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SINGLE-DEGREE-OF-FREEDOM SYSTEM WITH RANDOM STIFFNESS UNDER RANDOM KINEMATIC EXCITATION

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ABSTRACT

This paper is devoted to response analysis for mass-spring-dashpot system excited by support motion. System stiffness is supposed to be a uniformly distributed random value, and the support displacement — a stationary random process. Single-degree-of-freedom system with random stiffness excited by random force was studied earlier; now kinematic excitation is being considered. General formula (a finite expression) is obtained for vibration power spectral density (PSD). White noise excitation is considered first; PSD dependence on frequency differs slightly from that of the PSD for the force excitation case. When the damping is high the effect of random stiffness is small. Finite expression for response mean square value is obtained; it depends only on the mean value of the stiffness and not on its variance (unlike the random force excitation case). A stationary random process with a predominant frequency in its PSD is used as an excitation also: it is taken equal to filtered white noise. In that case PSD plot for deterministic system having two maxima may lose one of them after taking random stiffness into consideration; at the same time mean square value doubles. For greater damping accounting for stiffness randomness leads just to a slight drop in response.

Keywords: kinematic excitation, power spectral density, stationary random process, system with random parameters, uniform distribution, white noise

INTRODUCTION

The vibrations of structures are frequently described using the theory of random processes (Bendat & Piersol, 1980; Bolotin, 1984) because of both multiplicity of vibration sources and parameter indeterminacy of each. The simplest and most frequently used model of excitation is stationary zero-mean stochastic process described by its autocorrelation function or power spectral density (PSD). The problem of vibration calculation for a linear system with time-independent deterministic parameters under such an excitation has well known solution; response is a stationary zero-mean stochastic process also.

Uncertainty and random variability are inherent not only to dynamic excitation but also to the parameters of the vibratory system itself, especially when it includes geological media. Applying the law of total probability is recommended in this case (Bolotin, 1984). Some efforts were made to take into account uncertainty of parameters of structure in vibration. A brief review can be found in (Kholmyansky, 2000). The problem of response calculation for systems with random time-independent parameters under excitation described by the stationary stochastic process model was stated and solved (Kholmyansky, 1997). That allowed calculating of vibration of machine foundations under random dynamic excitation (Kholmyansky, 1998), rigid underground structures such as transportation tunnels (Kholmyansky, 1999) and some general considerations also (Kholmyansky, 2000). Results of some of these works are used below.

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THE PRINCIPLES OF THE THEORY

The system description

In this section we use the following problem statement (Kholmyansky, 2000). System parameters are supposed not to vary in time and to be random variables (not random functions of time). The system is supposed to be linear, therefore it is possible to describe its properties by the weighting function $h(t)$. This function is the system response to the impulse excitation $\delta(t)$; it depends not only on t, but on random system parameters. Therefore the function $h(t)$ is a random function of time t, or a random process (non-stationary).

The excitation $f(t)$ is assumed to be stationary random process with zero mean, as it was assumed in other works (Bolotin, 1984; Bendat & Piersol, 1980). This excitation may be either by force or by support motion (the both cases are compared in this paper). The random process $f(t)$ and the system random parameters are assumed to be stochastically independent. The autocorrelation function $K_f(t)$ of the random process $f(t)$ is given.

The expression for the response $u(t)$ has the same form as in the case of deterministic systems:

$$u(t) = \int_{-\infty}^{\infty} h(\xi) f(t-\xi) d\xi \tag{1}$$

Notice that for physically realizable system $h(t) = 0$ for $t < 0$.

The suppositions made allow concluding that $u(t)$ is zero-mean stationary random process. Its autocorrelation function $K_u(t)$ is given by the expression:

$$K_u(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_0(\xi) h_0(\eta) K_f(t+\xi-\eta) d\xi d\eta + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_h(\xi,\eta) K_f(t+\xi-\eta) d\xi d\eta \tag{2}$$

where $h_0(t)$ and $K_h(t)$ = mean value and autocovariance function of non-stationary random process $h(t)$. In the special case of deterministic system we have $K_h(t) = 0$ and (2) reduces to the well-known formula.

