes a good approximation to T^* for large c_1/c_2. This shows that if the replacement cost at time N^*T is lower that that at time T^*, number-dependent maintenance is more useful that time-dependent maintenance.

Periodic Replacement

Let's assume that the unit is replaced at a time NT and upon each failure, and that failures are detected immediately. From (46.10), the expected cost rate is

$$C_1(N) = \frac{c_1 M(NT) + c_2}{NT} \quad (N = 1, 2, \dots)\,. \tag{46.21}$$

Forming the inequality $C_1(N+1) - C_1(N) \geq 0$ implies that

$$NM[(N+1)T] - (N+1)M(NT) \geq \frac{c_2}{c_1} \quad (N = 1, 2, \dots)\,.$$

Next, let's suppose that a unit is replaced at time NT, but this time, if it fails, the unit is not replaced until the next scheduled replacement time. Then, from (46.13), the expected cost rate is

$$C_2(N) = \frac{c_1 \int_0^{NT} F(t)\,\mathrm{d}t + c_2}{NT} \quad (N = 1, 2, \dots)\,. \tag{46.22}$$

Table 46.1 Optimum T^*, N^* for $T = 1$ and percentile T_p when $F(t) = 1 - \exp(-t/100)^2$

c_1/c_2	T^*	N^*	T_p
2	110	109	64
4	59	59	47
6	46	45	39
10	34	34	31
20	23	23	22
40	16	16	16
60	13	13	13
100	10	10	10

Forming the inequality $C_2(N+1) - C_2(N) \geq 0$ implies that

$$\int_0^{NT} \overline{F}(t)\,\mathrm{d}t - N \int_{NT}^{(N+1)T} \overline{F}(t)\,\mathrm{d}t \geq \frac{c_2}{c_1}$$
$$(N = 1, 2, \dots) . \tag{46.23}$$

Since $\overline{F}(t)$ decreases to 0, the left-hand side of (46.23) increases to μ. Thus, if $\mu > c_2/c_1$, a finite and unique minimum N^* $(1 \leq N^* < \infty)$ exists which satisfies (46.23).

Finally, a unit is replaced at time NT and undergoes only minimal repair at failures between replacements. Then, from (46.16), the expected cost rate is

$$C_3(N) = \frac{c_1 H(NT) + c_2}{NT} \qquad (N = 1, 2, \dots) . \tag{46.24}$$

Forming the inequality $C_3(N+1) - C_3(N) \geq 0$ implies that

$$NH[(N+1)T] - (N+1)H(NT) \geq \frac{c_2}{c_1}$$
$$(N = 1, 2, \dots) . \tag{46.25}$$

When the failure rate $h(t)$ strictly increases to ∞, the left-hand side of (46.25) also strictly increases to ∞. In this case, a finite and unique minimum N^* $(1 \leq N^* < \infty)$ exists which satisfies (46.25).

46.2.2 Number-Dependent Replacement

We now consider three replacement models where a unit is replaced after a certain number of events (failures, PMs, repairs).

Number of Failures

Consider the periodic replacement in which a unit is replaced at failure N $(N = 1, 2, \dots)$ after its installation, and undergoes only minimal repair upon failure between replacements [46.6, 7]. The notation used is the same as in Sect. 46.1.3. Then, the expected cost rate is

$$C(N) = \frac{(N-1)c_1 + c_2}{\sum_{j=0}^{N-1} \int_0^\infty p_j(t)\,\mathrm{d}t} \qquad (N = 1, 2, \dots) , \tag{46.26}$$

where $p_j(t) \equiv \left\{[H(t)]^j / j!\right\} \mathrm{e}^{-H(t)}$ $(j = 0, 1, 2, \dots)$ represents the probability that j failures occur in an interval $(0, t]$.

From the inequality $C(N+1) - C(N) \geq 0$, we have

$$\frac{\sum_{j=0}^{N-1} \int_0^\infty p_j(t)\,\mathrm{d}t}{\int_0^\infty p_N(t)\,\mathrm{d}t} - (N-1) \geq \frac{c_2}{c_1}$$
$$(N = 1, 2, \dots) . \tag{46.27}$$

When $h(t)$ strictly increases, the left-hand side of (46.27) also strictly increases, since $\int_0^\infty p_N(t)\,\mathrm{d}t$ decreases to $1/h(\infty)$ [46.7]. Thus, if $h(\infty) > c_2/(\mu c_1)$, a finite and unique minimum N^* exists which satisfies (46.27).

Number of PM Events

Assume that the unit undergoes PM at the planned times kT $(k = 1, 2, \dots)$ and its operational age becomes x units of time younger at each PM event, where both x and T $(0 \leq x \leq T)$ are constant and have been specified previously. Only minimal repair is performed when the unit fails between replacements. Suppose that the unit is replaced if it operates for the time interval NT. Then, the expected cost rate is (from [46.5])

$$C(N) = \frac{1}{NT}\left[c_1 \sum_{j=0}^{N-1} \int_{j(T-x)}^{T+j(T-x)} h(t)\,\mathrm{d}t + (N-1)c_2 + c_3\right]$$
$$(N = 1, 2, \dots) ,$$

where c_1 is the cost of the minimal repair, c_2 is the cost of PM, and c_3 is the cost of replacement, with $c_3 > c_2$.

