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A CONSISTENT METHODOLOGY FOR THE IMPOSITION OF TIME VARYING BOUNDARY CONDITIONS IN SOIL DYNAMICS: THEORY AND APPLICATION

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ABSTRACT

The simulation of geotechnical structures within the context of soil dynamics requires an efficient, reliable and consistent methodology for the imposition of time varying boundary conditions. In the present paper a recently proposed methodology (Paraskevopoulos *et al*, 2006) is utilized to overcome drawbacks and inconsistencies introduced by frequently used techniques (e.g. quasi-static approaches) and is applied on a typical geotechnical structure subjected to dynamic loading.

This methodology, avoiding ad hoc procedures, is solely based on a weak form of the dynamic problem (generalized Hamilton's principle), employs the appropriate norm to the functional space where the weak formulation is posed and enforces the boundary conditions using the penalty method.

The above proposed methodology is firstly validated by a typical case of a time harmonic *p*-wave propagation in an elastic half-space; then it is utilized in order to estimate the optimum computational domain size embodying a tunnel structure, with respect to the expected accuracy and the computational cost, given the soil properties, the structure's geometrical data and the free-field response.

Keywords: Time-dependent boundary conditions, fem, soil dynamics, half-space, cavity.

INTRODUCTION

Problems of great interest in earthquake geotechnical engineering involve dynamic behavior of geotechnical structures such as tunnels and pipelines. These problems consist of structures embedded in a semi-infinite medium and the dynamic loading is usually given in terms of displacements and/or their time derivative fields. However within the context of the Finite Element Method is typically required that the computational domain is of finite extend while the dynamic loading degenerates to the appropriate time-dependent constraints, imposed on the boundaries of the finite element domain. Motivated by the above, the authors in the present work, utilize a methodology that involves a family of algorithms, based on a weak form of the dynamic problem (generalized Hamilton's principle).

In what follows the numerical treatment of time-dependent boundary conditions is presented, the validity of the proposed methodology is examined and finally a typical problem in earthquake geotechnical engineering is considered. The problem under consideration is that of determining the optimum computational domain size embodying a tunnel structure, with respect to the expected accuracy and the computational cost, given that the dynamic loading will be applied as time-dependent boundary conditions.

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NUMERICAL TREATMENT OF TIME DEPENDENT BOUNDARY CONDITIONS

Recently a consistent, rigorous and suitable within the context of Finite Element Method approach of imposing time-dependent boundary conditions has been proposed (Paraskevopoulos *et al*, 2006). It is based on a weak form of the dynamic problem with the boundary conditions enforced by the penalty or Lagrange multipliers method and applies to linear as well nonlinear problems, while it avoids the decomposition of the unknown vectors and the requirement of the superposition principle. It is also shown that the well known and frequently used technique of *Large Mass* may be derived from these considerations; furthermore plenty of algorithms may be stated after employing any time discretization scheme in the weak formulation of the problem. Finally in a subsequent work it will be shown that there is the possibility to adopt independent displacement and velocity fields; this approximation can be used to construct unconditionally stable algorithms in time.

As a starting point, the weak formulation of the dynamic problem of an elastic domain Ω is stated as:

Find

$$\left\{\mathbf{u},\mathbf{v},\mathbf{p}\right\}\in U\times V\times V \quad U=\left\{\mathbf{u}\in L_2(I,H^1(\Omega)):\dot{\mathbf{u}}\in L_2(I,H^1(\Omega))\right\},V=\left\{\mathbf{v}\in L_2(I,L_2(\Omega))\right\}, \qquad (1)$$

where (Reddy, 1986) $L_2(\Omega)$ is the space of square (Lebesque-)integrable functions in Ω, H^1 (*Sobolev* space of order 1) is the $L_2(\Omega)$ space together with all their weak first derivates and I is the time interval of interest, satisfying the relationship

