

SEISMIC WAVE PROPAGATION IN STOCHASTIC SOILS

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ABSTRACT

In recent years, civil engineering practice, and particularly geotechnical practice has seen an increasing emphasis on reliability. In earthquake engineering, the earthquake ground motions are usually given with certain probability of occurrence. However, what is missing is the actual quantification of these probabilities that come from point-wise and spatially uncertain soil layers. In this paper we investigated the influence of (a) point-wise uncertainty and (b) uncertain spatial variability of soil properties on seismic wave propagation. The spatial variability of soil properties in situ is usually modeled as stochastic field. Seismic wave propagation in spatially variable soil continuum can be described by partial differential equation (PDE) with stochastic coefficient. Stochastic elastic-plastic finite element method will be used to solve the stochastic PDE. The novelty of presented method is that it can handle any non-linear elastic-plastic stochastic constitutive model. Further this method can consider more than one soil properties as random. The various stochastic material properties (random fields) will be represented by their respective Karhunen-Loeve expansion and the solution field (displacement) will be represented by polynomial chaos expansion. At the constitutive level the non-linear probabilistic elastic-plastic equation will be solved using Eulerian-Lagrangian form of Fokker-Planck-Kolmogorov equation. A number of illustrative examples will show main features of developed methodology. It will be shown that some of the traditional results in elasto-plasticity, taken for granted, are not as certain as one might assume. In addition to that, the issue of (possibility of) failure will be discussed in the light of stochastic methodology developed.

Keywords: Soil uncertainty, Wave propagation, Probabilistic Elasto-plasticity, Stochastic finite element

INTRODUCTION

In recent years, civil engineering practice, and in particular the geotechnical engineering practice has seen an increasing emphasis on reliability. Modern building codes (regulations) are increasingly being based on reliability methods. Financial objectives by object owners tend to lead toward use of probabilistic theories in decision making. However, our modeling and simulations in geomechanics are still largely deterministic. Consistent development of probabilistic framework for geotechnical simulations (will) provide(s) a rational way to address our confidence (or lack of one) in simulated behavior. For example, probabilistic simulations (will) empower geotechnical engineers to take into account uncertain spatial variabilities of soil deposit, point-wise uncertainties in soil properties such as testing and transformation errors as well as uncertainties in applied forces such as earthquake motion in their simulations. In addition to that, inverse problems in geotechnical engineering, such as design of site characterization based on client specific expected probability of failure or optimal design of retrofit systems (e.g. for dams and port facilities) can be effectively addressed.

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Figure 1 (after Mayne et al., 2000) shows the variation of friction angle, obtained from different in-situ and laboratory testing methods, with depth at a typical site. In traditional deterministic simulation, one generally "assumes" the mean value of friction angle and use that value in further deterministic modeling and simulation and hence the uncertain spatial variability and testing errors are lost from the simulation results. In this context, one may note that typical coefficients of variation (COVs) for soil could be as high as 35%, and sometimes even higher (refer to Table 1, after Lacasse and Nadim, 1996). The usual remedy to this type of problem is higher factor of safety which is not only uneconomical but also unsafe in some cases (e.g. refer Duncan 2000).

