47. Risks and Assets Pricing

This chapter introduces the basic elements of risk and financial assets pricing. Asset pricing is considered in two essential situations, complete and incomplete markets, and the definition and use of a number of essential financial instruments is described. Specifically, stocks (as underlying processes), bonds and derivative products (and in particular call and put European and American options) are considered. The intent of the chapter is neither to cover all the many techniques and approaches that are used in asset pricing, nor to provide a complete introduction to financial asset pricing and financial engineering. Rather, the intent of the chapter is to outline through applications and problems the essential mathematical techniques and financial economic concepts used to assess the value of risky assets. An extensive set of references is also included to direct the motivated reader to further and extensive research in this broad and evolving domain of economic and financial engineering and mathematics that deals with asset pricing. The first part of the chapter (The Introduction and Sect. 47.1) deals with a definition of risk and outlines the basic terminology used in asset pricing. Further, some essential elements of the Arrow–Debreu framework that underlies the fundamental economic approach to asset pricing are introduced. A second part (Sect. 47.2), develops the concepts of risk-neutral pricing, no arbitrage and complete markets. A number of examples are used to demonstrate how we can determine a probability measure to which risk-neutral pricing can be applied to value assets when markets are complete. In this section, a distinction between complete and incomplete markets is also introduced. Sections 47.3, 47.4 and 47.5 provide an introduction to and examples of basic financial approaches and instruments. First, Sect. 47.3, outlines the basic elements of the consumption capital asset-pricing model (with the CAPM stated as a special case). Section 47.4 introduces the basic elements of net present value and bonds, calculating the yield curve as well as the term structure of interest rates and provides a brief discussion of default and rated bonds. Section 47.5 is a traditional approach to pricing of options using the risk-neutral approach (for complete markets). European and American options are considered and priced by using a number of examples. The Black–Scholes model is introduced and solved, and extensions to option pricing with stochastic volatility, underlying stock prices with jumps as well as options on bonds are introduced and solved for specific examples. The last section of the chapter focuses on incomplete markets and an outline of techniques that are used in pricing assets when markets are incomplete. In particular, the following problems are considered: the pricing of rated bonds (whether they are default-prone or not), engineered risk-neutral pricing (based on data regarding options or other derivatives) and finally we also introduce the maximum-entropy approach for calculating an approximate risk-neutral distribution.

Risk results from the direct and indirect adverse consequences of outcomes and events that were not accounted for or that we were ill prepared for, and concerns their effects on individuals, firms, financial markets or society at large. It can result from many reasons, internally, externally and strategically induced or resulting from risk externality – namely, when all costs or benefits are not incorporated by the market in the price of the asset, the product or the service received. A definition of risk involves several factors including: (i) consequences, (ii) their probabilities and their distribution, (iii) individual preferences, (iv) collective preferences and (v) sharing, contracts or risk-transfer mechanisms. These are relevant to a broad number of fields as well, each providing a different approach to the measurement, the valuation and the management of risk which *is motivated by psychological needs and the need to deal with problems that result from uncertainty and the adverse consequences they may induce* [47.1, 2]. In this chapter we are primarily concerned with risk and pricing and specifically financial assets pricing.

The definition of risk, risk measurement, risk pricing and risk management are intimately related, one feeding the other to determine the proper levels of risks that an individual seeks to sustain and the market's intended price [47.3–6]. Financial asset pricing has sought primarily to determine approaches and mechanisms for market pricing of these risks while financial risk management and engineering are concerned with the management of financial risks, seeking on the one hand to price private risks and on the other responding to the managerial finance considerations that these risks entail. Financial risk management, for example, deals extensively with hedging problems in order to reduce the risk of a particular portfolio through a trade or a series of trades, or contractual agreements reached to share and induce an efficient risk allocation by the parties involved [47.1, 7–12]. To do so, a broad set of financial instruments were developed, including bonds of various denominations, options of various types etc., in some cases broadly traded, thereby allowing a market mechanism to price these risks. For example, by a judicious use of options, contracts, swaps, insurance and investment portfolios etc. risks can be brought to bearable economic costs and shared by the parties involved in market transactions. These tools are not costless however, and require a careful balancing of the numerous factors that affect risk, the costs of applying these tools and a specification of tolerable risk. For example, options require that a premium be paid to limit the size of losses just as the insured are required to pay a premium to buy an insurance contract to protect them in case of unforeseen accidents, theft, diseases, unemployment, fire, etc. For this reason, private tools such as portfolio investment strategies, value at risk ([47.13–18] based on a quantile risk measurement providing an estimate of risk exposure) are used to manage individual risks. Financial engineering in particular has devoted a substantial attention to reconciling the management of individually priced risks and market-priced risks such that risks can be managed more efficiently. These concerns also reflect the basic approach of finance and the use of financial instruments, currently available through brokers, mutual funds, financial institutions, commodity and stock derivatives etc., which are motivated by three essential reasons [47.19–25]:

- To price the multiplicity of claims, accounting for risks and dealing with the adverse effects of uncertainty or risk (that can be completely unpredictable, partly or wholly predictable)
- To explain and account for investors' behavior. To counteract the effects of regulation and taxes by firms and individual investors (who use a wide variety of financial instruments to bypass regulations and

increase the amount of money investors can make while reducing the risk they sustain).
- To provide a rational framework for individuals' and firms' decision-making and to suit investors needs in terms of the risks they are willing to assume and pay for.

Financial instruments deal with uncertainty and the management of the risks they imply in many different ways. Some instruments merely transfer risk from one period to another and in this sense they reckon with the time phasing of events. One of the more important aspects of such instruments is to supply *immediacy* – i. e. the ability not to wait for a payment. Other instruments provide *spatial diversification* (in other words the distribution of risks across independent investments, classes or geography) and *liquidity*. By liquidity, we mean the cost to convert instantly an asset into cash at its fair price. This liquidity is affected both by the existence of a market (in other words, buyers and sellers) as well as the cost of transactions associated with the conversion of the asset into cash. As a result, essential financial risks include: (a) market-industry specific risks and (b) term structure – currency–liquidity risks. Throughout these problems financial engineering provides a comprehensive set of approaches, techniques and tools that seek to bridge the gap between theory and practice, between individual preferences and market pricing and seeks to provide numerical and computer-aided techniques that respond to the needs of individual investors and financial institutions. Further, it recognizes the centrality of money in decision-making processes: making money, not losing it and protecting investors from adverse consequences. To do so, asset pricing (valuation), forecasting, speculating and risk reduction through fundamental analysis, trading (hedging) are essential activities of traders, bankers and investors alike. Financial engineering, for example, deals extensively with the construction of portfolios, consisting of assets of broadly defined risk–return characteristics, derivatives assets etc. with risk profiles desired by individual investors [47.20, 26–38]. For these reasons, risk and financial engineering are applied not only to financial decision-making. Applications to engineering project valuation, management science, engineering risk economics and so on, are a clear indication of the maturity and the usefulness of financial asset pricing, financial engineering and financial risk management. The purpose of this chapter is to outline and explain the salient factors of these continually renewed and expanding fields of research and applications of asset pricing. At the same time we shall seek to bridge the gap between theory and practice while maintaining a mathematical level accessible to typical finance, risk, management and engineering students familiar with basic notions in probability and stochastic processes.

47.1 Risk and Asset Pricing

Asset pricing, broadly, seeks to reduce assets to the identification of *state prices*, a notion that Arrow has coined, and from which any security has an implied value as the weighted sum of its cash flows, state by state, time by time, with weights given by the associated state prices [47.19, 38, 39]. Such state prices may therefore be viewed as the marginal rates of substitution among state-time consumption opportunities, for an unconstrained investor, with respect to a numeraire good [47.23]. We shall focus first on complete markets, where state prices that make it possible to uniquely value assets do exist. Issues and topics relating to incomplete markets, stochastic volatility etc., are discussed as well. However, this is far too important and far too broad a field to be treated in this chapter's space. We begin by outlining some key terms commonly used in the language of asset pricing. A more explicit and detailed outline of such terms can be found in *Duffie* [47.22, 23], *Karatzas* and *Shreve* [47.40, 41]. We shall also outline the basic ideas of the Arrow–Debreu framework which underlies asset pricing.

47.1.1 Key Terms

In most financial models of asset pricing we use a set of states Ω, with associated probabilities to characterize the model underlying uncertainty. Such a set may be finite or infinite. A set of events is then expressed by $\Im$, also called a tribe and by some a σ-algebra. $\Im$ is a collection of subsets of Ω that can be assigned a probability $P(A)$ denoting the probability of a specific event A. In an intertemporal framework defined by the dates 0, 1, ... , T+1, there is at each date a tribe $\Im_t \subset \Im$, corresponding to the events based on the information available at that time t. Any event in $\Im_t$ is known at time t to be true or false. The convention $\Im_t \subset \Im_s$, $t \leq s$ is used at all times, meaning that events are never *forgotten* and therefore information accessed over time provides ever expanding *knowledge*. For simplicity, we let events in $\Im_0$ have probability 0 or 1, meaning that there is no information at time t=0. A filtration is defined by $\Phi = \{\Im_0, \Im_1, \Im_2, \ldots, \Im_T\}$, sometimes called an information structure, representing how information is revealed over time. For any random variable Y, we thus use at time t, $E_t(Y) = E(Y|\Im_t)$, to denote the conditional expectation of Y given $\Im_t$. For notational simplicity, we also let $Y = Z$ for any two random variables Y and Z, if the probability that $Y \neq Z$ is zero (for a review of essential elements in probability see for example, [47.42–48]).

An adapted process, defined by a sequence $X = \{X_0, X_1, X_2, \ldots, X_T\}$ such that, for each t, X_t is a random variable with respect to $(\Omega, \Im_t)$ means, informally, that X_t is observable at time t. An important characteristic of such processes in asset pricing is that an adapted process X is a martingale if, for any times t and $s > t$, we have $E_t(X_s) = X_t$. For example, if the conditional expectation of an asset price equals the currently observed price, then the adapted price process is a martingale. For this reason, important facets of financial asset pricing revolve around the notion of martingales. Another term of importance we use with respect to stochastic price processes is *non-anticipating*. This means that, for any time $t < s$ the function (price) is statistically independent of the *future uncertainty*, or the current price is independent of the future Wiener process $W(s) - W(t)$. These mathematical properties are extremely useful in proving basic results in the theoretical analysis of financial markets. However, in practice, underlying processes might not be martingales and further be anticipative processes. Of course, this will also imply a temporal dependence and our theoretical and financial constructs may be violated.

A security is a claim to an adapted (and non-anticipating) dividend process, say D, with D_t denoting the dividend paid by the security at time t. Each security has an adapted security-price process S, so that S_t is the price of the security, ex dividend, at time t. That is, at each time t, the security pays its dividend D_t and is then available for trade at price S_t. This convention implies that D_0 plays no role in determining ex-dividend prices. The cum-dividend security price at time t is $S_t + D_t$. A trading strategy is an adapted process $\boldsymbol{n}$ in $\mathbb{R}^N$. Here, n_t represents the portfolio held after trading at time t. The dividend process $D^{\boldsymbol{n}}$ generated by a trading strategy $\boldsymbol{n}$ is thus defined by $D_t^{\boldsymbol{n}} = n_{t-1}(S_{t-1} + D_t) - n_t S_t$ with n_{-1} taken to be zero by convention. Consider a portfolio that invests wealth in the security and in a bond B_{t-1}; and say that at time $t-1$, the portfolio wealth state is given by:

$$X_{t-1}^{\boldsymbol{n},m} = n_{t-1} S_{t-1} + m_{t-1} B_{t-1} \,.$$

We then state that a strategy is said to be self-financing if:

$$\begin{aligned} X_t^{\boldsymbol{n},m} - X_{t-1}^{\boldsymbol{n},m} &= n_{t-1}\left(S_t - S_{t-1}\right) \\ &\quad + m_{t-1}\left(B_t - B_{t-1}\right) \,. \end{aligned}$$

For example, a bonds-only strategy is defined by $n_{t-1} = 0$, while buy-and-hold (long) strategies (that do not depend on time) imply that $n_{t-1} = n > 0$. A short position is defined when $n_{t-1} < 0$. Finally, a strategy consisting of maintaining a constant proportion of our wealth in bonds and stock means that: $n_t S_t / X_t^{\boldsymbol{n},m} = \alpha$ while $m_t B_t / X_t^{\boldsymbol{n},m} = 1 - \alpha$.

The important notion of arbitrage in asset pricing is used to define a trading strategy that costs nothing to form, never generates losses, and, with positive probability, will produce strictly positive gains at some time. The notion of efficient markets in particular presumes that, for efficient markets to exist, there must not be arbitrage trading strategies. The search for profits by traders and investors is therefore motivated explicitly by the search for arbitrage opportunities. This is of course a rational investment approach, for in the presence of an arbitrage, any rational investor who prefers to increase his dividends would undertake such arbitrage without limit, so markets could not be in equilibrium (in a sense that we shall see later on). As a result, the notion of no arbitrage, and the associated concepts of martingales, risk-neutral pricing and complete markets are fundamental key terms that must be understood to appreciate the scope and the spirit of asset pricing, financial engineering and financial risk management. Further, we shall also distinguish clearly between individual and collective (market) valuation. In the former, the agent-investor is assumed to optimize an expected utility of consumption, subject to an endowment constraint, while in the latter case, that investor will be fully aware of the market valuation of risk, based on an equilibrium in state prices in order to tailor an appropriate and fitting strategy to his preferences. An essential objective of dynamic asset pricing, which deals with the intertemporal and risk effects of asset pricing is to link the collective (multi-agent) equilibrium valuation (pricing) of assets to macroeconomic variables, hopefully, observable. This latter field of study requires an extensive familiarity with economic theory, finance and stochastic calculus.

47.1.2 The Arrow–Debreu Framework

The Arrow–Debreu framework underlies the approach to asset pricing. It is therefore useful to present it, even if briefly (see also [47.19,22,49,50]). Assume that there are N securities, $S_1 \dots S_N$, each of which can be held long or short in a portfolio consisting of these securities. Let $n_i > 0$ be the number of securities S_i currently priced at p_i. Thus, the vectorial product $\boldsymbol{n} \cdot \boldsymbol{p}, \boldsymbol{n} \cdot \boldsymbol{p} = \sum_{i=1}^{N} n_i p_i$ denotes the value (price) of the portfolio held. To each security i, there are associated potential cash flows D_{ij}, $j = 1, 2, \dots, M$ where M is the number of all possible *states* of the market at the end of the trading period. For example, if over one period, the market can assume only two states (say high and low) then the market is binomial and $M = 2$. If it assumes three potential states, then $M = 3$ and the market is trinomial etc. For the portfolio as a whole, we thus have the cash flow matrix:

$$\boldsymbol{nD}_{.j} = \sum_{i=1}^{N} n_i D_{ij} ,$$
$$\boldsymbol{D} = (D_{ij}), \quad i = 1, \dots, n; \quad j = 1, \dots, M .$$

Where $\boldsymbol{D}_{.j}$ denotes the vector of cash flows for all securities held if state j occurs, while the j-th row of the matrix $\boldsymbol{D}$ represents all possible cash flows associated with holding one unit of the j-th security, including dividend payment and market profit/losses (in dollars). If $n_i > 0$, the investor is long and the investor collects $n_i D_{ij}$ at the end of the period. If $n_i < 0$, the investor is short and the investor has a liability at the end of the period (taken by borrowing securities and selling them at the market price). Further, we assume that the transaction costs, commissions, taxes, etc. are neglected. The cash flow of the portfolio at the j-th state is thus, $\boldsymbol{nD}_{.j}, \forall j \in [1, \dots M]$ as stated above. The cash flow thus defined allows us a formal definition of arbitrage.

Definition (Arbitrage)

An arbitrage portfolio is a portfolio $\boldsymbol{n}$ such that

1. Either $\boldsymbol{nP} = 0$, $\boldsymbol{nD}_{.j} \geq 0$, $\forall j \in [1, \dots M]$ and $\boldsymbol{nD}_{.j} > 0$ for some $j \in [1, \dots M]$,
2. or $\boldsymbol{nP} \leq 0$ and $\boldsymbol{nD}_{.j} \geq 0$ $\forall j \in [1, \dots M]$.

That is, an arbitrage portfolio is a position that either has zero initial cost or has no *downside* regardless of the market outcome, and thus offers the possibility to make money without investment and finally can realize an immediate profit that has no downside.

Inversely, there is no arbitrage if an arbitrage portfolio cannot be constructed. The implication of no arbitrage has a direct implication on asset pricing and to the definition of risk-neutral probabilities which are used to define a price linearly in terms of the markets' cash flows. This is summarized by the following Theorem 47.1 whose proof can be found in *Duffie* [47.23]:

Theorem 47.1 (The fundamental theorem of asset pricing with no arbitrage)

If there exist a vector of positive numbers (also called asset prices) $\pi_j (j = 1 \dots M)$ such that:

$$P_i = \sum_{j=1}^{M} D_{ij} \pi_j , \quad j = 1 \dots M$$

or in vector notation $\quad \boldsymbol{P} \equiv \boldsymbol{D\pi}$

Then there exist no arbitrage portfolios. And conversely, if there are no arbitrage portfolios, there exists a vector $\boldsymbol{\pi}$ with positive entries satisfying $\boldsymbol{P} \equiv \boldsymbol{D\pi}$ [47.23,50].

In this framework, risk-neutral probabilities or equivalently, risk-adjusted probabilities, are defined by

$$\hat{\pi}_j = \frac{\pi_j}{\sum_{j=1}^{M} \pi_k} , \quad j = 1, \dots M, \quad \hat{\pi}_j \geq 0 ,$$
$$\sum_{j=1}^{M} \hat{\pi}_j = 1 .$$

Using these probabilities we can write, based on Theorem 47.1, that an asset price is equal to the discounted value of future cash-flow payments, at a risk-less discount rate R_f, or:

$$P_i = \frac{1}{1 + R_\mathrm{f}} \sum_{j=1}^{M} D_{ij} \hat{\pi}_j .$$

To see that this is the case, suppose that there exists an investment opportunity (a pure, risk-free zero bond or a money market deposit) which guarantees (for sure) \$1 at the end of the period. The payoff of a bond is thus $1 \leq j \leq M$, $(1, 1, \dots 1)$ (for all states R in $\mathbb{R}^M$) and, according to Theorem 47.1, $P_{\mathrm{BOND}} = \sum_{j=1}^{M} \pi_j = 1/(1 + R_\mathrm{f})$. Thus, $1 + R_\mathrm{f} = 1 / \sum_{j=1}^{M} \pi_j$, and R_f is called the bond (in this case risk-free) yield. This can be written as $P_i = \frac{1}{1+R_\mathrm{f}} \sum_{j=1}^{M} D_{ij} \hat{\pi}_j$ as stated above, which is the expected value under the risk-adjusted (risk-neutral) probabilities, which we denote by $E_{\mathrm{RN}}\{\cdot\}$ (not to be confused with the historical distribution of prices). Namely, $E_{\mathrm{RN}}\{\cdot\}$ denotes the expectation associated with the operator associated with probabilities $\hat{\pi}_j$, $1 \le j \le M$. The implications of no arbitrage are summarized by the following Theorem 47.2.

Theorem 47.2 (The fundamental theorem of risk-neutral pricing)
Assume that the market admits no arbitrage portfolios and that there exists a risk-less lending/borrowing at rate R_{f}. Then, there exists a probability measure (risk neutral) defined on the set of feasible market outcomes, $\{1, 2, \ldots M\}$, such that the value of any security is equal to the expected value of its future cash flows discounted at the risk-less lending rate.

To calculate these risk-neutral probabilities, we use a portfolio replication which is defined as follows. Given a security S, and a set of securities $S_1, \ldots S_k$, we say that the portfolio $(n_1, n_2, \ldots n_N)$ *replicates* S if the security and the portfolio have identical cash flows. Further, given two identical cash flows, their price is, necessarily the same, as otherwise there would be an opportunity for arbitrage. This is also called the law of the single price. On the basis of the current analysis, we turn at last to a formal definition of complete markets.

Definition (Complete markets)
A securities market with M states is said to be *complete* if, for any vector cash-flow $(D_{.1} \ldots D_{.M})$, there exists a portfolio of traded securities $(n_1, n_2, \ldots n_N)$ which has cash flow D_j in state $j \in [1, \ldots M]$. Thus market completeness implies that:

$$\begin{aligned}
&\boldsymbol{n}\boldsymbol{D} \equiv \boldsymbol{D}, \quad \text{or} \quad \sum n_i D_{ij} = D_j\,, \\
&j \in [1, \ldots M] \quad \text{has a solution} \quad \boldsymbol{n} \in \mathbb{R}^N \\
&\text{for any} \quad D \in \mathbb{R}^M\,.
\end{aligned}$$

This is equivalent to the condition: rank $\boldsymbol{D} \equiv M$.

The implication of this definition is that, if a portfolio can be replicated uniquely (the rank condition $\boldsymbol{D} \equiv M$, providing a unique solution to the linear pricing equations) then there is one price and complete markets can exist. Inversely, the uniqueness of asset prices determines a complete market. This is summarized by the proposition below.

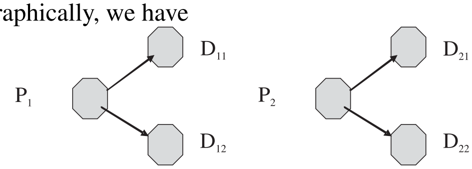
Proposition
Suppose that the market is complete. Then there is a unique set of state prices $(\pi_1 \ldots \pi_M)$ and hence a unique set of risk-neutral probabilities $(\hat{\pi}_1 \ldots \hat{\pi}_M)$. Conversely, if there is a unique set of state prices, then the market is complete, with risk-neutral probabilities $(\hat{\pi}_1 \ldots \hat{\pi}_M)$.

We shall find that the generality of these results is maintained in inter-temporal (dynamic) asset pricing. We consider first some simple examples to demonstrate the pricing implications of the Arrow–Debreu framework.

Example 47.1: Assume a binomial model for a security that assumes two possible future outcomes $\boldsymbol{D} = \{D_1, D_2\}$, $M = 2$. Consider as well two securities with current price P_i, $i = 1, 2$ (for example, a portfolio consisting of a stock and and a bond and a call option on the security with P_1 known and P_2 to be priced), each of these is generating a cash flow (D_{i1}, D_{i2}), $i = 1, 2$. We have no arbitrage if there are risk-neutral probabilities such that, for each security we have

$$P_i = \frac{1}{1+R_{\mathrm{f}}}\left(\hat{\pi}_1 D_{i1} + \hat{\pi}_2 D_{i2}\right), \quad i = 1, 2\,.$$

Graphically, we have



Since $\hat{\pi}_1 + \hat{\pi}_2 = 1$, both the risk-neutral probabilities are given by solving the system of equations in two unknowns for a unique solution (and therefore markets are complete):

$$\begin{aligned}
P_1 &= \frac{1}{1+R}\left[\hat{\pi}_1 D_{11} + (1-\hat{\pi}_1) D_{12}\right]; \\
P_2 &= \frac{1}{1+R}\left[\hat{\pi}_1 D_{21} + (1-\hat{\pi}_1) D_{22}\right],
\end{aligned}$$

where $\hat{\pi}_1$ and P_2 are to be calculated by a solution of these two equations, as:

$$\begin{aligned}
\hat{\pi}_1 &= \frac{(1+R)P_1 - D_{12}}{D_{11} - D_{12}}\,, \\
P_2 &= \frac{P_1}{D_{11} - D_{12}}(D_{21} - D_{22}) \\
&\quad + \frac{1}{1+R}\left(\frac{D_{11}D_{22} - D_{12}D_{21}}{D_{11} - D_{12}}\right).
\end{aligned}$$

Equivalently, the matrix $\boldsymbol{D}$ has to be of full rank 2. However, if both assets assume three potential states, leading to a trinomial model, then of course $M = 3$, while rank $(\boldsymbol{D}) = 2$ and therefore the market is not complete since the number of solutions to this system of equations is infinite. If we add a third asset whose price is based on the same events we obtain a 3×3 matrix with rank 3 whose solution is again unique and therefore the market is complete again. In other words, adding another asset has made it possible to obtain the risk-neutral probabilities, needed for asset pricing and *complete* the market. Similar examples can be used to price simple models of forward futures contracts as well as prove some fundamental equalities in options' finance (put–call parity for example, which we will see subsequently). The Arrow–Debreu framework underlies the basic approach of modern financial economics for asset pricing and therefore it is important to appreciate its basic assumptions as stated here.

47.2 Rational Expectations, Risk-Neutral Pricing and Asset Pricing

Rational (risk neutral) expectations, risk-neutral pricing, complete and incomplete markets, as shown in the Arrow–Debreu framework, underlie the valuation of risk and the use of financial engineering for asset pricing. Rational expectations imply that current prices reflect future uncertainties and their price, and also mean that current prices are based on the unbiased, minimum-variance mean estimate of future prices. This property provides the means to value assets and securities, although in this approach, bubbles are not possible, since they seem to imply a persistent error or bias in forecasting. Rational-expectation pricing will not allow investors to earn above-average returns without taking above-average risks. In such circumstances, arbitrageurs, those *smart investors* who seek to identify returns that have no risk and yet provide a return, will not be able to profit without assuming risks.

The concept of rational expectation is due to *Muth* [47.51], however, who formulated it as a decision-making hypothesis in which agents are informed, constructing a model of the economic environment and using all the relevant and appropriate information at a time a decision is made (see also [47.52], p. 23):

> *I would like to suggest that expectations, since they are informed predictions of future events, are essentially the same as the predictions of the relevant economic theory ... We call such expectations "rational" ... This hypothesis can be rephrased a little more precisely as follow: that expectations ... (or more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same information set, about the prediction of the theory (the objective probability of outcomes).*

In other words, if investors are "smart" and base their decisions on informed and calculated predictions, then, prices equal their discounted expectations. In other words, it implies that investors' subjective beliefs are the same as those of the real world – they are neither pessimistic nor optimistic. When this is the case and a *rational expectations equilibrium* holds, we say that markets are complete or efficient. Samuelson pointed out this notion in 1965 as the martingale property leading *Fama*, *Lucas* and *Harrison* and *Kreps* [47.53–56] to characterize such properties as market efficiency.

Lucas used a concept of rational expectations similar to Muth to confirm Milton Friedman's 1968 hypothesis of the long-run neutrality of monetary policy. Specifically, *Lucas* [47.54] and *Sargent* [47.57] have shown that economic agents alter both their expectations and their decisions to neutralize the effects of monetary policy.

Martingale and the concept of market efficiency are intimately connected, as shown in the Arrow–Debreu framework and pointed out by *Harrison* and *Pliska* [47.39] in their seminal paper. If prices follow a martingale, then only the information available today is relevant to make a prediction on future prices. In other words, the present price has all the relevant information embedding investors' expectations. This means that in practice (the weak form of efficiency) past prices should be of no help in predicting present prices or equivalently prices have no memory. Similarly, if prices follow a martingale and are unpredictable, markets are efficient. In this case, arbitrage is not possible and there is always a party to take on a risk irrespective of how high it is. Hence, risk can be perfectly diversified away and made to disappear. In such a world without risk, all assets behave as if they are risk-free and therefore prices can be discounted at a risk-free rate. This property, justifies our use of risk-neutral pricing (RNP). It breaks down however if any of the previous hypotheses (martingale, rationality, no arbitrage, absence of transaction costs etc.) are invalid. In such a case, prices can no longer be unique and markets are said to be incomplete.