From the inequality $C(N+1) - C(N) \geq 0$, we have

$$N \int_{N(T-x)}^{T+N(T-x)} h(t)\,\mathrm{d}t - \sum_{j=0}^{N-1} \int_{j(T-x)}^{T+j(T-x)} h(t)\,\mathrm{d}t \geq \frac{c_3 - c_2}{c_1}$$
$$(N = 1, 2, \dots) . \tag{46.28}$$

When $h(t)$ strictly increases, the left-hand side of (46.28) also strictly increases in N. Thus, if a finite N^* which satisfies (46.28) exists, it is unique and it minimizes $C(N)$.

Number of Repairs

Consider a single unit which is repaired upon failure and then returned to operation. It is assumed that the unit begins to operate at time 0 and that it has a failure distribution $F_1(t)$ with a finite mean μ_1, and after the $(j-1)$th repair $(j = 2, 3, \dots)$, it has a new distribution $F_j(t)$ with a mean μ_j, which is different and independent from the previous $F_{j-1}(t)$. The jth repair time has the distribution $G_j(t)$ with a mean β_j $(j = 1, 2, \dots)$.

A unit is replaced by a new one upon failure N after its installation; in other words, after the completion of the $(N-1)$th repair, the unit is not repaired – it is simply replaced with a new unit. Then, the expected cost rate is, from [46.5]:

$$C(N) = \frac{(N-1)c_1 + c_2}{\sum_{j=1}^{N} \mu_j + \sum_{j=1}^{N-1} \beta_j} \quad (N = 1, 2, \ldots), \tag{46.29}$$

Here c_1 is the cost of each repair and c_2 is the cost of the replacement.

From the inequality $C(N+1) - C(N) \geq 0$, we have

$$\frac{\sum_{j=1}^{N+1} \mu_j + \sum_{j=1}^{N} \beta_j}{\mu_N + \beta_N} - N \geq \frac{c_2}{c_1} \quad (N = 1, 2, \ldots). \tag{46.30}$$

If $\mu_{j+1} + \beta_j > \mu_{j+2} + \beta_{j+1}$ $(j = 0, 1, 2, \ldots)$ where $\beta_0 \equiv 0$ in other words the mean time of the cycle from one failure to the next decreases with the number of failures – then the optimum number N^* which satisfies (46.30) is unique.

In particular, suppose that $\mu_{j+1} \equiv a^{j-1}\mu$ and $\beta_j \equiv \beta$ $(j = 1, 2, \ldots; 0 < a < 1)$. Then, if $\mu/\beta > (1-a)(c_2/c_1)$, a finite and unique minimum N^* exists which minimizes $C(N)$.

46.2.3 Parallel System

Consider an n-unit parallel redundant system in which the system is replaced if all units have failed. First, we are interested in the number of units that is the most economical [46.8].

Suppose that each unit has an identical failure distribution $F(t)$ with a finite mean μ. Then an n-unit parallel system has a failure distribution of $[F(t)]^n$, and so the mean time to system failure is $\int_0^\infty [1 - F(t)^n]\,\mathrm{d}t$. Thus, the expected cost rate is

$$C(n) = \frac{nc_1 + c_2}{\int_0^\infty [1 - F(t)^n]\,\mathrm{d}t} \quad (n = 1, 2, \ldots), \tag{46.31}$$

where c_1 is the cost of one unit and c_2 is the cost of the replacement.

From the inequality $C(n+1) - C(n) \geq 0$, we have

$$\frac{\int_0^\infty [1 - F(t)^n]\,\mathrm{d}t}{\int_0^\infty [F(t)^n - F(t)^{n+1}]\,\mathrm{d}t} - n \geq \frac{c_2}{c_1} \quad (n = 1, 2, \ldots). \tag{46.32}$$

It is easy to see that the left-hand side of (46.32) strictly increases to ∞. A finite and unique minimum n^* exists which satisfies (46.32) and minimizes $C(n)$.

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, n^* is given by a finite and unique minimum such that

$$(n+1)\sum_{j=1}^{n} \frac{1}{j} - n \geq \frac{c_2}{c_1}.$$

Next, consider a parallel redundant system with n units in which units fail through shock at a mean interval of θ. It is assumed that the probability that an operating unit fails at shock j is p_j $(j = 1, 2, \ldots)$, depending on the number of shocks.

