<u>List of tasks</u>.

Some are quoted in the text (first number indicates the chapter, second is a running number)

Task 2.0 Gravitational constant G from vertical gravity and pressure profile in the ocean
Task 2.1 Gravity effect of a homogeneous sphere.
Task 2.1.1 Gravity effect of an ore body; estimate the amount of ore
Task 2.1.2 Gravimetric detection of cavities
Task 2.1.3 see Task 5.4 Deflection of the vertical in India
Task 2.1.4 Geoid undulations
Task 2.1.5 Potential and gravity outside and inside a Bouguer plate or above and below an equivalent stratum
Task 2.1.6 Orbits of satellites, moon and planets
Task 2.1.7 Circular orbit and leaving Earth gravity field
Task 2.1.8 Flattening of a point mass earth ellipsoid
Task 2.2 Gravity effect of a vertical rod, centre of gravity
Task 2.3 Gravity effect inside a homogeneous spherical shell.
Task 2.4 Mutual gravitational attraction of two hollow spheres in water.
Task 2.4.1 Mutual attraction of salt domes
Task 2.4.2 Mutual attraction of gold spheres
Task 2.5 Gravity effect of a thin rectangular vertical wall at its corner.
Task 2.6 Derive the vector dipole field from superposition of closely neighbouring polar fields

Task 4.1 Mass and moment of inertia of earth
Task 4.2 Gravity reductions along a mountainous profile

Task 5.1 Interpretation of the Helgoland gravity anomaly
Task 5.2 Upward and downward continuation.
Task 5.3 Qualitative estimates and comparisons of 2D and 3D gravity effects
Task 5.4 (2.1.3) Deflection of the vertical in India
Task 5.5 Isostasy

Task 6.1 Gravity effect of a horizontal square plate; application of solid angle

Task 7.1 Introduction to INVERT applied to the gravitational effect of a sphere
Task 7.2 Interpretation of INVERT results applied to the model of a cylinder
Task 7.3 Application of conditions between variables in INVERT applied to a 2D profile
Task 7.4 Inversion of the gravity anomaly across the SE Iceland shelf

- The tasks are designed to be solved more by thinking and simple calculations than by carrying through some recipes. Generally numerical results are meant to have the right order of magnitude or, say, 10 % (in the sense of qualitative interpretation). Tasks of Chap. 7 Inversion are exercises in applying INVERT. An executable version is contained on the disc to be installed on your PC.
- Some of the tasks are mathematical problems that require no numerical calculation but are meant to train your skills in principal problem solving.

Tasks to chapter 2

Task 2.0.

Describe a marine experiment that has the aim of an accurate determination of Newton's gravitational constant G from measuring gravity Δg and pressure Δp along a vertical profile. The pair of values must be measured with the highest possible precision at a set of vertically displaced stations. Consider, for example, two stations at depth z and $z + \Delta z$.

Task 2.1.

Calculate the gravitational effect of a uniform sphere ($r=R_k$) on P at the distance h from its surface. Show that the integral over the whole sphere is equal to the attraction of the whole mass m concentrated at the centre of the sphere. Additional task: calculate the surface area of the sphere

Suggestion: Take the solid angle approach for a uniformly massive lump (2.8.6.3). This is demonstrated in the solution section. Start for the gravitational effect from eq. (2.8.6 – 8).

You can calculate also the effect of a uniform, infinitesimally thin spherical shell and show that it is equal to its mass concentrated at its centre, applying the solid angle approach. If the relation is true for an arbitrary spherical shell, it is true for all such shells, hence for the whole uniformly massive sphere.

Task 2.1.1.

Calculate the gravity effect of a "compact" ore body, idealized as a uniform sphere (radius r = 100 m, depth of centre z = 200 m, density contrast $\Delta\rho = 1000\ kg/m^3$, reference density of the surrounding rock assumed $\rho_o = 2700\ kg/m^3$) directly above (x = 0) and at x = 200 m. What is the half width of the anomaly?

How much ore is present?

Apply Newton's law (2.4 – 3), especially in the form (2.6 – 5) and take into account that the effect of a sphere is equal to that of its mass concentrated at its centre (Task 2.1). See 5.6.3.1 and (5.6 – 2a) for half width, and (5.6 – 2b) for total mass; see also 2.6.6, last paragraph.

Task 2.1.2.

Detection of shallow cavities by gravity measurements

(a) How big is the smallest mass m at z = 1 m depth which can be detected by a field gravimeter? Assume a sensitivity limit of $\delta g = 0.03$ mGal.

Design a strategy to find the mass in an area of $100 \times 100\ m^2$. What are the main problems, limitations?

(b) Can one detect with a gravimeter a long horizontal tunnel of radius r = 2 m, z = 10 m deep in sandstone (density r = 2400 kg/m^3) below the surface, if the observational resolution is about 0.03 mGal?

Task 2.1.4.

Estimate the geoid undulation N (see 2.5) generated by the following mass anomalies:

(a) a gold sphere of radius r_k = 1 m at a depth z = 1, 2 or 5 m, density ρ = 19300 kg/m^3;

(b) an iron sphere of radius r_k = 1 km at depth z = 1 km; density contrast ρ = 5000 kg/m^3;

(c) a spherical mass anomaly of r_k = 10 km at z = 15 km depth and density contrast ρ = –300 kg/m^3;

(d) a spherical mass anomaly of r_k = 100 km at z = 110 km depth and density contrast ρ = 100 kg/m^3.

Remember Bruns' formula (2.5 – 7).

Compare with the gravity effects.

Discuss the sign of the anomaly.

Task 2.1.5.

Calculate the variation with z of δg and δW (gravity and potential effect) of (a) an equivalent stratum and (b) a Bouguer plate above, inside and below the stratum or plate.

Task 2.1.6.

Masses, orbits and periods of satellites, moon and planets.

(a) Calculate the (circular) orbit radius r of a geo-stationary satellite.

(b) Calculate the mass of Sun on the basis of the distances and revolution periods of Earth-Sun and Earth-Moon (approximately 1a, $1.5 \cdot 10^8$ km and 27.3 d, $3.84 \cdot 10^5$ km, respectively. The orbits are assumed circular.

Earth radius r_e = 6371 km; mean density ρ_e = 5520 kg/m^3; Earth mass $m_e = 4\pi/3\ r_e^3\ \rho_e = 5.98 \cdot 10^{24}$ kg; period of sidereal day τ_e = 24 3600 364/365 = 86163 s; angular velocity $\omega_e = 2\pi/\tau_e = 7.29 \cdot 10^{-5}$ s^{-1}.

Task 2.1.7.

Estimate the initial velocity of ballistic rockets in order (a) to reach a shallow orbit (radius r_o = 6371 km and at $r + \Delta r$, $\Delta r \ll r$) or (b) to leave the gravity field of Earth.

Neglect the time interval of rocket firing and acceleration, i.e. assume an instantaneous speed.

Take the velocity of the starting point of rotating Earth into account.

Task 2.1.8.

Calculate the flattening of a fictitious equipotential ellipsoid generated by the Earth mass concentrated at its centre point in a frame rotating with Earth angular velocity ω. The parameters are taken from 4.4.1 (2006 reference ellipsoid in Geodetic Reference System GRS80 (Moritz, 1980) and World Geodetic System WGS84):

- Earth equatorial radius a = 6 378 137 m,
- flattening f = (a–c)/a = 1/298.257 222, where c = polar radius (calculable from a and f);
- gravitational constant × mass Gm_e = 3 986 005 10^8 m^{3}s^{-2},
- angular frequency ω = 7 292 115 10^{-11} s^{-1}.

- volume of the reference ellipsoid: $1.083\ 10^{12}$ km^3,
- mean radius of equivalent sphere: $r_e = 6371$ km,
- mass, with $G = 6.6742\ 10^{-11}$ m^{3}kg^{-1}s^{-2}, $m_e = 5.972\ 10^{24}$ kg,
- mean density: $\underline{\rho} \approx 5\ 513.5$ kg/m^3.

Task 2.2.

Given P = (0, 0, 0) and a vertical rod extending from Q_o = (x, 0, 0) to infinite depth $z \to \infty$. Where does the gravitational attraction at P point to? (The barycentre of the rod is also at $z \to \infty$, but does δg point down vertically?)

Task 2.3.

(a) Calculate or estimate the gravity effect and the gravitational potential of a homogeneous spherical shell at a point inside the shell.

(b) Express g(r) along a radius within a sphere with given density ρ(r).

Task 2.4.

Two idealized rigid hollow spheres with massless walls of radius R are suspended with massless ropes in water at equal depth from the flat sea floor (spherical shells, ropes etc. massless). Both experience equal buoyancy in water of uniform density $\rho_w = 1000$ kg/m^3. They are anchored exactly at the horizontal distance of 2R such that the spheres would just touch each other; the earth is assumed flat, g = 10m/s^2.

What is their mutual influence? Do the massless spheres exert a force onto each other, any force at all, an attraction or repulsion? Would the ropes be diverted from verticality, if horizontally free? What is the disturbance of the gravitational potential and what the resulting deformation of the sea surface?

This is, of course, but a *gedankenexperiment*.

Task 2.4.1.

Estimate the gravitational force between to neighbouring equal cylindrical salt domes (distance between centres d = 5 km, radius r = 1 km, height h = 3 km, density contrast vs. wall rock $\Delta\rho = -400$ kg/m^3). Compare with the buoyancy.

Task 2.4.2.

Two gold spheres are suspended in a shaft. How much do they approach each other by their mutual gravitational attraction? The relative approach is called 2δ, where δ is the horizontal displacement of each sphere from its vertical suspension. How accurately could one measure the gravitational constant G?

Assume for the spheres: radius r = 5 cm, density of gold ρ = 19 300 kg/m^3 suspended at equal depth (length of suspension l = 3 km, horizontal distance of suspension points d = 2(r + δ). Gravity is assumed g = 10 m/s^2.

Task 2.5.

Calculate the gravity effect of a vertical wall onto a point directly on top (distance to surface zero) where an apparent singularity exists where the distance r between source and observation $\rightarrow 0$. The problem arises when the gravity effect of a finite vertical rod (2.8.3.2) or wall (2.8.3.4) or a rectangular shell or box is evaluated (2.8.5.1), in the latter case for the near sides.

The gravity effect of a uniform cuboid onto P at one corner was calculated in 2.8.6.1 by projecting the opposite rectangular boundaries onto the unit sphere about P. The solution can be based on the solid angle and logarithmic effect as applied to the corner point of a rectangular prism or cuboid (2.8.6 –1).

Begin with the thin, though finite vertical rod of density ρ and centred about x, y = 0 with the horizontal cross section $\Delta x \Delta y$, extending from z = 0 to some depth, e.g. in infinity, with P approaching (0,0,0).

Task 2.6.

Derive the vector dipole field from superposition of closely neighbouring polar vector fields.

The easier approach of superposition of two scalar polar fields, as that of the point gravitational potential to arrive at the potential dipole field of magnetostatics. That has been sketched in 2.10, (eqs. 2.10 – 1, 2.10 – 2) and shown in Fig. 2.10 – 1. The results are given in eqs. (2.10 – 3, 2.10 – 4).

Attempt a similar derivation for the vector field.

Tasks to chapter 4

Task 4.1.

Given:

$g = 9.81$ m/s^2, mean gravity at the Earth's surface,

$G = 6.67\ 10^{-11}$MKS, gravitaional constant,

$R = 6371$ km, the mean radius,

$\theta = 0.33\ MR^2$, the Earth's moment of inertia (instead of $0.4\ MR^2$ for a homogeneous sphere with the same total mass M).

The factor 0.33 instead of 0.4 in the moment of inertia is explained by mass concentration or densification with depth.

Calculate for the spherical uniform earth its mass M, mean density $\underline{\rho}$ and gravity g(r) or g(z) with radius r (z = 6371 – r) [km]; calculate for the cases A to C the density-depth functions ρ(r) or ρ(z) as well as g(r) or g(z) for M and $\underline{\rho}$ unchanged.

Case A: homogeneous sphere: ρ_c = const; $\theta = 0.33\ MR^2$ cannot be satisfied. How much mass m* would have to be concentrated at the Earth's centre, with the rest homogeneous (ρ*) to get the correct θ value?

Case B: two homogeneous spherical layers: 1. the mantle from surface, $R = r_m$, to the bottom at z_c = 2900 km, density ρ_m; 2. the core r = 0 to $r_c = r_m - z_c$, density ρ_c.

Case C: sphere (radius R) with a linear density increase with depth

Task 4.2.

Reductions are now usually carried out by computers, blind, possibly by technical personnel, according to technical rules, schematic typing in of data or from data loggers. It is, however, a useful exercise to carry out the reductions once by hand to better understand procedures and problems.

Given is a table of 10 observations along an N-S profile across an approx. 2D E-W mountain range that is at the southern margin of a major orogenic belt with its axis some 50 km further N. The required reduction parameters are listed. The reductions are to be carried out. Then the resulting anomalies are to be discussed, analyzed and qualitatively interpreted. Estimate the regional of the BA.

Latitude 45°N, vertical gradient –0.3086 mGal/m, Bouguer density 2670 kg/m^3

i	**x [km]**	**h [m]**	**obs [mGal]**	**$\Delta\delta g_{norm}$ [mGal]**	**δg_h [mGal]**	**δg_B [mGal]**	**δg_{relief} [mGal]**	**ΔFA [mGal]**	**ΔFA* [mGal]**	**ΔBA [mGal]**	**reg [mGal]**	**ΔBA–reg [mGal]**
1	0.00	99.98	-86.87									
2	1.83	510.32	-165.33									
3	2.99	691.40	-198.64									
4	4.06	980.87	-259.41									
5	5.50	550.38	-167.47									
6	6.72	810.51	-230.33									
7	8.11	150.29	-98.85									
8	9.09	153.08	-93.09									
9	10.24	155.36	-93.54									
10	12.13	120.44	-86.99									

Carry out the reductions for the FA, FA* and BA (FA* incorporates the relief reduction) and thus complete the data table.

Tasks to Chapter 5

Task 5.1.

Apply various **qualitative estimates of the source of the BA in the Helgoland region** (see Figs. 5.7 – 2 and 6.5 – 2). Apply half width rules (5.6.3), consider the Bouguer plate effect (5.6.1) and solid angle estimates (5.6.4), roughly estimate and discuss downward continuation of harmonic components

First define the gravity anomaly you want to interpret. Take two orthogonal gravity profiles over the gravity minimum, one across strike (x→ENE), the other along strike (y→NNW), at 1 km intervals. Take the coordinate origin (x, y = 0) at the gravity minimum.

Plot the BA values in the same diagram versus x and y. What do the anomalies (curves) tell? (symmetry, qualitative shape of source body in 3D, …). Discuss the curves and the contoured anomaly map of Fig. 5.7 – 2.

The curve BA(x) may be taken for 2D estimates (as in 5.7.2).

For approximate 3D estimates, roughly establish the "radial" average of the two profiles, i.e. symmetrise BA(x)+BA(y) – average of the BA values at the same distance from the minimum (x=0, y=0), and average the two symmetric zed curves.

Half widths 2D, 3D: discuss applicability, problems, meaning of results. Compare the 2D and 3D estimates. Rough estimate of the errors. Extrapolation of the "radial" anomaly? What does the background BA tell?

Bouguer plate and solid angle: which kind of information is needed?

Undulated density variations and downward continuation: problems of application.

x [km]	BA [mGal]	y [km]	BA [mGal]
-8	-18.9		
-7	-19.5		
-6	-20.2	-6	-22.5
-5	-20.9	-5	-24-5
-4	-22.0	-4	-23.3
-3	-23.1	-3	-24.0
-2	-24.0	-2	-24.5
-1	-24.9	-1	-25.8
0	-26.1	0	-26.1
+1	-23.7	+1	-25.1
+2	-22.3	+2	-24.0
+3	-20.6	+3	-23.6
+4	-19.5	+4	-22.9
+5	-19.0	+5	-22.5
+6	-18.7	+6	-22.3
+7	-18.4		
+8	-18.3		
+9	-18.2		
+10	-18.0		

Compare BA(x) with corresponding 2D effect of a line mass $(Gm^+z)/(r^2+z^2)$) and the BA(R) to the point mass curve $(Gmz/(R^2+z^2)^{3/2})$.

Task 5.2.

Sections 2.6.3, 2.9.5.3; 5.6.1, 5.6.5; eqs. (2.6 – 10), (2.9 – 15), (5.6 – 1)

Upward and downward continuation; simple example to demonstrate the attenuation of amplitudes by e^{-kz} ($k=2\pi/\lambda$); the equivalent stratum and its geological interpretation. If, for example, the dominant wavelength is 100 km with 3 mGal amplitude, and if it is generated by an undulating surface (density contrast $\Delta\rho$ = 300 kg/m^3) at 30 km average depth, what would be the amplitude of the undulation? What if $\Delta\rho$ = 100 kg/m^3?

Given the following table of data

n	x*)	δg [mGal]
1	1	–3.00
2	2	–0.18
3	3	+2.56
4	4	+1.13
5	5	–0.50
6	6	+1.63
7	7	+3.88
8	8	+1.32
9	9	–1.00
10	10	+0.68

*) choose a scale, i.e. a unit, e.g. 1m, 100m, 1km, 10 km, 100 km

Analyse the data for trend and harmonics (they are composed of a linear trend and two sine waves, all assumed with simple numbers – in contrast to nature where this is rarely the case).

Downward and upward continuation to $\pm z$ = 1, 10, 100, 1000m = 1km, 10, 100km. Within which limits can one expect plausible results?

Is the task unique? Describe the type of uniqueness or non-uniqueness.

Compare with models derived by inversion.

Task 5.3.

2D-3D comparison: how large are the errors if 3D structures are treated as 2D structures?

Discuss some of the consequences

(a) half width rule; compare also 3D circular disk to 2D strip (Fig. 5.6 – 1)

(b) maximum depth rules

(c) edge effect (Fig. 5.3 – 1)

(d) dipoles (Fig. 5.6 – 4)

(e) undulated surface

Task 5.4 (2.1.3).

Very roughly estimate the deflection of the vertical (or plumb line) by the topographical mass of the Himalaya and Tibet (mass above sea level) at points in India 200, 400 ands 600 km south of the mountain front. Simplify the attracting mass to the extreme of a point mass. How strong is the horizontal attraction a 1 kg mass?

The deflection of the vertical is the ratio of the horizontal gravitational attraction to normal gravity at the point of observation (see 2.6.4, 2.8.1.2). It was measured, for example, in India during the 19th century geodetic surveys and its relatively small values led to the discovery of the notion of isostasy independently by Airy (1801-1892) and Pratt (1809-1971). Their estimate of the attraction by the visible mass excess exceeded the observations are called for a compensating mass deficiency.

More sophisticated calculations can, of course be done, e.g. using the program INVERT, however, no more that the order of magnitude of the effect is wanted here.

Assume a mass body of dimensions a, b and h (1000, 500, 5 km, respectively) with a density ρ = 2500 kg/m^3

Task 5.5.

Isostasy:

(a) Estimate the depth to which the tectonic "block" of the ***Himalaya and Tibet*** (H/T) will be eroded until complete dynamic equilibrium (if not disturbed by additional endogenic processes). Initially assume the following parameters to describe the present situation (t = 0): elevation, $h_o = 5 \pm 0.2$ km, thickness of H/T crustal root, $r_o = 35 \pm 5$ km, lower crustal thickness, $c = 35 \pm 5$ km, H/T total crustal thickness, $d = h + c + r = 75 \pm 5$ km$^{*)}$, upper crustal density, $\rho_o = 2600 \pm 100$ kg/m^3, mean lower crustal density, $\rho_{low} = 2850 \pm 50$ kg/m^3, mantle-crust density contrast, $\Delta\rho_m = 300 \pm 100$ kg/m^3,

Two hypotheses may be made: (1) the collision does not or (2) does change the density of the affected H/T crustal block, i.e. mainly of the lower crust.

Take India for reference: $h_I = 0.5$ km, crustal thickness (Moho depth + h_I) = $c_I = 40 \pm 5$ km, upper crustal density, $\rho_{oI} = 2700 \pm 50$ kg/m^3, mean crustal density, $\rho_I = 2900 \pm 50$ kg/m^3, upper mantle density, $\rho_m = 3300 \pm 100$ kg/m^3.

Estimate the effect of a density increase of the mean crust in the collision process which would add a load. Causes may be mafic intrusions and/or uplift due to denudation such that rocks rise which have a higher density per se. Assume that the depth range which experiences a density increase of $\Delta\rho_c = 100$ kg/m^3 is the same as the present extent c.

*) The estimated "standard error" of ±5 km follows from the error propagation of the three terms ±0.2, ±5, ±5 km. But since h is small and $d \approx c + r$, its error is rather about $\sqrt{2}\times 5 \approx 7$ km.

(b) Estimate the ***thickness d_s of sediment accumulation*** below a continental margin where the initial (sediment-free) water depth is $d_w = 5$ km. Assume the density of water $\rho_w = 1030$ kg/m^3, of the accumulated sediments $\rho_s = 2500 \pm 100$ kg/m^3 and of uppermost mantle $\rho_m = 3200 \pm 100$ kg/m^3. To what depth or thickness d_m has the mantle to be taken into account? Why are the ocean crust density $\rho_c \approx 2900$ kg/m^3 and thickness $d_c = 5$ km not needed? In the final state the water is assumed to be completely replaced by sediments. Estimate also the effect of sediment density increased by mafic intrusions to $\rho_s + \Delta\rho$.

(c) Can the various models be tested by gravity?

Tasks to chapter 6

Task 6.1.

On the basis of the solid angle $\Delta\Omega$, the task is to calculate $\delta g \approx \Delta\Omega\rho*$ (ρ*=ρd, d = thickness of layer) at P which is at the elevation h (>>d) above the centre of a square plate of dimension $\rho\ d\ a^2$. The solid angle $\Delta\Omega$ depends on h. For a "reasonable" range of h/a values, $\Delta\Omega$(h,a) can be approximated: if h < a or particularly if h << a, $\Delta\Omega \rightarrow 2\pi$, and $\Delta h \approx \delta g/(2\pi G\Delta\rho)$; for h > a or h >> a, a better approximation would be $\underline{d \approx \delta g\ (4/3)\ (h^2/a^2)/(\pi G\Delta\rho)}$. This is to be shown.

