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# A NUMERICAL MODEL FOR DYNAMIC RESPONSE OF UNSATURATED SANDS

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## ABSTRACT

This paper includes a presentation of a numerical model for the description of the behavior of unsaturated sandy soils under dynamic loading. The model is developed within the theory of Biot and the formulation of Coussy. It is also based on laboratory observations. The formulation leads to a simplified model, which can deal both saturated and unsaturated sandy soils. After the presentation of the model, the paper describes its implementation in a finite element program; the model is validated on one dimensional dynamic consolidation problem and the performance of the model is illustrated by a numerical example of partially saturated soils.

Keywords: Biot, coupled, numerical model, unsaturated sands

## INTRODUCTION

Soil liquefaction constitutes a major cause of damage induced by earthquakes. It results from the interaction between the liquid and solid phases in porous media. This phenomenon may occur in both saturated and unsaturated soils; it results from a reduction in the effective mean stresses, which leads to a deterioration in the strength and stiffness of soils. Previous researches in liquefaction concerned mainly saturated condition. Since unsaturated soils are frequently encountered in geotechnical engineering (Fredlund, 1993), and the phreatic surface vary with the seasons and the climate, it is of major interest to investigate their behavior under dynamic loading with partially saturated conditions.

The hydro-mechanic coupled theory founded by Biot (1941, 1956, 1962) together with the concept of effective stress proposed by Terzaghi (1936) are used to study the liquefaction phenomena in saturated soils (Zienkiwicz et al, 1980, 1984, 1999). New developments in the area of unsaturated soil (Fredlund, 1993, 2006, Coussy, 1991, 2004) together with the generalization of the concept of effective stress to unsaturated soils (Bishop, 1959), are used in this paper for the elaboration of a numerical model for the dynamic response of unsaturated sandy soils. The paper presents successively the formulation of the numerical model, its implementation in a finite element program and finally validation examples.

## ASSUPTION OF UNSATURATED SANDY SOIL

In some processes, such as drying and drainage, the porous space is filled by several fluids so that the porous material is said to be unsaturated with regards to a reference fluid of principal concern,

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generally chosen as the liquid phase. In our case, the air and water phases coexist in the porous space; the reference fluid is water.

The main difference between saturated and unsaturated soil lies in the existence of soil suction, which is defined as the difference between the pore air pressure and the pore water pressure. It is an important parameter for unsaturated soils. However, for sandy soils, because of the big diameter of soil grains; the soil suction is so small that it can be neglected. Figure 1 shows the soil water characteristic curve for Hostun sand. It can be observed that the soil suction is very small over a very wide range of water saturation degree: suction remains lower than 5kPa when the degree of water saturation is higher than 20%. This value is so small that it can be neglected. In the following, the soil suction is neglected, which means: the pore water pressure equals to the pore air pressure ( $p_a = p_w$ ). As a consequence, the effective stress concept of Terzaghi can be used for such soils.

