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ANALYTICAL – NUMERICAL EXPERIMENTS FOR WAVE DISPERSION IN DRY SAND

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ABSTRACT

The phenomenon of wave dispersion in dry sand is studied both by purely analytical studies and by analytical – numerical experiments on the basis of gradient elastic and viscoelastic material models. These material models are employed in order to simulate the microstructural characteristics of dry sand. The analytical studies treat the material as a three-dimensional one and for both material models provide explicit expressions for the velocity of propagation of harmonic compressional (P) and shear (S) waves. These velocities are found to be functions of frequency, i.e., dispersive. The analytical – numerical studies treat the material as a one-dimensional one and try to simulate P and S wave propagation along the axial direction of cylindrical dry sand specimens. Thus, a sinusoidal pulse with a specific frequency is applied at one end of the specimen and the response is determined at some other point by solving a transient dynamic boundary value problem with the aid of numerical Laplace transform. This analytical – numerical experiment is repeated for various frequencies. Thus, one determines the velocities of P and S waves as functions of frequency, thereby proving again that wave propagation in dry sand is dispersive.

Keywords: Wave propagation; Gradient elasticity; Viscoelasticity; Compressional waves; Shear waves; Wave dispersion

INTRODUCTION

When the phase velocity of a wave propagating in a medium depends on frequency, the wave is called dispersive and the phenomenon dispersion. For a three-dimensional homogeneous and isotropic linear elastic material, the velocities of propagation of compression (P) and shear (S) waves are independent of frequency and depend only on the elastic moduli and the mass density of the material. This material is then called nondispersive. However, for materials with a distinct microstructure such as granular soil materials, microstructural effects are important and wave dispersion is observable.

On the basis of these relations between elastic moduli and dispersive or nondispersive velocities of wave propagation, various dynamic experimental methods for determining the small-strain elastic moduli of soil have been developed. A method of this category based on piezoelectric transducers has received considerable attention during the last decade or so. In some cases no wave dispersion has been observed (Rignoli et al, 1996; Lohani et al, 1999; Yamashita et al, 2001; Anh Dan et al, 2002), while in some others wave dispersion has indeed been noticed (Arulnathan et al, 1998; Greening & Nash, 2004; Donald & Butt, 2005). It is interesting to note that in some cases (Anh Dan et al, 2002) the compression and shear wave velocities are referred to as independent of frequency eventhough the measurements show the opposite, while in some others (Greening & Nash, 2004) the observed

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dispersion is attributed to the excitation of body modes of the specimen when, in reality, this is due to the material microstructure. Wave dispersion has been clearly measured in the works of other investigators dealing with the wave propagation of Rayleigh (R) waves in soil or rock media (Lai et al, 2002; Stavropoulou et al, 2003; Tokeshi et al, 2006).

In this work an attempt is made to prove by analytical studies and analytical-numerical experiments that there is wave dispersion during the compressional and shear wave propagation in granular soil (dry sand) specimens when one tries to determine the elastic and shear moduli by means of piezoelectric transducers. In the present studies the material behavior is assumed to be linear viscoelastic or gradient elastic. This material modeling is capable of taking into account microstructural effects, which are essential in realistically describing the behavior of granular soils, such as dry san (Chang & Gao, 1997; Tsepoura et al, 2002; Papargyri-Beskou et al, 2003; Stavropoulou et al, 2003; Papargyri-Beskou & Beskos, 2004; Wichtmann et al, 2005; Papargyri-Beskou, 2005).

More specifically, the analytical studies treat the material as a three-dimensional one and for both material models provide explicit expressions for the velocity of propagation of harmonic compressional (P) and shear (S) waves, which is found to be frequency dependent, i.e., dispersive. The analytical-numerical studies treat the material as a one-dimensional one and try to simulate P and S wave propagation along the axial direction of cylindrical dry sand specimens. Thus, a sinusoidal pulse with a specific frequency is applied at one end of the specimen and the response is determined at some other point by solving a transient dynamic boundary value problem with the aid of numerical Laplace transform. This analytical-numerical experiment is repeated for various frequencies. Thus, one determines the velocities of P and S waves as functions of frequency, thereby providing again that wave propagation in dry sand is dispersive. The above analytical and analytical-numerical studies serve to prove dispersion in wave propagation in dry san and help to resolve the existing controversy among experimentalists.