There is a confrontation between economists however, some of whom believe that markets are efficient and some who do not. Obviously, market efficiency fails to account for market anomalies such as bubbles and bursts, firms' performance and their relationship to size etc. As a result, behavioral finance has sought to provide an alternative dogma (based on psychology) to explain the behavior of financial markets. Whether these dogmas will converge back together as classical and Keynesian economics have, remains yet to be seen. In summary, however, some believe that the current price imbeds all future information, and some presume that past prices and behavior can be used to predict future prices. If the test is to make money, then the verdict is far from reached. Richard Roll, a financial economist and money manager argues:

> *I have personally tried to invest money, my clients and my own, in every single anomaly and predictive result that academics have dreamed up. And I have yet to make a nickel on these supposed market inefficiencies. An inefficiency ought to be an exploitable opportunity. If there is nothing that investors can exploit in a systematic way, time in and time out, then it is very hard to say that information is not being properly incorporated into stock prices. Real money investment strategies do not produce the results that academic papers say they should ... ... but there are some exceptions including long term performers that have over the years systematically beat the market (Burton Malkiel, The Wall Street Journal, December, 28, 2000).*

Information and power can also be sources of incompleteness. There are many situations when this is the case. Information asymmetries, insider trading and advantages of various sorts can provide an edge to individual investors and thereby violate the basic tenets of market efficiency. Further, the interaction of markets can lead to instabilities due to very rapid and positive feedback or to expectations that are becoming trader- and market-dependent. Such situations lead to a growth of volatility, instabilities and perhaps, in some special cases, to chaos. Nonetheless, whether it is fully right or wrong, it seems to work sometimes. Thus, it should be used carefully for making money. Of course, it is used for simple models for pricing options and derivatives in general. Throughout these approaches we shall use a known risk-free rate. In contrast, economic equilibrium theory based on the clearing of markets by equating supply to demand for all financial assets provides an equilibrium where interest rates are endogenous. It assumes however, that beliefs are homogenous, markets are frictionless (with no transaction costs, no taxes, no restriction on short sales and divisible assets) as well as competitive markets (in other words, investors are price takers) and finally it also assumes no arbitrage. Thus, general equilibrium is more elaborate than rational expectations (and arbitrage-free pricing) and provides more explicit results regarding market reactions and prices than the traditional finance-only approach [47.54].

47.2.1 Risk-Neutral Pricing and Complete Markets

In complete markets, we use risk-neutral probabilities which allow, conveniently, linear pricing measures. If there are such probabilities, and it is so in complete markets, then the current price ought to be determined by its future values. Since these probabilities are endogenous, based on an exchange between investors and speculators, *it is the market that determines prices and not uncertainty*. Uncertainty arises then only from an individual assessment of potential future events based on private information, on the one hand, or based on investors and speculators not aware of publicly available common information, on the other. In such situations the complete-markets mechanism (based on risk-neutral pricing) for determining asset prices is no longer viable.

Our ability to construct a unique set of risk-neutral probabilities depends on a number of assumptions which are of critical importance in finance and must be maintained theoretically and practically. These include: no arbitrage opportunities; no dominant trading strategies; and the law of the single price. No arbitrage occurs when it is not possible for an agent to make money for sure without having to invest any in the first place. The single-price hypothesis was elaborated by Modigliani and Miller in 1958, stating that two prospective future cash flows with identical risks must be priced equally. In other words, if we can replicate an asset price by a (synthetic) portfolio whose value can be ascertained, then if there is no arbitrage, the price of the asset and the synthetic portfolio are necessarily the same. This also implies market completeness, requiring that there be: no transaction costs; no taxes; infinitely divisible assets; that gents can borrow or lend at the same rates; no information asymmetry regarding future state prices; an impossibility to short sell; and finally, rational investors. Any violation or restrictions that will violate market completeness will open an opportunity for arbitrage. Although in practice, at least one of these assumptions is often violated, in theory and for many fundamental and useful results in finance theory, the assumption of no arbitrage is maintained. When this is not the case, and the assumption of market completeness is violated, it is no longer possible to obtain a unique set of *risk-*

neutral probabilities, but there may be several sets of such *risk-neutral* probabilities. To price an asset in such circumstances, statistical and numerical techniques are applied to select the martingale that best fits the observed behavior of financial markets and at the same time is consistent with basic economic and financial considerations. There are a number of approaches that we might follow in such circumstances, some of which will be used in the sequel. When markets are complete, replication of assets by a synthetic portfolio is a powerful tool to determine asset prices. Such as, pricing forward contracts, put–call parity, future prices and other derivatives that use replicating portfolios. For example, say that a stock spot price is S and let R_f be the period's risk-less lending rate. Next consider a forward contract consisting of an agreement to buy the stock at the end of the period at a set price F. If no dividends are paid and there is no arbitrage, then the contract profit (price) is $Q = S - F/(1+R_\mathrm{f}) = 0$ (since the contract is costless and no money exchanges hands initially). As a result, the forward price is $F = S(1+R_\mathrm{f})$. Similarly, for put–call parity we consider a stock and derived call and put contracts with the same strike price and time. The former gives the right to buy the stock at the strike price and the latter the right to sell at time T and price K. If their prices at maturity are $c_T = \max(S_T - K, 0)$ and $p_T = \max(K - S_T, 0)$, respectively, then the current put price can be replicated by a portfolio consisting of the call, a risk-less bond with price K at maturity T – the option's exercise price as well as the underlying stock. This will be considered later but it suffices for the moment to state that the put (and thereby its synthetic portfolio) price at time $t = 0$ is given by $p = c - S + K\,\mathrm{e}^{-R_\mathrm{f}T}$. Other cases will be considered subsequently. Below, we consider the implications and the mechanics of risk-neutral pricing using a number of stochastic price models.

47.2.2 Risk-Neutral Pricing in Continuous Time

The purpose of this section is to consider the mechanics of risk-neutral pricing in complete markets in a continuous time model. For simplicity, we restrict our attention to an underlying log-normal asset-price process, meaning that the rates of returns of the asset are normal with known mean and known variance. We set $S(t)$ as the asset price at time t with normally distributed rates of returns $\mathrm{d}S(t)/S(t)$. This can be written as an Ito stochastic differential equation as follows:

$$\frac{\mathrm{d}S}{S} = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}W, \quad S(0) = S_0 \quad \text{or}$$

$$S(t) = S(0)\,\mathrm{e}^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)}, \quad W(t) = \int_0^t \mathrm{d}W(t)\,,$$

where the rates of returns are normally distributed with mean and variance, both linear functions of time, given by αt and $\sigma^2 t$ respectively. Further, $W(t)$ denotes a Brownian motion, which is defined as normal (Wiener) prices with mean zero and variance t. Formally, we first note that with this probability measure, there is no risk-neutral pricing since:

$$\begin{aligned} S(0) &\neq \mathrm{e}^{-R_\mathrm{f}t} E\left[S(t)\right] \\ &= \mathrm{e}^{-R_\mathrm{f}t}\,\mathrm{e}^{\left(\alpha - \frac{\sigma^2}{2}\right)t} S(0) E\left(\mathrm{e}^{\sigma W(t)}\right) \\ &= \mathrm{e}^{(\alpha - R_\mathrm{f})t} S(0)\,. \end{aligned}$$

This is the case since for normal distributions $E\big(\exp[\sigma W(t)]\big) = \exp\left[(\sigma^2 t/2)\right]$. However, if we define a numeraire $W^*(t) = W(t) + \frac{\alpha - R_\mathrm{f}}{\sigma} t$ with respect to which the risk-neutral process will be defined, then we can write the price as

$$\begin{aligned} S(t) &= S(0)\,\mathrm{e}^{\left(R_\mathrm{f} - \frac{\sigma^2}{2}\right)t + \sigma\left[W(t) + \frac{\alpha - R_\mathrm{f}}{\sigma} t\right]} \\ &= S(0)\,\mathrm{e}^{\left(R_\mathrm{f} - \frac{\sigma^2}{2}\right)t + \sigma W^*(t)}\,, \end{aligned}$$

which, of course, corresponds to an underlying price process (where $\alpha = R_\mathrm{f}$) and therefore,

$$\frac{\mathrm{d}S}{S} = R_\mathrm{f}\,\mathrm{d}t + \sigma\,\mathrm{d}W^*(t), \quad S(0) = S_0\,,$$

and:

$$S(0) = \mathrm{e}^{-R_\mathrm{f}t} E^*\left[S(t)\right] = \mathrm{e}^{(R_\mathrm{f} - R_\mathrm{f})t} S(0) = S(0)\,,$$

where E^* is an expectation taken with respect to the (numeraire) process $W^*(t)$. Thus, the current price equals an expectation of the future price, just as the risk-neutral valuation framework indicated. It is important to remember however that the proof of such a result is based on our ability to replicate such a process by a risk-free process (and thereby value it by the risk-free rate, something that will be done later on). If such a numeraire can be defined, then of course, even an option can be valued, under the risk-neutral process, by a linear expectation. For example, for a European call option whose exercise price is K at time T, its price is necessarily:

$$C(0) = \mathrm{e}^{-R_\mathrm{f}T} E^*\left(\max\left[S(T) - K, 0\right]\right)\,.$$

In fact, under the risk-neutral framework, any asset price equals is risk-free rate discounted expectation under the risk-neutral distribution. For this reason, much effort is expanded, theoretically and practically, on determining the appropriate risk-neutral distribution (the martingale) that can be used to determine asset prices. In particular, it is worthwhile reconsidering the example treated by letting $\lambda = \frac{\alpha - R_f}{\sigma}$ define the market risk. In this case, the measure we have adopted above is equal to the original process plus the price of risk cumulated over time, or: $W^*(t) = W(t) + \lambda t$ and therefore: $(\alpha - R_f)\,dt + \sigma W(t) = \sigma W^*(t)$. Hence, $dS = S[\alpha\,dt + \sigma\,dW(t)] = S\sigma\,dW^*(t)$. Clearly, under the transformed measure $W^*(t)$, the stock process is a martingale but the remaining question is, can we treat this measure as a Wiener process The important theorem of Giranov allows such a claim under specific conditions, which we summarize below. Explicitly, say that Novikov's condition is satisfied, that is:

$$E\left(\exp\left(\frac{1}{2}\int_0^T \left|\frac{\alpha - R_f}{\sigma}\right| dt\right)\right) < +\infty\,.$$

Let the new measure be defined by the Radon–Nykodim derivative,

$$\frac{dP^*}{dP} = \mathbb{Z}, \quad E\mathbb{Z} = 1, \quad \text{where}$$

$$\mathbb{Z} \triangleq \exp\left[-\int_0^T \lambda\,dW(s) - \frac{1}{2}\int_0^T (\lambda)^2\,dW(s)\right],$$

$$E\mathbb{Z} = 1\,,$$

where P^* is the probability equivalent of the original measure. Note that: $P^*(A) = \int_A \mathbb{Z}(W)P(dW), \forall A \in \Im$. The Girsanov theorem then states that under these (Novikov) conditions and given the measure defined by the Radon–Nykodim derivative, the process $W^*(t)$ is a Wiener process under the measure P^*. This theorem is of course extremely important in asset pricing as it allows the determination of martingales to which risk-neutral pricing can be applied.

47.2.3 Trading in a Risk-Neutral World

Under a risk-neutral process, there is no trading strategy that can make money. To verify this hypothesis, we consider an investor's decision to sell an asset he owns (whose current price is S_0) as soon as it reaches an optimal (profit-rendering) price $S^* > S_0$. Let this profit be:

$$\pi_0 = E^* e^{-R_f\tau} S^* - S_0 \quad \text{with}$$
$$\tau = \text{Inf}\left\{t > 0,\, S(t) \geq S^*;\, S(0) = S_0\right\}\,,$$

where τ is the stopping (sell) time, defined by the first time that the optimal target sell price is reached. We shall prove that, under the risk-neutral framework, there is an *equivalence* between selling now or at a future date. Explicitly, we will show that $\pi_0 = 0$. Again, let the risk-neutral price process be:

$$\frac{dS}{S} = R_f\,dt + \sigma\,dW^*(t)$$

and consider the equivalent return process $y = \ln S$. By an application of Ito's calculus, we have

$$dy = \left(R_f - \frac{\sigma^2}{2}\right)dt + \sigma\,dW^*(t), \quad y(0) = \ln(S_0)$$

and

$$\tau = \text{Inf}\left\{t > 0,\, y(t) \geq \ln(S^*);\, y(0) = \ln(S_0)\right\}\,.$$

As a result, $E_S^*\left(e^{-R_f\tau}\right) = E_y^*\left(e^{-R_f\tau}\right)$, which is the Laplace transform of the sell stopping time when the underlying process has a mean rate and volatility given by $\mu = R_f - \sigma^2/2$ and σ respectively, [47.56, 58–60]:

$$g^*_{R_f}(S^*, \ln S_0)$$
$$= \exp\left[\frac{\ln S_0 - \ln S^*}{\sigma^2}\left(-\mu + \sqrt{\mu^2 + 2R_f\mu\sigma^2}\right)\right],$$
$$\sigma > 0,\, -\infty < \ln S_0 \leq \ln S^* < \infty\,.$$

The expected profit arising from such a transaction is thus

$$\pi_0 = S^* E^*\left(e^{-R_f\tau}\right) - S_0$$
$$= S^* g^*_{R_f}(\ln S^*, \ln S_0) - S_0\,.$$

Namely, such a strategy will, in a risk-neutral world, yield a positive return if $\pi_0 > 0$. Elementary manipulations show that this is equivalent to:

$$\pi_0 > 0 \quad \text{If} \quad \frac{\sigma^2}{2} > (R_f - 1)\,\mu \quad \text{or}$$
$$\frac{\sigma^2}{2} > (1 - R_f)\left(\frac{\sigma^2}{2} - R_f\right) \quad \text{if} \quad R_f > \frac{\sigma^2}{2}\,.$$

As a result,

$$\pi_0 = \begin{cases} > 0 & \text{If } R_f > \frac{\sigma^2}{2} \\ < 0 & \text{If } R_f < \frac{\sigma^2}{2}\,. \end{cases}$$

The decision to sell now or wait is thus reduced to the simple condition stated above. An optimal selling price in these conditions can be found by optimizing the return of such a sell strategy, which is found by noting that either it is optimal to have a selling price as large as possible (and thus never sell) or select the smallest price, implying selling now at the current (any) price. If the risk-free rate is *small* compared to the volatility, then it is optimal to wait, and vice versa, a small volatility will induce the holder of the stock to sell. In other words, for an optimal sell price:

$$\frac{d\pi_0}{dS^*} = \begin{cases} > 0 & R_f < \sigma^2/2 \\ < 0 & R_f > \sigma^2/2 \,. \end{cases}$$

Combining this result with the profit condition of the trade, we note that

$$\begin{cases} \frac{d\pi_0}{dS^*} > 0, \pi_0 < 0 & \text{If } R_f < \sigma^2/2 \,, \\ \frac{d\pi_0}{dS^*} < 0, \pi_0 > 0 & \text{If } R_f > \sigma^2/2 \,. \end{cases}$$

An therefore the only solution that can justify these conditions is: $\pi_0 = 0$, implying that whether one keeps the asset or sell is irrelevant, for under risk-neutral pricing, the profit realized from trading of maintaining the stock is equivalent. Say that $R_f < \sigma^2/2$ then a *wait-to-sell* transaction induces an expected trade loss and therefore it is best to obtain the current price. When $R_f > \sigma^2/2$, the expected profit from the trade is positive but it is optimal to select the lowest selling price, which is of course the current price and then again, the profit transaction, $\pi_0 = 0$, will be null, as our contention states. For a risk-sensitive investor (trader or speculator), however, whose utility for money is $u(.)$, a decision to sell will be defined in terms of his preference, given by the utility function. Buy–sell strategies differ therefore because investors have preferences (utilities) that are not the same.

Example 47.2 (Buying and selling on a trinomial random walk): Consider the risk-neutral log-normal risk process:

$$dS/S = R_f\, dt + \sigma\, dW, \quad S(0) = S_0$$

and apply Ito's lemma to the transformation $y = \ln(S)$ to obtain the rate of return process:

$$dy = \left(R_f - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW, \quad y(0) = y_0 \,.$$

Given this normal (logarithmic) price process, consider the trinomial random-walk approximation:

$$Y_{t+1} = \begin{cases} Y_t + f_1 & \text{w. p.} \quad p \\ Y_t + f_2 & \text{w. p.} \quad 1-p-q \\ Y_t + f_3 & \text{w. p.} \quad q \end{cases}$$

Where p, q and $1-p-q$ are the probabilities that returns increase (or decrease) by, f_1, f_3, f_2 respectively. The first two moments of this process are given by

$$\begin{aligned} E\left(Y_{t+1} - Y_t\right) &= f_2 + p\left(f_1 - f_2\right) + q\left(f_3 - f_2\right) \\ &\approx \left(R_f - \frac{1}{2}\sigma^2\right) , \\ E\left(Y_{t+1} - Y_t\right)^2 &= f_2^2 + p\left(f_1^2 - f_2^2\right) + q\left(f_3^2 - f_2^2\right) \\ &\approx \sigma^2 \,. \end{aligned}$$

Thus, an appropriate selection of the parameters p, q, f_1, f_2 and f_3 will provide an approximation to the risk-neutral pricing process. However, note that we have two known parameters (the mean and the variance of the process) while there are five parameters to choose. This corresponds to many potential discrete-time processes that can be considered as approximations to the continuous one. For this reason, a discretization of a risk-neutral process can often lead to incomplete processes (where risk-neutrality cannot be applied). In most cases, therefore, the underlying process has to be carefully applied [47.48]. For an asymmetric trinomial random walk we may set for simplicity $f_1 = 1$, $f_2 = 0$ and $f_3 = -1$, in which case

$$P\left(\Delta Y_i = +1\right) = p, \quad P\left(\Delta Y_i = -1\right) = q, \quad \text{and} \\ P\left(\Delta Y_i = 0\right) = r = 1 - p - q \,.$$

It is well known (for example see *Cox* and *Miller* [47.42], p. 75) that the probability of reaching one of the two boundaries in this case is given by,

$$P\left(Y_t = -a\right) = \begin{cases} \frac{1-(1/\lambda)^b}{1-(1/\lambda)^{a+b}} & \lambda \neq 1 \\ b/(a+b) & \lambda = 1 \end{cases} ,$$

$$P\left(Y_t = b\right) = \begin{cases} \frac{(1/\lambda)^b-(1/\lambda)^{a+b}}{1-(1/\lambda)^{a+b}} & \lambda \neq 1 \\ a/(a+b) & \lambda = 1 \end{cases} ,$$

where $\lambda = q/p$. Further, the expected first time to reach one of these two boundaries is

$$E\left(T_{-a,b}\right) = \begin{cases} \left(\frac{1}{1-r}\right)\left(\frac{\lambda+1}{\lambda-1}\right)\left(\frac{a(\lambda^b-1)+b(\lambda^{-a}-1)}{\lambda^b-\lambda^{-a}}\right) & \lambda = q/p \\ \frac{ab}{1-r} & \lambda = 1 \,. \end{cases}$$

Thus, if we own an asset whose current value is null and if it is sold either when the loss incurred is $-a$ or at b when a profit is realized, then the probability of making money is $P\left(S_{T(-a,b)} = b\right)$ while the probability of losing it is $P\left(S_{T(-a,b)}=-a\right)$, as calculated above. The expected amount of time the trade will be active is of course $E\left(T_{-a,b}\right)$. The trader profit or loss is thus a random variable given by

$$\tilde{\pi}=\begin{cases}-a & \text{w. p.} \begin{cases}\frac{1-(1/\lambda)^b}{1-(1/\lambda)^{a+b}} & \lambda\neq 1\\ b/(a+b) & \lambda=1\end{cases}\\ b & \text{w. p.} \begin{cases}\frac{(1/\lambda)^b-(1/\lambda)^{a+b}}{1-(1/\lambda)^{a+b}} & \lambda\neq 1\\ a/(a+b) & \lambda=1\,.\end{cases}\end{cases}$$

While its average return is (since the process can be considered a renewal process as well)

$$\bar{\pi}(-a,b)=\frac{bP\left(S_{T(-a,b)}=b\right)-aP\left(S_{T(-a,b)}=-a\right)}{E\left(T_{-a,b}\right)}\,.$$

In particular, when $\lambda=1$, the price process is a martingale and the long-run average profit will be null with a variance $2ab$ since:

$$E(\tilde{\pi})=\frac{(-ab+ba)}{a+b}=0\,,\qquad \operatorname{var}(\tilde{\pi})=2ab\,.$$

This also means that we cannot make money on the average with a worthless asset if the underlying price process is a (martingale) random walk (whether it is a binomial or a trinomial walk). In this circumstance, there is no free gift, an asset we receive that is worth nothing is indeed worth nothing. A risk-averse investor, will thus be better off getting rid of this asset and not sustaining the risk of losing money. For a binomial random walk, with $\lambda\neq 1$ and $r=0$ we have ([47.42], p. 31):

$$P\left(S_{T(-a,b)}=b\right)=\frac{\lambda^a-1}{\lambda^{a+b}-1}\,;$$
$$P\left(S_{T(-a,b)}=-a\right)=\frac{\lambda^{a+b}-\lambda^a}{\lambda^{a+b}-1}\,.$$

And therefore

$$E\left(T_{-a,b}\right)=\left(\frac{\lambda+1}{\lambda-1}\right)\left(\frac{a\left(\lambda^b-1\right)+b\left(\lambda^{-a}-1\right)}{\lambda^b-\lambda^{-a}}\right)\,.$$

The long-run average profit is thus

$$\bar{\pi}(-a,b)=\frac{\left[b\left(\lambda^a-1\right)-a\left(\lambda^{a+b}-\lambda^a\right)\right](\lambda-1)\left(\lambda^b-\lambda^{-a}\right)}{\left[b\left(\lambda^{-a}-1\right)+a\left(\lambda^b-1\right)\right](\lambda+1)\left(\lambda^{a+b}-1\right)}\,.$$

An optimization of the average profit over the parameters (a,b) when the underlying process is a historical process provides then an approach for selling and buying for a risk-prone trader. For a risk-neutral process $(\lambda=1)$, the expected profit will be null for all values a and b.

We can extend this example by considering a trader who owns now an asset worth i_0 dollars that he intends to sell at a later date, either at a preventive loss or at a given profit level. Technically, the sell strategy consists of selling at a price $b>i_0$ for a profit of $\kappa=b-i_0$ or at a price of $a<i_0$ for a loss of $\nu=i_0-a<0$ – whichever comes first. The problems we might be concerned with are:

1. What are the optimal parameters (a,b) for an individual investor if the investor uses a risk-adjusted discount rate and at a risk-free discount rate if the underlying process is a risk-neutral process?
2. What is the risk premium of such a strategy?
3. For an averages profit criterion, what are the optimal parameters (a,b) of the trading strategy?

As stated above, the rationality of such strategies are implicit in individual investors' risk aversion, seeking to make a profit by selling at a higher price, and inversely selling at a preventive loss in case prices fall too much, generating potentially a substantial losses.

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47.3 Consumption Capital Asset Price Model and Stochastic Discount Factor

Financial asset pricing is essentially based on defining an approach accounting for the time and risk preferences of future payoffs. To do so, we have sought to determine a discounting mechanism that would, appropriately, reflect the current value of uncertain payoffs to be realized at some future time. The risk-neutral asset pricing framework has provided a linear estimation rule based on the risk-free rate. Another approach considers an investor optimizing the expected utility of consumption and investment. This is also coined the consumption capital asset pricing model (CCAPM), also called the stochastic discount factor (SDF) approach [47.20]. This development will be presented through the application of Euler's equation in the calculus of variations applied to a consumption problem resulting in a pricing formula defined by:

$$p_t = E(\tilde{M}_{t+1}\tilde{x}_{t+1}), \quad \tilde{M}_{t+1} = \frac{1}{1+\tilde{R}_{t+1}},$$

where p_t is the current asset price at time t that we seek to value, $\tilde{x}_t$ is the next-period asset returns, a random variable and $\tilde{M}_t$ is a stochastic discount factor (also called a kernel). We shall show first through a simple two-period model how a pricing formula is derived. Subsequently, we consider a general multi-period problem. It is noteworthy that in this framework the current price equals the current discounted expectation of (only) future returns.

47.3.1 A Simple Two-Period Model

The rationality of the SDF approach can be explained simply by using the following example (subsequently generalized). Say that an investor owns at a certain time t, a certain amount of money s_t, part of which is invested to buy a quantity of stock y at a price p_t, while the residual is consumed. The utility of consumption is assumed to be $u(c_t)$ where $c_t = s_t - yp_t$. A period hence, at time $t+1$, the asset price is a random variable $\tilde{x}_{t+1}$, at which time it is sold and consumed. Thus, the next-period consumption is equal to the period's current income plus the return from the investment, namely $c_{t+1} = s_{t+1} + y\tilde{x}_{t+1}$. Let the discount factor be β, expressing the subjective discount rate of the consumer. Over two periods, the investor's problem consists then of maximizing the two periods' expected utilities of consumption, given by

$$U(c_t, c_{t+1}) = u(c_t) + \beta E_t u(c_{t+1}) \quad \text{or}$$
$$U(c_t, c_{t+1}) = u(s_t - yp_t) + \beta E_t u(s_{t+1} + y\tilde{x}_{t+1}).$$

The optimal quantity to invest (i. e. the number of shares to buy), found by maximizing the expected utility with respect to y, leads to:

$$\begin{aligned}\frac{\partial U}{\partial y} &= -p_t u'(s_t - p_t y) \\ &\quad + \beta E_t\left[\tilde{x}_{t+1}u'(s_{t+1} + \tilde{x}_{t+1}y)\right] \\ &= -p_t u'(c_t) + \beta E_t\left[\tilde{x}_{t+1}u'(c_{t+1})\right] = 0\end{aligned}$$

which yields for an optimum portfolio price:

$$p_t = E_t\left(\beta\frac{u'(c_{t+1})}{u'(c_t)}\tilde{x}_{t+1}\right).$$

If we set $\tilde{M}_{t+1} = \beta\frac{u'(c_{t+1})}{u'(c_t)}$, we obtain the pricing kernel (stochastic discount factor) used above to price the asset. This kernel expresses, as seen in our condition for optimality, the inter-temporal substitution of current and future marginal utilities of consumption. If we choose to write this term as a *discount rate*, then:

$$\tilde{R}_{t+1} = \left(1 - \beta\frac{u'(c_{t+1})}{u'(c_t)}\right)\Big/\beta\frac{u'(c_{t+1})}{u'(c_t)}.$$

This equation is particularly robust, and has many well-known results in finance expressed as special cases [47.20]. For example, if the utility function is of the logarithmic type, $u(c) = \ln(c)$ then, $u'(c) = 1/c$ and $\tilde{M}_{t+1} = \beta c_t/(c_{t+1})$, or $\tilde{R}_{t+1} = \beta[(1/\beta)c_{t+1} - c_t]/c_t$ and further, $p_t/c_t = \beta E_t(\tilde{x}_{t+1}/c_{t+1})$. In other words, if we write $\pi_t = p_t/c_t$; $\tilde{\pi}_{t+1} = \tilde{x}_{t+1}/c_{t+1}$, then we have: $\pi_t = \beta E_t(\tilde{\pi}_{t+1})$. Further, if the asset is a risk-less bond, worth 1 dollar at its exercise time a period hence, then $B_0 = E(\tilde{M}_1 B_1)$ or $B_0 = E(\tilde{M}_1)$. Since $B_0 = 1/(1+R_\mathrm{f})$, we obtain of course: $E(\tilde{M}_1) = 1/(1+R_\mathrm{f})$, providing thereby a relationship between the expected value of the kernel and the risk-free discount rate. An additional and particularly interesting case consists of using the linear CAPM model for pricing risky assets. In this case, let the kernel be

$$M_{t+1} = a_t + b_t R_{M,t+1},$$

where $R_{M,t}$ is the rate of return of the market portfolio (a market index for example). For a given stock, whose rate of return is $1 + R_{t+1} = p_{t+1}/p_t$, we have as stated earlier:

$$\begin{aligned}&1 = E\left[M_{t+1}(1+R_{t+1})\right], \quad \text{hence}\\ &E(1+R_{t+1}) = \frac{1}{E(M_{t+1})} - \frac{\mathrm{cov}\,(M_{t+1}, 1+R_{t+1})}{E(M_{t+1})}.\end{aligned}$$

Inserting the linear model for the kernel we have

$$\begin{aligned}&E(1+R_{t+1})\\ &= (1+R_{\mathrm{f},t})\left[1 - \mathrm{cov}\,(M_{t+1}, 1+R_{t+1})\right]\\ &= (1+R_{\mathrm{f},t+1})\left[1 - \mathrm{cov}\left(a + bR_{M,t+1}, 1+R_{t+1}\right)\right]\end{aligned}$$

which is reduced to

$$\begin{aligned}&E(R_{t+1} - R_{\mathrm{f},t+1})\\ &= \frac{\mathrm{cov}\left(R_{M,t+1} - R_{\mathrm{f},t+1}, R_{t+1} - R_{\mathrm{f},t+1}\right)}{\mathrm{var}\left(R_{M,t+1} - R_{\mathrm{f},t+1}\right)}\\ &\quad \times E_t\left(R_{M,t+1} - R_{\mathrm{f},t+1}\right).\end{aligned}$$

This can be written in the CAPM standard formulation (see also [47.61–63]):

$$E(R_{t+1} - R_{f,t+1}) = \beta E_t\left(R_{M,t+1} - R_{f,t+1}\right).$$

where the beta parameter is:

$$\beta = \frac{\operatorname{cov}\left(R_{M,t+1} - R_{f,t+1},\, R_{t+1} - R_{f,t+1}\right)}{\operatorname{var}\left(R_{M,t+1} - R_{f,t+1}\right)} .$$

Of course when the returns are normally distributed such calculations are straightforward and can be generalized further. *Stein* [47.64] has shown that, if market returns are some function $f(y)$, $y = R_{M,t+1} - R_{f,t+1}$ and, if the derivative $f'(.)$ exists, then

$$\operatorname{cov}[x,\, f(y)] = E\left[f'(y)\right]\operatorname{cov}(x,\, y) .$$

And as a result, the beta parameter is

$$\begin{aligned}\beta &= \frac{\operatorname{cov}\left[f(R_{M+1} - R_{f,t+1}),\, R_{t+1} - R_{f,t+1}\right]}{\operatorname{var}\left(R_{M+1} - R_{f,t+1}\right)} \\ &= \frac{E\left[f'(R_{M+1} - R_{f,t+1})\right]}{\operatorname{var}\left(R_{M+1} - R_{f,t+1}\right)} \\ &\quad \times \operatorname{cov}\left[f'(R_{M+1} - R_{f,t+1}),\, R_{t+1} - R_{f,t+1}\right] .\end{aligned}$$

For example, in a stochastic inflation world, *Roll* [47.65] extends the CAPM of Sharpe by setting:

$$E(RP) = R_f E(P) + \beta E\left(R_M P - R_f P\right) ,$$

where P is a stochastic purchasing power with

$$\beta = \frac{\operatorname{cov}\left(RP,\, R_M P\right)}{\operatorname{var}\left(R_M P\right)} .$$

The hypothesis that the kernel is linear may be limiting however. Recent studies have suggested that we use a quadratic measurement of risk with a kernel given by:

$$M_{t+1} = a_t + b_t R_{M,t+1} + c_t R^2_{M,t+1} .$$

In this case, the skewness of the distribution enters as well in determining the value of the asset. There is ongoing empirical research on this and related topics.