The system is replaced preventively before failure if the total number of failed units is $N+1$, $N+2$, ..., $n-1$, or it is replaced if all units have failed; otherwise it is left alone. Then, the expected cost rate is [46.8, 9]

$$C(N) = \frac{c_2 + (c_1 - c_2) \times \sum_{j=1}^{\infty} \sum_{r=0}^{N} \binom{n}{r} [p(j)]^{n-r} [P(j-1)]^r}{\theta \sum_{j=1}^{\infty} j \sum_{m=N+1}^{n} \binom{n}{m} [1 - P(j)]^{n-m} \times \sum_{r=0}^{N} \binom{m}{r} [p(j)]^{m-r} [P(j-1)]^r} \quad (N = 0, 1, 2, \ldots, n-1), \tag{46.33}$$

where c_1 is the cost of replacement for a failed system and c_2 is the cost of replacement for a system before failure with $c_2 < c_1$, $P(j) \equiv \sum_{i=1}^{j} p(i)$ $(j = 1, 2, \ldots)$ and $P(0) \equiv 0$.

If n and p_j are given, we can determine an optimum number N^* that minimizes the expected cost rate $C(N)$ in (46.33) by comparing $N = 0, 1, 2, \ldots, n-1$. If p_j is a geometric distribution, so $p_j = pq^{j-1}$ $(j = 1, 2, \ldots; q \equiv 1 - p, 0 < p < 1)$, then

$$C(N) = \frac{c_2 + (c_1 - c_2) \sum_{r=0}^{N} \binom{n}{r} (-1)^r p^{n-r} \times \sum_{i=0}^{r} (-1)^i \left[1/(1 - q^{n-i})\right]}{\theta \sum_{r=0}^{N} \binom{n}{r} (-1)^r \times \sum_{i=0}^{N-r} \binom{n-r}{i} \left[1/\left(1 - q^{n-i}\right)\right]} \quad (N = 0, 1, 2, \ldots, n-1). \tag{46.34}$$

In particular, when $n = 2$,

$$C(0) = \left[c_1 p^2 + c_2 \left(1 - p^2 - q^2\right)\right] / \theta, \tag{46.35}$$

$$C(1) = \frac{c_1}{\theta}\frac{1-q^2}{1+2q} \,. \tag{46.36}$$

Thus, if $c_2/c_1 < q/(1+2q)$, then the system is replaced when one unit fails, and if $c_2/c_1 \geq q/(1+2q)$ then it is replaced when two units have failed. Since $q/(1+2q) < 1/3$, if $c_1 \leq 3c_2$ then the system is replaced when two units have failed.

46.3 Amount-Dependent Maintenance

Some units are maintained preventively by monitoring their amount of wear, stress, fatigue, corrosion, erosion and garbage. A typical model in this case is the "cumulative damage model" or the "shock model", in which a unit fails when the cumulative amount of damage from shocks has exceeded a particular failure level [46.10].

This section discusses the replacement policy for a cumulative damage model where a unit is replaced before failure at damage Z, or upon failure, whichever occurs first. Then we consider the replacement policy for the scenario where a unit undergoes minimal repair upon failure. Optimum damage levels Z^* which minimize the expected cost rates are analytically derived for both replacement policies. The methods and results in this section can be applied to actual units by cmonitoring their deterioration, and using this to decide which form of maintenance is appropriate. If the amount of total damage is proportional to the total operating time and the number of uses, these correspond to the time- and number-dependent maintenance models, respectively.

46.3.1 Replacement Policies

Consider a unit that should operate for an infinite time span. It is assumed that shocks occur at time intervals of X_i and that each shock causes an amount of damage W_i to the unit. The total damage to the unit is additive. A unit fails when the total damage has exceeded a failure level K. Unit failure shoudl be avoided during actual operation if it is costly or dangerous. In such situations, it would be wise to replacement the unit or perform preventive maintenance before failure, since it would be less expensive.

It would be better to adopt the damage level as the trigger for replacement if we know the total damage at any time. We replace a unit before failure when the total damage has exceeded a threshold level Z $(0 \leq Z \leq K)$. That is, we can investigate the total damage immediately after each shock occurs. If the total damage exceeds Z and is less than K, we replace the unit before it fails. If the total damage has exceeded K, it has failed and is replaced; otherwise we leave it alone.

We denote that $F(t) \equiv \Pr(X_i \leq t)$ and $G(x) \equiv \Pr(W_i \leq x)$ $(i = 1, 2, \ldots)$, where $\theta \equiv E(X_i) < \infty$ and $\beta \equiv E(W_i) < \infty$. Therefore, the expected cost rate is (from [46.11]):

$$C(Z) = \frac{c_2}{\theta[1+M(Z)]} + \frac{(c_1-c_2)}{\theta[1+M(Z)]} \times \left\{1 - G(K) + \int_0^Z [1-G(K-x)]\,\mathrm{d}M(x)\right\} , \tag{46.37}$$

where c_1 is the cost of replacement at failure level K, c_2 is the cost of replacement at threshold level Z with $c_2 < c_1$, and $M(x) \equiv \sum_{j=1}^{\infty} G^{(j)}(x)$ represents the number of shocks expected before the total damage exceeds x. It is evident that

$$C(0) = \{c_1[1-G(K)] + c_2 G(K)\}/\theta \,,$$
$$C(\infty) = \frac{c_1}{\theta[1+M(K)]} \,.$$

We find an optimum threshold level Z^* which minimizes the expected cost rate $C(Z)$. Differentiating $C(Z)$ with respect to Z and setting it equal to zero,

$$\int_{K-Z}^{K} [1+M(K-x)]\,\mathrm{d}G(x) = \frac{c_2}{c_1-c_2} \,. \tag{46.38}$$

The left-hand side of (46.38) strictly increases from 0 to $M(K)$. Thus, if $M(K) > c_2/(c_1-c_2)$ then a finite and unique Z^* $(0 < Z^* < K)$ exists which satisfies (46.38), and the expected cost rate is

$$C(Z^*) = (c_1-c_2)[1-G(K-Z^*)]/\theta \,. \tag{46.39}$$

On the other hand, if $M(K) \leq c_2/(c_1-c_2)$ then $Z^* = K$, so the unit should be replaced after failure.

