

APPLICATIONS OF THERMODYNAMICAL APPROACHES TO DYNAMIC HARDIN-DRNEVICH MODEL FOR SOILS

Xiaoxia GUO¹, Shichun CHI² and Gao LIN³

ABSTRACT

Modern ideas of thermomechanics, based upon the use of internal variables, are used fully to model dynamic behaviour for soils with a systematic procedure. A central theme is that the constitutive behaviour is entirely determined by the knowledge of scalar potentials. Starting from the unloading and reloading hysteretic curves of dynamic Hardin-Drnevich model, fundamental thermodynamic state variable are defined and used to formulate different expressions of dissipation potential function in order to reconsider the energy dissipation mechanism of Hardin-Drnevich model. Since the plastic part of the free energy potential function is unknown, three kinds of different translation rules of the back stress are assumed. The first kind of the shift rule is the straight line; the second one is the skeleton curve; the third one is the product by the plastic modulus and the internal variable. Much use is made of Ziegler's orthogonality principle to obtain yield function expressions in the dissipative stress space. In addition, the shift stress is added to the dissipative stress and the form of the yield condition is deduced in the true stress space. The plotting results of yield curves in true stress space indicate that the form of the shear yield curve is straight line for the different shift rules hypothesis.

Keywords: thermodynamics, dynamic constitutive model, hysteresis, translation rule, dissipative stress

INTRODUCTION

At present, dynamic constitutive models for soils can be divided into two classes: one is experiential model obtained by fitting experimental data, which aim at improving fitting precision. Masing model is the representation of this kind of the model based on experiential nonlinear skeleton curves and Masing rule. This kind of model describes non-linearity and viscosity for soils from different points, moreover, proposes the method formulating dynamic stress-dynamic strain relationship (Iwan, 1967; Wang et al. 1980; Wu, 1988). The other is physical model. Iwan model is the representation of this kind of the model, which formulates constitutive equation based on mechanical elements of elastic spring, rigid-plastic slide and viscous damper (Li and Liao, 1989; Luan and Lin, 1992). In the Masing model, Hardin-Drnevich model and Ramberg-Osgood model are the most famous.

A basic requirement of all such models mentioned above is that they satisfy the basic laws of physics. The first law and the second law of thermodynamics are one of these basic laws that govern the dissipative behaviour of materials, but it is seldom invoked in geomechanical theories. Until recently thermodynamics was seldom seen as relevant to geotechnical problem. In the early days of development of the theory of elastic/plastic materials, quasi-thermodynamic postulates were

¹ Doctor, Institute of Earthquake Engineering, School of Civil and Hydraulic Engineering, Dalian University of Technology, China, Email: hanyuer2005@hotmail.com

² Professor, Institute of Earthquake Engineering, School of Civil and Hydraulic Engineering, Dalian University of Technology, China, Email: schchi@dlut.edu.cn

³ Professor, Academician of Chinese Academy of Science. Institute of Earthquake Engineering, School of Civil and Hydraulic Engineering, Dalian University of Technology, China, Email: gaolin@dlut.edu.cn

introduced in an effort to ensure that the dissipation of energy was always positive in a closed cycle of stress (Drucker's postulate) or strain (Il'iushin's postulate). However, it was soon realized that these postulates were actually classifications of types of material behaviour and not, in any sense, equivalent to the second law (Drucker, 1988; Lubliner, 1990).

The early developments of these general ideas were due to Ziegler (1983) and Ziegler and Wehrli (1987), who noted some applications of their general theory to the classical Coulomb model. A rigorous general theory was then developed by a number of researchers in France, accounts of which can be found in the books by Maugin (1999) and Maugin et al. (2000), Besseling and van der Giessen (1994) and Lemaitre and Chaboche (1990). Applications of these ideas to soil mechanics were pioneered by Houlsby (1981, 1982). A comprehensive analysis of the isothermal thermomechanics of geomaterials was given by Collins and Houlsby (1997) and Collins (2003), who demonstrated that a non-associated flow rule is necessary property of a 'frictional material', in which the plastic deformations are governed by stress ratios rather than by the magnitudes of certain yield stresses. Houlsby and Puzrin (2000) generalized some aspects of this work to non-isothermal conditions, and in a series of papers have developed a family of sophisticated models, based on the use of internal functions (e.g. puzrin and Houlsby, 2001b, 2001c).