The properties of the system response in the spectral domain

It is possible to derive the formula for the PSD function $S_u(\omega)$ of the response. For any $u(t)$ the corresponding PSD $S_u(\omega)$ is calculated from the autocorrelation function $K_u(t)$ with the aid of the Fourier transformation (Sveshnikov, 1968; Bendat & Piersol, 1980). Using (2) the result may be reduced to the form:

$$S_u(\omega) = \left[H_0(-\omega) H_0(\omega) + S_h(-\omega,\omega) \right] S_f(\omega) \tag{3}$$

where $H_0(\omega)$ = frequency response function of the deterministic system with weighting function $h_0(t)$;

$$S_h(\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(\lambda\xi+\mu\eta)} K_h(\xi,\eta) d\xi d\eta \tag{4}$$

Equation (3) sets a proportionality of the response PSD and the excitation PSD. The factor of proportionality depends on the properties of the system and consists of two terms. The first one is determined from the properties of the deterministic system with weighting function $h_0(t)$ and frequency response function $H_0(\omega)$. The second one depends on the autocovariance function of the random process $h(t)$ only.

In the case of deterministic system the second term in the factor of proportionality is zero and (3) is reduced to the well-known formula.

It would be possible to replace the frequency response function of the system with random parameters with its mean value. That would allow to return from the model of system with random parameters to the model of system with deterministic parameters having weighting function $h_0(t)$ and frequency response function $H_0(\omega)$. This approach however has a serious shortage. It is possible to deduce from a general property of autocovariance function (Sveshnikov, 1968) that $S_h(-\omega,\omega)$ is non-negative. Hence this approach underestimates the level of vibration at all frequencies.

Equation (3) can be written in another form:

$$S_u(\omega) = T_h(-\omega,\omega) S_f(\omega) \tag{5}$$

where $T_h(\lambda,\mu)$ = two-dimensional Fourier transform of autocorrelation function for non-stationary random process $h(t)$. The expression for $T_h(\lambda,\mu)$ is easily obtained

$$T_h(\lambda,\mu) = M[H(\lambda)H(\mu)] \qquad (\lambda = -\omega, \mu = \omega) \tag{6}$$

It is supposed that $H(\omega)$ implicitly depends on the random parameters.

SYSTEM WITH UNIFORMLY DISTRIBUTED STIFFNESS — FORCE EXCITATION

The general formula

Consider a simple example of system with random parameters — a single-degree-of-freedom (SDOF) system (linear oscillator). The stiffness k is uniformly distributed on $[k_1, k_2] = [k_0(1-\alpha), k_0(1+\alpha)]$. Stiffness mean value and coefficient of variation are k_0 and $v_k = \alpha/\sqrt{3}$ respectively. System mass m and damping b are deterministic parameters. The excitation is performed by a force applied to the oscillator mass.

In this simplest case the frequency response function of the deterministic system is well known. The closed-form solution for $T_h(-\omega,\omega)$ was obtained (Kholmyansky, 2000):

$$\begin{aligned} T_h(-\omega,\omega) &= \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} H(-\omega,k)H(\omega,k)d\omega \\ &= \frac{1}{2\alpha\omega k_0 b}\left[\arctan\frac{(1+\alpha)k_0 - m\omega^2}{\omega b} - \arctan\frac{(1-\alpha)k_0 - m\omega^2}{\omega b}\right] \end{aligned} \tag{7}$$

White noise excitation

Random process called white noise has a constant PSD: $S_f(\omega) = S_0$. In order to obtain the response mean square value Ψ_u^2 describing vibration power in the broad frequency range the formula is used that expresses it through the integral of PSD (Bolotin, 1984; Sveshnikov, 1968; Bendat & Piersol, 1980). The result obtained previously (Kholmyansky, 2000) is:

$$\Psi_u^2 = \frac{S_0}{4\alpha k_0 b}\ln\left(\frac{1+\alpha}{1-\alpha}\right) \tag{8}$$

Notice that the result does not depend on the mass of the system m. The response mean square value Ψ_u^2 rises when α increases; the change is however small (except for $\alpha \approx 1$).

