Cold Dark Matter halos based on collisionless Boltzmann-Poisson polytropes.

Juan Calvo, Estrella Florido, Oscar Sánchez, Eduardo Battaner, Juan Soler and Beatriz Ruiz-Granados

Abstract The aim of this work is to give some insight into the controversy between N-body simulations and observations of cold dark matter (CDM) halos by considering polytropic DM spheres associated to a collisionless gravitational Boltzmann-Poisson (BP) system. Our resulting polytrope model is used to make predictions on the behaviour of the CDM halos in those regions in which the numerical models cannot produce detailed results, i.e. near the center < 1 kpc, (due to resolution limitations) and at the rim (as halos cannot have infinite extent). These provides a complementary information where other models present difficulties to make predictions.

1 Introduction

N-body simulations produce mass density profiles which are almost universal for a large range of masses, from dwarf galaxies to rich clusters. One of the most popular formulae matching N-body simulations density profiles is the Navarro, Frenk and White (NFW).

$$\rho(r) = \frac{\rho_0(c)}{\left(\frac{r}{r_s}\right)\left(1+\frac{r}{r_s}\right)^2} \tag{1}$$

J. Calvo, O. Sánchez and J. Soler
Dpto. Matemática Aplicada, Univ. de Granada, e-mail: juancalvo@ugr.es, ossanchez@ugr.es, jsoler@ugr.es

E. Florido, E. Battaner and B. Ruiz-Granados
Dpto. Física Teórica y del Cosmos, Univ. de Granada e-mail: estrella@ugr.es, battaner@ugr.es, bearg@ugr.es

c is called the concentration parameter, $r_s = r_v/c$ a scale radius, and r_v the virial radius. Other possibilities are the so called ”isothermal non-singular” profile and the Burkert profile

$$\rho_{Iso}(r) = \frac{\rho_0}{1 + \left(\frac{r}{R_0}\right)^2} \tag{2}$$

$$\rho_{Burkert}(r) = \frac{\rho_0}{\left(1 + \frac{r}{R_0}\right)\left(1 + \frac{r}{R_0}\right)^2} \tag{3}$$

These models have some serious drawbacks: infinite radius, infinite mass and infinite density at the center. Oppositely, in the polytropic family (see below) we can find models with a wide variety of large but finite radii. We study DM halos using kinetic theory and statistical mechanics. As DM particles are collisionless, DM halos constitute ideal systems for the generalized kinetic theory to be applied, with a highly abstract mathematical handling.

One of the goals of this work concerns the central parts of galaxies where the resolution of simulations (≥ 1 kpc) avoids giving firm predictions and have a complex dynamics. In particular, if we define the slope in a log-log plot of density versus radius, as

$$\beta = -\frac{d \ln \rho}{d \ln R} \tag{4}$$

the NFW profiles gives $\beta_0 = \beta(R = 0) = 1$, while the isothermal and Burkert profiles give $\beta_0 = 0$. Observations also seem to suggest lower or vanishing values. Values of $\beta_0 > 0$ correspond to ”cuspy” halos, as the density becomes infinite for $R = 0$. Values of $\beta_0 = 0$ correspond to halos with core, being the core a region in which the density is nearly constant.

2 The mathematical model.

The self-consistent gravitational potential, ϕ, in the BP system is specified through Poisson’s equation $\Delta\phi = 4\pi G f(t,x,v)dv = 4\pi G\rho$, and then the evolution equation for the phase-space distribution f is the collisionless Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f - \nabla\phi \cdot \nabla_v f = 0 \tag{5}$$

Here x and v are the 6-dimension coordinates of the phase space and t the time. Jeans theorem assures that any steady-state solution of the BP equation depends on the phase space coordinates only through integrals of motion. In the case of spherical symmetry the dependence is only through $E = 1/2|v|^2 + \phi(|x|)$ and the square of the angular momentum, $F = |x \times v|^2$. A very popular family of such steady states are the so-called polytropes. The generalized polytropic ansatz reads $f(x,v) =$

$c(E_0 - E)_+^{\mu} F^k$ where μ, $k > -1$, $\mu + k + 3/2 \geq 0$, $c > 0$ and $E_0 < 0$ is a cut-off energy. These solutions are spherically symmetric and induce radial density profiles, which are isotropic if $k = 0$, with vanishing density at the origin if $k \neq 0$, finite mass if $\mu \leq 3k + 7/2$ and finite radius if $\mu < 3k + 7/2$.

