Name: ________________________

LINEAR PROGRAMMING MODULE
Part 1 - Model Formulation

INTRODUCTION

In general, a mathematical model is either *deterministic* or *probabilistic*. For example, the models and algorithms shown in the Graph-Optimization module are deterministic. On the other hand, the queuing model in the Probability Distribution module is probabilistic. A linear program in its basic form is a deterministic model, although probabilistic versions have evolved.

Linear programming is the simplest and most elegant of optimization procedures. There are some very convenient features that result from dealing with linear equations. Notice the word "programming" in linear programming does not necessarily mean an electronic computer program, but rather a set of procedures to arrive at a solution: an *algorithm*. The algorithm is tedious enough that computer programming is invariably required for other than the smallest “textbook” problems.

The development of linear programming proceeded quickly during World War II when large scale economic and military planning were needed. Linear programming typically deals with allocating *limited resources* (such as labor, time, machines etc.) between *competing activities* (such as deployment of a particular type of aircraft) and results in the best possible mix (say using an equal number of long range and medium range aircraft for the mission). George Dantzig was the person most responsible for developing the *simplex algorithm* in 1947 for solving linear-programming models.

Using linear programs for decision-making typically involves two steps:

1) the mathematical modeling of the process, and

2) the application of an algorithm to solve the mathematical model, arriving at a desirable solution or a set of feasible solutions.

This module is constructed to serve as an introduction to linear programming, After completing this module the reader should be able to:

a) use linear programming to *model* a real-world problem, where appropriate;

b) use the simplex algorithm to solve a specific type of linear programming problem.

MODELING WITH LINEAR PROGRAMMING

Consider a system under study. It may involve any large or small scale activities such as an amphibious strike force of the military, or a petrochemical plant, or a shipment of fruits from Florida to New York or an airline’s flight schedule. The basic *assumption* of linear programming is that the system as a whole can be *decomposed* into a number of elementary functions called activities. Each of the activities consumes varying amounts of the limited resources available. For example, consider an airline fleet consisting of Boeing 747 and a Boeing 777. If we just consider labor, then the Boeing 747 requires a larger flight and maintenance crew than the Boeing 777. In the linear programming formulation, the activities are usually represented by the variables X_1, X_2, X_3, ... For example, X_1 could be the number of flight crews assigned to the 747, and X_2 could be the number of flight crews assigned to the 777. These variables are in *linear* form and are not in such nonlinear forms as X^2, X^3, e^X, or log X, etc.

Another assumption of linear programming is that the limited resources are consumed by the activities in *direct proportions*. For example, if we initially have 20 maintenance crews assigned to one Boeing 747, then if we have two Boeing 747s, we need 40 maintenance crews, if we have three 747s, we need 60 maintenance crews, and so on. A related economics concept to this *proportionality* assumption is "constant returns to scale." It is also assumed that the resources consumed by the activities are *additive* in nature. For example, if the Boeing 747 consumes 5,000 gallons of fuel and the Boeing 777 consumes 3,000 gallons of fuel, then the total consumption is 8,000 gallons.

Finally, there is the "payoff" of the system. The payoff of a corporation is typically its profit; the payoff of a military strike force is usually its minimum time to deploy; the payoffs of a diet are usually its nutrients. The payoff is represented by a *linear objective function*. (The textbook also uses the term criterion function), The objective function is to be either *maximized* as in the case of corporation profit, or *minimized* as in the case of the time of deployment for the military strike force. There are other minor assumptions and they will be evident we work out some examples.

There are three steps in formulating a linear program (LP):

1. Formulate the "payoff" objective function.

2. Formulate the *constraint equations* representing the activity consumption of the limited resources.

3. Write the *non-negativity constraints*, i.e., all activities are available before they can be consumed; no overdraws are allowed.

Illustration 1)

PROBLEM: A builder is preparing a development plan for a 50-acre community. The zoning regulation requires ½ acre for a two-family unit and 1 acre for a four-family unit. The builder can construct a two-family unit at the cost of $250,000 and a four-family unit at the cost of $400,000. The builder could sell the two-family for the price of $300,000 and the four-family unit for the price of $480,000 to a real-estate agent. Thus, the builder could make a profit of $50,000 on a two-family unit and $80,000 on a four-family unit. The builder has a $15,000,000 loan from the bank. How many of each family unit should the builder construct?

FORMULATION:

Activities:
Let X_1 be the number of two-family units constructed.
Let X_2 be the number of four-family units constructed.

Objective function:

The builder wants to maximize profit.