$$\int_0^T\left(\iint_\Omega -\rho\mathbf{v}\cdot\delta\mathbf{v}d\Omega+\sum_{i=1}^m\iint_\Omega \mathbf{s}^i\cdot\frac{\partial\mathbf{w}}{\partial x_i}d\Omega-\int_{\Gamma_t}\tilde{\mathbf{t}}\cdot\mathbf{w}d\Omega-\int_{\Gamma_t}\mathbf{f}\cdot\mathbf{w}d\Omega\right)dt$$

$$+\int_0^T\left(\iint_\Omega\left(\rho(\delta\mathbf{v}-\frac{\partial\mathbf{w}}{\partial t})+\delta\mathbf{p}(\mathbf{v}-\frac{\partial\mathbf{u}}{\partial t})\right)d\Omega\right)dt+\iint_\Omega p\overline{\mathbf{v}}\cdot\mathbf{w}_{(t=0)}d\Omega=0$$

$$\forall\left\{\mathbf{w},\delta\mathbf{v},\delta\mathbf{p}\right\}\in W\times V\times V,$$

$$W=\left\{\mathbf{w}\in L_2(I,H^1(\Omega)):\dot{\mathbf{w}}\in L_2(I,H^1(\Omega)),\mathbf{w}(T)=\mathbf{0},\mathbf{w}=\mathbf{0}\in\Gamma_u\right\} \qquad (2)$$

with

$$\mathbf{u}(\mathbf{x},t)=\overline{\mathbf{u}}(\mathbf{x},t)\text{ or }\frac{\partial\mathbf{u}}{\partial t}(\mathbf{x},t)=\frac{\partial\overline{\mathbf{u}}}{\partial t}(\mathbf{x},t)\quad \textit{for}\quad (\mathbf{x},t)\in\Gamma_{\mathrm{u}}\times(0,T]$$

and

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_\mathbf{0}(\mathbf{x}),\ \frac{\partial\mathbf{u}}{\partial t}(\mathbf{x},0)=\frac{\partial\mathbf{u}_0}{\partial t}(\mathbf{x})\ \ \mathbf{x}\in\Omega$$

In the above, vector **x** and scalar t denote spatial coordinates and time respectively. The vectors **u**, $\mathbf{s_i}$ denote the displacements and stresses and the vectors **f**, $\mathbf{u_0}$ stand for prescribed functions. The vectors $\overline{\mathbf{u}}(x,t),\overline{\mathbf{v}}(x,t)$ and $\overline{\mathbf{t}}(x,t)$ denote time-dependent prescribed boundary conditions on the parts $\mathbf{\Gamma_u}$ and $\mathbf{\Gamma_t}$ of the boundary **Γ**. Furthermore, ρ denotes mass density and m equals the spatial dimensions of the problem. Finally, n_i denotes the component of the outward unit normal to $\mathbf{\Gamma_t}$.

Moreover the aforementioned system of equations is augmented by the constitutive equations and the flow rule in the presence of material nonlinearities. The constraint between the time derivative of displacements $\frac{\partial\mathbf{u}}{\partial t}$ and the weak velocities **v** is enforced in $L_2(\Omega)$ while the essential boundary conditions refer to the displacement variables $\mathbf{u}$ and their strong derivatives $\frac{\partial \mathbf{u}}{\partial t}$. The imposition of constraints can be achieved either through the penalty or the Lagrange multipliers method in an appropriate way which complies with the norm and the inner product of the functional space where the weak form has been posed (Ženišek, 1990).

In this work the authors utilize a member of the above family of algorithms which requires only velocities as input time-dependent boundary conditions and presents unconditional stability, which is very important for systems dealing with a large amount of degrees of freedom, such as problems in earthquake geotechnical engineering. The algorithm was implemented in *nemesis* (nemesis, 2006), a general purpose, experimental finite element code.

