49. Arithmetic and Geometric Processes

Section 49.1 introduces two special monotone processes. A stochastic process is an AP (or a GP) if there exists some real number (or some positive real number) such that after some additions (or multiplications) it becomes a renewal process (RP). Either is a stochastically monotonic process and can be used to model a point process, i. e. point events occurring in a haphazard way in time or space, especially with a trend. For example, the events may be failures arising from a deteriorating machine, and such a series of failures is distributed haphazardly along a time continuum.

Sections. 49.2–49.5 discuss estimation procedures for a number K of independent, homogeneous APs (or GPs). More specifically; in Sect. 49.2, Laplace's statistics are recommended for testing whether a process has a trend or K processes have a common trend, and a graphical technique is suggested for testing whether K processes come from a common AP (or GP) as well as having a common trend; in Sect. 49.3, three parameters – the common difference (or ratio), the intercept and the variance of errors – are estimated using simple linear regression techniques; in Sect. 49.4, a statistic is introduced for testing whether K processes come from a common AP (or GP); in Sect. 49.5, the mean and variance of the first average random variable of the AP (or GP) are estimated based on the results derived in Sect. 49.3.

Section 49.6 mentions some simulation studies performed to evaluate various nonparametric estimators and to compare the estimates, obtained from various estimators, of the parameters. Some suggestions for selecting the best estimators under three non-overlapping ranges of the common difference (or ratio) values are made based on the results of the simulation studies.

In Sect. 49.7, ten real data sets are treated as examples to illustrate the fitting of AP, GP, homogeneous Poisson process (HPP) and nonhomogeneous Poisson process (NHPP) models.

In Sect. 49.8, new repair–replacement models are proposed for a deteriorating system, in which the successive operating times of the system form an arithmetico-geometric process (AGP) and are stochastically decreasing, while the successive repair times after failure also constitute an AGP but are stochastically increasing. Two kinds of replacement policy are considered, one based on the working age (a continuous decision variable) of the system and the other determined by the number of failures (a discrete decision variable) of the system. These policies are considered together with the performance measures, namely loss (or its negation, profit), cost, and downtime (or its complement, availability). Applying the well-known results of renewal reward processes, expressions are derived for the long-run expected performance measure per unit total time, and for the long-run expected performance measure per unit operation time, under the two kinds of policy proposed.

In Sect. 49.9, some conclusions of the applicability of an AP and/or a GP based on partial findings of four real case studies are drawn.

Section 49.10 gives five concluding remarks. Finally, the derivations of some key results are outlined in the Appendix, followed by the results of both the APs and GPs summarized in Table 49.6 for easy reference.

Most of the content of this chapter is based on the author's own original works that appeared in *Leung* et al. [49.1–13], while some is extracted from *Lam* et al. [49.14–16].

In this chapter, the procedures are, for the most part, discussed in reliability terminology. Of course, the methods are valid in any area of application (see Examples 1, 5, 6 and 9 in Sect. 49.7), in which case they should be interpreted accordingly.

In the statistical analysis of a series of events, a common method is to model the series using a point process. To start with, it is essential to test whether the data of successive inter-event times, denoted by X_i $(i = 1, 2, \dots)$, demonstrate a trend. If there is no trend, we may model the data using a stationary point process (i. e. a counting process that has stationary, but not necessarily independent, increments), or using a sequence of independent and identically distributed (i.i.d.) random variables $X \equiv X_i$ for all i. For the latter, we may model the corresponding counts of events in time using a renewal process (RP). In particular, if X is exponentially distributed with a rate parameter λ, we may use a homogeneous Poisson process (HPP) with a constant rate λ to model the data. The HPP is one of the most common stochastic processes for modeling counts of events in time or area/volume. This process is a standard for *randomness*, as the assumptions involved state that events must occur independently and any two non-overlapping intervals of the same size have the same probability of capturing one of the events of interest. However, in practice the data of successive inter-event times usually exhibit a trend. We may model them using a nonstationary model, or using a nonhomogeneous Poisson process (NHPP) in which the rate at time t is a function of t. The NHPP is a popular approach used to model data with a trend. For more details of these methods, see *Cox* and *Lewis* [49.17], and *Ascher* and *Feingold* [49.18].

Most research on the maintenance of a repairable system has made either the perfect or minimal repair assumption. Perfect repair means that, after repair, a failed system is as good as new, i. e. a system's successive operating times constitute an RP, see *Barlow* and *Proschan* [49.19]. For a perfect repair model, if the time needed to repair a system is considered negligible, results of RPs can be applied to resolve the system's maintenance problems, see *Ross* [49.20]; if repair time has to be taken into account and the corresponding consecutive repair times constitute another RP, results of alternating RPs can be applied instead, see *Birolini* [49.21]. However, in practice, this is not always the case. Minimal repair means that a failed system will function, after repair, with the same rate of failure and the same effective age as at the epoch of the last failure. For a minimal repair model, where repair time is assumed negligible, an NHPP in which the rate of occurrence of failures (ROCOF) is monotone can provide at least a good first-order model for a deteriorating system, see *Ascher* and *Feingold* [49.18]. That is, failures constitute an NHPP with a suitable parametric form for ROCOF. If the repair time has to be taken into account, the NHPP approach cannot be used.

The popularity of the power-law process (PLP) is based on two features: firstly, it can model deteriorating or improving systems; secondly, point estimators for the parameters have simple closed-form expressions and hypotheses tests can be undertaken using existing tables. The PLP denoted and given by $r(t) = \lambda\beta t^{\beta-1}$ for $t \geq 0$ and $\lambda, \beta > 0$ is the most important ROCOF parametric form in an NHPP model. If $\beta > 1$, the ROCOF increases with time, as often happens with aging machinery. This is one of the two main conditions for a preventive replacement being worth carried out (the other condition is that the average cost c_p of a replacement is much greater than that c_r of a minimal repair). But if $0 < \beta < 1$, the ROCOF decreases with time; hence, the PLP can model reliability growth as well. The HPP, which has constant ROCOF, is a special case of the PLP with $\beta = 1$. *Rigdon* and *Basu* [49.22] gave a detailed discussion of the PLP.

Another two-parameter ROCOF form, widely quoted in the literature, that can also model deterioration and reliability growth is the log-linear process (LLP), in which the ROCOF at time t is modeled as $s(t) = \mathrm{e}^{\alpha_0+\alpha_1 t}$ for $t \geq 0$ and $-\infty < \alpha_0$, $\alpha_1 < \infty$, with $\alpha_1 > 0$ under deterioration. This process is less often used, possibly because it is seldom found to be applicable or mathematically less tractable.

Barlow and *Hunter* [49.23] first introduced the idea of minimal repair and proposed a system which is replaced with a regular period T and undergoes minimal repairs upon failures between the periodic replacements. *Muth* [49.24] then proposed a replacement model in which minimal repairs upon system failures are performed up to age T and the system is replaced at the first failure after T. Later, *Park* [49.25] proposed a modification of the model in which a system undergoes minimal repairs for the first $(N-1)$ failures and is replaced at the Nth failure. *Nakagawa* and *Kowada* [49.26] put the first and third types of policy together and constructed a replacement model in which a system is regularly replaced with a period T or at the Nth failure after its installation, whichever occurs first. The system undergoes only minimal repair upon failures between the periodic replacements. In *Leung* and *Cheng* [49.3], Nakagawa and Kowada's replacement model was employed, and the optimal replacement policy based on minimizing the long-run expected cost per month for each type of engine was determined.

The perfect repair model may be reasonable for a system with one simple unit only, and the minimal repair model seems plausible for systems consisting of many components, each having its own failure mode. In many practical instances, repair activities may not result in such extreme situations but in complicated intermediate ones. *Brown* and *Proschan* [49.27] considered the model of imperfect repair which, with probability p, is a perfect repair or, with probability $1-p$, is a minimal repair. *Kijima* [49.28, 29] studied a more general repair model which includes the imperfect repair model as a special case. For a review of imperfect maintenance models, see *Pham* and *Wang* [49.30], and *Wang* and *Pham* [49.31].

An arithmetic process (AP) or a geometric process (GP), which is a nonstationary model, can be used as an alternative to the NHPP in analyzing data of inter-event times that exhibit a trend. This appears to be a useful model for failure or repair data arising from a single system or a collection of independent, homogeneous systems. Consider the maintenance problems of a repairable system and bear in mind that most repairable systems, like engines and gearboxes, are deteriorative. Two basic characteristics of a deteriorating system are that, because of wear through operation or metal fatigue under stress, the system's successive operating times decrease and so the system's life is finite; and that, because it is more difficult and hence takes more time to rectify accumulated wear, the corresponding consecutive repair times increase until finally the system is beyond repair. Based on this understanding, an AP approach proposed by *Leung* [49.5] or a GP approach proposed by *Lam* [49.14] is considered more relevant, realistic and direct for the modeling of maintenance problems in a deteriorating system. Although all discussions in this chapter are in terms of deteriorating systems, they are also valid for improving systems (see Examples 2, 4 and 10 in Sect. 49.7).

The following main symbols in the text are adopted. For a fixed $k = 1, \dots, K$,

- $A_{n,k}$ (or $G_{n,k}$) denotes either the operating time after the $(n-1)$th repair for $n = 1, 2, \dots, N_k$ with $X_0 = 0$, or the repair time after the nth failure for $n = 1, 2, \dots, N_k$;
- d denotes either a common difference d_a of a decreasing arithmetic process such that $d_\mathrm{a} \in \left(0, \frac{\mu_{A_1,k}}{n-1}\right]$ or a common difference d_b of an increasing arithmetic process such that $d_\mathrm{b} < 0$;
- r denotes either a common ratio r_a of a decreasing geometric process such that $r_\mathrm{a} > 1$ or a common ratio r_b of an increasing geometric process such that $0 < r_\mathrm{b} < 1$;
- $\mu_{A_{n,k}}$ (or $\mu_{G_{n,k}}$) is the mean of $A_{n,k}$ (or $G_{n,k}$) for $n = 1, 2, \dots, N_k$;
- $\sigma^2_{A_{n,k}}$ (or $\sigma^2_{G_{n,k}}$) is the variance of $A_{n,k}$ (or $G_{n,k}$) for $n = 1, 2, \dots, N_k$;
- $\varepsilon_{\mathrm{A},n,k}$ (or $\varepsilon_{\mathrm{G},n,k}$) is an error term with mean 0 and constant variance denoted by $\sigma^2_{\mathrm{A},\varepsilon}$ (or $\sigma^2_{\mathrm{G},\varepsilon}$).

49.1 Two Special Monotone Processes

The work in this section is substantially based on *Leung* [49.4, 11].

49.1.1 Arithmetic Processes

Suppose that K independent, homogeneous APs are available. A definition of the kth AP for $k = 1, \dots, K$ is given below.

Definition 49.1
Given a sequence of random variables $A_{1,k}, A_{2,k}, \dots$, if for some real number d, $\{A_{n,k} + (n-1)d, n = 1, 2, \dots\}$ forms an RP, then $\{A_{n,k}, n = 1, 2, \dots\}$ is an AP for $k = 1, \dots, K$. The constant d is called the common difference of the AP.

Three specializations of an AP are given below.

If $d \in \left(0, \frac{\mu_{A_1}}{n-1}\right]$, where $n = 2, 3, \dots$ and $k = 1, \dots, K$; and $\mu_{A_{1,k}}$ is the mean of the first random variable $A_{1,k}$, then the AP is called a decreasing AP. If $d < 0$, then the AP is called an increasing AP. If $d = 0$, then the AP reduces to an RP.

The upper bound of d in the first specialization can be obtained as follows: by Definition 49.1, the expression for the general term of an AP is given by $A_{n,k} \stackrel{d}{=} A_{1,k} - (n-1)d$. Taking expectations of both sides of this expression, and remembering that $A_{n,k}$ is a nonnegative random variable and hence $\boldsymbol{E}(A_{n,k}) \equiv \mu_{A_{n,k}} \geq 0$ for $n = 1, 2, \dots$; we obtain, after transposition, the upper bound of d given by $\frac{\mu_{A_{1,k}}}{n-1}$ for $n = 2, 3, \dots$. Clearly, the positive integer n is limited for a decreasing AP. Moreover, if the value of d is close to its upper bound, we will obtain a short sequence of nonnegative random variables. However, such a subtractive process is likely to be useful in a deteriorating system (e.g. an engine or a gearbox), which fails rarely (e.g. two/three times) over its usual span of life (e.g. five years). This implicitly means that the system wears out, between two successive failures, to such an extraordinary extent that the corresponding system's successive operating time decreases dramatically.