Excitation with a predominant frequency (filtered white noise)
A stationary random process with a predominant frequency in its PSD was used as an excitation for the system studied above. The force transmitted to the fixed support of the SDOF system subject to the white noise excitation with the PSD S_0 (filtered white noise) was chosen as an excitation. The other SDOF system (with random parameters) under such excitation may be considered as a special case of a two-mass system under white noise excitation where the loaded mass (a part of the filter) is significantly less than the non-loaded one (see Figure 1). The expression for the force transmitted to the support (i.e. mass 2 on Figure 1) is well known (Richart et al., 1970, for example), so the PSD of the excitation is easily obtained:

$$S_f(\omega) = S_0 \frac{1 + 4D'^2 \left(\omega / \lambda'\right)^2}{\left[1 - \left(\omega / \lambda'\right)^2\right]^2 + 4D'^2 \left(\omega / \lambda'\right)^2} \tag{9}$$

where λ' and D' = undamped natural circular frequency and damping ratio of the filter. Unlike the previous case no closed-form solution for the response mean square value was found.

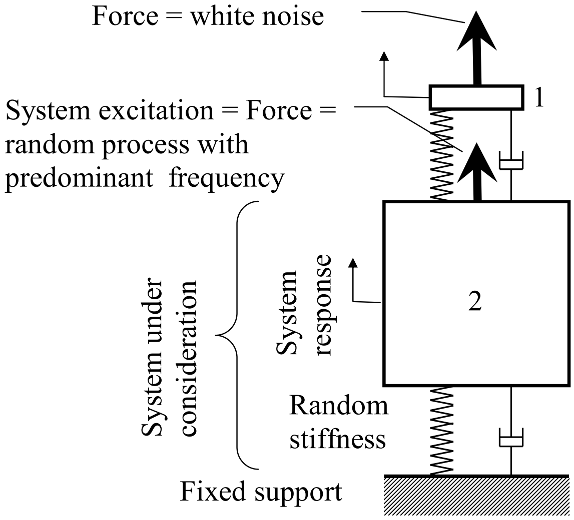

Figure 1. SDOF system with fixed support and force excitation with a predominant frequency ($m_1 << m_2$)

SYSTEM WITH UNIFORMLY DISTRIBUTED STIFFNESS — KINEMATIC EXCITATION

The general formula
Now consider another system with random parameters — a single-degree-of-freedom linear oscillator under kinematic excitation (that is support motion). The stiffness k is uniformly distributed again with the same parameters as above; system mass m and damping b are deterministic as above too. The excitation is represented by the support motion. Taking the known frequency response function of the deterministic system

$$H(\omega) = H(\omega, k) = \frac{k + i\omega b}{k - m\omega^2 + i\omega b} \tag{10}$$

we obtain

$$T_h(-\omega, \omega) = \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} H(-\omega, k) H(\omega, k) d\omega = (2\alpha k_0)^{-1} \left\{ F\left(k(1+\alpha), \omega\right) - F\left(k(1-\alpha), \omega\right) \right\} \tag{11}$$

where

$$F(k,\omega) = k - m\omega^2 + m\omega^2 \ln\left(\frac{\left(k - m\omega^2\right)^2 + \omega^2 b^2}{\omega^2 b^2}\right) + m^2\omega^3 b^{-1} \arctan\left(\frac{k - m\omega^2}{\omega b}\right) \tag{12}$$

White noise excitation
The PSD of the support motion is constant: $S_f(\omega) = S_0$, so

$$S_u(\omega) = S_0 T_h(-\omega,\omega) \tag{13}$$

where $T_h(-\omega,\omega)$ is given by (11). Response mean square value Ψ_u^2 is calculated using interchange of integrations:

$$\begin{aligned}\Psi_u^2 &= \frac{1}{\pi}\int_0^\infty S_u(\omega)d\omega = \frac{S_0}{\pi}\int_0^\infty T_h(-\omega,\omega)d\omega = \frac{S_0}{\pi}\int_0^\infty \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} H(-\omega,k)H(\omega,k)dkd\omega \\ &= \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} \Psi_{u,k}^2 dk = \frac{1}{2}\left(\frac{k_0}{b} + \frac{b}{m}\right)\end{aligned} \tag{14}$$

where

$$\Psi_{u,k}^2 = \frac{S_0}{\pi}\int_0^\infty H(-\omega,k)H(\omega,k)d\omega = \frac{1}{2}\left(\frac{k}{b} + \frac{b}{m}\right) \tag{15}$$

is response mean square value for the deterministic system with the value of the stiffness k.