VALIDATION

In order to verify the aforementioned algorithm a simple test case is examined. Consider a homogeneous elastic half-space where an incident time harmonic *p*-wave propagates, having an angle θ_0 w.r.t. to the *y*-axis (Figure 1(a)). The total displacement field consists of the superposition of the incident wave and the corresponding reflected field due to the free surface (Achenbach, 1990). The incident as well the reflected waves may be denoted by

$$\mathbf{u}^{(n)} = A_n \mathbf{d}^{(n)} \exp(i\eta_n), \tag{3}$$

where different values of index n serve to label the various types of waves that occur when a longitudinal wave is reflected on the free surface, and η_n is given as

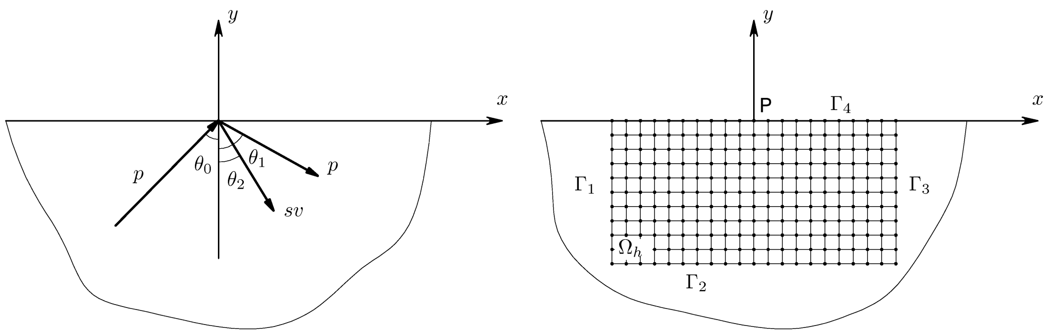
$$\eta_n = k_n (x p_x^{(n)} + y p_y^{(n)} - c_n t), \tag{4}$$

where k_n, $p^{(n)}$ and c_n stand for wave-number, propagation direction and wave velocity respectively. Sub/superscript n is assigned with the values 0, 1 and 2 for the *p*-incident, *p*-reflected and *sv*-reflected waves correspondingly, and $\mathbf{p}^{(n)}$ is given in terms of the propagation angle θ_n as follows:

$$\mathbf{p}^{(n)} = \sin\theta_n \hat{i}_x + \cos\theta_n \hat{i}_y . \tag{5}$$

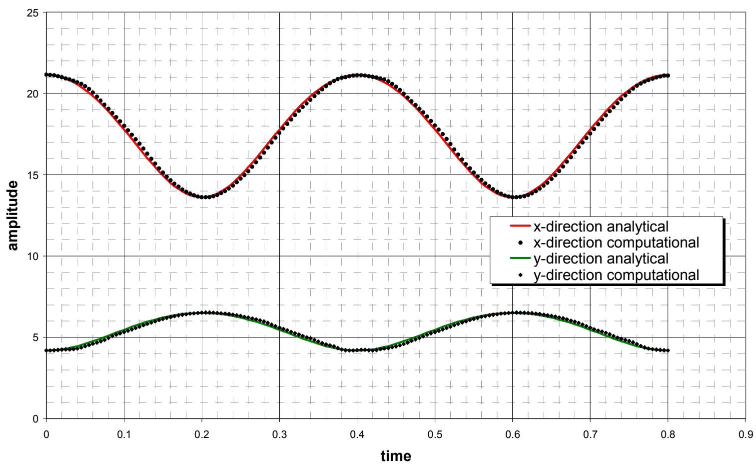
Now let Ω^h denote the discretized finite element domain with boundary $\Gamma = \bigcup_{i=1}^{4} \Gamma_i$ (see Figure 1(b)). The usual displacement based four-noded quadrilateral elements under the assumption of plain strain conditions, are used for the discretization of the domain. Zero natural boundary conditions are imposed on Γ_4 which belongs to the free surface, on boundaries Γ_i, i=1,..,3 the essential boundary conditions are applied as they are obtained from the velocity field and finally initial conditions are appropriately applied from the corresponding displacement/velocity fields.