Tasks to chapter 7

Task 7.1

The purpose of this task is to get acquainted with the work flow of INVERT calculations. Please read the Invert Users Guide (Invert-usersguide.pdf on CD), if you have questions. Any text written in courier-font (`example`) are commands to be entered in the console or output of INVERT.

(a) Create an observations file for a profile of 41 equally spaced observations points between x = −20 and +20, y = 0, height = 0 (all in km), gravity = 0 and standard deviation of gravity = 1 (both in mGal) and create a definitions-file for INVERT for a sphere at x = 0, y = 0, at a depth of 10 km below the surface (observation points), with radius 5 (all in km) and a density (contrast) of 500 kg/m^3.

(b) Calculate the forward gravity effect of this sphere and visualize it with in 2D and 3D.

(c) Change the density of the sphere to 250 kg/m^3 and give it a standard deviation of 125 mGal. Change the entry in line "s" of the offset constant to −1000 (mGal). Determine the density optimally fitting to the gravity calculated in section b. Explain why we suggest to enter these values in line "s".

(d) Disturb the gravity calculated in section b with a normally distributed error with a standard deviation of 1 mGal (hint: use the script `disturb`) and interpret the a-posteriori standard deviation of the residuals and the wRMSE of the observations.

(e) Change (only!) the value of the standard deviation of all observations to 0.5 mGal (thus underestimating the real scatter of the observations). Interpret the difference of the results of this model and the model of section e. Discuss the residuals, the a-posteriori standard deviation (absolute and relative) and the wRMSE's of the density and the model effect as well as the F-statistic.

(f) After how many iterations can you interpret a solution as sufficiently converged for the model and observations of section c, for each of the next cases of modifications to the definitions file:

  (1) Change standard deviation of the z-coordinate of the sphere to 7 km and the z-coordinate itself to −15 km.

  (2) Additionally to the changes of case 1 change the standard deviation of the x- and y-coordinate of the sphere to 5 km and their a-priori value also to 5 km.

Task 7.2

The purpose of this task is to learn how to interpret the correlation matrix and its eigenvalues (singular values) and eigenvectors as well as the residuals/F-statistic.

(a) Create a definitions file to invert the data in 72a.obs. These observations are the gravitational effect (with a standard deviation of about 0.1 mGal) of a cylindrically shaped density anomaly. According to the initial knowledge of the modeller this cylinder will have a radius of about 2.5 ± 0.5 km, an upper face at a depth of about 300 ± 75 m and lower face at about 600 ± 100 m depth. The centre lies at about x = 10, y = 10, both ± 0.25 km. The density contrast is about −200 ± 50 kg/m^3. All error bounds are marginally acceptable (extreme) values from the viewpoint of the modeller (95% interval). Try to find out the geometry and density contrast of the cylinder. Use settings `mscC` for "Output in definitions file" to answer the next questions.

(b) Which properties are giving new knowledge to the modeller, which ones only reproduce its initial assumptions and what inconsistency exists between the inverted model and its initial knowledge?

(c) Which variables can be determined independently and which variables can be determined only in combination?

(d) Up to what amount is it possible to determine if the cylinder is elliptical by chance?

Task 7.3

The purpose of this task is to learn how to define more or less weak conditions on variables.

(a) Create a definitions file to invert the data in 73a.obs. These observations are the gravitational effect (with a standard deviation of about 0.1 mGal) of a 2D layer with an assumed upper surface at a depth of about 2.5 km an a lower surface at about 5.5 km depth with a 95%-interval of 2 km around these values. The density is about 200 ± 25 kg/m^3. Define the layer by a PROFILE-body with nodes at the x-coordinates: −50, −20, −10, −5, 0, +5, +10, +20 and +50 km. Use settings `mscrCR` for “Output in definitions file” and compare the resulting residuals, a-posteriori standard deviations, the a-posteriori covariance matrix and the resolution matrix (and their eigenvectors) and the wRMSE / F-statistic with the results of the section b to d.

(b) Change the definitions file to reflect the assumption, that the lower surface is an exact mirror of the upper surface and compare. Why would we assume this “hard” condition, or why not?

(c) Change the definitions file to reflect the assumption, that for each surface there is a distance dependent Gaussian correlation (e.g. of power 2) between the depth at the nodes. The typical distance between nodes with still a strong correlation (the so called correlation length) is assumed to be 15 km. The depths of the nodes of different surfaces are assumed to be uncorrelated (hint: use CORRELATION MATRICES).

(d) Change the definitions file to reflect the the smoothness condition that two neighbouring nodes have to remain at the given (in line “M”) depth difference. This condition must be fulfilled better than 0.2 km (95%) for a horizontal node distance of 5 km and for neighbouring nodes at increased mutual distance with a proportionally relaxed condition (hint: use CONDITION MATRICES). What is the advantage of this (and that of section c) kind of a-priori information relative to that of section b?

(e) Modify the requirement of section b to the weaker condition that the lower surface should be only a mirror of the upper surface within a range of 0.2 km (95%) using the CONDITION MATRIX.

Task 7.4.

Several geological models were presented for the SE Iceland Shelf in Fig. 6.5 – 5.

In section 5.7.5 an edge effect between two contrasting density-depth distributions was offered as a qualitative interpretation. In 6.5.5 three more models of different geological nature were investigated, but only the edge or dipole model was followed up in 7.7.2 by applying inversion. The other geological ideas (models 2 – 4 of the figure) were suggested to be further worked out as a task.

Model 2 was fitted to the BA, while models 3 and 4 were fitted to the rBA. The data files are 74ba.gb and 74rba.gb. The original gravity observations had a precision of better that 0.1 mGal. As described in Chapter 7.4.2.1 the gravity data distribution of neighbouring profiles and compartimentation along the profile direction were considered to determine a representative precision of the reduced data set.

Before or while modelling reconsider the given standard deviations in the data files from the view point of the modelling accuracy and the modelling errors (e.g. wRMSE and F-statistic).

It is suggested to begin with the models that were presented in Fig. 6.5 – 5 and to continue by modifying those initial models as you may see fit. That may be followed up by fitting models 3 and 4 to the BA in which case the assumed a priori Moho density contrast must be changed accordingly. For further modelling modifications no limitations exist, except by geological constraints. The initial model definitions files are 74uplift.def, 74forward.def and 74rot.def) are listed below the data files. A brief description is given here.

Model 1 is discussed extensively in the text but can, of course, also be extended.

Model 2 (74uplift.def) is based on the idea of deep shelf erosion (e.g. by glaciers) and uplift with an associated lateral density increase (uplifted density increase with depth), supplemented by a light sediment body on the slope and deeper sea floor (possibly also a density depression in the crust depressed under the sediments, equivalent to the above increase). The lateral density variations are arbitrarily restricted to the upper crust (10 km). Horizontal variations are assumed gradual, realized in steps by nesting (see Fig. 6.1.1(d) in 6.1.5.1). The density depression below the sediments is (initially) commented out. It may be activated for tests.

Model 3 (74forward.def) is based on the hypothesis that a large part of the shelf was produced by massive mafic volcanisms when plume activity increased and also split the about 3 Ma old lithosphere east of Reykjanes. A heavy upper crustal body is assumed to have been added to the Iceland margin by massive volcanism. The assumed higher-density model has a similar geometry as that of model 2, however with no gradual variations; obviously the data and hypothetical nature of the a priori assumptions will hardly permit a discrimination of those models. An associated sediment body is also assumed.

Model 4 (74rot.def) is based on the idea that isostasy includes a balance of the torques related to the lateral density variation from Iceland to the deep sea. In the version presented here, the sediment wedge is commented out but can be activated any time. Other variants include a Moho density correction (0 ± 5, can be switched off by changing to 0 ± 0 kg/m^3 or by commenting out) and any idea invented by the modeller.

To zoom into the (small) interesting region of model 4 enter for instance:

```
plotprofile -d "set xrange [0:150]; set yrange[-3:0]" -e my74rot.def
```

Solutions

Task 2.0 Solution.

Assume at the sea surface at $z_o = 0$ (z positive downward) g_o and p_o.

At depth $z_1 > 0$, gravity is given by $g_1 = g_o + \partial g/\partial z \cdot z_1 + \delta g_{water} + \delta g_a$, where $\partial g/\partial z \cdot z_1$ refers to the normal earth gravity increase with depth, $\delta g_{water} \approx 2\pi G \rho z_1$ is the effect of the water slab (density $\rho = \rho_{water}$) between $z = 0$ and z_1 and δg_a is the sum of all "anomalous" effects deviating from the normal earth.

The difference $\Delta g = g_2 - g_1$ at $z + \Delta z$ and z is

$$\Delta g = \partial g/\partial z \cdot \Delta z - 4\pi G \rho_{water} \Delta z + \Delta \delta g_a$$

In this expression $\partial g/\partial z$ must be inferred from normal gravity and δg_a must be estimated as best as possible. Assume that $\Delta g_{water} = - 4\pi G \rho_{water} \Delta z = \Delta g - \partial g/\partial z \cdot \Delta z - \delta g_a$ can be determined accurately.

The effect of the water slab is mathematically well defined, but ρ_{water} varies with pressure, temperature and salinity and Δz may contain measuring errors. Both uncertainties can be more or less eliminated by measuring pressure p at the same stations (pressure can supposedly be measured very precisely):

$$\Delta p = \rho_{water}\, g\, \Delta z$$

where g is ambient gravity which is measured in any case.

In the ratio r of gravity and pressure the two uncertain values cancel each other (equality of heavy and inert mass assumed):

$$r = \Delta g_{water}/\Delta p = 4\pi G/g$$

which renders

$$G = rg/4\pi$$

The main task is to reduce the errors of g, $\partial g/\partial z \cdot \Delta z$ and Δg_a as much as possible. Among other aspects, the gravity field and sea surface topography in the region of this experiment should be very well known. The ocean floor and deeper mantle structure should be well mapped (marine seismic studies, mantle tomography); The region should be remote from major tectonic features as plate margins, volcanoes etc. And the ocean currents should be fairly well understood.

Task 2.1 Solution.

The starting point is (2.8.6 – 8):

$$\delta g = G\rho\int_S r \cos\varphi \, d\Omega$$

where φ is the angle of ***r*** relative to $\boldsymbol{\delta g}$. The direction of the integral δg effect toward the centre of the sphere is known beforehand for symmetry reasons.

Any mass line through the sphere along a ray bundle $\delta\Omega = \sin\vartheta \, d\lambda \, d\vartheta$ causes an attraction at P:

$$\Delta g = G\rho\int_S r \cos\vartheta \, \delta\Omega = G\rho\int_S r \cos\vartheta \sin\vartheta \, d\lambda,$$

independent from λ, hence $\int d\lambda = 2\pi$ and $d\Omega = 2\pi \sin\vartheta \, d\vartheta$

$$\Delta g = 2\pi G\rho\int_S r \sin\vartheta \cos\vartheta \, d\vartheta.$$

Thus, $d\Omega$ is specially chosen, ring-shaped on the unit sphere, Δg points along P-Q and r is the secant length of the rays.

This can also be seen from the surface area of a spherical cap:

$$\Omega_{cap} = 2\pi(1-\cos\vartheta),$$

hence

$$d\Omega = 2\pi \sin\vartheta \, d\vartheta.$$

The length $r(\vartheta) = 2\Delta r(\vartheta)$ enters the integral directly, and ϑ has to be taken from 0 to ϑ_o where the rays are grazing the sphere:

$$\Delta g = 4\pi G\rho \int_0^{\vartheta_o} \Delta r(\vartheta) \sin\vartheta \cos\vartheta \, d\vartheta$$

$\sin\vartheta_o = R/(R+h)$

$r_o = (R+h)\cos\vartheta_o = (R+h)(1-\sin^2\vartheta_o)^{1/2} = (R+h)(1-R^2/(R+h)^2)^{1/2} = ((R+h)^2-R^2)^{1/2} = (2Rh+h^2)^{1/2}$

$\Delta r^2 = R^2 - a^2$, with $a = (R+h)\sin\vartheta$

$\Delta r^2 = R^2 - (R+h)^2 \sin^2\vartheta = R^2(1-(R+h)^2/R^2 \sin^2\vartheta)$

$\Delta r = R(1-(R+h)^2/R^2 \sin^2\vartheta)^{1/2} = R(1-\sin^2\vartheta/\sin^2\vartheta_o)^{1/2} = R(\sin\vartheta_o - \sin\vartheta)^{1/2}/\sin\vartheta_o$

Hence

$$\Delta g = 4\pi G\rho(R+h) \int_0^{\vartheta_o} (R^2/(R+h)^2 - \sin^2\vartheta)^{1/2} \sin\vartheta \cos\vartheta \, d\vartheta$$

Let $\cos\vartheta \, d\vartheta = d(\sin\vartheta) = dx$ and $\sin\vartheta_o = R/(R+h)$

$$\Delta g = 4\pi G\rho(R+h) \int_0^{R/(R+h)} (R^2/(R+h)^2 - x^2)^{1/2} \, x dx$$

Gröbner & Hofreiter, p.52, case 236.5g, or Bronstein et. al., p 752, case 158:

$\int x(a^2-x^2)dx = -(a^2-x^2)^{3/2}/3$

$$\Delta g = -4/3 \, \pi G\rho(R+h) \, (R^2/(R+h)^2 - x^2)^{3/2} \Big|_0^{R/(R+h)} = -4/3 \, \pi G\rho(R+h) \, (0 - R^2/(R+h)^2)^{3/2} =$$

$$= -4/3 \, \pi G\rho R^3(R+h)/(R+h)^3$$

$$\boldsymbol{\Delta g = 4/3 \, \pi G\rho \, R^3/(R+h)^2} \qquad \textbf{q.e.d.}$$

Mass of sphere $M = 4/3\ \pi\ \rho\ R^3$, distance Q-P = R+h

Additional task: calculate surface of the sphere

Ring-shaped $dS = 2\pi R\sin\varphi R d\varphi = 2\pi R^2 \sin d\varphi$

$$S = 2\pi R^2 {}_o\int^{\pi} \sin\varphi d\varphi = 2\pi R^2 (-\cos\varphi\ d) \big|_0^{\pi} = 4\pi R^2$$

The alternative approach to δg on the basis of thin uniform spherical shells 2.8.5.3 would begin with eq. (2.8.5 – 5):

$$\delta g = G\rho\ \Delta h \int_S \cos\phi/\cos\vartheta\ d\Omega$$

where the direction of δg is known beforehand (from P to the centre of the sphere) and ϕ is the angle of $-\boldsymbol{r}$ relative to $\boldsymbol{\Delta g}$.

This requires the complication of taking ϕ into account. This is not carried through here but is suggested as an additional exercise.

Task 2.1.1 Solution.

From $\delta g_o = G\Delta m/z^2$ (x = 0) and $\Delta m = V\Delta\rho = 4\pi/3 \cdot r^3\ \Delta\rho$ ($\approx 4.2 \cdot 10^9$ kg) follows with the numerical values inserted:

$\Delta g_o \approx 7 \cdot 10^{-6}$ m/s^2 = <u>0.7 mGal</u>

and for x = 200 m

$\Delta g_{200} = G\Delta m z/(x^2 + z^2)^{3/2} = \Delta g_o z^3/(x^2 + z^2)^{3/2} = \Delta g_o(z/(x^2 + z^2)^{1/2})^3 = \Delta g_o/\sqrt{2}^3 \rightarrow$

$\Delta g_{200} \approx \Delta g_o/2.83 \approx$ <u>0.25 mGal</u>.

In the present case with z given, follows the half width $w \approx 3z/2$ = <u>300 m</u> from (5.6 2a).

Estimated total ore mass:

Assume that the estimate is based on the gravity effect δg_o and the half width w.

From (5.6 – 2b), $\Delta m \approx 4/9\ \delta g_o\ w^2/G \approx (4/9)\ 7 \cdot 10^{-6} \cdot 300^2/6.67 \cdot 10^{-11} =$

$= 4 \cdot 7 \cdot 10^{-6} \cdot 9 \cdot 10^4/(9 \cdot 6.67 \cdot 10^{-11}) \approx 4.2 \cdot 10^9$ kg, as already known from the input data.

The excess mass $\Delta m = V\Delta\rho$, based on the gravity effect, must be supplemented (if the body consists of 100% massive ore) by the "neutral" mass $m_o = V\rho_o$, where ρ_o is the reference density of the surrounding rock. In that case $m_{ore} = V\rho = V(\rho_o + \Delta\rho) = \Delta m(\rho_o/\Delta\rho + 1)$. As $\Delta\rho = 1000$ kg/m^3 is known, we could estimate $V = \Delta m/\Delta\rho \approx 4.2 \cdot 10^9$ kg/1000 kg/m^3 = $4.2 \cdot 10^6$ m^3. Hence, $m_{ore} \approx 4.2 \cdot 10^6$ m$^3 \cdot 3700$ kg/m$^3 \approx 15 \cdot 10^9$ kg =<u>15 Mt</u>.

Of course, this was the original input.

Task 2.1.2 Solution.

(a)

The estimate is based directly on Newton's law (2.4 – 3), or in the form (2.6 – 5). Directly above the mass m, $x = 0$ and distance $r = z$, whence $m = \delta g\, z^2/G$. Assume δg to be 0.03 mGal = $3 \cdot 10^{-7}$ m/s^2. Hence; $m \approx 3 \cdot 10^{-7}$ m/s^2 1m^2/6.67$\cdot 10^{-11}$ m^3/s^{2}kg = $4.5 \cdot 10^3$ kg.

Taking an extended volume of a sphere (radius r_k), its density would be $\Delta\rho = m/V_k = (r^2/G)(3/4\pi r_k^3)\delta g$. Taking $r_k = 1$ m (sphere touches the surface). We get $\Delta\rho \approx 1074$ kg/m^3 which could be a water body inside soil or a sedimentary rock of density a bit >2000 kg/m^3. An iron sphere of ~8000 kg/m^3 would have a radius of 0.5 m.

While the above result seems to suggest a manageable task, the problems of detecting such a mass in a 10 000 m2 area, because one would have to survey a very fine point grid, say 1m×1m, implying 10 000 points. Even if one designs a more ingenious strategy, the task would be excessive.

Another problem is evident in possible terrain and in surface-near density inhomogeneities which can significantly affect the gravity observations. This is not discussed further, but must be carefully assessed in any concrete application.

(b)

The gravity effect of an infinite horizontal line mass m^+ (kg/m) or a uniform circular cylinder around the line (same line mass with density $\rho = m^+/(\pi r^2)$) is given by (2.8.3 – 1), that of a limited line (cylinder half length λ) by (2.8.3 – 2).

For the extreme effect above the tunnel follows from $\delta g = 2Gm^+/z$: $m^+ = 0.5\, \delta g z/G$. With $\delta g = 3\ 10^{-7}$ m/s^2, z = 10 m, G = $6.67 \cdot 10^{-11}$ m^3/s^{2}kg, $m^+ \approx 3 \cdot 10^{-7} \cdot 10/6.67 \cdot 10^{-11} \approx 22500$ kg/m = $\rho\pi r^2$. From the latter expression follows, with r = 4 m, $\rho \approx 1800$ kg/m^3 which is less than the density contrast between air and sandstone; with $\Delta\rho = 2400$ kg/m^3, the extreme effect would be 0.04 mGal. If, on the other hand, the tunnel were water-filled, the extreme effect would be only 0.023 mGal.

If the tunnel is 200 m long, the central gravity effect would be slightly smaller; (2.8.3 – 2) renders the correction factor a ≈ 0.995, i.e. a reduction of 0.5 % of the effect of the infinite mass cylinder, quite insignificant.

The same cautioning remarks about terrain and density variations pertain to this example.

Task 2.1.3   **see Task 5.4.**

Task 2.1.4 Solution.

The geoid undulation N is the deviation from the reference geoid taken to be 0 everywhere.

The theoretical effect on the geoid is given by the potential anomaly

$\delta W = -Gm/r = -Gm/(x^2 + z^2)^{1/2}$, divided by gravity g_o: $N = \delta W/g_o$ (Bruns' formula). The mass is $m = 4\pi/3\ r_k^3\ \rho$, $g = 10\ m/s^2$.

Only the maximum value of N directly above the mass m is calculated.

(a) $Gm/g \approx 6.67\cdot10^{-11}\cdot4\cdot1\cdot1.93\cdot10^{4}\cdot10^{-1} \approx 5.15\ 10^{-7}\ m^2$.

For z = 1, 2, 5 m, N ≈ 5, 2.6, $1\cdot10^{-7}$ m, respectively (or ~0.5, 0.3, 0.1 µm)

These are very small distances. Multiplication with g/r renders the gravity effect,

i.e. $\delta g \approx 5, 1.3, 0.2\cdot10^{-6}\ m/s^2$ = 0.5, 0.13, 0.02 mGal, respectively. These values are well within the detection limits of gravimetry.

The cases (b) to (d) can be treated by simply by multiplying the result of N ≈ ***$5\cdot10^{-7}$ m*** of case (a) with $(r_k/r_{ka})^3\cdot z_a/z\cdot\rho/\rho_a$, where ***$r_{ka} = z_a = 1$ m*** and ***$\rho_a = 1.93\cdot10^4\ kg/m^3$*** (***gold***)

(b) ***1 km*** radius ***iron*** in sediment, ***1 km*** deep: $N \approx 5\cdot10^{-7}\cdot1000^3/1000\cdot5/19.3 \approx$ ***0.13 m***

(c) ***10 km rad, –300 kg/m³***, ***15 km*** deep: $N \approx -5\cdot10^{-7}\cdot10^{12}/1.5\cdot10^{4}\cdot3/193 \approx$ ***–0.52 m***

(d) ***100 km rad, 100 kg/m³***, ***110 km*** deep: $N \approx 5\cdot10^{-7}\cdot10^{10}/(1.1\cdot193) \approx$ ***24 m***

The gravity effect can be calculated directly from the expression Gm/r^2, but also by multiplying N with g/r.