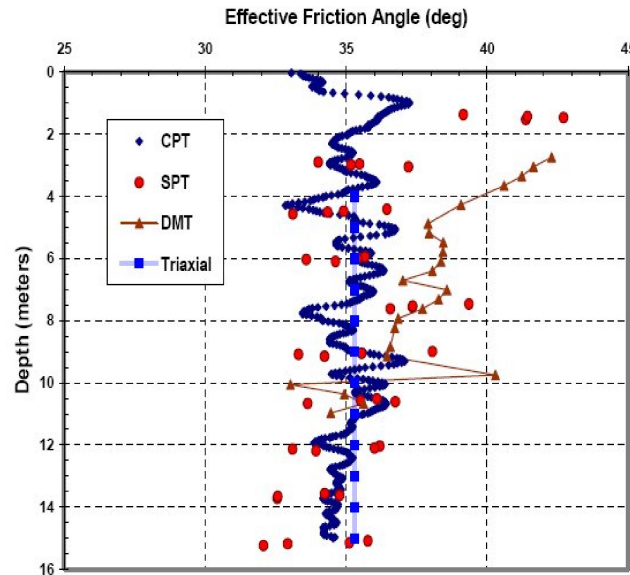


Figure 1: Typical ϕ profile

As these uncertainties in soil properties are inevitable in real life, probabilistic treatment of the problem could be a better choice as it would allow geotechnical engineers to quantify the uncertainties associated with the response based on the input uncertainties of the soil properties. Probabilistic simulation in geomechanics is twofold – quantification of soil properties uncertainties and propagation of those uncertainties through the governing differential equations (of motion for dynamic simulations and of equilibrium for static simulations). Under the framework of probability theory, uncertain soil parameters are modeled as random variable if they are specialized to a fixed location or as random field if they are specialized as a function of location in soil continuum. Uncertain spatial variability of soil deposit is modeled as random field, which can be completely described by mean, coefficient of variation (COV), and scale of fluctuation or covariance structure. Testing errors are extracted from field measurements using simple additive probabilistic model or are determined directly from comparative laboratory testing program.

Even-though there have been some valuable works in quantification of soil uncertainties and random field modeling of soil properties. (e.g. Fenton, 1999a and 1999b; Phoon and Kulhawy 1999a and 1999b), very few publications can be found on propagation of uncertainties through the governing equations in geomechanics. Most of the small numbers of studies on effects of material uncertainty have used repetitive deterministic models through Monte Carlo type simulations (Fenton and Griffiths 2005). While this approach might appear sound, it cannot be both computationally efficient and statistically consistent (produce statistically appropriate number of data points). Among alternate solution strategies, several formulations of the stochastic finite element method (SFEM) are popular. For example, perturbation method (Kleiber and Hien 1992), which uses Taylor series expansion about mean, introduces excessive errors for models with large COVs (which is common for soil properties) e.g. Sudret and Der Kiureghian, 2000. In addition to that, regular perturbation approach suffers from

"closure" problem (Kavvas, 2003), that is, higher order statistical moments are needed to solve for lower order ones. Spectral SFEM (Ghanem and Spanos, 2003; Keese 2003; Xiu and Karniadakis, 2003) overcomes the drawbacks of *Monte Carlo technique* and *perturbation approach* to a great extent but the present formulations are mainly for elastic materials. However, for geomechanics simulations, since elasto–plasticity is the most important aspect of soil behavior, proper probabilistic treatment is sorely needed.

TABLE 1. Typical COVs for Soil

Soil Property	Soil Type	PDF	Mean	COV (%)
Cone resistance	Sand Clay Clay	LN N/LN	*	*
Undrained shear strength	Clay (triaxial) Clay (index S_u) Clayey silt	LN LN N	*	5-20 10-35 5-15
Ratio S_u/σ'_{vo}	Clay	N/LN	*	5-15
Plastic limit	Clay	N	0.13-0.23	3-20
Liquid limit	Clay	N	0.30-0.80	3-20
Submerged unit weight	All soils	N	5-11 (kN/m ³)	0-10
Friction angle	Sand	N	*	2-5
Void ratio, porosity, initial void ratio	All soils	N	*	7-30
Over consolidation ratio	Clay	N/LN	*	10-35

The broader objective of this study is to develop a statistically consistent and efficient methodology to propagate the uncertain elastic–plastic soil properties through the governing differential equations in solid mechanics, with special emphasis to geomechanics problems. In this paper we outline the formulation of the probabilistic framework (spectral stochastic finite element), while the emphasis is on the constitutive level, probabilistic elastic–plastic formulation.

STOCHASTIC ELASTIC–PLASTIC FINITE ELEMENT METHOD

For static problems, the governing differential equations in mechanics are:

$$A\sigma = \Phi \quad ; \quad Bu = \epsilon \quad ; \quad \sigma = D\epsilon \quad (1)$$

where, σ denotes generalized stress, Φ the generalized forces, u the generalized displacements, [epsilon] the generalized strain and A , B , and D are operators which could be linear or non-linear. Upon elimination of σ and [epsilon] the field equation becomes,

$$ADB u = K u = \Phi \quad (2)$$

For problems involving uncertain soil properties, the operator D (stiffness tensor) is a random differential operator and hence Eq. (2) becomes a non–linear partial differential equation (PDE) with random coefficient. Existing analytical tools do not allow us to solve non–linear PDE with random coefficient analytically, especially for problems with irregular boundary condition. Alternative is to use numerical methods, among which the SFEM is quite popular. In particular, spectral method (Ghanem and Spanos, 2003; Keese 2003; Xiu and Karniadakis, 2003) is the most efficient in the SFEM family of methods, because of error minimizing property and optimal dimension of the problem.

