

Direction Cosines and Vector Geometry

12.1

Introduction

The orientations of structural elements are most readily obtained analytically by 3-D vector geometry. A line is represented by a vector of unit length and a plane by its pole vector or its dip vector. The direction cosines of a line describe the orientation of the unit vector parallel to the line. Structural information such as bearing and plunge is converted into direction cosine form, the necessary operations performed, and then the values converted back to standard geological format. This chapter gives the basic relationships and outlines some of the ways vector geometry can be applied to structural problems involving lines and planes.

12.2

Direction Cosines of Lines

Direction cosines define the orientation of a vector in three dimensions (Fig. 12.1). The direction angles between the line OC and the positive coordinate axes x, y, z are α, β, γ , respectively. The direction cosines of the line OC are

$$\cos \alpha = \cos \text{EOC} \quad , \quad (12.1a)$$

$$\cos \beta = \cos \text{GOC} \quad , \quad (12.1b)$$

$$\cos \gamma = \cos (90 + \text{AOC}) \quad . \quad (12.1c)$$

In the geological sign convention, an azimuth of 0 and 360° represents north, 90° east, 180° south, and 270° west. Dip (of a plane) or plunge (of a line) is an angle from the horizontal between 0 and 90° , positive downward. From the geometry of Fig. 12.1:

$$\cos \delta = \text{OA} / \text{OC} \quad , \quad (12.2a)$$

$$\cos \theta = \text{OG} / \text{OA} \quad , \quad (12.2b)$$

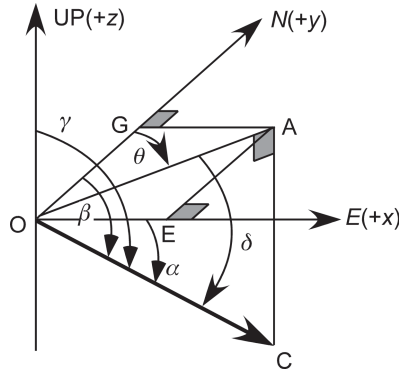
$$\sin \theta = \text{OE} / \text{OA} \quad , \quad (12.2c)$$

$$\cos \alpha = \text{OE} / \text{OC} \quad , \quad (12.2d)$$

$$\cos \beta = \text{OG} / \text{OC} \quad , \quad (12.2e)$$

$$\cos \gamma = \cos (90 + \delta) = -\sin (\delta) \quad . \quad (12.2f)$$

Fig. 12.1.
Orientation of vector OC in right-handed xyz space; θ is the azimuth of the vector and δ is the plunge. The direction angles are α, β, γ



12.2.1

Direction Cosines of a Line from Azimuth and Plunge

The direction cosines of a line in terms of its azimuth and plunge are obtained from the relationships in Eq. 12.2:

$$\cos \delta \sin \theta = (OA / OC) (OE / OA) = OE / OC = \cos \alpha \quad , \quad (12.3a)$$

$$\cos \delta \cos \theta = (OA / OC) (OG / OA) = OG / OC = \cos \beta \quad , \quad (12.3b)$$

$$-\sin \delta = \cos \gamma \quad . \quad (12.3c)$$

12.2.2

Azimuth and Plunge from Direction Cosines

To reverse the procedure and find the azimuth and plunge of a line from its direction cosines, divide Eq. 12.3a by 12.3b and solve for θ , then solve Eq. 12.3c for δ :

$$\theta' = \arctan (\cos \alpha / \cos \beta) \quad , \quad (12.4)$$

$$\delta = \arcsin (-\cos \gamma) \quad . \quad (12.5)$$

The value θ' returned by Eq. 12.4 will be in the range of $\pm 90^\circ$ and must be corrected to give the true azimuth over the range of 0 to 360° . The true azimuth, θ , of the line can be determined from the signs of $\cos \alpha$ and $\cos \beta$ (Table 12.1). The direction cosines give a directed vector. The vector so determined might point upward. If it is necessary to reverse its sense of direction, reverse the sign of all three direction cosines. Note that division by zero in Eq. 12.4 must be prevented.

An Excel equation that returns the azimuth in the correct quadrant has the form

$$=IF(\text{cell B} \leq 0, 180 + \text{Cell E}, IF(\text{cell A} > 0, \text{cell E}, 360 + \text{cell E})) \quad (12.6)$$

where cell A contains $\cos \alpha$, cell B contains $\cos \beta$, and cell E contains Eq. 12.4.