(a) $\delta g \approx 5\cdot10^{-7}\cdot10/1 = 5\cdot10^{-6}\ m/s^2$ = ***0.5 mGal***

(b) $\delta g \approx 0.13\cdot10/1000 = 1.3\cdot10^{-3}\ m/s^2$ = ***130 mGal***

(c) $\delta g \approx -0.52\cdot10/15\cdot10^{-3} = 0.35\cdot10^{-3}\ m/s^2$ = ***35 mGal***

(d) $\delta g \approx 24\cdot10/1.1\cdot10^{-5} = 21.6\cdot10^{-4}\ m/s^2$ = ***216 mGal***

A positive mass m generates a positive rise of the geoid, somewhat surprising in view of the minus sign in the definition of the potential. However, it means that the potential increases with increasing r, as does the reference potential of Earth ($W = -Gm_e/r$, $r = r_e$); in this sense, the two potentials are added, not subtracted. The potential work to lift a mass is greater than without the mass anomaly.

You may also think of the ocean with an attracting mass anomaly inside or below. The sea level is an equipotential surface. The additional attraction pulls water towards the mass and thus raises the water surface (in contrast to the erroneous idea that the surface is attracted and pulled own).

Task 2.1.5 Solution.

(a)

Assume a uniform surface density ρ^* on the plane z = 0, z pointing downward (such that normal $\boldsymbol{g} = \boldsymbol{g_o}$, positive).

The easiest approach (rule 1 in 2.8.1.1 and 2.8.3.3) is to start from the solid angle $\Delta\Omega = 2\pi$ under which the infinite plane is seen from above ($z < 0$), and the gravity effect is constant $\delta g = 2\pi G\rho^*$ (because of the infinite horizontal extent of the mass plane). Below the plane ($z > 0$) the effect is usually not considered but pointing upward: $\delta g = -2\pi G\rho^*$. On the plane z = 0, for reasons of symmetry, $\delta g = 0$. At z = 0, δg has a discontinuity of $4\pi G\rho^*$.

The potential δU follows from integration, e.g. from z = 0:

$$\delta U = {}_{(z)}\!\int \delta g dz = 2\pi G\rho^* z + C.$$

Below the plane, for $z > 0$, correspondingly:

$$\delta U - 2\pi G\rho^*.$$

The variation is continuous with a discontinuity in slope (gradient δg).

Alternatively one may follow rule 4 in 2.8.1.1 and calculate first the potential δU in vertical cylinder coordinates (2.8.2.2).

$$\delta U = \rho^* G\, {}_o\!\int^{2\pi} {}_o\!\int^{\infty} R dR d\varphi/(Z^2+R^2)^{1/2} = 2\pi G\rho^*\, {}_o\!\int^{\infty} R dR/(Z^2+R^2)^{1/2} = 2\pi G\rho^* \, (Z^2+R^2)^{1/2} \Big|_o^{\infty} =$$

$$= 2\pi G\rho^*[\infty - z]$$

where the constant ∞ is practically irrelevant. Hence

$$\delta U = 2\pi G\rho^* \, |z| \, .$$

The sign is a matter of convention, but the potential to do work increases linearly with increasing distance $|z|$ from the plane z = 0.

The gravity effect is

$$dg = dU/dz = \pm 2\pi G\rho^*,$$

depending on the location above or below z = 0.

(b)

The equivalent stratum is now replaced by a Bouguer plate of finite thickness D = 2d and density ρ with z = 0 at the centre of the plate and positive downward. We distinguish the cases outside the plate $|z| \geq d$ and inside the plate $|z| < d$.

Outside the mass plate the potential and gravity effects are essentially identical to the expressions for the stratum, for $|z| \geq d$:

$$-\delta U = \pm 2\pi G\rho D \, |z|$$

and

$$-\delta g = \pm 2\pi G\rho D.$$

Inside we can make use of the symmetries of mass around the observer at z, and here it is easier to begin with δg.

Inside the plate ($-d < z < d$): the contributions from the mass above and below, respectively, give

$$-\delta g = 2\pi G\rho(-(d-z) + (d+z)) = 2\pi G\rho\,(2z) = 4\pi G\rho z.$$

At the plate centre ($z = 0$): $\delta g = 0$, and at the bottom and top ($z = \pm d$) (and beyond, see above):

$$-\delta g = \pm 4\pi G\rho d = \pm 2\pi G\rho D.$$

The gravity effect has the constant negative (upward pointing) value $-2\pi G\rho D$ below the plate up to its bottom, from there on decreases linearly to zero at the plate centre and onward to $+2\pi G\rho D$ at the top, from where on it stays constant.

The potential δU can be integrated upward from $z = 0$ to $-z$ (or to $-d$):

$$\delta U = {}_{o}\!\int^{-z} \delta g d\zeta = 4\pi G\rho\,{}_{o}\!\int^{-z} \zeta d\zeta = 2\pi G\rho\zeta^2 \Big|_{o}^{-z} = 2\pi G\rho z^2.$$

At $z = d$:

$$\delta U = 2\pi G\rho d^2.$$

Downward from $z = 0$, the gravity effect changes sign, but the potential effect rises again with growing z (downward) and for symmetry reasons it has the same form as above the centre.

An alternative approach to the problem is to solve the Poisson equation inside the plate and the Laplace equation outside. The mass distribution in an infinite horizontal Bouguer plate renders $\partial/\partial x \equiv 0$ and $\partial/\partial y \equiv 0$, such that inside $\nabla^2\delta U = d^2\delta U/dz^2 = 4\pi G\rho$ and outside $\nabla^2\delta U = d^2/dz^2 = 0$.

Integration inside from $z = 0$ to z leads to

$$d\delta U/dz = \delta g = 2\pi G\rho z$$

and

$$\delta U = 2\pi G\rho z^2.$$

Outside ($z \geq d$):

$$d\delta U/dz = \text{const} = 2\pi G\rho d$$

and

$$\delta U = 2\pi G\rho d\,{}_{d}\!\int^{z} dz = 2\pi G\rho d\, z.$$

If z is replaced by $z' = z - d$ or $z = z' + d$,

$$\delta U = 2\pi G\rho d(z' + d) = 2\pi G\rho d(d^2 + d\,z).$$

At the boundary $z = d$, δU is continuous, as it should be.

Task 2.1.6 Solution.

(a)

Starting point is the equality of the gravitational and centrifugal accelerations, a_g and a_c, on the orbit , where $a_g = Gm_e/r^2$ and $a_c = \omega^2 r$ with $\omega = \omega_e$.

$r\omega_e^2 = Gm_e/r^2 \rightarrow r^3 = \mathbf{Gm_e/\omega_e^2} = G\rho_e\,(4\pi/3)\; r_e^3\,\tau_e^2/(4\pi^2) = \mathbf{r_e^3 G\rho_e\,\tau_e^2/(3\pi)} \approx r_e^3(6.67\cdot 5.52\cdot 8.6163^2\cdot 10^{-11+3+8}/9.425) \rightarrow$

$r \approx r_e\,(290)^{1/3} \approx 6.62\; r_e \approx$ **42 172 km**

(b)

The above relationship between orbital radius r, mass m and angular velocity ω or the orbital period τ of a satellite, planet or moon, $\mathbf{r^3 = Gm/\omega^2 = Gm\tau^2/(4\pi^2)}$, permits the solar mass m_s to be estimated. It does, of course, depend on the (poor) knowledge of G (with the index eo referring to the Earth orbit around Sun):

$m_s = \omega_{eo}^2 r^3/G\;[= 4\pi^2 r_{eo}^3/(\tau_{eo}^2 G)]$

$m_s \approx 4\cdot 10^{-14}\cdot 3.38\cdot 10^{33}/6.67\cdot 10^{-11} \approx 2\cdot 10^{30}$ kg

(Wikipedia: $1.9891\cdot 10^{30}$ kg)

Comparison of the lunar and terrestrial orbits with Earth and Sun, respectively, as attracting central bodies, gives the ratio of their masses which seems to offer a solar mass estimate independent from G. However, since m_e is known only from $m_e = gr^2/G$, the independence is only apparent.

From (with index mo referring to Moon orbit around Earth)

$(r_{mo}/r_{eo})^3 = m_e/m_s\;(\omega_{eo}/\omega_{mo})^2 = m_e\tau_{mo}^2/m_s\tau_{eo}^2$ and

$\rightarrow m_s/m_e = (r_{eo}/r_{mo})^3\;(\tau_{mo}/\tau_{eo})^2 \approx (15\cdot 10^7/3.84\cdot 10^5)^3\;(27.32/365.25)^2 \approx 59.6\cdot 10^6\cdot 0.0056 \approx$

$\approx 3.34\cdot 10^5$

$m_s \approx m_e\; 3.34\cdot 10^{24} \approx 5.98\cdot 3.34\cdot 10^{24+5} \approx 1.997\cdot 10^{30}$ kg

Task 2.1.7 Solution.

(a)

Assume a starting point at the equator which has the speed of $v_e = r_e\omega$

with sidereal $\omega = 2\pi/d\ (364.25/365.25)$ and $d = 24\cdot 60\cdot 60 = 86400$ s $\rightarrow \omega = 7.252\cdot 10^{-5}\ s^{-1}$ and

$v_e = 462$ m/s (0.462 km/s or 1663 km/h).

The "absolute" speed v_o of a rocket flying horizontally at Earth's surface ($r_o = 6.371\cdot 10^6$ m) such that the centrifugal acceleration $z_o = \omega^2 r_o$ equals gravity $g_o = 10\ m/s^2$ at $r = r_o$ follows from $g_o = \omega^2 r_o$ and $v_o = \omega r_o$ or $\omega = v_o/r_o \rightarrow g_o = v_o^2/r_o \rightarrow v_o = (g_o\cdot r_o)^{1/2}$

$v_o \approx (10\cdot 6.371\cdot 10^6)^{1/2} = 63.71^{1/2}\cdot 10^3$ m/s ≈ 8 km/s (7.982 km/s or about 28 800 km/h, i.e. ~1.4 h per orbit).

Hence the velocity v_h relative to Earth's surface would be 7.982 ± 0.462 km/s or 7.52 km/s to the E and 8.444 km/s to the W. The energy needed to accelerate the rocket deviates from the mean by about ±10 % depending on the direction of take-off.

Estimate the speed for a circular orbit above Earth's surface by the distance Δr, i.e. at a radial distance $r = r_o + \Delta r$. Expand the expressions and neglect small terms.

Both gravity and the centrifugal acceleration are affected.

$g = Gm/(r_o + \Delta r)^2 \approx GM/r_o^2\ (1 + \Delta r/r_o)^{-2} \approx g_o\ (1 - 2\ \Delta r/r_o)$

$v_h = (g\ r)^{1/2} \approx [g_o\ (1 - 2\ \Delta r/r_o)\ (r_o + \Delta r)]^{1/2} \approx (g_o r_o)^{1/2}\ [(1 - 2\ \Delta r/r_o)(1 + \Delta r/r_o)]^{1/2} \approx v_o(1-\Delta r/r_o)^{1/2} \approx v_o\ (1 - 1/2\ \Delta r/r_o)$.

Taking $\Delta r = 100$ km and $\Delta r/r_o \approx 0.0157$ gives $v_h \approx 7.92$ km/s.

Note that this is not the take-off speed at $r = r_o$ but the circular orbital speed at $r_o + \Delta r$ which must decrease as Δr increases, as both gravity from Earth and orbital curvature decrease. To get up into the orbit, the rocket must have a somewhat higher speed or an initial vertical velocity component on a ballistic parabola culminating at Δr height (in reality, the orbit would be a Keppler ellipse; to bring the rocket into a circular orbit, a correction has to be applied to its velocity).

To calculate the vertical or radial velocity v_r component needed to lift the rocket to $\Delta r = 100$ km height on a ballistic orbit, consider the free fall from $r_o + \Delta r$ to r_o. Assume $g = g_o$ everywhere and $v_{ro} = 0$ and $s_o = 0$ at $r_o + \Delta r$. Hence $v_r = g_o\bullet t$ and $s = g_o/2\bullet t^2$, from which follows $t = (2s/g_o)^{1/2}$. Insert this into $v_r = g_o\cdot t = (2s\cdot g_o)^{1/2}$.

The result is for $s = \Delta r = 100$ km: $v_r = \sqrt{2}\cdot 1000$ m/s ≈ 1.414 km/s.

The assumption of $g = g_o$ is incorrect and leads to some unwanted consequences. An improvement follows from assuming (as above) g to decrease from g_o to $g_o(1 - 2\Delta r/r_o)$ and from taking the mean of g at r_o and $r_o + \Delta r$: $g_o(1 - \Delta r/r_o)$. Inserted into $v_{ro} = (2\Delta r\cdot g_o)^{1/2}$, this renders $v_r = (2\Delta r\cdot g_o(1 - \Delta r/r_o))^{1/2} \approx v_{ro}(1 - 0.5\Delta r/r_o) \approx 0.992\ v_{ro} \approx 1.403$ km/s.

If the horizontal and vertical velocity components, v_r and v_h, are vectorially added, at take-off $v \approx (v_h^2 + v_r^2) \approx 8.045$ km/s (with the g correction: 8.043 km/s). The take-off angle is given by $\alpha = \mathrm{atan}(v_r/v_h) \approx 10°$.

Assessment of the take-off set-up. Take the time of free fall across 100 km, $t = (2s/g_o)^{1/2} \approx (2\cdot 10^4)^{1/2} \approx 141$ s. From this follows the horizontal distance $x = t\ v_h \approx 2000$ km from take-off to orbit culmination from where the rocket begins to descend across another 2000 km on a Keppler ellipse (spherical distance of 36° between start and end of the flight; most of the ellipse would be inside Earth with its centre being one focus of the ellipse).

An orbit tangent to Earth (at take-off and at its antipode point), would result with the same take-off velocity of 8.045 km/s, however at horizontal take-off (angle 0°). It follows also from equating the take-off (kinetic) energy with the energy sum (kinetic and potential) at culmination height Δr (assume $g = g_o$): $mv_o^2/2 = mv_{\Delta r}^2/2 + mg_o\Delta r$, from which follows $v_o = (v_{\Delta r}^2/2 + 2g_o\Delta r)^{1/2}$, identical to the above $v \approx (v_h^2 + v_r^2) \approx 8.045$ km/s.

However, there is a principal error in assuming an elliptical orbit in the form of an ellipse tangent with Earth at two opposite points and thus symmetric about Earth's centre, because it would not be a Keppler ellipse which must have one focus at Earth centre; (with $a = r_o + \Delta r$ and $b = r_o$, the focal distance from Earth centre, $e = (a^2 - b^2)^{1/2} \approx (2r_o\Delta r)^{1/2} \approx 1130$ km)

(b)

To leave the gravity filed of Earth, the rocket must have an initial kinetic energy $E_{kin} = mv_\infty^2/2$ equal to the extra potential energy $E_{pot\infty} = Gm_em/r_o$ when it has reached infinite distance ($r \rightarrow \infty$). From the equality follows $v_\infty = (2Gm_e/r_o)^{1/2}$.

This is a simple formula, however, you need to know (by heart?) m_e or Gm_e. You can transform the expression by inserting $g_o = Gm_e/r_o^2$ and obtain: $v_\infty = (2\ r_o\ g_o)^{1/2}$, with all quantities fairly well known. Hence, $v_\infty = v_o$ (of the circular orbit at $r = r_o$). Again, this expression can be evaluated fairly well without the aid of a calculator: $v_o = (2 \cdot 6.371 \cdot 10^6 \cdot 10)^{1/2} = v_o = (2 \cdot 63.71 \cdot 10^6)^{1/2} \approx \sqrt{2} \cdot 8 \cdot 10^3$ m/s ≈ 11.3 km/s.

Task 2.1.8 Solution.

Assume a mean radius $r_e = 6371$ km, $\omega = 7\,292 \cdot 10^{-11}\ s^{-1}$ and

$\boldsymbol{g} = Gm_e/r_e2 \cdot -\boldsymbol{r_e}/r_e + \omega^2\ r\cos\varphi\boldsymbol{p}$,

$W = -Gm_e/r_e + ½\ \omega^2\ r^2\ \cos^2\varphi$ and $N = \delta W/g$

where $\boldsymbol{p}$ points normal to $\boldsymbol{\omega}$ (axis) outward. The gravitational potential grows outward, the rotational potential grows inward toward the axis.

We assume further that the increase δW of the gravitational potential at the equator due to the additional "elevation" N equals the decrease δW of the centrifugal potential due to the shift N.

For $\varphi = 0$ simply calculate $\delta W = ½\ \omega^2\ r^2 \approx 0.5 \cdot 5.315 \cdot 10^{-9} \cdot 4.059 \cdot 10^{13}\ m^2/s^2 \approx 1.08 \cdot 10^5\ m^2/s^2$ and with $g = 9.81\ m/s^2$

$N \approx 11$ km

The corresponding flattening $f = (a - c)/a \approx 11/6380 \approx 1/580$

Which is about one half of the correct value of 1/298

Task 2.2 Solution.

Two components of the gravity effect are to be calculated, normal to the rod, δg_x, and parallel to it, δg_z. The latter is treated directly in 2.8.3.2 and given by (2.8.3 – 4):

$$\delta g_z = G\rho^+/r_o$$

where G is the gravitational constant, ρ^+ is the line density (kg/m), and r_o is the distance between P and Q_o, in this case: x.

The horizontal or normal component δg_x corresponds, after rotation of the whole geometry, to the usual vertical component of the effect of an infinite horizontal mass line directly below P (see 2.8.3.1; eq. 2.8.3 – 2), divided by 2:

$$\delta g_x = G\rho^+ z/r^{*2} = G\rho^+/r_o = \delta g_z$$

δg_x and δg_z are exactly equal. Hence, the vectorial effect points down to the rod at an angle of 45° and its "gravicentre" is thus at a depth of z = x, not at infinity where the commonly defined rod centre of gravity is located.

Task 2.3 Solution.

(a)

The simplest way is to apply the solid angle approach. Take an infinitely thin layer of surface density $\rho^* = \rho dr$. You may later integrate the effect for finite Δr, or generalize the result from dr to Δr, as the case may be.

For any point P within the shell you can imagine of draw any infinitesimal ray bundle with solid angle $d\Omega$ and its exact opposite from P. They will cut out of the shell symmetrically oriented pieces, ds of the surface s, as a ray pair will always be a secant that intersects the sphere (or circle) at equal angles on both sides. Since $s = r^2 dW/\cos\vartheta$ (ϑ = angle between ray and normal vector $\boldsymbol{ds}$), both mass elements will exert equal and opposite attractions on P and hence cancel each other. The whole shell will be covered this way if the one-sided rays cover half the sphere.

Hence, the gravity effect of the shell on the inside P is exactly zero. This will be true for any confocal shell, hence for a shell of finite thickness. Gravity effects of other external masses are not affected by the hollow sphere.

The potential is always defined as work/m (where m is the mass moved in the gravity field), i.e., the path integral $\delta W = \int_s \boldsymbol{\delta g} \cdot \boldsymbol{ds} + C$ along the path s. Since $\delta g \equiv 0$, $\delta W = C$. The constant is arbitrary, depending on the arbitrary reference point.

(b)

From (a) immediately follows that a point inside a sphere with $\rho(r)$ will "feel" only the attraction of the "inner sphere:

$$g(r^*) = Gm(r^*)/r^{*2};\ m(r^*) = 4\pi \int_o^{r^*} \rho(r) r^2 dr$$

Task 2.4 Solution.

The empty spheres ($\rho = 0$) of zero mass as such do, of course, exert no Newtonian force onto each other ($m = \rho V = 0$, $V=4\pi/3\ R^3$).

However, submersed in water, the massless spheres represent negative mass anomalies (density contrast $\Delta\rho = -\rho_w$, ρ_w = density of water, $\Delta m = -\rho_w V$) which distort the gravity field. We expect, that a mass anomaly Δm within a perfectly uniform body acts analogously to an ordinary mass in free space, such that Newton's law correctly describes the external gravitational effect of Δm. No intuitive reason seems to exist why this should not be the case: for an extra mass or positive mass anomaly it seems indisputable and hence it should equally hold for a negative mass anomaly. Then one may argue formally, that if for points P outside the sphere Newton's integral law (task 2.1) holds for a massive sphere of any uniform density ρ, it should hold also for $-\rho$, such that the superposition of such two spheres will have zero density and a zero gravitational effect. This implies that the two effects (of ρ and of $-\rho$) are real and identical except for their sign. We can thus operate with the negative gravity effect of the empty sphere (as we do that anyway successfully in gravity modelling and interpretation).

As a corollary, mutual gravity of the particles within a uniform mass (fluid) is zero because all effects mutually cancel pr are equal from all sides (directions). This is also implied by the perfect equilibrium of forces in the fluid: no forces exist to generate a permanent circulation. Furthermore, inside the hollow sphere, gravity should be zero (see task 2.3). All this is based on the assumption of infinite uniformity (the spheres excepted); a free sea surface introduced an asymmetry which generates variable gravity effects described by vertical gravity variation as generated by the Bouguer plate.

The "auto-effect" of Δm onto its own sphere is radially symmetric and its vectorial integral over the surface is zero. The vertical components of $\boldsymbol{\delta g}$ exerted by the mutually other sphere are perfectly symmetric about the horizontal middle plane; hence, their integral effect on buoyancy is zero. It seems anti-intuitive that zero mass bodies attract each other if immersed in a dense fluid. But a missing mass is negative relative to the surrounding uniform fluid which generates a repulsive gravitational effect on its particles, as the attraction from the massive side exceeds that from the empty side. In that field the outward directed force generates pressure field buoyancy directed towards the empty sphere.