In this paper, the elementary theories and general procedure of constituting models based on thermomechanics laws are introduced. Starting from the unloading and reloading hysteretic curves of Hardin-Drnevich model and considering different shift rules of the back stress, proper energy dissipation functions are formulated. The yield functions in dissipation stress space are deduced in terms of Ziegler's Orthogonality Condition. Then the different term between the dissipation stress and the true stress is introduced to obtain the yield function in the true stress space. In the final section, the plotting of yield curve in true stress space indicates that the form of the shear yield curve is reasonable.

OUTLINE OF THE THERMODYNAMICAL APPROACH

The First Law and the Second Law of Thermodynamic

The first law and the second law are the fundamental theories to formulate constitutive model of the soils by use of the thermomechanical principles (Houlsby, 1981; Collins and Houlsby, 1997; Houlsby and Puzrin, 2000).

The first law of thermodynamics can be written in a local rate form:

$$\dot{W} + \dot{Q} = \dot{u} \quad (1)$$

where $\dot{W} = \sigma_{ij} \dot{\epsilon}_{ij}$ is the input work ; $\dot{Q} = -q_{k,k}$ is the rate of heat supply to the material element from its surroundings ; \dot{u} is the internal energy per unit volume. The local heat source is assumed to be zero.

Similarly the local form of the second law of thermodynamics is:

$$\theta \dot{s} + q_{k,k} - \frac{q_k \theta_{,k}}{\theta} \geq 0 \quad (2)$$

where θ is the absolute temperature of the whole system; s is the entropy per unit volume; q_k is the heat flux vector. The first two terms $\theta \dot{s} + q_{k,k} = d$ are called the mechanical dissipation. The third term is called the thermal dissipation and it is always non-negative.

Legendre Transformations

The Legendre transformation is one of the most useful in applied mathematics, although its role is not always explicitly recognized. Well-known examples include the relation between the Lagrangean and Hamiltonian functions in analytical mechanics, between strain energy and complementary energy in elasticity theory, between thermodynamics potentials (Collins and Houlsby, 1997; Houlsby and Puzrin, 2000). Legendre transformation relationships between the four energy functions are as follows (Table 1):

Table 1. The four Legendre transformations relationships

Internal energy	Helmholtz free energy	Enthalpy	Gibbs free energy
$u = u(\varepsilon, s)$	$f = f(\varepsilon, \theta)$ $f = u - s\theta$	$h = h(\sigma, s)$ $h = u - \sigma\varepsilon$	$g = g(\sigma, \theta)$ $g = h - s\theta$
$\sigma = \partial u / \partial \varepsilon$ $\theta = \partial u / \partial s$	$\sigma = \partial f / \partial \varepsilon$ $s = -\partial f / \partial \theta$	$\varepsilon = -\partial h / \partial \sigma$ $\theta = \partial h / \partial s$	$\varepsilon = -\partial g / \partial \sigma$ $s = -\partial g / \partial \theta$

where ε , σ , η and θ are strain, stress, entropy and temperature, respectively. The choice of the formulations above is determined by application in hand. For example, the u and h formulations are particularly convenient for adiabatic problems, whilst the f and g formulations are appropriate for isothermal problems. The u and f formulations correspond to strain-space based plasticity models. Conversely the h and g formulations are particularly convenient for problems with prescribed stresses.