In spectral method, the input random field soil properties are discretized through truncated Karhunen–Loeve (KL) expansion, which represents the input random field of soil properties as eigen-modes of its

covariance kernel. For example if one models friction angle (ϕ) at a geotechnical site as random field, it can be discretized as:

$$\phi(x, \theta) = \bar{\phi}(x) + \sum_{n=1}^M \sqrt{\lambda_n} \xi_n(\theta) f_n(x) \quad (3)$$

where, $\phi(x, \theta)$ signifies friction angle (ϕ) is a spatial (x) random (θ) function. The variable $\bar{\phi}(x)$ represents spatially varying deterministic mean and the second-term on the r.h.s can be visualized as representation of variance w.r.t the deterministic mean ($\bar{\phi}(x)$). λ_n and $f_n(x)$ are eigenvalues and eigenfunctions of the covariance kernel, which can be obtained by solving the following generalized eigenvalue problem:

$$\int_D C(x_1, x_2) f(x_2) dx_2 = \lambda f(x_1) \quad (4)$$

where, $C(x_1, x_2)$ is the covariance function of the friction angle random field. Having solved the eigenvalue problem, the zero-mean random variables ($\xi_i(\theta)$) can be obtained as:

$$\xi_i(\theta) = \frac{1}{\sqrt{\lambda_i}} \int_D [\phi(x, \theta) - \bar{\phi}(x)] f_i(x) dx \quad (5)$$

Figure 2 shows KL approximation on an exponential covariance kernel ($C(x_1, x_2) = e^{-|x_1 - x_2|}$). One may note that four-terms of KL-expansion represent the covariance kernel pretty well. It is always prudent to minimize the number of terms of discretization expansion as each extra term of the expansion adds extra (stochastic) dimensions to the problem and in that sense KL-expansion is an optimal expansion as it minimizes the error of finite representation.

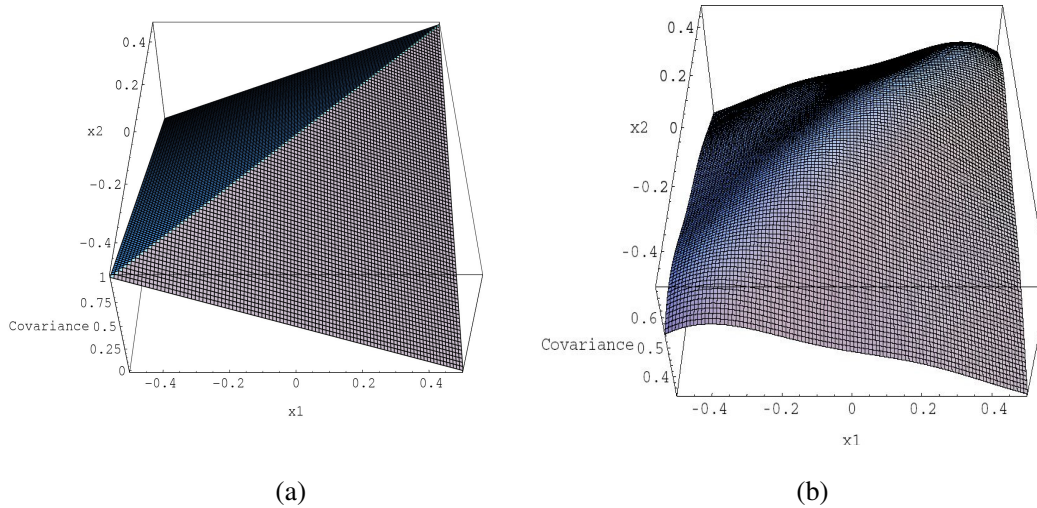


Figure 2: KL-expansion of (exponential) covariance kernel: (a) exact covariance surface, and (b) the four terms approximation