An empty sphere adds $\boldsymbol{\delta g} = G\Delta m\,(\boldsymbol{r}/r)/r^2$ (for $r \geq R$) to the normal vertical $\boldsymbol{g_o} = (0,0,g)$; $g=10\ m/s^2$; $\boldsymbol{\delta}g$ is positive outward along the radius vector $\boldsymbol{r}$, measured from the centre of the sphere. The vertical component is $\delta g_z = G\Delta m z/r^3$, the horizontal x-component is $\delta g_x = G\Delta m x/r^3$, where x points away from the centre of sphere 1. Sphere 2 has its centre at $x = 2R$. The horizontal integral force of sphere 1 on sphere 2 is $f = G\Delta m_1\Delta m_2/(4R^2)\boldsymbol{x}/x$ (see task 2.1) or $G(\Delta m/(2R))^2$, since $\Delta m_1=\Delta m_2=\Delta m$.

The repulsive force around the negative relative mass Δm can also be understood as resulting from the gravitation by the masses outside. Any mass point experiences gravitation from all other mass points around (within an infinite uniform mass cancelled, see above). The empty spheres represent asymmetries or irregularities, and the normal vertical field is superimposed by their effects. It is easier to integrate over the missing masses than over the infinite rest. The sum is $\boldsymbol{g_o} + \boldsymbol{\delta g}_1 + \boldsymbol{\delta g}_2$ with $\boldsymbol{\delta g_1} = G\Delta m(\boldsymbol{r_1}/r_1)/r_1^2$, $\boldsymbol{\delta g_2} = G\Delta m(\boldsymbol{r_2}/r_2)/r_2^2$ and $\boldsymbol{r_1} = (x, y, z)$, $\boldsymbol{r_2} = (x-2R, y, z)$.

Imagine sphere 2 (outside sphere 1) to be filled by "absolute" mass >0 with density $\rho_2 = \rho_w$; Sphere 2 is indistinguishable from the rest of the uniform fluid 2 or part of it. The surrounding water is in perfect equilibrium with the pressure forces at its surface an, if it exists, the conceptual force F, exerted by sphere 1 at the point of contact. We expect F = 0, because sphere 2 is not really identified and hence no point of contact can be defined. Since the gravitational force is repelling, the buoyant surface force due to the resulting pressure variation should be attracting and exactly cancel the gravitational force.

If, on the other hand, $\rho_2 > \rho_w$ or $\Delta\rho_2 = \rho_2 - \rho_w > 0$, F > 0, i.e. it is repulsive; and if $\rho_2 < \rho_w$, F < 0, i.e. it is attractive; e.g. $\Delta\rho_2 = -\rho_w$ as in the given task, $\mathbf{F = f = -G\Delta m_1\Delta m_2/(4R^2)}\boldsymbol{x}/\mathbf{x}$, directed towards sphere 1. This is because the buoyant force in the gravity field of sphere 1 exists independent from what is inside sphere 2, but no gravitational force on empty space inside sphere 2 exists (and as argued above, the buoyant force is equal to the gravitational force on mass with ρ_w). The physical effect of the disturbed gravity field $\boldsymbol{\delta g_1} = G\Delta m\,(\boldsymbol{r_1}/r_1)/r_1^2$ on sphere 2 is the buoyancy in the pressure disturbance field generated by $\boldsymbol{\delta g_1}$ with horizontal components which will generate a horizontal force between the two spheres. In other words, physically it is the mutual buoyancy the spheres experience in the mutually disturbed gravity field $\boldsymbol{g}$. It is the well-known basic Archimedean rule that buoyancy is the negative weight of the displaced fluid.

Buoyancy $\boldsymbol{b} = \Delta\rho V\boldsymbol{g}_o = \Delta m\boldsymbol{g}_o$ due to normal gravity is generated by the depth-dependent pressure p = rgz; acting everywhere normal to the spherical surface. The ratio F/b of the forces is:

$F/b = G\Delta m^2/(4R^2\Delta m\ g_o) = G\Delta m/(4R^2 g_o) = 4/3\ \pi\ R^3 G\Delta\rho/(4R^2 g_o) = G\Delta\rho R\pi/(3g_o) \approx G\Delta\rho R/g_o$. If we introduce $g_o = m_e G/R_e^2$ (R_e Earth mean radius, ρ_e mean density, Earth mass $m_e = \rho_e\ 4/3\ \pi\ R_e^3$), the ratio $F/b = \frac{1}{4}\ \Delta\rho/\rho_e\ R/R_e$. F/b is linear in $\Delta\rho$ and R.

Letting $\Delta\rho/\rho_e = 1/5.5$ and $R_e = 6.371\cdot10^6$ m, **F/b 7.13·10^{-9} R**[m]

It would be much more complicated to calculate the whole pressure field around a sphere (out to infinity), which would also cause the water surface to be deflected from horizontal resulting in a slight depression which can be estimated by Bruns' formula (2.5 – 7) $\delta h = \delta W/g_o$ where $\delta W = \Delta mG/r$ is the potential disturbance. The boundary condition at the deflected surface is zero pressure in equilibrium with the gravitational effects on the particles. The pressure on the sphere surface would have to be integrated.

Task 2.4.1 Solution.

This task is just a variant of Task 2.4. But it says also something about the forces and interactions between geological bodies.

The estimate can be rather crude, e.g. replace the actual cylinders by spheres of equal volume $V = \pi r^2 h$, centre of gravity and mass $m = V\Delta\rho$, or by the equivalent mass points.

Between the domes the horizontal gravitational force is

$f \approx Gm^2/d^2 \approx 6.67 \cdot 10^{-11}(\pi \cdot 10^6 \cdot 3 \cdot 10^3 \cdot (-4 \cdot 10^2))^2/25 \cdot 10^6 \approx \underline{3.8 \cdot 10^7 \text{ N}} = 3.8 \cdot 10^6$ kp (kilopond). This is an attraction which in absolute numbers is impressive. However, …

Compare to the buoyancy force b in the field of gravity with the density contrast -400 kg/m^3:

$b = \Delta\rho\ V\ g \approx 4 \cdot 10^2 \cdot 0.94 \cdot 10^{10} \cdot 10 \approx 3.8 \cdot 10^{13}$ N

Hence $f/b \approx 10^{-6}$

The mutual attraction of the salt domes is one millionth of their buoyancy.

Task 2.4.2 Solution.

The horizontal displacement $\delta = l \cdot f/w$. Thus, calculate the ratio f/w of attraction $f = Gm^2/d^2 \approx Gm^2/(4r^2)$ over weight $w = mg$ of each sphere.

$f/w \approx Gm/(4r^2g)$.

With $m = 4\pi/3\ r^3$, $f/w = G\pi r\rho/(3g) \approx Gr\rho/g \approx$

$\approx 6.67 \cdot 10^{-11} \cdot 5 \cdot 10^{-2} \cdot 1.93 \cdot 10^{4} \cdot 10^{-1} \approx 6.4\ 10^{-9}$.

Hence $\delta \approx l\ Gr\rho/g = 3 \cdot 10^3$ V 6.4v10^{-9} m $\approx 2 \cdot 10^{-5}$ m $= 0.02$ mm.

G would have to be measured with a relative error smaller than 10^{-4} or 10^{-5}, to improve current estimates. This alone would require lengths to be measurable with nanometre (nm) accuracy (laser?). But that would require constructions which would make the theoretical effects difficult to calculate.

The suspension wire must be strong enough to hold 10 kg of gold and its own weight, but for that its mass will probably have to exceed that of the gold sphere 100 times or more. A wire with diameter decreasing with depth might help, but it will certainly add to the mutual attraction (the reader may calculate such kind of effects; but that is no longer a trivial task).

Moreover, other error sources are the suspension mass and the inhomogeneities of the shaft and the density distribution of the surrounding rock. Corrections to be calculated for those effects may be uncertain.

Task 2.5 Solution.

(1) The square vertical rod.

P is at the centre of the top area (at z = 0) of a vertical rod with the rectangular shape or dimensions $\Delta x \Delta y$ z ($\Delta y = \Delta x$, assumed very small, though not zero). The geometry of the situation is shown in Fig. 2.8 – 3. For simplicity use the triple-index notation introduced in 2.8.3.4 and Fig. 2.8 – 6, and P is at 000.

Divide the rod into 4 equal quarter rods of sides $\Delta x/2$ and $\Delta y/2$, such that P at 000 lies at the corner of each, and all quarter rods have the same effect on P which is expressed by (2.8.6 – 1); the sides opposite P point to the three coordinates and are identified by the superscripts (x), (y) and (z) and each expands a tetrahedron with P which generates gravitational components in each coordinate direction. Here we need $\delta g_z^{rod} = 4(\delta g_z^{(z)}+\delta g_z^{(x)}+\delta g_z^{(y)})$; the factor 4 refers to the 4 quarter rods. Begin with

$\delta g_z^{(z)} = G\rho\ z\ (\arcsin[r_{100}r_{111}/r_{110}r_{101}] + \arcsin[r_{010}r_{111}/r_{011}r_{110}] - \pi/2)$

with $r_{100} = r_{010} = \Delta x/2$, $r_{110} = \sqrt{2}\Delta x/2$, $r_{001} = r_{101} = r_{111} \rightarrow z$

$\delta g_z^{(z)} \rightarrow G\rho\ z\ (\arcsin(1/\sqrt{2}) + \arcsin(1/\sqrt{2})) - \pi/2 \equiv 0$; i.e., the solid angle $\Delta\Omega^{(z)} \rightarrow 0$

Since $\Delta y = \Delta x$, the contributions $\delta g_z^{(y)} = \delta g_z^{(x)} = G\rho\ \Delta x/2\ \ln([(r_{010}+r_{110})\ r_{101}]/[(r_{010}+r_{111})\ r_{100}])$ where, with the above expressions, the argument of ln simplifies to $(1 + \sqrt{2})$ and $\ln(1 + \sqrt{2}) = 0.881$.

Summing all non-zero contributions of the 4 quarter rods gives

$\delta g_z^{(rod)} = 4G\rho\ \Delta x \ln(1 + \sqrt{2}) = G\rho s \ln(1 + \sqrt{2}) = 0.881\ s\ G\rho$ where $s = 4\Delta x$ is the circumference of the rod.

Remark: it seems to be a general rule that $\delta g_z^{(rod)}$ is proportional to its circumference; it hols also for the circular cylinder where the factor is 1 instead of $\ln(1+\sqrt{2})$.

(2) Vertical wall with P at a corner.

The dimensions are Δx b c.

The tetrahedron expanded by the "big" side bc and P has the effect (2.8 – 6):

$\delta g_z^{(x)} = G\rho\ \Delta x \ln([(r_{010}+r_{110})\ r_{101}]/[(r_{010}+r_{111})\ r_{100}])$

with $r_{100} = \Delta x$, $r_{010} = b$, $r_{001} = c$, $r_{110} \approx b$, $br_{101} \approx c$, $r_{011} \approx r_{111} \approx (b^2+c^2)^{1/2} = [b\&c]$,

$\delta g_z^{(x)} = G\rho\ \Delta x \ln(2bc/(b+[b\&c])\Delta x) = G\rho[\Delta x \ln(2bc/(b+[b\&c]) - \Delta x \ln\Delta x]$ or with c = 1

$\delta g_z^{(x)} = -G\rho\ \Delta x \ln((1+[1\&(1/b)])\Delta x/2)$

where $\lim_{\Delta x\rightarrow 0} \Delta x \ln\Delta x \rightarrow 0$, but it remains finite for $\Delta x > 0$.

There are two more tetrahedra expanded by P and the sides $c\Delta x$ and $b\Delta x$; they are much narrower than the above side bc.

Again according to (2.8 – 6):

$\delta g_z^{(y)} = G\rho\ b \ln([(r_{100}+r_{110})\ r_{011}]/[(r_{100}+r_{111})\ r_{010}]) \rightarrow G\rho\ b \ln(b\ [b\&c]/([b\&c]\ b) = G\rho b \ln 1 = 0$

with the above definitions and neglecting Δx against b and c.

$\delta g_z^{(z)} = G\rho c(\arcsin[r_{100}r_{111}/r_{110}r_{101}] + \arcsin[r_{010}r_{111}/r_{011}r_{110}] - \pi/2) \rightarrow$

$G\rho c(\arcsin(\Delta x[b\&c]/(bc) + \arcsin(b[b\&c]/([b\&c]b)$

The first term is proportional to $\Delta x \rightarrow 0$, the second term $\rightarrow \arcsin 1 = \pi/2$, such that $\delta g_z^{(z)} \rightarrow 0$.

The complete listing of all non-zero components of the $\boldsymbol{\delta g}$ vector exerted by a rectangular box onto P at one corner is given in eqs. (2.8.5 – 1 and 2a – c); see section 2.8.5.1 where also the present task is posed.

Task 2.6 Solution.

The vector dipole field, e.g. the magnetic dipole field, can be derived from superposition of closely neighbouring vector polar fields, as e.g. the gravity field.

Take two poles p_- and p_+ separated by the vector $\boldsymbol{l}$ from p_- to p_+. Define Cartesian coordinates x, y, z such that x is given by $\boldsymbol{l}$/l and y, z normal to x, otherwise arbitrary. Point P is defined by $\boldsymbol{r}$ = (x, y, z) or by the polar coordinates r, ϑ, where the origin 0 is at the centre of $\boldsymbol{l}$ (or in the end by letting lim $\boldsymbol{l} \to 0$) and ϑ is measured against $\boldsymbol{l}$ (or x). The separated poles, p_- and p_+, are at $x = \pm|\boldsymbol{l}|/2$, the vectors from p_- and p_+ to P are $\boldsymbol{r}_-$ and $\boldsymbol{r}_+$ and the unit vectors $\boldsymbol{e}_\pm = \boldsymbol{r}_\pm/r_\pm$.

Then the sum of the field vectors at P is given by

$$\boldsymbol{H}(r) = \boldsymbol{H}_+ - \boldsymbol{H}_- = p(\boldsymbol{e}_+/r_+^2 - \boldsymbol{e}_-/r_-^2).$$

The two triangles OPp_- and OPp_+ have at P the acute angles $\alpha_- = \vartheta - \vartheta_-$ and $\alpha_+ = \pi - \vartheta - \vartheta_+$, respectively.

Describing $\boldsymbol{H}$ by the radial and tangential components H_r and H_ϑ, respectively, we write

$$H_r = H_+\cos\alpha_+ - H_-\cos\alpha_-$$

and

$$H_\vartheta = H_+\sin\alpha_+ - H_-\sin\alpha_-.$$

With the cosine theorem and $\boldsymbol{l}$/r small we can write

$$r_+ = (r^2 - rl\cos\vartheta + (|\boldsymbol{l}|/2)^2)^{1/2} \approx r(1 - l/r\cos\vartheta)^{1/2}$$

and

$$r_- = (r^2 + rl\cos\vartheta + (|\boldsymbol{l}|/2)^2)^{1/2} \approx r(1 + l/r\cos\vartheta)^{1/2}.$$

$\alpha_+ = \arcsin([|\boldsymbol{l}|/2)] \times \sin\vartheta/r_+)$ and $\alpha_- = \arcsin([|\boldsymbol{l}|/2)] \times \sin\vartheta/r_-)$ follow from the sine theorem in the two triangles: $(|\boldsymbol{l}|/2)/\sin\alpha_+ = r_+/\sin\vartheta$; $(|\boldsymbol{l}|/2)/\sin\alpha_- = r_-/\sin(\pi-\vartheta) = r_-/\sin\vartheta$, and we obtain

$$1/r_\pm^2 \approx (r^2 - r|\boldsymbol{l}|\cos\vartheta -/+ (|\boldsymbol{l}|/2)^2)^{-1} \to r^{-2}(1 \pm |\boldsymbol{l}|/r\cos\vartheta)$$

$$H_r = p\{\cos\alpha_+/r_+^2 - \cos\alpha_-/r_-^2\} =$$
$$= p\{\cos\arcsin((|\boldsymbol{l}|/2) \times \sin\vartheta/r_+)/r_+^2 - \cos\arcsin((|\boldsymbol{l}|/2) \times \sin\vartheta/r_-)/r_-^2\};$$

with $\cos\arcsin x = (1 - x^2)^{1/2}$

$$H_r = p\{(1 - (|\boldsymbol{l}|/2)^2 \times \sin^2\vartheta/r_+^2)^{1/2}/r_+^2 - (1 - (|\boldsymbol{l}|/2)^2 \times \sin^2\vartheta/r_-^2)^{1/2}/r_-^2\} =$$
$$= p\{(1 - |\boldsymbol{l}|^2/(4r_+^2)\sin^2\vartheta)^{1/2}/r_+^2 - (1 - |\boldsymbol{l}|^2/(4r_-^2)\sin^2\vartheta)^{1/2}/r_-^2\} \approx p\{1/r_+^2 - 1/r_-^2\}$$
$$\approx p\,r^{-2}(1 + |\boldsymbol{l}|/r\cos\vartheta - 1 + |\boldsymbol{l}|/r\cos\vartheta) = 2p|\boldsymbol{l}|\cos\vartheta/r^3$$

With the definition of the magnetic moment, $p\boldsymbol{l} = \boldsymbol{m}$, this gives the magnetic radial component

$$\boldsymbol{H}_r = m/r^3\, 2\cos\vartheta\, \boldsymbol{e}_r$$

Similarly, with $\sin\arcsin x = x$ and $1/r_\pm^3 \approx r^{-3}(1 -/+ |\boldsymbol{l}|/r\cos\vartheta)^{-3/2} \approx r^{-3}(1 \pm 3/2\, |\boldsymbol{l}|/r\cos\vartheta)$

$$H_\vartheta = p(\sin\alpha_+/r_+^2 + \sin\alpha_-/r_-^2) = p|\boldsymbol{l}|/2\,\sin\vartheta(1/r_+^3 + 1/r_-^3) \approx$$
$$\approx p|\boldsymbol{l}|/2\,\sin\vartheta/r^3(1 + 3/2\,|\boldsymbol{l}|/r\cos\vartheta + 1 - 3/2\,|\boldsymbol{l}|/r\cos\vartheta),$$

hence the tangential component is

$$\boldsymbol{H}_{\vartheta} = m/r^3 \sin\vartheta \, \boldsymbol{e}_{\vartheta}.$$

Task 4.1 Solution.

From $g = GM/R^2 = G\underline{\rho}4/3\ \pi\ R^3/R^2 = 4/3\pi G\underline{\rho}R \rightarrow \underline{\rho} = 3g/(4\pi GR) = 5511$ kg/m^3 (a more accurate value is 5513.5 kg/m^3).

$g(r) = 4/3\ \pi G\underline{\rho}[s^{-2}]\ r[m] = 1.54\cdot10^{-6}\ [s^{-2}]\ r\ [m] = 1.54\cdot10^{-3}\ [s^{-2}]\ r\ [km]$

(Case A)

The simple answer is that the uniform mean density of the Earth's body must be reduced by 0.33/0.4 = 0.825 from the given approximate figures, since for a uniform sphere θ is linear in density:

Hence: $\rho^* = 5511 \cdot 0.825 = 4547$ kg/m^3, corresponding to a mass of $4.925\cdot10^{24}$ kg, less by $m^* = 1.045\cdot10^{24}$ kg than the total Earth mass of $5.97\cdot10^{24}$ kg. Roughly 1/6 of the Earth's mass would have to be concentrated to infinite density at the Earth's centre. Since it has no radial extension ($r \rightarrow 0$), its moment of inertia is zero and does not contribute to the total θ.

The gravity variation with radius inside this earth is the sum of the effect of the uniform body and that of the central point mass:

$g(r) = 4/3\ \pi G\underline{\rho}^*[s^{-2}]\ r[m] + Gm^*[m^3/s^2]/r^2[m^{-2}] = 1.27\cdot10^{-6}\ [s^{-2}]\ r\ [m] + 6.97\cdot10^{13}[m^3/s^2]/r^2[m^{-2}]$.

(Case B)

Given are the dimensions, the total mass $M = M_m + M_c$ and the total moment of inertia $\theta = \theta_m + \theta_c$. The densities of the mantle, ρ_m, and of the core, ρ_c, are each assumed constant. The contributions to M of the mantle, M_m, and the core, M_c, can be calculated from the dimensions up to the unknown densities. M and θ are linear in density.

Consider earth as a homogeneous sphere ($r = r_m$, ρ_m) with the core superimposed ($r_c = 0.545\ r$, $\Delta\rho_c = \rho_c - \rho_m$). Non-indexed quantities (V and W) belong to the whole Earth.

$M = gr_m^2/G = \mathbf{5.97\cdot10^{24}}$ kg $= V\underline{\rho}$, from which follows

$\underline{\rho} = \mathbf{5511}$ kg/m^3.