Ziegler's Orthogonality Condition

Ziegler's Orthogonality Condition (Ziegler, 1981, 1987; Houlsby and Puzrin, 2000; Einav, 2002) is determined by the two potential functions: one is the thermodynamics potential function; the other is the dissipation function. Starting from the two laws of the thermodynamic, considering that the internal energy is a function of state and using the definition of the dissipative stress, we obtain an expression for dissipation function.

$$d = \chi_{ij} \dot{\alpha}_{ij} \quad (3)$$

where d is dissipation function. χ_{ij} is the dissipative stress, the generalized stress or the residual stress. $\dot{\alpha}_{ij}$ is the rate of the internal variable.

For a rate-independent elastic/plastic deformation, d must be a homogeneous function of degree one in $\dot{\alpha}_{ij}$ since the material does not possess a characteristic time. Hence, from Euler's theorem for homogeneous functions it follows that

$$d = \frac{\partial d}{\partial \dot{\alpha}_{ij}} \dot{\alpha}_{ij} \quad (4)$$

Therefore

$$(\chi_{ij} - \frac{\partial d}{\partial \dot{\alpha}_{ij}}) \dot{\alpha}_{ij} = 0 \quad (5)$$

That is to say, The term $(\chi_{ij} - \frac{\partial d}{\partial \dot{\alpha}_{ij}})$ must be orthogonal to $\dot{\alpha}_{ij}$.

Further more, Ziegler argues that

$$\chi_{ij} = \frac{\partial d}{\partial \dot{\alpha}_{ij}} \quad (6)$$

This assumption is termed Ziegler's Orthogonality Condition. The principle can be viewed in a variety of ways, but perhaps the most useful is to view it as a stronger statement than the second law of thermodynamics (Ziegler, 1983). Some authors would accept the principle as 'true' (in the same sense that the second law of thermodynamics is almost universally regarded as 'true'), and these authors would accept the formulation derived here as having a rather general status. Others regard Ziegler's principle as unproven, so that materials obeying it are merely a subset of wider class of possible models. The authors do not intend to enter this debate, and are content therefore described here as a subset of all those possible. Nevertheless, this subset is certainly large, and is able to encompass a very wide range of rate-independent materials within a single framework.

The Thermomechanical Formulation of Elastic/Plastic Constitutive Models of Soils

The steps in constructing the ingredients of an elastic/plastic geomechanical model in a thermomechanical formulation are as follows (Collins and Kelly, 2002; Collins and Hilder, 2002):

(1) Define a free energy functions of the elastic and plastic strains, and a dissipation increment function will depend additionally on the plastic strain increments, and, possibly, on a number of true stress variables. It must be strictly positive whenever the plastic strain increments are non-zero.

$$g = g_1(\sigma_{ij}) - \sigma_{ij}\alpha_{ij} + g_2(\alpha_{ij}) \quad (7)$$

(2) Deduce the elasticity law from the elastic part of the free energy function.

$$\varepsilon_{ij}^e = \partial g_1(\sigma_{ij}) / \partial \sigma_{ij} \quad (8)$$

(3) Deduce the shift stresses from the plastic part of the free energy function.

$$\rho_{ij} = -\partial g_2(\alpha_{ij}) / \partial \alpha_{ij} \quad (9)$$

(4) Deduce the dissipative stresses from the dissipation increments function as in (6).

(5) Eliminate the plastic strain increments from the last expression and form the yield condition in dissipative stress space.

(6) Construct the flow rule for the plastic strain rates. This flow rule is always normal in the dissipative stress space.

(7) Add the shift stresses to the dissipative stresses and deduce the form of the yield condition in the true stress space. If the dissipative stress form of the yield condition does not involve the true stresses as parameters, then the flow rule remains normal. Else, the direction of the plastic strain increment vector is not normal to the final yield loci in the true stress space.

(8) If required, the incremental form of the constitutive law can be generated; as usual, by differentiating the yield condition to give the consistency equation.