Table 12.1.
Relationship between signs
of the direction cosines and
the quadrant of the azimuth
of a line

Azimuth	$\cos \alpha$	$\cos \beta$	θ
000+ to 090	+	+	θ'
090+ to 180	+	−	$180 + \theta'$
180+ to 270	−	−	$180 + \theta'$
270+ to 360	−	+	$360 + \theta'$

12.2.3

Direction Cosines of a Line on a Map

The direction cosines of a line defined by its horizontal, h , and vertical, v , dimensions on a map can be found by letting $\delta = \arctan (v/h)$ in Eqs. 12.3:

$$\cos \alpha = \cos (\arctan (v/h)) \sin \theta \quad , \quad (12.7a)$$

$$\cos \beta = \cos (\arctan (v/h)) \cos \theta \quad , \quad (12.7b)$$

$$\cos \gamma = -\sin (\arctan (v/h)) \quad . \quad (12.7c)$$

Both v and h are taken as positive numbers in Eqs. 12.7 and the resulting dip is positive downward.

The direction cosines of a line defined by the xyz coordinates of its two end points are obtained by letting point 1 be at O and point 2 be at C in Fig. 12.1. Then

$$x_2 - x_1 = OE \quad , \quad (12.8a)$$

$$y_2 - y_1 = OG \quad , \quad (12.8b)$$

$$z_2 - z_1 = AC \quad . \quad (12.8c)$$

Substitute Eqs. 12.8 into 12.3 to obtain

$$\cos \alpha = (x_2 - x_1) / OC \quad , \quad (12.9a)$$

$$\cos \beta = (y_2 - y_1) / OC \quad , \quad (12.9b)$$

$$\cos \gamma = -\sin \delta = -AC / OC = -(z_2 - z_1) / OC = (z_1 - z_2) / OC \quad , \quad (12.9c)$$

where $OC = L$, given by

$$L = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} \quad . \quad (12.10)$$

Using the convention that point 1 is higher and point 2 is lower, a downward-directed bearing is positive in sign.

12.2.4**Azimuth and Plunge of a Line from the End Points**

To find the azimuth and plunge of a line from the coordinates of two points, substitute Eq. 12.9 into 12.4 and 12.5:

$$\theta' = \arctan((x_2 - x_1)/(y_2 - y_1)) \quad , \quad (12.11)$$

$$\delta = \arcsin((z_2 - z_1)/L) \quad . \quad (12.12)$$

The value of L is given by Eq. 12.10. If $y_2 = y_1$, there is a division by zero in Eq. 12.13 which must be prevented. The value of θ is obtained from Table 12.1.

12.2.5**Pole to a Plane**

The pole to a plane defined by its dip vector has the same azimuth as the dip vector, and a plunge of $\delta_p = \delta + 90$. Substitute this into Eqs. 12.3 to obtain the direction cosines of the pole, p , in terms of the azimuth, θ , and plunge, δ , of the bedding dip:

$$\cos \alpha_p = \cos(\delta + 90) \sin \theta = -\sin \delta \sin \theta \quad , \quad (12.13a)$$

$$\cos \beta_p = \cos(\delta + 90) \cos \theta = -\sin \delta \cos \theta \quad , \quad (12.13b)$$

$$\cos \gamma_p = -\sin(\delta + 90) = -\cos \delta \quad . \quad (12.13c)$$

12.3**Attitude of a Plane from Three Points**

The general equation of a plane (Foley and Van Dam 1983) is

$$Ax + By + Cz + D = 0 \quad . \quad (12.14)$$

The equation of the plane from the xyz coordinates of three points is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 \quad . \quad (12.15)$$

This expression is expanded by cofactors to find the coefficients A , B , C , and D :

$$A = y_1 z_2 + z_1 y_3 + y_2 z_3 - z_2 y_3 - z_3 y_1 - z_1 y_2 \quad , \quad (12.16a)$$

$$B = z_2 x_3 + z_3 x_1 + z_1 x_2 - x_1 z_2 - z_1 x_3 - x_2 z_3 \quad , \quad (12.16b)$$

$$C = x_1y_2 + y_1x_3 + x_2y_3 - y_2x_3 - y_3x_1 - y_1x_2 , \quad (12.16c)$$

$$D = z_1y_2x_3 + z_2y_3x_1 + z_3y_1x_2 - x_1y_2z_3 - y_1z_2x_3 - z_1x_2y_3 . \quad (12.16d)$$

The direction cosines of the pole to this plane are (Eves 1984)

$$\cos \alpha_p = A / E , \quad (12.17a)$$

$$\cos \beta_p = B / E , \quad (12.17b)$$

$$\cos \gamma_p = C / E , \quad (12.17c)$$

where

$$E = (A^2 + B^2 + C^2)^{1/2} , \quad (12.18)$$

and the sign of $E = -\text{sign } D$ if $D \neq 0$; $= \text{sign } C$ if $D = 0$ and $C \neq 0$; $= \text{sign } B$ if $C = D = 0$.