The above test case is examined for a computational domain 20×10m^2, with unit square element size area, elastic properties $\mu=\lambda=180000$, ρ=2tn/m^3 (implying $c_L \cong$530m/s, c_T=300m/s) and an incident field with unit amplitude, angle $\theta_0=3\pi/8$ and frequency $2.5\cdot\pi$=1.25Hz. To each boundary node the corresponding velocity free-field solution is imposed according to its spatial coordinates as time-dependent constraint. In a similar manner to all nodes throughout the domain the free-field displacements/velocities serve as initial conditions. The response is recorded in node P (see Figure 1(b), and is compared to the analytical solution w.r.t. the velocity field (similar results are also obtained for the displacement field). In Figure 2(a,b) the amplitude and the phase angle vs. time are shown for the case of both the analytical and numerical approach. As it can be seen by Figure 2(c) the percentage error for the amplitude does not exceed 4%, implying the validity of the above formulation. It may be also noted here that similar results can be found in the literature when a BIEM approach is followed (Dineva & Saykov, 2005).

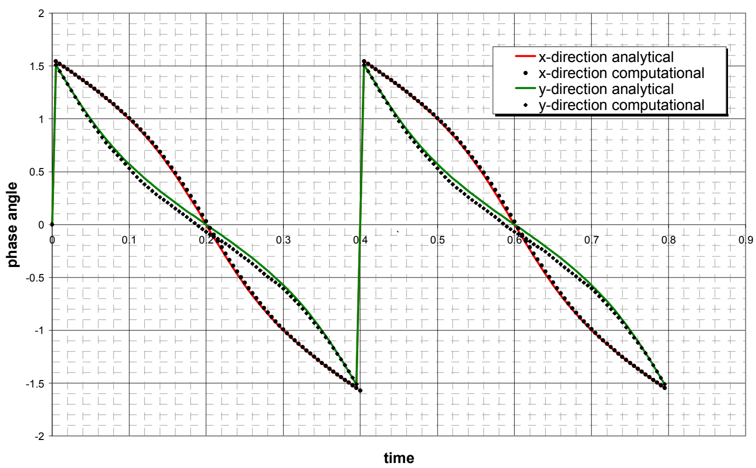


(a) (b)

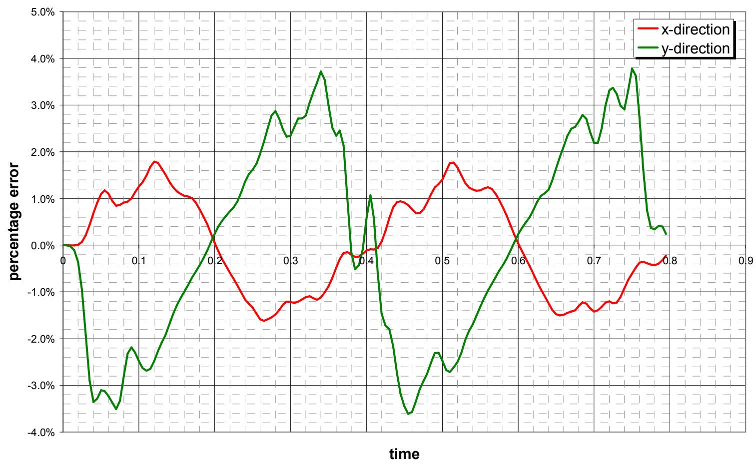
Figure 1. Wave propagation in an elastic homogeneous half space (a) and the fem discretization of a finite region situated in the half space (b). The response of the finite element solution is recorded at a point P which lays on the free surface.



(a)



(b)



(c)

Figure 2. Analytical and numerical velocity amplitudes recorded at point P in *x*, *y*-direction (a), corresponding phase angles (b) and percentage error for the velocity amplitudes (c).