Also, if the unit is replaced upon failure, at damage Z, at time T or upon shock number N (whichever occurs

Part E | 46.3

first), the expected cost rate is

$$C(Z,T,N)=\left(c_2+(c_1-c_2)\sum_{j=0}^{N-1}F^{(j+1)}(T)\right.$$
$$\times\int_0^Z[1-G(K-x)]\,\mathrm{d}G^{(j)}(x)$$
$$+(c_3-c_2)\sum_{j=0}^{N-1}\left[F^{(j)}(T)-F^{(j+1)}(T)\right]G^{(j)}(Z)$$
$$\left.+(c_4-c_2)F^{(N)}(T)G^{(N)}(Z)\right)$$
$$\times\left(\sum_{j=0}^{N-1}G^{(j)}(Z)\int_0^T\left[F^{(j)}(t)-F^{(j+1)}(t)\right]\mathrm{d}t\right)^{-1},$$

where c_3 is the cost of replacement at time T and c_4 is the cost of replacement at shock N.

46.3.2 Replacement with Minimal Repair

Suppose that a unit fails with probability $p(z)$ upon each shock (z is the total damage), where $p(0)\equiv 0$ and $p(\infty)=1$ [46.12], and that the unit undergoes only minimal repair upon failure. Further, a unit is replaced upon damage Z, at time T or upon shock number N (whichever occurs first). Therefore, the expected cost rate is (from [46.13]):

$$C(Z,T,N)=\frac{\begin{array}{l}c_1\sum_{j=1}^{N-1}F^{(j)}(T)\int_0^Z p(z)\,\mathrm{d}G^{(j)}(z)\\ +c_2\sum_{j=1}^{N}F^{(j)}(T)\left[G^{(j-1)}(Z)-G^{(j)}(Z)\right]\\ +c_3\sum_{j=0}^{N-1}\left[F^{(j)}(T)-F^{(j+1)}(T)\right]G^{(j)}(Z)\\ +c_4F^{(N)}(T)G^{(N)}(Z)\end{array}}{\sum_{j=0}^{N-1}G^{(j)}(Z)\int_0^T\left[F^{(j)}(t)-F^{(j+1)}(t)\right]\mathrm{d}t},\tag{46.40}$$

where c_1 is the cost of minimal repair upon failure, c_2 is the cost of replacement upon damage Z, c_3 is the cost of replacement at time T, and c_4 is the cost of replacement upon shock N.

The expected cost rate when a unit is only replaced at damage Z is

$$C(Z)\equiv\lim_{\substack{T\to\infty\\N\to\infty}}C(T,N,Z)=\frac{c_1\int_0^Z p(x)\,\mathrm{d}M(x)+c_2}{\theta[1+M(Z)]}.\tag{46.41}$$

Differentiating $C(Z)$ with respect to Z and setting it equal to zero, we have

$$\int_0^Z[1+M(x)]\,\mathrm{d}p(x)=\frac{c_2}{c_1}\tag{46.42}$$

which strictly increases in Z when $p(x)$ strictly increases. Thus, if a solution satisfying (46.42) exists then it is unique.

In particular, when $p(x)=1-\mathrm{e}^{-sx}$ for $s>0$,

$$\int_0^\infty[1+M(x)]\,\mathrm{d}p(x)=\frac{1}{1-G^*(s)},$$

where $G^*(s)$ is the Laplace–Stieltjes transform of $G(x)$. Therefore, if $1/[1-G^*(s)]>c_2/c_1$, then a finite and unique Z^* exists which satisfies (46.42), and the expected cost rate is $c_1p(Z^*)$. On the other hand, if $1/[1-G^*(s)]\le c_2/c_1$ then $Z^*=\infty$ and $C(\infty)=c_1/\theta$.

Similarly,

$$C(T)\equiv\lim_{\substack{N\to\infty\\Z\to\infty}}C(T,N,Z)=\frac{c_1\sum_{j=1}^{\infty}F^{(j)}(T)\int_0^\infty p(z)\,\mathrm{d}G^{(j)}(z)+c_3}{T},\tag{46.43}$$

$$C(N)\equiv\lim_{\substack{T\to\infty\\Z\to\infty}}C(T,N,Z)=\frac{c_1\sum_{j=1}^{N-1}\int_0^\infty p(z)\,\mathrm{d}G^{(j)}(z)+c_4}{\theta N}.\tag{46.44}$$

We can discuss the optimum T^* and N^*, which minimize the expected cost rates $C(T)$ and $C(N)$, respectively.