$V\rho_m + V_c\Delta\rho_c = M$

$W\rho_m + W_c\Delta\rho_c = \theta$

Earth volume $V = 4\pi/3\ r^3 = 4.189\ r^3 = \mathbf{1.083\cdot10^{21}\ m^3}$,

core volume $V_c = 4\pi/3\ r_c^3 = \mathbf{1.752\cdot10^{20}\ m^3}$; $V_c = (r/r_c)^3V = 0.162\ V$,

density-normalized Earth moment of inertia $W = \theta/\underline{\rho} = 2/5\ Vr^2 = 1.676\ r^5 = \mathbf{1.759\cdot10^{34}\ m^5}$

density-normalized core moment of inertia $W_c = \theta_c/\rho_c = 2/5\ V_c r_c^2 = 1.676\ r_c^5 =$ $\mathbf{8.444 \cdot 10^{32}\ m^5}$, $W_c = (r/r_c)^5 W = 0.048\ W$;

given moment of inertia $\theta = 0.33\ M\ r_m^2 = 1.451 \cdot 10^{34}\ [m^5]\ \underline{\rho}\ [kg/m^3] =$ $\mathbf{7.996 \cdot 10^{37}}[kgm^2]$

in contrast to $\theta^* = M\ \underline{\rho} = 9.695 \cdot 10^{37}\ [kgm^2]$; $\theta = 0.825\ \theta^*$ (= 0.33/0.4)

The above equations can be simplified:

$V(\rho_m + V_c/V\ \Delta\rho_c) = M \rightarrow \rho_m + V_c/V\ \Delta\rho_c = \underline{\rho} \rightarrow \rightarrow \rho_m + 0.162\ \Delta\rho_c = 5511 \quad [kg/m^3]$

$W(\rho_m + W_c/W\ \Delta\rho_c) = \theta \rightarrow \rho_m + W_c/W\ \Delta\rho_c = \theta/W \rightarrow \rho_m + 0.048\ \Delta\rho_c = 4545.8 \quad [kg/m^3]$

The solution renders the densities:

$\boldsymbol{\rho_m = 4219\ kg/m^3, \Delta\rho_c = 8465\ kg/m^3 \approx 2\rho_m; \rho_c = 12\ 684\ kg/m^3 \approx 3\rho_m.}$

Of course, the problem can be formulated also by considering the mantle and core individually, i.e.: $V_m\rho_m + V_c\rho_c = M$, $W_m\rho_m + W_c\rho_c = \theta$

as above with $V = 4\pi/3\ r^3 = 4.189\ r^3$ is volume $[m^3]$ and $W = 2/5\ Vr^2 = 1.676\ r^5\ [m^5]$ is the moment of inertia divided by (uniform) density

$V = 4\pi/3\ r_m^3 = 1.083 \cdot 10^{21}\ [m^3]$

$M = gr_m^2/G = 5.97 \cdot 10^{24}\ [kg]$

$\theta = 0.33\ M\ r_m^2 = 1.451 \cdot 10^{34}\ [m^5]\ \underline{\rho}\ [kg/m^3] = 7.996 \cdot 10^{37}[kgm^2]$

$V_m = 4\pi/3\ (r_m^3 - r_c^3) = 4.189(25.860 - 4.182) \cdot 10^{19}\ [m^3] = 9.080 \cdot 10^{20}\ [m^3]$

$V_c = 4\pi/3\ r_c^3 = 1.752 \cdot 10^{20}\ [m^3]$

$W_m = 1.676(r^5 - r_c^5) = 1.675 \cdot 10^{34}\ [m^5]$

$W_c = 1.676\ r_c^5 = 8.444 \cdot 10^{32}\ [m^5]$

Errors easely arise from the confusingly large terms. Simplify the equations in analogous fashion as above.

$\rho_c + ((r/r_c)^3 - 1)\rho_m = M/V_c \quad \rho_c + 5.184\rho_m = 34\ 075$

$\rho_c + ((r/r_c)^5 - 1)\rho_m = \theta/W_c \quad \rho_c + 19.834\rho_m = 94\ 694$

with the solution $\boldsymbol{\rho_c = 12\ 625,\ \rho_m = 4138\ kg/m^3}$

The discrepancies of <2% are due to rounding errors.

Gravity as a function of radius, g(r), is expressed as sum of the two contributions of the spheres ($R = r_m$, ρ_m) and (r_c, $\rho_m + \Delta\rho_c$).

Inside the core ($0 \le r \le r_c$): $g = 4\pi/3\ G(\rho_m + \Delta\rho_c)\ r = 4\pi/3\ G\rho_c r$

In the mantle ($r_c < r \le R$): $g = 4\pi/3\ G\rho_m r + 4\pi/3\ r_c^3 G\Delta\rho_c/r^2 = g = 4\pi/3\ G(\rho_m r + \Delta\rho_c r_c^3/r^2)$

Outside Earth ($r > R$): $g = 4\pi/3\ G(\rho_m R^3 + \Delta\rho_c r_c^3)/r^2 = GM/r^2$

(Case C)

A sphere of radius R and a linear density increase with depth $\rho = \rho_o - \Delta\rho/R\cdot r$ has the mass M = ${}_o\!\int^R \rho dV = \underline{\rho}V$ = **5.97·10^24 kg**, with dV= $4\pi r^2 dr$, and the moment of inertia $\theta = {}_o\!\int^R \rho dW = \rho^*W$ = **$7.996\cdot10^{37}$ kgm²**,with $W = 8\pi R^5/15 = 1.676\ R^5$ = **$1.759\cdot10^{34}$ m⁵**, $dW = 2/3\cdot\rho dVr^2dr = 8/3\cdot\pi r^4\rho dr$ (integration of infinitesimally thin shells of uniform density: $m_s = 4\pi r^2 dr\cdot\rho$, $\theta_s = 2/3\ mr^2$).

With ${}_o\!\int^R r^n dr = r^{n+1}/(n+1)$ occur with n = 2, 3, 4, 5.

$M_o = 4/3\cdot\pi R^3\rho_o$, $M_r = -\pi R^3\Delta\rho_o$, $\theta_o = 8/15\cdot\pi R^5\rho_o$, $\theta_r = -8/18\cdot\pi R^5\Delta\rho$

The equations are

$M_o + M_r = M \rightarrow M_o/V + M_r/V = M/V \rightarrow \rho_o - 3/4\ \Delta\rho = M/V = \underline{\rho} = 5511\ kg/m^3$

$\theta_o + \theta_r = \theta \rightarrow \theta_o/W + \theta_r/W = \theta/W \rightarrow \rho_o - 5/6\ \Delta\rho = \theta/W = \rho^* = 4546\ kg/m^3$

The solution is **ρ_o = 14 198, $\Delta\rho$ = 11582** and thus **ρ_R = 2615 kg/m³**.

These numbers seem plausible.

Comment: the linear $\rho(r)$ comes close to the surface density (2670 kg/m³) but considerably exceeds the PREM density in the lower mantle (CMB: 8925 kg/m³). On average the core density comes out somewhat low.

Comments

Comment: The results of Task 4.1 (Case B) are surprisingly close to recent solutions for $\rho(r)$ as PREM (mantle: 3 380 – 5 566, core 9 900 – 13 100 kg/m³).

Averaging the two solutions (which would only slightly contradict the conditions) leads to even closer agreement of $\rho(r)$ with PREM: mantle: 3 400 – 6500, core 10 800 – 13 400 kg/m³, at r = 1800 km: PREM: 11 810, model: 11790). The deviation is generally <10%, the core density is closely approximated, the mantle density is, however, systematically overestimated.

Most of the gross characteristic numbers describing the Earth are remarkably close to one-digit numbers if expressed in SI units, only the radius of 6371 km does not fit this “rule”:

gravity at surface $g \approx 10\ m/s^2$ (-2%)

surface $A \approx 5\cdot10^{14}\ m^2$ (+2%)

volume $V \approx 1\cdot10^{21}\ m^3$ (+8%)

mass $M \approx 6\cdot10^{24}$ kg (-0.5%)

moment of inertia $\theta \approx 8\cdot10^{37}\ kgm^2$ (-0.05%)

Task 4.2 Solution.

The normal reduction can be carried out with the appropriate horizontal northward gradient, with the value of 0.8 mGal/km (the reference is station 1 in the North), the height and Bouguer reductions with the standard parameters. The relief reduction may be estimated by assuming a 2D topography (no near-field reduction possible without the proper knowledge). To achieve this, plot the topography without vertical exaggeration; estimate the 2D terrain effect graphically by taking the solid angle approach (general: 2.8.3.3; simple diagram or template: 5.6.4). Place a compass on the graphic profile such that you can read or estimate the angle a under which you see the non-occupied part of the Bouguer plate (BP) (mass free, below the station elevation) or the mass which lies above the BP (i.e. above the station elevation).

It should be like this. The regional, reg, can, however, not be estimated before ΔBA has been calculated; reg is not precisely fixed and may deviate a little. Plot topography and the anomalies (you may choose different gravity scales). Estimate the trend of ΔBA and subtract it.

i	x [km]	h [m]	obs [mGal]	$\Delta\delta g_{norm}$ [mGal]	δg_h [mGal]	δg_B [mGal]	δg_{relief} [mGal]	ΔFA [mGal]	ΔFA* [mGal]	ΔBA [mGal]	reg [mGal]
1	0.00	99.98	-25.97	0.00	-30.85	11.20	-6.52	4.88	11.40	0.2	-60.9
2	1.83	510.32	-105.83	-1.46	-157.48	57.16	-5.73	53.11	58.84	1.7	-59.5
3	2.99	691.40	-139.64	-2.39	-213.37	77.44	-5.72	76.12	81.84	4.4	-59.0
4	4.06	980.87	-201.01	-3.25	-302.70	109.86	-12.92	104.94	117.86	8.0	-58.4
5	5.50	550.38	-109.67	-4.40	-169.85	61.64	-6.66	64.58	71.24	9.6	-57.8
6	6.72	810.51	-173.43	-5.38	-250.12	90.78	-14.71	82.07	96.78	6.0	-56.9
7	8.11	150.29	-43.55	-6.49	-46.38	16.83	-10.51	9.32	19.83	3.0	-55.3
8	9.09	153.08	-38.19	-7.27	-47.24	17.14	-2.42	16.32	18.74	1.6	-54.9
9	10.24	155.36	-39.54	-8.19	-47.94	17.40	-1.61	16.59	18.20	0.8	-54.0
10	12.13	120.44	-33.99	-9.70	-37.17	13.49	-1.21	12.88	14.09	0.6	-53.0

Note that all values listed are effects (to be subtracted), not reductions (to be added).

The FA and FA* clearly reflect the topography; no other effect is evident.

The BA is a bell-shaped anomaly with a maximum near the intermediate valley between the two peaks, superimposed on a simple linear trend; it can be constructed by connecting the ends of the profile by a straight line; this is simplistic and may be affected by the accidental choice of the first and last point, but it plausibly reflects the crustal thickening, probably thickest to the north (left) below the axis of the mountain belt. Linearity is an approximate assumption, but lack of information on the Moho allows no more.

Next, check the Bouguer density ρ_B assumption of 2670 kg/m^3 with the aid of a Nettleton analysis (3.6.3.6). The Bouguer density may be changed in 100 kg/m^3 steps. With (5.6 – 1), we calculate the corresponding change of the Bouguer reduction as ΔBA[mGal] = 0.00418 h[m]. For simplicity the change in the terrain reduction is neglected. Starting point is the de-trended BA. The results are tabulated

i	h	ΔBA	BA: 2870	BA: 2770	BA: 2670	BA: 2570	BA: 2470	BA: 2370
1	100	0.4			+0.2	-0.2	-0.6	-1.0
2	510	2.3			+1.8	-0.5	-2.8	-5.1
3	691	2.9			+4.0	+1.1	-1.8	-4.7
4	981	4.1	15.9	11.8	+7.7	+3.6	-1.1	-5.2
5	550	2.3	13.6	11.3	+9.0	+6.7	+4.4	+2.1
6	811	3.4	12.2	+8.8	+5.4	+2.0	-1.4	-4.8
7	150	0.6	+4.5	+3.9	+3.3	+2.7	+2.1	+1.5
8	153	0.6			+1.5	+0.9	+0.3	-0.3
9	155	0.6			+1.0	+0.4	-0.2	-0.8
10	120	0.5			+0.7	+0.2	-0.7	-1.2

A composite plot of topography h and the various BAs shows that the overall correlation with h changes sign at ρ_B 2470±50 kg/m^3. But the correlation varies considerably from point triple to point triple: For points 2 – 4: 2540, 3 – 5: 2590, 4 – 6: 2850, 5 – 7: 2720, 6 – 8: 2670 kg/m^3. A variable ρ_B is a possibility to explain the BA. Interestingly, the overall near-zero correlation occurs at a lower density value than any of the point triple values. At 2670 kg/m^3, the BA forms a simple bell-shaped variation with an apparent half width of about 3.7 km, possibly a little more if at the ends of the profile, the BA>0. For a 2D mass line (cylinder axis) it indicates a ~2 km depth located horizontally between points 4 and 5 (see half width interpretation, 5.6.3).

Originally assumed was a slightly different regional effect calculated for an assumed Moho.

i	x [km]	h [m]	obs [mGal]	$\Delta\delta g_{norm}$ [mGal]	δg_h [mGal]	δg_B [mGal]	δg_{relief} [mGal]	ΔFA [mGal]	ΔFA* [mGal]	ΔBA [mGal]	reg [mGal]
1	0.00	99.98	-25.97	0.00	-30.85	11.20	-6.52	4.88	11.40	0.2	-60.9
2	1.83	510.32	-105.83	-1.46	-157.48	57.16	-5.73	53.11	58.84	1.7	-59.5
3	2.99	691.40	-139.64	-2.39	-213.37	77.44	-5.72	76.12	81.84	4.4	-59.0
4	4.06	980.87	-201.01	-3.25	-302.70	109.86	-12.92	104.94	117.86	8.0	-58.4
5	5.50	550.38	-109.67	-4.40	-169.85	61.64	-6.66	64.58	71.24	9.6	-57.8
6	6.72	810.51	-173.43	-5.38	-250.12	90.78	-14.71	82.07	96.78	6.0	-56.9
7	8.11	150.29	-43.55	-6.49	-46.38	16.83	-10.51	9.32	19.83	3.0	-55.3
8	9.09	153.08	-38.19	-7.27	-47.24	17.14	-2.42	16.32	18.74	1.6	-54.9
9	10.24	155.36	-39.54	-8.19	-47.94	17.40	-1.61	16.59	18.20	0.8	-54.0
10	12.13	120.44	-33.99	-9.70	-37.17	13.49	-1.21	12.88	14.09	0.6	-53.0

Task 5.1 Solution.

Half width estimates.

The curves are fairly peaked. It means that the source mass distribution is poorly represented by a mass point (or sphere) or a mass line (or cylinder). Nevertheless, start with these idealisations. From half widths and amplitudes get depth and size of mass for the 2D and 3D cases.

Compare the two results.

2D (5.6.3.2)

Take the x-Profile; reference BA: –18 mGal, peak –26 mGal; half width at –22 mGal: $w \approx 6$ km $z \approx w/2 \approx 3$ km.

amplitude ≈ 8 mGal; at x = 0.

$\delta g_e = 2Gm^+/z \rightarrow m^+ \approx \delta g_e z/(2G) = 8{\cdot}10^{-5}\ 3\ 10^3/(2 \cdot 6.67{\cdot}10^{-11})$ [MKS] $\rightarrow m^+ \approx 1.8{\cdot}10^9$ kg/m (negative)

A cylinder with $R_c = 1$ km, cross section $F = \pi R_c^2 \approx 3.1\ 10^6\ \mathrm{m}^2$ would have a density contrast $\Delta\rho^+ = m^+/F \approx -600\ \mathrm{kg/m^3}$.

This is the right order of magnitude for rock salt, but on the high side, partly due to crude approximation. The cross section or radius should be bigger: e.g. $R_c = 2$ (1.75) km $\rightarrow$ $\Delta\rho^+ \approx -150$ (–200) kg/m^3 which is fairly realistic.

Effect of 2D mass line: $\delta g = 2Gm^+z/(x^2+z^2)$; $\delta g_e = 2Gm^+/z \rightarrow \delta g = \delta g_e z^2/(x^2+z^2)$.

Listed as x [km], δg [mGal]: 0, 8; 1, 7.2; 3, 4; 5, 2.1, 9, 0.8. Plot this.

3D (5.6.3.1)

Form the average radial curve: $BA(R_i) = \frac{1}{4}(BA(x_i)+BA(x_{-i})+BA(y_i)+BA(y_{-i}))$; $R_i = (x_i^2+y_i^2)^{1/2}$.

x [km]	BA [mGal]	y [km]	BA [mGal]	symm.mean of x and y
-6	-20.2	-6	-22.5	-20.9
-5	-20.9	-5	-24.5	-21.7
-4	-22.0	-4	-23.3	-21.9
-3	-23.1	-3	-24.0	-22.8
-2	-24.0	-2	-24.5	-23.7
-1	-24.9	-1	-25.8	-24.9
0	-26.1	0	-26.1	-26.1
+1	-23.7	+1	-25.1	-24.9
+2	-22.3	+2	-24.0	-23.7
+3	-20.6	+3	-23.6	-22.8
+4	-19.5	+4	-22.9	-21.9
+5	-19.0	+5	-22.5	-21.7
+6	-18.7	+6	-22.3	-20.9

Reference BA – 20 mGal, peak –26 mGal, half width –23 mGal: $w \approx 5.2$ km $\rightarrow z \approx 2w/3 \approx 3.5$ km; amplitude ≈ 6 mGal. What does it mean that the background BA, instead of –18 in 2D is here –20 mGal? Short answer: it demonstrates its uncertainty (~10%).

At R=0, $\delta g_e = Gm/z^2$ with $m = \rho V_s$ and $V_s = 4/3\ \pi\ r_s^3 \approx 4\ r_s^3 \rightarrow \Delta\rho \approx \delta g_e z^2/(4Gr_s^3)$

$\Delta\rho \approx 6{\cdot}10^{-5} \cdot 12{\cdot}10^6/(4 \cdot 6.67{\cdot}10^{-11}{\cdot}10^{-9})$ [MKS] $\approx 2.7{\cdot}10^3$ kg/m^3 (negative).

This is a too large contrast; with radii of 2, 2.4 or 3 km, the sphere volume increases by a factor of 8, 13.5 or 27, respectively, hence $\Delta\rho \approx 330$, 200 or 100 kg/m^3 (negative), which represents a range of plausible density contrast of rock salt versus the Mesozoic sediments.

Effect of 3D mass point: $\delta g = Gmz/(x^2+z^2)^{3/2}$; $\delta g_e = Gm/z^2 \rightarrow \delta g = \delta g_e z^3/(x^2+z^2)^{3/2}$.

Listed as x [km], δg [mGal]: 0, 6; 1, 5.5; 2.6, 3; 4, 1.8, 8, 0.4. Plot this.

Discussion 2D/3D

The contours suggest that both 2D and radial symmetry are obvious oversimplifications.

The cross profile (x) is more asymmetric and suggests that the next step of more realistic approximation could be a triangular cross section of differently steep sides (could be a task for inversion of the x profile).

The source characteristics are fairly similar (for $\delta\rho \approx -200$ kg/m^3), the variation about their mean is of the order of 10%):

2D: $z \approx 3$ km, $r_s \approx 2.4$ km –8 mGal

3D: $z \approx 3.5$ km, $r_s \approx 1.75$ km –6 mGal

The 2D model is shallower and more massive, partly because of the different amplitudes resulting from the different reference gravity value –18 instead of –20 mGal.

The reference value of an anomaly is generally difficult to estimate, and its meaning is problematic and could be assessed only in a more regional study (scale 100 – 1000 km). BA values are expected zero on average for regions approximately at sea level; here we are in the epicontinental North Sea of some 100 water depth. The expected mean value of +6 mGal (Bouguer reduction) means that the FA is about –25 mGal, suggesting a broad scale mass deficit in the crust and/or upper mantle. But as said, this is a very preliminary assessment.

If the **Bouguer plate** and **solid angle $\Delta\Omega$** approach (5.6.1, 5.6.4) is applied to the extreme value Δg_e (–6 to –8 mGal above the seafloor depression "Görtel"), an estimate can be attempted on the basis of the expression for the Bouguer plate, $\Delta g_e \approx G\Delta\Omega\Delta\rho d$ (2.8.1.1, rule 1; (2.8.3 – 1), (2.8.7 – 1), Fig 2.8 – 1). The numerical estimation may be facilitated by applying (5.6 – 1) for the Bouguer plate: $\Delta g \approx 0.1 \times d \times \Delta\rho/2390$, where Δg in mGal, d in m, $\Delta\rho$ in kg/m^3. In order to incorporate the limited solid angle $\Delta\Omega$ (instead of 2π) immediately, multiply it with $\Delta\Omega/2\pi$ from which follows $\mathbf{d \approx \Delta g \times 23900/\Delta\rho \times 2\pi/\Delta\Omega}$. This method is most easily applied to the 2D case where $\Delta\Omega = 2\Delta\alpha$, where α is the plane angle under which the layer or body of thickness d and effective density $\Delta\rho$ is seen from the station. Beside the estimate of the solid angle $\Delta\Omega$, d and $\Delta\rho$ are needed (assumed, estimated?) a priori. Neither quantity can be derived directly from the gravity anomaly. Beside δg_e, the variation of $\delta g(x)$ cannot be estimated on the basis of the solid angle, only roughly on the Bouguer plate effect.

This would then become a crude trial and error approach and could lead to an initial model for more quantitative modelling and inversion.

Sketch the essential geometrical parameters of the 2D section; it can clarify the situation and facilitate the estimate of $\Delta\Omega$. From the sketch we may read/estimate the following numbers. For the thickness of the sediments and the salt layer we have only the area of near-outcrop 1 km to the SW and the borehole penetrating into *Zechstein* salt (?) at 718 m depth; the bottom had not yet been reached at 3010 m. The vertical extent of the density contrast boundary between the top sediments and the rock salt is greater than the 718 m where it still dips and deepens to the NE; for the dip angle an average of 17° NE has been estimated; if extrapolated to the distance where the BA levels off, 5 to 6 km on both sides. This would lead to a sediment thickness between 2 and 2.5 km. If the dip becomes continuously gentler, the lower estimates may be more appropriate.

Extrapolation of 17° dip upwards towards the outcrop in the SW projects the intersection of a straight line further SW: the boundary must thus be steeper near the surface: The far side wall of the salt stock should be less steep on the basis of the gravity anomaly gradient on both sides (although details are unknown). The SW-ward gradient is nearly constant ~1.1 mGal/km to 6 km distance, suggesting a constant dip shallower than 17°. For a long gentle ramp the horizontal gravity gradient is $\partial g/\partial x \approx \pi G\Delta\rho\sin 2\alpha$ (α = dip angle; the expression follows from differentiating (2.8.7 – 13) after x and letting $\varphi_A - \varphi_B \approx \pi$ and $r_A/r_B \approx 1$; see Fig. 2.8 – 14c); this would suggest a dip of ~10°; on the other hand, 10° would correspond to only a little more than 1 km sediment thickness increase over 6 km distance or a smaller density contrast. The discrepancy partly results from the approximate relationship and possibly from lateral density variations.