ANALYTICAL STUDIES

Consider first the case of the propagation of plane compressional waves along the x direction in a three-dimensional linear viscoelastic medium. Then the equation of motion is (Christensen, 1971)

$$2\sigma_{,x} = \rho \ddot{u} \tag{1}$$

where ρ is the mass density, u the displacement along the x direction and σ the stress along x given by

$$\sigma = \int_{-\infty}^{t} \left[\lambda(t-\tau) + 2\mu(t-\tau)\right] \dot{u}_{,x}\, d\tau \tag{2}$$

In the above, commas indicate differentiation with respect to x, overdots differentiation with respect to time t or τ, while $\lambda(t)$ and $\mu(t)$ are the time dependent Lamé constants.

Assuming harmonic waves of the form

$$u(x,t) = \bar{u}\, e^{i(n_p x + \omega t)} \tag{3}$$

where $\bar{u}$ is amplitude, ω the circular frequency and n_p a parameter related to phase velocity and attenuation of P waves, Eqs (1) and (2) become (Christensen, 1971)

$$1/n_p^2 = \left(1/\rho\omega^2\right) \int_{-\infty}^{t} M(t-\tau)\, de^{-i\omega(t-\tau)} \tag{4}$$

where $M(t) = \lambda(t) + 2\mu(t)$.

Thus, on the assumption that $M(t) = \sum_{j=1}^{9} M_j e^{-t/t_j} H(t)$, where H(t) is the unit step function and M_j and t_j material constants, Eq. (4) becomes

$$\frac{\omega^2}{n_p^2} = \frac{2}{\rho} M(0) - \frac{1}{\rho} \sum_{j=1}^{9} \frac{M_j}{1+\omega^2 t_j^2} + \frac{i\omega}{\rho} \sum_{j=1}^{9} \frac{M_j t_j}{1+\omega^2 t_j^2} \tag{5}$$

or symbolically as

$$n_p = n_{p1} + i\, n_{p2} \tag{6}$$

where n_{p1} and n_{p2} are real quantities corresponding to dispersion and attenuation, respectively, of the propagating wave. The phase velocity V_p can then be calculated from

$$V_p = \omega / n_{p1} \tag{7}$$

For the case of shear waves, one has simply to repeat the above procedure with n_s in place of n_p, M(t)=$\mu(t)$ and obtain the phase velocity $V_s = \omega / n_{s1}$.

According to the simplified dipolar gradient theory of elasticity of Mindlin (Mindlin, 1964; Georgiadis et al, 2004; Polyzos, 2005), one can have in indicial notation the governing equation of motion of a linear dipolar gradient elastic body as

$$\mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + g^2 \left(\mu u_{i,jj} + (\lambda+\mu) u_{j,ji}\right)_{,kk} = \rho \ddot{u}_i - \rho h^2 \ddot{u}_{,kk} \tag{8}$$

where λ and μ are the classical Lamé constants, g and h are the gradient coefficients of volumetric strain energy and velocity, respectively and i, j and k receive the values 1, 2, 3. Assuming harmonic waves of the form

$$u = \nabla \phi + \nabla x A \tag{9}$$

$$\begin{aligned} \nabla \phi &= m e^{-i[(\omega / c_p) m \cdot x + \omega t]} \\ \nabla x A &= d\, e^{-i[(\omega / c_s) m \cdot x + \omega t]} \end{aligned} \tag{10}$$

where $c_p^2 = (\lambda + 2\mu)/\rho$ and $c_s^2 = \mu / \rho$ are the compressional and shear wave propagation velocities in a homogeneous and isotropic linear elastic body and **m**, **d** and **x** are directional vectors, one can prove that both longitudinal and shear waves are here dispersive and the $V_{p,s}$ versus $f = \omega / 2\pi$ relation takes the form (Aggelis et al, 2004; Papargyri-Beskou, 2005)

$$V_{p,s}^2 = 8\pi^2 c_{p,s} g^2 f^2 / \left[-\left(c_{p,s}^2 - 4\pi^2 h^2 f^2\right) + \sqrt{\left(c_{p,s}^2 - 4\pi^2 h^2 f^2\right)^2 + 16\pi^2 c_{p,s}^2 g^2 f^2} \right] \tag{11}$$

