Name: _______________

LINEAR PROGRAMMING MODULE
Part 2 - Solution Algorithm

THE SIMPLEX SOLUTION METHOD

It should be obvious to the reader by now that the graphical method for solving linear programs is limited to models of 2 variables such as in Illustration 1). The models such as Illustration 3), with 3 or more variables, are not easily graphed on two-dimensional graph paper. Thus, an algebraic technique is needed to solve LPs with numerous variables and equations. Also, such an algebraic technique is conducive to computer solution. One algebraic technique for solving linear programs is called the simplex algorithm. It was created by George Dantzig in 1947, and its theoretical foundation was established in 1948 by Gale, Kuhn and Tucker in the working paper “Extremum Problems with Inequalities as Subsidiary Conditions.” The interested reader is referred to the classic volume, *Linear Programming and Extensions*, by George Dantzig, Princeton University Press, 1963.

Illustration 9)

Consider the linear program:

Objective function: max $Z = 10\ X_1 + 40\ X_2$

Constraints:

$$X_1 + X_2 \leq 10$$
$$2\ X_1 + 5\ X_2 \leq 30$$
$$X_1 \geq 0,\ X_2 \geq 0.$$

1) The simplex algorithm begins by moving all the terms on the right-hand side of the objective function to the left: $Z = 10\ X_1 + 40\ X_2$ becomes

$$\boxed{Z - 10\ X_1 - 40\ X_2 = 0}$$

2) The *inequalities* of the constraint equations are changed into *equalities* by the addition of the *slack variables*, X_3 and X_4. Slack variables represent resources that are not used. For example, if we have 10 cars and our motor pool requires 6 cars, then we have 4 cars not being used: a slack of 4 cars. Six is obviously less than10, but if we add the slack of 4 to 6 we have the equality 6 + 4 = 10.

$$X_1 + X_2 \leq 10 \text{ becomes}$$

$$\boxed{X_1 + X_2 + X_3 = 10}$$

and

$$2X_1 + 5X_2 \leq 30 \text{ becomes}$$

$$\boxed{2X_1 + 5X_2 + X_4 = 30}$$

3) The three adjusted equations are:

$$\begin{array}{l} Z - 10X_1 - 40X_2 = 0 \\ X_1 + X_2 + X_3 = 10 \\ 2X_1 + 5X_2 + X_4 = 30 \end{array}$$

Note that the constants on the right-hand-side (RHS) of the equations are 0, 10, 30.

4) **a** - We make a chart, called the *tableau*, with Z, X_1, X_2, X_3, X_4 as variables and the constants on the RHS of the columns.
b - We enter the *coefficients* of Z, X_1, X_2, X_3, X_4 and the RHS at appropriate places in this tableau.

	Z	X_1	X_2	X_3	X_4	RHS
objective function row	1	- 10	- 40	0	0	0
1st constraint row	0	1	1	1	0	10
2nd constraint row	0	2	5	0	1	30

5) We indicate the column in the tableau with the *smallest number* in the *objective function row*.

Z	X_1	X_2	X_3	X_4	RHS
1	- 10	- 40	0	0	0
0	1	1	1	0	10
0	2	5	0	1	30

The column with -40 has the smallest value in the objective function row. The indicated column, X_2, is called the *pivot column*.

6) **a** - We pick only the *positive numbers* in the pivot column, i.e., no.'s bigger than 0.
b - We divide the corresponding RHS by these numbers.

Z	X_1	X_2	X_3	X_4	RHS	
1	- 10	- 40	0	0	0	
0	1	1	1	0	10	10 / 1 =10
0	2	5	0	1	30	30 / 5 = 6

7) We indicate the row with the smallest RHS/X ratio. The row is called the *pivot row*. 30 / 5 = 6 is smaller than 10 / 1 = 10.

	Z	X_1	X_2	X_3	X_4	RHS
	1	- 10	- 40	0	0	0
	0	1	1	1	0	10
pivot row	0	2	5	0	1	30
			pivot column			

The number "5" common to both pivot row and pivot column is the *pivot element*.

8) We change the pivot element, 5, into 1 by multiplying the pivot row by 1/5.