The nearby NE-ward gravity gradient out to 4 km is on average ~1.6 mGal/km; applying the above expression for α, we obtain ~14°, which is acceptable. For 1.5 km from the minimum to the NE, the gradient is much steeper, ~4.7 mGal/km, and the approximate expression fails ($\sin\alpha > 1$), which is no surprise in view of the approximations made.

Finally, taking the above expression for d with the values $\Delta g_e \approx 8$ mGal, $\Delta\rho \approx 150$ kg/m^3 and $\pi/\Delta\Omega \approx 1.8$, we obtain d ≈ 2300 m, well within the above estimate of 2 – 2.5 km.

The cross profile BA is taken as crudely the sum of **harmonic functions** (sine waves with wavelength or wave number (spatial frequency), amplitude and phase:

$BA(x) \approx c + a_1 \sin(k_1[x + x_1]) + a_2 \sin(k_2[x + x_2]) + a_3 \sin(k_3[x + x_3]) + \ldots$

Estimate graphically a few (maximum 3) phases, not by a formal harmonic analysis which would result in an approximate Fourier series. Rather estimate a few waves which, when superimposed will approximate the cross profile roughly, especially near the gravity minimum.

First guesses can be done "by hand" or by inspection; since such guesses are very uncertain, the estimated waves should be calculated and checked, how closely they fit the observations. The whole given anomaly profile need not be fitted, it may suffice that the shortest wave fits only the central part.

Dropping the units mGal and km, we may get:

$c \approx -21$,

$\lambda_1 \approx 14$, $a_1 \approx 2$, $x_1 \approx -2.5$;

$\lambda_2 \approx 8$, $a_2 \approx 1.5$, $x_2 \approx -2.3$;

$\lambda_3 \approx 4$, $a_3 \approx 1.5$, $x_3 \approx -1.1$.

The corresponding wave numbers $2\pi/\lambda$ are $k_1 \approx 0.45$, $k_2 \approx 0.8$, $k_3 \approx 1.6$ [km^{-1}], or in degree $360/\lambda$ are $k_1 \approx 26$, $k_2 \approx 45$, $k_3 \approx 90$ [$°/km^{-1}$].

The trial would start with

$BA(x) \approx -21 + 2 \sin(26[x - 2.5]) + 1.5 \sin(45[x - 2.3]) + 1.5 \sin(90[x - 1.1])$

The results are poor at the sides, but they roughly depict the dominant wavelengths of the anomaly (which is anyway not harmonic and repeated infinitively, as by the way, no anomaly is on Earth).

Downward continuation is problematic, since it would penetrate the source region where the density is not uniform in the depth range considered. The exercise has thus no more value than a demonstration of the effects. A glance on the amplitude magnification with depth (phase is, of course unchanged and uninteresting) can be judged from the following table of factors $f = \exp(2\pi z/\lambda)$ and amplitudes for the depths shown (depending on the wavelength). The last three columns are the depth variations or (double) amplitudes [in km] of the corresponding undulations of density contrast surfaces (to be superimposed, at least near the gravity minimum) of mean depth z for λ_1, λ_2, λ_3, where the density contrast has been assumed (for simplicity) to be 240 kg/m^3.

f	z		z [km] for				hence [mGal]				layer double amplitude [km]		
			$\lambda_{1(14)}$	$\lambda_{2\,(8)}$	$\lambda_{3\,(4)}$		a_1	a_2	a_3		λ_1	λ_2	λ_3
1	0		0	0	0		2	1.5	1.5		0.4	0.3	0.3
1.65	$\lambda/4\pi$		1.1	0.64	0.32		3.3	2.4	2.4		0.66	0.48	0.48
2.72	$\lambda/2\pi$		2.2	1.3	0.64		5.4	4.1	4.1		1.1	0.8	0.8
7.4	λ/π		4.5	2.5	1.3		15	11	11		3	2.2	2.2
20	$3\lambda/2\pi$		6.7	3.8	1.9		40	30	30		8	6…	6

Depth amplitudes (half values) that are larger than mean depths z are marked blue, those larger than half the mean depth are marked green. The ratios of amplitude/mean depth are listed below:

z		z [km] for				layer double amplitude [km]				ratios a/z		
		$\lambda_{1(14)}$	$\lambda_{2\,(8)}$	$\lambda_{3\,(4)}$		λ_1	λ_2	λ_3		λ_1	λ_2	λ_3
0		0	0	0		0.4	0.3	0.3		∞	∞	∞
$\lambda/4\pi$		1.1	0.64	0.32		0.66	0.48	0.48		0.3	0.38	0.75
$\lambda/2\pi$		2.2	1.3	0.64		1.1	0.8	0.8		0.27	0.31	0.62
λ/π		4.5	2.5	1.3		3	2.2	2.2		0.33	0.44	0.88
$3\lambda/2\pi$		6.7	3.8	1.9		8	6…	6		0.6	0.8…	1.6

Note that the ratio goes through a minimum. There the approximation is least deceitful; this is at ~2, ~1 and ~1.8 km depth (but in the latter case, 0.62, is bad anyway). The amplitudes are inversely proportional to the assumed density contrast and thus less critical for larger values, however, they cannot be realistically much larger than assumed.

Task 5.2 Solution.

The data, as described, can be written

$f(x) = b_o+a_ox + a_1\sin(2\pi/\lambda_1 x–b_1) + a_2\sin(2\pi/\lambda_2 x–b_2) = b_o+a_ox + a_1\sin(k_1x–b_1) + a_2\sin(k_2x–b_2)$, with a constant, a linear trend and two sine waves of amplitudes a_i, wavelengths λ_i and phases b_i.

Call the length unit L.

To determine the components, plot the given values. Then you may approach the solution by simple arguments, trial and error iteration and considering "simple" numbers.

The short wave is obvious: the distance Δx between maxima and minima is 2L, hence $\lambda_1 = 4L$.

The intervening points describe something like a half sine wave over 8 or 9 units, hence take for $\lambda_2 \approx$ 16L (if you choose 18L, a check will reveal the choice to be wrong).

A positive trend (rise) to the right is evident.

The amplitude of the short wave can be estimated with the aid of straight lines connecting the neighbouring maxima or minima and measuring the δg difference of the intervening extrema; take the mean value (about 4 mGal).

For the long wave take the intermediate points (of the short wave), connect the end points and estimate the maximum gravity difference: about 1.5 mGal.

The linear trend may be guessed by looking at the whole data set, as about 2 mGal across the whole distance range, i.e. about 0.25 mGal/L.

Of no importance for upward or downward continuation are the phase relations, but they are, of course needed for the reconstruction of the gravity effects at different elevations or depths and for interpretations of the equivalent strata.

Minima and maxima of the short wave occur at x = 1, 3, 5, etc.; hence, at least roughly, for x = 0: $\sin(–b_1) = 0$, hence $–b_1 = \arcsin(0) = 0$ modulo π; for x = 1, $\sin(2\pi/4 – n\pi) = –1$; $\arcsin(–1) = – \pi/2$ modulo 2π. It is a negative sine wave or a sine wave shifted by the phase angle $\pm\pi$, with the amplitude 2: $–2\sin(\pi x/2) = 2\sin(–\pi x/2) = 2\sin\pi(x/2 \pm 1)$.

For the long wave it looks as though it is about zero at the first value x = 1, thus at x = 0 the value would be $–1.5\sin(\pi/8)$ and the phase shift would be –1L or –1. Hence with the amplitude 1.5: $1.5 \sin(2\pi x/16 – \pi/8) = 1.5 \sin[\pi/8(x – 1)]$.

The linear trend of about 0.25x is represented by a straight line which intersects the abscissa somewhere; the mean value of $\delta g = 0.69$,which mainly results from the long wave, very slightly counteracted by the short wave; from this one may guess that the straight line hardly contributes to the average, implying the intersection is somewhere in the middle of the segment from 1 to 10; assume x = 5; if true: $a_o = –1.25$: $y = 0.25x – 1.25$.

<u>The full expression</u> is:

$\delta g(x) = –1.25 + 0.25x – 2\sin(\pi x/2) + 1.5 \sin[\pi/8(x – 1)]$

This expression should be verified by a few test calculations.

n	x	δg		lin.trend	shortwave	longwave
1	1	-3.00		-1.0	-2.	0.0
2	2	-0.18		-0.75	0.	0.57
3	3	+2.56		-0.50	2.	1.06
4	4	+1.13		-0.25	0.	1.38
5	5	-0.50		0.0	-2.	1.50
6	6	+1.63		+0.25	0.	1.38
7	7	+3.88		+0.50	2.	1.06
8	8	+1.32		+0.75	0.	0.57
9	9	-1.00		+1.0	-2.	0.0
10	10	+0.68		1.25	0.	-0.57

For upward or downward continuation subtract the linear trend and define $z > 0$ downward. Then the amplitudes of the two waves vary with z and k as e^{zk}; $zk = 2\pi z/\lambda$, where $z^* = z/\lambda$ is the decisive parameter: $\delta g(z^*) = \delta g(0)e^{2\pi z^*}$.

With the λ and z values $(10^{0\text{-}5})$ m, above, the ratio z/λ varies from 10^{-5} to 10^{5}, i.e. 10 orders of magnitude: $e^{2\pi z^*} = (535.5)^{z^*}$. Even with the $z^* = 1$ this is a very large factor, 535, with $z^* = -1$, the factor is $1.9\cdot10^{-3}$; with $z^* = 10$, the factor becomes about $2\cdot10^{27}$, and with $z^* = -10$, it is $0.5\cdot10^{-27}$!

One example: choose L=1 km, z=2 km (up: –2 km), λ_1 = 4 km,: dg values: –46.3, 0, +46.3 (–0.09, 0, +0.09);

λ_2 = 16 km, $e^{2\pi z^*} = e^{\pi/4} = 2.2$ ($e^{-2\pi z^*} = e^{-\pi/4} = 0.46$).

				2 km down				2 km up		
n	x	δg		lin.trend	shortwave	longwave		δg	shortwave	longwave
1	1	–47.3		–1.0	–46.28	0.00		–1.09	–0.09	+0.00
2	2	–0.5		–0.75	0.	1.25		–0.49	0.	+0.26
3	3	+48.1		–0.50	+46.28	2.32		–0.11	+0.09	+0.48
4	4	+2.8		–0.25	0.	3.03		+0.38	0.	+0.63
5	5	–43.0		0.0	–46.28	3.29		+0.59	–0.09	+0.68
6	6	+3.3		+.25	0.	3.03		+0.88	0.	+0.63
7	7	+49.1		+.50	46.28.	2.32		+1.07	+0.09	+0.48
8	8	+2.0		+.75	0.	1.25		+1.01	0.	+0.26
9	9	–45.3		+1.0	–46.28	0.00		+0.95	–0.09	+0.00
10	10	+0.0		1.25	0.	–1.25		+0.99	0.	–0.26

Taking the above values λ = 100 km, δg = 3 mGal; $\Delta\rho$ = 300 kg/m^3; z = 30km, $z^* = 0.3$, $e^{2\pi z^*} \approx 6.6$, and the gravity anomaly downward continued is about 20 mGal.

If generated by an equivalent stratum directly below, its surface density would be ρ^* [kg/m^2] ≈ 23900 δg [mGal] ≈ $4.8\cdot10^5$ kg/m^2. If the equivalent stratum is replaced be a (moderate) undulation of amplitude d [m] = $\rho^*/\rho = 4.8\cdot10^5/3\ 10^2 = 1.6\ 10^3$ m = 1.6 km (this is “moderate” in view of 100 km wavelength). If r = 100 kg/m^3, d ≈ 4.8 km, still "moderate". You may check the results by upward continuation back to the surface.

If, as in the above data set the shorter wavelength is 1/4 of the 100, i.e. λ = 25 km, δg = 2 mGal and z = 30 km, z* = 1.2, the result of downward continuation is totally unrealistic. The amplitude factor $e^{2\pi z^*} = e^{7.5} \approx 1881$, i.e. 285 times the above value of 6.6, and all derived quantities must be multiplied by 285. The amplitude of 460 km would greatly exceed the mean depth of 30 km, and thus the result is purely formal or meaningless. Alternatively, the vertical amplitude can be kept moderate, e.g. 460/100 = 4.6 km, on the expense of raising the density by a factor of 100, i.e. 30000 or about 10 times the absolute density at the depth of 30 km, certainly also totally unrealistic. – This example demonstrates the power of the exponential function governing the downward continuation. It is, of course, also true in the opposite direction, away from the source (equivalent stratum), usually in upward continuation: the gravity amplitudes quickly decay to essentially zero.

Although no strict rules can be formulated, magnification (or decay) factors of 10 to100 (0.1 to 0.01) may be chosen as limits of plausibility. The corresponding $z^* = z/\lambda = \ln 10/2\pi = 0.37$ or $\ln 100/2\pi = 0.73$ (down or up). The cases of scales and continuation depths mentioned in the task encompass values $10^{-5} < z^* < 10^5$; only the ones close to z* = 0 (± 0.7) may be sensible!

Task 5.3 Solution.

The classical comparison is between effects of the point mass and the mass line. However, this is not exact for all cases which are usually more complicated.

(a)

The classical half width rules: 3D point mass – 2D line mass

Expressions:

3D (5.6 – 2a), (5.6 – 2b)

Half width w as function of depth z: $w = 2(4^{1/3} - 1)^{1/2} z = 2 \cdot 0.76642 \cdot z \approx 3z/2 \rightarrow z \approx 2w/3$,

mass anomaly $\Delta m = \delta g_e z^2/G \approx (4/9)\delta g_e w^2/G$

2D: depth $z = w/2$; $\delta g_e = 2G\Delta m^+/z \approx 4G\rho^+/w \rightarrow \Delta m^+ = \delta g_e z/2G = \delta g_e w/4G$

3D and 2D can be compared by ratios of the "results" z_3/z_2 and $\Delta m/\Delta m^+$.

$\mathbf{z_3/z_2 \approx 4/3 = 1.3333}$ (33%) or $\mathbf{z_2/z_3 \approx 3/4 = 0.75}$ (-25%)

$\Delta m_3/\Delta m^+_2 = 2z = 16/9\ w$. What does it mean, especially in view of the fact that $\Delta m_2 \rightarrow \infty$?. Conversion to densities is difficult because they depend arbitrarily on the assumed radii r; the sphere (3D, radius r) and a short cylinder (2D, length l = 4/3 r) have the same volume and hence the same density ratio as above ($V = 4/3\ \pi r^3$, $\Delta\rho_3/\Delta\rho_2^* = 16/9\ w$), but again, this means little.

How about comparing the density of the sphere (3D) touching the surface ($r = z \approx 2w/3$) with the density of the equivalent cylinder (2D: $r = z = w/2$), which means that w and δg_e are the same (the same anomaly is interpreted!) and the depths and radii differ. In this case $\Delta\rho_2 = \delta g_e/(\pi w G)$ and $\Delta\rho_3 = 9\delta g_e/(8\pi w G)$, and the ratio $\mathbf{\Delta\rho_3/\Delta\rho_2 = 9/8 = 1.125}$ (12.5%) or $\mathbf{\Delta\rho_2/\Delta\rho_3 = 8/9 = 0.889}$ (-11%).

2D applied to 3D situations, both depth and equivalent density (as defined above) are underestimated!

Finite length λ of a mass line reduces the effect by a factor $a = \lambda/(x^2 + \lambda^2 + z^2)^{1/2}$ (correction for finite length) where $0 < a < 1$. The errors of applying 2D formulae to such 3D cases are reduced accordingly.

Extend the comparison of point and line masses thin plates: to a 3D circular disk and a 2D infinite strip at depth z and base the comparison on **Fig. 5.6 – 1**: gravity effect increasing with increasing body *half width* w from zero to infinity, w/z from 0 (point and line) to ∞ (Bouguer plate). Here the sources are given and their effects are compared as functions of w/z, in contrast to the anomaly being given and z and mass to be found.

The figure somewhat schematically shows both effects (2D strip and 3D disk) rising with w linearly from w = 0 for narrow widths (w/z << 1). However, consider that the mass of the 2D strip effectively grows linearly with its width w (~ρ*w); for small w, the effect, as the solid angle $\Omega_2 = 2\alpha$, grows also linearly ($\alpha/2 = \arctan(w/2z) = \arcsin(w/2r) = \arccos(z/r)$, see Fig. 5.6 – 1 and note that r = 1, z = $\cos(\alpha/2)$ and for small w, $\alpha/2 \approx w/2z \approx w/2r$). In contrast, the mass of the circular 3D disk and the effect grows, with the solid angle Ω_3, from 0 as $R^2 = w^2$. Thus, the growth rates of Ω_2 and Ω_2 with w differ. At $w = 0$, $\Omega_2/\Omega_3(w) \to \infty$, dropping rapidly and approaching 1, as both approach 2π with $w \to \infty$.

$\Omega_2 = 2\alpha = 4 \arctan(w/2z) \approx 2w/z \approx 2w$ for small w ($z \approx r = 1$).

$\Omega_3 = 2\pi rh = 2\pi r(r - z) = 2\pi r^2(1 - \cos(\alpha/2)) = 2\pi r^2(1 - \cos(\arctan(w/2z))) \approx$

$2\pi r^2(1 - 1 + \arctan^2(w/2z)) \approx 2\pi(w/2z)^2 \approx \pi w^2/2$ for small w ($z \approx r = 1$).

The absolute difference has a maximum at $w/z = 1.68$ as shown below.

Starting with the same expressions for Ω_2 and Ω_3 as above, we find the maximum by differentiating $(\Omega_2 - \Omega_3)$ after w:

$\partial\Omega_2/\partial w = 4/[(1+(w/2z)^2 \cdot 2z] = 2/[z((1+(w/2z)^2]$

$\partial\Omega_3/\partial w = 2\pi r^2 \sin(\arctan(w/2z)/[(1+(w/2z)^2 \cdot 2z]$

$\partial(\Omega_2 - \Omega_3)/\partial w = 1/[z(1+(w/2z)^2] (2 - \pi r^2 \sin(\arctan(w/2z)) = 0$

Let $r = 1$ and $\arctan(w/2z) = \alpha/2$

$\to \alpha/2 = \arcsin(2/\pi) \approx 40° \to$ $\mathbf{w/z = 2\tan(\alpha/2) \approx 1.68}$

$(\Omega_2 - \Omega_3)/(2\pi) = 0.44 - 0.23 = 0.21$ and $\Omega_2/\Omega_3 \approx 1.9$ at $\alpha/2 = 40°$.

(b)

Maximum depth rules have not been treated in detail, because the authors are sceptical about their value (see 5.6.7). If the point or line mass of infinite density is considered the maximum depth where the mass must be found, the same ratios apply as above.

(c)

For the ***edge of a disk*** discuss Fig. 5.3 – 1. The edge is a vertical 1 km step with a density contrast $\rho = 238.5$ kg/m^3 (the Bouguer plate effect is 10 mGal). The 2D step is compared to the edge of vertical cylindrical disks of 10 and 3 km radius (cyl10 and cyl3) along gravity profiles to 5 km distance from the edge to both sides.

The limited disks have, of course, smaller effects than the half-infinite 2D disk. The middle part of the figure shows that the difference (step – cyl) has a maximum above the edge (0.5 mGal or 5 % of the Bouguer effect for cyl10 and 1 mGal or 10 % for cyl3). Away from the edge of cyl10, the difference drops (at 5 km distance to 2 % outside and 3 % on the disk); the asymmetry results from the approach to the opposite disk edge. For cyl3 the difference is everywhere larger and has nearly a plateau above the disk, rising toward the opposite disk edge to 20 % at 5 km distance. If the profile extends to only 3 km from the edge, one might be easily be mislead in the interpretation.

It may be more revealing to consider a lsq fit of an assumed 2D step model to the "3D reality" of cyl10 and cyl3. Top of Fig. 5.1 – 3 shows that the density-adjusted 2D step effects (lines: solid cyl10, dashed: cyl3) fit the 3D "observations" (dots) quite well. The residuals ("observed" – 2D model, opposite to "difference", above) are shown in the centre part of the figure. They are much smaller then the above differences and add up to zero (due to the lsq fitting). The minimum near the edge is reduced to 0.2 mGal (cyl10) and 0.25 (cyl3). The lsq densities are, expectedly, smaller (cyl10: 2.36 kg/m^3, cyl3: 2.17 kg/m^3) by 1 and 9 %. – The "errors" are of the same order of those discussed for the above comparison of the sphere and the cylinder.

Applying INVERT, one may be tempted to fit a more complex model by giving the model geometry some freedom to adjust to the given disk effect. The reader is invited to extend the task accordingly.

(d)

Dipoles: 2D line dipoles and 3D point dipoles; discuss Fig. 5.6 – 4 (solid lines: depth ratio of dipole masses 2, dashed lines: depth ratio 1.5 – to be referred to by *r2* and *r1.5*).

What are the consequences of interpreting a given 3D anomaly with a 2D model?

The comparison is more difficult than in the case of line vs. point effect.

The line dipole effect has a more pronounced negative side lobe, but especially if the observations are not extended far enough (say to $x > 4z$, where z is the depth of the top mass), the negative sides may be taken as reference (which is usually uncertain) and the 3D curve will look very similar to a 2D curve, which however, has still a lower maximum by 13 % (*r2*) and even 30 % (*r1.5*). Since the amplitude of the maximum is proportional to the mass, it or density would come out too small, however, consider the mass or density comparison of point and line masses, above. Moreover, the zero-crossing of the curves which are closer to the maximum for 2D than for 3D and which might be taken as indicator of the depth ratio cannot be used.

Apply INVERT for a more detailed comparison of 2D and 3D dipoles.

(e)

Undulated density contrast surfaces. If two "plane 2D" undulations ($\delta g_1(x) = \Delta g_1 \exp(k_1 x)$ and $\delta g_2(y) = \Delta g_2 \exp(k_2 y)$, wavelengths $\lambda_1 = 2\pi/k_1$ and $\lambda_2 = 2\pi/k_2$) are superimposed at right angles, the contours may become elliptical, even circular and the effective wavelength λ of the 3D undulation is shorter than that of either wave and $k = 2\pi/\lambda < \lambda_1$ and λ_2. One may think that combining the waves in wave number space may be achieved by letting $k = (k_1^2 + k_2^2)^{1/2}$.