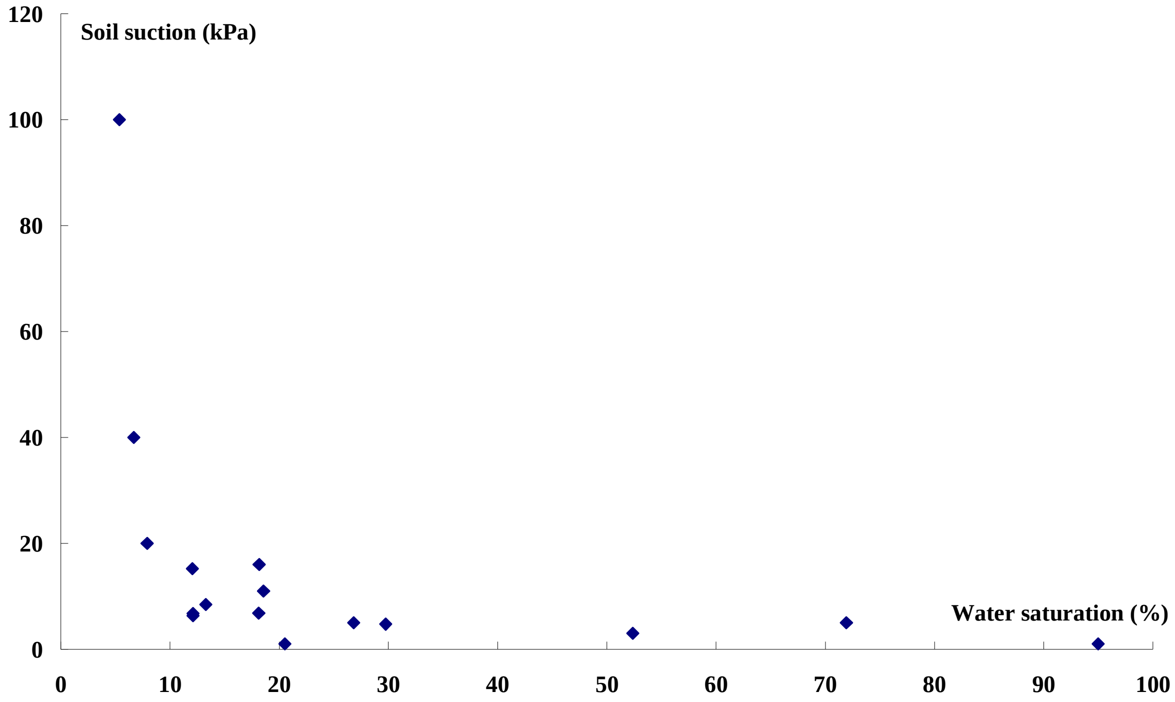


**Figure 1. The soil water characteristic curve of Hostun sand**

## THE GENERAL EQUATIONS FOR UNSATURATED SANDY SOIL

According to the theory of Coussy (1991, 2004), the potential of porous media under isothermal conditions assuming small perturbation can be expressed as:

$$\psi = \sigma_0 : \varepsilon + g_{mi}^0 m_i + \frac{1}{2}\varepsilon : C : \varepsilon - \left(\frac{m}{\rho_0}\right)_j M_{jk} B_k \varepsilon_v + \frac{1}{2}\left(\frac{m}{\rho_0}\right)_j M_{jk} \left(\frac{m}{\rho_0}\right)_k \quad (1)$$

Where $M_{jk}$ is the Biot modulus, and $B_k$ is the Biot coefficient, $C$ is the undrained elastic matrix, $m_j$ is the fluid mass content per unit volume of fluid $j$, $\varepsilon_v$ and $\varepsilon$ are the volumetric deformation and the deformation tensor of the skeleton, $\sigma_0$ is the initial stress tensor, $\rho_{0j}$ is the initial density of fluid $j$, $g_{mi}^0$ is the initial free enthalpy in fluid $i$. And as a result, the constitutive equation can be derived as follows:

$$\sigma = \sigma_0 + C : \varepsilon - \left(\frac{m}{\rho_0}\right)_j M_{jk} B_k I \tag{2}$$

Where $I$ is the unit tensor with two orders and the free enthalpy per mass of fluid $i$ can be expressed as:

$$g_{mi} = g_{mi}^0 - (\frac{\delta_{ij}}{\rho_{0j}}) M_{jk} B_k \varepsilon_v + \frac{1}{2}\left[ (\frac{\delta_{ij}}{\rho_{0j}}) M_{jk} (\frac{m_k}{\rho_{0k}}) + (\frac{m_j}{\rho_{0j}}) M_{jk} (\frac{\delta_{ki}}{\rho_{0k}}) \right] \tag{3}$$

Where $\delta_{ij}$ is the Kronecker's delta. The free enthalpy is defined as (Coussy 1991):

$$g_{mi} = g_{mi}^0 + \left(\frac{p - p_0}{\rho_0}\right)_i - \left(\frac{(p - p_0)^2}{2\rho_0 K}\right)_i \tag{4}$$

The linear form is used here as:

$$g_{mi} = g_{mi}^0 + \left(\frac{p - p_0}{\rho_0}\right)_i \tag{5}$$

Combination of equation(5) and (3) leads to the expression of pore pressure:

$$p_i = p_{0i} - M_{ij} \left\{ B_j \varepsilon_v - \left(\frac{m}{\rho_0}\right)_j \right\} \tag{6}$$