Pivot row:	(0	2	5	0	1	30) × 1/5 =
New pivot row:	(0	2/5	1	0	1/5	6)

New tableau:

	Z	X_1	X_2	X_3	X_4	RHS
	1	- 10	- 40	0	0	0
	0	1	1	1	0	10
new pivot row	0	2/5	1	0	1/5	6

9) We change the other entries, -40 and 1 in the objective-function row and 1st constraint row respectively, in the pivot column to *zeros* by arithmetic procedures.

X_2 (before)	X_2 (after)
−40	0
1	0

Note: We always want to keep the pivot element at "1" and change all the other numbers in the pivot column to "0."

a - We multiply the new pivot row by 40:
(0 2/5 1 0 1/5 6) × 40 =
(0 16 40 0 8 240)
and add this to the objective-function row
(1 -10 -40 0 0 0)
and the sum is:
(1 6 0 0 8 240) ← new objective function row

New tableau:

	Z	X_1	X_2	X_3	X_4	RHS
new obj-fn row	1	6	0	0	8	240
	0	1	1	1	0	10
new pivot row	0	2/5	1	0	1/5	6

b - We multiply the new pivot row by -1:
(0 2/5 1 0 1/5 6) x (-1) =
(0 −2/5 −1 0 −1/5 −6)
and add this to the 1st constraint row:
(0 1 1 1 0 10)
and the sum is:
(0 3/5 0 1 −1/5 4),

New tableau:

Z	X_1	X_2	X_3	X_4	RHS
1	6	0	0	8	240
0	3/5	0	1	−1/5	4
0	2/5	1	0	1/5	6

pivot
column

The pivot column has zeros and one "1". In the addition or subtraction operations between rows, we *always want to keep the RHS nonnegative*.

10) Are the entries in the objective-function row (the top row) zero or positive? Yes (the next illustration deals with a "no" answer). We're at the end of the simplex algorithm if the answer is "yes". The final and *optimal* tableau is:

Z	X_1	X_2	X_3	X_4	RHS
1	6	0	0	8	240
0	3/5	0	1	−1/5	4
0	2/5	1	0	1/5	6

The solution to the problem is read off the tableau as follows:

a - Pick all columns with zeros and one "1".

b - The corresponding RHS on the same row of the "1" is the optimal answer of the variable.

Z	X_1	X_2	X_3	X_4	RHS
1	6	0	0	8	240
0	3/5	0	1	−1/5	4
0	2/5	1	0	1/5	6

Z= 240, X_2 = 6, X_3 = 4 and all the other variables are equal to zero, X_1= 0, X_4 = 0.

The optimal answer:
Maximum value of Z = 240 at X_1= 0, and X_2 = 6. There is a slack of 4 for the 1st constraint equation, i.e., 4 units of the limited resource represented by the 1st constraint equation is not used.

NOTE: This specific procedure of the simplex algorithm works only for these conditions:
a - maximizing the objective function.
b - less-than-equal-to inequalities in all the constraint equations
c - nonnegativity

Illustration 10)

Consider:
$\max Z = 4 X_1 + 3 X_2 + 6 X_3$

subject to:

$$3 X_1 + X_2 + 3 X_3 \leq 30$$
$$2 X_1 + 2 X_2 + 3 X_3 \leq 40$$
$$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0$$

1) Transform objective function:

$$\boxed{Z - 4 X_1 - 3 X_2 - 6 X_3 = 0}$$

2) Add slack variable, X_4, to the 1st constraint equation:

$$\boxed{3 X_1 + X_2 + 3 X_3 + X_4 = 30}$$

3) Add slack variable, X_5, to the 2nd constraint equation:

$$\boxed{2 X_1 + 2 X_2 + 3 X_3 + X_5 = 40}$$

4) Write tableau. (Please fill in the blank entries in the following tableau.)

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	− 4	− 3	− 6	0	0	0
0	3				0	30
0	2			0		

5) Pick pivot column, the column with the smallest numerical value in the objective row. (Please fill in the blank entries in the following tableau.)

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	- 4	- 3	- 6	0	0	0
0	3		3		0	30
0	2		3	0		

6) Divide the numbers in the RHS column by the numbers in the X_3 column, the pivot column.

X_3	RHS
- 6	0
3	30 → 30 / 3 = 10
3	( ) → ( ) = ( )

The ( ) constraint equation has the smallest RHS/X_3 ratio. This is the pivot row.

7)

	Z	X_1	X_2	X_3	X_4	X_5	RHS
	1	- 4	- 3	- 6	0	0	0
pivot row	0	3	1	3	1	0	30
	0	2	2	3	0	1	40
				pivot column			

The pivot element is 3.