CONSTRUCTION OF THE DISSIPATION FUNCTION FOR HARDIN-DRNEVICH MODEL

Masing Rule of Hardin-Drnevich Model

The hyperbolic skeleton curve of Hardin-Drnevich model can be written in the following form (Xie, 1988)

$$\tau_m = \frac{\gamma_m}{\frac{1}{G_0} + \frac{\gamma_m}{\tau_y}} \quad (10)$$

where τ_y is the maximum dynamic shear stress; G_0 is the maximum dynamic shear modulus; τ_m and γ_m are shear stress and shear strain on the skeleton curve, respectively.

Masing assumes that the form of reloading and unloading curves is accordant with that of the skeleton curve. However, the scale of dynamic stress and strain coordinate is twice that of the skeleton curve. Moreover, shear modulus when load is reversed is equal to initial shear modulus. Therefore, hysteretic curves are given as follows:

$$\tau - \tau_m = (\gamma - \gamma_m) \left/ \left(\frac{1}{G_0} + \frac{\gamma - \gamma_m}{2\tau_y} \right) \right. \quad (11)$$

In the equation (10) and (11), due to the skeleton curve in the first quadrant and the reloading curve as the same trend as the skeleton curve in the first quadrant, the value of maximum dynamic shear stress τ_y is given as positive; However, due to the skeleton curve in the third quadrant and the unloading curve as the same trend as the skeleton curve in the third quadrant, the value of the maximum dynamic shear stress τ_y is given as minus (Xie, 1988; Zhang et al., 1997). The skeleton curve and Masing's hysteretic curve of Hardin-Drnevich model are shown in Figure 1.

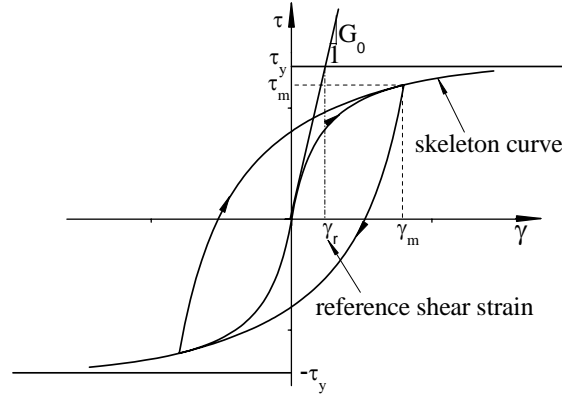


Figure 1. The skeleton curve and Masing's hysteretic curve of Hardin-Drnevich model

Construction Procedure of Dissipation Function under the Different Shift Rules

In order to illustrate dissipative mechanism of Hardin-Drnevich model, we adopted the following considerations:

Firstly, the maximum dynamic shear modulus G_0 defined through the consolidation pressure-dependent expression describes the skeleton curves under the different consolidation pressures. Secondly, the accumulation of the internal variable α_{ij} can be expressed through the accumulation of plastic strain on the skeleton curve. Thirdly, the rate of the internal variable $\dot{\alpha}_{ij}$ can be expressed by use of $d\dot{\gamma}^p$. Fourthly, the shift stress is deduced from the plastic part of the free energy function, which is unknown for this problem. So adopt three assumptions: straight line, skeleton curve and the product of plastic modulus and internal variable. Based on discussed above, the construction of dissipation functions are discussed in the following sections.

The straight line back stress

When the back stress is assumed as straight line, the slope of the straight line can be expressed as τ_m / γ_m . So the back stress can be obtained as follows:

$$\tau = \frac{\gamma}{\frac{1}{G_0} + \frac{\gamma_m}{\tau_y}} \quad (12)$$

Formula (12) subtracts from formula (11) and integrates along the rate of the strain, and then the dissipation function d can be expressed as

$$d = \left[\frac{\gamma}{1/G_0 + \gamma_m/\tau_y} - \tau_m - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] d\dot{\gamma} = \left[\frac{\gamma}{1/G_0 + \gamma_m/\tau_y} - \tau_m - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] (d\dot{\gamma}^e + d\dot{\gamma}^p) \quad (13)$$