The above two material models were applied to the case of dry sand with grains of mean radius r_g=0.275mm, volume fraction V_g=0.6, Lamé constants $\lambda_g = 3050\,\text{MPa}$ and $\mu_g = 558\,\text{MPa}$ and mass density $\rho_g = 2633\,\mathrm{Kg/m^3}$. For this mixture one can obtain the effective values $\mu_e = 137\,\mathrm{MPa}$, $\lambda_e = 120\,\mathrm{MPa}$ and $\rho_e = \rho_g V_g = 1580\ \mathrm{Kg/m^3}$. For the linear viscoelastic model it was assumed that $M(t) = \sum_{j=1}^{9} M_j e^{-(t/t_j)} H(t)$, with $\nu = 0.233$ and $M_j = \lambda_j + 2\mu_j$ with $\lambda_1 = 16.98, \lambda_2 = 33.87, \lambda_3 = 18.28, \lambda_4 = 16.78, \lambda_5 = 13.32$, $\lambda_6 = 10.39, \lambda_7 = 5.23$, $\lambda_8 = 4.044$, $\lambda_9 = 0.679\,\mathrm{MPa}$, $\mu_1 = 19.46$, $\mu_2 = 38.82$, $\mu_3 = 20.92$, $\mu_4 = 19.24$, $\mu_5 = 15.63$, $\mu_6 = 11.91$, $\mu_7 = 5.99$, $\mu_8 = 4.63$, $\mu_9 = 0.778\,\mathrm{MPa}$ and $t_1 = 0.0$, $t_2 = 1.5\times10^{-10}$, $t_3 = 1.5\times10^{-9}$, $t_4 = 1.5\times10^{-8}$, $t_5 = 1.5\times10^{-7}$, $t_6 = 1.5\times10^{-6}$, $t_7 = 1.5\times10^{-5}$, $t_8 = 1.5\times10^{-4}$, $t_9 = 1.5\times10^{-3}$ secs. The dipolar gradient elastic model was used with material constants $g_1 = 55\times10^{-5}\,\mathrm{m}$ and $h_1 = g_1/1.034\,\mathrm{m}$ as well as $g_2 = 16.5\times10^{-5}\,\mathrm{m}$ and $h_2 = g_2/1.030\,\mathrm{m}$.

Figures 1 and 2 depict the phase velocity versus frequency relations for the above two models and the case of P and S waves, respectively. One can observe that the classical elasticity results (c_p=499.20 m/s, c_s=294.46 m/s) for both types of waves (P and S) show no dispersion, as expected.

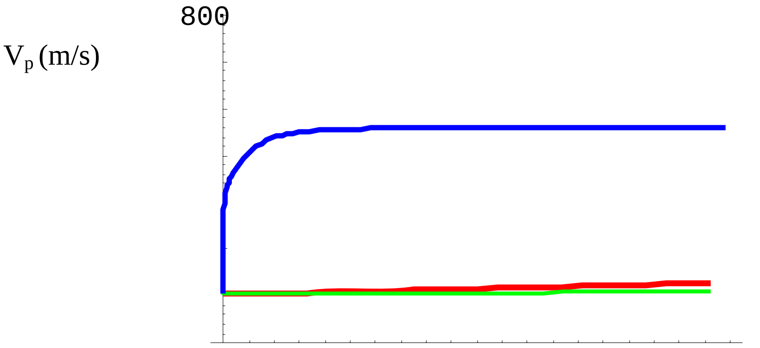

Figure 1. P – wave velocity V_p versus frequency f for viscoelastic (v) and gradient elastic (g_1,g_2) material models