Given an AP $\{A_{n,k}, n = 1, 2, \dots\}$ for $k = 1, \dots, K$, we have $A_{n,k} \stackrel{d}{=} A_{1,k} - (n-1)d$ by Definition 49.1. Therefore, the means and variances of $A_{n,k}$ can respectively be written as

$$\begin{aligned}\mu_{A_{n,k}} &\equiv \boldsymbol{E}(A_{n,k}) = \boldsymbol{E}(A_{1,k}) - (n-1)d \\ &\equiv \mu_{A_{1,k}} - (n-1)d \end{aligned} \tag{49.1}$$

and

$$\begin{aligned}\sigma^2_{A_{n,k}} &\equiv \boldsymbol{V}(A_{n,k}) = \boldsymbol{V}[A_{1,k} - (n-1)d] = V(A_{1,k}) \\ &\equiv \sigma^2_{A_{1,k}} . \end{aligned} \tag{49.2}$$

Consider K independent, homogeneous APs $\{A_{n,k}, n = 1, \dots, N_k$ and $k = 1, \dots, K\}$ together. Without loss of generality, we assume $N_1 \geq N_2 \geq \dots \geq N_K$. Denote

$$\bar{A}_n \equiv \frac{\sum_{k=1}^{K^*} A_{n,k}}{K^*} ,$$

$$\mu_{\bar{A}_1} \equiv \frac{\sum_{k=1}^{K} \mu_{A_{1,k}}}{K} = \frac{\sum_{k=1}^{K^*} \mu_{A_{1,k}}}{K^*} , \quad \text{and}$$

$$\sigma^2_{\bar{A}_1} \equiv \frac{\sum_{k=1}^{K} \sigma^2_{A_{1,k}}}{K} = \frac{\sum_{k=1}^{K^*} \sigma^2_{A_{1,k}}}{K^*} ,$$

where

$$\begin{aligned} K^* &= K && \text{for } n = 1, \dots, N_K \text{ or,} \\ &= K-1 && \text{for } n = N_K + 1, \dots, N_{K-1} \text{ or,} \\ &= K-2 && \text{for } n = N_{K-1} + 1, \dots, N_{K-2} \text{ or,} \\ &\ \ \vdots \\ &= 1 && \text{for } n = N_2 + 1, \dots, N_1 . \end{aligned}$$

To clarify the definition of $\bar{A}_n$, let us consider $K = 3$, $N_1 = 5$, $N_2 = 3$, $N_3 = 2$. Then we have

$$\begin{aligned} \bar{A}_1 &= \frac{A_{1,1} + A_{1,2} + A_{1,3}}{3} , \\ \bar{A}_2 &= \frac{A_{2,1} + A_{2,2} + A_{2,3}}{3} , \\ \bar{A}_3 &= \frac{A_{3,1} + A_{3,2}}{2} , \\ \bar{A}_4 &= A_{4,1} \text{ and } \bar{A}_5 = A_{5,1} . \end{aligned}$$

Therefore, using (49.1) and (49.2), the means and variances of $\bar{A}_n$ can respectively be written as

$$\mu_{\bar{A}_n} \equiv \boldsymbol{E}(\bar{A}_n) = \frac{\sum_{k=1}^{K^*} E(A_{n,k})}{K^*} = \frac{\sum_{k=1}^{K^*} E[A_{1,k} - (n-1)d)]}{K^*} \equiv \frac{\sum_{k=1}^{K^*} \mu_{A_{1,k}} - K^*(n-1)d}{K^*} = \mu_{\bar{A}_1} - (n-1)d \tag{49.3}$$

and

$$\sigma^2_{\bar{A}_n} \equiv \boldsymbol{V}(\bar{A}_n) = \frac{\sum_{k=1}^{K^*} V(A_{n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} V[A_{1,k} - (n-1)d]}{(K^*)^2} \equiv \frac{\sum_{k=1}^{K^*} \sigma^2_{A_{1,k}}}{(K^*)^2} = \frac{\sigma^2_{\bar{A}_1}}{K^*} . \tag{49.4}$$

49.1.2 Geometric Processes

Suppose that K independent, homogeneous GPs are available. A definition of the kth GP for $k = 1, \ldots, K$ is given below.

Definition 49.2
Given a sequence of random variables $G_{1,k}, G_{2,k}, \ldots,$ if for some $r > 0$, $\{r^{(n-1)} G_{n,k}, n = 1, 2, \ldots\}$ forms an RP, then $\{G_{n,k}, n = 1, 2, \ldots\}$ is a GP for $k = 1, \ldots, K$. The constant r is called the common ratio of the GP.

Three specializations of a GP are given below.

If $r > 1$, then the GP is called a decreasing GP. If $0 < r < 1$, then the GP is called an increasing GP. If $r = 1$, then the GP reduces to an RP.

Given a GP $\{G_{n,k}, n = 1, 2, \ldots\}$ for $k = 1, \ldots, K$; we have $G_{n,k} \overset{d}{=} \frac{G_{1,k}}{r^{(n-1)}}$ from Definition 49.2. Therefore, the means and variances of $G_{n,k}$ can be written as

$$\mu_{G_{n,k}} \equiv \boldsymbol{E}(G_{n,k}) = \frac{\boldsymbol{E}(G_{1,k})}{r^{(n-1)}} \equiv \frac{\mu_{G_{1,k}}}{r^{(n-1)}} \tag{49.5}$$

and

$$\sigma^2_{G_{n,k}} \equiv \boldsymbol{V}(G_{n,k}) = \boldsymbol{V}\left(\frac{G_{1,k}}{r^{(n-1)}}\right) = \frac{\boldsymbol{V}(G_{1,k})}{r^{2(n-1)}} \equiv \frac{\sigma^2_{G_{1,k}}}{r^{2(n-1)}} . \tag{49.6}$$

Consider K independent, homogeneous GPs $\{G_{n,k}, n = 1, \ldots, N_k$ and $k = 1, \ldots, K\}$ together. Without loss of generality, we assume $N_1 \geq N_2 \geq \ldots \geq N_K$. Denote

$$\bar{G}_n \equiv \frac{\sum_{k=1}^{K^*} G_{n,k}}{K^*} ,$$

$$\mu_{\bar{G}_1} \equiv \frac{\sum_{k=1}^{K} \mu_{G_{1,k}}}{K} = \frac{\sum_{k=1}^{K^*} \mu_{G_{1,k}}}{K^*} , \quad \text{and}$$

$$\sigma^2_{\bar{G}_1} \equiv \frac{\sum_{k=1}^{K} \sigma^2_{G_{1,k}}}{K} = \frac{\sum_{k=1}^{K^*} \sigma^2_{G_{1,k}}}{K^*} ,$$

where K^* has previously been defined and $\bar{G}_n$ has a similar meaning to $\bar{A}_n$.

Therefore, using (49.5) and (49.6), the means and variances of $\bar{G}_n$ can respectively be written as

$$\mu_{\bar{G}_n} \equiv \boldsymbol{E}(\bar{G}_n) = \frac{\sum_{k=1}^{K^*} \boldsymbol{E}(G_{n,k})}{K^*} = \frac{\sum_{k=1}^{K^*} \boldsymbol{E}\left[\frac{G_{1,k}}{r^{(n-1)}}\right]}{K^*} \equiv \frac{\sum_{k=1}^{K^*} \frac{\mu_{G_{1,k}}}{r^{(n-1)}}}{K^*} = \frac{\mu_{\bar{G}_1}}{r^{(n-1)}} \tag{49.7}$$

and

$$\sigma^2_{\bar{G}_n} \equiv \boldsymbol{V}(\bar{G}_n) = \boldsymbol{V}\left(\frac{\sum_{k=1}^{K^*} G_{n,k}}{K^*}\right) = \frac{\sum_{k=1}^{K^*} V(G_{n,k})}{(K^*)^2} ,$$

as $G_{n,k}$s are independent

$$= \frac{\sum_{k=1}^{K^*} V\left[\frac{G_{1,k}}{r^{(n-1)}}\right]}{(K^*)^2} \equiv \frac{\sum_{k=1}^{K^*} \frac{\sigma^2_{G_{1,k}}}{r^{2(n-1)}}}{(K^*)^2} = \frac{\sigma^2_{\bar{G}_1}}{r^{2(n-1)} K^*} . \tag{49.8}$$

According to the three specializations of an AP (or a GP), for a deteriorating system, it is reasonable to assume that the successive operating times of the system form a decreasing AP (or GP), whereas the corresponding consecutive repair times constitute an increasing AP (or GP). However, the replacement times for the system are usually stochastically the same no matter how old the used system is; hence, these will form an RP. This is the motivation behind the introduction of the AP (or GP) approach.

Thus, d, $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ (or r, $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$) are the most important parameters in K APs (or GPs) because the means and variances of the $\bar{A}_n$s (or $\bar{G}_n$s) are completely determined by these three parameters. In view of this fact, in this chapter the procedure is defined for applying the AP (or GP) approach in a reliability context and the functions of estimators are derived for the three fundamental parameters. Now, there are three questions. The first is, given a set of data of successive inter-event times of a point process, how do we test whether this is consistent with an AP (or a GP)? The second question is, if the data do come from a common AP (or GP), how can we estimate the parameters d, $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ (or r, $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$)? The third question is, after fitting an AP (or a GP) model to the data set, how good is the fit?

In this chapter, the statistical inference for K independent, homogeneous APs (or GPs) is investigated and the first two questions are answered using well-known statistical methods. In Sect. 49.3, the parameters d, α_A and $\sigma^2_{\mathrm{A},\varepsilon}$ (or r, α_G and $\sigma^2_{\mathrm{G},\varepsilon}$) are estimated using simple linear regression techniques. In Sect. 49.5, first $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ (or $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$) are estimated based on the results derived in Sect. 49.3, and then $\mu_{\bar{A}_n}$ and $\sigma^2_{\bar{A}_n}$ (or $\mu_{\bar{G}_n}$ and $\sigma^2_{\bar{G}_n}$) are correspondingly estimated using (49.3) and (49.4) [or (49.7) and (49.8)], respectively.

49.2 Testing for Trends

Much of the work in this section is based on *Leung* [49.4, 11].

49.2.1 Laplace Test

Suppose now that K independent, homogeneous series are available with periods of observation $T_1, T_2, \ldots, T_K$. The numbers of events in the different series are denoted by $N_1, N_2, \ldots, N_K$ and the times of occurrence of events by $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,K}$. Given the data $\{A_{n,k}$ (or $G_{n,k}$), $n = 1, 2, \ldots$ and $k = 1 \ldots, K\}$ of successive inter-event times of a point process, first of all we need to test whether the $A_{n,k}$s are identically distributed by checking for the existence of a trend. To do this, many techniques discussed in *Ascher* and *Feingold* [49.18] can be used. Laplace's trend test is used for ease of manipulation and interpretation.

Null hypothesis H_0: $A_{n,k}$s (or $G_{n,k}$) are identically distributed.

Alternative hypothesis H_1: $A_{n,k}$s (or $G_{n,k}$) are not identically distributed, i. e. there is a trend.

Laplace's test statistic for a time-truncated data set (i. e. when the data are time truncated, the time of the conclusion of observation is fixed and the number of events is random) is given by

$$L_k = \frac{\sum_{n=1}^{N_k} Y_{n,k} - \frac{N_k T_k}{2}}{\sqrt{\frac{N_k T_k^2}{12}}} \quad \text{for } k = 1, \ldots, K\,, \tag{49.9}$$

where $Y_{1,k}, \ldots, Y_{N_k,k}$, with $Y_{n,k} = \sum_{i=1}^{n} A_{i,k}$ (or $Y_{n,k} = \sum_{i=1}^{n} G_{i,k}$) are the event times for a process observed in $(0, T_k]$, and T_k is the pre-specified time of observation.

Laplace's test statistic for an event-truncated data set (i. e. when the data are event-truncated, the number of events is fixed before observation begins and the time of the conclusion of the observation is random) is given by

$$L_k = \frac{\sum_{n=1}^{N_k-1} Y_{n,k} - \frac{(N_k-1)Y_{N_k,k}}{2}}{\sqrt{\frac{(N_k-1)Y^2_{N_k,k}}{12}}} \quad \text{for } k = 1, \ldots, K\,, \tag{49.10}$$

where $Y_{n,k} = \sum_{i=1}^{n} A_{i,k}$ (or $Y_{n,k} = \sum_{i=1}^{n} G_{i,k}$) is the time of the nth failure for $n = 1, 2, \ldots, N_k$, and N_k is the pre-specified number of events.

L_k is approximately distributed as the standard normal for $N_k \geq 3$, time-truncated data, or $N_k \geq 4$, event-truncated data, at the 5% level of significance, see *Ascher* and *Feingold* [49.18]. If $|L_k| > 1.96$, then H_0 is rejected at the 5% level of significance, i. e. the event data set $\{A_1, A_2, \dots, A_{N_k}\}$ (or $\{G_1, G_2, \dots, G_{N_k}\}$) exhibits a trend.

Rigdon and *Basu* [49.22], on p. 259, reach the conclusion that

> *using any model for event times, one clearly indicates the time that data collection started and the time that it ceased. This is necessary so that the appropriate analysis, that is, an analysis based on event-truncated or time-truncated data, can be applied and maximum information can be obtained from the data. For time-truncated data, the time between the last event and the termination of the test contains some information that should not be wasted.*

Suppose, however, that a pooled test is required. It would often be best to take the null hypothesis to be that the series individually follow stationary point processes which possibly differ for different series. We can then make a combined test for trend in the data and this can be done using (49.11) or (49.12), which are given below.

Laplace's test pooled statistic for a time-truncated data set is given by

$$L = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} Y_{n,k} - \frac{1}{2}\sum_{k=1}^{K} N_k T_k}{\sqrt{\frac{\sum_{k=1}^{K} N_k T_k^2}{12}}} \tag{49.11}$$

and Laplace's test pooled statistic for an event-truncated data set is given by

$$L = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k-1} Y_{n,k} - \frac{1}{2}\sum_{k=1}^{K} (N_k-1) Y_{N_k,k}}{\sqrt{\frac{\sum_{k=1}^{K} (N_k-1) Y_{N_k,k}^2}{12}}}\,, \tag{49.12}$$

which under the null hypothesis has zero mean, unity variance and very nearly a normal distribution.

Note that the series can be tested individually for trend using (49.9) or (49.10); however, it is worth making a combined trend test in the data using (49.11) or (49.12). *Cox* and *Lewis* [49.17], on p. 50, note that there are other ways the separate trend tests could be combined, for example by forming

$$L = \frac{\sum_{k=1}^{K} L_k}{\sqrt{12}} \quad \text{or} \quad L = \sum_{k=1}^{K} L_k^2$$

the former would be tested as a standardized normal variable, the latter as chi-squared with $(K-1)$ degrees of freedom. These tests take no account of the very different numbers of observations in the different series.

49.2.2 Graphical Techniques

Another possible approach is to use simple linear regression techniques.

Arithmetic Processes

To start with, let

$$W_{A,n,k} = A_{n,k} + (n-1)d\,. \tag{49.13}$$

From Definition 49.1, $W_{A,n,k}$s are i.i.d. and can be written as

$$W_{A,n,k} = \alpha_A + \varepsilon_{A,n,k}\,, \tag{49.14}$$

where

$$E(W_{A,n,k}) = \alpha_A \tag{49.15}$$

and $\varepsilon_{A,n,k}$s are also i.i.d. (not necessarily normally distributed if our objective is estimation only, e.g. see *Gujarati* [49.32], p. 281) with

$$E(\varepsilon_{A,n,k}) = E(\varepsilon_{A,n}) = 0, \text{ irrespective of } k \tag{49.16}$$

and

$$V(\varepsilon_{A,n,k}) = V(\varepsilon_{A,n}) \equiv \sigma^2_{A,\varepsilon_n}\,, \tag{49.17}$$

irrespective of k and $\sigma^2_{A,\varepsilon_n} = \sigma^2_{A,\varepsilon}$, equal variance irrespective of n.

Combining (49.13) and (49.14) yields

$$A_{n,k} = -d(n-1) + \alpha_A + \varepsilon_{A,n,k} \quad \text{for } n = 1, \dots, N_k \quad \text{and } k = 1, \dots, K\,, \tag{49.18}$$

which is a simple linear regression equation.