Where $p_{0i}$ is the initial pore pressure of fluid $i$. By using the conception of stress partition, under the assumption of infinitesimal transformations, the incremental average stress is expressed in function of the average matrix stress and the pore pressure (Coussy, 2004):

$$d\sigma_m = (1-\phi) d\sigma_m^s - \phi_j dp_j \tag{7}$$

Where $d\sigma_m$ is the incremental average stress defined as $d\sigma_m = tr(d\sigma)/3$, $\phi_j$ is the porosity of fluid $j$ and $\phi$ is the current porosity, $d\sigma_m^s$ is the incremental average stress of the matrix, which can be expressed as:

$$d\sigma_m^s = K_s d\varepsilon_v^s \tag{8}$$

Where $K_s$ is the matrix bulk modulus, and $d\varepsilon_v^s$ is the incremental volumetric deformation of matrix. Also with the strain partition, the incremental volumetric deformation of matrix is linked with the incremental deformation of skeleton and the variation of porosity (Coussy 1991):

$$(1-\phi) d\varepsilon_v^s = (1-\phi) d\varepsilon_v - d\phi \tag{9}$$

Where $d\phi = \phi_0 - \phi$ is the variation of porosity, $\phi_0$ is the initial porosity.
The incremental constitutive equation of the saturating fluid may be expressed as:

$$dp_j = K_j \frac{d\rho_j}{\rho_j} \tag{10}$$

Generally, the pore air is considered as an ideal gas, so the modulus of compressibility of pore air is associated with the absolute pore air pressure:

$$K_a = \overline{p}_a = p_a + p_{a0} \tag{11}$$

Where $\overline{p}_a$ is the absolute pore air pressure, and $p_a$ is the measured pore air pressure, $p_{a0}$ is the atmospheric pressure. While for the pore water, the modulus of compressibility is about 2000MPa, compared with the compressibility of pore air, the linear property is assumed for the pore water.
From equations(7), (8), (9)and(10), the incremental macroscopic means stress can be expressed by:

$$d\sigma_m = (1-\phi) K_s d\varepsilon_v - \phi_j K_j \frac{d\rho_j}{\rho_j} - K_s d\phi \tag{12}$$

The conservation of mass ($dm_j = J\rho_j\phi_j - \rho_{0j}\phi_{0j}$) together with the infinitesimal transformation ($J \approx 1 + d\varepsilon_v$) allow the derivation of the following expression for the variation of the porosity:

$$d\phi = \sum_{j=1}^{n} d\phi_j = \sum_{j=1}^{n} \left(\frac{dm_j}{\rho_j} - \frac{d\rho_j}{\rho_j}\phi_j - \phi_j d\varepsilon_v\right) \tag{13}$$

In the soil mechanics, especially for sandy soil, the soil grain is always supposed as incompressible. We suppose in the sandy soil, the fluids are composed of ideal dry air and pure water. The combination of equations(9),(10) and (13) leads to:

$$d\varepsilon_v = \frac{dm_a}{\rho_a} + \frac{dm_w}{\rho_w} - \frac{\phi_a}{K_a} dp_a - \frac{\phi_w}{K_w} dp_w \tag{14}$$

Remember the assumption for unsaturated sand soil, according to equation(14), the pore air pressure and pore water pressure maybe expressed as:

$$dp_a = dp_w = M\frac{dm_a}{\rho_a} + M\frac{dm_w}{\rho_w} - Md\varepsilon_v \tag{15}$$

With the Biot modulus for unsaturated soil:

$$\frac{1}{M} = \frac{\phi_a}{K_a} + \frac{\phi_w}{K_w} = \frac{\phi(1-S_w)}{p_w + p_{a0}} + \frac{\phi S_w}{K_w} \tag{16}$$

Using equation(6), the constitutive equations for unsaturated sandy soil become:

$$\begin{aligned} dp_a &= -M_{aa}B_a d\varepsilon_v - M_{aw}B_w d\varepsilon_v + M_{aa}\frac{dm_a}{\rho_a} + M_{aw}\frac{dm_w}{\rho_w} \\ dp_w &= -M_{wa}B_a d\varepsilon_v - M_{ww}B_w d\varepsilon_v + M_{wa}\frac{dm_a}{\rho_a} + M_{ww}\frac{dm_w}{\rho_w} \end{aligned} \tag{17}$$

Comparison of equation (15)and (17) yields:

$$M_{aa} = M_{aw} = M_{wa} = M_{ww} = M;\, B_a + B_w = 1 \tag{18}$$

Neglecting the air flux, the constitutive equation is governed by the following expressions:

$$\begin{aligned} \sigma &= \sigma_0 + C:\varepsilon - M\frac{m_w}{\rho_w} I \\ p_w = p_a &= p_0 - M\varepsilon_v + M\frac{m_w}{\rho_w} \end{aligned} \tag{19}$$

Which are similar to that used for saturated soil, but with a Biot modulus depending on the porosity, pore pressure and water saturation, while for saturated soil, the modulus is constant. It is of interest to indicate that, in equation(16), if the water saturation is unit, that means the soil is saturated, the formulation for saturated soil is recovered. It also means that this formulation can be used for both saturated and unsaturated problems.