On the other hand, the solution (displacement) random field ($u(x, \theta)$) can be expressed as:

$$u(x, \theta) = \sum_{j=1}^L e_j \chi_j(\theta) b_j(x) \quad (6)$$

However, since the covariance kernel of $u(x, \theta)$ is not known a priori, the eigenvalues (e_j) and eigenvectors ($b_j(x)$) in Eq. (6) are also unknown. The random displacement field $u(x, \theta)$ can be expressed as a functional of known random variables (of input random field) and unknown deterministic function as:

$$u(x, \theta) = \zeta[\xi_i(\theta), x] \quad (7)$$

Therefore, one needs a basis of known random variables – the polynomial chaos (PC) expansion. Using PC expansion one can expand the zero-mean random variables ($\chi_j(\theta)$) in Eq. (6) as:

$$\chi_j(\theta) = \sum_{i=0}^P \gamma_i^{(j)} \psi_i[\{\xi_r\}] \quad (8)$$

where, number of terms (P) in Eq. (8) depends on number of terms of KL-expansion (M , refer to Eq. (3)) and order of PC. Substituting Eq. (8) in Eq. (6), one can expand $u(x, \theta)$ as:

$$u(x, \theta) = \sum_{j=1}^L \sum_{i=0}^P \gamma_i^{(j)} \psi_i[\{\xi_r\}] e_j b_j(x) = \sum_{i=0}^P \psi_i[\{\xi_r\}] d_i(x) \quad (9)$$

where, one can obtain the unknown deterministic coefficients, $d_i(x) = \sum_{j=1}^L \gamma_i^{(j)} e_j b_j(x)$ by minimizing norm of error of finite representation (e.g. using Galerkin technique). Applying Galerkin scheme and standard discretization technique (shape function method) of deterministic finite element method, one may obtain the final equation as:

$$\sum_{n=1}^N K_{mn} d_{ni} + \sum_{n=1}^N \sum_{j=0}^P d_{nj} \sum_{k=1}^M C_{ijk} K'_{mnk} = \langle F_m \psi_i[\{\xi_r\}] \rangle \quad (10)$$

where, K is the standard deterministic stiffness matrix, and K' can be viewed as stochastic stiffness matrix, given in the following forms:

$$K_{mn} = \int_D B_n D B_m dV \quad (11)$$

$$K'_{mnk} = \int_D B_n \sqrt{\lambda_k} f_k B_m dV \quad (12)$$

where B is the derivative of shape function similar to the deterministic finite element method. The coefficient C in the stochastic stiffness matrix is a third order tensor given by

$$C_{ijk} = \langle \xi_k(\theta) \psi_i[\{\xi_r\}] \psi_j[\{\xi_r\}] \rangle \quad (13)$$

In Eq. (10), F is the distributed force vector (similar to deterministic FEM) given as:

$$F_m = \int_D \Phi N_m dV \quad (14)$$

where, N are the shape functions. One may also note that in Eq. (10), d is the vector of unknown deterministic coefficients of PC, which may be visualized as generalized degrees of freedom. The number of generalized degrees of freedom (DOFs) depends on number of terms in KL-expansion (KL-dimension), order of PC, and number of deterministic degrees of freedom as follows:

$$\# \text{ of Generalized DOF} = \left(1 + \sum_{s=1}^p \frac{1}{s!} \prod_{r=0}^{s-1} (M + r) \right) \times \# \text{ of Deterministic DOF} \quad (15)$$

where, M is the KL-dimension and p is the order of PC. After solving Eq. (10) for the generalized DOFs d and substituting them in Eq. (9), one may obtain an expression for solution (displacement) random field as:

$$u = \sum_{j=0}^P d_j \psi_j[\{\xi_r\}] \quad (16)$$

However, one might note that for soils (and other materials in general) the constitutive equations are non-linear (elastic-plastic), and that the deterministic stiffness (K) and stochastic stiffness matrix (K') are nonlinear functions of generalized displacements and need regular updating. This triggers the need for constitutive level integrations of probabilistic elastic-plastic incremental equations at the integration (for example Gauss) points.