The direction cosines of the pole may be converted to the direction cosines of the dip vector. If $\cos \gamma_p$ is negative, the pole points downward. Reverse the signs on all three direction cosines to obtain the upward direction. If $\cos \gamma_p$ is positive, the pole points upward, and the dip azimuth is the same as that of the pole and the dip amount is equal to 90° plus the angle between the pole and the z axis:

$$\cos \alpha = A / E , \quad (12.19a)$$

$$\cos \beta = B / E , \quad (12.19b)$$

$$\cos \gamma = \cos (90 + \arccos (C / E)) . \quad (12.19c)$$

To find the azimuth and plunge of the dip direction, use Eqs. 12.4 and 12.5 and Table 12.1 or the Excel equation 12.6.

12.4

Vector Geometry of Lines and Planes

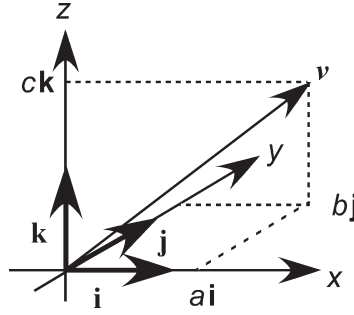
Vector geometry (Thomas 1960) is an efficient method for the analytical computation of the relationships between lines and planes. A 3-D vector has the form

$$\mathbf{v} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k} , \quad (12.20)$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ = unit vectors parallel to the x, y, z axes, respectively (Fig. 12.2) and the coefficients are the direction cosines l, m, n of the line with respect to the corresponding axes. The vector \mathbf{v} is a unit vector if its length is equal to one, and gives the orientation of a line. The length of the vector is

$$|l\mathbf{i} + m\mathbf{j} + n\mathbf{k}| = (l^2 + m^2 + n^2)^{1/2} , \quad (12.21)$$

Fig. 12.2.
Vector components in the
 xyz coordinate system. Vector
components i, j, k each have
lengths equal to one



where the vertical bars = the absolute value of the expression between them. The direction cosines of any vector can be normalized to generate a unit vector by dividing each direction cosine (l, m , and n) by the right-hand side of Eq. 12.21.

12.4.1

Angle between Two Lines or Planes

The angle, Θ , between two lines, is given by the scalar or dot product of the two unit vectors with the same orientations as the lines. If the two vectors represent the poles to planes or the dip vectors of planes, then the dot product is the angle between the planes, normal to their line of intersection (Fig. 12.3). The *dot product* is defined as

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1| |\mathbf{v}_2| \cos \Theta \quad , \quad (12.22)$$

where $|\mathbf{v}| = 1$ for a unit vector and $\mathbf{v}_1 \cdot \mathbf{v}_2$ is evaluated as

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = l_1 l_2 + m_1 m_2 + n_1 n_2 \quad , \quad (12.23)$$

where the subscript “1” = the direction cosines of the first vector and subscript “2” = the direction cosines of the second vector. Equating 12.22 and 12.23 gives the angle between two unit vectors:

$$\cos \Theta = l_1 l_2 + m_1 m_2 + n_1 n_2 \quad . \quad (12.24)$$

12.4.2

Line Perpendicular to Two Vectors

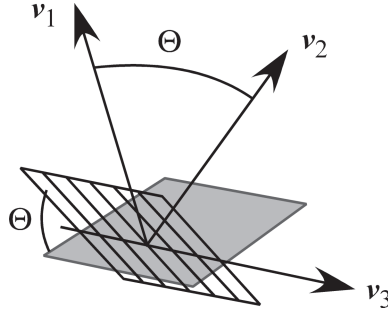
The orientation of a line perpendicular to two other vectors is given by the vector product, also called the cross product (Fig. 12.3). The *cross product* of two vectors is defined as

$$\mathbf{v}_1 \times \mathbf{v}_2 = n |\mathbf{v}_1| |\mathbf{v}_2| \sin \Theta \quad , \quad (12.25)$$

where n = a unit vector perpendicular to the plane of \mathbf{v}_1 and \mathbf{v}_2 . When the vectors are unit vectors given in terms of the direction cosines, the cross product is:

Fig. 12.3.