## VARIATION OF WATER SATURATION

The water saturation in the constitutive equation should be solved. Generally, for unsaturated soil, the water saturation is determined by using the soil water characteristic curve, however for sandy soil, as indicated above this will cause difficulties. The ideal gas law will used to determine the water saturation. The assumption of zero air flux which is used for deriving the constitutive equation, will also be used for determine the water saturation. Neglecting the flux, all the air will rest in its original place. We consider an unsaturated soil specimen, with a volume $V$ and initial measured pore pressure $p_{0w}$. The initial porosity is $\phi_0$ and the initial water saturation is $S_w^0$. So the initial pore air and pore water volume are:

$$V_a^0 = \phi_0(1 - S_w^0)V; \quad V_w^0 = \phi_0 S_w^0 V \tag{20}$$

After a small perturbation, under drained or undrained conditions (for pore water only; while for pore air the condition is always undrained), the measured pore pressure becomes $p_w$, the volumetric deformation is $\varepsilon_v$. Because the soil grain is incompressible, the variation of the soil volume is equal to the variation of the porosity volume:

$$dV = \varepsilon_v V = (\phi - \phi_0)V \tag{21}$$

Because the soil grain is incompressible, so the variation of the soil volume is composed of the variation of the pore air volume and pore water volume as:

$$dV = dV_a + dV_w \tag{22}$$

According to the ideal gas law, we can calculate the variation of the pore air volume as:

$$dV_a = \left( \frac{p_{0w} + p_{a0}}{p_w + p_{a0}} - 1 \right) V_a^0 \tag{23}$$

Current pore water volume will change because of the drained condition or the compressibility of water, or the two in the same time. Combination of equation(20), (22) gives the current pore water and pore air volume:

$$V_w = V_w^0 + dV - dV_a; \quad V_a = V_a^0 + dV_a \tag{24}$$

So the current water saturation can be expressed as its definition:

$$S_w = \frac{V_w}{V_w + V_a} \tag{25}$$

Combination of the equations from(20) to(25), we get the expression of current water saturation in function of pore pressure and the volumetric deformation of skeleton as:

$$S_w = \frac{\phi_0 S_w^0 + \varepsilon_v - \left( \frac{p_{0w} + p_{a0}}{p_w + p_{a0}} - 1 \right) \phi_0 (1 - S_w^0)}{\phi_0 + \varepsilon_v} \tag{26}$$

## FINITE ELEMENT MODELLING

The finite element method is used to solve the dynamic problem which is governed by the following expressions:

1) The balance equation as:

$$div(\sigma) - \rho \ddot{u} = 0 \tag{27}$$

Where $u$ is the displacement vector of soil skeleton and $\rho$ is the soil density, which depends on soil porosity and the densities of the solid grain and water:

$$\rho = (1 - \phi)\rho_s + \phi S_w \rho_w \tag{28}$$

The density of pore air is neglected here.

2) The diffusion law

For instance the generalized Darcy's law is used:

$$\frac{\vec{w}}{\rho_w} = -\frac{k}{\mu}\left[ grad(p_w) + \rho_w \ddot{u} \right] \tag{29}$$

Where $\vec{w}$ is the water mass flux, $k$ is the soil permeability, $\mu$ is the water viscosity, $\ddot{u}$ is the acceleration vector of soil skeleton.

3) The conservation of mass:

$$\dot{m}_w = div(\vec{w}) \tag{30}$$

The constitutive equations(19) are used to solve the coupled dynamic problem, by eliminating the water mass content in first equation; we get the total stress in function of pore water pressure and the deformation as:

$$\sigma = \sigma_0 + C_0 : \varepsilon - (p_w - p_0) I \tag{31}$$

Where $C_0$ is the drained elastic matrix. Combination of equation(29), (30) and the second equation in(19) leads to the diffusion equation of pore water as:

$$div\left\{\frac{k}{\mu}[grad(p_w) + \rho_w \ddot{u}]\right\} + \frac{1}{M}\frac{\partial p_w}{\partial t} + \frac{\partial \varepsilon_v}{\partial t} = 0 \tag{32}$$