Geometric processes

To start with, let

$$W_{G,n,k} = r^{(n-1)} G_{n,k} \tag{49.19}$$

or

$$\ln W_{G,n,k} = (n-1)\ln r + \ln G_{n,k} \,. \tag{49.20}$$

From Definition 49.2, $W_{G,n,k}$s are i.i.d. and can be written as

$$\ln W_{G,n,k} = \alpha_G + \varepsilon_{G,n,k} \tag{49.21}$$

or

$$W_{G,n,k} = e^{\alpha_G + \varepsilon_{G,n,k}} \,, \tag{49.22}$$

where

$$E(\ln W_{G,n,k}) = \alpha_G \tag{49.23}$$

and $\varepsilon_{G,n,k}$s are also i.i.d. (not necessarily normally distributed if our objective is estimation only, e.g. see *Gujarati* [49.32], p. 281) with

$$E(\varepsilon_{G,n,k}) = E(\varepsilon_{G,n}) = 0, \text{ irrespective of } k \tag{49.24}$$

and

$$V(\varepsilon_{G,n,k}) = V(\varepsilon_{G,n}) \equiv \sigma^2_{G,\varepsilon_n} \,, \tag{49.25}$$

irrespective of k and $\sigma^2_{G,\varepsilon_n} = \sigma^2_{G,\varepsilon}$, equal variance irrespective of n.

Combining (49.20) and (49.21) yields

$$\ln G_{n,k} = -\ln r(n-1) + \alpha_G + \varepsilon_{G,n,k} \quad \text{for } n = 1, \ldots, N_k \quad \text{and } k = 1, \ldots, K \,, \tag{49.26}$$

which is a simple linear regression equation.

According to (49.18) [or (49.26)], we can plot $A_{n,k}$ (or $\ln G_{n,k}$) against $(n-1)$ for $n = 1, \ldots, N_k$ and $k = 1, \ldots, K$ to see whether there is a linear relationship between them. Clearly, this is also useful for testing whether the observations $\{A_{n,k}$ (or $G_{n,k}$), $n = 1, \ldots, N_k$ and $k = 1, \ldots, K\}$ come from a common AP (or GP) as well as whether they share a common trend.

49.3 Estimating the Parameters

The work in this section is substantially based on *Leung* [49.4, 11].

49.3.1 Estimate Parameters d, α_A and $\sigma^2_{A,\varepsilon}$ of K APs (or r, α_G and $\sigma^2_{G,\varepsilon}$ of K GPs)

We can estimate the parameters d, α_A and $\sigma^2_{A,\varepsilon}$ using the simple linear regression method. The least-squares point estimates $\hat{d}$, $\hat{\alpha}_A$ and $\hat{\sigma}^2_{A,\varepsilon}$ of the parameters d, α_A and $\sigma^2_{A,\varepsilon}$ are calculated respectively using the following formulae:

$$\hat{d} = \frac{\dfrac{\left(\sum\limits_{k=1}^{K} N_k^2 - N\right)\left(\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} A_{n,k}\right)}{2N} - \sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k}(n-1)A_{n,k}}{\sum\limits_{k=1}^{K}\dfrac{(N_k-1)N_k(2N_k-1)}{6} - \dfrac{\left(\sum\limits_{k=1}^{K} N_k^2 - N\right)^2}{4N}} \,, \tag{49.27}$$

$$\hat{\alpha}_A = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} A_{n,k}}{N} + \frac{\hat{d}\left(\sum\limits_{k=1}^{K} N_k^2 - N\right)}{2N} \tag{49.28}$$

and

$$\hat{\sigma}^2_{A,\varepsilon} = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} A_{n,k}^2 - \dfrac{\left(\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} A_{n,k}\right)^2}{N}}{N-2} + \frac{\hat{d}\left(\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k}(n-1)A_{n,k}\right)}{N-2} - \frac{\hat{d}\left(\dfrac{\left(\sum\limits_{k=1}^{K} N_k^2 - N\right)\left(\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} A_{n,k}\right)}{2N}\right)}{N-2} \,, \tag{49.29}$$

where

$$N = \sum_{k=1}^{K} N_k \,. \tag{49.30}$$

The derivations of (49.27) to (49.29) are given in the Appendix.

The least-squares point estimates $\hat{r}$, $\hat{\alpha}_G$ and $\hat{\sigma}^2_{G,\varepsilon}$ of the parameters r, α_G and $\sigma^2_{G,\varepsilon}$ can be obtained simply by replacing $\hat{d}$ with $\ln \hat{r}$ and $A_{n,k}$ with $\ln G_{n,k}$ on the right-hand side of (49.27) to (49.29).

49.3.2 Estimating the Parameters of a Single AP (or GP)

When $K=1$, (49.30) becomes $N=N_1$. Note that

1. for a single AP or GP, we simply use N to represent the number of successive events, and
2. the equations given below are consistent with those derived in *Leung* [49.8] and *Lam* [49.15].

Then (49.27) to (49.29) become

$$\hat{d} = \frac{6(N-1)\sum_{n=1}^{N} A_n - 12\sum_{n=1}^{N}(n-1)A_n}{(N-1)N(N+1)}\,, \tag{49.27.1}$$

$$\hat{\alpha}_{\mathrm{A}} = \frac{2(2N-1)\sum_{n=1}^{N} A_n - 6\sum_{n=1}^{N}(n-1)A_n}{N(N+1)} \tag{49.28.1}$$

and

$$\hat{\sigma}^2_{\mathrm{A},\varepsilon} = \frac{\sum_{n=1}^{N} A_n^2 - \frac{1}{N}\left(\sum_{n=1}^{N} A_n\right)^2}{N-2} - \frac{\hat{d}\left[\frac{(N-1)}{2}\sum_{n=1}^{N} A_n - \sum_{n=1}^{N}(n-1)A_n\right]}{N-2}\,. \tag{49.29.1}$$

For a single GP, $\hat{d}$, $\hat{\alpha}_{\mathrm{A}}$, $\hat{\sigma}^2_{\mathrm{A},\varepsilon}$ and A_n are replaced by $\ln \hat{r}$, $\hat{\alpha}_{\mathrm{G}}$, $\hat{\sigma}^2_{\mathrm{G},\varepsilon}$ and $\ln G_n$ in (49.27.1) to (49.29.1).

49.4 Distinguishing a Renewal Process from an AP (or a GP)

Much of the work in this section is based on *Leung* [49.4, 11]. We test whether the data comes from an RP or AP.

Null hypothesis $\mathbf{H}_0$: $d=0$

Alternative hypothesis $\mathbf{H}_1$: $d \neq 0$

The t-test statistic is denoted and given by

$$t_{\mathrm{A}} = \frac{-\hat{d}\sqrt{\sum_{k=1}^{K}\frac{(N_k-1)N_k(2N_k-1)}{6} - \frac{\left(\sum_{k=1}^{K} N_k^2 - N\right)^2}{4N}}}{\hat{\sigma}_{\mathrm{A},\varepsilon}}\,, \tag{49.31}$$

where t_{A} is distributed as a Student's t with $(N-2)$ degrees of freedom. If $|t_{\mathrm{A}}|$ is larger than the critical value $t_{N-2,0.025}$, then H_0 is rejected at the 5% level of significance, i. e. the data set $\{A_{n,k},\ n=1,\dots,N_k$ and $k=1,\dots,K\}$ comes from a common AP. The derivation of (49.31) is given in the Appendix.

To test whether the data comes from an RP or a GP, H_0 becomes $\ln r = 0$ or its equivalence $r=1$ and H_1 becomes $\ln r \neq 0$ or its equivalence $r \neq 1$, and the t-test statistic is obtained simply by replacing t_{A}, $\hat{d}$ and $\hat{\sigma}^2_{\mathrm{A},\varepsilon}$ with t_{G}, $\ln \hat{r}$ and $\hat{\sigma}^2_{\mathrm{G},\varepsilon}$ in (49.31).

One point worth noting is that, for testing purposes, each $\varepsilon_{\mathrm{A},n,k}$ (or $\varepsilon_{\mathrm{G},n,k}$) is essentially normally distributed, e.g. see *Gujarati* [49.32], p. 282. It is difficult to evaluate the normality assumption for a sample of only 20 observations, and formal test procedures are presented in *Ramsey* and *Ramsey* [49.33].

49.5 Estimating the Means and Variances

The work in this section is substantially based on *Leung* [49.4, 11].

49.5.1 Estimating $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ of $\bar{A}_n$s

First, the mean and variance of $\bar{A}_1$ are estimated using the relevant estimators with the formulae given below.

First we denote

$$\bar{W}_{\mathrm{A},n} \equiv \frac{\sum_{k=1}^{K^*} W_{\mathrm{A},n,k}}{K^*} \quad \text{and} \quad \mu_{\bar{A}_1} \equiv \frac{\sum_{k=1}^{K^*} \mu_{A_1,k}}{K^*}\,.$$

From Definition 49.1, $W_{\mathrm{A},n,k}$s are i.i.d., we have

$$\mathrm{E}(\bar{W}_{\mathrm{A},n}) \equiv \frac{\sum_{k=1}^{K^*} \mathrm{E}(W_{\mathrm{A},n,k})}{K^*} = \frac{\sum_{k=1}^{K^*} \mu_{A_1,k}}{K^*} \equiv \mu_{\bar{A}_1}$$

and

$$\mathrm{V}(\bar{W}_{\mathrm{A},n}) \equiv \frac{\sum_{k=1}^{K^*} \mathrm{V}(W_{\mathrm{A},n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} \sigma^2_{A_1,k}}{(K^*)^2} \equiv \frac{\sigma^2_{\bar{A}_1}}{K^*}\,.$$

From (49.14), (49.15), (49.16) and (49.17), we obtain

$$\mathrm{E}(\bar{W}_{\mathrm{A},n}) \equiv \frac{\sum_{k=1}^{K^*} \mathrm{E}(W_{\mathrm{A},n,k})}{K^*} = \frac{K^*\alpha_{\mathrm{A}}}{K^*} = \alpha_{\mathrm{A}}$$

and

$$\begin{aligned}\mathrm{V}(\bar{W}_{\mathrm{A},n}) &\equiv \frac{\sum_{k=1}^{K^*} \mathrm{V}(W_{\mathrm{A},n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} \mathrm{V}(\alpha_A + \varepsilon_{\mathrm{A},n,k})}{(K^*)^2} \\ &= \frac{\sum_{k=1}^{K^*} \mathrm{V}(\varepsilon_{\mathrm{A},n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} \sigma^2_{\mathrm{A},\varepsilon}}{(K^*)^2} \\ &= \frac{K^*\sigma^2_{\mathrm{A},\varepsilon}}{(K^*)^2} = \frac{\sigma^2_{\mathrm{A},\varepsilon}}{K^*} .\end{aligned}$$

Therefore, the first estimators for $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ are denoted and given by

$$\hat{\mu}_{\bar{A}_1,1} = \hat{\alpha}_{\mathrm{A}} \tag{49.32}$$

and

$$\hat{\sigma}^2_{\bar{A}_1,1} = \hat{\sigma}^2_{\mathrm{A},\varepsilon} . \tag{49.33}$$

Alternatively, since $W_{\mathrm{A},n,k}$s are i.i.d. with mean $\mu_{W_{\mathrm{A},n,k}} = \mu_{A_1,k}$ and variance $\sigma^2_{W_{\mathrm{A},n,k}} = \sigma^2_{A_1,k}$, it is plausible to estimate $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ by the sample mean and sample variance of $\hat{W}_{\mathrm{A},n,k}$s, where $\hat{W}_{\mathrm{A},n,k} = A_{n,k} + (n-1)\hat{d}$. Hence, the second estimators for $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ are denoted and given by

$$\begin{aligned}\hat{\mu}_{\bar{A}_1,2} &= \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} \hat{W}_{\mathrm{A},n,k}}{N} = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} [A_{n,k} + (n-1)\hat{d}]}{N} \\ &= \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} A_{n,k}}{N} + \frac{\hat{d}\left(\sum_{k=1}^{K} N_k^2 - N\right)}{2N}\end{aligned} \tag{49.34}$$

and

$$\begin{aligned}\hat{\sigma}^2_{\bar{A}_1,2} &= \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} \left[A_{n,k} + (n-1)\hat{d}\right]^2}{N-1} \\ &\quad - \frac{\left\{\sum_{k=1}^{K}\sum_{n=1}^{N_k} [A_{n,k} + (n-1)\hat{d}]\right\}^2}{N(N-1)} .\end{aligned} \tag{49.35}$$

Notice that the second estimator $\hat{\mu}_{\bar{A}_1,2}$ given by (49.34) is the same as the first estimator $\hat{\mu}_{\bar{A}_1,1}$ given by (49.32) or (49.28).

It is also plausible to obtain the third estimator for $\mu_{\bar{A}_1}$ provided $N_k \cong N_0$ for $k = 1, \ldots, K$ as follows.

Let $S_N = \sum_{k=1}^{K}\sum_{n=1}^{N_k} A_{n,k} = \sum_{k=1}^{K}\sum_{n=1}^{N_k} [A_{1,k} - (n-1)d]$. Then

$$\begin{aligned}\mathrm{E}(S_N) &= \sum_{k=1}^{K} N_k \mathrm{E}(A_{1,k}) - \frac{d\left(\sum_{k=1}^{K} N_k^2 - N\right)}{2} \\ &= \sum_{k=1}^{K} N_k \mu_{A_{1,k}} - \frac{d\left(\sum_{k=1}^{K} N_k^2 - N\right)}{2} .\end{aligned}$$

If $N_k \cong N_0$ for $k = 1, \ldots, K$; then

$$\begin{aligned}\mathrm{E}(S_N) &\cong N_0 \sum_{k=1}^{K} \mu_{A_1,k} - \frac{dK\left(N_0^2 - N_0\right)}{2} \\ &= KN_0\mu_{\bar{A}_1} - \frac{dKN_0\left(N_0 - 1\right)}{2} .\end{aligned}$$

After transposition, we have

$$\mu_{\bar{A}_1} \cong \frac{\mathrm{E}(S_N)}{KN_0} + \frac{d\left(N_0 - 1\right)}{2} .$$

Hence, the third estimator for $\mu_{\bar{A}_1}$, provided $N_k \cong N_0$, for $k = 1, \ldots, K$, is denoted and given by

$$\begin{aligned}\hat{\mu}_{\bar{A}_1,3} &\cong \frac{S_N}{KN_0} + \frac{\hat{d}\left(N_0 - 1\right)}{2} \\ &= \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_0} A_{n,k}}{KN_0} + \frac{\hat{d}\left(N_0 - 1\right)}{2} .\end{aligned} \tag{49.36}$$

In fact, we can deduce (49.36) directly from (49.28) by putting $N_k \cong N_0$ and $N \cong KN_0$. In other words, the third estimator for $\mu_{\bar{A}_1}$ is indeed the first estimator but calculated approximately using (49.36). As a whole, only one estimator for $\mu_{\bar{A}_1}$, namely $\hat{\mu}_{\bar{A}_1,1}$ has been derived so far. It is furthermore plausible to obtain the second (*fourth*) and third (*fifth*) estimators for $\mu_{\bar{A}_1}$, provided $N_k \cong N_0$ for $k = 1, \ldots, K$, as follows.