In the borderline case $\mu = 7/2$ we recover a well-known model, the so-called Plummer sphere, whose associated density profile can be written down explicitly as $\rho(r) = (1 + r^2/3)^{-5/2}$. Even more, in the range $0 \leq \mu \leq 7/2$ these models are found to be minima of certain energy-Casimir (or q-entropy) functionals under suitable contrains, and as a consequence several non-linear stability results are derived. The functional form of these polytropes ensures that they solve the kinetic equation. To conclude with this approach of constructing steady states, we must check that ϕ solves the induced Poisson equation. Using spherical coordinates this reduces to Endem-Fowler's equation

$$\begin{aligned} &\tfrac{1}{r^2}\left(r^2\phi'\right)' = c\bar{c}_\mu \left(E_0 - \phi(r)\right)_+^{\mu+\frac{3}{2}} \\ &\phi(0) < 0 \\ &\phi'(0) = 0 \end{aligned} \tag{6}$$

Here $\bar{c}_\mu = 2^{\frac{7}{2}}\pi^2 G\beta\left(\frac{1}{2},1\right)\beta\left(\mu+1,\frac{3}{2}\right)$.

3 Numerical results

For the fitting procedure we rest mainly in the crucial fact that all profiles under consideration assume the form $r \to a\rho'(br)$, being ρ' an universal function characterizing the model. This allows us to work with normalized profiles, and then it's an easy matter to perform a least squares fit for the exponent μ held fixed. Thus the fit determines two out of three free parameters of the polytrope whenever the values of a and b of the model under study are given.

To perform our fits we have only considered profiles with $\mu \leq \beta \leq 7/2$. Values higher than 7/2 could produce better fits but the halo mass becomes infinity.

- Fits to NFW profiles in a wide range of radii: we pick up the first eight profiles (the ones which give rise to the least massive halos of the sample) that adjust the N-body simulations of Navarro et al. (1996) to compare them against the whole, three-parametric, family of polytropes in the range comprised between the virial radius over 100 and the virial radius itself.

- Fits to NFW profiles assuming bounded halos: as before, but we impose an upper bound on the radius of 300 kpc, which rules out Plummer's profile. We see that for μ held fixed the obtained profile reaches the maximum radius prefixed. This fact can be used to obtain numerically the best exponents

- Fits to the Isothermal model: we compare the normalized (a = b = 1) isothermal density profile with ours for all the radii this is, $r \in [0, \infty]$

- Fits to the Burkert model: in the same vein as before, we compare the normalized (a = b = 1) Burkert density profile with ours for $r \in [0, \infty]$

Table 1 Fits to NFW profiles (lengths in kpc, masses in 10^{12} solar masses)

Virial radius	Fitting parameter c	Virial mass	Relative error$^{(1)}$	Best exponen$^{(2)}$t	Relative error$^{(2)}$
177	19.230	0.319	0.0206624	3.33428	0.0212227
172	17.543	0.293	0.0219044	3.33413	0.0224761
193	12.195	0.414	0.0275006	3.29355	0.0282747
209	21.739	0.525	0.0191086	3.31307	0.0197279

$^{(1)}$Unrestricted NFW fits
$^{(2)}$Truncated NFW fits

Table 2 Isothermal model

Exponent	3.4	3.2	3.0	2.8
Relative error	0.0120101	0.0124264	0.0128845	0.0133903

Table 3 Burkert model

Exponent	3.4	3.2	3.0	2.8
Relative error	0.0103225	0.0106748	0.0110647	0.0114982

Concerning the behaviour at the origin, using Endem-Fowler's equation we get that for polytropes $\ln\rho \sim (\mu+3/2)(1-r^2/3!)$, so that $\beta(r) \sim (\mu+3/2)r/3$ and thus we get $\beta(0)=0$ as expected.

4 Conclusions

We have shown that a unified theory of cold dark matter halos based on collisionless BP polytropes is a powerful complementary method for studying galactic and cluster halos.

We confirm the results obtained by numerical simulations (NFW universal profiles) but other profiles (Isothermal and Burkert) cannot be disregarded. Once the agreement between our results and the N-body simulations is stablished, we are able to explore complementary problems out of the range of validity of simulations. We are able to investigate the polytropic solutions for finite mass and size halos and, mainly, we are able to find the profiles in the inner region where simulations are limited by resolution problems. We therefore provide the shape of the profile in the $\beta \leq 1$ kpc- region to which NFW-profiles must converge. Future extensions of this work are the use of angular momentum invariants, the use of models with other symmetries, and to introduce realistic baryonic disk and bulge, and to calculate the associated rotation curves.