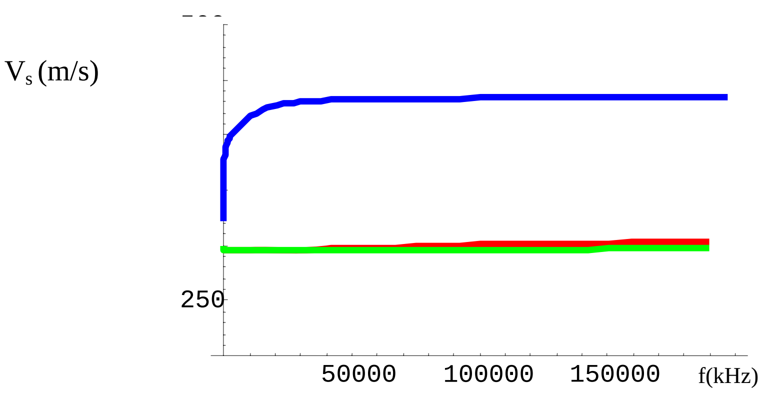

Figure 2. S – wave velocity V_s versus frequency f for viscoelastic *(v)* and gradient elastic *(g_1,g_2)* material models

The gradient elasticity results are close to the classical ones and indicate a very small dispersion, which however, shows a slight increase for high frequencies. The behavior of the classical viscoelastic model also shows dispersion, which is more pronounced for high frequencies than the one for the gradient elasticity model.

ANALYTICAL - NUMERICAL EXPERIMENTS

In this section analytical-numerical experiments are performed in an effort to simulate P and S wave propagation along the axial direction of cylindrical dry sand specimens. This wave propagation in real experiments is realized with the aid of piezoelectric transducers at the two ends of the cylindrical specimen with one of them being a wave generator and the other a receiver. Thus, a P or S wave is generated at one end of the specimen, propagates along its length and is received at the other end. Measuring the travel time it takes for the wave to propagate along the specimen on the basis of the received signal, one is able to calculate the wave phase velocity. This is usually done for a number of generated sinusoidal pulses each one characterized by its specific frequency. Thus, velocity values are determined for a sequence of pulses (frequencies) and the possibility of the velocity to be frequency dependent, i.e., the wave to be dispersive, is investigated.

Consider first the case of the linear viscoelastic one-dimensional medium for which the governing equations of P and S wave propagation have respectively, the form

$$\begin{aligned} E * u_{,xx} &= \rho \ddot{u} \\ G * u_{,xx} &= \rho \ddot{u} \end{aligned} \tag{12}$$

where u=u(x,t), E=E(t) and G=G(t) are the time dependent moduli of elasticity and the $*$ denotes time convolution. This one-dimensional medium is assumed to be fixed at one end (x=0) and subjected to a sinusoidally varying with time displacement pulse at the other end (x=L), i.e.,

$$u(0,t) = 0, \qquad u(L,t) = \alpha \sin \omega t \tag{13}$$

where α and ω are the amplitude and the characteristic circular frequency of the pulse.

Applying Laplace transform with respect to time under zero initial conditions onto Eqs (12) and invoking the viscoelastic-elastic correspondence principle (Christensen 1971) one obtains the representative equation

$$s\overline{Q}(s)\overline{u}_{,xx} = \rho s^2 \overline{u} \tag{14}$$

where s is the Laplace transform parameter, an overbar denotes transformed quantities, $\overline{Q}(s)$ stands for either $\overline{E}(s)$ or $\overline{G}(s)$, i.e., the Laplace transformed moduli and $\overline{u} = \overline{u}(x,s)$. Application of Laplace onto Eqs (13) results in the transformed boundary conditions of the form

$$\overline{u}(0,s) = 0, \quad \overline{u}(L,s) = \alpha\, \omega \frac{1 - e^{-2\pi s/\omega}}{s^2 + \omega^2} = u_o \tag{15}$$

Solving Eq. (14) one obtains

$$\overline{u}(x,s) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{16}$$

where

$$\lambda_{1,2} = \pm\sqrt{\rho s / \overline{Q}(s)} \tag{17}$$

and the constants of integration c_1 and c_2, in view of the boundary conditions (15), can be explicitly expressed as

$$c_1 = -c_2 = -u_o / \left(e^{\lambda_2 L} - e^{\lambda_1 L}\right) \tag{18}$$

The Laplace transformed solution $\bar{u}$ of Eq. (16) is obtained at x=L/2 and is numerically inverted with the aid of the algorithm of Durbin (1974) to obtain the response u(L/2, t), where t should be less than the time required for the wave to travel a distance less than L in order to avoid wave reflection at the fixed end.