47.3.2 Euler's Equation and the SDF

The CCAPM model, in its inter-temporal framework can be formulated as a problem in the calculus of variations and the SDF determined by applying the Euler condition for optimal consumption utility [47.1]. Explicitly, let an investor maximize the expected utility of consumption over a horizon $[0,\, T]$:

$$V_t = \max \sum_{j=0}^{T-1} \beta^j u(c_{t+j}) + \beta^T G(S_T) ,$$

where $u(c_{t+j})$ is the utility of consumption c_{t+j} at time $t+j$, T is the final time and $G(S_T)$ expresses the terminal utility of the wealth state at time T. The investor's discount rate is β. At time t, the investor's wealth is given by $S_t = S_{t-1} - q_t c_t + R_t$, where consumption is priced q_t, while returns from investments are R_t. As a result:

$$c_{t+j} = \frac{R_{t+j} - \Delta S_{t+j}}{q_{t+j}} , \quad \Delta S_{t+j} = S_{t+j} - S_{t+j-1} .$$

The investor's utility is therefore:

$$V_t = \max \sum_{j=0}^{T-1} \beta^j u\left(\frac{R_{t+j} - \Delta S_{t+j}}{q_{t+j}}\right) + \beta^T G(S_T) .$$

Application of Euler's equation (the calculus of variations) yields:

$$\frac{\partial V_t}{\partial S_{t+j}} - \Delta\left(\frac{\partial V_t}{\partial \Delta S_{t+j}}\right) = 0 .$$

Since $\partial V_t / \partial S_{t+j} = 0$, $\Delta\left(\partial V_t / \partial \Delta S_{t+j}\right) = 0$ and therefore we have the following *equilibrium* results:

$$\frac{\partial V_t}{\partial \Delta S_{t+j}} = \frac{\beta^j}{q_{t+j}} \frac{\partial u\left(c_{t+j}\right)}{\partial \Delta S_{t+j}} = A \text{ constant} .$$

For two consecutive instants of time ($j = 0$, $j = 1$):

$$\begin{aligned}&\frac{\partial V_t}{\partial \Delta S_t} = \frac{\partial V_t}{\partial \Delta S_{t+1}} \quad \text{and therefore} \\ &\frac{1}{q_t}\frac{\partial u\left(c_t\right)}{\partial \Delta S_t} = \frac{\beta}{q_{t+1}}\frac{\partial u\left(c_{t+1}\right)}{\partial \Delta S_{t+1}} \quad \text{and} \\ &\frac{\partial u\left(c_t\right)}{\partial \Delta S_t} = \beta E\left(\frac{q_t}{q_{t+1}}\frac{\partial u\left(c_{t+1}\right)}{\partial \Delta S_{t+1}}\right) .\end{aligned}$$

In other words, the marginal utility of wealth (savings) equals the discounted inflation-adjusted marginal utilities of consumption. In particular, if wealth is invested in a portfolio of assets such that:

$$\begin{aligned}&\Delta S_t = (N_t - N_{t-1})p_t = p_t \Delta N_t \quad \text{and} \\ &\frac{\partial V_t}{\partial \Delta S_{t+1}} = p_t \frac{\partial u\left(c_{t+1}\right)}{\partial \Delta N_{t+1}} \quad \text{then} \\ &\frac{\partial u\left(c_t\right)}{\partial \Delta N_t} p_{t-1} = E\left[\beta \frac{q_t}{q_{t+1}}\left(\frac{\partial u\left(c_{t+1}\right)}{\partial \Delta N_{t+1}} p_t\right)\right] ,\end{aligned}$$

which is reduced to the previous condition in two periods, or

$$\begin{aligned}&p_{t-1} = E\left(\beta \frac{q_t}{q_{t+1}} \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} p_t\right) ; \\ &u'\left(c_{t+1}\right) = \frac{\partial u\left(c_{t+1}\right)}{\partial \Delta N_{t+1}} .\end{aligned}$$

We can write this expression in terms of the stochastic factor M_t, expressing again the *consumption impatience*

$$p_{t-1} = E\left(M_t p_t\right); \quad M_t = \beta \frac{q_t}{q_{t+1}} \frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)} .$$

Again, if we set $1+R_t = p_t/p_{t-1}$, this equation can also be written as follows:

$$1 = E\left(M_t \frac{p_t}{p_{t-1}}\right), \quad \text{hence}$$
$$1 = E\left\{M_t(1+R_t)\,|\Phi_t\right\} ,$$

which is the standard form of the SDF equation while Φ_t is a filtration at time t. Here too, we see that to price a default-free zero-coupon bond paying one dollar for sure at time 1, then applying the known risk-free discount rate R_f, we have

$$\frac{1}{1+R_\mathrm{f}} = (1)E\left(M_t\right). \quad \text{And therefore ,}$$
$$E\left(M_t\right) = \frac{1}{1+R_\mathrm{f}} \quad \text{and finally}$$
$$p_{t-1} = \frac{1}{1+R_\mathrm{f}} E_t^*\left(p_t\right) ,$$

where E_t^* assumes the role of a modified (subjective) probability distribution. When the utility function is assumed known, some simplifications can be reached. For example, for $u(.) = \ln(.)$, $u'\big(c_t\big) = 1/c_t$ and therefore $v_t = E\big\{\beta(q_t/q_{t+1})v_{t+1}\big\}$, $v_t = p_{t-1}/c_t$.

47.4 Bonds and Fixed-Income Pricing

The financial valuation of assets, real or financial, deals with streams of cash such as dividends, coupon payments, investment in engineering projects etc. which occur in a random manner or not, paid in at deterministic or random times. In some cases, there may be a default in such payments due to delays, lost and partially recuperated payments etc. For example, investing in a portfolio might result in future returns and dividends that are at best defined in terms of random cash flows. Traditionally, a number of techniques were applied to value such cash streams, spanning a broad set of subjective techniques such as: payback, internal rate of return (IRR), cost–benefit analysis (CBA), net present value (NPV) etc. Bonds pricing is often used to value these cash flows. Here we shall see how bonds, whether risk-free, rated or default-prone, are used to value these cash flows. The simplest bond is the zero-coupon risk-free bond paying 1 dollar a year from now. An investor can have an individual valuation of such a payment, say $B_\mathrm{IND} = 1(1+r)^{-1}$, in which case r represents the discount factor that the investor is willing to associate to such a payment. Buying such a bond is an investment in a risk-free payment which cannot earn anything else but the risk-free rate (otherwise there would be arbitrage). Say that the *market price* for such a bond is currently quoted at \$ 0.90. In this case, the discount rate that the market associates to this bond would be:

$$B_\mathrm{IND} = 0.90 = \frac{100}{(1+R_\mathrm{f})} \quad \text{or}$$
$$R_\mathrm{f} = \frac{1.0}{0.90} - 1 = 0.1111 ,$$

where R_f is used to denote the fact that this is a risk-free rate (since the bond payment has no risk). These rates are usually specified by US government bonds when they are assumed to be risk-less (a currency trader might not think this is the case, however). The price of this bond is usually specified by a function $B_\mathrm{f}(t, T)$, which is the price at time t for a bond paying 1 dollar for sure at time T when the going risk-free rate $R_\mathrm{f}(t, T)$ expresses the time structure of interest rates. For a stream of payments, say a project defined by a set of payments and returns in the future, corporate firms may use a discount rate r for the time value of money. In this case, the present value of such a project at the initial time $t = 0$, NPV(0), is written as follows:

$$\mathrm{NPV}(0) = -\sum_{i=0}^{n} \frac{I_i}{(1+r)^i} + \sum_{i=0}^{N} \frac{C_i}{(1+r)^i} ,$$

where I_i denotes the investment (or costs sustained at time i, while C_i is a certain (risk-free) cash flow generated by the project. If $\mathrm{NPV}(0) = 0$ then the solution of this equation yields the IRR, the corporate entity uses to rank and value investment projects. There are many problems with this valuation however that open an opportunity for arbitrage by investment funds. For example, the investments and return (or the Is and Cs) may be random, potentially involving defaults, payments delays and so on. Further, the discount rate used might not reflect the cost of borrowing of the firm and its risk rating (potentially given by rating firms such as Moody's, Fitch, Standard and Poor, and their like). In addition, the discount rate does not reflect the time at which these payments occur (the term structure). As a result, such a valuation (pricing) is quite naïve and arbitrage on these firms can be used to provide the same cash flows at a lower price, thereby cashing in the difference. If payments are known for sure, a market-sensitive valuation would use the term structure risk-free discount rate $R_{\mathrm{f}}(0, i)$ for the payment i-periods hence, the value of such a cash stream would be:

$$\mathrm{NPV}_{\mathrm{f}}(0) = -\sum_{i=0}^{n} \frac{I_i}{[1+R_{\mathrm{f}}(0, i)]^i} + \sum_{i=0}^{n} \frac{C_i}{[1+R_{\mathrm{f}}(0, i)]^i} .$$

While at any time t, it is given by:

$$\mathrm{NPV}_{\mathrm{f}}(t) = -\sum_{i=t}^{n} \frac{I_i}{[1+R_{\mathrm{f}}(t, i)]^{i-t}} + \sum_{i=t}^{n} \frac{C_i}{[1+R_{\mathrm{f}}(t, i)]^{i-t}} .$$

This latter expression can of course be written in terms of zero-coupon risk-free bonds as follows:

$$\mathrm{NPV}_{\mathrm{f}}(t) = -\sum_{i=t}^{n} I_i B_{\mathrm{f}}(t, i) + \sum_{i=t}^{n} C_i B_{\mathrm{f}}(t, i) .$$

If all pure bonds are priced by the market then of course the NPV is determined by the market. In practice however, pure bond prices are available for only a given subset of times and therefore the NPV has to be priced in some other manner. Subsequently, we shall see that this leads to an important technical problem in financial engineering – one of calculating the yield of the bond (or any portfolio).

Next, say that payments are made by a firm rated k, in which case, the project NPV can be written as follows:

$$\mathrm{NPV}_k(0) = -\sum_{i=0}^{n} \frac{I_i}{[1+R_{\mathrm{f}}(0, i)]^i} + \sum_{i=0}^{N} \frac{\hat{C}_i}{[1+R_k(0, i)]^i}, \quad \hat{C}_i = E\left(\tilde{C}_i\right) ,$$

where $R_k(0, i)$ is the discount rate applied for expected receipts $\hat{C}_i$, i periods hence for a firm whose risk notation is k (for example, AAA, BB, B+, C etc.). Note that in the above expression, we have maintained the payments I_i deterministic and therefore they ought to be discounted at the risk-free rate in effect at time $t = 0$ for time i (if this is not the case, then of course, it will be necessary to select the appropriate discount rate as well). Note that $R_k(0, i)$ expresses the k-rated firm's term structure used for discounting its future returns which includes the risk premium $\Delta_k(0, i)$ in the firm's cash flows, or $R_k(0, i) = R_{\mathrm{f}}(0, i) + \Delta_k(0, i)$. For example, say that the firm is currently rated k. The implication of such a rating is that an obligation of the firm to pay in i periods 1 dollar is currently priced by the market at $B_k(0, i)$. As a result, the set of future prospective investment returns of the firm can be priced by:

$$\mathrm{NPV}_k(0, n) = -\sum_{i=0}^{n} I_i B_{\mathrm{f}}(0, i) + \sum_{i=0}^{n} \hat{C}_i B_k(0, i) ,$$

and at time t,

$$\mathrm{NPV}_k(t, n) = -\sum_{i=t}^{n} I_i B_{\mathrm{f}}(t, i) + \sum_{i=t}^{n} \hat{C}_i B_k(t, i) .$$

This NPV includes of course the discount rate that the market applies to the firm's obligations. For fixed and secured payments the firm is obliged to use the risk-free rate, or equivalently it equals a risk-free coupon bond paying one dollar i-periods hence and denoted by $B_{\mathrm{f}}(0, i)$. When we use the same discount rate for certain payments and uncertain (valued at expectation) costs and returns, the traditional approach may overestimate (or underestimate) the net present value of the investment. For example, a firm which is highly rated may be tempted to borrow more money because it is cheaper than say another firm perceived as risky. By the same token, investment in some projects (ports, highways etc.) may be less expensive when they are performed by a government, that can tax its citizen to repay a loan taken to build such a project, than say, a firm who would invest to self-build the project. Of course, it is for these reasons that private investors in national projects require some government assurance and insurance to reduce the risk (and thereby the risk premium) which they have to pay for building such projects.

The approach outlined above can be generalized further and applied to value all kinds of assets; we consider some examples. Again let $\mathrm{NPV}_k(t)$ be the net present value of an investment project at time t when the firm is rated k. Such a firm may however switch to being rated j with probability p_{kj} a period of time (year) later. These probabilities define a Markov chain $\boldsymbol{P} = [p_{kj}]$, usually specified by rating firms (Moody's, Standard and Poor, Fitch etc.). As a result, over two consecutive periods, we have

$$\mathrm{NPV}_k(t) = -I_t + C_t + B_k(t, t+1) \times \sum_{\ell=1}^{m} p_{k\ell} \mathrm{NPV}_\ell(t+1)$$

and at the final time n, the NPV is given as a function of the rating (risk) state the firm will be in and specified by $\mathrm{NPV}_\ell(n)$. Further, note that due to the potential (or non-)transition of the firm's rating, the price of the bond may be altered over time as well. Explicitly, the price $B_k(t, T)$ of a coupon paying bond $c_{k,t}$ at time t and l_k dollars at maturity when it is rated k equals the expected present value of the bond in the next period, discounted at a rate associated with its rating in the next period. This is given by the following recurrence equation

$$B_k(t, T) = c_{k,t} + \sum_{j=1}^{m} \frac{p_{kj}}{1+R_{jt}} B_j(t+1, T);$$
$$B_k(T, T) = 1_k, \quad k = 1, 2, 3, \dots m .$$

However, note that for the NPV valuation, we used a zero-coupon bond paying one dollar in the next period, and therefore,

$$B_k(t, t+1) = \sum_{j=1}^{m} \frac{p_{kj}}{1+R_{jt}} B_j(t+1, t+1) = \sum_{j=1}^{m} \frac{p_{kj}}{1+R_{jt}} 1_j .$$

For example, if the bond pays one dollar in all circumstances, except if it is rated m, in which case it pays nothing, then

$$B_k(t, t+1) = \sum_{j=1}^{m-1} \frac{p_{kj}}{1+R_{jt}} 1 = \sum_{j=1}^{m-1} \frac{p_{kj}}{1+R_{jt}} .$$

Now assume that at some future time t we have the option to stop the bond payments at a price of say $-Q_t$. In other words, the actual net present value at time t with an option to stop at this time would be:

$$\mathrm{NPV}_k^{(o)}(t) = \max\left[-Q_t, \mathrm{NPV}_k(t)\right] ,$$
$$\mathrm{NPV}_k^{(o)}(T) = \mathrm{NPV}_k(T)$$

and Q_t is the cost associated with implementing the option (for example, the cost to the firm of discontinuing a service, etc.). Note that at the final time, the option is worthless if the project has been terminated. A stopping time (at which the option is exercised) occurs thus at time τ when $-Q_t \geq \mathrm{NPV}_k(\tau)$. A number of situations may arise then. For example, for a firm that is down-rated, the cost of borrowing would increase and it might lead it to decide to exercise the option because of its cost in capital. And vice versa, a firm that is up-rated and is trapped in a costly investment might decide to either stop it or refinance it to reduce its cost or have the current cash flow of the project to be more in concordance with its improved rating. The price of such a *real* option can be valued by noting that, if it is exercised at time $t+1$, when the firm is rated j, resulting in a savings of $-Q_{t+1} - \mathrm{NPV}_j(t+1)$, with ($\mathrm{NPV}_j < 0$), this saving is worth today

$$\mathrm{NPV}_k^{(1)}(t) = B_k(t, t+1) \sum_{\ell=1}^{m} p_{k\ell} \times \max\left(-Q_{t+1} - \mathrm{NPV}_j(t+1), 0\right)$$
$$\mathrm{NPV}_k^{(1)}(\tau) = -Q_\tau - \mathrm{NPV}_j(\tau), \quad \tau \leq T .$$

These equations are of course meaningful only when the discount rate associated to a given rating is specified. If this is the case, then of course, our equations are simple to calculate. A potential for arbitrage may thus occur when these discount rates are not appropriately specified. Later on, we shall be concerned with the determination of these rates based on the construction of a bonds portfolio of various ratings. Examples that demonstrate how calculations are performed will also be used.

Finally, it is worth noting that, when a stream of payments are random, given by $\tilde{C}_i$, and subjectively valued by an investor whose utility of money $u(.)$ is known, then we can calculate the certainty equivalent CE_i of the uncertain payment, in which case: $u\big(CE_i\big) = Eu\big(\tilde{C}_i\big)$ and $CE_i = u^{-1}\big[Eu\big(\tilde{C}_i\big)\big]$. The NPV can then be calculated by applying the risk-free rate:

$$\mathrm{NPV}_\mathrm{f}(0) = -\sum_{i=0}^{n} \frac{I_i}{[1+R_\mathrm{f}(0,i)]^i} + \sum_{i=0}^{n} \frac{u^{-1}\left[Eu\left(\tilde{C}_i\right)\right]}{[1+R_\mathrm{f}(0,i)]^i} .$$

Unfortunately, this valuation is also subjective for it is based on a utility function which might not be available. Alternatively, a market valuation, can be used when the price of risk is known, or we establish some mechanism for appropriately accounting for the risk implied in an uncertain cash flow. This is done by calculating the yield of a bond. There are numerous techniques, inspired both theoretically and numerically, that are applied to calculate the yield. Such a problem is the topic of commercial and theoretical research. Nonetheless we shall consider a number of approaches to calculating the yield because of its importance in financial engineering.

47.4.1 Calculating the Yield of a Bond

The yield of a bond is the discount rate y_T applied to holding the bond for T periods. This yield is often difficult to calculate because data pertaining to the term structure of zero-coupon bonds is simply not available, or available only for certain periods. For example, say that we have a bond at time whose exercise price occurs at time t, or $B(0, t) = B(t)$. To each time t, we associate the rate $y(t)$ and therefore the bond price is $B(t) = \exp\left[-y(t)t\right]$. The function $y(t)$ is called the yield curve. Of course, if zero-coupon bonds are price for time $t = 1$, we then have $y(1) = R_\mathrm{f}$, which is the current spot rate. However, if there are no zero-coupon bonds for $t = 6$, the yield for such a bond can only be inferred by some numerical or estimation technique. In other words, unless a zero-coupon bond exists for every maturity for which the discount factor is desired, some estimation technique will be needed to produce a discount factor for any *off-maturity* time. Since zero-coupon bonds are available for only some and other maturities, a lack of liquidity may prevent the determination of the true bond yield [47.66] (see also [47.67–69]. The *yield engineering* problem consists then in determining some technique and appropriate sources of information to estimate the yield for all maturities (also called the yield curve). Approaches to this problems are of course varied. *Nelson* and *Siegel* [47.70] for example suggest the following four-parameter equation, which can be estimated numerically by fitting to the appropriate data

$$\begin{aligned} y(t) = a_0 + (a_1 + a_2) \left(\frac{1 - \exp(-a_3 t)}{a_3} \right) \\ - a_2 \exp(-a_3 t) \; ; \end{aligned}$$

$y(t)$ is the spot rate while a_0, a_1, a_2, a_3 are the model's parameters (for related studies and alternative models see [47.66, 71–79], www.episolutions.com) suggest however a zero curve solution, which uses a combination of liquid securities, both zero-coupon and coupon-bearing bonds for which prices are readily available, and consisting of an application of bootstrapping techniques to calculate the yield curve. Explicitly, the Wets approach is based on an approximation, and in this sense it shares properties with purely spline methods. It is based upon a Taylor series approximation of the discount function in integral form. Some prevalent methods for computing (extracting) the zeros, curve-fitting procedures, and equating the yield curve to observed data in central banks include, among others: in Canada the use of the Svensson procedure and David Bolder (Bank of Canada), in Finland the Nelson–Siegel procedure, in France the Nelson–Siegel, Svensson procedures, while in Japan and the USA, the banks use smoothing splines etc. (see [47.71–73, 80]). A critical appreciation of the zero-curve approach is provided by [47.81] and [47.80] (essentially based on the structural form of the polynomial used in the episolutions approach). Other approaches span numerical techniques, smoothing techniques, [47.82], (such as least-squares approaches as we shall see later on when introducing rated bonds), kernel smoothing (SDF) techniques etc. [47.83–86]. For example, consider the price $B(t, T - t)$ of a bond at time t whose maturity is at time T. The next-period price of the bond is in fact unknown (depending on numerous factors including random interest rates). Applying the SDF approach, we can state that

$$B(t, T - t) = E_t\left[M_{t+1} B(t + 1, T - t - 1)\right] ,$$

where M_{t+1} is the pricing kernel. Rearrange this term as follows

$$1 = E_t\left(M_{t+1} \frac{B(t+1, T-t-1)}{B(t, T-t)}\right) , \quad \text{where}$$
$$\frac{B(t+1, T-t-1)}{B(t, T-t)} = 1 + y_{t+1,T}$$

with $y_{t+1,T}$ denoting the yield of the bond whose maturity is at time T, at time $t + 1$. As a result,

$$\begin{aligned} 1 &= E_t\left[M_{t+1}\left(1 + y_{t+1,T}\right)\right] , \\ E_t\left[M_{t+1}\right] &= \frac{1}{1 + R_{t,\mathrm{f}}} . \end{aligned}$$

To calculate the yield some model is needed for both the kernel and of course the yield distributions. Since these variables are both random and dependent an appropriate model has to be constructed on the basis of which an empirical econometric verification can be reached. Alternatively, if information regarding the marginal distributions of the kernel and the returns is available, then we may also construct copulas to represent the statistical covariation effects of both the kernel and the returns. These are problems that require further research however.