It is, however, known that principally each wave is to be treated individually and independently, e.g., in downward continuation (this is just the advantage of the harmonic functions!). Amplitude growth with depth is given by $\exp(k_1 z)$ and $\exp(k_2 z)$, and the waves sum to $\Delta g_1(x)\exp(k_1 z) + \Delta g_2(y)\exp(k_2 z)$ (amplitudes $\Delta g_1 + \Delta g_2 = \Delta g_o$). Letting, for simplicity, k_1 and $k_2 = k$ and $\Delta g_1 = \Delta g_2(x) = \Delta g_o/2$, means that the combined 3D wave grows in amplitude with depth exactly as the individual waves grow. – Compare this to $\delta g_o \exp(kz)$ with $k = k\sqrt{2}$ according to the above suggestion. Obviously the amplitude growth with depth would be proportional to $\exp(z\, k\sqrt{2})$, and generally, $\frac{1}{2} e^{zk} + \frac{1}{2} e^{zk} = e^{zk} \neq e^{zk\sqrt{2}}$. It is true only for $z = 0$, i.e. at the surface.

Keep the problems of the harmonic functions in mind which are infinitely spread out in space; a closed "3D" anomaly is not really harmonic and the wave approximation is too crude.

Task 5.4 (2.1.3) Solution.

The “equivalent” mass point m lies 250 km N of the mountain front. The mass of the block is $m = abh\rho = 10^3 \cdot 5\cdot10^2 \cdot 5\ \text{km}^3 \cdot 2.5\cdot10^3\ \text{kg/m}^3 \cdot 109\ (\text{m/km})^3 = 6.25\cdot10^{18}$ kg.

With the distance x, the horizontal gravitation is $g_h \approx Gm/x^2$, where $G = 6.67\cdot10^{-11}$, m as above and, respectively, x = 450, 650 and 850 km,

hence

$1/450^2\ \text{km}^{-2} = 0.45^{-2}\cdot10^{-12}\ \text{m}^{-2} \approx 5\cdot10^{-12}\ \text{m}^{-2}$,

$1/650^2\ \text{km}^{-2} = 0.65^{-2}\cdot10^{-12}\ \text{m}^{-2} \approx 2.4\cdot10^{-12}\ \text{m}^{-2}$,

$1/850^2\ \text{km}^{-2} = 0.85^{-2}\cdot10^{-12}\ \text{m}^{-2} \approx 1.4\cdot10^{-12}\ \text{m}^{-2}$.

The results are for 200 (450) km distance $g_h \approx 2\cdot10^{-3}\ \text{m/s}^2 = 200$ mGal

for 400 (650) km distance $g_h \approx 1\cdot10^{-3}\ \text{m/s}^2 = 100$ mGal

for 600 (850) km distance $g_h \approx 0.6\cdot10^{-3}\ \text{m/s}^2 = 60$ mGal

The corresponding deflections of the vertical are, respectively, $\gamma_{rad} = g_h/g \approx 2, 1, 0.6\ 10^{-4}$

or $\gamma' = \gamma_{rad}\ 60\cdot180/\pi \approx 0.7', 0.34', 0.2'$ (i.e. in minutes).

Task 5.5 Solution.

(a)

The simplest argument to find an answer would be to assume that the final equilibriun state of the ***Himalaya/Tibet*** crust after complete peneplanation to zero elevation ($h_\infty \rightarrow 0$) would be the restoration of the previously subducted Indian crust to its pre-subduction position at the surface, but is presently in approximate equilibrium in the elevated Himalaya/Tibet block. That implies a crustal thickness reduction from about 75 to about 35 or 40 km by denudation or erosion and removal of the upper 35 ± 5 km, and the rocks ultimately exposed at the surface would be those presently buried 35 km deep.

For a somewhat quantified treatment, assume simple Airy isostasy which implies the equilibrium: $h_o \rho_o = r_o \Delta\rho$ or $r_o = h_o \rho_o/\Delta\rho$ or at any point in time $r = h \rho_o/\Delta\rho$.

This equally holds for changes: $\Delta r = \Delta h \rho_o/\Delta\rho$.

The lower crust is not involved.

From $d = h + c + r$ follows that the thickness Δd of the removed layer can be written for any point in time t: $\Delta d = \Delta h + \Delta r = \Delta h(1 + \rho_o/\Delta\rho)$.

For $t \rightarrow \infty$, $h \rightarrow h_\infty = 0$, $\Delta h = h$, and the ultimate crustal thickness is $c_\infty = h - h + c + r - r = c_o$, while the total thickness removed is $d_\infty = h + r = h(1 + \rho_o/\Delta\rho)$.

First test the assumed parameter set for compatibility with the H/T situation and Airy isostasy. If all the above assumed parameters are inserted, a discrepancy results. With h_o, ρ_o and $\Delta\rho$, the crustal root thickness r_o would come out 8 km greater than assumed: $r_o = 5 \cdot 2600/300 \approx 43$ km (instead of the assumed 35). The assumptions are not quite consistent with Airy isostasy: r_o and $\Delta\rho$ may be too small, ρ_o and h_o may be too large, and the density ρ_c of the lower crust may have a deficit (laterally, estimated to be about $\Delta\rho \cdot 8/r \approx 75$ kg/m^3 relative to India, which indeed agrees with the above assumptions). Without that lateral density deficit and if small changes are kept within their assumed uncertainty limits, compatible parameter sets could be, e.g. [$h_o = 4.8$ km, $r_o = 40$ km, $\rho_o = 2500$ kg/m^3, $\Delta\rho = 300$ kg/m^3] or [$h_o = 5$ km, $r_o = 36$ km, $\rho_o = 2550$ kg/m^3, $\Delta\rho = 350$ kg/m^3].

The interesting question would be, if the process affects the subducted Indian crust, e.g., by increasing its mean density through intrusions and/or uplift.

Since the three sections of the vertical are not defined by internal boundaries (but by comparison with India), only the "lower crust" of thickness c is considered (which in the mass balance through time, as was discussed above, and remains constant; it could be thus neglected. An additional load is proportional to $c \cdot \Delta\rho_c$, and the mass balance through time is:

$h\rho_o + c\Delta\rho_c = r \cdot \Delta\rho$, from which follows: $r = h \cdot \rho_o/\Delta\rho + c \cdot \Delta\rho_c/\Delta\rho$.

There are now two downward loads balanced by the upward buoyancy of the crustal root which must thus be larger than in the above case. It means for the end of peneplanation ($h_\infty \rightarrow 0$) that the root does not completely disappear, a reduced root remains and the thickness of the layer removed by erosion is smaller than estimated above.

Take as a numerical example: $\rho_o/\Delta\rho \approx 9$, $\Delta\rho_c \approx 100$ kg/m^3 $\rightarrow \Delta\rho_c/\Delta\rho \approx 0.3$:

$r = 9 \cdot h + 0.3 \cdot c$, and equally for any change: $\Delta r = 9 \cdot \Delta h + 0.3 \cdot c$.

For $t \to \infty$, $\Delta r_\infty = r_\infty \to 0.3\ c \approx 10$ km (this is proportional to $\Delta\rho_c$ and to c.

Hence $d_\infty = r_o - r_\infty \approx 35 - 10 = 25$ km.

An original crust of 35 to 40 km thickness would then after peneplanation have grown to about 50 km thickness. That is a value close to the crustal thickness of Proterozoic to parts of Finland.

(b)

Sediment accumulation at a continental margin

First, the relevant upper mantle layer thickness follows from (sketch!) $d_w + d_c + d_m = d_s + d_c$, whence: $d_m = d_s - d_w$.

Equilibrium through time, i.e. also between the initial and final stages is expressed as:

$d_w\rho_w + d_c\rho_c + d_m\rho_m = d_s\rho_s + d_c\rho_c \to d_s(\rho_m - \rho_s)\rho_m = d_w(\rho_m - \rho_w)$ whence:

$d_s = d_w(\rho_m - \rho_w)/(\rho_m - \rho_s)$

Inserting the above numbers with their errors and assuming Gaussian error propagation, we get $d_s \approx 5\ (2170 \pm 100)/(700 \pm 140) \approx 5\ (2170 \pm 5\%)/(700 \pm 20\%) \approx$

$\approx 5 \cdot 3.1 \pm 21\% \approx 5 \cdot (3.1 \pm 0.7) \approx 15.3 \pm 3.5$.

Likely solutions could range from 12 to 18 km sediments. The simple isostatic model gives the right order of magnitude for the thickness of orogenic sequences. All there is needed is a receptacle for sediments.

(c)

Gravity modelling will always offer possibilities to test isostatic models. Both aspects must be compatible with the observations.

However the above isostatic models are extremely schematic and will lead only to a rough agreement of mean anomalies, not to a detailed one-to-one fit of observed gravity. Extreme values, e.g. at block centres or near the edges may be discriminative. Gravity is, of course, not applicable to comparison with initial and final stages, because not observed; however, geological structures which are at different stages of development can be compared with each other also in their gravity anomalies.

As the isostatic approach essentially compares large blocks with each other, edge effects are important. It must, however, be taken into account that they are strongly modified by geometrical details of the edges.

Detailed modelling is left to the reader, and the use of INVERT (Chap. 7) is recommended. This is not treated further in the Task section.

Related problems are discussed in Chaps. 5, 6 and 7, such as:

- edge effects in 5.6.8 (Fig. 5.6 – 3),
- vertical dipoles in 5.6.9, especially crustal root dipoles in 5.6.9.3,
- the example of the Southeast Iceland shelf in 5.7.5, 6.5.5 and 7.7.2.

Task 6.1 Solution.

Start with (2.8.4 – 9, 2.8.4.4). P = (x_P, y_P,h). The square is on the plane z = 0 and is divided into 8 equal triangles with the sides a/2, a/2 and 2/√2 and the central azimuthal angle π/4 between the radii from P' = (0,0,0) to the corners and side centres of the square. A triangular element [P'Q_1Q_2] with Q_1=(a/2,0,0) and Q_2=(a/2, a/2, 0) is the projection of the triangle [PQ_1Q_2], and the solid angle ΔΩ is the projection of [P'Q_1Q_2] onto the unit sphere around P. The dip angle of the ray from P to Q_i is φ_i; and the distance P – Q_i (with $\Delta x_i = x_{Qi} - x_P$, $\Delta y_i = y_{Qi} - y_P$) is $r_i = (\Delta x_i^2+\Delta y_i^2+h^2)^{1/2}$; $r_1 = (h^2+a^2/2)^{1/2}$, $r_2 = (h^2+a^2/4)^{1/2}$.

$$\Delta\Omega \approx \Delta\lambda(1 - (\sin\varphi_i + \sin\varphi_{i+1})/2) = \Delta\lambda[2 - h(r_i+r_{i+1})/(r_i r_{i+1})]/2 \qquad (2.8.4 – 9).$$

The expression is exact for a sector of a spherical cap bounded by a small circle; the average of two caps of different radius vector $\boldsymbol{r_i}$ is an approximation.

$$\Delta\Omega \approx \Delta\lambda[(2 - h(r_1+r_2)]/r_1r_2 = \pi/4\ (2 - h/r_2 - h/r_1).$$

$h/r_1 = h/(h^2+a^2/2)^{1/2} \approx 1 - a^2/(4h^2)$; $h/r_2 = h/(h^2+a^2/4)^{1/2} \approx 1 - a^2/(8h^2)$;

$$\Delta\Omega \approx \pi/4\ ((1/4 + 1/8)a^2/h^2)$$

$$\delta g^{square} \approx 3/4\ \pi\ G\rho^*\ a^2/h^2,$$

$\rho^* = \Delta\rho\ d$

$$d \approx \delta g\ (4/3)\ (h^2/a^2)/(\pi G\Delta\rho) \quad \text{q.e.d.}$$

This is the arithmetic mean of the effects of similarly derived approximate expressions for the effectsof the inscribed (radius a/2) and circumscribed (radius a/√2) circles around P:

inscribed:

$$\delta g \approx 1/2\ \pi\ G\rho^*\ a^2/h^2;$$

circumscribed:

$$\delta g \approx \pi\ G\rho^*\ a^2/h^2.$$

The direct inverse problem (6.3.1.2 (5)) is to calculate d from ρ*=ρd, but with increasing h, the influence of the neighbouring mass elements becomes detrimental to the purpose of direct interpretation. (see 6.3.1.2 (5)), but an iterative approach may work. However, there are limiting conditions. One is that the density contrast must not be so small, that d=ρ*/ρ grows beyond the space available, but even much before that, the non-linearity of the gravity effect with h would lead to large errors.

Task 7.1 Solution

(a)

1. Windows user click on <u>initinvert.bat</u> in the "windows"-directory copied from the CD. Linux users call `.initinvert.sh` (in a bash-shell) or `source initinvert.csh` (in a tcsh-shell) inside the "linux"-directory copied from the CD. In this way you initialize search-paths, etc.
2. In your favourite editor (windows-users can use the automatically started "notepad2"-editor) you create a file similar to <u>71a.obs</u>. Alternatively you can call `creategrid -x -20 -X 20 -n 41 my71a.obs`, to create the profile data more easily. The file name can be chosen differently, naturally. We will assume your files are called *my7[1-4][a-z].** according to the task- and section number. On disk example files without the prefix "*my*" are stored, to compare your results with. In this text names of files available on the disk are marked by underscores, files you will create yourself are written in italics.
3. Then you call `invert` without any options and hit the return or enter key on the question after the name of the observation file. In this way a template definitions file will be created with the name "def". You should rename this file to e.g. *my71a.def* and load it into you editor.
4. Using this editor, insert the keyword START in an empty line before OFFSET FUNCTION, insert STOP before BOUGUER REDUCTION, add START before SPHERE, and insert STOP before ROD. In this way only the OFFSET FUNCTION and the SPHERE are read by INVERT because only they are in between START and STOP keywords. To improve the readability of the definitions file, you can delete all bodies outside the START/STOP-pair and also any STOP/START-pairs without any body in between.
5. Delete all values behind the first column of numbers in line "n", "s" and "M" of the OFFSET FUNCTION, because only a single offset variable is needed. Alternatively you can enter an exclamation mark behind the value "201" in line "n": this comments out the rest of that line and the program then only reads the columns in lines "s" and "M" corresponding to uncommented n-entries.
6. With an exclamation mark, you comment out line "t" of the sphere.
7. Then you change the value for "Radius" in line "p" to 5 and the values for "Density" and "Z" in line "M" of the sphere to 500 and −10. Your file should be similar to <u>71a.def</u>.

(b)

1. Windows users enter `invert` or `invert -P my71a.inp` (thus saving your entries into *my71a.inp*, so that the call can be repeated with the same answers by calling `invert -p my71.inp` the next time) in the "INVERT Console". Linux enter the same command in their shell console. Enter `my71a.obs` at the question for the observations file and `my71a.def` for the definitions file. For all other questions the default settings are confirmed by hitting the return or enter key, except for:
    1. "Number of iterations", which must be `0` because the gravity effect of the initial model (the forward model) will already be calculated in the 0^{th} iteration.
    2. "Output in definitions file", which should be `ms`, so that after the computations the inverted model variables, as well as the a-posteriori standard deviation of the variables will be shown.
    3. "Output in separate file", which should be `M` to show the a-priori model in the plots.
    4. "Output of observation results" which should be `Ro` to output the residuals and the a-posteriori standard deviation of the model effect in a file called *my71a.res* and to output the model effect (forward modelling effect in this case) in *my71a.mod*.

2. The output should show:

```
INVERSION...


Iteration    F-statistic        Chi^2      RMSE  mult.   Model diff.   Damping
   nr.    actual    optimum  improvement               prior    post    10^

   0      87.177     87.177       -       9.337   1.00   -       -        -
   -                                                                    none
```

You can inspect the model effect numerically by loading *my71a.mod* into your editor.

3. Then you call `plotprofile my71a.def` in the "INVERT Console" or shell console to plot the model and its gravity effect in 2D. The red line in the upper plot represents the forward model effect. With the additional option `-e`, the standard deviations of the observations are plotted too. To see a 3D plot you call `plotmap my71a.def`. You can rotate this plot with the mouse. Three 3D plots will be created: one underlain with observations, one with model effect and one with residuals. If you are only interested in the forward model effect you can call `plotmap -m my71a.def`. To close the plot window (and the simultaneously opened gnuplot command window too), you have press the key "x", while that window is the active one. By entering `plotprofile -h` or `plotmap -h` you get a list of all valid options.

(c)

1. Copy the contents of *my71a.mod* to a new file called *my71c.obs*. In this way the model effect of the forward model will serve as the observations for the modified input model to be inverted.
2. Enter 125 in line "s" of the density of the SPHERE and 250 in line "M". Enter −1000 in line "s" of the OFFSET and save this def-file as *my71c.def*.
3. Call `invert` again, with the same answers as in (b) except:
   1. "Adaptive damping factor" can be `0`, because the observations depend linearly on the only variables offset and density and thus no damping is necessary. (You can also leave the damping at its default value, but then you will need at least 4 iterations.)
   2. "Number of iterations" must be `1`, because in 1 iteration the solution of a linear problem can be calculated. (You can also enter a higher number, but you will see, that the solution will not change after the first one any more.)
4. The output should show:

```
INVERSION...


Iteration    F-statistic        Chi^2      RMSE  mult.   Model diff.   Damping
   nr.    actual    optimum  improvement               prior    post    10^

   0      22.339     22.339       -       4.668   1.00   -       -        -
   1       0.099      0.099     889.60    0.036   1.00  1.395  21.090   none
   -                                                                    none
```

If you reload your *my71c.def* into your editor, you will see the output lines appended by INVERT behind the keyword END, starting with == RESULTS ===. An overview of the input data is shown, the iteration output as shown on the screen and the model results. Line "M" contains the variable values after inversion. For the inverted offset you will read:

```
M          0.052
R         -0.052
S          0.272
```

And for the density

```
 M        496.616
 R       -246.616
 S         14.526
```

So the “correct” density is almost completely recovered.

The standard deviation of the density is chosen to be 125 kg/m^3, because even in this synthetic example the modeller is assumed to have some ideas about the range of possible values for the density of the sphere. And as in real cases we hope that his (although in detail wrong) assumption of a most likely value of 250 kg/m^3 should still include the possibility of the correct value of 500 kg/m^3. According to the 2σ-rule the standard deviation should be chosen about equal to one half of the maximum acceptable difference: (500−250)/2 = 125 kg/m^3.

The offset in this particular case is 0, because the observation values are created synthetically. In reality you will (usually) have only gravity values relative to an arbitrarily chosen reference value. Even if you have absolute gravity values, the INVERT-model will not model the whole earth , resulting in a unknown offset of the observed values from the calculated ones. The OFFSET variable is necessary to account for this difference. Since we are not really interested in this variable, we want to allow any necessary value for this offset to achieve a best fit. A positive s-value defines a standard deviation of the a-priori values (line “m” or line ”M” if “m” is missing), but this would constrain the variable. A negative s-value, however, defines no standard deviation, but a normalization value for a “per iteration standard deviation”, i.e. for a damping, without constraining the solution, as long as enough iteration steps are preformed. So we have chosen a negative s-value for the offset to ensure that we won't constrain this variable, neither if no damping is applied nor after several iteration steps if damping is turned on . The strength of the damping will be adapted per iteration by the program. This single damping parameter is scaled to each individual variable by the magnitude of the s-value. For the offset variable this magnitude must be chosen such that any reasonable value can be reached even with moderate damping. The mean difference of our gravity data and our model effect can be estimated (even if we wouldn't have created them synthetically) to be much less than, say, 1000 mGal. So we will reach any reasonable offset value, if we allow changes per iteration of about that magnitude. Summary: With an s-value of −1000 the offset value will not be constrained in any way if no damping is applied. And even if damping were active its value will not be constrained after several iteration steps; only changes per iteration step larger than about 1000 mGal will be damped depending on the damping level.

If our synthetic modeller would have no idea about the likely range of densities, he could also have entered −125 for the s-value of the density. Then, because of the negative value, any density would have been accepted and only the damping would be of the order of 125 kg/m^3 if it would have been active. Because then there is no “pull towards 250 kg/m^3 any more, the correct solution of 500 kg/m^3 would result:

```
 s       -125.000
 m        250.000
 M        499.992
 R       -249.992
 S         14.625
```

(the difference of 0.002 kg/m^3 results from the 3-digit accuracy in the my71c.obs).

But in non-synthetic cases it will be usually be preferred to slightly pull the solution (in this case resulting in only 4 kg/m^3 difference) by entering a positive value in the s-line: The modeller enters a standard deviation of 125 kg/m^3 because he wouldn't trust a solution outside the range 0-500 kg/m^3; he would prefer to assume errors in the observations, or he would assume his his model to be incomplete. Only because this is a synthetic example we know that the “correct” solution is 500 kg/m^3, but if you had been the “synthetic” modeller, you would have only reluctantly accepted this value, because it is at the very border of your geologically acceptable range of density values. From this point of view a value of 496.6 kg/m^3 is a very good compromise.

(d) The standard deviation entered in *my71c.obs* was 1 mGal, which indicates, that either the measurement process has only this precision or that the (almost certainly simplified) model cannot be expected to be more exact than this value on the average. But the observations are calculated by the forward model in (b) and thus are actually without any error (except for rounding).

You can synthetically disturb the error-free observations of *my71c.obs* with a standard deviation of 1 mGal (normally distributed) by calling e.g. `disturb -s 1 -d 1 my71c.obs my71d.obs` on the console. And use the resulting *my71d.obs* as observations file for the next inversion.