Considering the known conditions $d\gamma = d\gamma^e + d\gamma^p$, so

$$\frac{d\tau}{H} = \frac{d\tau}{G_0} + \frac{d\tau}{H'} = \left(\frac{H'}{G_0} + 1 \right) d\gamma^p \quad (14)$$

For

$$\frac{1}{H} = \frac{1}{G_0} + \frac{1}{H'} \Rightarrow \frac{H'}{H} = \frac{H'}{G_0} + 1 = \frac{G_0}{G_0 - H} \quad (15)$$

As we can know, from the formula (11), the following formula can be obtained.

$$H = \frac{\partial \tau}{\partial \gamma} = \frac{1}{G_0 \left(\frac{1}{G_0} + \frac{\gamma - \gamma_m}{2\tau_y} \right)^2} \quad (16)$$

Therefore,

$$\frac{H'}{G_0} + 1 = \frac{G_0}{G_0 - H} = \frac{\left(1 + \gamma - \gamma_m / 2\gamma_r \right)^2}{\left(1 + \gamma - \gamma_m / 2\gamma_r \right)^2 - 1} \quad (17)$$

Combine formula (13) till formula (17), we can get the dissipation function of the straight line back stress.

$$d = \left[\frac{\gamma - \gamma_m}{1/G_0 + \gamma_m/\tau_y} - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] \frac{\left(1 + \gamma - \gamma_m / 2\gamma_r \right)^2}{\left(1 + \gamma - \gamma_m / 2\gamma_r \right)^2 - 1} d\dot{\gamma}^p \quad (18)$$

where, H is elastic-plastic modulus; H' is plastic modulus; γ_r is reference shear strain, which is obtained from the formula $\gamma_r = \tau_y / G_0$.

The skeleton curve back stress

When the back stress is supposed to be the skeleton curve, then it can be written as

$$\tau = \frac{\gamma}{\frac{1}{G_0} + \frac{\gamma}{\tau_y}} \quad (19)$$

The dissipation function can be obtained by use of the same derivation as the above mentioned.

$$d = \left(\frac{\gamma}{1/G_0 + \gamma/\tau_y} - \frac{\gamma_m}{1/G_0 + \gamma_m/\tau_y} - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right) \frac{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2}{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2 - 1} d\dot{\gamma}^p \quad (20)$$

Considering the condition that the back stress can be written as $\rho = H\alpha$

Referring to the plastic part of the free energy function by Puzrin and Houlsby (2001), assume that the back stress is the product of plastic modulus and the internal variable, then it can be expressed as follows

$$\rho = H\alpha = \frac{\alpha G_0}{G_0^2 \left(1/G_0 + (\gamma - \gamma_m)/2\tau_y\right)^2 - 1} \quad (21)$$

where α is the internal variable, as above, we can still choose the internal variable α_{ij} to be the plastic strain.

Adopt the same considerations as discussed above, the dissipation function can be written as

$$d = \left[\frac{\gamma^p \left(1/G_0 + (\gamma - \gamma_m)/2\tau_y\right)^2}{G_0 - 1/G_0 \left(1/G_0 + (\gamma - \gamma_m)/2\tau_y\right)^2} - \frac{\gamma_m}{1/G_0 + \gamma_m/\tau_y} - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] \frac{\left(1 + (\gamma - \gamma_m)/2\gamma_r\right)^2}{\left(1 + (\gamma - \gamma_m)/2\gamma_r\right)^2 - 1} d\dot{\gamma}^p \quad (22)$$

The Yield Functions in the Dissipative Stress Space and the True Stress Space

Starting from the dissipation function proposed above, obtain yield function expressions in the dissipative stress space by use of Ziegler's orthogonality principle. Moreover, obtain yield function in the true stress space through adding the shift stress to the dissipative stress. Take the straight line back stress as an example to carry on the analysis.