In view of the fact that $\mathrm{E}(\bar{W}_{\mathrm{A},n}) = \mu_{\bar{A}_1}$, we can write

$$\bar{W}_{\mathrm{A},n} = \mu_{\bar{A}_1}(1 + \delta_{\mathrm{A},n}) . \tag{49.37}$$

1. We have

$$\mathrm{E}\left(\frac{\bar{W}_{\mathrm{A},n}}{\mu_{\bar{A}_1}}\right) = 1 + \mathrm{E}(\delta_{\mathrm{A},n})$$

and so it follows that

$$\mathrm{E}(\delta_{\mathrm{A},n}) = 0\,. \tag{49.38}$$

2. We obtain

$$\mathrm{V}\left(\frac{\bar{W}_{\mathrm{A},n}}{\mu_{\bar{A}_1}}\right) = \mathrm{V}(1+\delta_{\mathrm{A},n})\,,$$
$$\frac{\mathrm{V}(\bar{W}_{\mathrm{A},n})}{\mu^2_{\bar{A}_1}} = \mathrm{V}(\delta_{\mathrm{A},n})$$

and so it follows that

$$\mathrm{V}(\delta_{\mathrm{A},n}) = \frac{\sigma^2_{\bar{A}_1}}{K^*\mu^2_{\bar{A}_1}}\,. \tag{49.39}$$

3. Taking the logarithm of $\bar{W}_{\mathrm{A},n} \equiv \frac{\sum_{k=1}^{K^*} W_{\mathrm{A},n,k}}{K^*}$, using equation (49.14) and the fact that $\varepsilon_{\mathrm{A},n,k} \equiv \varepsilon_{\mathrm{A},n}$, irrespective of k, and taking the logarithm for (49.37), we obtain

$$\ln \bar{W}_{\mathrm{A},n} = \ln\left[\alpha_{\mathrm{A}}\left(1+\frac{\varepsilon_{\mathrm{A},n}}{\alpha_{\mathrm{A}}}\right)\right] = \ln\alpha_{\mathrm{A}} + \ln\left(1+\frac{\varepsilon_{\mathrm{A},n}}{\alpha_{\mathrm{A}}}\right) \tag{49.40}$$

and

$$\ln \bar{W}_{\mathrm{A},n} = \ln\mu_{\bar{A}_1} + \ln(1+\delta_{\mathrm{A},n})\,. \tag{49.41}$$

Taking the expectations of (49.40) and (49.41), equating them, and expanding the logarithm series, we have

$$\ln\alpha_{\mathrm{A}} + \mathrm{E}\left(\frac{\varepsilon_{\mathrm{A},n}}{\alpha_{\mathrm{A}}} - \frac{\varepsilon^2_{\mathrm{A},n}}{2\alpha^2_{\mathrm{A}}} + \frac{\varepsilon^3_{\mathrm{A},n}}{3\alpha^3_{\mathrm{A}}} - \dots\right)$$
$$= \ln\mu_{\bar{A}_1} + \mathrm{E}\left(\delta_{\mathrm{A},n} - \frac{\delta^2_{\mathrm{A},n}}{2} + \frac{\delta^3_{\mathrm{A},n}}{3} - \dots\right)\,,$$
$$\ln\alpha_{\mathrm{A}} + \frac{1}{\alpha_{\mathrm{A}}}\mathrm{E}(\varepsilon_{\mathrm{A},n}) - \frac{1}{2\alpha^2_{\mathrm{A}}}\mathrm{E}(\varepsilon^2_{\mathrm{A},n})$$
$$\cong \ln\mu_{\bar{A}_1} + \mathrm{E}(\delta_{\mathrm{A},n}) - \frac{1}{2}\mathrm{E}(\delta^2_{\mathrm{A},n})\,,$$
$$\ln\alpha_{\mathrm{A}} - \frac{1}{2\alpha^2_{\mathrm{A}}}\mathrm{V}(\varepsilon_{\mathrm{A},n})$$
$$= \ln\mu_{\bar{A}_1} - \frac{1}{2}\mathrm{V}(\delta_{\mathrm{A},n}) \quad \text{by (49.16) and (49.38)}$$
$$\ln\alpha_{\mathrm{A}} - \frac{\sigma^2_{\mathrm{A},\varepsilon}}{2\alpha^2_{\mathrm{A}}}$$
$$= \ln\mu_{\bar{A}_1} - \frac{\sigma^2_{\bar{A}_1}}{2K^*\mu^2_{\bar{A}_1}} \quad \text{by (49.17) and (49.39);}$$

4. $\mu_{\bar{A}_1}$ must satisfy the equation

$$\ln\left(\frac{\mu_{\bar{A}_1}}{\alpha_{\mathrm{A}}}\right) - \frac{1}{2}\left(\frac{\sigma^2_{\bar{A}_1}}{K^*\mu^2_{\bar{A}_1}} - \frac{\sigma^2_{\mathrm{A},\varepsilon}}{\alpha^2_{\mathrm{A}}}\right) = 0\,.$$

5. We can estimate $\mu_{\bar{A}_1}$, provided $N_k \cong N_0$ for $k = 1, \dots, K$, i.e., $K = K^*$, by $\hat{\mu}_{\bar{A}_1,2}$, which satisfies the equation

$$\ln\left(\frac{\mu_{\bar{A}_1}}{\hat{\alpha}_{\mathrm{A}}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}^2_{\bar{A}_1,1}}{K\mu^2_{\bar{A}_1}} - \frac{\hat{\sigma}^2_{\mathrm{A},\varepsilon}}{\hat{\alpha}^2_{\mathrm{A}}}\right) = 0 \tag{49.42}$$

or by $\hat{\mu}_{\bar{A}_1,3}$, which satisfies the equation

$$\ln\left(\frac{\mu_{\bar{A}_1}}{\hat{\alpha}_{\mathrm{A}}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}^2_{\bar{A}_1,2}}{K\mu^2_{\bar{A}_1}} - \frac{\hat{\sigma}^2_{\mathrm{A},\varepsilon}}{\hat{\alpha}^2_{\mathrm{A}}}\right) = 0\,, \tag{49.43}$$

where $\hat{\alpha}_{\mathrm{A}}, \hat{\sigma}^2_{\mathrm{A},\varepsilon}, \hat{\sigma}^2_{\bar{A}_1,1}$ and $\hat{\sigma}^2_{\bar{A}_1,2}$ are given by (49.28), (49.29), (49.33) and (49.35), respectively.

Clearly, if $d = 0$, the parameters $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ can be estimated using the sample mean and sample variance, which are given by

$$\hat{\mu}_{A_1,4} = \frac{\sum\limits_{n=1}^{N} A_n}{N} \quad \text{and} \quad \hat{\sigma}^2_{A_1,3} = \frac{\sum\limits_{n=1}^{N} (A_n - \hat{\mu}_{A_1,4})^2}{N-1}\,. \tag{49.44}$$

Secondly, we use (49.3) and (49.4), and let $N_1 \geq N_2 \geq \dots \geq N_K$, the means and variances of $\bar{A}_n$ for $n = 2, 3, \dots, N_k$ and $k = 1, \dots, K$ are estimated using the following formulae:

$$\hat{\mu}_{\bar{A}_n} = \hat{\mu}_{\bar{A}_1} - (n-1)\hat{d} \quad \text{and}$$
$$\hat{\sigma}^2_{\bar{A}_n} = \frac{\hat{\sigma}^2_{\bar{A}_1}}{K^*} \quad \text{for } n = 2, 3, \dots, N_1\,. \tag{49.45}$$

49.5.2 Estimating $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$ of $\bar{G}_n$s

First, the mean and variance of $\bar{G}_1$ are estimated using the relevant estimators with the formulae given below.

First we denote

$$\bar{W}_{\mathrm{G},n} \equiv \frac{\sum\limits_{k=1}^{K^*} W_{\mathrm{G},n,k}}{K^*}\,.$$

From Definition 49.2, $W_{G,n,k}$s are i.i.d., we have

$$\mathrm{E}(\bar{W}_{\mathrm{G},n}) \equiv \frac{\sum_{k=1}^{K^*} \mathrm{E}(W_{\mathrm{G},n,k})}{K^*} = \frac{\sum_{k=1}^{K^*} \mu_{G_1,k}}{K^*} \equiv \mu_{\bar{G}_1}$$

and

$$\mathrm{V}(\bar{W}_{\mathrm{G},n}) \equiv \frac{\sum_{k=1}^{K^*} \mathrm{V}(W_{\mathrm{G},n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} \sigma^2_{G_1,k}}{(K^*)^2} \equiv \frac{\sigma^2_{\bar{G}_1}}{K^*} .$$

From (49.22), (49.24) and (49.25), we obtain

$$\begin{aligned}\mathrm{E}(\bar{W}_{\mathrm{G},n}) &\equiv \frac{\sum_{k=1}^{K^*} \mathrm{E}(W_{\mathrm{G},n,k})}{K^*} = \frac{\sum_{k=1}^{K^*} \mathrm{E}(\mathrm{e}^{\alpha_\mathrm{G}+\varepsilon_{\mathrm{G},n,k}})}{K^*} \\ &= \frac{\mathrm{e}^{\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{E}(\mathrm{e}^{\varepsilon_{\mathrm{G},n,k}})}{K^*} \\ &= \frac{\mathrm{e}^{\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{E}\left(1+\varepsilon_{\mathrm{G},n,k}+\frac{\varepsilon^2_{\mathrm{G},n,k}}{2!}+\cdots\right)}{K^*} \\ &\cong \frac{\mathrm{e}^{\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \left[1+\mathrm{E}(\varepsilon_{\mathrm{G},n,k})+\frac{1}{2}\mathrm{E}(\varepsilon^2_{\mathrm{G},n,k})\right]}{K^*} \\ &= \frac{\mathrm{e}^{\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \left[1+\frac{1}{2}\mathrm{V}(\varepsilon_{\mathrm{G},n,k})\right]}{K^*} \\ &= \mathrm{e}^{\alpha_\mathrm{G}}\left(1+\frac{\sigma^2_{\mathrm{G},\varepsilon}}{2}\right) .\end{aligned}$$

From (49.22) and (49.25), we obtain

$$\begin{aligned}\mathrm{V}(\bar{W}_{\mathrm{G},n}) &\equiv \frac{\sum_{k=1}^{K^*} \mathrm{V}(W_{\mathrm{G},n,k})}{(K^*)^2} = \frac{\sum_{k=1}^{K^*} \mathrm{V}\left(\mathrm{e}^{\alpha_\mathrm{G}+\varepsilon_{\mathrm{G},n,k}}\right)}{(K^*)^2} \\ &= \frac{\mathrm{e}^{2\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{V}\left(\mathrm{e}^{\varepsilon_{\mathrm{G},n,k}}\right)}{(K^*)^2} \\ &= \frac{\mathrm{e}^{2\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{V}\left(1+\varepsilon_{\mathrm{G},n,k}+\frac{\varepsilon^2_{\mathrm{G},n,k}}{2!}+\cdots\right)}{(K^*)^2} \\ &\cong \frac{\mathrm{e}^{2\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{V}(1+\varepsilon_{\mathrm{G},n,k})}{(K^*)^2} \\ &= \frac{\mathrm{e}^{2\alpha_\mathrm{G}} \sum_{k=1}^{K^*} \mathrm{V}(\varepsilon_{\mathrm{G},n,k})}{(K^*)^2} = \frac{\mathrm{e}^{2\alpha_\mathrm{G}}\sigma^2_{\mathrm{G},\varepsilon}}{K^*} .\end{aligned}$$

Therefore, the first estimators for $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$ are denoted and given by

$$\hat{\mu}_{\bar{G}_1,1} = \mathrm{e}^{\hat{\alpha}_\mathrm{G}}\left(1+\frac{\hat{\sigma}^2_{\mathrm{G},\varepsilon}}{2}\right) \tag{49.46}$$

and

$$\hat{\sigma}^2_{\bar{G}_1,1} = \mathrm{e}^{2\hat{\alpha}_\mathrm{G}}\hat{\sigma}^2_{\mathrm{G},\varepsilon} . \tag{49.47}$$

Alternatively, since $W_{\mathrm{G},n,k}$s are i.i.d. with mean $\mu_{W_{\mathrm{G},n,k}} = \mu_{G_1,k}$ and variance $\sigma^2_{W_{\mathrm{G},n,k}} = \sigma^2_{G_1,k}$, it is plausible for us to estimate $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$ by the sample mean and sample variance of $\hat{W}_{\mathrm{G},n,k}$s, where $\hat{W}_{\mathrm{G},n,k} = \hat{r}^{(n-1)}G_{n,k}$. Hence, the second estimators for $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$ are denoted and given by

$$\hat{\mu}_{\bar{G}_1,2} = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} \hat{W}_{\mathrm{G},n,k}}{N} = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} (\hat{r}^{(n-1)}G_{n,k})}{N} \tag{49.48}$$

and

$$\hat{\sigma}^2_{\bar{G}_1,2} = \frac{\sum_{k=1}^{K}\sum_{n=1}^{N_k} (\hat{r}^{(n-1)}G_{n,k})^2 - \frac{\left[\sum_{k=1}^{K}\sum_{n=1}^{N_k} (\hat{r}^{(n-1)}G_{n,k})\right]^2}{N}}{N-1} . \tag{49.49}$$

It is also plausible for us to obtain the third estimator for $\mu_{\bar{G}_1}$ provided $N_k \cong N_0$ for $k = 1, \ldots, K$.