47.4.2 Bonds and Risk-Neutral Pricing in Continuous Time

In many situations, we use stochastic models of interest rates to value bonds. Below we shall consider some examples and provide as well some general results for the valuation of such bonds. In practice however, it is extremely difficult to ensure that such models do indeed predict very well the evolution of interest rates and therefore there is a broad range of techniques for calculating the *yields* of various bonds – of both risk denomination and term structures. In continuous time, let $r(t)$ be the known spot interest rate. A risk-free bond paying one dollar at T with a compounded interest rate $r(t)$ is then given by:

$$B(0,t)=\exp\left[\int_0^t r(u)\,\mathrm{d}u\right], \quad B(T,T)=1\,.$$

The interest rate process may be deterministic or stochastic as stated above. Since bonds depend intimately on the interest rate process, it is not surprising that much effort is devoted to constructing models that can replicate and predict reliably the evolution of interest rates, as one process values the other. There are many interest rate models however, each expressing an economic rationale for the evolution of interest rates. Generally, and mostly for convenience, an interest rates process $\{r(t), t \geq 0\}$ is represented by an Ito stochastic differential equation:

$$\mathrm{d}r=\mu(r,t)\,\mathrm{d}t+\sigma(r,t)\,\mathrm{d}w\,,$$

where μ and σ are the drift and the diffusion function of the process which may or may not be stationary. Various authors consider alternative models in their analysis [47.1, 87, 88]. The Vasicek model in particular provides a straightforward rationality for interest rates movements (also called the Ornstein–Uhlenbeck process). In other words, it states that the rate of change in interest rates fluctuates around a long-run rate α. This fluctuation is subjected to random and normal perturbations of mean zero and variance $\sigma\Delta t$, or

$$\mathrm{d}r=\beta(\alpha-r)\,\mathrm{d}t+\sigma\,\mathrm{d}w\,.$$

Without much difficulty, it can be shown (see also [47.48]) that this equation has a solution expressed in terms of the current interest rates and the model's parameters given by

$$r(\tau)=\alpha+[r(t)-\alpha]\,\mathrm{e}^{-\beta(\tau-t)} +\sigma\int_t^\tau \mathrm{e}^{-\beta[\tau-y]}\,\mathrm{d}w(y)\,.$$

The value of a bond with variable interest rates is thus:

$$\begin{aligned}B(t,\tau)&=E\exp\left[\int_t^\tau r(u)\,\mathrm{d}u\right]\\&=E\exp\left\{\int_t^\tau \alpha+[r(t)-\alpha]\,\mathrm{e}^{-\beta(u-t)}\right.\\&\left.\quad+\sigma\int_t^u \mathrm{e}^{-\beta[u-y]}\,\mathrm{d}w(y)\,\mathrm{d}u\right\}\,;\\B(T,T)&=1\end{aligned}$$

with $\mathrm{d}w(y)$ denoting the risk source (a normally distributed random variable of zero mean and variance $\mathrm{d}y$). Interest rates are therefore also normal with a mean and variance (volatility) evolution we can easily compute. In particular note that:

$$\begin{aligned}\ln B(t,\tau)&=\ln\left\{\int_t^\tau \alpha+[r(t)-\alpha]\,\mathrm{e}^{-\beta(u-t)}\,\mathrm{d}u\right\}\\&\quad+\ln E\exp\left[\int_t^\tau \sigma\int_t^u \mathrm{e}^{-\beta[u-y]}\,\mathrm{d}w(y)\,\mathrm{d}u\right]\end{aligned}$$

which can be written as a linear function in the current interest rate, or

$$\ln B(t,\tau)=A(t,\tau)r(t)+D(t,\tau)\,.$$

This is a general property called the affine property which, is found in some general Markov processes X in a state space $D\subset\mathbb{R}^d$. Namely, it states that the bond return is linear in the process X, or $R(x)=a_0+a_1X$. Explicitly, we have the characteristic function:

$$\begin{aligned}&E\left\{\mathrm{e}^{\mathrm{i}uX(t)}|X(s)\right\}\\&=\exp[(\varphi t-s,u)+\psi(t-s,u)X(s)]\,.\end{aligned}$$

The logarithm is of course a linear function with $a_0=\varphi(t-s,u)$ and $a_1=\psi(t-s,u)$ deterministic coefficients. *Duffie* et al. [47.89] show that for a time-homogeneous affine process X with a state space of the form $\mathbb{R}_+^n\times\mathbb{R}^{d-n}$, provided the coefficients $\varphi(\cdot)$ and $\psi(\cdot)$

of the characteristic function are differentiable and their derivatives are continuous at 0. The affine process X must a be a jump-diffusion process in that

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t + \mathrm{d}J_t$$

for standard Brownian motion W in $\mathbb{R}^d$ and a pure jump process J, with J affine dependent on X. A related property is of course

$$E_t\left(\mathrm{e}^{\int_t^s -R[x(u)]\,\mathrm{d}u + wX(s)}\right) = \mathrm{e}^{\alpha(s-t)+\beta(s-t)X(t)}\,,$$

where $\alpha(\cdot)$, $\beta(\cdot)$ satisfy a generalized Ricatti ordinary differential equation (with real boundary conditions). To see this property (in a specific case) consider the following example. Let interest rates be given by the following stochastic differential equation

$$\mathrm{d}r = \beta(\alpha - r)\,\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}w\,.$$

Application of Ito's differential rule to $B(t,r) = \exp\left[-\int_t^T r(u)\,\mathrm{d}u\right]$ yields

$$\frac{E\,\mathrm{d}\ln B(t,r)}{\mathrm{d}t} = -r - (T-t)\beta(\alpha - r) + \frac{1}{2}(T-t)^2\sigma^2 r\,,$$

which is clearly a linear function of the current interest rate. Elementary mathematical treatment will also show that the mean and the variance of the interest rates are given by

$$E\{r(t)\mid r_0\} = c(t)\left(\frac{4\beta\alpha}{\sigma^2} + \xi\right)\,;$$

$$\operatorname{var}\{r(t)\mid r_0\} = c(t)^2\left(\frac{8\beta\alpha}{\sigma^2} + 4\xi\right)\,, \quad \text{where}$$

$$c(t) = \frac{\sigma^2}{4\beta}\left(1 - \mathrm{e}^{-\beta t}\right)\,,$$

$$\xi = \frac{4r_0\beta}{\sigma^2[\exp(\beta t) - 1]}\,.$$

In this case, interest rates are not normal. Nonetheless the Laplace transform can be calculated and applied to price the bond as we have shown it above.

When interest rate models include stochastic volatility, the valuation of bonds is incomplete. Therefore, it is necessary to turn to appropriate mechanisms that can help us to price bonds. For example, denote by $V = \sigma^2(r,t)$, a *stochastic volatility model* consisting of two stochastic differential equations, with two sources of risk (W_1, W_2), which may be correlated or not. An example would be

$$\mathrm{d}r = \mu(r,t)\,\mathrm{d}t + \sqrt{V(r,t)}\,\mathrm{d}W_1\,;$$
$$\mathrm{d}V = \nu(V,r,t)\,\mathrm{d}t + \gamma(V,r)\,\mathrm{d}W_2\,,$$

where the variance V appears in both equations. Due to market incompleteness, there may be an infinite number of prices. A special case provided by *Hull* and *White* [47.90] is reproduced below. Note that the interest rate model is the square-root model we saw earlier. However, since the variance is subject to stochastic variations as well, it is modeled separately as a stochastic differential equation which is *mean–variance* reverting.

$$\frac{\mathrm{d}r}{r} = \mu\,\mathrm{d}t + \sqrt{V}\,\mathrm{d}W_1\,;$$
$$\mathrm{d}V = \alpha(\beta - V)\,\mathrm{d}t + \gamma r V^{\lambda}\,\mathrm{d}W_2\,.$$

In this case, when stock prices increase, volatility increases, while when volatility increases, interest rates (or the underlying asset we are modeling) increases also. These problems will be considered subsequently when we treat incomplete markets.

47.4.3 Term Structure and Interest Rates

If $r(t,T)$ is the interest rate applied at t for a payment at time T, then at $t+1$, the relevant rate for this period T would be $r(t+1,T)$. If these interest rates are not equal, there may be an opportunity for refinancing [47.91]. As a result, the evolution of interest rates for different maturity dates is important. Further, since bonds may have various maturities, the interest rates applied to value these bonds require necessarily that we assess the interest rates term structure. Below, we shall see how the term structure is implicit in bonds valuation. Say that an interest rate model for maturity at T is:

$$\mathrm{d}r(t,T) = \mu(r,T)\,\mathrm{d}t + \sigma(r,T)\,\mathrm{d}W\,.$$

A bond price with the same maturity is therefore a function of such interest rates, leading to:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)} = \alpha(r,t,T)\,\mathrm{d}t + \beta(r,t,T)\,\mathrm{d}W\,.$$

The parameters $\alpha(.)$ and $\beta(.)$ are easily found by application of Ito's lemmas to $B(t,T) = \exp\left[-r(t,T)(T-t)\right]$,

$$\mathrm{d}B(t,T) = \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu(r,T) + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,T)\right]\mathrm{d}t + \frac{\partial B}{\partial r}\sigma(r,T)\,\mathrm{d}w\,.$$

Equating these two bond price equations, we have:

$$\alpha(r,t,T)B = \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu(r,T) + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,T)\right];$$

$$\beta(r,t,T)B = \frac{\partial B}{\partial r}\sigma(r,T).$$

Now assume that the risk premium is proportional to their returns standard deviation and let the price of risk be a known function of r and time t:

$$\alpha(r,t,T) = r + \lambda(r,t)\frac{1}{B}\frac{\partial B}{\partial r}.$$

Inserted into the bond equation derived above, this leads to

$$rB + \lambda(r,t)\frac{\partial B}{\partial r} = \left[\frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}\mu(r,T) + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,T)\right]$$

and finally to the partial differential equation:

$$0 = \frac{\partial B}{\partial t} + \frac{\partial B}{\partial r}[\mu(r,T) - \lambda(r,t)] + \frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,T) - rB; \quad B(r,T,T) = 1.$$

The solution of this equation, although cumbersome, can be determined. For example, if we set the constants $\left[\mu(r,T) - \lambda(r,t)\right] = \theta$; $\sigma^2(r,T) = \rho^2$, then the following solution can be verified:

$$B(r,t,T) = \exp\left[-r(T-t) - \frac{1}{2}\theta(T-t)^2 + \frac{1}{6}\rho^2(T-t)^3\right].$$

In general these equations are difficult to solve analytically or numerically and require therefore a certain amount of mathematical and numerical ability. Alternatively, if we set

$$[\mu(r,T) - \lambda(r,t)] = k(\theta - r);\ \sigma^2(r,T) = \rho^2 r.$$

Then we can show that the solution for the bond value is of the affine structure form and therefore given by

$$\ln B(r,t,T) = A(T-t) + rD(T-t).$$

A solution for the function $A(.)$ and $D(.)$ can then be found by substitution.

47.4.4 Default Bonds

There are various models for default-prone bond, falling into one of two categories: structural models and reduced-form models [47.74, 92, 93]. Structural models of default specify a particular value process and assume that default occurs when the value falls below some explicit threshold (for example, default may occur when the debt-to-equity ratio crosses a given threshold). In this sense, default is a *stopping time* defined by the evolution of a representative stochastic process. These models determine both equity and debt prices in a self-consistent manner via arbitrage, or contingent-claims pricing. These models assume often that debt-holders get back a fraction of the face value of the debt, sometimes called the recovery ratio at default. Such an assumption is observed largely in practice with bondholders recovering 20–80% of their investment. This recovery ratio is known a priori, however, in their models. Structural models have a number of additional drawbacks. For example, they cannot incorporate credit-rating changes that occur frequently for default prone (risky) corporate debts. Many corporate bonds undergo credit downgrades by credit-rating agencies before they actually default, and bond prices react to these changes either in anticipation or when they occur. Thus, any valuation model should take into account the uncertainty associated with credit-rating changes as well as the uncertainty surrounding default.

Reduced-form models instead, specify the default process explicitly, interpreting it as an exogenously motivated jump process, usually given as a function of the firm value. This class of models has been investigated for example by *Jarrow* and *Turnbull* [47.92], *Jarrow*, *Lando* and *Turnbull* [47.93, 94], *Duffie* and *Singleton* [47.78], and others. Although these models are useful when fitting default to observed credit spreads, neglecting the underlying value process of the firm renders it less useful when it is necessary to determine credit-spread variations. There are numerous publications regarding default-prone bonds and therefore we only consider some classical and simple examples.

Example 47.3 (structural models): Longstaff–Schwartz [47.95] assume a risk-free interest rate two-factor model with interest rates given by a *Vacicek* [47.96] model. Let V_t and r_t be the time-t values of the firm's assets and the risk-free interest rate, respectively. The dynamics of these two factors is written in terms of the following equations

$$dV/V = (r - \delta)\,dt + \sigma_1\,dZ_1\,;$$
$$dr = (\alpha - \beta r)\,dt + \sigma_2\,dZ_2\,,$$

where δ, σ_1, α, β and σ_2 are constants, and Z_1 and Z_2, two standard Brownian motion processes with constant correlation coefficient ρ.

In their model, default occurs when the value of the firm declines to a pre-specified boundary (with the par value of the bond – the face amount due on the maturity date – taken as the boundary). As a result, the default boundary is specified exogenously. In the event of default, bondholders recover a constant fraction of the par value of the bond. In the Longstaff–Schwartz model, a risky coupon bond is valued as a simple portfolio of a risky zero-coupon bond whose value for a \$ 1 face value is given by

$$P(V_t\,|F, r_t, T) = D(r_t, T)\,[1 - (1 - w)Q(V_t\,|F, r_t, T)]\ ,$$

where $D(.)$ denotes the value of a default-free discount bond given by the *Vacicek* [47.96] model, $Q(.)$ represents the forward default probability while w is the recovery rate.

Example 47.4 (reduced-form models): Failure of structural models to adequately price risky bonds found in the marketplace led to another approach based on reduced-form models of default risk. These make no attempt to define default as an endogenous event (arising from a low level of firm value or cash flow), but rather specify default as an exogenous event and thus do not explicitly incorporate any relationship between leverage and firm value into the model. These models, such as those of *Duffie* and *Singleton* [47.78], *Jarrow* and *Turnbull* [47.92], and *Jarrow*, *Lando* and *Turnbull* [47.93], are based on parameters that can be estimated with readily available data, such as default rates or bond spreads. The model of *Jarrow*, *Lando*, and *Turnbull* ([47.93], for example) assumes that the value of a default-free zero-coupon bond is known at time t. This bond will mature at time T and pay one dollar on maturity. $p(t, T)$ is the value of this bond. If $v_i(t, T)$ denotes the value of a defaultable zero-coupon bond of a firm that currently has credit rating i (for example, AAA) at time t, will mature at time T, and has a promised payoff of \$ 1 at maturity, then Jarrow, Lando, and Turnbull show that:

$$v_i(t, T) = p(t, T)[\phi + (1 - \phi)q_i(t, T)]\ ,$$

where ϕ is the recovery ratio, the fraction of the face value (\$ 1) that is recovered at time T after default, and $q_i(t, T)$ denotes the probability of a default occurring after T given that the debt has credit rating i as of time t. To arrive at the valuation formula, *Jarrow*, *Lando*, and *Turnbull* [47.93] assume that default is independent of the level of interest rates. However, this assumption is not critical. The independence assumption can be relaxed so that the model of *Jarrow* et al. extended so that the default relaxes the independence assumption and extends the model of Jarrow et al. so that the default probability can depend on the level of interest rates.

47.5 Options

Options are instruments that give the buyer of the option (the long side) the *right to exercise*, for a price, called the *premium*, the delivery of a commodity, a stock, a foreign currency etc. at a given price, called the *strike price*, at *(within)* a given time period, also called the *exercise date*. Such an option is called a *European* (*American) call* for the buyer. The seller of such an option (the short side), has by contrast the *obligation* to sell the option at the stated strike and exercise date. A *put* option (the long side) provides the option to sell, while for the short seller there is an obligation to buy. There are many types of options however and considerable research on the pricing of options (for example, see [47.24, 32, 97–106]. We shall consider in particular call and put options. Options are traded on many trading floors and mostly, they are defined in a standard manner. Nevertheless, there are also over-the-counter options, which are not traded in specific markets but are used in some contracts to fit specific needs. For example, there are Bermudan and Asian options. The former option provides the right to exercise the option at several specific dates during the option lifetime, while the latter defines the exercise price of the option as an average of the value attained over a certain time interval. Of course, each option, defined in a different way, will lead to alternative valuation formulas. There can be options on real assets, which are not traded but used to define a contract between two parties (real options). The valuation of options has attracted a huge amount of interest and for this reason it will also be a substantial issue we shall deal within this chapter.

47.5.1 Options Valuation and Martingales

When the underlying price process is a martingale and risk-neutral pricing of financial assets applies, then the price of a cash flow $\tilde{S}_n$ realized at time n is

$$S_0 = \frac{1}{(1+R_\mathrm{f})^n} E^*\left(\tilde{S}_n\right) .$$

Hence the forward price is:

$$V_0\,(1+R_\mathrm{f})^n = E^*\left(\tilde{S}_n\,|\Phi_0\right) ,$$

where Φ_0 is a filtration, representing the initial information on the basis of which the expectation is taken (under the risk-neutral distribution where expectation is denoted by $E^*(*)$). If K is the exercise price of a call option for exercise at some time T, then, the price C_t of such an option (as well as a broad variety of other options) under risk-neutral pricing is

$$C_t = \frac{1}{(1+R_\mathrm{f})^{T-t}} E^*\left[\max\left(K-\tilde{S}_T, 0\right)|\Phi_t\right] .$$

A simple example often used is the binomial option model. For simplicity, assume, a stock whose current price is 1 \$ and consider two asset state prices one period hence (H, L), $H > 1$, $L < 1$. Under risk-neutral pricing, then of course $1 = \frac{1}{1+R_\mathrm{f}}\left[pH+(1-p)L\right]$, where p is a risk-neutral probability. To determine this probability we construct a replicating portfolio for the call option whose state prices are $(C_H, C_L) = (H-K, 0)$, $H > K$, $L < K$. Let this portfolio consist of the stock and a risk-less zero-coupon bond paying one dollar one period hence and be given initially by $P = a + b$. One period hence, the portfolio state prices are necessarily $(P_H, P_L) = \left[aH + b(1+R_\mathrm{f}), aL + b(1+R_\mathrm{f})\right]$. It is a replicating portfolio if $(P_H, P_L) = (C_H, C_L)$. A solution of these replicating asset prices yields both a^* and b^* – the replicating portfolio composition. Since two assets with identical cash flows have the same price, the portfolio price and the call option ought, in complete markets, have the same price and therefore $C = P^* = a^* + b^*$, which provides the desired solution. Since the call option under risk-neutral pricing equals the discounted (at the risk-free rate) value of the call option at its exercise, or:

$$\begin{aligned} C &= \frac{1}{1+R_\mathrm{f}}\left[pC_H + (1-p)C_L\right] \\ &= \frac{1}{1+R_\mathrm{f}}\left[p\,(H-K)\,,(1-p)\,(0)\right] \end{aligned}$$

we can solve this equation and obtain the risk-neutral probability

$$\begin{aligned} p &= \frac{(1+R_\mathrm{f})\,(1)-L}{H-L} ; \\ q &= 1-p = \frac{H-(1+R_\mathrm{f})(1)}{H-L} . \end{aligned}$$

This analysis can be repeated for several periods. Explicitly, for an option whose exercise is at time n we obtain by induction

$$\begin{aligned} C &= \frac{1}{(1+R_\mathrm{f})^n} \\ &\quad \times \left[\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}(H^jL^{n-j}x-K)^+\right] , \\ x &= 1 . \end{aligned}$$

We can write this expression in still another form

$$\begin{aligned} C_n &= \frac{1}{(1+r)^n}E^*\left[(x_n-K)^+\right] , \quad \text{where} \\ P\left(x_n = H^jL^{n-j}x\right) &= \binom{n}{j}p^j(1-p)^{n-j} . \end{aligned}$$

47.5.2 The Black–Scholes Option Formula

In continuous time and continuous state, the pricing of Black–Scholes options are obtained in a similar manner, albeit using stochastic calculus. The traditional approach is based on the replication of the option value by the construction of a portfolio consisting of the underlying asset (the security) and a risk-free bond. Let $S(t)$ be a security-stock price at time t, distributed as a log-normal process and let V be the value of an asset derived from this stock, which we can write by the following function $V = f(S, t)$, assumed to be differentiable with respect to time and the security-stock $S(t)$. For simplicity, let the security price be given by a log-normal process:

$$\frac{\mathrm{d}S}{S} = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}W, \quad S(0) = S_0 ,$$

where $\left\{W(t), t \geq 0\right\}$, $W(0) = 0$ is a standard Brownian motion. Let P be a replicating portfolio consisting of bonds and investment in the given stock, $P = B + aS$ or $B = P - aS$, in which case the price of a risk-less bond and the price of a portfolio $P - aS$ is necessarily the same. A perfect hedge is thus constructed by setting: $\mathrm{d}B = \mathrm{d}P - a\,\mathrm{d}S$ where $\mathrm{d}B = R_\mathrm{f}B\,\mathrm{d}t$. Now, let $V = C = f(S, t)$ be the option price. Setting the replicating portfolio, we have $P = C$ and $\mathrm{d}P = \mathrm{d}C$, which is used to obtain a partial differential equation of the option price with appropriate boundary conditions and constraints, providing thereby the solution to the Black–Scholes option price. Each of these steps is translated into mathematical manipulations. First, note that:

$$\mathrm{d}B = \mathrm{d}f - a\,\mathrm{d}S = R_\mathrm{f} B\,\mathrm{d}t\,.$$

By an application of Ito's differential rule we obtain the option price:

$$\begin{aligned}\mathrm{d}C &= \mathrm{d}f\\ &= \left(\frac{\partial f}{\partial t} + \alpha S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t\\ &\quad + \left(\sigma S\frac{\partial f}{\partial S}\right)\mathrm{d}W\end{aligned}$$

and therefore,

$$\begin{aligned}&\left(\frac{\partial f}{\partial t} + \alpha S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}\right)\mathrm{d}t\\ &\quad + \left(\sigma S\frac{\partial f}{\partial S}\right)\mathrm{d}W - a\,\mathrm{d}S = R_\mathrm{f}(f - aS)\,\mathrm{d}t\,.\end{aligned}$$

Thereby,

$$\begin{aligned}&\left[\frac{\partial f}{\partial t} + \alpha S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2}\right.\\ &\quad \left. - aS\,(\alpha - R_\mathrm{f}) - R_\mathrm{f} f\right]\mathrm{d}t\\ &\quad + \sigma S\left(-a + \frac{\partial f}{\partial S}\right)\mathrm{d}W = 0\,,\end{aligned}$$

or

$$a = \frac{\partial f}{\partial S};\ \left(\frac{\partial f}{\partial t} + R_\mathrm{f} S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} - R_\mathrm{f} f\right) = 0$$

and finally

$$\begin{aligned}&-\frac{\partial f}{\partial t} = R_\mathrm{f} S\frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 f}{\partial S^2} - R_\mathrm{f} f;\quad f(0,t) = 0\,,\\ &\forall t \in [0,T]\,,\qquad f(S,T) = \max\left[0, S(T) - K\right]\,.\end{aligned}$$

The boundary conditions are specified by the fact that the option cannot be exercised until the exercise time (unlike an American option, as we shall see below) and therefore it is worthless until that time. At the exercise date T however, it equals $f(S,T) = \max[0, S(T) - K]$. The solution was shown by Black and Scholes to be

$$W = f(S,t) = S\Phi(d_1) - K\,\mathrm{e}^{-R_\mathrm{f} t}\Phi(\mathrm{d}_2)\,,$$

where

$$\begin{aligned}\Phi(y) &= (2\pi)^{-1/2}\int_{-\infty}^{y}\mathrm{e}^{-u^2/2}\,\mathrm{d}u\,;\\ d_1 &= \left(\frac{\log(S/K) + (T-t)(R_\mathrm{f} + \sigma^2/2)}{\sigma\sqrt{T-t}}\right);\\ d_2 &= d_1 - \sigma\sqrt{T-t}\,.\end{aligned}$$

This result is remarkably robust and holds under very broad price processes. Further, it can be estimated by simulation very simply. There are many computer programs that compute these option prices as well as their sensitivities. The price of a put option is calculated in a similar manner (see also [47.107, 108]).

47.5.3 Put–Call Parity

The put–call parity relationship establishes a relationship between the price of a put and that of a call. It can be derived by a simple arbitrage argument between two equivalent portfolios, yielding the same payoff regardless of the stock price. Their value must therefore be the same. To do so, construct the following two portfolios at time t:

Time t	Time T
	$S_T < K$
(1) $c + K\,\mathrm{e}^{-R_\mathrm{f}(T-t)}$	K
(2) $p + S_t$	$K = (K - S_T) + S_T$

Time t	Time T
	$S_T > K$
(1) $c + K\,\mathrm{e}^{-R_\mathrm{f}(T-t)}$	$(S_T - K) + K = S_T$
(2) $p + S_t$	S_T

We see that at time T, the two portfolios yield the same payoff $\max(S_T, X)$ which implies the same price at time t. Thus

$$c + K\,\mathrm{e}^{-R_\mathrm{f}(T-t)} = p + S_t\,.$$

If this is not the case then there would be some arbitrage opportunity. In this sense, computing European options prices is simplified, since knowing one leads necessarily to knowing the other.

When we consider dividend-paying options, the put–call parity relationships are slightly altered. Let D denote the present value of the dividend payments during the lifetime of the option (occurring at the time of its ex-dividend date), then:

$$c > S - D - K\,\mathrm{e}^{-R_\mathrm{f}(T-t)}\,,$$
$$p > D + K\,\mathrm{e}^{-R_\mathrm{f}(T-t)} - S\,.$$

Similarly, for put–call parity in a dividend-paying option, we have the following bounds

$$S - D - K < C - P < S - K\,\mathrm{e}^{-R_\mathrm{f}(T-t)}\,.$$

Put–call parity can be applied similarly between securities denominated in different currencies. For example, let α be the euro/dollar exchange rate (discounted at the dollar risk-free rate) and let $R_{\mathrm{f,E}}$ be the euro-area discount rate. Then, by put–call parity, we have

$$c + \frac{K}{1+R_{\mathrm{f,E}}} = p + \alpha\,,$$

which can be used as a regression equation to determine the actual exchange rate based on options data on currencies exchange.

47.5.4 American Options – A Put Option

American options, unlike European options, may be exercised prior to the expiration date. The price of such options is formulated in terms of stochastic dynamic programming arguments. As long as the option is *alive* we may either exercise it or maintain it, continuing to hold it. In a continuation region, the value of the option is larger than the value of its exercise and therefore, it is optimal to wait. In the exercise region, it is optimal to exercise the option and cash in the profits. If the time to the option's expiration date is t, then the exercise of the option provides a profit $K - S(t)$. In this latter case, the exercise time is a *stopping time*, and the problem is terminated. Another way to express such a statement using dynamic programming arguments is:

$$\begin{aligned} &f(S,t)\\ &= \max\left[K - S(t),\ \mathrm{e}^{-R_\mathrm{f}\,\mathrm{d}t} Ef(S+\mathrm{d}S, t+\mathrm{d}t)\right]\,, \end{aligned}$$

where $f(S,t)$ is the option price at time t when the underlying stock price is S and one of the two alternatives holds at equality. At the contracted strike time of the option, we have necessarily, $f(S,0) = K - S(0)$. The solution of the option's exercise time is difficult however and has generated a large number of studies seeking to solve the problem analytically or numerically. Noting that the solution is of barrier type, meaning that there is some barrier $X^*(t)$ that separates the exercise and continuation regions, we have

$$\begin{cases} \text{If } K - S(t) \geq X^*(t) & \text{exercise region:} \\ & \text{stopping time} \\ K - S(t) < X^*(t) & \text{continuation region}\,. \end{cases}$$

The solution of the American put problem consists then of selecting the optimal exercise barrier [47.109, 110]. A number of studies have attempted to do so, including [47.111] as well as many other authors. Although the analytical solutions of American put options are hard to reach, there are some problems that have been solved analytically. For most practical problems, numerical and simulation techniques are used.