Unlike force excitation the result does depend on all the parameters of the system including the mass m. Stiffness randomness effects the vibration power at all frequencies but in such a way that the total power is independent on stiffness coefficient of variation. That makes the difference from the force excitation where the response mean square value increases with α (and v_k) and even tends to infinity with $\alpha \to 1$, as it seen from (8).

Excitation with a predominant frequency
It is reasonable to consider kinematic excitation to be the displacement of another SDOF oscillator, which is subject to white noise kinematic excitation in turn (see Figure 2). This random process (filtered white noise) is often used in such problems (Newmark & Rosenblueth, 1973). It can be easily shown that such excitation is the same as for force excitation; its PSD is given by (10).

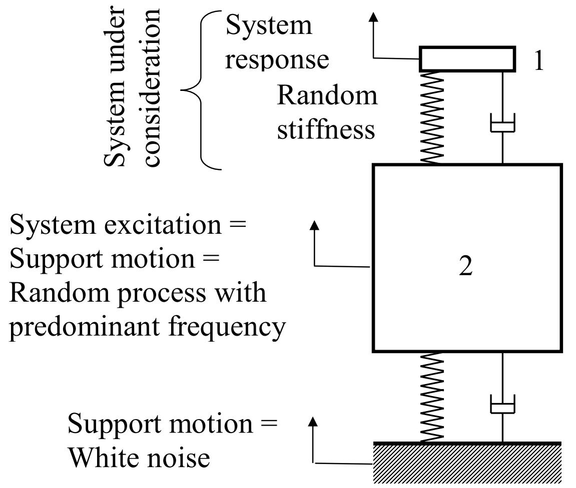

Figure 2. SDOF system excited by moving support ($m_1 << m_2$)

RESULTS AND DISCUSSION

This section contains numerical results obtained with the formulae derived above. In all the cases $\alpha = 0.2$ ($v_k = 0.115$) is taken. The deterministic SDOF oscillator with the stiffness k_0 is supposed to have undamped natural circular frequency $\lambda_0 = \sqrt{(k_0/m)}$ and damping ratio $D_0 = b/\{2\sqrt{(k_0 m)}\}$.

White noise excitation

Consider first kinematic excitation of an oscillator with damping ratio $D_0 = 0.025$; the curves for two deterministic systems are plotted also (see Figure 3). Semilogarithmic net is used for this and some other diagrams. As for force excitation (Kholmyansky, 2000) random variability leads to a significant decrease of PSD but near its peak only; for other frequencies random variability leads to decrease of PSD. This makes difference with the influence of growing damping that can only lower PSD.

The results for a higher damping level $D_0 = 0.25$ are shown on Figure 4. It can be seen that in a qualitative sense the results are the same, yet the influence of randomness becomes very small.

Consider vibrations under force excitation for comparison (Figure 5). The difference between PSD values for the two excitation types is moderate even for low damping value.