The application of the finite element approach to equation(32) and the combination of (31)and (27) leads the following expressions:

$$\begin{aligned} & M_{ss}\ddot{U} + RU + C_{up}P = F_u \\ & M_{pu}\ddot{U} + C_{pu}\dot{U} + C_{pp}\dot{P} + K_{pp}P = F_p \end{aligned} \tag{33}$$

$U$ and $P$ denote the nodal displacement and pore water pressure vectors, respectively. The appendix gives the mathematical expression of different matrix in equation(33). The generalized Newmark method is used to integrate the equations in the time domain. The incremental displacement and pore water pressure are calculated at each time increment as:

$$\begin{bmatrix} \overline{R} & \overline{C}_{up} \\ \overline{M} & \overline{C}_{pp} \end{bmatrix} \begin{Bmatrix} \Delta U \\ \Delta P \end{Bmatrix} = \begin{Bmatrix} \overline{F}_u \\ \overline{F}_p \end{Bmatrix} \tag{34}$$

Where $\Delta U$ and $\Delta P$ are the incremental displacement and pore pressure vectors at element nodes, the other matrixes are list in the appendix.

## VALIDATION OF PROGRAM

The Terzaghis's one dimensional consolidation problem is used to validate the program (Coussy, 1991, 2004; Wang, 2000). Figure 2 and Table 1 show the geometric and mechanical characteristics used in this example. The base and the lateral boundaries are assumed to impervious, while free flow condition is assumed at the top. A constant step pressure ( $P_s$ =100kPa) is applied at the top of the soil column. The water is assumed compressible ( $K_w = 100 MPa$ ) and the initial water saturation is unit.