APPLICATION

In this section the aforementioned technique is applied to a typical plain wave propagation problem, that of an elastic half space containing a circular cavity. This problem is of great interest among geotechnical structures such as tunnels and pipelines, and within that context is typically stated as: Given the soil properties, the structure's geometrical data and the free-field response, find the appropriate finite element domain, to which boundaries the free-field conditions should be imposed as time-dependent constraints.

The free-field response is usually obtained by analyzing existing measurements on the bedrock using several techniques like one-dimensional analyses or by using transfer functions (Kramer, 1996). From a numerical point of view, assuming that the influence of the scattered field vanishes in an infinite distance from the scatterer, a measure of that field near the boundaries could serve as an indication whether the distance between the scatterer and the boundaries of the computational domain is large enough. The objective of the following parametric studies is to determine the optimum size of the computational domain in view of the respective computational cost.

Consider a tunnel with circular cross-section of radius r=3.0, with its center located $5\cdot r$=15.0m below the ground surface. It is assumed that the soil mass around the tunnel is elastic and homogeneous with Lamé coefficients $\lambda = \mu$=180000 and mass density ρ=2tn/m^3. The computational domain is assumed to expand with respect to a scalar n as shown in Figure 3, while the soil structure is subjected to an incident time harmonic p-wave, described by a unit amplitude and a frequency ω=2.5π.

The total displacement/velocity fields are the superposition of the free-field and the scattered ones, i.e.

$$\mathrm{u}^{\mathrm{total}} = \mathrm{u}^{\text{free-field}} + \mathrm{u}^{\mathrm{scattered}}, \tag{6}$$

$$\mathrm{v}^{\mathrm{total}} = \mathrm{v}^{\text{free-field}} + \mathrm{v}^{\mathrm{scattered}}. \tag{7}$$

While the boundaries of the computational domain are moving away from the scatterer the respective wave field is expected to decay and therefore, it may be accepted that by using only the free-field velocities as time-dependent boundary conditions in the aforementioned algorithm does not introduce any significant error. This error is estimated by means of an appropriate norm, consistent with the space where the problem is stated, i.e.

$$\|v\| = \sqrt{\int_t \iint_\Omega \left({}^{analytical}\mathbf{v} - {}^{numeric}\mathbf{v} \right)^2} \tag{8}$$

or, with its discrete equivalent,

$$\|v\| = \sqrt{\frac{1}{2} dt \sum_{k=0}^{steps-1} \left[\left(\sum_{j=1}^{nodes} \sum_{i=1}^{dim} \left({}^{analytical}v_i^j - {}^{numeric}v_i^j \right)^2 \right)_k + \left(\sum_{j=1}^{nodes} \sum_{i=1}^{dim} \left({}^{analytical}v_i^j - {}^{numeric}v_i^j \right)^2 \right)_{k+1} \right]} \tag{9}$$

in some spatially fixed nodes adjacent to boundary (Figure 3). It must be also noted that under the assumption that at t_0=0 the scattered field is zero, initial conditions on displacements and velocities are those of the free-field.

A series of parametric analyses have been performed with n ranging from 5.0 to 25.0 with a step of 2.5 ending up in 9 different domain sizes (see e.g. Figure 4). The angle θ_0 that the wave impinges on the cavity is taking the values 0, π/8, π/4 and 3π/8. The normalized results are plotted in Figure 5, where also a measure of the computational cost is given. As it can be seen the difference between the free-field and the numerical results, measured by the above defined norm, diminishes as the computational domain expands. These plots of the norms accompanied with the computational cost, may serve as an indicator of the computational domain size when dealing with similar problems. As an example, for the given configuration one may say that any domain size with n>15 will increase unnecessarily the computational cost, while beyond that size, the effects of the scatterer are still significant reducing the reliability of the numerical results throughout the domain.