CONSTITUTIVE INTEGRATION: PROBABILISTIC ELASTO-PLASTICITY

One can write the general form of 3-D elastic-plastic constitutive equation, which needs to be solved at each Gauss point as:

$$\frac{d\sigma_{ij}(x_t, t)}{dt} = D_{ijkl}(\sigma_{ij}, D_{ijkl}^{el}, f, U, q_*, r_*; x_t, t) \frac{d\epsilon_{kl}(x_t, t)}{dt} \quad (17)$$

where, D_{ijkl} is the non-linear elastic-plastic coefficient tensor which is a function of stress tensor (σ_{ij}), elastic moduli tensor (D_{ijkl}^{el}), yield function (f), plastic potential function (U), internal variables (q_*) and direction of evolution of internal variables (r_*). The internal variables (q_*) could be scalar (for perfectly plastic and isotropic hardening models), or second-order tensor (for translational and rotational kinematic hardening models), or fourth-order tensor (for distortional hardening models) or combinations of above. The same classification applies to the direction of evolution of internal variables (r_*).

Due to uncertainties in material properties, the non-linear coefficient in Eq. (17) becomes random and hence Eq. (17) is an ordinary differential equation (ODE) with random coefficient, which can not be integrated in standard Riemannian sense. Anders and Hori (2000) used *perturbation approach* to solve the elastic-plastic constitutive equation. However, in dealing with geomaterials, the perturbation approach is limited because of high COVs associated with material properties. In addition to that, difficulty arises if one tries to extend the method for problems having more than one uncertain soil property. To overcome the difficulty associated with *perturbation approach* and *Monte Carlo technique* (statistically consistent but computationally inefficient) Jeremić et al. (2006) recently developed an Eulerian-Lagrangian for of the Fokker-Planck-Kolmogorov (FPK) equation

corresponding to any general 1-D elastic–plastic constitutive rate equation. Following Jeremić et al. (2006), one may obtain the FPK-equation corresponding to 3-D general elastic–plastic constitutive rate equation (Eq. 17) as:

$$\begin{aligned}
\frac{\partial P(\sigma_{ij}(x_t, t), t)}{\partial t} = & \frac{\partial}{\partial \sigma_{mn}} \left[\left\langle \eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t)) \right\rangle \right. \\
& + \int_0^t d\tau Cov_0 \left[\frac{\partial \eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t))}{\partial \sigma_{ab}}; \right. \\
& \left. \left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t, t), t) \left. \right] \\
& + \frac{\partial^2}{\partial \sigma_{mn} \partial \sigma_{ab}} \left[\left\{ \int_0^t d\tau Cov_0 \left[\eta_{mn}(\sigma_{mn}(x_t, t), D_{mnrs}(x_t), \epsilon_{rs}(x_t, t)); \right. \right. \right. \\
& \left. \left. \left. \eta_{ab}(\sigma_{ab}(x_{t-\tau}, t-\tau), D_{abcd}(x_{t-\tau}), \epsilon_{cd}(x_{t-\tau}, t-\tau)) \right] \right\} P(\sigma_{ij}(x_t, t), t) \right]
\end{aligned} \tag{18}$$

where, $P(\sigma)$ is the probability density of stress (σ). The FPK equation (Eq. 18) is exact to second-order covariance of time. The advantage of solving the constitutive equation in probability density space is that the FPK equation is linear and deterministic, whereas the constitutive equation is non-linear and stochastic in real space. Having solved Eq. (18) for probability densities of stress (σ), one may obtain mean and standard deviation of stress (σ) and statistical properties of elastic–plastic modulus (D) by standard techniques.