Explicitly, assume that an American put option derived from a security is exercised at time $\tau < T$ where T is the option exercise period while the option exercise price is K. Let the underlying stock price be a risk-neutral process:

$$\frac{\mathrm{d}S(t)}{S(t)} = R_\mathrm{f}\,\mathrm{d}t + \sigma\,\mathrm{d}W(t),\ S(0) = S_0\,.$$

Under risk-neutral pricing, the value of the option equals the discounted value (at the risk-free rate) at the optimal exercise time $\tau^* < T$, namely:

$$J(S,T) = \max_{\tau \leq T} E_S\,\mathrm{e}^{-R_\mathrm{f}\tau}\,[K - S(\tau), 0]\,.$$

Thus,

$$\begin{aligned} &J(S,t)\\ &= \begin{cases} K - S(t) & \text{exercise region: stopping time} \\ \mathrm{e}^{-R_\mathrm{f}\,\mathrm{d}t} EJ(S+\mathrm{d}S, t+\mathrm{d}t) & \\ & \text{continuation region}\,. \end{cases} \end{aligned}$$

In the continuation region we have explicitly:

$$\begin{aligned} J(S,t) &= \mathrm{e}^{-R_\mathrm{f}\,\mathrm{d}t} EJ(S+\mathrm{d}S, t+\mathrm{d}t)\\ &\approx (1 - R_\mathrm{f}\,\mathrm{d}t)\\ &\quad\times E\left[J(S,t) + \frac{\partial J}{\partial t}\mathrm{d}t + \frac{\partial J}{\partial S}\mathrm{d}S + \frac{1}{2}\frac{\partial^2 J}{\partial S^2}(\mathrm{d}S)^2\right] \end{aligned}$$

which is reduced to the following partial differential equation

$$-\frac{\partial J}{\partial t} = -R_\mathrm{f} J(S,t) + \frac{\partial J}{\partial S} R_\mathrm{f} S + \frac{1}{2}\frac{\partial^2 J}{\partial S^2}\sigma^2 S^2\,,$$

while in the exercise region:

$$J(S,t) = K - S(t)\,.$$

For a perpetual option, note that the option price is not a function of time but of price only and therefore $\frac{\partial J}{\partial t}=0$ and the option price is given by an ordinary differential equation of second order

$$0=-R_{\mathrm{f}}J(S)+\frac{\mathrm{d}J}{\mathrm{d}S}R_{\mathrm{f}}S+\frac{1}{2}\frac{\mathrm{d}^2J}{\mathrm{d}S^2}\sigma^2S^2\,.$$

Assume that an interior solution exists, with an exercise at price S^*, $S(t)\le S^*$, $S^*\le K$. These specify the two boundary conditions required to solve our equation. In the exercise region $J(S^*)=K-S^*$, while for optimal exercise price $\frac{\mathrm{d}J(S)}{\mathrm{d}S}\big|_{S=S^*}=-1$. Let the solution be of the type $J(S)=qS^{-\lambda}$. This reduces the differential equation to an equation we solve for $\lambda:\sigma^2\frac{\lambda(\lambda+1)}{2}-\lambda R_{\mathrm{f}}-R_{\mathrm{f}}=0$ and $\lambda^*=2R_{\mathrm{f}}/\sigma^2$. At the exercise boundary S^* however: $J(S^*)=qS^{*-\lambda^*}=K-S^*$; $\mathrm{d}J(S^*)/\mathrm{d}S^*=-\lambda^*qS^{*-\lambda^*-1}=-1$. These two equations are solved for q and $S^*[0,1]$ leading to: $S^*=\lambda^*K/(1+\lambda^*)$ and $q=(\lambda^*)^{\lambda^*}K^{1+\lambda^*}/(1+\lambda^*)^{1+\lambda^*}$ and the option price is thus:

$$J(S)=\left(\frac{(\lambda^*)^{\lambda^*}K^{1+\lambda^*}}{(1+\lambda^*)^{1+\lambda^*}}\right)S^{-\lambda^*}\,,$$

$$\lambda^*=\frac{2R_{\mathrm{f}}}{\sigma^2},\qquad S^*=\frac{\lambda^*}{1+\lambda^*}K\,.$$

Thus the solution of the perpetual American put is explicitly given by:

$$\begin{cases}\text{Sell if} & S\le S^*\\ \text{Hold if} & S>S^*\,.\end{cases}$$

When the option time is finite, say T, this problem is much more difficult to solve however. Further, for an American call, it is easily demonstrated that it equals in fact the price of the European call.

In discrete time, a similar approach may be applied if risk-neutral pricing can be applied. For example, consider again the binomial option model considered earlier. The stock can assume at time n the following prices: $H^iL^{n-i}x-K, i=0,1,2,\dots n+1$, where p is the probability of the price increasing (and $1-p$, the probability that it decreases) and x is the initial price (at time $t=0$). The price of a put option with an option maturity at time n is then: $P_n(i)=\max(K-H^iL^{n-i}x,0)$. Suppose that at time t the put is exercised, then the profit is $P_t(i)$. Alternatively, say that the option is not sold at t. In this case, by risk-neutral pricing, the price of the option is

$$P^*_{t+1}(i)=\frac{1}{1+R_{\mathrm{f}}}\left[pP_{t+1}(i+1)+(1-p)P_{t+1}(i)\right]\,.$$

Thus, by the recurrence (Bellman) equation for this problem, we have:

$$P_t(i)=\max\left[K-H^iL^{n-i}x,\,P^*_{t+1}(i)\right]$$

with boundary condition

$$P_n(i)=\max\left(K-H^iL^{n-i}x,0\right)\,,$$

and a solution can be found by numerical techniques.

47.5.5 Departures from the Black–Scholes Equation

Any departure from the basic assumptions underlying the Black–Scholes model will necessarily alter the Black–Scholes (BS) solution. For example, if volatility is stochastic, if interest rates are stochastic, if stock prices are not log-normal, etc. the solution will not be necessarily a BS solution. For many cases however, it is possible to construct replicating portfolios and thereby remain within the assumptions that markets are complete. Below we shall consider a number of such cases to demonstrate how we might proceed in different manners. These approaches however, are based on a valuation based on risk-neutral pricing (for example, *Hull* [47.32], *Jarrow* and *Rudd* [47.112]).

The BS option price depends, of course, on the assumptions made regarding the underlying price process. Further, it depends essentially on the stock volatility, which cannot be observed directly. For this reason, the relationships between the option price and volatility have been taken to reflect one or the other. In other words, given the options price and other observables (interest rates, strike price, etc.), the implied volatility is that volatility that solves the BS price equation: $\hat{C}=W\left(.\middle|\sigma_{\mathrm{imp}}\right)$, where $\hat{C}$ is the current option price and $C=W\left(.\middle|\sigma\right)$ is the theoretical option price with an implied volatility $\sigma=\sigma_{\mathrm{imp}}$. Importantly, when the volatility is constant then σ_{imp} does not change as a function of T and K and it equals the true historical volatility. However, in practice when we calculate this implied volatility as a function of (T,K), we observe that there are some variations and therefore the BS model cannot be considered as the true market option price. Further, when the underlying price changes, the implied volatility can be a function of time as well and as a result, the implied volatility is a function $\sigma_{\mathrm{imp}}(t)$. When we consider the options price variations as a function of the strike K, we observe a volatility skew which is the well-known volatility smile. Skewness is smaller however for at-the-money options (in which case, the BS model is a good predictor of option price). The valuation of options in these circumstances is more difficult and there are, commensurably, numerous studies and extensions that calculate option prices. For example, [47.113, 114] consider transaction costs, [47.90, 115–117] consider option prices with stochastic volatility. *Nelson* and *Ramaswamy* [47.118] use discretized approximations and, even in physics, option pricing is considered as an application [47.119]. We consider below some well-known cases.

Option Valuation and Stochastic Volatility

When the underlying process has a stochastic volatility the replication of an option price by a portfolio requires special attention. We may proceed then by finding an additional asset to use (for example, another option with different maturity and strike price). Consider the following stochastic volatility process as an example

$$\begin{aligned} \mathrm{d}p/p &= \alpha\,\mathrm{d}t + \sqrt{V}\,\mathrm{d}w, \qquad && p(0) = p_0 \\ \mathrm{d}V/V &= \mu\,\mathrm{d}t + \xi\,\mathrm{d}z, && V(0) = v_0\,; \\ E(\mathrm{d}w\,\mathrm{d}z) &= \rho\,\mathrm{d}t\,, \end{aligned}$$

where (w, z) are two Brownian motions with correlation ρ. A call option would in this case be a function of both p and V, as we saw earlier for the BS option model, or $C(t, p, V)$. Application of Ito's lemma yields

$$\begin{aligned} \mathrm{d}C = &\left[\frac{\partial C}{\partial t} + \frac{\partial C}{\partial p}\alpha p + \frac{\partial C}{\partial V}\mu V + \frac{1}{2}\frac{\partial^2 C}{\partial p^2}(\sigma p)^2 \right. \\ &\left. + \frac{1}{2}\frac{\partial^2 C}{\partial V^2}(\xi V)^2 + \frac{\partial^2 C}{\partial p \partial V}(\mu V + \rho\xi\sigma p V)\right]\mathrm{d}t \\ &+ \frac{\partial C}{\partial p}(\sigma p\,\mathrm{d}W) + \frac{\partial C}{\partial V}(\xi V\,\mathrm{d}Z)\ . \end{aligned}$$

The first term in the brackets is the deterministic component while the remaining ones are stochastic terms that ought to be nullified by an appropriate portfolio (i. e. hedged) if we are to apply a risk-neutral framework. Since there are two sources of risk, we require two assets in addition to the underlying asset price. For this reason, we construct a replication portfolio by: $X = n_1 p + n_2 C_2 + B$, $B = (X - n_1 p - n_2 C_2)$, where n_1, n_2 are the number of stock shares and another option with different maturity. In this case, proceeding as we have for the BS model, we have $\mathrm{d}C_1 = \mathrm{d}X$ and therefore

$$\begin{aligned} &\mathrm{d}C_1 - n_1\,\mathrm{d}p - n_2\,\mathrm{d}C_2 = rB\,\mathrm{d}t \\ &= r(C_1 - n_1 p - n_2 C_2)\,\mathrm{d}t \quad \text{or} \\ &(\mathrm{d}C_1 - rC_1\,\mathrm{d}t) - n_1(\mathrm{d}p - rp\,\mathrm{d}t) \\ &- n_2(\mathrm{d}C_2 - rC_2\,\mathrm{d}t) = 0 \end{aligned}$$

which provides the equations needed to determine a hedging portfolio given by

$$\begin{aligned} \mathrm{d}\Phi_1 &= (\mathrm{d}C_1 - rC_1\,\mathrm{d}t)\,, \qquad && \text{hence} \quad \Phi_1 = \mathrm{e}^{-rt}C_1\,; \\ \mathrm{d}\Phi_2 &= (\mathrm{d}p - rp\,\mathrm{d}t)\,, && \text{hence} \quad \Phi_2 = \mathrm{e}^{-rt}p\,, \\ \mathrm{d}\Phi_3 &= (\mathrm{d}C_2 - rC_2\,\mathrm{d}t)\,, && \text{hence} \quad \Phi_2 = \mathrm{e}^{-rt}C_2\,; \\ \mathrm{d}\Phi_1 &= n_1\,\mathrm{d}\Phi_2 + n_2\,\mathrm{d}\Phi_3\,. \end{aligned}$$

As a result,

$$\mathrm{d}\left(\mathrm{e}^{-rt}C_1\right) = n_1\,\mathrm{d}\left(\mathrm{e}^{-rt}p\right) + n_2\,\mathrm{d}\left(\mathrm{e}^{-rt}C_2\right)\ .$$

Further,

$$\begin{aligned} &\frac{\mathrm{d}C_1}{C_1} = \mu_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1, \quad \text{with} \quad \lambda_1 = \frac{(\mu_1 - r)}{\sigma_1}\,, \\ &\text{hence} \quad \frac{\mathrm{d}C_1}{C_1} - r\,\mathrm{d}t = \sigma_1(\lambda_1\,\mathrm{d}t + \mathrm{d}W_1) = \sigma_1\,\mathrm{d}\tilde{W}_1 \end{aligned}$$

while $(\lambda_1\,\mathrm{d}t + \mathrm{d}W_1) = \mathrm{d}\tilde{W}_1$, is the risk-neutral measure. If we apply a CAPM risk valuation, we have then:

$$\begin{aligned} \frac{1}{\mathrm{d}t}E\left(\frac{\mathrm{d}C_1}{C_1} - r\right) &= \sigma_1\lambda_1 \\ &= \left[\left(R_p - r\right)\beta_{cp} + (R_V - r)\,\beta_{cV}\right]\,, \end{aligned}$$

where R_p is the stock mean return, $\beta_{cp} = \frac{p}{C_1}\frac{\partial C_1}{\partial p}\beta_p$ is the stock beta, R_V is the volatility drift while $\beta_{cV} = \frac{V}{C_1}\frac{\partial C_1}{\partial V}\beta_V$ is the beta due to volatility. We therefore obtain the following equations:

$$\begin{aligned} &\frac{1}{\mathrm{d}t}E\left(\frac{\mathrm{d}C_1}{C_1}\right) \\ &= r + \left[(\alpha - r)\,\frac{p}{C_1}\frac{\partial C_1}{\partial p}\beta_p + (\mu - r)\,\beta_V\frac{V}{C_1}\frac{\partial C_1}{\partial V}\right]\,; \\ &R_p = \alpha; \quad R_V = \mu, \quad \lambda_V = (\mu - r)\,V\beta_V\,, \end{aligned}$$

where λ_V is the risk premium associated with the volatility. Thus,

$$\frac{1}{\mathrm{d}t}E\left(\frac{\mathrm{d}C_1}{C_1}\right) = r + \left[(\alpha - r)\,\frac{p}{C_1}\frac{\partial C_1}{\partial p} + \frac{\lambda_V}{C_1}\frac{\partial C_1}{\partial V}\right]$$

which we equate to the option we are to value. Since,

$$\frac{1}{\mathrm{d}t}E\left(\frac{\mathrm{d}C}{C}\right) = r + \left[(\alpha - r)\,\frac{p}{C}\frac{\partial C}{\partial p} + \frac{\lambda_V}{C}\frac{\partial C}{\partial V}\right]$$

and obtain at last:

$$\begin{aligned} &\frac{1}{\mathrm{d}t}E\left(\frac{\mathrm{d}C}{C}\right) \\ &= \frac{1}{\mathrm{d}t}E\left[\frac{\partial C}{C\partial t}\mathrm{d}t + \frac{\partial C}{C\partial p}\mathrm{d}p + \frac{\partial C}{C\partial V}\mathrm{d}V \right. \\ &\quad + \frac{1}{2}\frac{\partial^2 C}{C\partial p^2}(\mathrm{d}p)^2 + \frac{1}{2}\frac{\partial^2 C}{C\partial V^2}(\mathrm{d}V)^2 \\ &\quad \left. + \frac{\partial^2 C}{C\partial p\partial V}(\mathrm{d}p\,\mathrm{d}V)\right] \end{aligned}$$

which leads to a partial differential equation we might be able to solve numerically. Or:

$$E\left[\frac{\partial C}{\partial t}\,\mathrm{d}t+\frac{\partial C}{\partial p}\,\mathrm{d}p+\frac{\partial C}{\partial V}\,\mathrm{d}V+\frac{1}{2}\frac{\partial^2 C}{\partial p^2}\,(\mathrm{d}p)^2 +\frac{1}{2}\frac{\partial^2 C}{\partial V^2}\,(\mathrm{d}V)^2+\frac{\partial^2 C}{\partial p\partial V}\,(\mathrm{d}p\,\mathrm{d}V)\right] =rC+\left[(\alpha-r)\,p\frac{\partial C}{\partial p}+\lambda_V\frac{\partial C}{\partial V}\right]$$

and explicitly,

$$\left[\frac{\partial C}{\partial t}+\frac{\partial C}{\partial p}rp+\frac{\partial C}{\partial V}\,(\mu V-\lambda_V)+\frac{1}{2}\frac{\partial^2 C}{\partial p^2}Vp^2 +\frac{1}{2}\frac{\partial^2 C}{\partial V^2}V^2\xi^2+\frac{\partial^2 C}{\partial p\partial V}pV^{3/2}\xi\rho-rC\right]=0\,,$$

where $\lambda_V=(\mu-r)V\beta_V$, as stated earlier. Of course the boundary constraints are then $C(T,\,p)=\max(p-K,0)$. The analytical treatment of such problems is clearly unlikely however (see also [47.120].

Options and Jump Processes [47.121]

The valuation of an option with a jump price process also involves two sources of risk, the diffusion and the jump. Merton considered such a problem for the following price process:

$$\frac{\mathrm{d}p}{p}=\alpha\,\mathrm{d}t+\sigma\,\mathrm{d}w+K\,\mathrm{d}Q\,,$$

where $\mathrm{d}Q$ is an adapted Poisson process with parameter $q\Delta t$. In other words, $Q(t+\Delta t)-Q(t)$ has a Poisson distribution function with mean $q\Delta t$ or for infinitesimal time intervals

$$\mathrm{d}Q=\begin{cases}1 & \text{w. p.}\quad q\,\mathrm{d}t\\ 0 & \text{w. p.}\quad (1-q)\,\mathrm{d}t\,.\end{cases}$$

Let $F=F(p,t)$ be the option price. When a jump occurs, the new option price is $F[p(1+K)]$. As a result,

$$\mathrm{d}F=\{F[p(1+K)]-F\}\,\mathrm{d}Q$$

when no jump occurs, we have

$$\mathrm{d}F=\frac{\partial F}{\partial t}\,\mathrm{d}t+\frac{\partial F}{\partial p}\,\mathrm{d}p+\frac{1}{2}\frac{\partial^2 F}{\partial p^2}(\mathrm{d}p)^2$$

and explicitly, letting $\tau=T-t$ be the remaining time to the exercise date, we have

$$\mathrm{d}F=\left(-\frac{\partial F}{\partial \tau}+\alpha p\frac{\partial F}{\partial p}+\frac{1}{2}p^2\sigma^2\frac{\partial^2 F}{\partial p^2}\right)\,\mathrm{d}t +p\sigma\frac{\partial F}{\partial p}\,\mathrm{d}w\,.$$

Combining these two equations, we obtain

$$\begin{aligned}&\mathrm{d}F=a\,\mathrm{d}t+b\,\mathrm{d}w+c\,\mathrm{d}Q\,,\\ &a=\left(-\frac{\partial F}{\partial \tau}+\alpha p\frac{\partial F}{\partial p}+\frac{1}{2}p^2\sigma^2\frac{\partial^2 F}{\partial p^2}\right)\,;\\ &b=p\sigma\frac{\partial F}{\partial p};\quad c=F[p(1+K)]-F\end{aligned}$$

with

$$E(\mathrm{d}F)=(a+qc)\,\mathrm{d}t\quad\text{since}\quad E(\mathrm{d}Q)=q\,\mathrm{d}t\,.$$

To eliminate the stochastic elements (and thereby the risks implied) in this equation, we shall construct a portfolio consisting of the option and a stock. To eliminate the *Wiener risk*, i.e. the effect of "$\mathrm{d}w$", we let the portfolio Z consist of a future contract whose price is p, for which a proportion v of stock options is sold (which will be calculated such that this risk disappears). In this case, the value of the portfolio is

$$\mathrm{d}Z=p\alpha\,\mathrm{d}t+p\sigma\,\mathrm{d}w+pK\,\mathrm{d}Q -(va\,\mathrm{d}t+vb\,\mathrm{d}w+vc\,\mathrm{d}Q)\,.$$

If we set $v=p\sigma/b$ and insert in the equation above (as done by Black–Scholes), then we will eliminate the Wiener risk since:

$$\mathrm{d}Z=p(\alpha-\sigma a/b)\,\mathrm{d}t+(p\sigma-vb)\,\mathrm{d}w +p(K-\sigma c/b)\,\mathrm{d}Q$$

or

$$\mathrm{d}Z=p(\alpha-\sigma a/b)\,\mathrm{d}t+p(K-\sigma c/b)\,\mathrm{d}Q\,.$$

In this case, if there is no jump, the evolution of the portfolio follows the differential equation

$$\mathrm{d}Z=p(\alpha-\sigma a/b)\,\mathrm{d}t\quad\text{if there is no jump}\,.$$

However, if there is a jump, then the portfolio evolution is

$$\mathrm{d}Z=p(\alpha-\sigma a/b)\,\mathrm{d}t+p(K-\sigma c/b)\,.$$

Since the jump probability equals $q\,\mathrm{d}t$, we obviously have

$$\frac{E(\mathrm{d}Z)}{\mathrm{d}t}=p(\alpha-\sigma a/b)+pq(K-\sigma c/b)\,.$$

There remains a risk in the portfolio due to the jump. To eliminate it we can construct another portfolio using an option F' (with exercise price E') and a future contract such that the terms in $\mathrm{d}Q$ are eliminated as well. Then, constructing a combination of the first (Z) portfolio and the second portfolio (Z'), both sources of uncertainty will be reduced. Applying the arbitrage argument (stating that there cannot be a return to a risk-less portfolio which is greater than the risk-less rate of return r) we obtain the proper proportions of the risk-less portfolio.

Alternatively, finance theory [and in particular, application of the capital asset pricing model (CAPM)] state that any risky portfolio has a rate of return in a small time interval $\mathrm{d}t$ which is equal the risk-less rate r plus a return premium for the risk assumed, which is proportional to its effect. Thus, using the CAPM we can write

$$E\frac{\mathrm{d}Z}{Z\,\mathrm{d}t}=r+\lambda\frac{p(K-\sigma c/b)}{Z}\,,$$

where λ is assumed to be a constant and expresses the market price for the risk associated with a jump. This equation can be analyzed further, leading to the following partial differential equation which remains to be solved (once the boundary conditions are specified):

$$-\frac{\partial F}{\partial \tau}+\left((\lambda-q)\left\{pK\frac{\partial F}{\partial p}-F[p(1+K)-F]\right\}\right)+\frac{1}{2}\frac{\partial^2 F}{\partial p^2}p^2\sigma^2-rF=0$$

with boundary condition

$$F(T)=\max\,[0,\,p(T)-E]\ .$$

Of course, for an American option, it is necessary to specify the right to exercise the option prior to its final exercise date, or

$$F(t)=\max\left[F^*(t),\,p(t)-E\right]\,,$$

where $F^*(t)$ is the value of the option which is not exercised at time t and given by the solution of the equation above. The solution of this equation is of course much more difficult than the Black–Scholes partial differential equation. Additional papers and extensions include for example, [47.122–124] as well as [47.125, 126].

Call Options on Bonds

Options on bonds are popular products traded in many financial markets. To value these options requires both an interest rate model and a term-structure bond price process. The latter is needed to construct the evolution over time of the underlying bond (say a T bond), which confers the right to exercise it at time $S<T$, in other words, the bond value at time S, whose value is given by an S-bond. To do so, we proceed in two steps: first we evaluate the term structure for a T and an S bond and then proceed to determine the value of a T bond at time S, which is used to replace the spot price at time S in the plain option model of Black–Scholes.

First we construct a hedging portfolio consisting of the two maturities S and T bonds ($S<T$). This portfolio will provide a synthetic rate, equated to the spot interest rate so that no arbitrage is possible. We denote by $k(t)$ this synthetic rate. For example, let the interest process

$$\mathrm{d}r=\mu(r,t)\,\mathrm{d}t+\sigma(r,t)\,\mathrm{d}W$$

and construct a portfolio of these two bonds, whose value is V, with:

$$\frac{\mathrm{d}V}{V}=n_S\frac{\mathrm{d}B(t,S)}{B(t,S)}+n_T\frac{\mathrm{d}B(t,T)}{B(t,T)}\,.$$

The T and S bond values are however, given by:

$$\frac{\mathrm{d}B(t,T)}{B(t,T)}=\alpha_T(r,t)\,\mathrm{d}t+\beta_T(r,t)\,\mathrm{d}W\,,$$

where as seen earlier in the previous section, the term structure is

$$\begin{aligned}&\alpha_T(r,t)\\&=\frac{1}{B(t,T)}\left[\frac{\partial B}{\partial t}+\frac{\partial B}{\partial r}\mu(r,T)+\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,T)\right];\\&\beta_T(r,t)=\frac{1}{B(t,T)}\frac{\partial B}{\partial r}\sigma(r,T)\,.\end{aligned}$$

Similarly, for an S-Bond,

$$\frac{\mathrm{d}B(t,S)}{B(t,S)}=\alpha_S(r,t)\,\mathrm{d}t+\beta_S(r,t)\,\mathrm{d}W$$

with

$$\begin{aligned}&\alpha_S(r,t)\\&=\frac{1}{B(t,S)}\left[\frac{\partial B}{\partial t}+\frac{\partial B}{\partial r}\mu(r,S)+\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\sigma^2(r,S)\right];\\&\beta_S(r,t)=\frac{1}{B(t,S)}\frac{\partial B}{\partial r}\sigma(r,S)\,.\end{aligned}$$

Replacing the terms for the mean rate of growth in the bond value and its diffusion, we have

$$\frac{\mathrm{d}V}{V}=(n_S\alpha_S+n_T\alpha_T)\,\mathrm{d}t+(n_S\beta_S+n_T\beta_T)\,\mathrm{d}W\,.$$

For a risk-less portfolio we require that the portfolio volatility be null. Further, since initially the portfolio was worth only one dollar, we obtain two equations in two unknowns (the portfolio composition), which we can solve

$$\begin{cases}n_S\beta_S+n_T\beta_T=0\\ n_S+n_T=1\end{cases}\Rightarrow\begin{cases}n_S=\frac{\beta_T}{\beta_T-\beta_S}\\ n_T=-\frac{\beta_S}{\beta_T-\beta_S}\,.\end{cases}$$

The risk-less portfolio thus has a rate of growth which we call the synthetic rate, or

$$\frac{\mathrm{d}V}{V}=\left(\frac{\beta_T\alpha_S-\beta_S\alpha_T}{\beta_T-\beta_S}\right)\mathrm{d}t=k(t)\mathrm{d}t\,.$$

This rate is equated to the spot rate as stated above, providing thereby the following equality:

$$\left(k(t)=\frac{\beta_T\alpha_S-\beta_S\alpha_T}{\beta_T-\beta_S}\right)\Rightarrow k(t)=r(t)\quad\text{or}$$
$$\frac{r(t)-\alpha_S}{\beta_S}=\frac{r(t)-\alpha_T}{\beta_T}=\lambda(t)$$

with $\lambda(t)$ the price of risk per unit volatility. Each bond with maturity T and S has at its exercise time a one dollar denomination, the value of each of these (S and T) bonds is given by

$$\begin{aligned}0&=\frac{\partial B_T}{\partial t}+\frac{\partial B_T}{\partial r}\left[\mu(r,T)-\lambda\beta_T\right]\\&\quad+\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\beta_T^2-rB_T\,;\\B(r,T)&=1\,,\\0&=\frac{\partial B_S}{\partial t}+\frac{\partial B_S}{\partial r}\left[\mu(r,S)-\lambda\beta_S\right]\\&\quad+\frac{1}{2}\frac{\partial^2 B}{\partial r^2}\beta_S^2-rB_S\,;\\B(r,S)&=1\,.\end{aligned}$$

Given a solution to these two equations, we define the option value of a call on a T bond with $S<T$ and strike price K, to be:

$$X=\max\left[B(S,T)-K,0\right]$$

where $B(S,T)$ is the price of the T bond at time S. $B(S,T)$ is of course found by solving for the term structure and then equating $B(r,S,T)=B(S,T)$. To simplify matters, say that the solution (valued at time t) for the T bond is given by $F(t,r,T)$, then at time S, this value is $F(S,r,T)$, to which we equate $B(S,T)$. In other words,

$$X=\max\left[F(S,r,T)-K,0\right]\,.$$

Now, if the option price is $P(.)$, then as we have seen in the plain vanilla model in the previous chapter, the value of the bond is found by solving for P in the following partial differential equation

$$\begin{aligned}0&=\frac{\partial P}{\partial t}+\mu(r,t)\frac{\partial P}{\partial r}+\sigma^2\frac{1}{2}\frac{\partial^2 P}{\partial r^2}-rP\,;\\P(S,r)&=\max\left[F(S,r,T)-K,0\right]\,.\end{aligned}$$

Although this might be a difficult problem to solve numerically, there are mathematical tools that allow the finding of such solution. A special case of interest consists of using the term structure model in the problem above, also called the affine term structure (ATS) model, which was indicated earlier in the previous section. In this case, we have:

$$F(t,r,T)=\mathrm{e}^{A(t,T)-rD(t,T)}\,,$$

where $A(.)$ and $D(.)$ are calculated by the term structure model while the option valuation model becomes

$$\begin{aligned}0&=\frac{\partial P}{\partial t}+\mu(r,t)\frac{\partial P}{\partial r}+\sigma^2\frac{1}{2}\frac{\partial^2 P}{\partial r^2}-rP\,;\\P(S,r)&=\max\left[\mathrm{e}^{A(S,T)-rD(S,T)}-K,0\right]\,.\end{aligned}$$

Again, these problems are mostly solved by numerical or simulation techniques.