Consider next the case of the linear dipolar gradient elastic one-dimensional medium. For this medium the governing equations of P and S wave propagation, as obtained from Eqs (8), have respectively, the form

$$\begin{aligned} E\left(u_{,xx} - g^2 u_{,xxxx}\right) &= \rho\ddot{u} - \rho h^2 \ddot{u}_{,xx} \\ G\left(u_{,xx} - g^2 u_{,xxxx}\right) &= \rho\ddot{u} - \rho h^2 \ddot{u}_{,xx} \end{aligned} \tag{19}$$

The classical boundary conditions of the problem are again Eqs (13). However, in gradient elasticity two more (non-classical) boundary conditions are necessary. Following Tsepoura et al (2002), these conditions read

$$u(L,t)_{,x} = 0, \qquad g^2 u(0,t)_{,xx} = 0 \tag{20}$$

Introducing constant Q to represent either E or G and applying Laplace transform with respect to time under zero initial conditions onto (19), (13) and (20) one receives

$$Q\left(\bar{u}_{,xx} - g^2 u_{,xxx}\right) = \rho s^2 \bar{u} - \rho h^2 s^2 \bar{u}_{,xx} \tag{21}$$

$$\begin{aligned} &\bar{u}(0,s) = 0, \qquad \bar{u}(L,s) = u_o \\ &\bar{u}(L,s)_{,x} = 0, \quad \bar{u}(0,s)_{,xx} = 0 \end{aligned} \tag{22}$$

Solution of Eq. (21) under boundary conditions (22) leads to

$$\bar{u}(x,s) = \sum_{i=1}^{4} c_i e^{q_i x} \tag{23}$$

where

$$q_{1,2,3,4} = \pm\sqrt{\frac{E + \rho h^2 s^2}{2Eg^2} \pm \sqrt{\frac{E + \rho h^2 s^2}{2Eg^2} - \frac{\rho s^2}{Eg^2}}} \tag{24}$$

and the constants c_i are obtained by menas of the boundary conditions (22) numerically. Numerical inversion of the transformed solution $\bar{u}(x,s)$ with the aid of the algorithm of Durbin (1974) provides the time domain response u(L/2, t), where t should be less than the time required for the wave to travel a distance less than L in order to avoid wave reflection at the fixed end.

Figures 3 and 4 depict the received response at x=L/2 for the case of the viscoelastic and the dipolar gradient elastic material behaviour, respectively, to a sinusoidal P wave pulse with

$\alpha = 3.015 \times 10^{-9}\,\mathrm{m}$ and $f = \omega / 2\pi = 20\,\mathrm{kHz}$ at x=L=0.1 m as obtained on the basis of the material data of the previous section. From these figures it is easy to determine the P wave travel time between points x=L and x=L/2 for the two material models. Thus, one can find $t_v = 1.08 \times 10^{-4}\,\mathrm{sec}$ and t_g = $1.03 \times 10^{-4}\,\mathrm{sec}$ from Figs 3 and 4, respectively and hence compute the corresponding P-wave velocities as $V_p = 462.96\,\mathrm{m/s}$ and $V_p = 485.00\,\mathrm{m/s}$ for the viscoelastic and the dipolar gradient elastic (with g_1 and h_1) cases, respectively. Repeating the same process for a sequence of values of pulse frequencies, one can obtain the velocity versus frequency relations. Figures 5 and 6 show the P wave velocity versus frequency relations for the cases of viscoelastic and dipolar gradient elastic cases, respectively. It is observed that for both models the velocity depends on frequency, thereby establishing the existence of wave dispersion. It is noteworthy the fact that at zero frequency one recovers the value of the wave velocity obtained on the basis of the classical theory of elasticity (c_p=462.50 m/s).