Let

$$\begin{aligned}S_N &= \sum_{k=1}^{K}\sum_{n=1}^{N_k} G_{n,k} = \sum_{k=1}^{K}\sum_{n=1}^{N_k}\left(\frac{G_{1,k}}{r^{(n-1)}}\right) \\ &= \sum_{k=1}^{K} G_{1,k} \sum_{n=1}^{N_k} \frac{1}{r^{(n-1)}} \\ &= \frac{1}{1-r^{-1}} \sum_{k=1}^{K}\left(1-r^{-N_k}\right) G_{1,k} .\end{aligned}$$

Then

$$\mathrm{E}(S_N) = \frac{1}{1-r^{-1}} \sum_{k=1}^{K}\left(1-r^{-N_k}\right) \mu_{G_1,k} .$$

If $N_k \cong N_0$ for $k = 1, \ldots, K$, then

$$\mathrm{E}(S_N) \cong \frac{K\mu_{\bar{G}_1}\left(1-r^{-N_0}\right)}{1-r^{-1}} .$$

After transposition, we have

$$\mu_{\bar{G}_1} \cong \frac{\mathrm{E}(S_N)\left(1-r^{-1}\right)}{K(1-r^{-N_0})} .$$

Hence, the third estimator for $\mu_{\bar{G}_1}$, provided $N_k \cong N_0$, for $k = 1, \ldots, K$, is denoted and given by

$$\hat{\mu}_{\bar{G}_1,3} \cong \frac{S_N\left(1-\hat{r}^{-1}\right)}{K(1-\hat{r}^{-N_0})} = \frac{\left(1-\hat{r}^{-1}\right)\sum_{k=1}^{K}\sum_{n=1}^{N_0} G_{n,k}}{K(1-\hat{r}^{-N_0})} . \quad (49.50)$$

It is furthermore plausible for us to obtain the fourth and fifth estimators for $\mu_{\bar{G}_1}$ provided $N_k \cong N_0$ for $k = 1, \ldots, K$, as follows:

Since $\mathrm{E}(\bar{W}_{\mathrm{G},n}) = \mu_{\bar{G}_1}$, we can write

$$\bar{W}_{\mathrm{G},n} = \mu_{\bar{G}_1}(1+\delta_{\mathrm{G},n}) . \quad (49.51)$$

1. We have
$$\mathrm{E}\left(\frac{\bar{W}_{\mathrm{G},n}}{\mu_{\bar{G}_1}}\right) = 1+\mathrm{E}(\delta_{\mathrm{G},n})$$
and so it follows that
$$\mathrm{E}(\delta_{\mathrm{G},n}) = 0 . \quad (49.52)$$
2. We obtain
$$\mathrm{V}\left(\frac{\bar{W}_{\mathrm{G},n}}{\mu_{\bar{G}_1}}\right) = \mathrm{V}(1+\delta_{\mathrm{G},n}) ,$$
$$\frac{\mathrm{V}(\bar{W}_{\mathrm{G},n})}{\mu^2_{\bar{G}_1}} = \mathrm{V}(\delta_{\mathrm{G},n})$$
and so it follows that
$$\mathrm{V}(\delta_{\mathrm{G},n}) = \frac{\sigma^2_{\bar{G}_1}}{K^*\mu^2_{\bar{G}_1}} . \quad (49.53)$$
3. Taking the logarithm of $\bar{W}_{\mathrm{G},n} \equiv \frac{\sum_{k=1}^{K^*} W_{\mathrm{G},n,k}}{K^*}$, using (49.21) and the fact that $\varepsilon_{\mathrm{G},n,k} \equiv \varepsilon_{\mathrm{G},n}$, irrespective of k, and taking the logarithm of (49.51), we obtain
$$\ln \bar{W}_{\mathrm{G},n} = \alpha_{\mathrm{G}} + \varepsilon_{\mathrm{G},n} \quad (49.54)$$
and
$$\ln \bar{W}_{\mathrm{G},n} = \ln \mu_{\bar{G}_1} + \ln(1+\delta_{\mathrm{G},n}) \quad (49.55)$$

Taking the expectations of (49.54) and (49.55), equating them, and expanding the logarithm series, we have

$$\begin{aligned}\alpha_{\mathrm{G}} + \mathrm{E}(\varepsilon_{\mathrm{G},n}) &= \ln \mu_{\bar{G}_1} \\ &\quad + \mathrm{E}\left(\delta_{\mathrm{G},n} - \frac{\delta^2_{\mathrm{G},n}}{2} + \frac{\delta^3_{\mathrm{G},n}}{3} - \cdots\right) , \\ \alpha_{\mathrm{G}} &\cong \ln \mu_{\bar{G}_1} + \mathrm{E}(\delta_{\mathrm{G},n}) - \frac{1}{2}\mathrm{E}(\delta^2_{\mathrm{G},n}) \\ &= \ln \mu_{\bar{G}_1} - \frac{1}{2}\mathrm{V}(\delta_{\mathrm{G},n}) \\ &= \ln \mu_{\bar{G}_1} - \frac{\sigma^2_{\bar{G}_1}}{2K^*\mu^2_{\bar{G}_1}} \quad \text{by (49.53)} .\end{aligned}$$

4. $\mu_{\bar{G}_1}$ must satisfy the equation
$$2K^*(\ln \mu_{\bar{G}_1} - \alpha_{\mathrm{G}})\mu^2_{\bar{G}_1} - \sigma^2_{\bar{G}_1} = 0 .$$
5. We can estimate $\mu_{\bar{G}_1}$, provided $N_k \cong N_0$ for $k = 1, \ldots, K$, i.e., $K = K^*$, by $\hat{\mu}_{\bar{G}_1,4}$ which satisfies the equation
$$2K(\ln \mu_{\bar{G}_1} - \hat{\alpha}_{\mathrm{G}})\mu^2_{\bar{G}_1} - \hat{\sigma}^2_{\bar{G}_1,1} = 0 \quad (49.56)$$
or by $\hat{\mu}_{\bar{G}_1,5}$ which satisfies the equation
$$2K(\ln \mu_{\bar{G}_1} - \hat{\alpha}_{\mathrm{G}})\mu^2_{\bar{G}_1} - \hat{\sigma}^2_{\bar{G}_1,2} = 0 , \quad (49.57)$$
where $\hat{\alpha}_{\mathrm{G}}, \hat{\sigma}^2_{\mathrm{G},\varepsilon}, \hat{\sigma}^2_{\bar{G}_1,1}$ and $\hat{\sigma}^2_{\bar{G}_1,2}$ are given by (49.28), (49.29) (where $\hat{d}$, $A_{n,k}$ are replaced by $\ln \hat{r}$ and $\ln G_{n,k}$), (49.47) and (49.49), respectively.

Clearly, if $\ln r = 0$ or $r = 1$, the parameters $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$ can be estimated using the sample mean and sample variance, which are given by

$$\hat{\mu}_{G_1,6} = \frac{\sum_{n=1}^{N} G_n}{N} \quad \text{and} \quad \hat{\sigma}^2_{G_1,3} = \frac{\sum_{n=1}^{N}(G_n - \hat{\mu}_{G_1,6})^2}{N-1} . \quad (49.58)$$

Secondly, we use (49.7) and (49.8), and let $N_1 \geq N_2 \geq \ldots \geq N_K$, the means and variances of $\bar{G}_n$ for $n = 2, 3, \ldots, N_k$ and $k = 1, \ldots, K$ are estimated using the following formulae:

$$\hat{\mu}_{\bar{G}_n} = \frac{\hat{\mu}_{\bar{G}_1}}{\hat{r}^{(n-1)}} \quad \text{and} \quad \hat{\sigma}^2_{\bar{G}_n} = \frac{\hat{\sigma}^2_{\bar{G}_1}}{\hat{r}^{2(n-1)}K^*}$$
$$\text{for } n = 2, 3, \ldots, N_1 . \quad (49.59)$$

49.5.3 Estimating the Means and Variances of a Single AP or GP

When $K = 1$, (49.30) becomes $N = N_1$. Note again that

1. for a single AP or GP, we simply use N to represent the number of successive events, and
2. the results listed in the next two subsections are consistent with those derived in *Leung* [49.8] and *Lam* [49.15].

A Single AP

First, the mean and variance of A_1 are estimated using the relevant estimators with the formulae given below.

The first estimators for μ_{A_1} and $\sigma^2_{A_1}$ are denoted and given by

$$\hat{\mu}_{A_1,1} = \hat{\alpha}_{\mathrm{A}} \tag{49.32.1}$$

and

$$\hat{\sigma}^2_{A_1,1} = \hat{\sigma}^2_{\mathrm{A},\varepsilon}\,, \tag{49.33.1}$$

where $\hat{\alpha}_{\mathrm{A}}$ and $\hat{\sigma}^2_{\mathrm{A},\varepsilon}$ are given by (49.28.1) and (49.29.1).

The second estimator for $\sigma^2_{A_1}$ is denoted and given by

$$\hat{\sigma}^2_{A_1,2} = \frac{\sum\limits_{n=1}^{N} [A_n + (n-1)\hat{d}]^2 - \dfrac{\left\{\sum\limits_{n=1}^{N} [A_n+(n-1)\hat{d}]\right\}^2}{N}}{N-1}\,, \tag{49.35.1}$$

where $\hat{d}$ is given by (49.27.1).

The second $\hat{\mu}_{A_1,2}$ and third $\hat{\mu}_{A_1,3}$ estimators for μ_{A_1}, respectively, satisfy the equations

$$\ln\left(\frac{\mu_{A_1}}{\hat{\alpha}_{\mathrm{A}}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}^2_{A_1,1}}{\mu^2_{A_1}} - \frac{\hat{\sigma}^2_{\mathrm{A},\varepsilon}}{\hat{\alpha}^2_{\mathrm{A}}}\right) = 0 \tag{49.42.1}$$

and

$$\ln\left(\frac{\mu_{A_1}}{\hat{\alpha}_{\mathrm{A}}}\right) - \frac{1}{2}\left(\frac{\hat{\sigma}^2_{A_1,2}}{\mu^2_{A_1}} - \frac{\hat{\sigma}^2_{\mathrm{A},\varepsilon}}{\hat{\alpha}^2_{\mathrm{A}}}\right) = 0\,, \tag{49.43.1}$$

where $\hat{\alpha}_{\mathrm{A}}$, $\hat{\sigma}^2_{\mathrm{A},\varepsilon}$, $\hat{\sigma}^2_{A_1,1}$ and $\hat{\sigma}^2_{A_1,2}$ are given by (49.28.1), (49.29.1), (49.33.1) and (49.35.1), respectively.

Clearly, if $d = 0$, the parameters μ_{A_1} and $\sigma^2_{A_1}$ can be estimated using (49.44).

Secondly, using (49.3) and (49.4), the means and variances of A_n for $n = 2, 3, \ldots, N$ are estimated using the following formulae:

$$\hat{\mu}_{A_n} = \hat{\mu}_{A_1} - (n-1)\hat{d} \quad \text{and} \quad \hat{\sigma}^2_{A_n} = \hat{\sigma}^2_{A_1} \quad \text{for } n = 2, 3, \ldots, N\,. \tag{49.45.1}$$

A Single GP

First, the mean and variance of G_1 are estimated using the relevant estimators with formulae given below.

The first estimators for μ_{G_1} and $\sigma^2_{G_1}$ are denoted and given by

$$\hat{\mu}_{G_1,1} = \mathrm{e}^{\hat{\alpha}_{\mathrm{G}}}\left(1 + \frac{\hat{\sigma}^2_{\mathrm{G},\varepsilon}}{2}\right) \tag{49.46.1}$$

and

$$\hat{\sigma}^2_{G_1,1} = \mathrm{e}^{2\hat{\alpha}_{\mathrm{G}}}\hat{\sigma}^2_{\mathrm{G},\varepsilon}\,, \tag{49.47.1}$$

where $\hat{\alpha}_{\mathrm{G}}$ and $\hat{\sigma}^2_{\mathrm{G},\varepsilon}$ are given by (49.28.1) and (49.29.1) with $\hat{d}$ and A_n replaced by $\ln\hat{r}$ and $\ln G_n$.

The second estimators for μ_{G_1} and $\sigma^2_{G_1}$ are denoted and given by

$$\hat{\mu}_{G_1,2} = \frac{\sum\limits_{n=1}^{N} \hat{r}^{(n-1)}G_n}{N} \tag{49.48.1}$$

and

$$\hat{\sigma}^2_{G_1,2} = \frac{\sum\limits_{n=1}^{N} \left(\hat{r}^{(n-1)}G_n\right)^2 - \dfrac{\left(\sum\limits_{n=1}^{N} \hat{r}^{(n-1)}G_n\right)^2}{N}}{N-1}\,, \tag{49.49.1}$$

where $\hat{r}$ is given by (49.27.1) with A_n replaced by $\ln G_n$.

The third estimator for μ_{G_1} is denoted and given by

$$\hat{\mu}_{G_1,3} = \frac{\left(1 - \hat{r}^{-1}\right)\sum\limits_{n=1}^{N} G_n}{1 - \hat{r}^{-N}}\,. \tag{49.50.1}$$

The fourth $\hat{\mu}_{G_1,4}$ and fifth $\hat{\mu}_{G_1,5}$ estimators for μ_{G_1} respectively satisfy the equations

$$2(\ln\mu_{G_1} - \hat{\alpha}_{\mathrm{G}})\mu^2_{G_1} - \hat{\sigma}^2_{G_1,1} = 0 \tag{49.56.1}$$

and

$$2(\ln\mu_{G_1} - \hat{\alpha}_{\mathrm{G}})\mu^2_{G_1} - \hat{\sigma}^2_{G_1,2} = 0\,, \tag{49.57.1}$$

where $\hat{\sigma}^2_{G_1,1}$ and $\hat{\sigma}^2_{G_1,2}$ are given by (49.47.1) and (49.49.1).

Clearly, if $r = 1$, the parameters μ_{G_1} and $\sigma^2_{G_1}$ can be estimated using (49.58).

Secondly, using (49.7) and (49.8), the means and variances of A_n for $n = 2, 3, \ldots, N$ are estimated using the following formulae:

$$\hat{\mu}_{G_n} = \frac{\hat{\mu}_{G_1}}{\hat{r}^{(n-1)}} \quad \text{and} \quad \hat{\sigma}^2_{G_n} = \frac{\hat{\sigma}^2_{G_1}}{\hat{r}^{2(n-1)}}$$
$$\text{for } n = 2, 3, \ldots, N\,. \tag{49.59.1}$$

49.6 Comparison of Estimators Using Simulation

Much of the work in this section is based on *Leung* [49.7], and *Leung* and *Lai* [49.13].