Geometry of the dot product and cross product. Θ is the angle between v_1 and v_2 ; v_3 is perpendicular to the plane of v_1 and v_2 . The *shaded plane* is perpendicular to v_1 and the *ruled plane* is perpendicular to v_2



$$v_3 = \begin{vmatrix} i & j & k \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = (m_1 n_2 - n_1 m_2)i + (n_1 l_2 - l_1 n_2)j + (l_1 m_2 - m_1 l_2)k \quad (12.26)$$

In Eq. 12.26, v_3 is the vector perpendicular to both v_1 and v_2 . The order of multiplication changes the direction of v_3 but not the orientation of the line or its magnitude.

12.4.3

Line of Intersection between Two Planes

The properties of the cross product make it suitable for solving a number of problems in structural analysis. For example, two dip domains can be defined by their poles (or their dip vectors) and the cross product of these vectors will give the orientation of the hinge line between them. If the pole to a plane is defined by the azimuth (θ) and dip (δ) of a vector pointing in the dip direction, the orientation of this vector in terms of direction cosines is given by Eqs. 12.13, which are

$$l = \cos \alpha_p = -\sin \delta \sin \theta \quad (12.27a)$$

$$m = \cos \beta_p = -\sin \delta \cos \theta \quad (12.27b)$$

$$n = \cos \gamma_p = -\cos \delta \quad (12.27c)$$

The subscript “p” = the pole to a plane. Let the subscript “1” = the first domain dip, 2 = the second domain dip, and h = the hinge line, then substitute Eqs. 12.27 into 12.26 to obtain a vector parallel to the hinge line in terms of the direction cosines:

$$\cos \alpha = \sin \delta_1 \cos \theta_1 \cos \delta_2 - \cos \delta_1 \sin \delta_2 \cos \theta_2 \quad (12.28a)$$

$$\cos \beta = \cos \delta_1 \sin \delta_2 \sin \theta_2 - \sin \delta_1 \sin \theta_1 \cos \delta_2 \quad (12.28b)$$

$$\cos \gamma = \sin \delta_1 \sin \theta_1 \sin \delta_2 \cos \theta_2 - \sin \delta_1 \cos \theta_1 \sin \delta_2 \sin \theta_2 \quad (12.28c)$$

The direction cosines must be normalized to ensure that the sum of their squares equals 1, which is done by dividing through by N_c where

$$N_c = (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)^{1/2} . \quad (12.29)$$

The final equation for the line of intersection between two planes is

$$\cos \alpha_h = (\cos \alpha) / N_c , \quad (12.30a)$$

$$\cos \beta_h = (\cos \beta) / N_c , \quad (12.30b)$$

$$\cos \gamma_h = (\cos \gamma) / N_c . \quad (12.30c)$$

The azimuth and dip of the line of intersection are given by Eqs. 12.4, 12.5 and Table 12.1.

12.4.4

Plane Bisecting Two Planes

The axial surface of a constant thickness fold hinge is a plane that bisects the angle between the two adjacent bedding planes and can be found as a vector sum. A vector sum is the diagonal of the parallelogram formed by two vectors. If the two vectors forming the sum have the same lengths, as do unit vectors, then the diagonal bisects the angle between them (Fig. 12.4). The sum of two vectors is the sum of their corresponding components:

$$\mathbf{v}_1 + \mathbf{v}_2 = (l_1 + l_2)\mathbf{i} + (m_1 + m_2)\mathbf{j} + (n_1 + n_2)\mathbf{k} . \quad (12.31)$$

The unit vector that bisects the angle between the two vectors \mathbf{v}_1 and \mathbf{v}_2 is

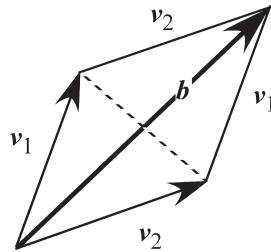
$$\mathbf{v}_3 = (1/N_b)(l_1 + l_2)\mathbf{i} + (1/N_b)(m_1 + m_2)\mathbf{j} + (1/N_b)(n_1 + n_2)\mathbf{k} , \quad (12.32)$$

where

$$N_b = ((l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2)^{1/2} \quad (12.33)$$

serves to normalize the length to make the resultant a unit vector. Depending on the directions of \mathbf{v}_1 and \mathbf{v}_2 , this vector could bisect either the acute angle or the obtuse

Fig. 12.4.
Bisecting vector (\mathbf{b}) of the
angle between two vectors in
the plane of the two vectors



angle. If the two vectors are the poles to planes, the vector bisector is the pole to one of the two bisecting planes. The two bisecting planes are separated by 90° .

