MA2ROTMAS

1. Rotation matrix and matrix algebra

Rotation matrix $\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix}$

The transposed matrix is the inverse matrix

$$\begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

The product of rotation matrix and its transposed matrix is the unit matrix

$$\begin{pmatrix} \cos(\phi)^2 + \sin(\phi)^2 & 0 \\ 0 & \cos(\phi)^2 + \sin(\phi)^2 \end{pmatrix}$$

For the demonstration of matrix properties: The product of two matrices is equal to the product of the transposed matrixces in reverse order

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi)\cdot\cos(\theta) - \sin(\phi)\cdot\sin(\theta) & -\cos(\phi)\cdot\sin(\theta) - \sin(\phi)\cdot\cos(\theta) \\ \sin(\phi)\cdot\cos(\theta) + \cos(\phi)\cdot\sin(\theta) & \cos(\phi)\cdot\cos(\theta) - \sin(\phi)\cdot\sin(\theta) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \cdot \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

$$\begin{pmatrix} \cos(\phi)\cdot\cos(\theta) - \sin(\phi)\cdot\sin(\theta) & \sin(\phi)\cdot\cos(\theta) + \cos(\phi)\cdot\sin(\theta) \\ -\cos(\phi)\cdot\sin(\theta) - \sin(\phi)\cdot\cos(\theta) & \cos(\phi)\cdot\cos(\theta) - \sin(\phi)\cdot\sin(\theta) \end{pmatrix}$$

2. Transformation of a matrix and the eigenvalue problem

The matrix corresponds to an ellipse with cross terms. We want to transform the coordinate system to a new one, in which the ellipse appears without cross terms. (This is called "Principal axis transformation").

$$\begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

The product is calculated "symbolically"

$$\begin{pmatrix} \cos(\phi)^2 + \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \sin(\phi)^2 & \frac{1}{2}\cdot\sin(\phi)^2\cdot\sqrt{2} - \frac{1}{2}\cdot\cos(\phi)^2\cdot\sqrt{2} \\ \frac{1}{2}\cdot\sin(\phi)^2\cdot\sqrt{2} - \frac{1}{2}\cdot\cos(\phi)^2\cdot\sqrt{2} & \sin(\phi)^2 - \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \cos(\phi)^2 \end{pmatrix}$$

The general equation of the ellipse is obtained by multiplication with the vector (E1, E2)

$$(\mathrm{E1} \quad \mathrm{E2})\cdot\begin{pmatrix} \cos(\phi)^2 + \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \sin(\phi)^2 & \frac{1}{2}\cdot\sin(\phi)^2\cdot\sqrt{2} - \frac{1}{2}\cdot\cos(\phi)^2\cdot\sqrt{2} \\ \frac{1}{2}\cdot\sin(\phi)^2\cdot\sqrt{2} - \frac{1}{2}\cdot\cos(\phi)^2\cdot\sqrt{2} & \sin(\phi)^2 - \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \cos(\phi)^2 \end{pmatrix}\begin{pmatrix} \mathrm{E1} \\ \mathrm{E2} \end{pmatrix}$$

The matrix elements are given as y(i,k)

The matrix is diagonal if the y(1,2) and y(2,1) elements are zero

$$\frac{1}{2}\cdot\sin(\phi)^2\cdot\sqrt{2} - \frac{1}{2}\cdot\cos(\phi)^2\cdot\sqrt{2}$$

The result is

$\phi = \pi/4$

3. Eigenvalue Problem y21 and y12 are zero

For $\phi := \frac{\pi}{4}$ We calculate

$$y11 := \cos(\phi)^2 + \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \sin(\phi)^2 \qquad y11 = 1.707$$

$$y22 := \sin(\phi)^2 - \cos(\phi)\cdot\sin(\phi)\cdot\sqrt{2} + \cos(\phi)^2 \qquad y22 = 0.293$$

1.707 = 1+1/sq(2) and .293 is 1-1/sq(2)

We have now for the matrix

$$\begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}$$

We obtain the equation of the ellipse by multiplication with the electrical field vectors in the new coordinate system, called now EE1 and EE2

$$(EE1 \quad EE2)\cdot\begin{pmatrix} 1 + \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 - \frac{1}{\sqrt{2}} \end{pmatrix}\cdot\begin{pmatrix} EE1 \\ EE2 \end{pmatrix}$$

$$EE1^2 + \frac{1}{2}\cdot EE1^2\cdot\sqrt{2} + EE2^2 - \frac{1}{2}\cdot EE2^2\cdot\sqrt{2}$$

$$EE1^2\left(1 + \frac{\sqrt{2}}{2}\right) + EE2^2\left(1 - \frac{\sqrt{2}}{2}\right) = CONSTANT$$

This is the equation of the ellipse in the EEy and EEz coordinates without cross terms.The ellipse is on principal axes.

4. Rotation matrix

We know the angle for the rotation matrix $\phi := \frac{\pi}{4}$

and get

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The product of the three matrices is

$$\begin{bmatrix} 1 + \frac{1}{2}\cdot\sqrt{2} & 0 \\ 0 & \frac{1}{2}\cdot\left(\sqrt{2} - 1\right)\cdot\sqrt{2} \end{bmatrix} \quad \text{that is} \quad \begin{bmatrix} 1 + \frac{1}{2}\cdot\sqrt{2} & 0 \\ 0 & \left(1 - \frac{1}{2}\cdot\sqrt{2}\right) \end{bmatrix}$$

The principal axis transformation has been performed by rotation of the coordinate system