47.6 Incomplete Markets and Implied Risk-Neutral Distributions

Markets are incomplete when we cannot generate any random cash flow by an appropriate portfolio strategy. The market is then deemed *not rich enough*. Technically, this may mean that the number of assets that make up a portfolio is smaller than the number of risk sources plus one. In the Arrow–Debreu framework seen earlier, this corresponds to rank condition $\boldsymbol{D}\equiv M$, providing a unique solution to the linear pricing equation. If markets are not complete or close to it, financial markets cannot uniquely value assets and there may be opportunities for arbitrage. In such circumstances, financial markets may be perceived as too risky, perhaps *chaotic* and therefore profits may be too volatile, the risk premium would then be too high and investment horizons smaller, thereby reducing investments. Finally, contingent claims may have an infinite number of prices (or equivalently an infinite number of martingale measures). As a result, valuation becomes, forcibly, utility based or based on some other mechanism, which is *subjective* rather than based on the market mechanism. *Ross* [47.104] has pointed out that

> *it is a truism that markets are not complete in the obvious sense that there exist contingencies that have no clearly associated market prices, but, it is not always immediately clear how meaningful this is*

for either pricing or efficiency. Some contingencies may have no markets but may be so trivial as to be insurable in the sense that their associated events are small and independent of the rest of the economy, others, may be replicable while not directly traded

Thus, even if markets are incomplete, we may be able to determine some mechanism which will still allow an approach to asset pricing. Of course, this will require the exact sense of the market incompleteness and determine a procedure to complete it. Earlier, we pointed out some sources of incompleteness, but these are not the only ones. It may arise because of lack of liquidity (leading to market-makers bid/ask spreads for which trading micro-models are constructed); it may be due to excessive friction defined in terms of taxes, indivisibility of assets, varying rates for lending and borrowing (such as no short sales and various portfolio constraints); it could be due to transaction costs and to information asymmetries (insider trading, leading to mis-pricing) indicating that one dimension along which markets are clearly incomplete is that of time. Most traded derivatives markets – futures and options – extend only a few years at most in time and, even when they are formally quoted further out, there is generally little or no liquidity in the far contracts. Yet, it is becoming increasingly common in the world of derivatives to be faced with long-run commitments while liquid markets only provide trading opportunities over shorter run horizons. These are by no means the only situations that lead to incomplete markets. Choice – too much or too little of it – may also induce incompleteness. Rationality implies selecting the best alternative but, when there are too many or the search cost is too high, often investors seek "satisficing solutions" (in the sense of Herbert A. Simon). Barry Schwartz in an article in Scientific American (April 2004) points out, for example, that too much choice may induce an ill feeling and therefore to suboptimal decisions. In addition, regrets [47.17, 36, 37, 127–134], search and other costs can also affect investors' rationality (in the sense of finance's fundamental theory). The Financial Times has pointed out that some investment funds seek to capitalize on human frailties to make money. For example: are financial managers human? Are they always rational, mimicking Star Trek's Mr. Spock? Are they devoid of emotions and irrationality? Psychological decision-making processes integrated in economic rationales have raised serious concerns regarding the rationality axioms of decision-making (DM) processes. There are of course, many challenges to reckon with in understanding human behavior. Some of these include: thought processes based on decision making-approaches focusing on the one hand on the big pictures versus compartmentalization; the effects of under- versus overconfidence on decision making; the application of heuristics of various sorts applied in trading and DM processes. These heuristics are usually based on simple rules. In general, the violation of the assumptions made regarding the definition of rational decision makers and decision makers' psychology are very important issues to reckon with when asset prices in incomplete markets are to be defined (some related references include [47.87, 133, 135–143] and [47.144]).

Networks of hedge funds, communicating with each other and often coordinated explicitly and implicitly into speculative activities can lead to market inefficiencies, thus contradicting a basic hypothesis in finance which assumes that agents are price-takers. In networks, information exchange provides a potential for information asymmetries or at least delays in information [47.145, 146]. In this sense, the existence of networks in their broadest and weakest form may also be a symptom of market breakdown. Analysis of competition in the presence of moral hazard and adverse selection emphasizes the substantial differences between trading of contracts and of contingent commodities. The profit associated with the sale of one unit of a (contingent) good depends then only on its price. Further, the profitability of the sale of one contract may also depend on the identity of the buyer. Identity matters either because the buyer has bought other contracts (the exclusivity problem) or because profitability of the sales depends on the buyer's characteristics, which is also known as the screening problem. Do these issues relate to financial intermediation? Probably yes. Thus financial markets theory has to give a key role to informational and power asymmetries to better understand prices and how they differ from the social values of commodities.

In the presence of proportional transaction costs, no perfect replication strategy is in general available. It is necessary then to define other pricing criteria. Some explicit solutions to the multivariate super-replication problem under proportional transaction costs using a utility maximization problem have been suggested, however. The implication of these and related studies are that super-replication prices are highly expensive and are not acceptable for practical purposes.

Quantitative modeling provides also important sources of incompleteness and at the same time seeks to represent such incompleteness. Research in modeling uncertainty and studies that seek to characterize mathematically randomness, which can be also sources of incompleteness. We use a number of quantitative approaches including: Brownian motion; long-run memory models and chaos-related approaches ([47.147]; heavy (fat) tails (stable) distributions that, unlike the normal distribution, have very large or infinite variance. These approaches underpin some confusion regarding the definition of uncertainty and how it can be structured in a theory of economics and finance. "G-D does not play dice" (Einstein), "Probabilities do not exist" (Bruno de Finetti) etc. are statements that may put some doubt on the commonly used random-walk hypothesis which underlies martingales finance and markets' efficiency (for additional discussion see [47.148, 149]). These topics are both important and provide open-ended avenues for further and empirical research. In particular, issues of long-run memory and chaos (inducing both very large variances and skewness) are important sources of incompleteness that have been studied intensively [47.150–157]. To deal formally with these issues, *Mandelbrot* and co-workers have introduced both a methodology based on fractal stochastic processes and application of Hurst's 1951 R/S (a range to starndard deviation statistic) methodology (for example, see [47.158–168]). Applications to finance include [47.169–178]). A theoretical extension based on the range process and R/S analysis based on the inverse range process can be found in [47.179, 180] as well as [47.181–183] and [47.184]. Finally, continuous stochastic processes to which risk-neutral pricing can be applied may become incomplete when they are discretized for numerical analysis purposes. Below we shall consider a number of pricing problems in incomplete markets to highlight some of the approaches to asset pricing.

47.6.1 Risk and the Valuation of a Rated Bond

Bonds are not always risk-free. Corporations emitting bonds may default, governments can also default in the payment of their debts, etc. For this reason, rating agencies sell their services and rate firms to assure buyers of the risks they assume when buying the bond. For this reason, the pricing of rated bonds is an important aspect of asset pricing. Below we shall show how such bonds may be valued (see also the earlier bond section). Consider first a non-default coupon-bearing rated bond with a payment of one dollar at maturity T. Further, define the bond m-ratings matrix by a Markov chain $[p_{ij}]$ where $0 \le p_{ij} \le 1$, $\sum_{j=1}^{m} p_{ij} = 1$ denotes the probability that a bond rated i in a given year will be rated j the following one. Discount factors are a function of the rating states, thus a bond rated i has a spot yield R_{it}, $R_{it} \le R_{jt}$ for $i < j$ at time t. As a result, a bond rated i at time t and paying a coupon c_{it} at this time has, as we saw earlier, a value given by

$$B_i(t,T) = c_{it} + \sum_{j=1}^{m} \frac{p_{ij}}{1+R_{jt}} B_j(t+1,T);$$
$$B_{i,T} = \ell_i, \quad i = 1,2,3,\dots m,$$

where ℓ_i is the nominal value of a bond rated i at maturity. Usually, $\ell_i = 1$, $i = 1,2,\dots m-1$, and $\ell_m = 0$ where m is the default state, and there is no recovery in case of default. In vector notation, we have

$$\boldsymbol{B_t} = \boldsymbol{c_t} + \boldsymbol{F_t}\boldsymbol{B_{t+1}}; \quad \boldsymbol{B_T} = \boldsymbol{L},$$

where the matrix $\boldsymbol{F_t}$ has entries $[p_{ij}/(1+R_{jt})]$ and $\boldsymbol{L}$ is a diagonal matrix of entries ℓ_i, $i = 1,2,\dots,m$. For a zero-coupon bond, we have $\boldsymbol{B_t} = \prod_{k=t}^{T} \boldsymbol{F_k}$. By the same token, rated bonds discounts $q_{it} = 1/(1+R_{it})$ are found by solving the matrix equation

$$\begin{pmatrix} q_{1t} \\ q_{2t} \\ \dots \\ \dots \\ q_{mt} \end{pmatrix} = \begin{pmatrix} p_{11}B_{1,t+1} & p_{12}B_{2,t+1} & \dots & \dots & p_{1m}B_{m,t+1} \\ p_{21}B_{1,t+1} & p_{22}B_{2,t+1} & & & p_{2m}B_{m,t+1} \\ \dots & & & & \\ \dots & & & & \\ p_{m1}B_{1,t+1} & p_{m2}B_{2,t+1} & & & p_{mm}B_{m,t+1} \end{pmatrix}^{-1} \times \begin{pmatrix} B_{1,t}-c_{1t} \\ B_{2,t}-c_{2t} \\ \dots \\ \dots \\ B_{m,t}-C_{mt} \end{pmatrix},$$

where at maturity T, $B_i(T,T) = \ell_i$. Thus, in matrix notation, we have: $\bar{\boldsymbol{q}}_{\boldsymbol{t}} = \boldsymbol{\Gamma}_{t+1}^{-1}(\boldsymbol{B_t} - \boldsymbol{c_t})$. Note that, one period prior to maturity, we have: $\bar{\boldsymbol{q}}_{T-1} = \boldsymbol{\Gamma}_T^{-1}(\boldsymbol{B}_{T-1} - \boldsymbol{c}_{T-1})$, where $\boldsymbol{\Gamma}_T$ is a matrix with entries $p_{ij}B_j(T,T) = p_{ij}\ell_j$.

In order to price the rated bond, consider a portfolio of rated bonds consisting of N_i, $i = 1,2,3\dots,m$ bonds rated i, each providing ℓ_i dollars at maturity. Let the portfolio value at maturity be equal one dollar. Namely,

$$\sum_{i=1}^{m} N_i \ell_i = 1.$$

One period (year) prior to maturity, such a portfolio would be worth $\sum_{i=1}^{m} N_i B_i(T-1,T)$ dollars. By the same token, if we denote by $R_{\mathrm{f},T-1}$ the risk-free discount rate for one year, then assuming no arbitrage, one period prior to maturity, we have:

$$\sum_{i=1}^{m} N_i B_i(T-1,T) = \frac{1}{1+R_{\mathrm{f},T-1}}\,;$$
$$B_i(T-1,T) = c_{it} + \sum_{j=1}^{m} q_{jt} p_{ij} B_j(T,T)\,;$$
$$B_i(T,T) = \ell_i, \quad i = 1,2\ldots,m$$

with $q_{jt} = 1/(1+R_{j,t})$ and $R_{j,t}$ is the one-period discount rate applied to a j rated bond. Assuming no arbitrage, such a system of equations will hold for any of the bonds periods and therefore, we can write the following no-arbitrage condition

$$\sum_{i=1}^{m} N_i B_i(T-k,T) = \frac{1}{\left(1+R_{\mathrm{f},T-k}\right)^k}$$
$$k = 0, 12, 3, \ldots, T\,,$$

where $R_{\mathrm{f},T-k}$, $k = 1, 2, 3, \ldots$, is the risk-free rate term structure which provides a system of $T+1$ equations spanning the bond life. In matrix notation this is given by

$$\boldsymbol{N}\boldsymbol{B}_{\boldsymbol{T}-k} = \frac{1}{\left(1+R_{\mathrm{f},T-k}\right)^k},$$
$$k = 0, 1, 2, \ldots, T; \quad \boldsymbol{N} = (N_1, N_2, \ldots, N_m);$$
$$\boldsymbol{B}_{\boldsymbol{T}-k} = (B_{1,T-k}, B_{2,T-k}, \ldots, B_{m,T-k})\,.$$

As a result, assuming that the bond maturity is larger than the number of ratings ($T \geq m+1$), the hedging portfolio of rated bonds is found by a solution of the system of linear equations above, leading to the unique solution:

$$\boldsymbol{N}^* = \mathfrak{I}^{-1}\boldsymbol{\Omega}\,,$$

where $\mathfrak{I}$ is the matrix transpose of $[B_{i,T-j+1}]$ and $\boldsymbol{\Omega}$ is a column vector with entries $\left[1/(1+R_{\mathrm{f},T-s})^s\right]$, $s = 0, 1, 2\ldots m-1$. Explicitly, we have:

$$\begin{pmatrix} N_1 \\ N_2 \\ \ldots \\ \ldots \\ N_m \end{pmatrix} = \begin{pmatrix} B_{1,T} & B_{2,T} & B_{3,T} & \ldots & B_{m,T} \\ B_{1,T-1} & B_{2,T-1} & B_{3,T-1} & \ldots & B_{m,T-1} \\ \ldots & & & & \ldots \\ \ldots & & & & \ldots \\ B_{1,T-m} & B_{2,T-m} & B_{3,T-m} & \ldots & B_{m,T-m} \end{pmatrix}^{-1} \times \begin{pmatrix} 1 \\ 1/(1+R_{\mathrm{f},T-1})^1 \\ \ldots \\ \ldots \\ 1/(1+R_{\mathrm{f},T-m})^m \end{pmatrix}.$$

Thus, a condition for no arbitrage is given by the system of nonlinear equations

$$\mathfrak{I}^{-1}\boldsymbol{\Omega}\boldsymbol{B}_{\boldsymbol{T}-k} = \frac{1}{\left(1+R_{\mathrm{f},T-k}\right)^k}$$
$$k = m, m+1, \ldots, T\,.$$

For example, for a zero-coupon rated bond and stationary short discounts, we have $\boldsymbol{B}_{t-k} = (\boldsymbol{F})^k$ and therefore, the no-arbitrage condition becomes:

$$\mathfrak{I}^{-1}\boldsymbol{\Omega}\boldsymbol{F}^k = \frac{1}{\left(1+R_{\mathrm{f},T-k}\right)^k} \quad k = m, m+1, \ldots, T$$

where $\boldsymbol{F}$ has entries $q_j p_{ij}$. This system of equations therefore provides $T+1-m$ equations applied to determining the bond ratings short (one-period) discount rates q_j. Our system of equations may be over- or under-identified for determining the ratings discount rates under our no-arbitrage condition, however. Of course, if $T+1-m = m$, we have exactly m additional equations we can use to solving the discount rates uniquely (albeit, these are nonlinear equations and can only be solved numerically).

Otherwise, the rated bond market is incomplete and we must proceed to some approach that can, nevertheless, provide an estimate of the discount rates. We use for convenience a sum of squared deviations from the rated bond arbitrage condition, in which case we minimize the following expression:

$$\min_{0\leq q_1,q_2,\ldots q_{m-1},q_m\leq 1} \sum_{k=m}^{T}\left(\mathfrak{I}^{-1}\boldsymbol{\Omega}\boldsymbol{B}_{\boldsymbol{T}-k} - \frac{1}{\left(1+R_{\mathrm{f},T-k}\right)^k}\right)^2.$$

Further additional constraints, reflecting expected and economic rationales of the ratings discounts q_j, might be added, such as:

$$0 \leq q_j \leq 1 \quad \text{and}$$
$$0 \leq q_m \leq q_{m-1} \leq q_{m-2} \leq q_{m-3}, \ldots \leq q_2 \leq q_1 \leq 1\,.$$

These are typically nonlinear optimization problems however. A simple two-rating example highlights some of the complexities in determining both the hedging portfolio and the ratings discounts provided the risk-free term structure is given. When the bond can default, we have to proceed as shown below.

47.6.2 Valuation of Default-Prone Rated Bonds

Let the first time n, a bond rated initially i, is rated j and let the probability of such an event be $f_{ij}(n)$. This probability equals the probability of not having gone through a j-th rating in prior transitions and be rated j at time n. For transition in one period, this is equal the transition bond rating matrix (S&P or Moody's matrix, as stated earlier), while for a transition in two periods it equals the probability of transition in two periods conditional on not having reached rating j in the first period. In other words, we have:

$$f_{ij}(1)=p_{ij}(1)=p_{ij}; \quad f_{ij}(2)=p_{ij}(2)-f_{ij}(1)p_{jj}$$

and generally, by recursion,

$$f_{ij}(n)=p_{ij}(n)-\sum_{k=1}^{n-1} f_{ij}(k)p_{jj}(n-k)\,.$$

The probability of a bond defaulting (and not defaulting) prior to time n is thus,

$$F_{km}(n-1)=\sum_{j=1}^{n-1} f_{km}(j)\,;$$
$$\bar{F}_{km}(n-1)=1-F_{km}(n-1)\,.$$

At present, denote by $\Phi_i(n)$ the probability that the bond is rated i at time n. In vector notation we write $\bar{\Phi}(n)$. Thus given the rating matrix $[\boldsymbol{P}]$, we have:

$$\bar{\Phi}(n)=[\boldsymbol{P}]'\bar{\Phi}(n-1)\,,$$
$$n=1,2,3,\ldots \quad \text{and} \quad \bar{q}(0) \text{ given}\,,$$

where $[\boldsymbol{P}]'$ is the matrix transpose. Thus, at n, $\bar{\Phi}(n)=[\boldsymbol{P}']^n\bar{\Phi}(0)$. The present value of a coupon payment at time n (given that there was no default at this time) is therefore discounted at the yield $R_{j,n}$, $q_{j,n}=1/(1+R_{j,n})$ if the bond is rated j. In other words, its present value is

$$\sum_{j=1}^{m-1} c_{j,n}q_{j,n}^n\Phi_{j,n}; \quad \Phi_{j,n}=\sum_{i=1}^{m-1}\Phi_{i,0}p_{ij}^{(n)}\,,$$

where $p_{ij}^{(n)}$ is the ij-th entry of the transposed power matrix $[\boldsymbol{P}']^n$ and $\Phi_{i,0}$ is the probability that initially the bond is rated i.

When a coupon-bearing default bond rated i at time s defaults at time $s+1$, $T-(s+1)$ periods before maturity with probability $f_{im}(s+1-s)$, we have a value:

$$V_{s,i}=\left(c_{i,T-s}+q_i\ell_{m,T-(s+1)}\right) \text{ w. p. } f_{im}(1)\,.$$

If such an event occurs at time $s+2$, with probability $f_{im}(s+2-s)=f_{im}(2)$, we have:

$$V_{s,i}=\left(c_{i,T-s}+\sum_{k=1}^{m-1}q_kc_{k,T-(s+1)}\Phi_{k,(s+1)-s}\right.$$
$$\left.+q_i^2\ell_{m,T-(s+2)}\right) \text{ w. p. } f_{im}(2)\,,$$

where $\Phi_{k,1}=\sum_{i=1}^{m-1}\Phi_{i,0}p_{ik}^{(1)}$ and $\Phi_{i,0}$ is a vector whose entries are all zero except at i (since at s we conditioned the bond value to a rating i). By the same token three periods hence and prior to maturity, we have

$$V_{s,i}=\left(c_{i,T-s}+\sum_{k=1}^{m-1}q_kc_{k,T-(s+1)}\Phi_{i,0}p_{ik}^{(1)}\right.$$
$$+\sum_{k=1}^{m-1}q_k^2c_{k,T-(s+2)}\Phi_{i,0}p_{ik}^{(2)}$$
$$\left.+q_i^3\ell_{m,T-(s+3)}\right) \text{ w. p. } f_{im}(3)\,.$$

And generally, for any period prior to maturity,

$$V_{s,i}=\left(c_{i,T-s}+\sum_{\theta=1}^{\tau-1}\sum_{k=1}^{m-1}q_k^\theta c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right.$$
$$\left.+q_i^\tau\ell_{m,T-(s+\tau)}\right) \text{ w. p. } f_{im}(\tau)\,.$$

In expectation, if the bond defaults prior to its maturity, its expected price at time s,

$$EB_{i,\mathrm{D}}(s,T)=c_{i,T-s}+\sum_{\tau=1}^{T-s}\left(q_i^\tau\ell_{m,T-(s+\tau)}\right.$$
$$\left.+\sum_{\theta=1}^{\tau-1}\sum_{k=1}^{m-1}q_k^\theta c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)$$
$$\times f_{im}(\tau)\,,$$

where $\ell_{m,T-j}$ is a portion of the bond nominal value that the bondholder recuperates when the bond defaults and which is assumed to be a function of the time remaining for the bond to be redeemed. And therefore, the price of such a bond is:

$$B_{i,\mathrm{ND}}(s,T)=\left(c_{i,T-s}+\sum_{k=1}^{m-1} q_k^{T-s}\ell_k\Phi_{i,0}p_{ik}^{(T-s)}+\sum_{\theta=1}^{T-s-1}\sum_{k=1}^{m-1} q_k^{\theta}c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)\times\left[1-\sum_{u=1}^{T-s} f_{im}(u)\right]$$

where ℓ_i denotes the bond nominal value at redemption when it is rated i. Combining these sums, we obtain the price of a default-prone bond rated i at time s

$$\begin{aligned}B_i(s,T)=c_{i,T-s}&+\left(c_{i,T-s}+\sum_{k=1}^{m-1} q_k^{T-s}\ell_k\Phi_{i,0}p_{ik}^{(T-s)}+\sum_{\theta=1}^{T-s-1}\sum_{k=1}^{m-1} q_k^{\theta}c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)\times\left[1-\sum_{u=1}^{T-s} f_{im}(u)\right]\\&+\sum_{\tau=1}^{T-s}\left(q_i^{\tau}\ell_{m,T-(s+\tau)}+\sum_{\theta=1}^{\tau-1}\sum_{k=1}^{m-1} q_k^{\theta}c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)f_{im}(\tau)\,.\end{aligned}$$

Finally, for a zero-coupon bond, this is reduced to

$$B_i(s,T)=c_{i,T-s}+\left(\sum_{k=1}^{m-1} q_k^{T-s}\ell_k\Phi_{i,0}p_{ik}^{(T-s)}\right)\times\left[1-\sum_{u=1}^{T-s} f_{im}(u)\right]+\sum_{\tau=1}^{T-s}\left(q_i^{\tau}\ell_{m,T-(s+\tau)}\right)f_{im}(\tau)\,.$$

To determine the price (discounts rates) for a default-prone rated bond we can proceed as before by constructing a hedging portfolio consisting of $N_1, N_2, \dots, N_{m-1}$ shares of bonds rated $i=1,2,\dots,m-1$. Again, let $R_{\mathrm{f},T-u}$ be the risk-free rate when there are u periods left to maturity. Then, assuming no arbitrage and given the term structure risk-free rate, we have:

$$\sum_{i=1}^{m-1} N_i B_i(s,T)=\frac{1}{\left(1+R_{\mathrm{f},T-s}\right)^s}\,,\qquad s=0,1,2,\dots$$

with $B_i(s,T)$ defined above. Note that the portfolio consists of only $m-1$ rated bonds and therefore, we have in fact $2m-1$ variables to be determined based on the risk-free term structure. Assuming that our system is over- (or under-)determined, we are reduced to solving the following minimum squared deviations problem:

$$\min_{\substack{0\le q_1\le q_2\le\dots\le q_{m-1}\le 1;\\ N_1,N_2,N_3\dots,N_{m-1}}}\sum_{s=0}^{T}\left[\sum_{k=1}^{m-1} N_k B_k(s,T)-\frac{1}{\left(1+R_{\mathrm{f},T-s}\right)^s}\right]^2$$

subject to:

$$\begin{aligned}B_i(s,T)=c_{i,T-s}&+\left(c_{i,T-s}+\sum_{k=1}^{m-1} q_k^{T-s}\ell_k\Phi_{i,0}p_{ik}^{(T-s)}+\sum_{\theta=1}^{T-s-1}\sum_{k=1}^{m-1} q_k^{\theta}c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)\\&\times\left[1-\sum_{u=1}^{T-s} f_{im}(u)\right]+\sum_{\tau=1}^{T-s}\left(q_i^{\tau}\ell_{m,T-(s+\tau)}+\sum_{\theta=1}^{\tau-1}\sum_{k=1}^{m-1} q_k^{\theta}c_{k,T-(s+\theta)}\Phi_{i,0}p_{ik}^{(\theta)}\right)f_{im}(\tau)\,.\end{aligned}$$

This is of course a nonlinear optimization problem which can be solved analytically with respect to the hedged portfolio, and use the remaining equations to calculate the ratings discount rates. A solution can be found numerically. Such an analysis is a straightforward exercise however. Below we consider some examples.

Example 47.5 (a two-rated default bond): Consider a two-rated zero-coupon bond and define the transition matrix

$$\boldsymbol{P}=\begin{pmatrix}p & 1-p\\ 0 & 1\end{pmatrix}\quad\text{with}\quad \boldsymbol{P}^n=\begin{pmatrix}p^n & 1-p^n\\ 0 & 1\end{pmatrix}.$$

The probability of being in one of the two states after n periods is $(p^n, 1-p^n)$. Further,

$$\begin{aligned}f_{12}(1)&=1-p\,;\\ f_{12}(2)&=p_2^{(2)}-(1)f_{12}(1)=1-p^2-(1-p)\\&=p(1-p)\,.\end{aligned}$$

Thus, for a non-coupon-paying bond, we have:

$$B_i(s,T) = c_{i,T-s} + q^{T-s}\ell\left[1 - \sum_{u=1}^{T-s} f_{12}(u)\right] + \sum_{\tau=1}^{T-s}\left(q^{\tau}\ell_{m,T-(s+\tau)}\right) f_{12}(\tau)\,.$$

In particular,

$$\begin{aligned}
B_1(T,T) &= \ell\,,\\
B_1(T-1,T) &= q\ell\,[1-f_{12}(1)] + q\ell_{m,0} f_{12}(1)\,,\\
B_1(T-2,T) &= q^2\ell\,[1-f_{12}(1)-f_{12}(2)]\\
&\quad + q\ell_{m,1} f_{12}(\tau) + q^2\ell_{m,0} f_{12}(2)\,,\\
B_1(T-3,T) &= q^3\ell\,[1-f_{12}(1)-f_{12}(2)-f_{12}(3)]\\
&\quad + \left(q\ell_{m,2}\right) f_{12}(1) + q^2\ell_{m,1} f_{12}(2)\\
&\quad + q^3\ell_{m,0} f_{12}(3)\,.
\end{aligned}$$

If we have a two-year bond, then the condition for no arbitrage is

$$\begin{aligned}
NB_1(T,T) &= N\ell = 1 \quad \text{and} \quad N = 1/\ell\,,\\
NB_1(T-1,T) &= \frac{1}{1+R_{\mathrm{f},T-1}} \Rightarrow 1 + R_{1,T-1}\\
&= \frac{1+R_{\mathrm{f},T-1}}{1-\left(1-\ell_{m,0}/\ell\right) f_{12}(1)}\,.
\end{aligned}$$

If we have a two-year bond, then the least quadratic deviation cost rating can be applied. Namely,

$$\min_{0\le q\le 1} \mathbb{Q} = \left[(1/\ell)B\,(T-1,T) - \left(q_{f,T-1}\right)\right]^2 + \left[(1/\ell)B\,(T-2,T) - \left(q_{f,T-2}\right)\right]^2\,.$$

Subject to:

$$\begin{aligned}
B_1(T-1,T) &= q\ell\,[1-f_{12}(1)] + q\ell_{m,0} f_{12}(1),\\
B_1(T-2,T) &= q^2\ell\,[1-f_{12}(1)-f_{12}(2)]\\
&\quad + q\ell_{m,1} f_{12}(\tau) + q^2\ell_{m,0} f_{12}(2)\,.
\end{aligned}$$

Leading to a cubic equation in q that we can solve by the usual methods. Rewriting the quadratic deviation in terms of the discount rate yields:

$$\begin{aligned}
\min_{0\le q\le 1} &\left(q\left\{1-f_{12}(1)\left[1-(\ell_{m,0}/\ell)\right]\right\} - \left(q_{f,T-1}\right)\right)^2\\
&+ \left\{q^2\left[1-f_{12}(1) - (1-\ell_{m,0}/\ell) f_{12}(2)\right]\right.\\
&\left.+ q(\ell_{m,1}/\ell) f_{12}(1) - \left(q_{f,T-2}\right)\right\}^2\,.
\end{aligned}$$

Set

$$\begin{aligned}
a &= \left\{1-f_{12}(1)\left[1-(\ell_{m,0}/\ell)\right]\right\}\,;\\
b &= \left[1-f_{12}(1) - (1-\ell_{m,0}/\ell) f_{12}(2)\right]\,;\\
c &= (\ell_{m,1}/\ell) f_{12}(1)\,.
\end{aligned}$$

Then an optimal q is found by solving the equation

$$2q^3b^2 + 3q^2bc + q\left(a^2 - 2bq_{f,T-2} + c^2\right) - \left(aq_{f,T-1} + cq_{f,T-2}\right) = 0\,.$$

Assume the following parameters,

$$\begin{aligned}
&R_{\mathrm{f},T-1} = 0.07; \quad &&R_{\mathrm{f},T-1} = 0.08, \quad &&p = 0.8,\\
&\ell = 1, &&\ell_{m,0} = 0.6, &&\ell_{m,1} = 0.4\,.
\end{aligned}$$

In this case, $f_{12}(1) = 1 - p = 0.2$ and $f_{12}(2) = p(1-p) = 0.16$. For a one-period bond, we have

$$1 + R_{1,T-1} = \frac{1+0.07}{1-(0.084)} = 1.168$$

and therefore we have a 16.8% discount, $R_{1,T-1} = 0.168$. For a two-period bond, we have instead (using the minimization technique): $a = 0.92$, $b = 0.736$, $c = 0.084$ and therefore,

$$q^3 + 0.171\,129\,q^2 - 0.470\,28\,q - 0.865\,33 = 0\,.$$

Whose solution provides q and therefore $1 + R_{1,T-1}$.