49.6.1 A Single AP or GP

Some simulation studies were performed to evaluate various estimators given in Sect. 49.5.3 and to compare the different estimates of μ_{A_1} and $\sigma^2_{A_1}$ (or μ_{G_1} and $\sigma^2_{G_1}$).

For each realization $\{A_n, n = 1, \ldots, 20\}$, the estimates $\hat{\mu}_{A_1,i}$, $i = 1, 2, 3, 4$ are ranked using three criteria. First, if our objective is to estimate the value of μ_{A_1}, we can compute the deviation ϕ of $\hat{\mu}_{A_1,i}$ from μ_{A_1}, i.e. $\phi = \left|\hat{\mu}_{A_1,i} - \mu_{A_1}\right|$. Secondly, if our objective is to fit A_ns values only, we can calculate the mean square error (MSE) between the fitted values $\hat{A}_{n,i} = \hat{\mu}_{A_1,i} - (n-1)\hat{d}$s and observations A_ns, i.e. $\text{MSE} = \sum_{n=1}^{N} \left(\hat{A}_{n,i} - A_n\right)^2 / N$. Thirdly, if our objective is to estimate μ_{A_1} as well as fitting values of A_ns, then we can use $\Phi = \phi + \sqrt{\text{MSE}}$. Moreover, the estimates $\hat{\sigma}^2_{A_1,1}$ and $\hat{\sigma}^2_{A_1,2}$ can be compared by their standard deviations (s.d.) from $\sigma^2_{A_1}$. The recommended estimators based on the simulation studies are summarized in Table 49.1 (for more details, see *Leung* et al. [49.7]).

Similarly, for each realization $\{G_n, n = 1, \ldots, 101\}$, the estimates $\hat{\mu}_{G_1,i}$, $i = 1, 2, 3, 4, 5, 6$ are ranked using the aforementioned three criteria, and the estimates $\hat{\sigma}^2_{G_1,1}$ and $\hat{\sigma}^2_{G_1,2}$ can be compared by their s.d. from $\sigma^2_{G_1}$. The recommended estimators based on the simulation studies are summarized in Table 49.2 (for more details, see *Lam* [49.15]).

49.6.2 *K* Independent, Homogeneous APs or GPs

Some simulation studies were also performed to evaluate various estimators given in Sect. 49.5.1 (or Sect. 49.5.2) and to compare the different estimates of $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$ (or $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$).

For each realization $\{A_{n,k}, n = 1, \ldots, 20$ and $k = 1, \ldots, 10\}$ (or $\{G_{n,k}, n = 1, \ldots, 101$ and $k = 1, \ldots, 10\}$), the estimates $\hat{\mu}_{\bar{A}_1,i}$, $i = 1, 2, 3, 4$ (or $\hat{\mu}_{\bar{G}_1,i}$, $i = 1, 2, 3, 4, 5, 6$) are ranked using the first criterion, namely the deviation ϕ of $\hat{\mu}_{\theta,i}$ from μ_θ, i.e. $\phi = \left|\hat{\mu}_{\theta,i} - \mu_\theta\right|$, and the estimates $\hat{\sigma}^2_{\theta,1}$ and $\hat{\sigma}^2_{\theta,2}$

Table 49.1 Recommended estimators for μ_{A_1} and $\sigma^2_{A_1}$

d	ϕ μ_{A_1}	MSE A_n	Φ μ_{A_1} & A_n	s. d. $\sigma^2_{A_1}$
$= 0$	$\hat{\mu}_{A_1,4}$	$\hat{\mu}_{A_1,2}$ or $\hat{\mu}_{A_1,4}$	$\hat{\mu}_{A_1,3}$ or $\hat{\mu}_{A_1,4}$	$\hat{\sigma}^2_{A_1,3}$
< 0	$\hat{\mu}_{A_1,1}$ or $\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\sigma}^2_{A_1,2}$
$\in \left(0, \frac{\mu_{A_1}}{n-1}\right]$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\mu}_{A_1,2}$	$\hat{\sigma}^2_{A_1,2}$

Table 49.2 Recommended estimators for μ_{G_1} and $\sigma^2_{G_1}$

r	ϕ μ_{G_1}	MSE G_n	Φ μ_{G_1} & G_n	s. d. $\sigma^2_{G_1}$
$= 1$	$\hat{\mu}_{G_1,6}$	$\hat{\mu}_{G_1,3}$ or $\hat{\mu}_{G_1,6}$	$\hat{\mu}_{G_1,6}$	$\hat{\sigma}^2_{G_1,3}$
$\in (0, 1)$	$\hat{\mu}_{G_1,5}$	$\hat{\mu}_{G_1,3}$	$\hat{\mu}_{G_1,3}$	$\hat{\sigma}^2_{G_1,2}$
> 1	$\hat{\mu}_{G_1,3}$	$\hat{\mu}_{G_1,3}$	$\hat{\mu}_{G_1,3}$	$\hat{\sigma}^2_{G_1,2}$

Table 49.3 Recommended estimators for $\mu_{\bar{A}_1}$ and $\sigma^2_{\bar{A}_1}$, and $\mu_{\bar{G}_1}$ and $\sigma^2_{\bar{G}_1}$

d	ϕ $\mu_{\bar{A}_1}$	s. d. $\sigma^2_{\bar{A}_1}$	r	ϕ $\mu_{\bar{G}_1}$	s. d. $\sigma^2_{\bar{G}_1}$
$=0$	$\hat{\mu}_{A_1,4}$	$\hat{\sigma}^2_{A_1,3}$	$r=1$	$\hat{\mu}_{G_1,6}$	$\hat{\sigma}^2_{G_1,3}$
<0	$\hat{\mu}_{\bar{A}_1,1}$	$\hat{\sigma}^2_{\bar{A}_1,1}$ or $\hat{\sigma}^2_{\bar{A}_1,2}$	$\in (0,1)$	$\hat{\mu}_{\bar{G}_1,2}$	$\hat{\sigma}^2_{\bar{G}_1,2}$
$\in \left(0, \frac{\mu_{\bar{A}_1}}{n-1}\right]$	$\hat{\mu}_{\bar{A}_1,1}$	$\hat{\sigma}^2_{\bar{A}_1,1}$ or $\hat{\sigma}^2_{\bar{A}_1,2}$	>1	$\hat{\mu}_{\bar{G}_1,2}$ or $\hat{\mu}_{\bar{G}_1,3}$	$\hat{\sigma}^2_{\bar{G}_1,2}$

can be compared by their s.d. from σ^2_θ, where $\theta =$ either $\bar{A}_1$ or $\bar{G}_1$. The recommended estimators based on the simulation studies are summarized in Table 49.3 (for more details, see *Leung* and *Lai* [49.13]).

49.6.3 Comparison Between Averages of Estimates and Pooled Estimates

Having obtained the estimates $\hat{d}$, $\hat{\mu}_{A_1}$ and $\hat{\sigma}^2_{A_1}$ (or $\hat{r}$, $\hat{\mu}_{G_1}$ and $\hat{\sigma}^2_{G_1}$) using the relevant estimators suggested in Table 49.1 (or Table 49.2), of the parameters d, μ_{A_1} and $\sigma^2_{A_1}$ (or r, μ_{G_1} and $\sigma^2_{G_1}$) of a single system, we can compute the averages of the respective estimates for a collection of homogeneous systems and then use these averages to estimate $\hat{\mu}_{A_n}$ and $\hat{\sigma}^2_{A_n}$ (or $\hat{\mu}_{G_n}$ and $\hat{\sigma}^2_{G_n}$). *Leung* and *Lai* [49.13] drew the conclusion that, in any cases, the pooled estimates obtained using the pooled estimators for APs or GPs suggested in Table 49.3 are better than the respective averages of estimates.

49.7 Real Data Analysis

Lam et al. [49.16] presented ten examples, each analyzing a real data set using four models:

1. the GP model with a nonparametric method,
2. the HPP model,
3. the NHPP model with PLP and
4. the NHPP model with LLP.

Example 1 examines 190 data of the intervals in days between successive coal-mining disasters in Great Britain, which have been used by a number of researchers to illustrate various techniques that can be applied to point processes; see, for example, *Cox* and *Lewis* [49.17], pp. 42–43. The data set can be found in *Hand* et al. [49.34], p. 155 or *Andrews* and *Herzberz* [49.35], pp. 51–56, in which the data are recorded in more detail.

Examples 2–4 study 29, 30 and 27 data of the intervals in operating hours between successive failures of air-conditioning equipment in aircrafts 3, 6 and 7. The 13 data sets tabulate on p. 6 of *Cox* and *Lewis* [49.17], and the data sets being examined are the largest three.

Example 5 investigates 257 failure times of a computer in unspecified units. The data are given in *Cox* and *Lewis* ([49.17], p. 11).

Example 6 examines 245 arrival times of patients at an intensive care unit in a hospital. The data are given in *Cox* and *Lewis* ([49.17], p. 14 and pp. 254–255).

Examples 7 and 8 study 71 and 56 data of the arrival times to unscheduled overhauls for the no. 3 and no. 4 main propulsion diesel engines for two submarines. The two data sets tabulate on pp. 75–76 of *Ascher* and *Feingold* [49.18].

Example 9 investigates the times that 41 successive vehicles traveling northwards along the M1 motorway in England passed a fixed point near junction 13 in Bedfordshire on Saturday 23 March 1985. The data are given in *Hand* et al. ([49.34], p. 3).

Example 10 examines 136 failure times [in central processing unit (CPU) seconds, measured in terms of execution time] of a real-time command-and-control software system. The data are given in *Hand* et al. ([49.34], p. 10–11).

Lam et al. [49.16] concluded that, on average, the GP model is the best model for fitting these ten real data sets among the four models based on the MSE criterion (as defined in Sect. 49.5.1). This is the reason why the GP model can be applied to the maintenance problems.

Furthermore, *Lam* and *Chan* [49.36] applied the GP model to fit the three real data sets in Examples 1, 7 and 8 using a parametric method with one of the lognormal, exponential, gamma and Weibull distributions. The numerical results also conclude that all three data sets can be well fitted by the GP model based on the MSE criterion.

The author is currently investigating the following:

1. The ten data sets are analyzed using the AP model with a nonparametric method and the numerical results are compared with those in *Lam* et al. [49.16].
2. The data sets in Examples 2–4 (or even the 13 data sets) and the data sets in Examples 7 and 8 are respectively pooled together to estimate the parameters using the methods suggested in Sects. 49.2–49.5 for AP and GP, and the methods used in *Leung* and *Cheng* [49.3] for HPP and NHPP. The numerical results are also compared with those in *Lam* et al. [49.16] plus those obtained using the AP model.

49.8 Optimal Replacement Policies Determined Using Arithmetico-Geometric Processes

The work in this section is substantially based on *Leung* [49.5].

49.8.1 Arithmetico-Geometric Processes

A definition of an AGP is given below.

Definition 49.3
Given a sequence of random variables $H_1, H_2, \ldots,$ if for some real number d and some $r > 0$, $\{[H_n + (n-1)d]r^{(n-1)}, n = 1, 2, \ldots\}$ forms an RP, then $\{H_n, n = 1, 2, \ldots\}$ is an AGP. The two parameters d and r are called the common difference and the common ratio of the AGP respectively.

Three specializations of an AGP are given below.

If $r > 1$ and $d \in \left(0, \frac{\mu_{H_1}}{(n-1)r^{(n-1)}}\right]$, where $n = 2, 3, \ldots$ and μ_{H_1} is the mean of the first random variable H_1, then the AGP is called a decreasing AGP. If $d < 0$ and $0 < r < 1$, then the AGP is called an increasing AGP. If $d = 0$ and $r = 1$, then the AGP reduces to an RP.

Two immediate remarks concerning the characteristics of an AGP are as follows:

1. An AGP is the name given to a series in which the general term is the product of the general term of an AP and of a GP; we take this term to be, in general,
$$H_n = \frac{H_1}{r^{(n-1)}} - (n-1)d\,.$$
2. It is evident that, if we put $r = 1$ but $d \neq 0$, or $d = 0$ but $r \neq 1$ into the above expression, the process obtained becomes an AP, or a GP. Hence, an AGP extends and generalizes an AP or a GP.

Therefore, for a deteriorating system, it is reasonable to assume that the successive operating times of the system form a decreasing AGP, whereas the corresponding consecutive repair times constitute an increasing AGP. However, the replacement times for the system are usually stochastically the same no matter how old the used system is; hence, they will form an RP. This is the motivation behind the introduction of the AGP approach.

49.8.2 Model

Before deriving new repair–replacement models, the following assumptions are stated.

1. At the beginning, a new system is used.
2. Whenever the system fails, it can be repaired. Let X_n be the survival time after the $(n-1)$th repair, then a sequence $\{X_n, n = 1, 2, \ldots\}$ forms a decreasing AGP with parameters $d_a > 0$ and $r_a > 1$ such that $\mu_{X_1}/(n-1)r_a^{(n-1)} \geq d_a$, where $E(X_1) \equiv \mu_{X_1} > 0$.
3. Let Y_n be the repair time after the nth failure, then a sequence $\{Y_n, n = 1, 2, \ldots\}$ forms an increasing AGP with parameters $d_b < 0$ and $0 < r_b < 1$, and $E(Y_1) \equiv \mu_{Y_1} \geq 0$. $\mu_{Y_1} = 0$ means that the repair time is negligible.
4. A sequence $\{X_n, n = 1, 2, \ldots\}$ and a sequence $\{Y_n, n = 1, 2, \ldots\}$ are independent.
5. An average operating cost rate is c_o, an average repair cost rate is c_f, and an average revenue rate of a working system is w.
6. The system may be replaced at some time by a new and identical one. An average replacement cost rate under policy T or N is c_{RT} or c_{RN}, respectively, and an average replacement downtime under policy T or N is u_{RT} or u_{RN}, respectively. Two kinds of replacement policy are considered in this model.
   a) A replacement policy T is a policy in which we replace the system whenever the working age of the system reaches T, a continuous decision variable, see *Barlow* and *Proschan* [49.19]. The working age T of a system at time t is the

cumulative survival time by time t, i. e.

$$T=\begin{cases} t-V_n,\ U_n+V_n\le t<U_{n+1}+V_n \\ U_{n+1},\ U_{n+1}+V_n\le t<U_{n+1}+V_{n+1}\ , \end{cases} \tag{49.60}$$

where $U_n=\sum_{i=1}^{n}X_i$, $V_n=\sum_{i=1}^{n}Y_i$ and $U_0=0$, $V_0=0$.

b) A replacement policy N is a policy in which we replace the system at the time of Nth failure since the last replacement, a discrete decision variable, see *Nakagawa* [49.37].