46.4 Other Maintenance Models

We now introduce two maintenance models that are interesting statistically. One is the *repair limit policy* and the other is *inspection with human errors*. Further, we give an example of the practical application of a maintenance policy, for a phased array radar.

46.4.1 Repair Limit Policy

In the previous sections, we have dealt with replacement and PM policies in which a unit undergoes maintenances at time T or upon failure (whichever occurs first). One alternative, considered here, is to repair a failed unit if the repair time is short but to replace it if the repair time is long. That is, if the estimated repair time of a failed unit is greater than a specified time T, which is called the *repair limit time*, then it is replaced.

It is assumed that the repair time has a general distribution $G(t)$ with a finite mean β. Let c_1 be the replacement cost of the failed unit and $c_r(t)$ be the expected repair cost during $(0, t]$ when the failed unit undergoes repair. Suppose that when the unit fails, its repair time is estimated. If the repair time is estimated to be less than T, it is repaired; otherwise it is replaced. The expected cost rate is, from [46.14],

$$C(T) = \frac{c_1\overline{G}(T) + \int_0^T c_r(t)\,\mathrm{d}G(t)}{\mu + \int_0^T t\,\mathrm{d}G(t)} \,. \tag{46.45}$$

Evidently

$$C(0) = c_1/\mu \,,$$
$$C(\infty) = \frac{\int_0^\infty c_r(t)\,dG(t)}{\mu + \beta} \,.$$

In particular, when $c_r(t) = ct$, the expected cost rate is

$$C(T) = \frac{c_1\overline{G}(T) + c\int_0^T t\,\mathrm{d}G(t)}{\mu + \int_0^T t\,\mathrm{d}G(t)} \,. \tag{46.46}$$

Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$\frac{\mu c}{c_1} = \frac{\mu + \int_0^T \overline{G}(t)\,\mathrm{d}t}{T} \tag{46.47}$$

whose right-hand side strictly decreases from ∞ to 0. Thus, a finite and unique optimum repair limit time T^* exists which satisfies (46.47).

46.4.2 Inspection with Human Errors

Suppose that an operating unit is checked at times kT $(k = 1, 2, \ldots)$ for $0 < T < \infty$, and that failed units are detected only through inspection and are then replaced. Now, two types of human error can occur when the standby unit is checked at periodic times kT $(k = 1, 2, \ldots)$ [46.15]:

1. *Type I human error*: The operational unit is judged to have failed.
2. *Type II human error*: The failed unit is judged to be operational.

It is assumed that the probabilities of type I and type II errors occurring are, respectively, p_1 and p_2, where $0 \le p_1 + p_2 < 1$. In this case, the number of inspections needed to detect and replace a failed unit is

$$\sum_{j=0}^{\infty} j\,{p_2}^{j-1}(1 - p_2) = \frac{1}{1 - p_2} \,.$$

Consider one cycle from time $t = 0$ to the time when a failed unit is detected by perfect inspection or a good unit is replaced in a type I error, whichever occurs first. Let c_1 be the cost of each inspection and c_2 be the cost of the lost operational time elapsed between a failure and its detection per unit of time. Then, the total expected cost of one cycle is

$$\begin{aligned}
C(T) = {} & \sum_{j=0}^{\infty}(1-p_1)^j \Bigg\{ \int_{jT}^{(j+1)T} \Bigg[c_1\left(j + \frac{1}{1-p_2}\right) \\
& + c_2\left(jT + \frac{T}{1-p_2} - t\right)\Bigg]\,\mathrm{d}F(t) \\
& + p_1 c_1 (j+1)\overline{F}((j+1)T)\Bigg\} \\
= {} & (c_1 + c_2 T)\Bigg\{\frac{1}{1-p_2}\sum_{j=0}^{\infty}(1-p_1)^j \\
& [\overline{F}(jT) - \overline{F}((j+1)T)] \\
& + \sum_{j=0}^{\infty}(1-p_1)^j\overline{F}[(j+1)T]\Bigg\} \\
& - c_2\sum_{j=0}^{\infty}(1-p_1)^j\int_{jT}^{(j+1)T}\overline{F}(t)\,\mathrm{d}t \,.
\end{aligned} \tag{46.48}$$

When $p_1 = p_2 = 0$ (the inspection is perfect), the expected cost is

$$C(T) = (c_1 + c_2 T)\sum_{j=0}^{\infty}\overline{F}(jT) - c_2\mu \tag{46.49}$$

which agrees with that of the standard inspection policy [46.1].