47.6.3 "Engineered" Risk-Neutral Distributions and Risk-Neutral Pricing

When a market is complete, an asset price can be defined as follows:

$$S_t = \mathrm{e}^{-R_{\mathrm{f}}(T-t)} E_t^*\left(S_T\,|\Omega_t\right)\,,$$

where Ω_t is a filtration, meaning that the expectation is calculated on the basis of all the information available up to time t and the probability distribution with respect to which the expectation is taken is a risk-neutral distribution. That is:

$$E_t^*\left(S_T\,|\Omega_t\right) = \int S_T\,\mathrm{d}F_{T|t}\,, \quad T > t$$

where $F_{T|t}$ is the asset risk-neutral distribution at time T based on the data available at time t. If the underlying price process is given by a stochastic process, then $E_t^*\left(S_T\,|\Omega_t\right)$ is the optimal forecast (filter) estimate of the asset price using the distribution $F_{T|t}$. In such circumstances, and if such a distribution exists, then derived assets such as call and put options are also priced by

$$\begin{aligned}
C_t &= \mathrm{e}^{-R_{\mathrm{f}}(T-t)} E_t^*\left(C_T\,|\Omega_t\right) = \int_K C_T\,\mathrm{d}F_{T|t}\,(S_T)\,,\\
C_T &= \max(S_T - K, 0), \quad T > t\,,
\end{aligned}$$

$$P_t = \mathrm{e}^{-R_\mathrm{f}(T-t)} E_t^* \left(P_T \,|\Omega_t\right) = \int_0^K P_T \,\mathrm{d}F_{T|t}\,(S_T)\ ,$$
$$P_T = \max(K - S_T, 0), \quad T > t\ .$$

Of course, if the distribution happens to be normal then the assets prices equal the discounted best mean forecast of the future asset price. When this is not the case and markets are incomplete, asset pricing in practice seeks to determine the risk-neutral distribution that allows application of risk-neutral pricing whether markets are in fact complete or incomplete (for an empirical study see [47.185, 186], for example). There are numerous sources of information, and approaches used to *engineer* such a distribution. For example, let there be m derived assets x_t^j and let there be some data points up to time t regarding these assets. In this case, for each time t, the optimal least-square estimate of the risk-neutral distribution is found by minimizing the least squares below by a selection of the appropriate parameters defining the underlying price process:

$$\sum_{i=0}^{t} \left\{ \sum_{j=1}^{m} \left[x_i^j - \mathrm{e}^{-R_\mathrm{f}(T-i)} E_i^* \left(x_T^j \,|\Omega_i \right) \right]^2 + \left[S_i - \mathrm{e}^{-R_\mathrm{f}(T-i)} E_i^* \left(S_T \,|\Omega_i\right) \right]^2 \right\} .$$

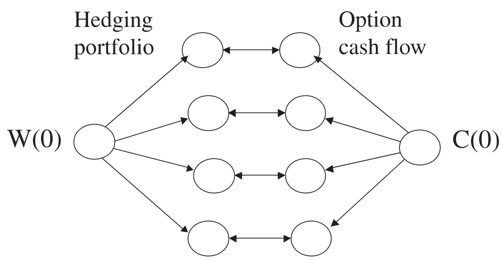
Here x_i^j is an actual observation of asset j taken at time i. If prices are available over several specific time periods (for example, an option for three months, six months and a year), then summing over available time periods we will have:

$$\sum_{i=0}^{t} \left\{ \sum_{\ell=1}^{L} \sum_{j=1}^{m} \left[x_i^j - \mathrm{e}^{-R_\mathrm{f}(T_\ell-i)} E_i^* \left(x_{T_\ell}^j \mid \Omega_i \right) \right]^2 + \sum_{k=i+1} \left[S_i - \mathrm{e}^{-R_\mathrm{f}(k-i)} E_i^* \left(S_k \mid \Omega_i\right) \right]^2 \right\} .$$

Of course, other techniques can be taken in this spirit, providing thereby the optimal distribution forecast estimate. This is a problem that is over-parameterized however and therefore some assumptions are often made to reduce the number of parameters that define the presumed risk-neutral distribution. Examples and applications are numerous. Some authors assume a general multi-parameter distribution (such as the Burr distribution) for the risk-neutral distribution and calculate the parameters. Others seek the distribution outright while others assume an underlying process and calculate the best fit parameters. Both discrete-time and continuous-time models are used. Other models assume a broader framework such as a stochastic process with or without stochastic volatility with parameters to be estimated based on data availability. Below we shall consider a number of such cases (see also [47.187, 188].

Example 47.6 (mean variance replication hedging): This example consists of constructing a hedging portfolio in an incomplete (stochastic volatility) market by equating *as much as possible* cash flows resulting from a hedging portfolio and option prices. We shall do so while respecting the basic rules of rational expectations and risk-neutral pricing. This implies that at all times the price of the portfolio and the option price are the same. Let $W(t)$ be the portfolio price and $C(t)$ be the option price. At time $t = 0$, we evidently have as well $W(0) = C(0)$, similarly at some future date.

$$\begin{array}{ccc} W(0) & \Leftarrow & \tilde{W}(1) \\ \Downarrow & & \Updownarrow \\ C(0) & & \tilde{C}(1) \end{array}$$



$$\tilde{W}(1) = \tilde{C}(1)$$

However, under risk-neutral pricing we have:

$$C(0) = \frac{1}{1+R_\mathrm{f}} E\tilde{C}(1); \quad W(0) = \frac{1}{1+R_\mathrm{f}} E\tilde{W}(1)$$
$$\text{and} \quad C(0) = W(0) \quad \text{and} \quad E\tilde{W}^2(1) = \tilde{E}C^2(1)\ .$$

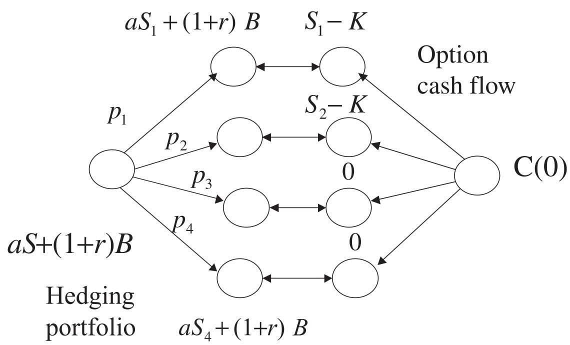
These provide three equations only. Since a hedging portfolio can involve a far greater number of parameters, it might be necessary to select an objective to minimize. A number of possibilities are available.

Rubinstein [47.189], as well as *Jackwerth* and *Rubinstein* [47.190] for example, suggested a simple quadratic optimization problem by minimizing the quadratic difference of the probabilities associated to the binomial tree. Alternatively, a quadratic objective that leads to the minimization of a hedging portfolio and the option ex-post values of some option contract with risk-neutral pricing leads to:

$$\min_{p_1,\dots p_n} \Phi = E\left(\tilde{W}(1)-\tilde{C}(1)\right)^2 . \quad \text{Subject to:}$$

$$W(0) = C(0) \quad \text{or}$$

$$\frac{1}{1+R_\mathrm{f}}\left[E\tilde{W}(1)\right] = \frac{1}{1+R_\mathrm{f}}\left[E\tilde{C}(1)\right] \quad \text{and}$$
$$E\tilde{W}^2(1) = \tilde{E}C^2(1)$$



Of course the minimizing objective can be simplified further to:

$$\min_{p_1,\dots p_n} \Phi = E\tilde{C}^2(1) - E\tilde{W}(1)\tilde{C}(1) \quad \text{or}$$

$$\min_{p_1,\dots p_n} \Phi = \sum_{i=1}^{n} p_i\left(C_{1i}^2 - W_{1i}C_{1i}\right),$$

where W_{1i}, C_{1i} are the hedging portfolio and option outcomes associated with each of the events i, which occur with probability $p_i, i = 1, 2, \dots n$.

Example 47.7: Define by S_j, $j = 1, 2, \dots n$ the n states a stock can assume at the time an option can be exercised. We set, $S_0 < S_1 < S_2 < \dots < S_n$ and define the buying and selling prices of the stock by S^a, S^b, respectively. By the same token, define the corresponding observed call option prices C^a, C^b. Let p be the probability of a price increase. Of course, if the ex-post price is S_n, this will correspond to the stock increasing each time period with probability $P_n = (n, n)^\top p^n(1-p)^{n-n} = p^n$.

By the same token, the probability of the stock having a price S_j corresponding to the stock increasing j times and decreasing $n-j$ times is given by the binomial probability

$$P_j = \binom{n}{j} p^j(1-p)^{n-j} .$$

As a result, we have under risk-neutral pricing:

$$\begin{aligned} S &= \frac{1}{1+R_\mathrm{f}}\sum_{j=0}^{n} P_j S_j \\ &= \frac{1}{1+R_\mathrm{f}}\sum_{j=0}^{n} S_j\binom{n}{j} p^j(1-p)^{n-j} ; \end{aligned}$$
$$S^a \le S \le S^b$$

while the call option price is

$$\begin{aligned} C &= \sum_{j=0}^{n} C_j \\ &= \frac{1}{(1+R_\mathrm{f})^n}\sum_{j=0}^{n}\binom{n}{j} p^j(1-p)^{n-j} \\ &\quad \times \max(S_j - K, 0) \end{aligned}$$

with an appropriate constraint on the call option value $C^a \le C \le C^b$. Note that S, C as well as p are the only unknown values so far. While the buy and sell values for stock and options, the strike time n and its price K as well as the discount rate and future prices S_j are given. Our problem at present is to select an objective which will make it possible to obtain risk-neutral probabilities. We can do so by minimizing the quadratic distance between a portfolio of a unit of stock and a bond B. At n, the portfolio is equal $aS_j + (1+R_\mathrm{f})^n B$ if the price is S_j. Of course, initially, the portfolio equals:

$$S = \frac{1}{(1+R_\mathrm{f})^n}\left[a\sum_{j=0}^{n} P_j S_j + (1+R_\mathrm{f})^n B\right] .$$

As a result, the least-squared replicating portfolio is given by:

$$\Phi = \sum_{j=1}^{n} P_j\left[aS_j + (1+R_\mathrm{f})^n B - \max(S_j - K, 0)\right]^2$$

which leads to the following optimization problem

$$\min_{1\geq p\geq 0,C,S} \Phi = \sum_{j=1}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \left[aS_j + (1+R_f)^n B - \max(S_j - K, 0)\right]^2 .$$

Subject to:

$$S = \frac{1}{(1+R_f)^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} S_j ;$$
$$S^a \leq S \leq S^b ,$$
$$C = \frac{1}{(1+R_f)^n} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \times \max(S_j - K, 0); \quad C^a \leq C \leq C^b ,$$
$$aS + B = C .$$

The numerical solution of this problem is straightforward.

Example 47.8 (fitting continuous risk-neutral distributions): The simplest such model of course consists of using an underlying binomial stock process and using options data (call and put) to estimate the stock process parameters and risk-neutral distribution. Assume that we assume a theoretical mixture price model given by:

$$\frac{dS}{S} = \begin{cases} \alpha_j \, dt + \sigma_j \, dW_j & \pi_j \\ j = 1, 2, \ldots, m & \\ \pi_j \geq 0 & \sum_{j=1}^{m} \pi_j \end{cases} , \quad S(0) = S_0 .$$

The solution of this model in terms of a risk-neutral numeraire is, (as we saw earlier):

$$S(t) = S(0) \sum_{j=1}^{m} \pi_j \, e^{\left(R_f - \frac{\sigma_j^2}{2}\right)t + \sigma_j W_t^*(t)} ,$$
$$W_t^*(t) = W_j(t) + \frac{\alpha_j - R_f}{\sigma_j} t .$$

Let there be call and put options prices given by:

$$C_t = e^{-R_f(T-t)} E_t^* \left(C_T \,|\Omega_t\right) ,$$
$$C_T = \max(S_T - K, 0), \quad T > t ,$$
$$P_t = e^{-R_f(T-t)} E_t^* \left(P_T \,|\Omega_t\right) ,$$
$$P_T = \max(K - S_T, 0), \quad T > t .$$

A simple least-squares optimization problem that seeks to calculate the underlying process parameters is then:

$$\min_{\alpha_j,\sigma_j,\pi_j} \sum_{i=0}^{t} \left\{ \left[C_i - e^{-R_f(T-i)} E_i^* \left(C_T \,|\Omega_i\right)\right]^2 + \left[P_i - e^{-R_f(T-i)} E_i^* \left(P_T \,|\Omega_i\right)\right]^2 \right\} ,$$

where

$$C_t = e^{-R_f(T-t)} E_t \left\{ \max\left[S(0) \times \sum_{j=1}^{m} \pi_j \, e^{\left(R_f - \frac{\sigma_j^2}{2}\right)t + \sigma_j W^*(t)} - K, 0 \right] |\Omega_t \right\} ,$$

$$P_t = e^{-R_f(T-t)} E_t \left\{ \max\left[K - S(0) \times \sum_{j=1}^{m} \pi_j \, e^{\left(R_f - \frac{\sigma_j^2}{2}\right)t + \sigma_j W^*(t)} , 0 \right] |\Omega_t \right\} .$$

Of course such problems can be solved by MATLAB or by nonlinear optimization routines. Further refinements can be developed by noting that the stock price must also meet the risk-neutral condition at each time prior to time t. A simple case is obtained when we consider the mixture of two such log-normal models. Such an assumption will imply of course that prices are skewed. Explicitly, assume a mixture of the log-normal distributions

$$f\left(S_T\right) = \begin{cases} L(\alpha_1, \beta_1) & \text{w.p. } \theta \\ L(\alpha_2, \beta_2) & \text{w.p. } 1-\theta \end{cases} ;$$
$$\hat{f}\left(S_T\right) \sim \theta L(\alpha_1, \beta_1) + (1-\theta) L(\alpha_2, \beta_2) .$$

For such a process we can use the normal mixtures as follows:

$$P(\tilde{R} \leq R) = \theta N\left(\frac{\log R - \mu_1}{\sigma_1}\right) + (1-\theta) N\left(\frac{\log R - \mu_2}{\sigma_2}\right) .$$

Elementary but tedious analysis of the moments can provide an estimate of the variance, the distribution skewness and its kurtosis, or

$$\mathrm{var} = \theta_1\sigma_1^2 + \theta_2\sigma_2^2 + \theta_1\theta_2\left(\mu_1-\mu_2\right)^2 ,$$

$$\mathrm{Skewness} = \frac{\theta_1\theta_2\left(\mu_1-\mu_2\right)}{(\mathrm{var})^{3/2}} \times\left[3\left(\sigma_1^2-\sigma_2^2\right)+\left(\theta_2-\theta_1\right)\left(\mu_1-\mu_2\right)^2\right],$$

$$\mathrm{Kurtosis} = \frac{3\left(\theta_1\sigma_1^4+\theta_2\sigma_2^4\right)}{(\mathrm{var})^2} + \frac{6\theta_2\theta_1\left(\mu_1-\mu_2\right)^2\left(\theta_2\sigma_1^1+\theta_1\sigma_2^2\right)}{(\mathrm{var})^2} + \frac{\theta_2\theta_1\left(\mu_1-\mu_2\right)^4\left(\theta_1^3+\theta_2^3\right)}{(\mathrm{var})^2} .$$

These moments of the mixture process indicate the behavior of the underlying risk-neutral distribution, highlighting the *market* intention as it is reflected in these moments (skew to the right or to the left of the uncertainty regarding future asset prices). To estimate the mixture process parameters we may then fit the data to say, call and put data, freely available on the appropriate market. Let:

$$\hat{C}_{i,T} = \text{European Call option price at } i = 1,\ldots m ;$$

$$\hat{P}_{i,T} = \text{European Put option price at } i = 1,2,\ldots m$$

with

$$\hat{C}(S,K_i,T) = \mathrm{e}^{-R_\mathrm{f}T}\int_{K_i}^{\infty}(S_T-K_i)\hat{f}(S_T)\,\mathrm{d}S_T ,$$

$$\hat{P}(S,K_i,T) = \mathrm{e}^{-R_\mathrm{f}T}\int_{0}^{K_i}(K_i-S_T)\hat{f}(S_T)\,\mathrm{d}S_T .$$

Table 47.1 Comparison of the log-normal and bi-log-normal model

Date	Number of observations	Fit (sum of squares) Log-normal	Bi-log-normal
09/06/2002	24	0.121	0.121
10/06/2002	24	0.060	0.060
17/06/2002	26	0.022	0.011
15/06/2002	23	0.070	0.004
10/07/2002	19	0.038	0.008
12/08/2002	16	0.022	0.005
Average	21.7	0.056	0.038

The resulting (data fit) optimization problem is then

$$\min_{\alpha_1,\beta_1;\alpha_2,\beta_2;\theta}\left[\sum_{i=1}^{m}\left(\hat{C}_{i,T}-C_{i,T}\right)^2 + \sum_{j=1}^{m}\left(\hat{P}_{j,T}-P_{j,T}\right)^2\right] .$$

Subject to the call and put theoretical price estimates. In other words, introducing the time-dependent prices for the call and put options, we have

$$\min_{\alpha_1,\beta_1;\alpha_2,\beta_2;\theta}\sum_{t=0}^{T}\left[\sum_{i=1}^{m}\left(\hat{C}_{i,t}-C_{i,t}\right)^2 + \sum_{j=1}^{m}\left(\hat{P}_{j,t}-P_{j,t}\right)^2\right] . \quad \text{Subject to:}$$

$$C_{it} = \mathrm{e}^{-R_\mathrm{f}(T-t)}\int_{K_i}^{\infty}(S_T-K_i)\hat{f}_t(S_T)\,\mathrm{d}S_T \quad \text{and}$$

$$\hat{P}_{it} = \mathrm{e}^{-R_\mathrm{f}(T-t)}\int_{0}^{K_i}(K_i-S_T)\hat{f}_t(S_T)\,\mathrm{d}S_T ,$$

where $\hat{f}_t(S_T)$ is given by the underlying multi-parameter mixture log-normal process. *Stein* and *Hecht* (Bank of Israel, Monetary Devision) use instead the following objective, which they have found more stable using data of the Israeli shekel and the American dollar.

$$\min_{\alpha_1,\beta_1;\alpha_2,\beta_2;\theta} \sum_{t=0}^{T}\left[\sum_{i=1}^{m}\left(1-\frac{\hat{C}_{i,t}}{C_{i,t}}\right)^2 + \sum_{j=1}^{m}\left(1-\frac{\hat{P}_{i,t}}{P_{i,t}}\right)^2\right]$$

Using Israeli and US currency data and comparing a log-normal and a bi-log-normal model they show that the bi-log-normal model provides a better fit, as shown in Table 47.1.

For each of these periods, they calculated as well the parameters of the underlying exchange rate process. As a result, they were able to estimate the probability of currency devaluation and that of appreciation of the shekel versus the dollar. Clearly, the result in Table 47.1 point out to a better fit when the bi-lognormal model is used.

Some authors simplify the computation of implied parameters by considering a multi-parameter distribution. In other words, this approach assumes outright that the risk-neutral distribution can be approximated by a known distribution. For simplicity, assume first a two-parameter (c, ζ) Weibull distribution given by:

$$
\begin{aligned}
f(\tau) &= \frac{c}{\zeta}\left(\frac{\tau}{\zeta}\right)^{c-1} \mathrm{e}^{-(\tau/\zeta)^c}, \quad \tau \geq 0, \zeta, \quad c > 0\,;\\
F(\tau) &= 1 - \mathrm{e}^{-(\tau/\zeta)^c}\,,\\
E(\tau) &= \zeta\Gamma\left(\frac{1}{c}+1\right)\,,\\
\mathrm{var}(\tau) &= \zeta^2\left[\Gamma\left(\frac{2}{c}+1\right) - \Gamma^2\left(\frac{1}{c}+1\right)\right].
\end{aligned}
$$

For the call option, calculations are easily applied and we have after some elementary manipulations:

$$
\begin{aligned}
\hat{C}_{it} &= \mathrm{e}^{-R_\mathrm{f}(T-t)} \int_{K_i}^{\infty} (S_T - K_i)\hat{f}_t(S_T)\,\mathrm{d}S_T\\
&= \mathrm{e}^{-R_\mathrm{f}(T-t)} c \int_{K_i}^{\infty} \left(\frac{S_T}{\zeta}\right)^c \mathrm{e}^{-(S_T/\zeta)^c}\,\mathrm{d}S_T\\
&\quad - \mathrm{e}^{-R_\mathrm{f}(T-t)} K_i\,\mathrm{e}^{-(K_i/\zeta)^c}\,.
\end{aligned}
$$

If we set:

$$
\left(\frac{S_T}{\zeta}\right)^c = u, \quad c\left(\frac{1}{\zeta}\right)\left(\frac{S_T}{\zeta}\right)^{c-1} \mathrm{d}S_T = \mathrm{d}u \quad \text{then}\,,
$$

$$
\begin{aligned}
\hat{C}_{it} &= \mathrm{e}^{-R_\mathrm{f}(T-t)} c \int_{\left(\frac{K_i}{\zeta}\right)^c}^{\infty} u\,\mathrm{e}^{-u}\,\mathrm{d}S_T\\
&\quad - \mathrm{e}^{-R_\mathrm{f}(T-t)} K_i\,\mathrm{e}^{-(K_i/\zeta)^c}\,.
\end{aligned}
$$

For the put option, we have similarly:

$$
\begin{aligned}
\hat{P}_{it} &= \mathrm{e}^{-R_\mathrm{f}(T-t)} \int_{0}^{K_i} (K_i - S_T)\hat{f}_t(S_T)\,\mathrm{d}S_T\\
&= \mathrm{e}^{-R_\mathrm{f}(T-t)} K_i \left(1 - \mathrm{e}^{-(K_i/\zeta)^c}\right)\\
&\quad - \mathrm{e}^{-R_\mathrm{f}(T-t)} \zeta \int_{0}^{\left(\frac{K_i}{\zeta}\right)^c} u^{\frac{1}{c}}\,\mathrm{e}^{-u}\,\mathrm{d}u\,.
\end{aligned}
$$

Note that:

$$
\int_{0}^{u_i} u^{\frac{1}{c}}\,\mathrm{e}^{-u}\,\mathrm{d}u = \gamma\left(1+\frac{1}{c}, u_i\right) \quad \text{and}
$$

$$
\int_{0}^{\left(\frac{K_i}{\zeta}\right)^c} u^{\frac{1}{c}}\,\mathrm{e}^{-u}\,\mathrm{d}u = \gamma\left[1+\frac{1}{c}, \left(\frac{K_i}{\zeta}\right)^c\right]
$$

and therefore

$$
\begin{aligned}
\hat{P}_{it} &= \mathrm{e}^{-R_\mathrm{f}(T-t)} K_i \left(1 - \mathrm{e}^{-(K_i/\zeta)^c}\right)\\
&\quad - \mathrm{e}^{-R_\mathrm{f}(T-t)} \zeta\gamma\left[1+\frac{1}{c}, \left(\frac{K_i}{\zeta}\right)^c\right].
\end{aligned}
$$

The underlying price is then

$$
\begin{aligned}
\hat{S}_t &= \mathrm{e}^{-R_\mathrm{f}(T-t)} E_{\mathrm{RND,W}}(S_T) \quad \text{and}\\
\hat{S}_t &= \mathrm{e}^{-R_\mathrm{f}(T-t)} \zeta\Gamma\left(\frac{1}{c}+1\right).
\end{aligned}
$$

We can use other distributions as well. For example, several authors like to use the Burr distribution because it includes as special cases numerous and well-known distributions. In his case, we have

$$
\begin{aligned}
F(S_T, T) &= 1 - \frac{1}{\left(1+S_T^c\right)^q}, \; S_T \geq 0, c, q > 1\,;\\
f(S_T, T) &= \frac{qcS_T^{c-1}}{\left(1+S_T^c\right)^{q+1}}
\end{aligned}
$$

or using the following notation

$$
\begin{aligned}
f(S_T) &= \frac{\alpha\nu S_T^{\alpha-1}}{\left(S_T^\alpha+\delta\right)^{\delta+1}}\,,\\
E\left(S_T^m\right) &= \frac{1}{\alpha}\delta^{(m+\alpha)/\alpha} B\left(1+\frac{m}{\alpha}, \delta-\frac{m}{\alpha}\right),
\end{aligned}
$$

which can be used to fit an available data set to the distribution and optimize to obtain parameter estimates. The problem with this technique, however, is that it is mostly appropriate for an estimation of a specific risk-neutral distribution for a specific instant of time rather than to the evolution of the risk-neutral distribution over a stochastic process. Similar considerations are applied when we use the Burr distribution. In this case, for a Burr III distribution we have

$$
\begin{aligned}
F_{\mathrm{BR}}(S_T, T) &= \left(1 - \frac{1}{1+(S_T/\beta)^c}\right)^\alpha,\\
S_T &\geq 0, c > 0, \alpha > 0, \beta > 0
\end{aligned}
$$

while the probability distribution is

$$
f(S_T, T) = \frac{c\alpha S_T^{c\alpha-1}\left(S_T^c+\beta^c\right) - c\alpha S_T^{c(\alpha-1)}}{\left(S_T^c+\beta^c\right)^{1+\alpha}}.
$$

Then by simply minimizing the sum of squared differences between a model premia conditional upon the parameters of the distribution and the observed option premia, an estimate of the approximate risk-neutral distribution can be obtained (for references see [47.191, 192]).