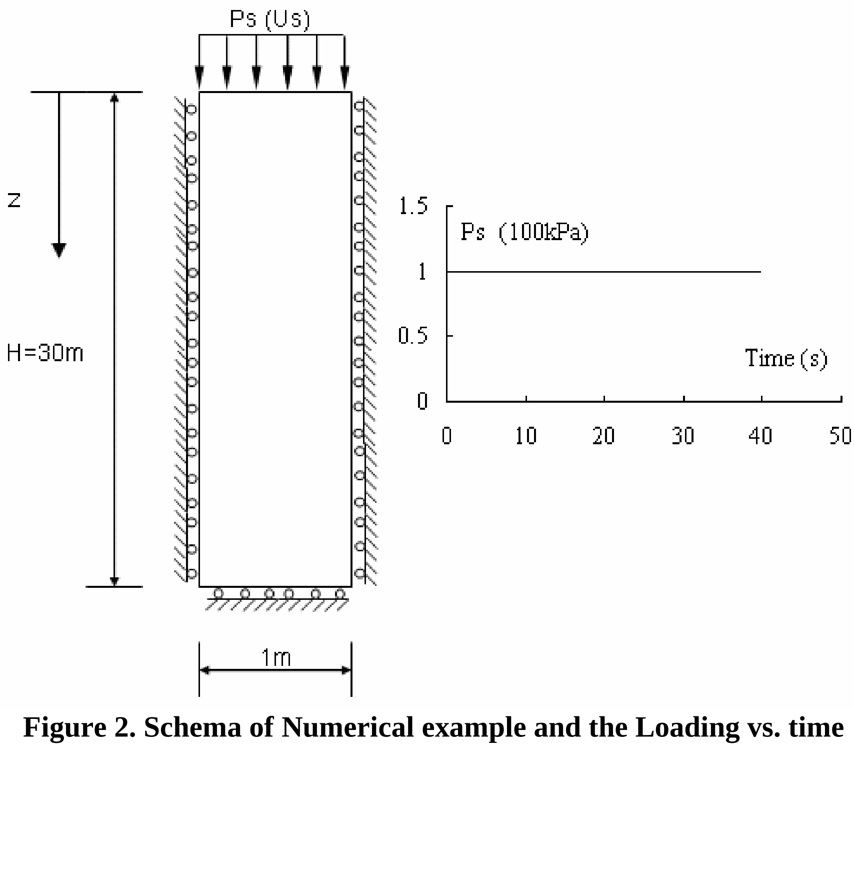


**Figure 2. Schema of Numerical example and the Loading vs. time**

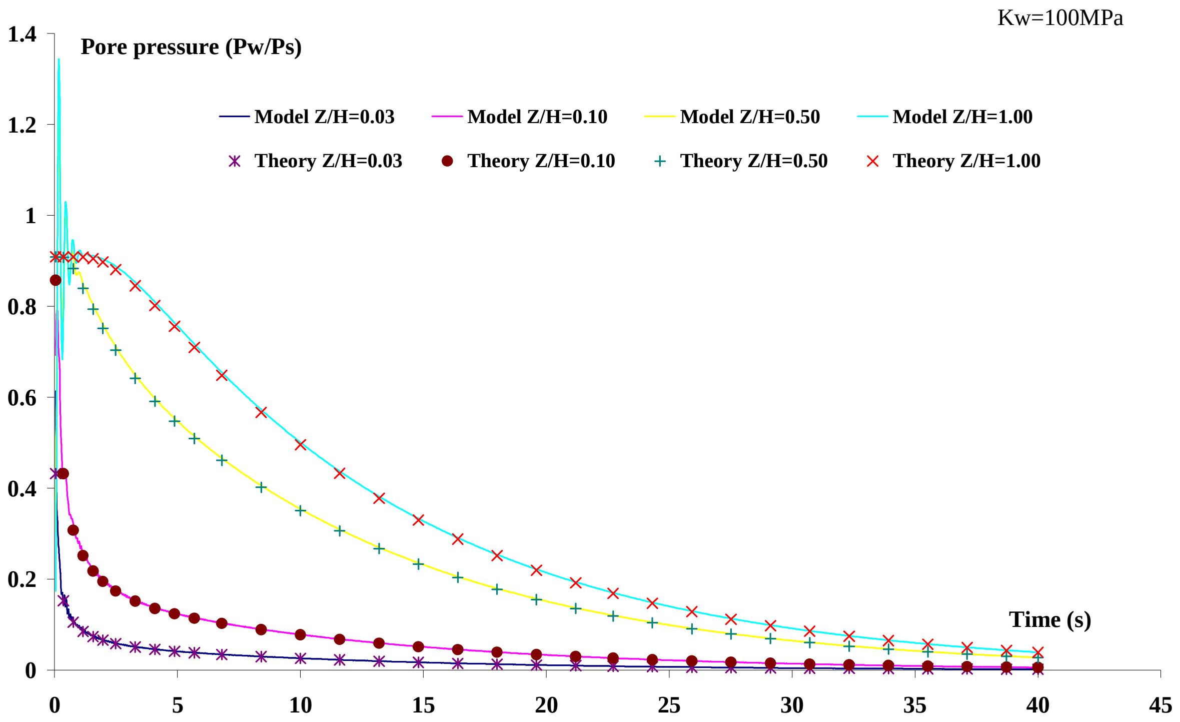


**Figure 3. Pore water pressure Vs time of saturated sandy soil**

Figure 3 presents numerical and the analytic results of the variation of pore water pressure at different depths vs. time for initial water saturation of 1.0. The pore pressure and the position (in Y direction) are normalized by the constant step pressure $P_s$ and the high of the soil column $H$, respectively. It can be observed that the numerical simulation agree well with the analytic results. The oscillation of numerical results at the beginning is due to the compressibility of the pore water and the influence of inertial effects. It is also of interest to indicate that although the problem is saturated; however the water with a small compressibility, about 100MPa, means that it includes some air; consequently this example concerns a partially saturated problem.

**Table 1. Material properties of numerical examples**

| | | |
|---|---|---|
| Young's Modulus | $E$ | 30000kPa |
| Poisson Ratio | $v$ | 0.2 |
| Mass density of solid | $\rho_s$ | 2000kg/m$^3$ |
| Mass density of fluid | $\rho_w$ | 1000kg/m$^3$ |
| Permeability | $K$ | 0.01m/s |
| Porosity | $\phi$ | 0.3 |
| Newmark parameter | $a$ | 0.5 |
| Newmark parameter | $b$ | 0.5 |

While for the unsaturated case, the same geometric and mechanical characteristics are used, only the initial water saturation is changed from saturated to 0.99. And the water compressibility of pure water is used ($K_w = 2000MPa$). Figure 4 presents the variation of pore water pressure at different depths vs. time for initial water saturation of 0.99. It can be observed that the water saturation has an important influence on both generation of pore pressure and its dissipation. A decrease of 1% in the degree of saturation reduces the pore pressure generation to 50%. The dissipation of excess pore pressure is slow compared to the saturated case. Figure 5 illustrates the distribution of water saturation at different time during the dynamic consolidation. It can be seen that, at first the base of the soil column, the water saturation is the highest, just like the distribution of excess pore water pressure; and the water saturation approaches the initial state. This is due to the assumption of zero air flux. However, from the two numerical examples, it can be seen that the water saturation has great influence on the generation of excess pore pressure, even with a little perturbation of water saturation (for example 1% variation). Consequently, it is necessary to consider the unsaturated condition in the practice.