Under replacement policy T or N, the problem is to determine an optimal replacement policy T^* or N^*, respectively, such that the long-run expected loss per unit total time or per unit operation time is minimized.

49.8.3 The Long-Run Expected Loss Rate

Let T_n be the time between the $(n-1)$th replacement and the nth replacement with $T_0=0$, then $\{T_n,\ n=1,2,\dots\}$ forms an RP. Applying known results from renewal theory (see e.g. *Ross* [49.20], pp. 51–54), the long-run expected loss per unit time is obtained by

$$l(T)\text{ or }l(N)=\frac{\mathrm{E(loss\ incurred\ in\ a\ cycle)}}{\mathrm{E(length\ of\ a\ cycle)}}\ , \tag{49.61}$$

where the loss is defined as the total cost minus total revenue, and a cycle is the time between two consecutive replacements.

Under the replacement policy T, denote the length of a cycle by W, then

$$\begin{aligned} &W=T+V_n \quad \text{and}\\ &U_n<T\le U_{n+1} \quad \text{for } n=0,1,\dots\ . \end{aligned} \tag{49.62}$$

From Definition 49.3 and Assumption 3, it follows that $\mathrm{E}(Y_n)=\frac{\mu_{Y_1}}{r_\mathrm{b}^{(n-1)}}-(n-1)d_\mathrm{b}$. Then

$$\begin{aligned}\mathrm{E}(W)&=\mathrm{E}\left(T+\sum_{j=1}^{n}Y_j\right)\\ &=\mathrm{E}\left[T+\sum_{j=1}^{\infty}Y_jI(n\ge j)\right]\ ,\end{aligned}$$

where I is the indicator random variable defined as

$$I(n\ge j)=\begin{cases}1 & \text{if } n\ge j\\ 0 & \text{if } n<j\end{cases}\ .$$

Since $n\ge j\Leftrightarrow U_j\le T$, we have

$$\begin{aligned}\mathrm{E}(W)&=\mathrm{E}\left[T+\sum_{j=1}^{\infty}Y_jI(U_j\le T)\right]\\ &=\mathrm{E}(T)+\sum_{j=1}^{\infty}\mathrm{E}(Y_j)\mathrm{E}[I(U_j\le T)]\\ &\quad\text{as } Y_j \text{ and } U_j \text{ are independent}\\ &=T+\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]\\ &\quad\times[1\times\Pr(U_j\le T)+0\times\Pr(U_j>T)]\\ &=T+\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]F_j(T)\ ,\end{aligned}$$

where F_j is the cumulative distribution function of U_j.

Thus, from (49.61) the long-run expected loss per unit total time $l(T)$ under policy T is given by

$$\begin{aligned}l(T)=&\frac{(c_\mathrm{o}-w)T+c_{\mathrm{R}T}u_{\mathrm{R}T}}{T+\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]F_j(T)+u_{\mathrm{R}T}}\\ &+\frac{c_\mathrm{f}\left\{\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]F_j(T)\right\}}{T+\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]F_j(T)+u_{\mathrm{R}T}}\end{aligned} \tag{49.63}$$

and also from (49.61) the long-run expected loss per unit total time $l(N)$ under policy N is given by

$$\begin{aligned}l(N)=&\left\{(c_\mathrm{o}-w)\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_\mathrm{a}^{(j-1)}}-(j-1)d_\mathrm{a}\right]\right.\\ &\left.+c_\mathrm{f}\sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]+c_{\mathrm{R}N}u_{\mathrm{R}N}\right\}\\ &\bigg/\left\{\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_\mathrm{a}^{(j-1)}}-(j-1)d_\mathrm{a}\right]\right.\\ &\left.+\sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_\mathrm{b}^{(j-1)}}-(j-1)d_\mathrm{b}\right]+u_{\mathrm{R}N}\right\}\ .\end{aligned} \tag{49.64}$$

Further, from (49.61) the long-run expected loss per unit operation time $l_{op}(T)$ under policy T or $l_{op}(N)$ under policy N is respectively given by

$$l_{op}(T) = \frac{(c_o - w)T + c_{RT}u_{RT}}{T} + \frac{c_f\left\{\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right]F_j(T)\right\}}{T} \quad (49.65)$$

or

$$l_{op}(N) = \frac{(c_o - w)\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_a^{(j-1)}} - (j-1)d_a\right]}{\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_a^{(j-1)}} - (j-1)d_a\right]} + \frac{c_f\sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right] + c_{RN}u_{RN}}{\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_a^{(j-1)}} - (j-1)d_a\right]}\,. \quad (49.66)$$

To simplify the optimization tasks, first operating cost is excluded ($c_o = 0$) because this belongs to the account of production (rather than maintenance) costs; secondly, without loss of generality, w can be set equal to HK\$ 1 because money can be measured in an arbitrary scale (c_f, C_{RT}, and C_{RN} are on the same scale of Hong Kong dollars). Hence, putting $c_o = 0$ and $w = 1$ into (49.63) to (49.66) and then adding unity to the right-hand side of (49.63) to (49.66), we obtain the simplified versions of $l(T)$ and $l(N)$, and $l_{op}(T)$ and $l_{op}(N)$ namely

$$l_1(T) = \frac{(c_f+1)\left\{\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right]F_j(T)\right\}}{T + \sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right]F_j(T) + u_{RT}} + \frac{(c_{RT}+1)u_{RT}}{T + \sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right]F_j(T) + u_{RT}} \quad (49.67)$$

and

$$l_1(N) = \left\{(c_f+1)\sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right] + (c_{RN}+1)u_{RN}\right\} \Bigg/ \left\{\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_a^{(j-1)}} - (j-1)d_a\right] + \sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right] + u_{RN}\right\}\,, \quad (49.68)$$

$$l_{op1}(T) = \frac{c_f\left\{\sum_{j=1}^{\infty}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right]F_j(T)\right\}}{T} + \frac{c_{RT}u_{RT}}{T} \quad (49.69)$$

and

$$l_{op1}(N) = \frac{c_f\sum_{j=1}^{N-1}\left[\frac{\mu_{Y_1}}{r_b^{(j-1)}} - (j-1)d_b\right] + c_{RN}u_{RN}}{\sum_{j=1}^{N}\left[\frac{\mu_{X_1}}{r_a^{(j-1)}} - (j-1)d_a\right]}\,, \quad (49.70)$$

respectively.

Finally, the optimal replacement policy, denoted in general by T^* or N^*, can be determined by minimizing $l(T)$, $l_{op}(T)$, $l(N)$ or $l_{op}(N)$ [or alternatively $l_1(T)$, $l_{op1}(T)$, $l_1(N)$ or $l_{op1}(N)$], respectively. Furthermore, the minimization procedure can be achieved using analytical or numerical methods.

In practice, we prefer to adopt the optimal policy N^* rather than use the optimal policy T^*, because of the much simpler form of $l(N)$, $l_{op}(N)$, $l_1(N)$, and $l_{op1}(N)$. Moreover, under some mild conditions, *Lam* [49.38] has proved that the optimal policy N^* is better than any policy T; in particular, it is better than the optimal policy T^*.

Note that the replacement policy that minimizes l or l_{op} also maximizes p or p_{op} because the long-run expected profit p or p_{op} per unit total time or per unit operation time is equal to the negation of the long-run expected loss l or l_{op} per unit total time or per unit operation time, respectively. The expressions for l and l_g include the performance measure of cost or downtime as a special case.

The expressions for c and c_{op}, the long-run expected cost per unit total time and per unit operation time, are obtained by substituting $w = 0$ or $w = c_o = 0$ into the expressions for l and l_{op}, respectively.

The expressions for u and u_{op}, the long-run expected downtime per unit total time and per unit operation time, are obtained by substituting $w = c_o = 0$, $c_f = c_{RT} = c_{RN} = 1$ into the expressions for l and l_{op} respectively. Also, availability $= (1 - u) \times 100\%$ or availability $= (1 - u_{op}) \times 100\%$.

49.9 Some Conclusions on the Applicability of an AP and/or a GP

As concluded in the last paragraph before Sect. 49.1 AP and/or a GP approach is considered to be relevant, realistic and direct to the modeling of deteriorating system maintenance problems. If a decreasing GP (or an increasing AP) does not fit the system's successive operating times (or repair times), a decreasing AP (or an increasing GP) may be attempted instead. Equation (49.64) with $d_b = 0$, $r_a = 1$ then becomes

$$l_{A,G}(N) = \left\{(c_o - w)\left\{\frac{N}{2}\left[2\mu_{X_1} - (N-1)d_a\right]\right\} + c_f\mu_{Y_1}\sum_{j=1}^{N-1}\frac{1}{r_b^{(j-1)}} + c_{RN}u_{RN}\right\} \Bigg/ \left\{\frac{N}{2}\left[2\mu_{X_1} - (N-1)d_a\right] + \mu_{Y_1}\sum_{j=1}^{N-1}\frac{1}{r_b^{(j-1)}} + u_{RN}\right\} \tag{49.71}$$

which is one of the four replacement models using an AP or a GP or both. This means that there are altogether four options to resolve a replacement problem when using model (49.64).

The reliability findings for the 6LXB type of engine analyzed using GPs (*Leung* and *Lee* [49.1]) and APs (*Leung* and *Kwok* [49.6]) are summarized in Table 49.4.

The reliability findings for the Benz type of gearbox analyzed using GPs (*Leung* and *Fong* [49.2]) and APs (*Leung* and *Lai* [49.10]) are summarized in Table 49.5.

In Table 49.5, the parameter $\hat{d}_b$ or $\hat{r}_b$ is larger than zero or unity for the Benz gearbox. $\hat{d}_b > 0$ or $\hat{r}_b > 1$ indicates that the repair times of the gearboxes decrease and will tend towards zero. The reasons for this phenomenon are:

1. The Kowloon Motor Bus (KMB) Company Limited spends a lot of time on the following when a gearbox first fails (see *Leemis* [49.39], p. 148)
   a) Diagnosis time: time used for fault finding, including adjustment of test equipment, carrying out checks, interpretation of information gained, verification of the conclusions drawn and deciding corrective action.
   b) Logistic time: time used in waiting for spare parts, test gears, additional tools and manpower to be transported to the system.
   c) Administrative time: time used in the allocation of repair tasks, manpower changeover due to

Table 49.4 Estimated values of common difference and ratio, and means for the 6LXB engine

Survival times	$\hat{\mu}_{X_1}$ **(y)**	$\hat{\mu}_{X_2}$ **(y)**	$\hat{\mu}_{X_3}$ **(y)**	$\hat{\mu}_{X_4}$ **(y)**
AP with $\hat{d}_a = 1.6$ y	3.4	1.8	0.2	–
GP with $\hat{r}_a = 4.533$	3.79	0.8361	0.1844	0.0407
Repair times	$\hat{\mu}_{Y_1}$ **(d)**	$\hat{\mu}_{Y_2}$ **(d)**	$\hat{\mu}_{Y_3}$ **(d)**	$\hat{\mu}_{Y_4}$ **(d)**
AP with $\hat{d}_b = -12.17$ d	8.86	21.03	33.20	–
GP with $\hat{r}_b = 0.524$	9.881	18.86	35.99	68.68

Table 49.5 Estimated values of common difference and ratio, and means for the Benz gearbox

Survival times	$\hat{\mu}_{X_1}$ **(y)**	$\hat{\mu}_{X_2}$ **(y)**	$\hat{\mu}_{X_3}$ **(y)**	$\hat{\mu}_{X_4}$ **(y)**
AP with $\hat{d}_a = 0.97$ y	3.05	2.08	1.11	0.14
GP with $\hat{r}_a = 2.004$	1.969	0.9825	0.4903	0.2447
Repair times	$\hat{\mu}_{Y_1}$ **(d)**	$\hat{\mu}_{Y_2}$ **(d)**	$\hat{\mu}_{Y_3}$ **(d)**	$\hat{\mu}_{Y_4}$ **(d)**
AP with $\hat{d}_b = 34.07$ d	85.46	51.39	17.32	–
GP with $\hat{r}_b = 2.096$	37.25	17.77	8.479	4.045

demarcation arrangements, official breaks, disputes, etc.
2. KMB gains repair experience from the first failure, which is used to improve their time management, so repair time decreases.
3. When a gearbox is taken out of a bus, there is no follow-up tracing of the gearbox and hence we are unable to find exact consecutive repair times.

The optimal replacement policy based on minimum cost is to replace the engine or gearbox after the second or third failure using the AP or GP approach. Notice that theoretically it is replaced after the ninth failure of the engine or gearbox using the GP approach; this is possible since a decreasing GP converges to zero (but a decreasing AP produces negative values, which are nonexistent in a reliability context).

Based on the four real case studies, we observe that both approaches are applicable in solving reliability problems. As to which one is more appropriate to a given set of reliability data, some criteria have to be established. Once we have criteria comparing the results using the AP and GP approaches, we can separately compare the findings obtained in *Leung* and *Lee* [49.1] with those in *Leung* and *Kwok* [49.6], and *Leung* and *Fong* [49.2] with *Leung* and *Lai* [49.10].

49.10 Concluding Remarks

There follow five notes concerning the application of the models given in the previous sections.

The first note concerns the third question: after fitting an AP (or a GP) model to the data set, how good is the fit?

Estimation of parameters is properly only a precursor to further analysis. The techniques outlined in Sects. 49.2– 49.5 may be extended to provide a basis for confidence bounds, tests for comparing different sets of event counts, and so on. *Lam* et al. [49.16] obtained the asymptotic distributions of the nonparametric estimators of r, μ_{G_1} and $\sigma^2_{G_1}$. By a parametric approach, *Lam* and *Chan* [49.36] also obtained the estimators of r, μ_{G_1} and $\sigma^2_{G_1}$ and their asymptotic distributions.