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, the expected cost can be rewritten as

$$\begin{aligned}
C(T) = {} & (c_1 + c_2 T)\frac{\left(1 - \mathrm{e}^{-\lambda T}\right)/(1-p_2) + \mathrm{e}^{-\lambda T}}{1 - (1-p_1)\mathrm{e}^{-\lambda T}} \\
& - \frac{c_2}{\lambda}\,\frac{1 - \mathrm{e}^{-\lambda T}}{1 - (1-p_1)\mathrm{e}^{-\lambda T}} \,.
\end{aligned} \tag{46.50}$$

Differentiating $C(T)$ with respect to T and setting it equal to zero,

$$\frac{e^{\lambda T}-1}{\lambda}[1-p_2(1-p_1)e^{-\lambda T}]-(1-p_1-p_2)T = \frac{c_1}{c_2}(1-p_1-p_2)\,. \tag{46.51}$$

It is evident that the left-hand side of (46.51) strictly increases from 0 to ∞. Therefore, a finite and unique T^* exists which satisfies (46.51).

46.4.3 Phased Array Radar

Finally, we consider an example scenario of the maintenance of a phased array radar (PAR) [46.16]. A PAR consists of a large number of small and homogeneous elements, and it steer the electromagnetic wave used for detection by shifting the signal phases of waves that are radiated from these individual elements [46.17].

Keithely [46.18] showed that the maintenance model applied to a PAR with 1024 elements had a strong influence on its availability. *Heresh* [46.19] discussed the following three maintenance models for a PAR in which all failed elements were detected immediately, calculated the average time to failures of the equipment, and derived its availability:

1. *Immediate maintenance*: Failed elements are detected, localized and replaced immediately.
2. *Cyclic maintenance*: Failed elements are detected, localized and replaced periodically.
3. *Delayed maintenance*: Failed elements are detected and localized periodically, and replaced when they have exceeded a prespecified managerial number.

In real world scenarios, immediate maintenance is rarely adopted because frequent maintenance degrades the availability of the system. Cyclic or delayed maintenance are the most common approaches.

In this section, we investigate the periodic detection of failed elements of the PAR. The PAR consists of N_0 elements, and failures are detected at periodic times kT $(k=1\,,2\,,\dots)$ for a given T $(0<T<\infty)$. If the number of failed elements has exceeded a failure number n $(0<n\le N_0)$, the PAR cannot maintain the required level of radar performance, resulting in operational errors such as target oversight

Cyclic Maintenance

We consider the following cyclic maintenance of the PAR:

1. The PAR consists of N_0 elements which have an identical constant failure rate λ_0. The number of failed elements during $(0\,,t]$ has a binomial distribution with a mean $N_0[1-\exp(-\lambda_0 t)]$. Since N_0 is large and λ_0 is very small, it can be assumed that failures occur approximately according to a Poisson process with a mean $\lambda\equiv N_0\lambda_0$. That is, the probability that j failures occur during $(0\,,t]$ is
$$p_j(t)\equiv\frac{(\lambda t)^j}{j!}e^{-\lambda t}\qquad (j=0\,,1\,,2\,,\dots)\,.$$
2. When the number of failed elements has exceeded a failure number n, the PAR cannot maintain the required level of radar performance.
3. Failed elements are checked at periodic times kT $(k=1,2,\dots)$, and the checking time is negligible.
4. Failed elements are replaced by new ones at time KT $(K=1,2,\dots)$ or at the time when they exceed n, whichever occurs first.

We now introduce some costs. Cost c_1 is the replacement cost of one failed element; c_2 is the cost of the operational loss during replacement, and c_3 is the cost of the degradation of radar performance per unit time. Then, the expected cost to replace the failed element is [46.16]

$$\begin{aligned}
&\sum_{j=0}^{n-1}(jc_1+c_2)p_j(KT)+\sum_{k=1}^{K}\sum_{j=0}^{n-1}p_j[(k-1)T]\\
&\quad\times\sum_{i=n-j}^{\infty}\left\{[(i+j)c_1+c_2]p_i(T)\right.\\
&\qquad\left.+c_3\int_{(k-1)T}^{kT}(kT-t)\,\mathrm{d}p_i[t-(k-1)T]\right\}\\
&=c_2+c_1\lambda T\sum_{k=0}^{K-1}\sum_{j=0}^{n-1}p_j(kT)+\frac{c_3}{\lambda}\sum_{k=0}^{K-1}\sum_{j=0}^{n-1}p_j(kT)\\
&\quad\times\sum_{i=n-j}^{\infty}(i+j-n)p_i(T)
\end{aligned}$$

and the mean time to replace the element is

$$\begin{aligned}
&\sum_{j=0}^{n-1}(KT)p_j(KT)+\sum_{k=1}^{K}\sum_{j=0}^{n-1}p_j[(k-1)T]\\
&\quad\times\sum_{i=n-j}^{\infty}(kT)p_k(T)\\
&=T\sum_{k=0}^{K-1}\sum_{j=0}^{n-1}p_j(kT)\,.
\end{aligned}$$

Thus, the expected cost is

$$C_1(K) = c_1\lambda + \frac{c_2}{T\sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT)} + \frac{(c_3/\lambda)}{T\sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT)} \times \sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT) \sum_{i=n-j}^{\infty}(i+j-n)p_i(T)\,, \quad (K=1,2,\ldots)\,. \tag{46.52}$$