An alternative approach consists of recovering the risk-neutral distribution associated with an asset based on the information available on its derived products. For example, under risk-neutral pricing, for a vanilla call option with exercise price K and exercise date T, we have by definition:

$$C(K,T) = \mathrm{e}^{-R_\mathrm{f}T} \int\limits_K (S-K)f(S)\,\mathrm{d}t\,,$$

where $f(S)$ is the underlying asset risk-neutral density function of the price at its exercise time. Note that:

$$\frac{\partial C(K,T)}{\partial K} = -\mathrm{e}^{-R_\mathrm{f}T} \int\limits_K f(S)\,\mathrm{d}t$$
$$= -\mathrm{e}^{-R_\mathrm{f}T}\,[1-F(K)]\;,$$
$$\frac{\partial^2 C(K,T)}{\partial K^2} = \mathrm{e}^{-R_\mathrm{f}T} f(S)\,,$$
$$f(S) = \mathrm{e}^{R_\mathrm{f}T} \frac{\partial^2 C(K,T)}{\partial K^2}\,,$$

where $f(S)$, the RND is explicitly stated above as the second partial derivative of the call option. Of course, although such a relationship may be of theoretical value, it has little use if there is no data that can be used to calculate the underlying risk-neutral distribution. Other approaches and techniques might therefore be needed. This is a broad and difficult area of research which we can consider here only briefly. One approach that may be used is based on the constrained maximization of entropy to determine candidate risk-neutral distributions. This is considered next.

47.6.4 The Maximum-Entropy Approach

When some characteristics, data or other information regarding the risk-neutral distribution are available, it is possible to define its underlying distribution by selecting that distribution which assumes the least, that is the distribution with the greatest variability, given the available information. One approach that allows the definition of such distributions is defined by the maximum-entropy principle.

Entropy is based essentially on a notion of randomness. Its origins are in statistical physics. Boltzmann observed that entropy relates to missing information inasmuch as it pertains to the number of alternatives which remain possible to a physical system after all the macroscopically observable information concerning it has been recorded. In this sense, information can be interpreted as that which changes a system's state of randomness (or equivalently, as that quantity which reduces entropy). For example, for a word, which has k letters assuming zeros and ones, and one two, define a sequence of k letters, $(a_0, a_1, a_2, \ldots, a_k)$,

$$a_i = \begin{cases} 1 \\ 0 \end{cases}$$

for all $i \neq j$, $a_j \neq a_i = 2$ and one j. The total number of configurations (or strings of $k+1$ letters) that can be created is N where, $N = 2^k(k+1)$. The logarithm to the base 2 of this number of configurations is the information I, or $I = \log_2 N$ and in our case, $I = k + \log_2(k+1)$. The larger this number I, the larger the number of possible configurations and therefore the larger the randomness of the word. As a further example, assume an alphabet of G symbols and consider messages consisting of N symbols. Say that the frequency of occurrence of a letter is f_i, i. e., in N symbols the letter G occurs on average $N_i = f_i N$ times. There may then be W different possible messages, where

$$W = \frac{N!}{\prod\limits_{i=1}^{N} N_i!}\,.$$

The uncertainty of an N-symbol message is simply the ability to discern which message is about to be received. Thus,

$$W = \mathrm{e}^{HN} \quad \text{and}$$
$$H = \lim_{N\to\infty} \frac{1}{N} \log(W) = \sum p_i \log\left(1/p_i\right)\,,$$

which is also known as Shannon's entropy. If the number of configurations (i. e. W) is reduced, then the *information* increases. To see how the mathematical and statistical properties of entropy may be used in defining the risk-neutral distribution, we shall outline below a number of problems.

Discrimination and divergence

Consider for example two probability distributions given by $[F, G]$, one a theoretical risk-neutral distribution and another empirical expressing observed prices for example. We want to construct a measure that makes it possible to discriminate between these distributions. An attempt may be reached by using the following function we call the discrimination information [47.193]:

$$I(F,G) = \int F(x) \log \frac{F(x)}{G(x)}\,\mathrm{d}x\,.$$

In this case, $I(F, G)$ is a measure of *distance* between the distributions F and G. The larger the measure, the more we can discriminate between these distributions. For example, if $G(.)$ is a uniform distribution, then we have Shannon's measure of information. In this sense, it also provides a measure of departure from the random distribution. Selecting a distribution which has a maximum entropy (given a set of assumptions which are made explicit) is thus equivalent to the principle of insufficient reason proposed by Laplace. Thus, selecting a distribution with the largest entropy will imply a most conservative (risk wise) distribution. By the same token, we have

$$I(G, F) = \int G(x) \log \frac{G(x)}{F(x)} \mathrm{d}x$$

and the *divergence* between these two distributions is defined by

$$\begin{aligned} J(F, G) &= I(F, G) + I(G, F) \\ &= \int [F(x) - G(x)] \log \frac{F(x)}{G(x)} \mathrm{d}x , \end{aligned}$$

which provides a *symmetric measure* of distributions' *distance* since $J(F, G) = J(G, F)$.

For a discrete-time distribution (p, q), discrimination and divergence are given by

$$I(p, q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} ;$$

$$I(q, p) = \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i} , \sum_{i=1}^{n} p_i = 1 ,$$

$$\sum_{i=1}^{n} q_i = 1, \quad p_i \geq 0, \quad q_i \geq 0 ,$$

$$\begin{aligned} J(p, q) &= \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i} + \sum_{i=1}^{n} q_i \log \frac{q_i}{p_i} \\ &= \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i} . \end{aligned}$$

For example, say that q_i, $i = 1, 2, 3, \ldots n$ is a known empirical distribution and say that p_i, $i = 1, 2, 3, \ldots n$ is a theoretical distribution given by the geometric distribution: $p_i = (n, i)^{\top} p^i (1-p)^{n-i}$, whose parameter p we seek to estimate by minimizing the divergence, then the problem is:

$$\begin{aligned} &\min_{0 \leq p \leq 1} J(p, q) \\ &= \sum_{i=1}^{n} \left[\binom{n}{i} p^i (1-p)^{n-i} - q_i \right] \\ &\quad \times \log \frac{\binom{n}{i} p^i (1-p)^{n-i}}{q_i} , \end{aligned}$$

which can be minimized with respect to the parameter p. This approach can be generalized further to a multi-variable setting. For a bi-variate state discrete distribution, we have similarly:

$$I(p, q) = \sum_{j=1}^{m} \sum_{i=1}^{n} p_{ij} \log \left(\frac{p_{ij}}{q_{ij}} \right) ;$$

$$J(p, q) = \sum_{j=1}^{m} \sum_{i=1}^{n} (p_{ij} - q_{ij}) \log \left(\frac{p_{ij}}{q_{ij}} \right) .$$

On other hand for, for continuous distributions, we also have

$$I(F, G) = \int \int F(x, y) \log \frac{F(x, y)}{G(x, y)} \mathrm{d}x \, \mathrm{d}y$$

as well as the divergence:

$$\begin{aligned} &J(F, G) \\ &= \int \int [F(x, y) - G(x, y)] \log \frac{F(x, y)}{G(x, y)} \mathrm{d}x \, \mathrm{d}y . \end{aligned}$$

This distribution may then be used to provide divergence–distance measures between empirically observed and theoretical distributions.

When the underlying process is time-varying, we have for each time period:

$$I(p, q) = \sum_{t=1}^{T} \sum_{i=1}^{n} p_{it} \log \left(\frac{p_{it}}{q_{it}} \right) \quad \text{and}$$

$$J(p, q) = \sum_{t=1}^{T} \sum_{i=1}^{n} (p_{it} - q_{it}) \log \left(\frac{p_{it}}{q_{it}} \right)$$

where, obviously,

$$\sum_{i=1}^{n} p_{it} = 1; \quad \sum_{i=1}^{n} q_{it} = 1; \quad p_{it} \geq 0, q_{it} \geq 0 .$$

Moments condition as well as other constraints may also be imposed, providing a *least-divergent* risk-neutral pricing approximation to the empirical (incomplete) distribution considered.

Example 47.9: Assume that a non-negative random security price $\{\theta\}$ has a known mean given by $\hat{\theta}$, the maximum-entropy distribution for a continuous state distribution is given by solution of the following optimization problem:

$$\max H = -\int_0^\infty f(\theta)\log[f(\theta)]\,\mathrm{d}\theta \quad \text{Subject to:}$$

$$\int_0^\infty f(\theta)\,\mathrm{d}\theta = 1,$$

$$\hat{\theta} = \int_0^\infty \theta f(\theta)\,\mathrm{d}\theta\,.$$

The solution of this problem, based on the calculus of variations, yields an exponential distribution. In other words,

$$f(\theta) = \frac{1}{\hat{\theta}}\mathrm{e}^{-\theta/\hat{\theta}}, \quad \theta \geq 0\,.$$

When the variance of a distribution is specified as well, it can be shown that the resulting distribution is the normal distribution with specified mean and specified variance. This approach can be applied equally when the probability distribution is discrete, bounded, and multivariate with specified marginal distributions etc. In particular, it is interesting to point out that the maximum entropy of a multivariate distribution with specified mean and known variance–covariance matrix also turns out to be multivariate normal, implying that the normal is the most random distribution that has a specified mean and a specified variance. Evidently, if we also specify leptokurtic parameters, the distribution will not be normal. Potential applications are numerous, for example, let S and V be a stock price and its volatility, each of which is assumed to have observable prices and volatility. If we apply the conditions for a risk-neutral price, we then have:

$$\begin{aligned} S(t) &= \mathrm{e}^{-R_\mathrm{f}(T-t)}E_{\mathrm{RN}}S(T) \\ &= \int_0^\infty\int_0 \mathrm{e}^{-R_\mathrm{f}(T-t)}S(T)f(S,V,T)\,\mathrm{d}V\,\mathrm{d}S\,, \end{aligned}$$

where $f(S,V,T)$ is the probability distribution of the stock price with volatility V at time T. Adding data regarding the observed volatility at various times, prices of call and puts derived from this security, a theoretical optimization problem can be constructed that will indicate potential candidate distributions as implied risk-neutral distributions.

Example 47.10: consider the following random volatility process:

$$x_{t+1}^\alpha = x_t^\alpha + z_t; \quad z_t = \begin{cases} +3 & (0.5)(1-q) \\ +1 & (0.5)q \\ -1 & (0.5)q \\ -3 & (0.5)(1-q)\,. \end{cases}$$

A three-stage standard binomial process with probability π leads to

$$x_{t+1}^\alpha = x_t^\alpha + z_t\,;$$

$$z_t = \begin{cases} +3 & 0.5(1-q) \leftrightarrow \pi^3 \\ +1 & 0.5q \leftrightarrow 3\pi^2(1-\pi) \\ -1 & 0.5q \leftrightarrow 3\pi(1-\pi)^2 \\ -3 & 0.5(1-q) \leftrightarrow (1-\pi)^3\,. \end{cases}$$

As a result, we can calculate the probability π by minimizing the divergence J, which is given by by an appropriate choice of π

$$\begin{aligned} J &= \left[\pi^3 - (0.5)(1-q)\right]\log\left(\frac{\pi^3}{0.5(1-q)}\right) \\ &+ \left[3\pi^2(1-\pi) - (0.5)q\right]\log\left(\frac{3\pi^2(1-\pi)}{0.5q}\right) \\ &+ \left[3\pi(1-\pi)^2 - (0.5)q\right]\log\left(\frac{3\pi(1-\pi)^2}{0.5q}\right) \\ &+ \left[(1-\pi)^3 - 0.5(1-q)\right]\log\left(\frac{(1-\pi)^3}{0.5(1-q)}\right)\,. \end{aligned}$$

Example 47.11: the problem based on forward and option prices be given by:

$$\max_{f(.)}\int_0^\infty f(S)\ln\left(\frac{1}{f(S)}\right)\mathrm{d}S \quad \text{Subject to:}$$

$$\int_0^\infty f(S)\,\mathrm{d}S = 1; \quad F(0,T) = \int_0^\infty Sf(S)\,\mathrm{d}S \quad \text{and}$$

$$C_i(S,K,T) = \mathrm{e}^{-R_\mathrm{f}T}\int_0^\infty c_i(x)f(x)\,\mathrm{d}x,$$

$$i = 1, 2, \ldots m\,,$$

where C_i is the price of the option at time t with payoff at T given by $c_i(x)$. The solution of this problem is

$$f(S) = \frac{1}{\mu} \exp\left[\lambda_0 S + \sum_{i=1}^{m} \lambda_i c_i(S)\right] \quad \text{with}$$
$$\mu = \int_0^\infty \exp\left[\lambda_0 S + \sum_{i=1}^{m} \lambda_i c_i(S)\right] \mathrm{d}S$$

which can be used as a candidate risk-neutral distribution where the parameters are to be determined based on the available data.

Example 47.12 (a maximum-entropy price process): Consider a bivariate probability distribution (or a stochastic price process) $h(x,t)$, $x \in [0,\infty)$, $t \in [a,b]$ the maximum-entropy criterion can be written as an optimization problem, maximizing the entropy as follows:

$$\max H = -\int_0^\infty \int_a^b h(x,t) \log\left[h(x,t)\right] \mathrm{d}t\, \mathrm{d}x\,,$$

subject to partial information regarding the distribution $h(x,t)$, $x \in [0,\infty)$, $t \in [a,b]$. Say that, at the final time b, the price of a stock is for sure X_b, while initially it is given by X_a. Further, let the average price over the relevant time interval be known and be given by $\bar{X}_{(a,b)}$, this may be translated into the following constraints:

$$h(X_a,a) = 1, \quad h(X_b,b) = 1 \quad \text{and}$$
$$\frac{1}{b-a} \int_a^b \int_0^\infty x h(x,t)\, \mathrm{d}x\, \mathrm{d}t = \bar{X}_{(a,b)}$$

which are to be accounted for in the entropy optimization problem. Of course, we can add additional constraints when more information is available. Thus, the maximum-entropy approach can be used as an *alternative rationality* for the construction of risk-neutral distributions when the burden of explicit hypotheses formulation or the justification of the model at hand is too heavy. Theoretical justifications as well as applications to finance may be found in *Avellaneda* et al. [47.194] (see also [47.195–197]). Below we consider a simple example to highlight some of the practical issues we may have to address when dealing with such problems.

Example 47.13 (engineered bond pricing): consider a Vasicek model of interest rates, fluctuating around a long-run rate α. This fluctuation is subjected to random and normal perturbations of mean zero and variance $\sigma\, \mathrm{d}t$, or.

$$\mathrm{d}r = \beta(\alpha - r)\, \mathrm{d}t + \sigma\, \mathrm{d}w$$

whose solution at time t when the interest rate is $r(t)$ is, as seen earlier:

$$r(u;t) = \alpha + \mathrm{e}^{-\beta(u-t)}\left[r(t) - \alpha\right] + \sigma \int_t^u \mathrm{e}^{-\beta(u-\tau)}\, \mathrm{d}w(\tau)\,.$$

In this theoretical model we might consider the parameters set $\Lambda \equiv (\alpha, \beta, \sigma)$ as determining a number of martingales (or bond prices) that obey the model above, namely bond prices at time $t = 0$ can theoretically equal the following:

$$B_{\text{th}}(0,T;\alpha,\beta,\sigma) = E\left(\mathrm{e}^{\int_0^T r(u;\alpha,\beta,\sigma)\,\mathrm{d}u}\right)\,.$$

In this simple case, interest rates have a normal distribution with a mean and variance (volatility) evolution stated above and therefore $\int_0^T r(u,\alpha,\beta,\sigma)\,\mathrm{d}u$ also has a normal probability distribution with mean and variance give by

$$m[r(0),T] = \alpha T + \left(1 - \mathrm{e}^{-\beta T}\right) \frac{r(0) - \alpha}{\beta}\,,$$
$$v(r(0),T) = v(T) = \frac{\sigma^2}{2\beta^3}\left(4\mathrm{e}^{-\beta T} - \mathrm{e}^{-2\beta T} + 2\beta T - 3\right)\,.$$

Note that in these equations the variance is independent of the interest rate while the mean is a linear function of the interest, which we write as:

$$m[r(0),T] = \alpha\left(T - \frac{\left(1 - \mathrm{e}^{-\beta T}\right)}{\beta}\right) + r(0)\frac{\left(1 - \mathrm{e}^{-\beta T}\right)}{\beta}\,.$$

This property is called an affine structure as we saw earlier and is of course computationally desirable for it will allow a simpler calculation of the desired martingale. As a result, the theoretical zero-coupon bond price paying one dollar T periods hence can be written as:

$$\begin{aligned}B_{\text{th}}(0, T; \alpha, \beta, \sigma) &= E\left(\mathrm{e}^{\int_0^T r(u,\alpha,\beta,\sigma)\,\mathrm{d}u}\right)\\ &= \mathrm{e}^{-m(r(0),T)+v(T)/2}\\ &= \mathrm{e}^{A(T)-r_0 D(T)},\\ A(T) &= -\alpha\left(T - \frac{\left(1-\mathrm{e}^{-\beta T}\right)}{\beta}\right)\\ &\quad + \frac{\sigma^2}{4\beta^3}\left(4\mathrm{e}^{-\beta T} - \mathrm{e}^{-2\beta T}\right.\\ &\quad \left. + 2\beta T - 3\right),\\ D(T) &= \frac{\left(1-\mathrm{e}^{-\beta T}\right)}{\beta}.\end{aligned}$$

Assume now that continuous series affine bond values are observed and given by $B_{\text{obs}}(0, T)$ which we write for convenience as $B_{\text{obs}}(0, T) = \mathrm{e}^{-R_T T}$. Without loss of generality we can consider the yield error term given by

$$\Delta_T = R_T - [A(T) - r_0 D(T)]$$

and thus select the parameters (i. e. the martingale) that is closest in some sense to the observed values. For example, a least-squares solution of n observed bond values yields the following optimization problem:

$$\min_{\alpha,\beta,\sigma} \sum_{i=1}^{n} (\Delta_i)^2$$

Alternatively, we can also minimize the divergence between the theoretical and the observed series. For a continuous-time function $B_{\text{obs}}(0, T) = \mathrm{e}^{-R_T T}$, we have

$$\begin{aligned}&\min_{\alpha,\beta,\sigma} J(\Upsilon)\\ &= \int_0^{\Upsilon} [B_{\text{th}}(0, u) - B_{\text{obs}}(0, u)] \ln\left(\frac{B_{\text{th}}(0, u)}{B_{\text{obs}}(0, u)}\right) \mathrm{d}u\,.\end{aligned}$$

In this case, it is easy to show that, given the continuous-time observed bond function $\left[B_{\text{obs}}(0, u), 0 \le u \le \Upsilon\right]$, the optimal parameters satisfy the following three equalities

$$\begin{aligned}&\int_0^{\Upsilon} \left(\frac{\partial \ln B_{\text{th}}(0, u)}{\partial\theta} [B_{\text{obs}}(0, u) - B_{\text{th}}(0, u)]\right)\\ &= \int_0^{\Upsilon} \frac{\partial B_{\text{th}}(0, u)}{\partial\theta} \ln\left(\frac{B_{\text{th}}(0, u)}{B_{\text{obs}}(0, u)}\right) \mathrm{d}u\,,\end{aligned}$$

where $\theta = \alpha, \beta, \sigma$. These problems can be solved numerically of course.

When the model has time-varying parameters, the problem we faced above turns out to have an infinity of unknown parameters and therefore the yield-curve estimation problem we considered above might be grossly under-specified. Explicitly, let the interest mode be defined by:

$$\mathrm{d}r(t) = \beta[\alpha(t) - r(t)]\,\mathrm{d}t + \sigma\,\mathrm{d}w$$

The theoretical bond value still has an affine structure and therefore we can write

$$\begin{aligned}B_{\text{th}}(t, T; \alpha(t), \beta, \sigma) &= E\left(\mathrm{e}^{\int_0^T r(u,\alpha,\beta,\sigma)\,\mathrm{d}u}\right)\\ &= \mathrm{e}^{A(t,T)-r(t)D(t,T)}.\end{aligned}$$

The integral interest rate process is still normal with mean and variance leading to

$$\begin{aligned}D(t, T) &= \frac{1}{\beta}\left(1-\mathrm{e}^{-\beta(T-t)}\right),\\ A(t, T) &= \int_t^T \left[\frac{1}{2}\sigma^2 D^2(s, T) - \beta\alpha(s)D(s, T)\right]\mathrm{d}s\end{aligned}$$

or

$$\begin{aligned}\frac{\mathrm{d}A(t, T)}{\mathrm{d}t} &= \alpha(t)\left(1-\mathrm{e}^{-\beta(T-t)}\right)\\ &\quad - \frac{\sigma^2}{2\beta^2}\left(1-\mathrm{e}^{-\beta(T-t)}\right)^2, \quad A(T, T) = 0\end{aligned}$$

in which $\alpha(t)$, β, and σ are unspecified. If we equate this equation to the available bond data we will obviously have far more unknown variables than data points and therefore the yield-curve estimate will depend again on the optimization technique we use to generate the best fit functions $\alpha^*(t)$, β^*, and σ^*. Such problems can be formulated as standard problems in the calculus of variations. For example, if we consider the observed bond prices $B_{\text{obs}}(t, T)$, $t \le T < \infty$, for a specific time T and minimize the squared error, the following problem results

$$\begin{aligned}\min_{\alpha(u)} J(\alpha, A) = \int_0^t &\left[\mathrm{e}^{A(u,T)-r(u)\left[\frac{1}{\beta}\left(1-\mathrm{e}^{-\beta(T-u)}\right)\right]}\right.\\ &\left. - B_{\text{obs}}(u, T)\right]^2 \mathrm{d}u,\end{aligned}$$

$$\frac{\mathrm{d}A(u,T)}{\mathrm{d}u} = \alpha(u)\left(1-\mathrm{e}^{-\beta(T-u)}\right) - \frac{\sigma^2}{2\beta^2}\left(1-\mathrm{e}^{-\beta(T-u)}\right)^2 ,$$
$$A(T,T) = 0$$

which can be solved by the usual techniques in optimal control. Note that this problem can be written as the linear quadratic optimization problem where we have purposely given greater weight to data observed close to time t, and given less importance to data that are farther away from the current time t, or:

$$\min_{\alpha(u)} = \int_0^t \mathrm{e}^{\upsilon u}\,[A(u,T)-c(u,T)]^2\,\mathrm{d}u\,,$$
$$\frac{\mathrm{d}A(u,T)}{\mathrm{d}u} = \alpha(u)a(u,T) - b(u,T)\,,\quad A(T,T)=0\,,$$
$$c(u,t) = y_{\text{obs}}(u,T) + r(u)\left[\frac{1}{\beta}\left(1-\mathrm{e}^{-\beta(T-u)}\right)\right],$$
$$a(u,t) = \left(1-\mathrm{e}^{-\beta(T-u)}\right);$$
$$b(u,t) = \frac{\sigma^2}{2\beta^2}\left(1-\mathrm{e}^{-\beta(T-u)}\right)^2 .$$

The solution of this problem is a standard optimal control problem which may be either a boundary solution (called bang–bang, bringing the control parameter $\alpha(u)$ to an upper or lower constraint value) or that can be singular (in which case its calculation is found by tests based on a higher-order derivative). Using the deterministic dynamic programming framework, we have an optimal solution given by:

$$-\frac{\partial J}{\partial u} = \min_{\alpha(u)}\left\{\mathrm{e}^{\upsilon u}\,[A(u,T)-c(u,T)]^2 + \frac{\partial J}{\partial A}\,[\alpha(u)a(u,T)-b(u,T)]\right\}$$

On a singular strip, $\frac{\partial J}{\partial A}=0$ where $a(u,t)\neq 0$ and thus, in order to calculate $\alpha(u)$, we can proceed by a change of variables and transforming the original control problem into a linear quadratic control problem that can be solved by the standard optimal control methods. Explicitly, set:

$$y(u) = \mathrm{e}^{\upsilon u/2}\,[A(u,T)-c(u,T)]\,,$$
$$\frac{\mathrm{d}w(u)}{\mathrm{d}u} = \alpha(u)\quad\text{and}$$
$$z(u) = y(u) - \mathrm{e}^{\upsilon u/2}a(u,T)w(u)\,.$$

Thus, the problem objective is reduced to

$$\min_{\alpha(u)} = \int_0^t \left[z(u)+e^{\upsilon u/2}a(u,T)w(u)\right]^2 \mathrm{d}u$$

while the constraint is

$$\frac{\mathrm{d}A(u,T)}{\mathrm{d}u} = \alpha(u)a(u,T) - b(u,T)\,,$$
$$z(u) + \mathrm{e}^{\upsilon u/2}a(u,T)w(u) = \mathrm{e}^{\upsilon u/2}\,[A(u,T)-c(u,T)]\;;$$
$$\frac{\mathrm{d}w(u)}{\mathrm{d}u} = \alpha(u)\,.$$

After some elementary manipulations, we have

$$\dot{z}(u) = \frac{\upsilon}{2}z(u) - \mathrm{e}^{\upsilon u/2}\dot{a}(u,T)w(u) - \mathrm{e}^{\upsilon u/2}\left[b(u,T) - \mathrm{e}^{\upsilon u/2}\dot{c}(u,T)\right].$$

This defines a linear quadratic cost-control problem

$$\min_{w(u)} = \int_0^t \left[z(u)+\mathrm{e}^{\upsilon u/2}a(u,T)w(u)\right]^2 \mathrm{d}u\,.$$

Subject to:

$$\frac{\mathrm{d}z(u)}{\mathrm{d}u} = \frac{\upsilon}{2}z(u) - \mathrm{e}^{\upsilon u/2}\dot{a}(u,T)w(u) - \mathrm{e}^{\upsilon u/2}\left[b(u,T) - \mathrm{e}^{\upsilon u/2}\dot{c}(u,T)\right].$$

Inserting the original problem parameters we have:

$$\min_{w(u)} \int_0^t \left[z(u) + \mathrm{e}^{\upsilon u/2}\left\{1-\mathrm{e}^{-\beta(T-u)}\right\} \times w(u)\right]^2 \mathrm{d}u.\quad \text{Subject to:}$$
$$\frac{\mathrm{d}z(u)}{\mathrm{d}u} = \dot{z}(u) = \frac{\upsilon}{2}z(u) + \beta\,\mathrm{e}^{-\beta(T-u-\upsilon u/2\beta)}w(u) - \mathrm{e}^{\upsilon u/2}\left[\frac{\sigma^2}{2\beta^2}\left(1-\mathrm{e}^{-\beta(T-u)}\right)^2 - \mathrm{e}^{\upsilon u/2}\dot{c}(u,T)\right],$$
$$\frac{\mathrm{d}c(u)}{\mathrm{d}u} = \dot{c}(u,t) = \dot{y}_{\text{obs}}(u,T) + \dot{r}(u)\left[\frac{1}{\beta}\left(1-\mathrm{e}^{-\beta(T-u)}\right)\right] - r(u)\,\mathrm{e}^{-\beta(T-u)}\,,$$

which is a control problem linear in the state and in the control with a quadratic objective. As a result, the solution for the control $w(u)$ is of the linear feedback form

$$w(u) = Q(u) + S(u)z(u) \quad \text{or}$$
$$\alpha(u) = \dot{Q}(u) + S(u)\dot{z}(u) + \dot{S}(u)z(u)\,.$$

Finally, when bond data are available over multiple periods dates T, the optimal control problem we have considered above can be extended by solving

$$\min_{\alpha(u)} \sum_{j=1}^{N_T} \int_0^t e^{vu} \left[A(u, T_j) - c(u, T_j)\right]^2 \mathrm{d}u$$

$$\frac{\mathrm{d}A(u, T_j)}{\mathrm{d}u} = \alpha(u)a(u, T_j) - b(u, T_j)\,,$$
$$A(T_j, T_j) = 0$$

and in continuous time:

$$\min_{\alpha(u)} \int_0^\infty \int_0^t \mathrm{e}^{vu} \left[A(u, T) - c(u, T)\right]^2 \mathrm{d}T\, \mathrm{d}u\,,$$
$$\frac{\partial A(u, T)}{\partial u} = \alpha(u)a(u, T) - b(u, T)\,,$$
$$A(T, T) = 0\,.$$

The solution of these problems are then essentially numerical problems, however.

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