To bisect a given pair of planes, begin with the poles of the planes to be bisected (Eqs. 12.27). Substitute Eqs. 12.27 for both planes into 12.32 and use the same subscripting convention as above, with the subscript "B" = bisector, to obtain

$$\cos \alpha_{B1} = -(1/N_B)(\sin \delta_1 \sin \theta_1 + \sin \delta_2 \sin \theta_2) \quad , \quad (12.34a)$$

$$\cos \beta_{B1} = -(1/N_B)(\sin \delta_1 \cos \theta_1 + \sin \delta_2 \cos \theta_2) \quad , \quad (12.34b)$$

$$\cos \gamma_{B1} = -(1/N_B)(\cos \delta_1 + \cos \delta_2) \quad , \quad (12.34c)$$

where, from Eq. 12.33:

$$N_B = [(\cos \delta_1 \sin \theta_1 + \cos \delta_2 \sin \theta_2)^2 + (\sin \delta_1 \cos \theta_1 + \sin \delta_2 \cos \theta_2)^2 + (\cos \delta_1 + \cos \delta_2)^2]^{1/2} \quad . \quad (12.35)$$

Equations 12.34 and 12.35 give the pole to one of the bisecting planes. The direction cosines of the pole should be converted to the dip vector. If $\cos \gamma_{B1}$ is negative, the pole points downward. Reverse the signs on all three direction cosines to obtain the upward direction. If $\cos \gamma_{B1}$ is positive, the pole points upward, and the dip azimuth is the same as that of the pole and the dip amount is equal to 90° plus the angle between the pole and the z axis:

$$\cos \alpha_1 = \cos \alpha_{B1} \quad , \quad (12.36a)$$

$$\cos \beta_1 = \cos \beta_{B1} \quad , \quad (12.36b)$$

$$\cos \gamma_1 = \cos(90 + \arccos(\cos \gamma_{B1})) \quad . \quad (12.36c)$$

The procedure given above finds one of the two bisectors of the two planes. The other bisecting plane includes the line normal to both the line of intersection of the two planes (hinge line, Eqs. 12.30) and to the first bisecting line (Eqs. 12.36). Find the pole to the second plane by substituting Eqs. 12.30 and 12.36 into the cross product 12.26 to obtain

$$\cos \alpha_{b2} = (1/N_p)(\cos \beta_h \cos \gamma_{B1} - \cos \gamma_h \cos \beta_{B1}) \quad , \quad (12.37a)$$

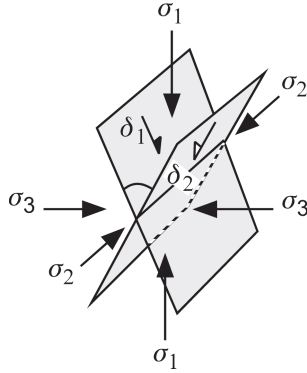
$$\cos \beta_{b2} = (1/N_p)(\cos \gamma_h \cos \alpha_{B1} - \cos \alpha_h \cos \gamma_{B1}) \quad , \quad (12.37b)$$

$$\cos \gamma_{b2} = (1/N_p)(\cos \alpha_h \cos \beta_{B1} - \cos \beta_h \cos \alpha_{B1}) \quad , \quad (12.37c)$$

where

$$N_p = [(\cos \beta_h \cos \gamma_{B1} - \cos \gamma_h \cos \beta_{B1})^2 + (\cos \gamma_h \cos \alpha_{B1} - \cos \alpha_h \cos \gamma_{B1})^2 + (\cos \alpha_h \cos \beta_{B1} - \cos \beta_h \cos \alpha_{B1})^2]^{1/2} + 1 \quad . \quad (12.38)$$

Fig. 12.5.
Geometry of conjugate faults.
 δ_i : Dip vectors of fault planes;
 σ_i : principal stress directions



Equation 12.38 serves as a check on the calculation: it should always equal 1 because the starting vectors are orthogonal and of unit length. Convert this pole to the bisecting plane (Eqs. 12.37) into the dip vector of the plane as done above. First reverse the direction of the vector if it points downward ($\cos \gamma$ negative) by reversing the signs of all the direction cosines. Then

$$\cos \alpha_2 = \cos \alpha_{b2} \quad , \quad (12.39a)$$

$$\cos \beta_2 = \cos \beta_{b2} \quad , \quad (12.39b)$$

$$\cos \gamma_2 = \cos (90 + \arccos (\cos \gamma_{b2})) \quad . \quad (12.39c)$$

The azimuth and dip of the planes in Eqs. 12.36 and 12.39 are given by Eqs. 12.4, 12.5 and Table 12.1.

If the two planes being bisected are conjugate faults (Fig. 12.5), then the line of intersection of the faults is the intermediate principal compressive stress axis (σ_2), the line that bisects the acute angle is the maximum principal compressive stress axis (σ_1), and the line that bisects the obtuse angle is the least principal compressive stress axis (σ_3).