By use of the formula (6), the dissipative stress is obtained as follows:

$$\chi_\tau = \left[\frac{\gamma - \gamma_m}{1/G_0 + \gamma_m/\tau_y} - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] \frac{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2}{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2 - 1} \quad (23)$$

Add the shift stress to the dissipative stress and deduce the form of the yield condition in the true stress space. The back stress defines the difference between the true and the generalized stress variables. The relationship between them is as follows:

$$\sigma_{ij} = \chi_{ij} + \rho_{ij} \quad (24)$$

where ρ_{ij} plays the role of a 'back', 'shift' or 'drag' stress in simple kinematic hardening models.

So the true stress can be expressed as

$$\tau = \chi_\tau + \rho_\tau = \left[\frac{\gamma - \gamma_m}{1/G_0 + \gamma_m/\tau_y} - \frac{\gamma - \gamma_m}{1/G_0 + (\gamma - \gamma_m)/2\tau_y} \right] \frac{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2}{\left(1 + \gamma - \gamma_m/2\gamma_r\right)^2 - 1} + \frac{\gamma}{1/G_0 + \gamma/\tau_y} \quad (25)$$

THE PLOT OF THE YIELD CURVES

Selection of Parameters

Triaxial dynamic testing parameters of a core-fill dam are utilized for the evaluation. The maximum dynamic shear modulus and the reference shear strain are determined by use of the following formulas respectively (Zhang et al., 1997).

$$\left. \begin{aligned} G_0 &= k_2 p_a \left(\frac{\sigma_m}{p_a} \right)^n \\ \gamma_r &= K_r \left(\frac{\sigma_m}{p_a} \right)^{n_r} \end{aligned} \right\} \quad (26)$$

where σ_m is the initial average static stress of the soil. K_2 , n , k_r , and n_r are the parameters which are used to define the maximum dynamic shear modulus and the reference shear strain respectively. The values of these parameters are given in Table 2.

Table 2. The parameters for computation

The parameters for G_0		The parameters for γ_r	
K_2	n	k_r	n_r
1000	0.5	0.0005	0.20

The Yield Curves under the Different Back Stresses

This section discusses the form of the shear yield curves in p - q plane under the different shift rules. For the hardening yield function, a certain plastic deformation corresponds to a yield surface. Thus, for a given plastic shear strain, the yield locus is the line connecting those changing constantly stress points. In the following section, discuss that the change law of the yield surface with the accumulation of the plastic deformation on the hysteretic curve for a certain plastic shear deformation on the skeleton curve.

The yield surfaces under the different back stresses are shown in Figure 2 till Figure 4.

Figures above show that the form of the yield curves under all the three shift rules is the straight line. When the stress point moves along the skeleton curve, the yield curves have a counterclockwise rotation trend, which shows that the yield surfaces are increasing along the skeleton curves. This indicates that energy dissipation is increasing constantly with the larger hysteresis loop.

CONCLUSIONS

The establishment of a theoretical framework within thermodynamics is regarded as an important step towards a fuller understanding of soils behaviour. This paper carries on dynamic constitutive modelling by use of thermodynamic approach. Construct different expressions of dissipation potential functions of Hardin-Drnevich model starting from the unloading and reloading hysteretic curves. A central theme is that the constitutive behaviour is entirely determined by the knowledge of scalar potentials. A very important result of this approach is that it automatically satisfy thermodynamical requirement. The plotting results of yield curves in true stress space indicate that the form of the shear yield curve is straight line for the different shift laws hypothesis. Moreover, energy dissipation is increasing constantly with the larger hysteresis loop. This result applies good reference for application of dynamic constitutive model in the Geotechnical engineering. For simplicity and relevance, this paper has dealt only with the formulation of dynamic Hardin-Drnevich model based on Masing rule. It is not difficult to generalize the general theory to the modified Masing rule and realistic hysteresis loop in order to model the variation of shear modulus and damping with strain level.