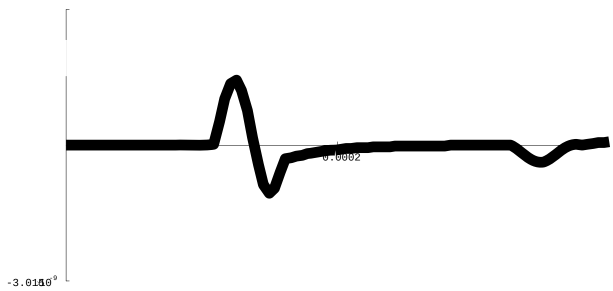

Figure 3. P – wave displacement versus time *t* at *x=L/2* due to a sinusoidal pulse at *x=L* in a 1-D viscoelastic medium

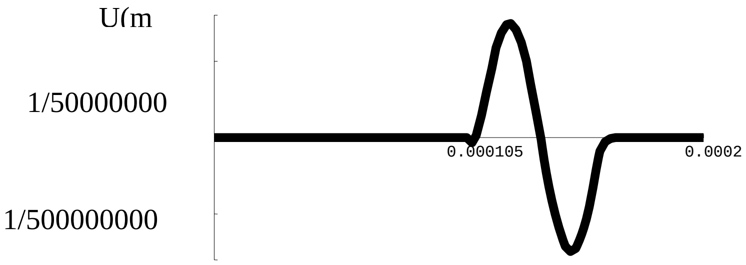

Figure 4. P – wave displacement versus time *t* at *x=L/2* due to a sinusoidal pulse at *x=L* in a 1-D gradient elastic medium

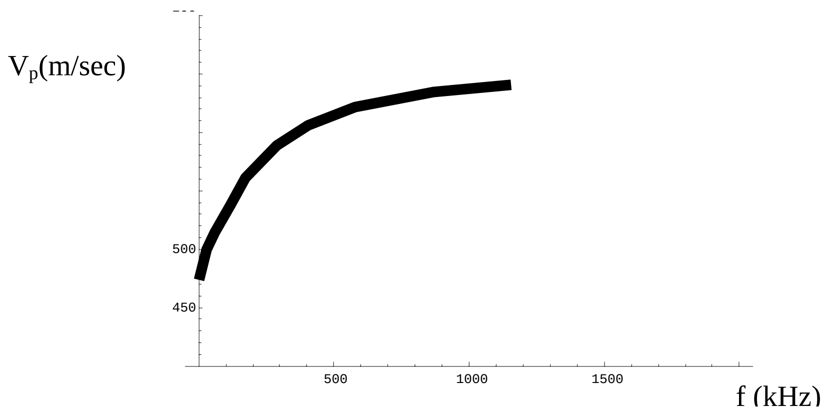

Figure 5. P – wave velocity *V_p* versus frequency f for 1-D viscoelastic medium

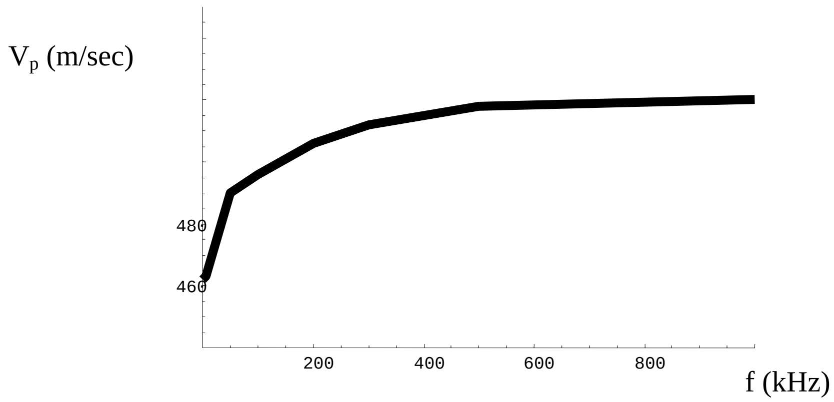

Figure 6. P – wave velocity *V_p* versus frequency f for 1-D gradient elastic medium

Figures 7 and 8 show the velocity versus frequency relations for S-waves propagating in viscoelastic and gradient elastic, respectively, one-dimensional media. The frequency dependence of the S-wave velocity is apparent, thereby again demonstrating wave dispersion. For zero frequency one again recovers the classical elasticity wave velocity (c_s=294.46 m/s).