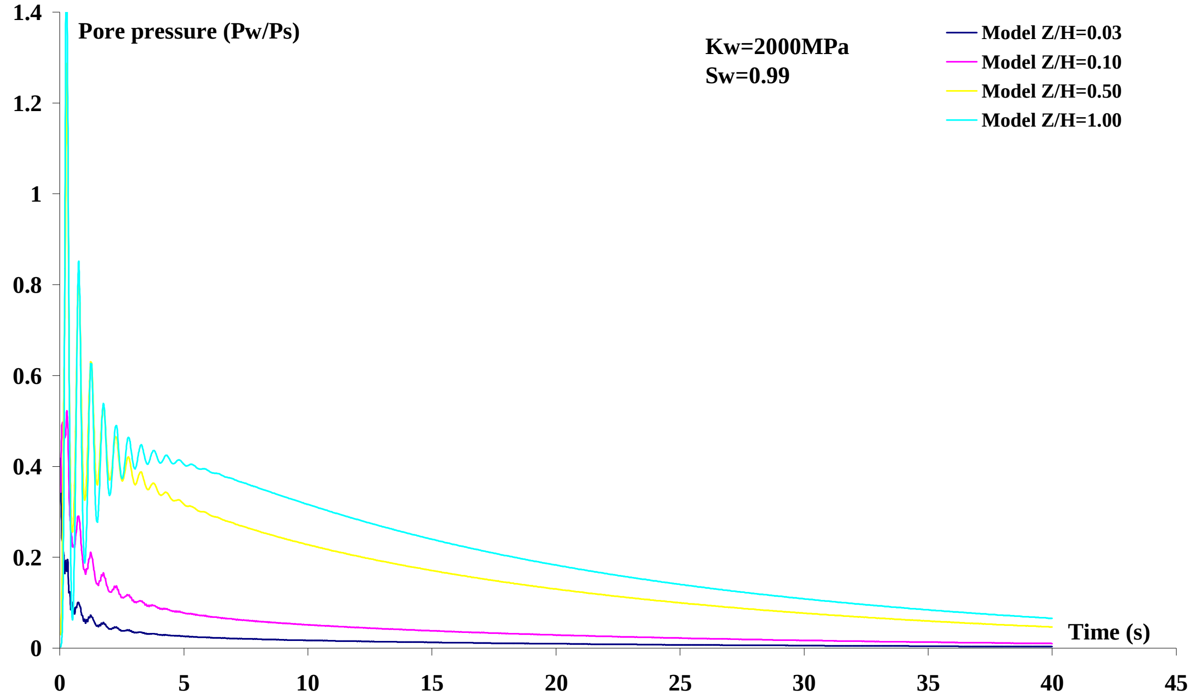


**Figure 4. Pore water pressure Vs time for unsaturated sandy soil**

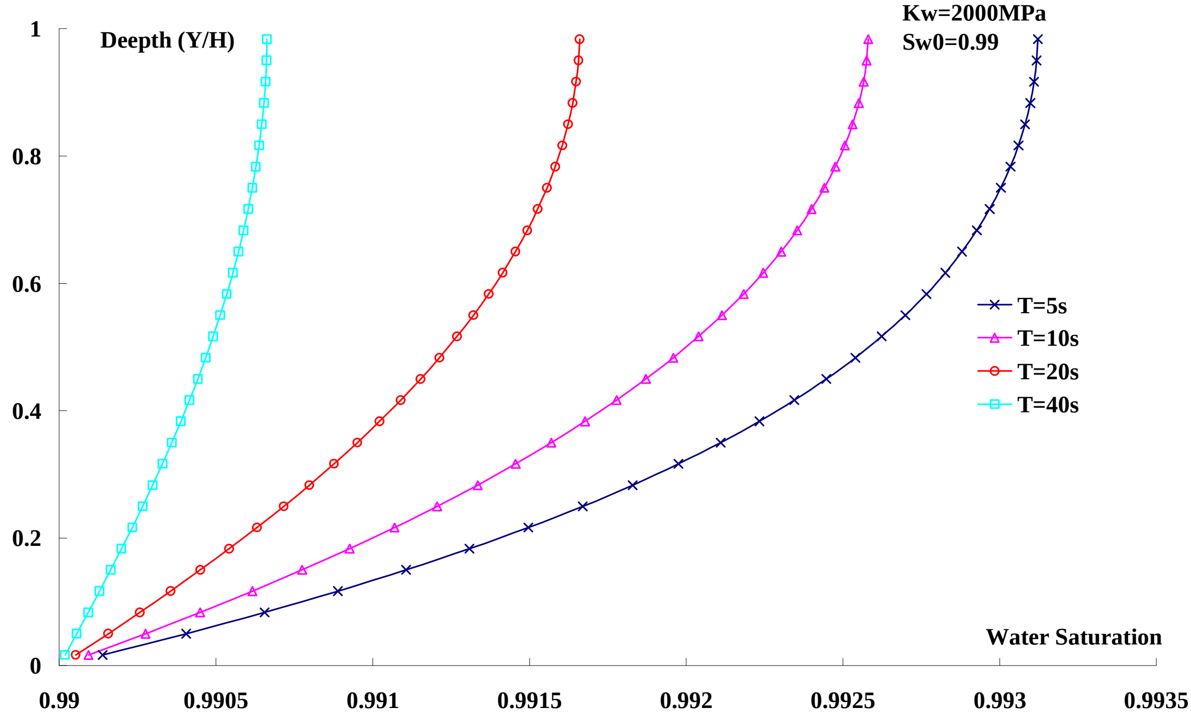


**Figure 5. The distribution of water saturation at different times**

## CONCLUSION

This paper included the presentation of a numerical model for the analysis of the response of unsaturated sandy soils to dynamic loading. The model is based on the theory of Biot and the formulation of Coussy together with the hypothesis of low suction in the sandy soils. The model was implemented in a finite element program. Tests carried assuming elastic behavior for the soil material showed that the soil saturation degree largely affects the dynamic response of soils. Work is under progress to implement a cyclic elastoplastic model in the finite element program to study the influence of the saturation on the soil liquefaction.

## AKNOWLEDGEMENTS

The measurement of soil water characteristic curve of Hostun sand in laboratory has been done in the Laboratory of Ashikaga Institute of Technology in Japan. The authors thank Tomoyoshi Nishimura for technical assistance.

## APPENDIX A

The matrixes of finite element method are listed here:

$$M_{ss} = \int_\Omega N^t \rho N d\Omega \qquad R = \int_\Omega B^t C_0 B d\Omega \qquad C_{up} = -\int_\Omega B^t mbN' d\Omega$$

$$F_u = \int_\Gamma N^t f_s d\Gamma \qquad C_{pp} = \int_\Omega N'^t \frac{1}{M} N' d\Omega \qquad C_{pu} = \int_\Omega N'^t mbBd\Omega$$

$$K_{pp} = \int_\Omega B'^t \frac{k}{\mu} B' d\Omega \qquad M_{pu} = \int_\Omega B'^t \rho_w \frac{k}{\mu} mNd\Omega \qquad F_p = -\int_\Gamma N' f_{flux} d\Gamma$$

By using the General Newmark method, the formulation will became:

$$\overline{R}\Delta U + \overline{C}_{up}\Delta P = \overline{F}_u$$
$$\overline{M}\Delta U + \overline{C}_{pp}\Delta P = \overline{F}_p$$

With

$$\overline{R} = M_{ss}\frac{2}{\Delta t^2} + bR; \overline{C}_{up} = bC_{up}$$

$$\overline{M} = M_{pu}\frac{2}{b\Delta t} + C_{pu}\frac{2a}{b}; \overline{C}_{pp} = C_{pp} + \Delta t K_{pp}$$

$$\overline{F}_u = bF^u_{t+\Delta t} - bC_{up}P_t + M_{ss}\left(\frac{2}{\Delta t}\dot{U}_t + (1-b)\ddot{U}_t\right) - bRU_t$$

$$\overline{F}_p = \Delta t F^p_{t+\Delta t} - \Delta t K_{pp} P_t + \left[M_{pu}\frac{2}{b} + \Delta t C_{pu}(\frac{2a}{b} - 1)\right]\dot{U}_t + \left\{\Delta t M_{pu}\frac{(1-b)}{b} + \Delta t^2 C_{pu}\frac{(a-b)}{b}\right\}\ddot{U}_t$$

Here $a$ and $b$ are Newmark coefficients.

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