8) Multiply the pivot row by ( ) to change the pivot element into "1". (Please fill in the blanks in square brackets below.)

pivot row (0 3 1 3 1 0 30) × ( ) =

(0 [ ] [ ] 1 [ ] 0 [ ])

9) New tableau:

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	- 4	- 3	- 6	0	0	0
0	1	1/3	1	1/3	0	10
0	2	2	3	0	1	40

We want to change

X_3 (old)	X_3 (new)
- 6	0
1	1
3	0

a) Multiply new pivot row by ( ) and add to objective function row to change -6 of the pivot column to 0.
b) Multiply new pivot row by ( ) and add to 2nd constraint equation row to change 3 of the pivot column to 0.
c) Do the arithmetic. (Please show the calculations.)

10) The new tableau after the arithmetic procedures of step 9 is:

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	2	- 1	0	2	0	60
0	1	1/3	1	1/3	0	10
0	− 1	1	0	− 1	1	10

Are all the entries in the objective function row (top row) zero or positive? 'No". The entry "−1" in the X_2 column is not zero or positive. Therefore, X_2 column is the *new* pivot column. And again, we calculate the RHS/X_2 ratios, except for the objective-function row.

X_2	RHS
- 1	60
1/3	10 → 10 / (1/3) = ( )
1	10 → ( ) = ( )

11) The ( ) constraint equation row has the smallest RHS/X_2 ratio, thus it becomes the new pivot row.

	Z	X_1	X_2	X_3	X_4	X_5	RHS
	1	2	- 1	0	2	0	60
	0	1	1/3	1	1/3	0	10
new pivot row	0	− 1	1	0	− 1	1	10

new pivot column

The new pivot element is 1.

We want to change the X_2 column as follows:

X_2 (old)	X_2 (new)
- 1	0
1/3	0
1	1

Note: we always want to keep the pivot element at "1" and change all the other numbers in the pivot column to “0.”

a) Multiply new pivot row by 1 and add to objective-function row to change -1 to 0.
b) Multiply new pivot row by ( ) and add to 1st constraint-eqn row to change 1/3 to 0.
c) Do the arithmetic. (Please show the calculations.)

12) The new tableau after the arithmetic procedures of step 11:

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	1	0	0	1	1	70
0	4/3	0	1	2/3	−1/3	20/3
0	− 1	1	0	− 1	1	10

Are all the entries in the objective function row (top row) zero or positive? "Yes". We have arrived at the final and optimal tableau.

Z	X_1	X_2	X_3	X_4	X_5	RHS
1	1	0	0	1	1	70
0	4/3	0	1	2/3	−1/3	20/3
0	− 1	1	0	− 1	1	10

The optimal answer:

Z = 70 X_2 = ( ) X_3 = ( )
X_1 = ( ) X_4 = ( ) X_5 = ( )

Illustration 11)

Apply the simplex algorithm to the ice-cream plant illustration. (Please show all the calculations in all the tableaux.)

X_1 = number of quarts of chocolate produced
X_2 = number of quarts of vanilla produced

$$\begin{array}{lll}
\max Z = 13 X_1 + 10 X_2 & & \\
X_1 + X_2 \le 1{,}000 & & \text{(Plant capacity constraint)} \\
X_1 \le 600 & & \text{(Chocolate sale constraint)} \\
X_2 \le 800 & & \text{(Vanilla sale constraint)} \\
X_1 \ge 0,\ X_2 \ge 0. & &
\end{array}$$

CONCLUSION

In this simplex and the model formulation modules, we have studied various LP models and the simplex solution algorithm. It can be seen that LPs are used frequently to configure processes, programs, and plans. These examples are used for the sole purpose of learning the basic techniques. The simplex algorithm illustrated in this module is limited to LP models formulated in a particular format. In real life situations, LP models usually assume many different forms, and they have hundreds and thousands of variables and constraint equations. Thus, a more advanced understanding of LP and computer programs are essential in solving these LP models. Chapter 4 (entitled “Prescriptive Tools”) and Appendix 4 of this text (entitled “Optimization Schemes”) will provide more depth in linear programming. Many available software packages are quite efficient in handling LP models of large size. For convenience, the author has included a software survey in both Chapter 7 and 8 of the text.