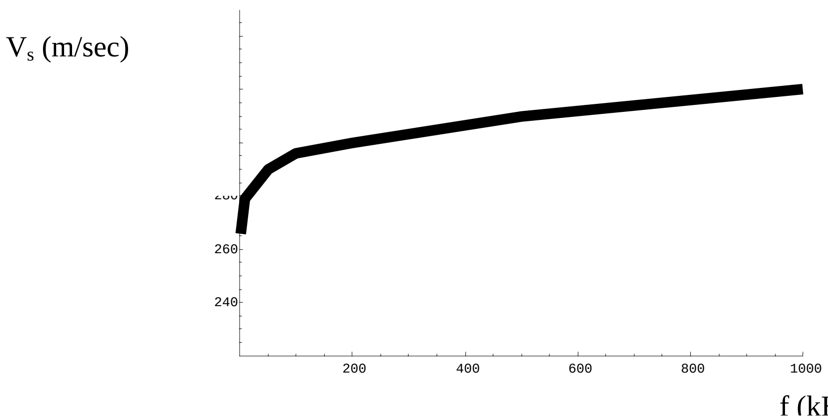

Figure 7. S – wave velocity *V_s* versus frequency f for 1-D viscoelastic medium

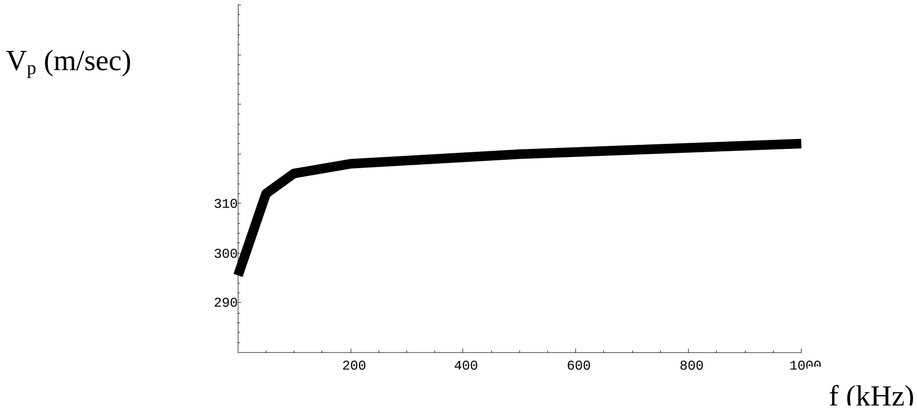

Figure 8. S – wave velocity V_s versus frequency f for 1-D gradient elastic medium

CONCLUSIONS

Purely analytical studies and analytical-numerical experiments on the basis of linear viscoelastic and gradient elastic material models have demonstrated the existence of wave dispersion during P and S wave propagation in cylindrical specimens of dry sand. It is hoped that these results will help to resolve the existing controversy among experimentalists about the existence or not of wave dispersion during wave propagation in dry sand specimens.

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REFERENCES

Aggelis, D.G., Philippidis, T.P., Tsinopoulos, S.V., and Polyzos, D., "Wave dispersion in concrete due to microstructure," CD ROM Proc. of 2004 International Conference on Computational & Experimental Engineering & Sciences, Madeira, Portugal, S.N. Atluri (ed.), Tech. Science Press. 2004.

AnhDan LQ, Koseki J and Sato T. "Comparison of young's moduli of dense sand and gravel measured by dynamic and static methods," Geotechnical Testing Journal, 25(4): 1-20, 2002.

Arulnathan R, Boulanger RW, and Riemer MF. "Analysis of bender element tests," Geotechnical Testing Journal, 21: 120-131,1998.