The output shows:

```
INVERSION...

Iteration    F-statistic         Chi^2      RMSE   mult.   Model diff.    Damping
   nr.     actual    optimum  improvement                  prior    post    10^

   0      24.287     24.287        -        4.868   1.00    -        -        -
   1       0.921      0.921     934.64      0.887   1.00  1.507   21.618    none
   -                                                                         none
```

The much higher value of the F-statistic (quadratic weighted error of variables AND observations divided by degrees of freedom) and the (weighted) RMSE (of the observations) in this listing indicate that the solution doesn't fit as well as before. Both values indicate that the error is of about the size of the standard deviation of variables and observations (as expected, as we disturbed them with errors of this size). With several calls of `disturb` without option -d, i.e. with different disturbances of the same size, you will see many values in the range of about 0-4 for the F-statistic and 0-2 for the RMSE (why?). For this particular disturbance of the observations the density needs to pushed up even above 500 kg/m^3 to achieve an optimal fit.

(e)

1. You can either change the 5th column in *my71d.obs* to 0.5 or you can enter 0.5 on the question “Multiplier of model st.dev.” (since all standard deviations have to be changed to the same amount).
2. Now that the observations have to be fitted with lesser standard deviation, the density is even pushed up slightly higher (from 516 to 519 kg/m^3), but not significantly. The residuals (in file *my71e.res*) are correspondingly slightly smaller, but also not significantly.
3. However, the (absolute) a-posteriori standard deviation of the density has been halved, because the propagated uncertainty of the observation has been halved. This is independent of the realized residuals of the observations, which are, as mentioned before, almost unchanged! This half uncertainty of the density propagates further to the standard deviation of the model effect (listed in the last column of *my71e.res*), which also shows half of the value of section d.

4. The relative standard deviations of the observations (listed in *my71e.def*) are the same, since both a-priori and a-posteriori standard deviations are halved. The relative standard deviations of the density is halved, because of the reduced uncertainty of the observations.
5. The weighted RMSE (wRMSE) of the observations is doubled, because the residuals are almost unchanged, but they are now normalized with the halved a-priori standard deviation. The weighted RMSE of the density is almost the same as before, because the solution, as well as the normalizing standard deviation are (almost) unchanged for this variable.
6. The F-statistic is strongly increased, since it is calculated by summing up the weighted (with the a-priori standard deviation) squared residuals of both the observations and the variables (density), and the contribution of the observations is increased. Now the F-statistic (3.339) exceeds its 5% criterion (1.394). Values larger that this criterion value will occur with a probability of only 5% if all the model assumptions are correct. These assumptions are: the errors in observations and variables are normally distributed with the given standard deviation (the standard deviations (and correlations) are correct and there are no outliers) and the geometrical and gravitational relationship between the variables and observations is correct and complete. Because the F-statistic exceeds the criterion value, some of these assumptions is likely to be violated. For the synthetic example we know that the standard deviation of the observations is indeed too small or that the data errors are too large for the given standard deviation.

(f)

1.
    1. Because now the observations are no longer linearly dependent on all the variables (caused by the additional z-coordinate), you must iterate the solution. In this case it is best to turn damping on. The relevant settings are:
        1. “Adaptive damping factor”: best set to its default value (a damping of 0.1 times the inverse of the a-posteriori variance equal to the inverse of 10 times the a-posteriori variance) by hitting the return or enter key.
        2. “Number of iterations”: initially also best set to to its default value (30) the same way.
    2. The output in *my71f.def* shows:

```
Iteration     F-statistic        Chi^2      RMSE  mult.   Model diff.   Damping
   nr.    actual    optimum  improvement                  prior    post    10^

     0       48.757     48.757        -        6.897   1.00   -       -        -
     1 2031494.635     48.757*********** 1407.816   1.00  1.568  24.669   -1.0
     2       18.268     18.268    1219.56     4.213   1.00  1.023  21.061    0.0
...
    29        0.095      0.095       0.00     0.112   1.00  0.000   0.000    1.5
    30        0.095      0.095      -0.00     0.112   1.00  0.000   0.002    1.0
     -                                                                      none


 Iteration limit reached.
```

The last line indicates, that the iterations are halted, because the maximum number of iterations is reached, but that the solution has not converged already (an important convergence criterion is defined by the answer to “Minimal Chi^2 improvement”, see below). So you have to allow more iterations by setting “Number of iterations” e.g. to `60`.

Now the listing shows:

```
    35        0.095      0.095      -0.00     0.115   1.00  0.006   0.013   -4.0
    36        0.095      0.095      -0.00     0.112   1.00  0.000   0.001   none
```

```
   -                                                                   none

 Local optimum found.
```

So, after 36 iterations the last line indicates that the convergence criteria are met.

By the way: the reason that the solution doesn't converge very well, is because of the strong trade-off between the density and the z-coordinate of the sphere. In the range of all a-priori standard deviations both variables are about equally well suited to fit the observations. The fitting function to be minimized is quite “flat” near the actual solution, so the “best” solution is not easily found (see Note below).

2.

1. Hint: For easier comparison, INVERT will merge the output of the previous run with next one, if you move the keyword END to the end of the “RESULTS” section written by INVERT in the last run. Then the output of the next run will be appended behind this END entry. Check that there is a STOP keyword (and no START behind it) before the first “RESULT” section, otherwise the bodies listed in this output are interpreted as additional input bodies.
2. The y-coordinates of the second model show even more trade-off with especially density, because the profile of observation points doesn't constrain the location of the sphere perpendicular to the profile. Therefore the solution has not converged even after 60 iterations.
3. If you double that number to `120`, the algorithm stops after 115 iterations:

```
 105      0.136      0.136       0.00     0.115   1.00  0.000    0.000     3.5
 106      0.136      0.136       0.00     0.115   1.00  0.000    0.001     2.5
 107      0.136      0.136      -0.00     0.115   1.00  0.000    0.002     1.5
 108      0.136      0.136      -0.00     0.115   1.00  0.000    0.005     0.5
 109      0.136      0.136      -0.00     0.114   1.00  0.001    0.013    -0.5
 110      0.136      0.136      -0.00     0.116   1.00  0.007    0.022    -1.5
 111      0.137      0.136      -0.04     0.125   1.00  0.034    0.047    -2.5
 112      0.141      0.136      -0.17     0.139   1.00  0.053    0.059    -3.5
 113      0.136      0.136      -0.00     0.115   1.00  0.000    0.000     2.5
 114      0.136      0.136      -0.00     0.115   1.00  0.000    0.000     3.5
 115      0.136      0.136      -0.00     0.115   1.00  0.000    0.000     4.5
   -                                                                   none

 Local optimum found.
```

But you should look at the final damping value: 4.5 means that each variable is not allowed to change more than $1/10^{4.5}$ times its a-posteriori variance (i.e. $1/10^{2.25} = 0.0056$ of its a-posteriori standard deviation). A solution can, however, only be considered as well converged, if it is reached with (almost) no damping, Otherwise the reason that no changes occur any more, is only due to the strong damping. In iteration 107 to 112 the solution deteriorated only very, very slightly while reducing the damping form 1.5 to −3.5. But at this value in iteration113 the solution deteriorates even to 0.17, which is above the “Minimal Chi^2 improvement”-criterion value of 0.1 as defined by the (default) settings. Because the last improvement (a positive “Chi^2 improvement”) occurred with damping 1.5 and less damping didn't result in a better solution, now stronger dampings are tried. But even at the almost infinite damping of −4.5 the solution deteriorates. Because no better value can be found any more, the iterations are stopped. So no real convergence is achieved. (Note, that this situation is deliberately evoked here to explain how to interpret the iteration information and how to recognize (failing) convergence. It is not (or should not be) a representative case; it is caused by intentionally providing insufficient a-priori information on the variables in order to reduce convergence.)

4. However, if you loosen the criteria a little, you can try to find an iteration step, which wasn't followed by any significant change of the solution any more:

```
59      0.139     0.136     -0.10    0.124   1.00  0.041   0.043   -4.0
60      0.136     0.136      0.00    0.115   1.00  0.001   0.012   -1.0
61      0.136     0.136     -0.00    0.114   1.00  0.002   0.009   -1.5
62      0.137     0.136     -0.01    0.118   1.00  0.018   0.025   -2.5
63      0.137     0.136     -0.03    0.121   1.00  0.029   0.032   -3.5
64      0.137     0.136     -0.03    0.122   1.00  0.031   0.033   -4.5
65      0.139     0.136     -0.12    0.125   1.00  0.041   0.042   none
```

   We can consider:

   1. At iteration 65 the solution only deteriorated (negative “Chi^2 improvement”) slightly above the “Minimal Chi^2 improvement”-criterion even without any damping.
   2. Even at iterations with only slight damping (less than −2 or none) the “Chi^2 improvement”'s (only counting positive values, i.e. real improvements) are very small after iteration 18 and even don't sum up to a significant value any more after iteration 42. Note that at iterations with strong damping the improvement will always be small, because it is not allowed to change strongly, so values at these iterations should not be considered.
   3. Considering again iterations with only slight damping, the variables only changed by a factor of 0.042 times the a-posteriori standard deviation (“Model diff. post”) on the average (which means 0.042*62.7 kg/m^3 for density or 0.042*4.7 m for the y-coordinate). For the modeller, who defined the range of acceptable values of the variables to be 1/0.041 = 24 times as large, this is no significant change to the model any more (as long as these changes do not continue in the same direction for many iterations).
   4. The F-statistic is already far below its 5% criterion value, so changes in the solution are also statistically not relevant.

5. Taking these 4 criteria into account we could have halted the program after about 60-70 iterations.
6. Linux users can pause the computations by hitting the return or enter key during iterations and decide to continue or to finalize the program from that moment on.

Task 7.2 Solution

(a) Check your def-file with 72a.def. The results for the cylinder should be similar to:

```
     Density [g/cm3]  Zmin [m]    Zmax [m]       X [m]       Y [m]  Radius [m]

M        -191.789      -0.550      -0.323       9.301      10.534       2.975
R          -8.211      -0.050       0.023       0.699      -0.534      -0.475
S          24.525       0.039       0.040       0.034       0.031       0.031
```

With a calculated offset of 0.395 mGal.

(b)

(1) The a-posteriori standard deviation (line "S") of the density is almost not reduced, and mainly reproduces its initial knowledge. The lower face is just slightly better resolved. And the three other variables have got a much higher precision from the gravity observations, than the modeller would give them according to his a-priori information.

(2) The residuals of the variables are mainly within the range of 2 times the a-priori standard deviation, only the radius is calculated at the very edge: −0.475 km is only just inside a 95% interval of ±0.5 km.

(3) The wRMSE of the 6 variables is 1.143 which summarizes this interpretation of the residuals of the variables: On average the residuals are just slightly bigger that the a-piori standard deviation.

(4) The wRMSE of the observations is 1.314, which indicates that the residuals of the gravitational model effect are 30% higher than expected with the assumed standard deviation of 0.1 mGal. Either the model has a significant shortcoming or the standard deviation of the observation is of the order 0.13 mGal.

(5) Because the size of the F-statistic is usually of the order of the squared maximum of the wRMSE of variables and observations, this quantity is also much larger than one: 1.823. which also exceeds its 5% criterion (1.245), indicating again a likely error in the modelling. (Have a look in 72a.def to find out what the correct model would have been).

(c) Take a look at the correlation matrix in the output (please read also the Users Guide how to interpret the layout of these data (Chapter 3, Section "Results in definitions file").

```
Correlation matrix of variables

         1  2  3  4  5  6  7

1   201 *0
       ---------------------
2   701 -1|\a
3   702 -4|-2 \8
4   703 -4| 6  7 \1
5   704 -1|-1 -2 -2 \1
6   705 -1| 0  0  0  2 \1
7   706  4| 0  0  0 -5 -3 \1
```

Variable 1 (201) represents the offset, others are density, zmin (lower face depth), zmax (upper face depth), x, y (both centre coordinates), and radius of the cylinder respectively.

(1) The main diagonal shows again the reduction of standard deviation for each variable (the letter "a" represents 10/10, "8" = 8/10, etc).

(2) A quite strong correlation between the upper face and density (6/10) as well as between upper face and lower face(7/10) can be recognized. This can be expected, as all these variables can have a similar effect on the observed gravity. Also the radius seems to be slightly correlated

with x coordinate of the centre: unevenly located observations points will make it difficult to discriminate between small variations radius and centre coordinates at high accuracy levels.

The eigenvalues (singular values) and eigenvectors also provide interesting information:

```
Eigenvalues, resolution and residuals of eigenvectors of covariance matrix

          1   2   3   4   5   6   7

EVal    1.0E+00 1.3E-01 5.5E-02 1.0E-05
Resol   5.0E-05 8.7E-01 9.4E-01 1.0E+00
Resid   2.8E-02 2.0E+00 4.3E-01 5.4E-04

Eval        7.6E-01 5.9E-02 8.9E-03
Resol       2.4E-01 9.4E-01 9.9E-01
Resid       1.1E+00 1.5E+00 1.9E-01


 1   201   0   0   0   0   0   0  a_
        ------------------------------
 2   701  a_  29   0   0   1  -7   0
 3   702 -30  a_  -1   0   2 -11   0
 4   703   4  14  -1   0  -2  a_   0
 5   704   0  -2 -33 -23  a_   2   0
 6   705   0   0 -17  a_  20   1   0
 7   706   0   0  a_   9  36   1   0
```

(3) The first eigenvector (column) has an “EVal” of 1.0E+00, which indicates no reduction of precision by the observations. Its biggest components (in the lower part) are “a”, which means 100/100 in the direction of variable 2 (701, density) and −30/100 in the direction of variable 3 (702, zmin = lower face depth). This means that 100/100 times a density change of $1\sigma_{density}$ plus −30/100 times a zmin-change of $1\sigma_{zmin}$ is not resolved by the observations. Since both density and zmin are negative and the a-priori standard deviations are 25 kg/m^3 and 0.05 km respectively this can be reformulated: An increase of density (i.e. a decrease of density contrast) of 100/100 * 25 kg/m^3 can be compensated by a decrease of the depth lower face (i.e. a increase in depth) of −30/100 * 0.05 km. (or at least with the same proportionality). This corresponds to our expectation, that a slightly thicker cylinder with a slightly less density contrast will have about the same gravity effect.

(4) The second eigenvector is already better resolved: the reduction of its standard deviation is 7.8E-01. This is however still not very good, but is is not as badly resolved as the density/-zmin combination of the first eigenvector. This one represents a sum of 29/100 times a change of 1σ in density, plus 100/100 times a change of 1σ in zmin, plus 14/100 times a change of 1σ in zmax (depth of upper face). Neglecting the factors 29 and 14, the sum of zmin and zmax is proportional to the mean depth (zmin+zmax)/2). So this eigenvector represents the sum of density and mean depth of the cylinder. Here we learn that increasing density (decreasing deity contrast, since the density is negative) and simultaneously increasing the mean z-coordinate (i.e. increasing depth) is not very well resolved too. This corresponds to our expectation, that moving a relatively small cylinder with negative density up can be quit well compensated by a less negative density. That this compensation is not as complete as for the density/thickness compensation is reflected by the lesser eigenvalue (singular value) 0.78 of the former, relative to 1.00 for the latter.

(5) The next three eigenvectors show a reasonable to very good resolution (reduction of standard deviation) for combinations of centre coordinates and radius, since the main components are at variables 5,6 and 7.

(6) Very well resolved (EVal = 8.9E-03 = 0.0089) is an eigenvector which is mainly characterized by the depth of the upper face.

(7) The best resolved eigenvector represents the offset, which 1) is physically of no interest and 2) has this good resolution only, because we compare it with a precision of 1000 mGal a-

priori, which was deliberately set that high, because we do not have information about it a-priori and any information is almost infinitely better than almost no information.

(8) What is the difference of the information we get from these eigenvectors compared with the information we get from the a-priorira-posteriori standard deviations (section b-1 above)? Above we only saw, that density and zmin are not well resolved. Here we see that a certain combination of density and zmin (density increase, with zmin decrease) are even extremely badly resolved, while other combinations (e.g. density and mean depth) are even slightly better resolved than we would expect from the individual standard deviations. With this information we might be able to reconsider and refine our a-priori information on this set of variables.

(d) Enter a non-zero "Mode"-parameter to line "p" of the cylinder: with value 2 you define a deviation from a cylinder-shape by 2 extensions and 2 reductions of radius (compare 72d.def). This corresponds to an ellipse (with mode 3 to a triangular deviation from the circular or elliptical shape, etc.). The radius deviation and the azimuth of the "first" positive deviation for each mode are additional variables, that can be inverted.

If you leave the values in line "M" at zero for both the elliptical difference radius and the azimuth and enter in line "s" for the difference radius e.g. 0.5 km and 45 degrees for the azimuth, you get:

The results:

	dR1 [m]	Azimuth1
M	-0.076	-8.5
R	0.076	8.5
S	0.037	13.0

This shows that the a-posteriori 95% interval includes a radius difference of 0. Which would imply no ellipticity. This means: if the given observations are correct, than it is quite certain, that the body is (almost) not elliptical (have a look at at the original body in 72d.def again!).

Task 7.3 Solution

(a) Check your def-file with 73a.def. The last node is effectively a copy of the first one, to close the polygon. To make sure, that this (copied) node will also move exactly the same way as its original (903, 904) during inversion, the negative n-entries (defining the connected variables) and the (+1)-s-entries (defining the exactly proportional changes) for the last node are required.

(b) Check your def-file with 73b.def. By changing the numbers in line "n" of the nodes of the lower surface to minus the number of the corresponding nodes of the upper surface, the lower nodes become linearly dependent on the upper ones. For these dependent nodes the entry in line "s" must be −1 for the depths, so that each depth-increase of an upper node will imply a depth-decrease to the same amount of a lower node. The horizontal coordinates are not allowed to change, even for the upper nodes, but in case the standard deviations are changed to a non-zero value, the horizontal coordinates of the lower nodes are connected to the upper ones too, so that they will move together. With an s-value of +1 this horizontal shift will be in the same direction.

Note that the solution equals the original data almost perfectly. Obviously, this condition represents so much information, that no room for other solutions exists any more.

A good reason to choose such a hard condition is, that the number of resolved variables is only 9 (of 19) free variables. If the geological boundary conditions are in favour of such a model, it can be an easy and effective way to reduce the number of free variables. On the other hand such information is not very likely in real situations and in case it conflicts with the geological situation, the solution will be biased. The next two types of assumptions will be more realistic.

(c) Check your def-file with 73c.def. The CORRELATION MATRIX (one for the upper and one for the lower nodes) must be entered with an exponent of 2. On the main diagonal it contains the correlation length. The other elements are the (horizontal) distances for each pair of nodes. Note that the solution is effectively smoother than in section a.

(d) Check your def-file with 73d.def. All pairs of neighboured nodes enter into one column of the condition matrix, one with the coefficient +1 and the other one with −1. The first column represents thus this condition equation on the change of the variable values (dv) after inversion: $1*dv_{904} + (-1)*dv_{906} + 0*dv_{908} + ... +0*dv_{920} = 0 \pm 0.3$. The last two values come from lines "r" and "s" of that column. The meaning of the s-value is the standard deviation of the result of this equation and its use should be intuitive. But please read the Users Guide very carefully if you want to enter a different value as 0 for "r", which is the initial residual of the equation! Since we have eight pairs of neighboured nodes, there are 8 conditions.

(e) Check your def-file with 73e.def. The result is very similar to that of section b. But by selecting an appropriate value in line "s" of the condition matrix, you can come closer to the actual content of your a-priori information on the shape of the body.

Also, if you take a look at the lower part of the resolution matrix, you recognize the pairs of connected variables. But although the a-priori information is equal for all pairs of nodes, the mutual influence for the central pair 912/930 is much more asymmetric (-7/-2) than for the lateral pairs 904/938 or 920/922 (-5/-4). The asymmetry tells us that, if the observations require a change of (only) the lower node of $+1\sigma$, the "mirror"-condition will cause a change of the upper node of -0.7σ. On the other hand, if the observations require a change of the upper node of $+1\sigma$, the lower node will only change will only change by -0.2σ. The reason is the different strength of the gravitational signal from changes in different depths and therefore the different possibility to smear the required changes over different variables.

The eigenvalues (singular values) / eigenvectors of the correlation matrix show a quite small standard deviation ("EVal") for all linear combinations except for the 1st and 2nd eigenvector. But even the eigenvectors 3 to 8, which represent small wavelength variations (many sign-changes) of the lower surface, have obviously a less precise solution than those of the upper surface. For eigenvectors 3 to 11 the resolution ("Resol") deteriorates (smaller is worse) although the precision ("Eval") improves (smaller is better): the linear combinations describe increasingly longer wavelengths (with improving precision) but they also describe more and more parallel movements of both surfaces, which are strongly bounded by the mirror condition, which implies the bad resolution.

The eigenvalues/eigenvectors of the resolution matrix show this aspect in another way: The badly resolved eigenvectors describe either parallel movements (e.g. eigenvector 1) or asymmetric movements of upper and lower surface (e.g. eigenvector 2). These changes are mainly prohibited by the a-priori information, not by the observations. Starting from eigenvector 12, the eigenvectors represent symmetric changes, but now the resolution improves, because the wavelength of the changes increases.

Task 7.4 Solution

The modelling itself is left over to the reader. Here we only discuss the selection of the standard deviation of the observations:

The file 74baNom.gb and 74rbaNom.gb contains the nominal standard deviation of the observations, but not at the original locations. You will recognize a very high wRMSE and F-Statistic, because the the model cannot satisfy these narrow error bounds. But even the much more realistic standard deviations derived form neighbouring profiles and compartimentations do not account for the ability of the model to fit these data. Therefore we have to incorporate the "modelling precision" into the standard deviation of the observations.

In 74baRed.gb and 74rbaRed1.gb the standard deviations of observations at lateral parts of the model, where no model variables are available to fit them, were increased to reduce their influence on the solution, the wRMSE and the F-Statistic. Note that if observations do not fit the inversion process will also allow deviations of the variables from their a-priori informations of about the same amount. Therefore it is important to estimate the possibilities of the model to fit the observations.

The data in rba.gb show a striking maximum at x = 150 km, where (initially) no model variable is available to fit them. Therefore in 74rbaRed2.gb the standard deviation of these observations is also increased. It depends on the intent of the modelling if it is more important for the interpretation of the model to fit these data well and to add model variables to fit them, or if these observations can be neglected by reducing their standard deviation.