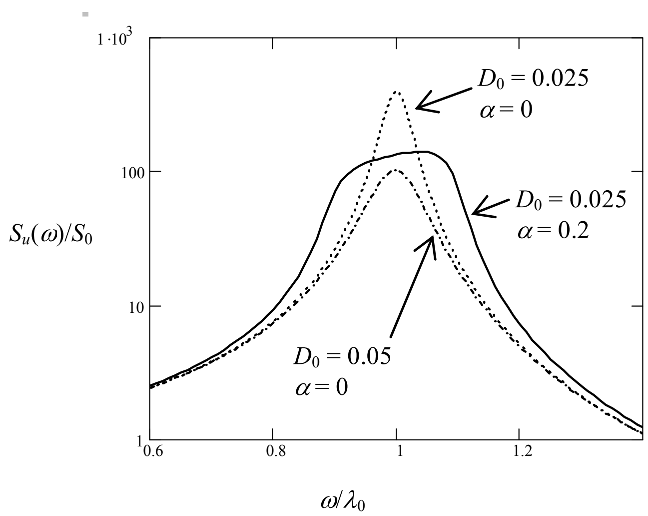

Figure 3. SDOF system with D_0 = 0.025 under white noise excitation

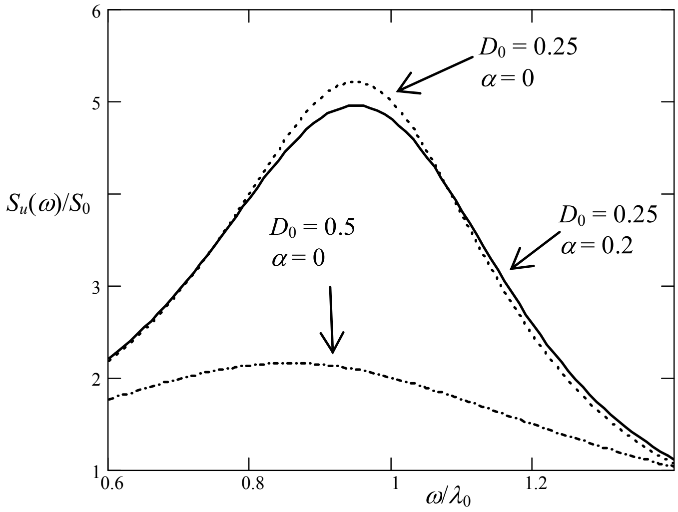

Figure 4. SDOF system with D_0 = 0.25 under white noise excitation

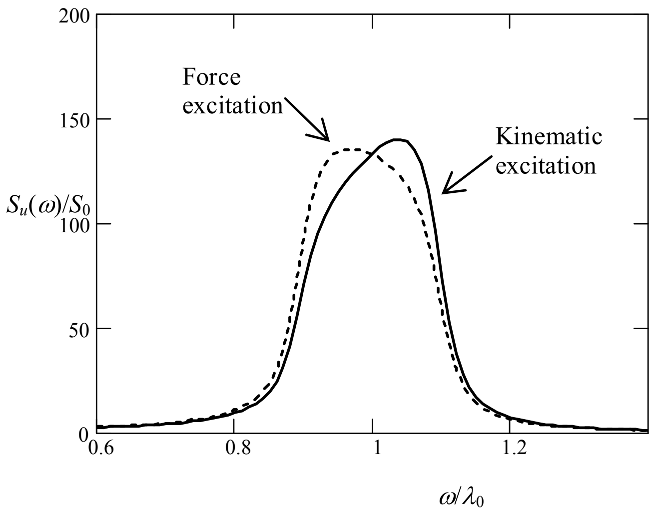

Figure 5. SDOF systems with D_0 = 0.025 under force and kinematic white noise excitations

Excitation with a predominant frequency

PSD diagrams for SDOF with damping ratio $D_0 = 0.025$ are shown on Figure 6 and Figure 7. To obtain the most interesting result the excitation predominant frequency is chosen $\lambda' = 0.9\lambda_0$; parameter $D' = 0.01$ (Figure 6); for Figure 7 we take $D' = 0.1$.

It can be seen form Figure 6 how PSD plot for deterministic system having two maxima may lose one of them after taking randomness into consideration; at the same time mean square value doubles. For less accented frequency dependence (Figure 7) the effect of randomness resembles those one for white noise excitation (see Figure 3).

The results for systems with higher damping ratio $D = 0.25$ are shown on Figure 8 and Figure 9. It is seen that for greater damping accounting for stiffness randomness leads just to a slight drop in response.

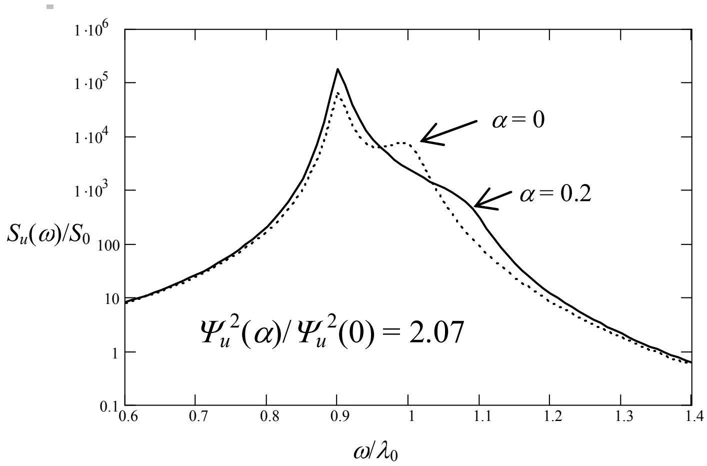

Figure 6. SDOF system with D_0 = 0.025 under excitation with predominant frequency λ' = 0.9λ_0; D' = 0.01