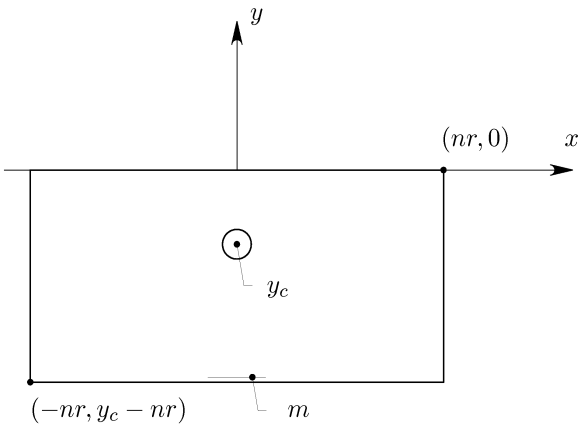


Figure 3. Geometrical data of the computational domain used in the parametric studies. The tunnel radius is denoted by *r*, y_c is the distance of the center of the tunnel from the ground surface, while *n* is used as a factor to expand the domain. Nodes on line *m* are used for the norm evaluation.

CONCLUSIONS

A consistent methodology for imposing time-dependent boundary conditions, which is suitable for problems in soil dynamics is presented, validated and applied to a typical earthquake engineering problem. This methodology is based on a weak form of the dynamic problem and is implemented within the context of the Finite Element Method in a straight forward manner. A simple case is chosen to validate the methodology, where a good agreement with the existing analytical solution is observed. Finally the aforementioned formulation is applied to a problem of great interest in the field of earthquake engineering, that of determining the appropriate computational domain size for a tunnel structure under a certain free-field response. Through the results an optimum domain size may be identified, in accordance to the expected accuracy and the computational cost.

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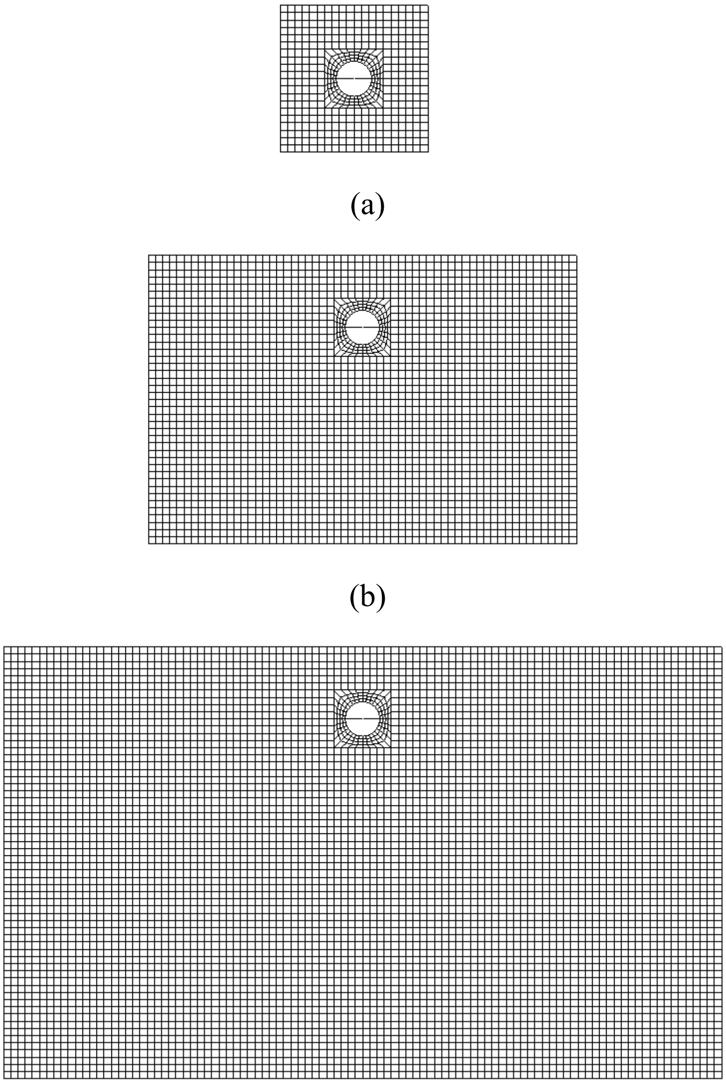

(a)

(b)

(c)

Figure 4. The finite element discretized domain for *n* = 2.5 (a), 12.5 (b) and 25.0 (c).