Chang CS and Gao J. "Wave propagation in granular rod using high-gradient theory," Journal of Engineering Mechanics, ASCE, 123: 52-59, 1997.

Christensen RM. Theory of Viscoelasticity, Academic Press, New York, 1971.

Donald JA and Butt SD. "Experimental technique for measuring phase velocities during triaxial compression tests," International Journal of Rock Mechanics and Mining Sciences, 42: 307-314, 2005.

Durbin F. "Numerical inversion of Laplace transform: an efficient improvement to Dubner and Abate's method," Computer Journal, 17: 371-376, 1974.

Georgiadis HG, Vardoulakis I and Velgaki EG. "Dispersive Rayleigh – wave propagation in microstructured solids characterized by dipolar gradient elasticity," Journal of Elasticity, 74: 17-45, 2004.

Greening PD and Nash DFT. "Frequency domain determination of G_0 using bender elements," Geotechnical Testing Journal, 27(3): 1-7, 2004.

Lai CG, Rix GJ, Foti S and Roma V. "Simultaneous measurement and inversion of surface wave dispersion and attenuation curves," Soil Dynamics and Earthquake Engineering, 22: 923-930, 2002.

Lohani, T.N., Imai, G., and Shibuya, S., "Determination of shear wave velocity in bender element test," Proc. of 2nd International Conference on Earthquake Geotechnical Engineering, S. Pinto (ed.), A.A. Balkema, Rotterdam, 1: 101-106, 1999.

Mindlin RD. Microstructure in Linear Elasticity, Archive of Rational Mechanics and Analysis, 16: 51-78, 1964.

Papargyri-Beskou, S., "Comparison of four constitutive models for wave propagation in dry sand," CD ROM Proc. of the International Conference on Computational & Experimental Engineering & Sciences, Chennai (Madras), India, S.N. Atluri (ed.), Tech. Science Press, 2005.

Papargyri-Beskou S and Beskos DE. "Response of gradient viscoelastic bar to static and dynamic axial load," Acta Mechanica, 170: 199-212, 2004.

Papargyri-Beskou S, Polyzos D and Beskos DE. "Dynamic analysis of gradient elastic flexural beams," Structural Engineering and Mechanics, 15: 705-716, 2003.

Polyzos D. "3D frequency domain BEM for solving dipolar gradient elastic problems," Computational Mechanics, 35: 292-304, 2005.

Rignoli EGM, Gotti M and Stokoe KH II. "Measurement of shear waves in laboratory specimens by means of piezoelectric tranducers," Geotechnical Testing Journal, 19: 384-397, 1996.

Stavropoulou M, Exadaktylos G, Papamichos E, Larsen I and Ringstad C. "Rayleigh wave propagation in intact and damaged geomaterials," International Journal of Rock Mechanics and Mining Sciences, 40: 377-387, 2003.

Tokeshi JC, Karkee MB and Sugimura Y. "Reliability of Rayleigh wave dispersion curve obtained from f-k spectral analysis of microtremor array measurement," Soil Dynamics and Earthquake Engineering, 26: 163-174, 2006.

Tsepoura KG, Papargyri-Beskou S, Polyzos D and Beskos DE. "Static and dynamic analysis of gradient elastic bars in tension," Archive of Applied Mechanics, 72: 483-497, 2002.

Yamashita, S., and Suzuki, T., "Small strain stiffness on anisotropic consolidated state of sands by bender elements and cyclic loading tests," Proc. of 15th International Conference on Soil Mechanics & Geotechnical Engineering, Istanbul, A.A. Balkema, Lisse, 1: 325-328, 2001.

Wichtmann, T., Sellountos, E.I., Tsinopoulos, S.V., Niemunis, A., Papargyri-Beskou, S., Polyzos, D., Triantafyllidis, T., and Beskos, D.E., "Wave dispersion in dry sand by experimental, analytical and numerical methods," Proc. of 5th GRACM International Congress on Computational Mechanics, Limassol, Cyprus, Georgiou G., Papanastasiou P., Papadrakakis M. (eds), University of Cyprus, Nicosia, 487-493, 2005.