We seek an optimum number K^* which minimizes (46.52). From the inequality $C_1(K+1)-C_1(K)\geq 0$, we have

$$\sum_{k=0}^{K-1}\sum_{m=0}^{n-1} p_m(kT)\sum_{j=0}^{n-1} p_j(KT)\sum_{i=n-j}^{\infty}(i+j-n)p_i(T) - \sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT)\sum_{i=n-j}^{\infty}(i+j-n)p_i(T) \geq \frac{\lambda c_2}{c_3}\,, \quad (K=1,2,\ldots)\,. \tag{46.53}$$

Denoting the left-hand side of (46.53) by $Q_1(K)$, it is clear that $Q_1(K)$ increases to $Q_1(\infty)$ in K. Thus, if $Q_1(\infty) > \lambda c_2/c_3$ then a finite and unique minimum K^* exists which satisfies (46.53).

Delayed Maintenance

We now consider a delayed maintenance model for the PAR, which is similar to the model before, but:

4. Failed elements are replaced by new ones only when they have exceeded a managerial number N ($N \leq n$).

Other assumptions are the same as the ones used for cyclic maintenance.

The expected cost to replace the element is [46.16]

$$\sum_{k=1}^{\infty}\sum_{j=0}^{N-1} p_j[(k-1)T]\sum_{i=N-j}^{n-j+1}[(i+j)c_1+c_2]p_i(T) + \sum_{k=1}^{\infty}\sum_{j=0}^{N-1} p_j[(k-1)T]\sum_{i=n-j}^{\infty}\left\{[(i+j)c_1+c_2] \times p_i(T) + c_3\int_{(k-1)T}^{kT}(kT-t)\,\mathrm{d}p_i[t-(k-1)T]\right\}$$

$$= c_2 + c_1\lambda T\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT) + \frac{c_3}{\lambda}\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT) \times \sum_{i=n-j}^{\infty}(i+j-n)p_i(T)$$

and the mean time to replacement is

$$\sum_{k=1}^{\infty}\sum_{j=0}^{N-1} p_j[(k-1)T]\sum_{i=N-j}^{n-j+1}(kT)p_i(T) + \sum_{k=1}^{\infty}\sum_{j=0}^{N-1} p_j[(k-1)T]\sum_{i=n-j}^{\infty}(kT)p_i(T) = T\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)\,.$$

Thus, the expected cost rate is

$$C_2(N) = c_1\lambda + \frac{c_2}{T\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)} + \frac{(c_3/\lambda)}{T\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)} \times \sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)\sum_{i=n-j}^{\infty}(i+j-n)p_i(T)\,, \quad (N=1,2,\ldots,n)\,. \tag{46.54}$$

We seek an optimum number N^* which minimizes (46.54). From the inequality $C_2(N+1)-C_2(N)\geq 0$, we have

$$\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)\sum_{i=1}^{\infty} i[p_{n+i-N}(T)-p_{n+i-j}(T)] \geq \frac{\lambda c_2}{c_3} \quad (N=1,2,\ldots,n)\,. \tag{46.55}$$

Denoting the left-hand side of (46.55) by $Q_2(N)$, it is clear that $Q_2(N)$ increases to $Q_2(\infty)$ in N. Therefore, if $Q_2(\infty) > \lambda c_2/c_3$ then a finite and unique minimum N^* exists which satisfies (46.55).

We now show a numerical example for when $c_1 = 0$, because it does not affect K^* and N^*. Table 46.2 gives the optimum numbers K^* and N^* and the expected costs $C_1(K^*)$ and $C_2(N^*)$ for $T = 24$, $48, 72, \ldots, 168$ h and $\lambda_0 = 1, 2, 3, \ldots, 10\times 10^{-4}$ h, when $N_0 = 1000$, $n = 100$ and $c_2 = c_3 = 1$. Table 46.2 indicates that both K^* and N^* decrease when T

Table 46.2 Optimum replacement number K^*, failed element number N^*, and the expected costs $C_1(K^*)$ and $C_2(N^*)$

λ_0	T	K^*	K^*T	$C_1(K^*)$	N^*	$C_2(N^*)$	$C_1(K^*)/C_2(N^*)$
	24	31	744	1.38×10^{-3}	93	1.07×10^{-3}	1.29
	48	15	720	1.41×10^{-3}	89	1.10×10^{-3}	1.28
	72	10	720	1.42×10^{-3}	86	1.13×10^{-3}	1.26
1×10^{-4}	96	7	672	1.49×10^{-3}	83	1.16×10^{-3}	1.28
	120	6	720	1.42×10^{-3}	80	1.18×10^{-3}	1.20
	144	5	720	1.42×10^{-3}	77	1.21×10^{-3}	1.17
	168	4	672	1.49×10^{-3}	74	1.24×10^{-3}	1.20
2×10^{-4}		16	384	2.75×10^{-3}	90	2.19×10^{-3}	1.26
3×10^{-4}		11	264	4.19×10^{-3}	87	3.34×10^{-3}	1.25
4×10^{-4}		8	192	5.41×10^{-3}	85	4.54×10^{-3}	1.19
5×10^{-4}		6	144	6.98×10^{-3}	82	5.77×10^{-3}	1.21
6×10^{-4}	24	5	120	8.36×10^{-3}	80	7.03×10^{-3}	1.19
7×10^{-4}		5	120	1.04×10^{-2}	77	8.34×10^{-3}	1.25
8×10^{-4}		4	96	1.06×10^{-2}	75	9.69×10^{-3}	1.09
9×10^{-4}		3	72	1.39×10^{-2}	73	1.10×10^{-2}	1.26
10×10^{-4}		3	72	1.39×10^{-2}	71	1.26×10^{-2}	1.10