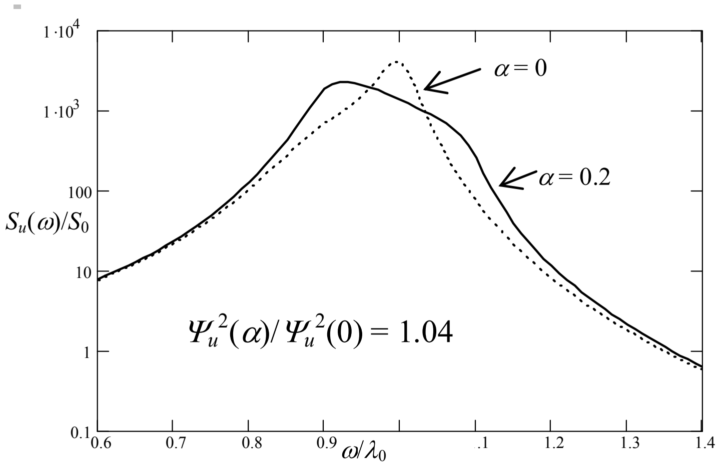

Figure 7. SDOF system with D_0 = 0.025 under excitation with predominant frequency λ' = 0.9λ_0; D' = 0.1

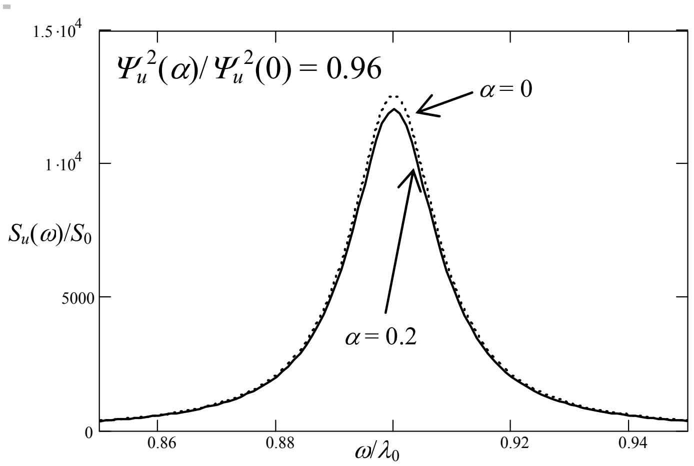

Figure 8. SDOF system with D_0 = 0.25 under excitation with predominant frequency λ' = 0.9λ_0; D' = 0.01

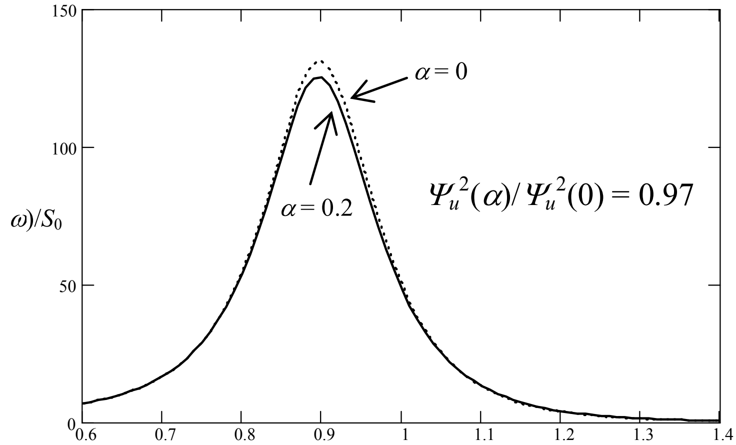

Figure 9. SDOF system with D_0 = 0.25
under excitation with predominant frequency $\lambda' = 0.9\lambda_0$; D' = 0.1

CONCLUSIONS

The paper studies random vibrations of system with random parameters. The results for kinematic excitation show some difference with the results for force excitation. Neglecting system parameters randomness may lead to significant underestimation of response level.

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