In demonstrating the application of the Eulerian-Lagrangian FPK equation approach in probabilistic constitutive simulation for geomechanics problems, here we have present the solution of 1-D modified Cam Clay shear constitutive equation (σ_{12} -[epsilon]₁₂ relationship) with random (variable) soil parameters. Figure 3 shows the contours of the probability density function (PDF) as well as mean, mode, and standard deviations of shear stress respectively with time/shear strain for a (a) low and high (b) OCR modified Cam Clay material model. The material parameters are defined with their distribution, namely, normally distributed random shear modulus (G) with a mean value of 2.5 MPa and a standard deviation of 0.5 MPa and normally distributed slope of critical state line (M) with a mean value of 0.6 and a standard deviation of 0.1. The other material properties are considered deterministic and are as follows: overconsolidation pressure for low OCR sample $p_0=0.2$ MPa, while for high OCR it is $p_0=0.8$ MPa; applied confinement pressure (p) = 0.1 MPa; slope of the unloading–reloading and normal compression lines in v - $\ln(p')$ space $\kappa=0.05$ and $\lambda=0.25$, cam clay model parameter $\kappa = 0.05$; and initial void ratio (e_0) = 2.18. The deterministic solution was obtained by neglecting the standard deviations of shear modulus G and slope of critical state line (M) and using their respective mean values.

Among many interesting feature in resulting contours (distributions) is the difference between deterministic solution (exclusively used nowadays) and the mode (most likely) and mean of the probabilistic solutions. This difference effectively means that the deterministic solution for this simple modified Cam–Clay model *is not the most likely solution*. In other words, deterministic solution can be on either *safe* or *unsafe* side in the solution space. This discrepancy of most likely and deterministic solution will entirely depend on the type of statistical distribution of material input parameters and the

(wrong, but currently exclusive) choice to neglect the available statistics, and use mean of multiple measurements.

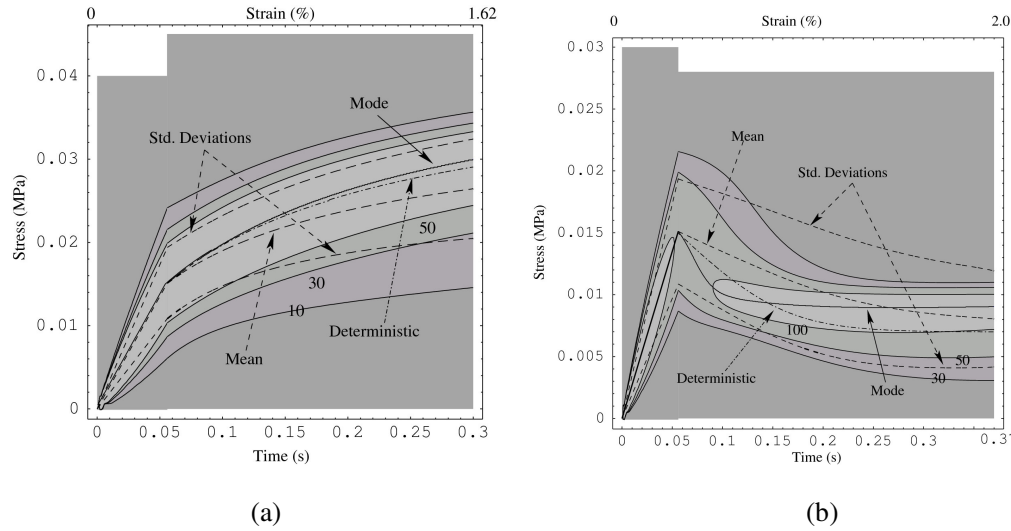


Figure 3: Evolution of Contours of PDF, Mean, Mode, Standard Deviations, and Deterministic Solution of Shear Stress (σ_{12}) with Time (t) /Shear Strain ($[\epsilon]_{12}$ for (a) low and (b) high OCR modified Cam Clay Response with Random Normally Distributed Shear Modulus (G) and Random Normally Distributed Slope of Critical State Line (M)

CONCLUSIONS

In this paper, a probabilistic computational framework - spectral stochastic elastic-plastic finite element method - is outlined for simulation of geotechnical engineering problems. Spectral method, advocated here, overcomes the drawbacks of *Monte Carlo technique* and *perturbation approach* to a great extent because of its error minimizing property and optimal expansion of any problem. In addition, a method is presented to carry out probabilistic integration at the constitutive level. Eulerian-Lagrangian form of Fokker-Planck-Kolmogorov (FPK) equation is proposed for probabilistic elastic-plastic constitutive integration. The advantage of FPK equation based approach is evident as it transforms the original stochastic non-linear constitutive rate equation into a linear deterministic partial differential equation. The application of FPK equation approach is shown in an illustrative example solving constitutive behavior of low and high OCR modified Cam Clay models probabilistically.

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