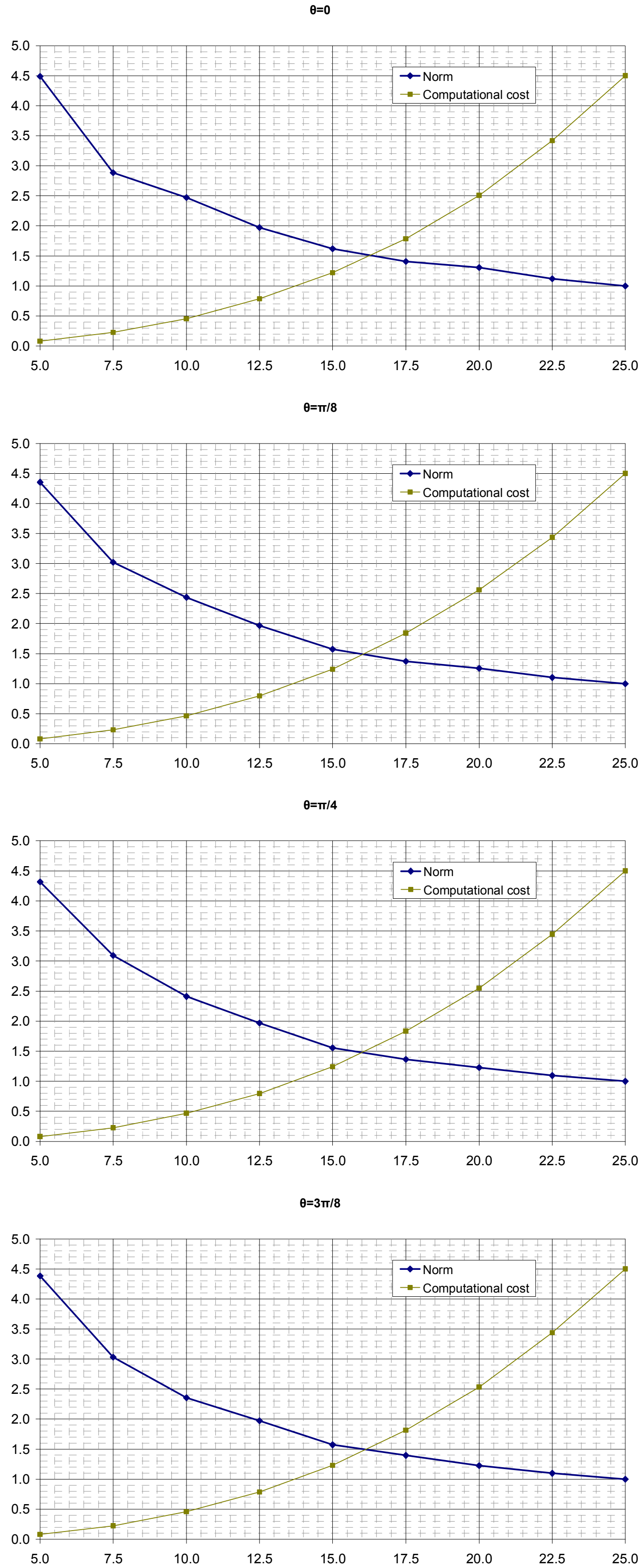

Fig.5. Norms and computational cost (normalized) for different angles of propagation and different domain sizes.

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