and λ_0 increase. It is interesting that the value of K^*T are approximately 720 h when $\lambda_0 = 1\times10^{-4}$. In this example, $C_1(K^*)$ is always greater than $C_2(N^*)$ and $C_1(K^*)/C_2(N^*) \approx 1.2$. Therefore, in this case it is clear that delayed maintenance is more efficient than cyclic maintenance from an economic point of view.

Up to now, we have only discussed the best policies for minimizing costs, without considering another important factor: the availability of the system. To finish this chapter, we now obtain the availabilities for these two maintenance models. Let T_0 be the time required for checks at times kT $(k = 1, 2, \ldots)$ and T_1 be the time required for each replacement that occurs at time kT when the number of failed elements exceeds a failure number n or a managerial number N. Then, in a similar way to the way that the expected costs were derived, we can obtain the availabilities. For cyclic maintenance, the availability is given by

$$A_1(K) = \frac{T\sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT) - (1/\lambda) \times \sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT)\sum_{i=n-j}^{\infty}(i+j-n)p_i(T)}{T_1 + (T+T_0)\sum_{k=0}^{K-1}\sum_{j=0}^{n-1} p_j(kT)}, \quad (K = 1, 2, \ldots) \tag{46.56}$$

and for delayed maintenance it is given by

$$A_2(N) = \frac{T\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT) - (1/\lambda) \sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)\sum_{i=n-j}^{\infty}(i+j-n)p_i(T)}{T_1 + (T+T_0)\sum_{k=0}^{\infty}\sum_{j=0}^{N-1} p_j(kT)} \quad (N = 1, 2, \ldots, n). \tag{46.57}$$

References

46.1 T. Nakagawa: Maintenance and optimum policy. In: *Handbook of Reliability Engineering*, ed. by H. Pham (Springer, London 2003)

46.2 R. E. Barlow, F. Proschan: *Mathematical Theory of Reliability* (Wiley, New York 1965)

46.3 T. Nakagawa, M. Kowada: Analysis of a system with minimal repair and its application to replacement policy, Eur. J. Oper. Res. **17**, 176–182 (1983)

46.4 T. Nakagawa: Modified, discrete replacement models, IEEE Trans. Reliab. **R-36**, 243–245 (1987)

46.5 T. Nakagawa: A summary of discrete replacement policies, Eur. J. Oper. Res. **17**, 243–245 (1984)
46.6 H. Morimura: On some preventive maintenance policies for IFR, J. Oper. Res. Soc. Jpn. **12**, 94–124 (1970)
46.7 T. Nakagawa: Generalized models for determining optimal number of minimal repairs before replacement, J. Oper. Res. Soc. Jpn. **24**, 325–337 (1981)
46.8 T. Nakagawa: Optimal number of units for a parallel system, J. Appl. Prob. **21**, 431–436 (1984)
46.9 T. Nakagawa: Replacement problem of a parallel system in random environment, J. Appl. Prob. **16**, 203–205 (1979)
46.10 J. D. Esary, A. W. Marshall, F. Proschan: Shock and wear process, Ann. Prob. **1**, 627–649 (1973)
46.11 T. Nakagawa: On a replacement problem of a cumulative damage model, Oper. Res. Q. **27**, 895–900 (1976)
46.12 S. D. Chikte, S. D. Deshmukh: Preventive maintenance and replacement under additive damage, Nav. Res. Logist. Q. **28**, 36–46 (1981)
46.13 T. Nakagawa, M. Kijima: Replacement policies for a cumulative damage model with minimal repair at failure, IEEE Trans. Reliab. **38**, 581–584 (1989)
46.14 H. Mine, T. Nakagawa: Interval reliability and optimum preventive maintenance policy, IEEE Trans. Reliab. **26**, 131–133 (1977)
46.15 J. J. Coleman, I. J. Abrams: Mathematical model for operational readiness, Oper. Res. **10**, 126–138 (1962)
46.16 T. Nakagawa, K. Ito: Comparison of cyclic and delayed maintenances for a phased array radar, J. Oper. Res. Soc. Jpn. **47**, 51–61 (2004)
46.17 E. Brookner: *Phased-array radars* (Artech House, Boston 1991)
46.18 H. M. Keithley: Maintainability impact on system design of a phased array radar, Annual New York Conf. on Electronic Reliab. **7**, 1–10 (1966)
46.19 A. H. Heresh: Maintainability of phased array radar systems, IEEE Trans. Reliab. **16**, 61–66 (1967)