Scarf [49.40], on p. 498, has recommended that, if the assumptions of a simple AP or GP model are not

Table 49.6 Summary of useful results of both AP and GP processes

APs equation given by	GPs equation given by
$\hat{d}$ (49.27) or (49.27.1)	$\ln \hat{r}$ (49.27) or (49.27.1) with $A_{n,k}$ replaced by $\ln G_{n,k}$
$\hat{\alpha}_{\mathrm{A}}$ (49.28) or (49.28.1)	$\hat{\alpha}_{\mathrm{G}}$ (49.28) or (49.28.1) with $A_{n,k}$ replaced by $\ln G_{n,k}$
$\hat{\sigma}_{\mathrm{A},\varepsilon}$ (49.29) or (49.29.1)	$\hat{\sigma}_{\mathrm{G},\varepsilon}$ (49.29) or (49.29.1) with $A_{n,k}$ replaced by $\ln G_{n,k}$
t_{A} (49.31)	t_{G} (49.31) with $A_{n,k}$ replaced by $\ln G_{n,k}$
For $d \neq 0$	**For $r \neq 1$**
$\hat{\mu}_{\bar{A}_1,1}$ (49.32) or (49.32.1)	$\hat{\mu}_{\bar{G}_1,1}$ (49.46) or (49.46.1)
–	$\hat{\mu}_{\bar{G}_1,2}$ (49.48) or (49.48.1)
–	$\hat{\mu}_{\bar{G}_1,3}$ (49.50) or (49.50.1)
$\hat{\mu}_{\bar{A}_1,2}$ (49.42) or (49.42.1)	$\hat{\mu}_{\bar{G}_1,4}$ (49.56) or (49.56.1)
$\hat{\mu}_{\bar{A}_1,3}$ (49.43) or (49.43.1)	$\hat{\mu}_{\bar{G}_1,5}$ (49.57) or (49.57.1)
$\hat{\sigma}^2_{\bar{A}_1,1}$ (49.33) or (49.33.1)	$\hat{\sigma}^2_{\bar{G}_1,1}$ (49.47) or (49.47.1)
$\hat{\sigma}^2_{\bar{A}_1,2}$ (49.35) or (49.35.1)	$\hat{\sigma}^2_{\bar{G}_1,2}$ (49.48) or (49.48.1)
For $d = 0$	**For $r = 1$**
$\hat{\mu}_{A_1,4}$, $\hat{\sigma}^2_{A_1,3}$ (49.44)	$\hat{\mu}_{G_1,6}$, $\hat{\sigma}^2_{G_1,3}$ (49.58)
$\hat{\mu}_{\bar{A}_n}$, $\hat{\sigma}^2_{\bar{A}_n}$ (49.45) or (49.45.1)	$\hat{\mu}_{\bar{G}_n}$, $\hat{\sigma}^2_{\bar{G}_n}$ (49.59) or (49.59.1)
Replacement model	**Replacement model**
$l_{\mathrm{A}}(T)$ (49.72)	$l_{\mathrm{G}}(T)$ (49.74)
$l_{\mathrm{A}}(N)$ (49.73)	$l_{\mathrm{G}}(N)$ (49.75)

valid, and in practice this is usually so, then there are two possible routes:

1. extend the model with extra parameters, here d and r, making greater demands on the available data;
2. use the simple model to obtain a crude approximation to the optimum policy.

The second note relates to route 1. The author is focusing his efforts on developing a procedure of statistical inference for an AGP, since fitting a model to failure and/or repair data is preliminary to the utilization of an optimization model, from which an optimal maintenance policy based on minimizing loss, cost or downtime may be found. Naturally, the development of such a procedure involves much more mathematics than that for a GP by *Lam* [49.15] or for an AP by *Leung* [49.8]. Once the procedure is warranted, two parallel case studies using an AGP approach for the same set of real maintenance data of engines and gearboxes will be carried out and then findings will be compared with those obtained in *Leung* et al. [49.1, 4, 5, 10]; these case studies will be presented in two future papers.

The third note relates to route 2. Estimation of parameters is also a precursor to practical use of an optimization model. Two AP models used in resolving replacement problems are obtained by putting $r_\mathrm{b}=1$ in (49.63) and $r_\mathrm{a}=r_\mathrm{b}=1$ in (49.64), namely the long-run expected loss per unit total time under policy T, which is given by

$$l_\mathrm{A}(T)=\frac{(c_\mathrm{o}-w)T+c_{\mathrm{R}T}u_{\mathrm{R}T}}{T+\sum\limits_{j=1}^{\infty}[\mu_{Y_1}-(j-1)d_\mathrm{b}]F_j(T)+u_{\mathrm{R}T}} +\frac{c_\mathrm{f}\left\{\sum\limits_{j=1}^{\infty}[\mu_{Y_1}-(j-1)d_\mathrm{b}]F_j(T)\right\}}{T+\sum\limits_{j=1}^{\infty}[\mu_{Y_1}-(j-1)d_\mathrm{b}]F_j(T)+u_{\mathrm{R}T}} \tag{49.72}$$

and the long-run expected loss per unit total time $l_\mathrm{A}(N)$ under policy N, which is given by

$$l_\mathrm{A}(N)=\left\{(c_\mathrm{o}-w)\left\{\frac{N}{2}\left[2\mu_{X_1}-(N-1)d_\mathrm{a}\right]\right\} +c_\mathrm{f}\left\{\frac{N-1}{2}\left[2\mu_{Y_1}-(N-2)d_\mathrm{b}\right]\right\} +c_{\mathrm{R}N}u_{\mathrm{R}N}\right\} \Big/ \left\{\frac{N}{2}\left[2\mu_{X_1}-(N-1)d_\mathrm{a}\right] +\frac{N-1}{2}\left[2\mu_{Y_1}-(N-2)d_\mathrm{b}\right]+u_{\mathrm{R}N}\right\}. \tag{49.73}$$

Correspondingly, two GP models obtained by *Lam* [49.14] by putting $d_\mathrm{b}=0$ in (49.63) and $d_\mathrm{a}=d_\mathrm{b}=0$ in (49.64) are given by

$$l_\mathrm{G}(T)=\frac{(c_\mathrm{o}-w)T+c_\mathrm{f}\mu_{Y_1}\sum\limits_{j=1}^{\infty}\frac{F_j(T)}{r_\mathrm{b}^{(j-1)}}+c_{\mathrm{R}T}u_{\mathrm{R}T}}{T+\mu_{Y_1}\sum\limits_{j=1}^{\infty}\frac{F_j(T)}{r_\mathrm{b}^{(j-1)}}+u_{\mathrm{R}T}} \tag{49.74}$$

and

$$l_\mathrm{G}(N)=\frac{(c_\mathrm{o}-w)\mu_{X_1}\sum\limits_{j=1}^{N}\frac{1}{r_\mathrm{a}^{(j-1)}}}{\mu_{X_1}\sum\limits_{j=1}^{N}\frac{1}{r_\mathrm{a}^{(j-1)}}+\mu_{Y_1}\sum\limits_{j=1}^{N-1}\frac{1}{r_\mathrm{b}^{(j-1)}}+u_{\mathrm{R}N}} +\frac{c_\mathrm{f}\mu_{Y_1}\sum\limits_{j=1}^{N-1}\frac{1}{r_\mathrm{b}^{(j-1)}}+c_{\mathrm{R}N}u_{\mathrm{R}N}}{\mu_{X_1}\sum\limits_{j=1}^{N}\frac{1}{r_\mathrm{a}^{(j-1)}}+\mu_{Y_1}\sum\limits_{j=1}^{N-1}\frac{1}{r_\mathrm{b}^{(j-1)}}+u_{\mathrm{R}N}}, \tag{49.75}$$

where F_j is the cumulative distribution function of $\sum_{i=1}^{j}X_i$; μ_{X_1} is the mean operating time after installation; μ_{Y_1} is the mean repair time after the first failure; d_a (or r_a) and d_b (or r_b) are the common differences (or ratios) corresponding to the failure and repair processes of a system, respectively; c_o is the average operating cost rate; c_f is the average repair cost rate; $c_{\mathrm{R}T}$ (or $c_{\mathrm{R}N}$) is the average replacement cost rate under policy T (or N); $u_{\mathrm{R}T}$ (or $u_{\mathrm{R}N}$) is the average replacement downtime under policy T (or N); and w is the average revenue rate of a working system.

Notice that model (49.72) (or (49.74)) only depends on the AP (or GP) through the parameters d_b (or r_b) and μ_{Y_1}, and model (49.73) (or (49.75)) only on d_b (or $\mathrm{r_b}$) and μ_{Y_1} plus d_a (or $\mathrm{r_a}$) and μ_{X_1}. When $K>1$, μ_{X_1} and μ_{Y_1} are replaced by $\mu_{\bar{X}_1}$ and $\mu_{\bar{Y}_1}$. In practice, model (49.73) or (49.75) is adopted because of its much simpler form. Moreover, under some mild conditions, *Lam* [49.38] has proved that the optimal policy N^* is better than the optimal policy T^*. Note that, under the same conditions, *Zhang* [49.41] has showed that the optimal bivariate replacement policy $(T, N)^*$ is better than N^*, which in turn is better than T^* (see also *Leung* [49.12]).

The fourth note is that an AP, GP or AGP approach has not incorporated the dependency of data on maintenance actions. If a GP model is appropriate, then the dependency of data upon maintenance actions should be modeled, i. e. the common ratios r_a and r_b of two distinct GPs are two functions of some preventive maintenance (PM) policy, where the subscripts 'a' and 'b' correspond to the failure and repair processes of a system respectively. *Leung* [49.9] established one type of the relationships between the common ratios r_a and r_b and a nonperiodic PM policy.

The fifth note is that a GP model has been widely used in maintenance problems of one-component systems, and two-component series, parallel and standby systems; see *Lam* [49.42], *Lam* and *Zhang* [49.43, 44], and *Zhang* [49.45] for details. *Lam* et al. [49.46] proved that the monotone process model for the multi-state system is equivalent to a GP model for a two-state one-component system by showing that two systems will have the same long-run expected loss per unit total time and the same optimal policy N^*. Furthermore, *Lam* [49.15], *Lam* and *Chan* [49.36], and *Lam* et al. [49.16] also applied the GP to the analysis of data from a series of events. *Lam* [49.47] gave a brief review and more references for the GP. For more properties and applications of GP, see *Lam* et al. [49.16, 48, 49] and *Zhang* et al. [49.50–53]. Finally, the author considers that almost all variants of GP formulation are also valid for AP or AGP.

49.A Appendix

To determine the line of best fit to the N paired-observations, we minimize the sum of squared errors $(S_{\varepsilon\varepsilon})$ given by

$$\begin{aligned} S_{\varepsilon\varepsilon} &= \sum_{k=1}^{K}\sum_{n=1}^{N_k}(y_{n,k}-\hat{y})^2 \\ &= \sum_{k=1}^{K}\sum_{n=1}^{N_k}(y_{n,k}-\hat{\beta}_0-\hat{\beta}_1 x_{n,k})^2 \,. \end{aligned} \tag{49.A1}$$

Denote

$$N = \sum_{k=1}^{K} N_k \,, \tag{49.30}$$

$$\bar{x} = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}}{N} \quad \text{and} \quad \bar{y} = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} y_{n,k}}{N} \,, \tag{49.A2}$$

$$\begin{aligned} S_{xx} &= \sum_{k=1}^{K}\sum_{n=1}^{N_k} x_{n,k}^2 - N\bar{x}^2 \,, \\ S_{yy} &= \sum_{k=1}^{K}\sum_{n=1}^{N_k} y_{n,k}^2 - N\bar{y}^2 \quad \text{and} \\ S_{xy} &= \sum_{k=1}^{K}\sum_{n=1}^{N_k} x_{n,k}y_{n,k} - N\bar{x}\bar{y} \,. \end{aligned} \tag{49.A3}$$

Differentiating (49.A1) with respect to $\hat{\beta}_0$ and $\hat{\beta}_1$, setting them equal to zero and solving the associated equations simultaneously, we obtain

$$\hat{\beta}_0 = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} y_{n,k} - \hat{\beta}_1 \sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}}{N} = \bar{y} - \hat{\beta}_1\bar{x} \tag{49.A4}$$

and

$$\hat{\beta}_1 = \frac{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}y_{n,k} - \bar{y}\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}}{\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}^2 - \bar{x}\sum\limits_{k=1}^{K}\sum\limits_{n=1}^{N_k} x_{n,k}} = \frac{S_{xy}}{S_{xx}} \,. \tag{49.A5}$$

Substituting (49.A4) and (49.A5) into (49.A1), and after some manipulation, we obtain

$$S_{\varepsilon\varepsilon} = S_{yy} - \hat{\beta}_1 S_{xy} \,. \tag{49.A6}$$

It can be shown that

$$\hat{\sigma}_\varepsilon^2 = \frac{S_{\varepsilon\varepsilon}}{N-2} \,, \tag{49.A7}$$

usually called the mean squared error (MSE), provides a good estimator for σ_ε^2, and that

$$t = \frac{\hat{\beta}_1 - \beta_{1,0}}{\hat{\sigma}_\varepsilon/\sqrt{S_{xx}}} \,, \tag{49.A8}$$

a Student's t distribution with $(N-2)$ degrees of freedom. This statistic is used to test a hypothesis that β_1 equals some particular numerical value, say $\beta_{1,0}$.

Now, putting $y = A_{n,k}$ or $\ln G_{n,k}$, $x_{n,k} = n-1$, $\beta_0 = \alpha_A$ or α_G, $\beta_1 = -d$ or $-\ln r$, $\beta_{1,0} = 0$, $\hat{\sigma}_\varepsilon^2 = \hat{\sigma}_{A,\varepsilon}^2$ or $\hat{\sigma}_{G,\varepsilon}^2$, $t = t_A$ or t_G, and using (49.A2) through (49.A8), we obtain (49.27) to (49.29) and (49.31) accordingly. The final forms require the following

$$\bar{x} = \frac{\sum_{k=1}^{K} \sum_{n=1}^{N_k} (n-1)}{N} = \frac{\sum_{k=1}^{K} N_k(N_k-1)}{2N} = \frac{\sum_{k=1}^{K} N_k^2 - N}{2N}$$

and

$$S_{xx} = \sum_{k=1}^{K} \sum_{n=1}^{N_k} (n-1)^2 - N\bar{x}^2 = \sum_{k=1}^{K} \frac{(N_k-1)N_k(2N_k-1)}{6} - N\bar{x}^2 .$$

Note that there are three and two estimators for $\mu_{\bar{A}_1}$ and $\sigma_{\bar{A}_1}^2$, respectively, when $d \neq 0$, but five and two estimators for $\mu_{\bar{G}_1}$ and $\sigma_{\bar{G}_1}^2$, respectively, when $r \neq 1$.

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