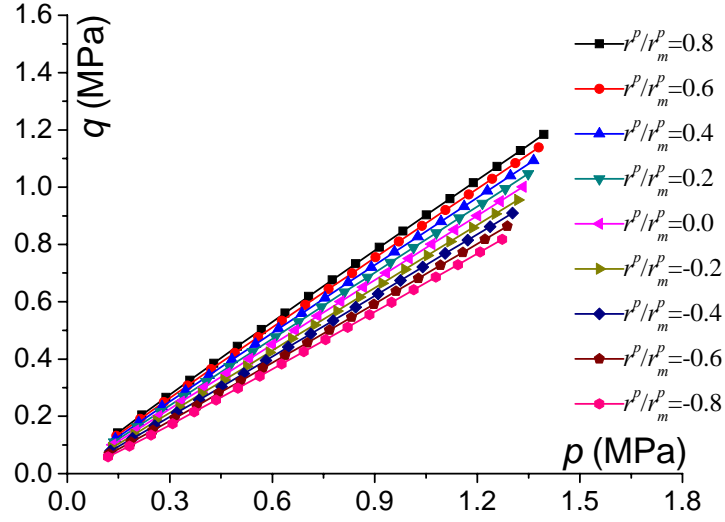


Figure 2. Yield curves when the shift stress is the straight line

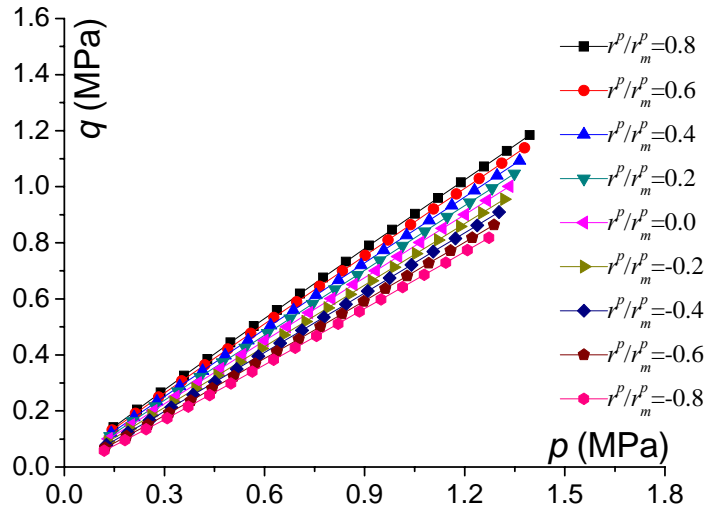


Figure 3. Yield curves when the shift stress is the skeleton curve

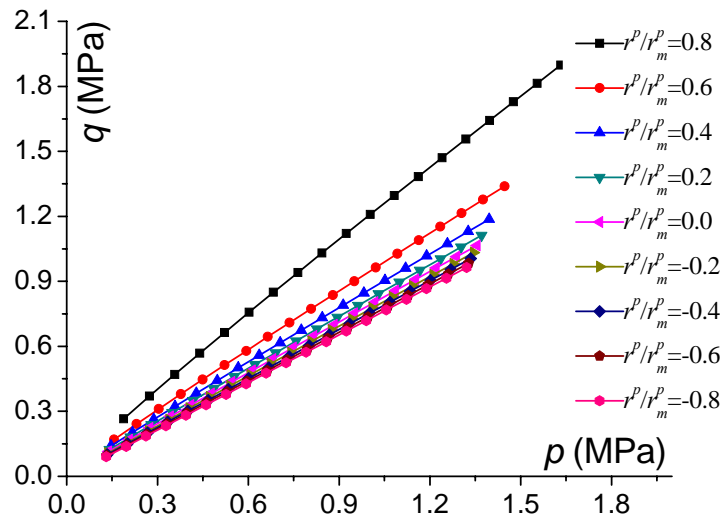


Figure 4. Yield curves when the shift stress is the product of the plastic modulus and the plastic deformation

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