Maximize, the profit: $Z = 50{,}000\ X_1 + 80{,}000\ X_2$

Constraint equations:

$\frac{1}{2} X_1 + X_2 \leq 50$ (zoning law and land constraint)

$250{,}000\ X_1 + 400{,}000\ X_2 \leq 15{,}000{,}000$ (monetary constraint)

Nonnegativity:
$X_1 \geq 0,\ X_2 \geq 0.$

The whole model, in standard notation is:

$$
\begin{aligned}
&\text{Max } Z = 50{,}000\ X_1 + 80{,}000\ X_2 \\
&\text{subject to} \\
&\qquad \tfrac{1}{2} X_1 + X_2 \leq 50 \\
&\qquad 250{,}000\ X_1 + 400{,}000\ X_2 \leq 15{,}000{,}000 \\
&X_1 \geq 0,\ X_2 \geq 0.
\end{aligned}
$$

Illustration 2) (Please fill in the blanks)

PROBLEM: The Peekskill Electronic Corp. has a product line of three transistor-radios, models DH1, DH2, DH3. Three types of transistors in varying amounts are used in each model. The unit profit of each model is also in varying amounts. These facts are summarized in the following Table:

Radio model	Transistor required for each model			Unit profit
	Type 1	Type 2	Type 3	
DH1	6	3	1	20
DH2	5	3	2	14
DH3	3	1	3	10

An inventory shows 5,000 Type 1, 3,000 Type 2, and 2,000 Type 3 transistors are available for the next month. How many of each radio model should the firm manufacture in the next month?

Activities:
X_1 = the number of DH1 radio models produced in one month
X_2 = the number of DH2 radio models produced in one month
X_3 = the number of DH3 radio models produced in one month

Objective function: maximize profit $Z = 20\ X_1 + (\quad)\ X_2 + (\quad)\ X_3$

Constraint equations:

$6X_1 + 5X_2 + 3X_3 \leq 5{,}000$ (Type 1 transistor constraint)

$3X_1 + 3X_2 + X_3 \leq 3{,}000$ (Type 2 transistor constraint)

$(\quad)X_1 + (\quad)X_2 + (\quad)X_3 \leq 2{,}000$ (Type 3 transistor constraint)

Nonnegativity :
$X_1 \geq 0$, $X_2 \geq 0$, $X_3 \geq 0$.

Illustration 3)

PROBLEM: A dietician is planning a special diet for a patient consisting of only rice, steak, and tomato. The following Table summarizes the nutritional values and cost for each food item.

Food item	Units of carbohydrates	Units of protein	Unit cost
Rice	4	2	1
Steak	3	8	2
Tomato	2	10	3

The minimum dietary requirements are 6 units of carbohydrates and 8 units of protein. How many daily units of each food item should the dietician plan for the patient to *minimize* cost?

FORMULATION:

Activities:
Let X_1 = units of rice
Let X_2 = daily units of steak
Let X_3 = daily units of tomato

Objective function:
minimize cost $Z = 1\ X_1 + (\quad)X_2 + (\quad)(\quad)$

Constraint equations:
$4X_1 + 3X_2 + 2X_3 \geq 6$
(Minimum carbohydrate requirement; the inequality is “greater than or equal to” because 6 is the minimum requirement.)
$2X_l + (\quad)(\quad) + (\quad)(\quad) \geq (\quad)$
(Minimum protein requirement)
$X_1 \geq 0$, $X_2 \geq 0$, $X_3 \geq 0$.

(Nonnegativity)

Illustration 4)

PROBLEM: A company has two factories and two regional warehouses. The cost per unit of shipment from each factory to each warehouse is shown below:

Shipment to	Warehouse 1	Warehouse 2
Factory 1	20	25
Factory 2	30	16

Factory 1 produces 800 units and factory 2 produces 600 units. All of the units must be shipped out. Warehouse 1 has a capacity of 750 units and warehouse 2 has a capacity of 650 units. How many units should be shipped from each factory to each warehouse to *minimize* the shipment cost?

FORMULATION

Activities:
Let X_{11} = the number of units shipped from factory 1 to warehouse 1. (X_{11} is read "X one one," really "X sub one one," but we frequently leave off “sub,” which stands for subscript.)
Let X_{12} = the number of units shipped from factory 1 to warehouse 2.
Let X_{21} = the number of units shipped from factory 2 to warehouse 1.
Let X_{22} = ( )

Objective function:
Minimize cost Z = 20 X_{11} + 25 X_{12} + ( ) ( ) + ( ) ( )

Constraint equations:
$X_{11} + X_{12} = 800$ (The total shipment *from* factory 1 is 800 units, hence the equality.)
X_{21} + ( ) = ( ) (The total shipment *from* factory 2 to the two warehouses.)
$X_{11} + X_{21} \leq 750$ (The total shipment *to* warehouse 1 is limited to 750, hence the inequality.)
$X_{12} + X_{22} \leq 650$ (Shipment to warehouse 2.)
$X_{11} \geq 0$, $X_{12} \geq 0$, $X_{21} \geq 0$, $X_{22} \geq 0$.

Illustration 5)

PROBLEM: An airline company is considering the purchase of new aircraft. The price for a long-range plane is $220 million, for a medium-range plane is $195 million and for a short-range plane is $110 million. The company can get total finance of $5.5 billion for these purchases. The estimated net annual profit for each plane is:

long range: $13.75 million

medium range: $11 million
short range $6.875 million

There are sufficient trained pilots available next year to fly 40 new aircraft. Maintenance facilities next year would be able to handle 45 aircraft. The company wants to maximize profit. Formulate a linear program model to get a preliminary estimate of the numbers of each type of aircrafts to purchase.

FORMULATION:

Activities :
X_1= the number of long-range planes purchased
X_2= the number of medium-range planes purchased
X_3= the number of short-range planes purchased

Objective function:

Maximize profit Z = ( )X_1 + ( )( ) + ( )( )

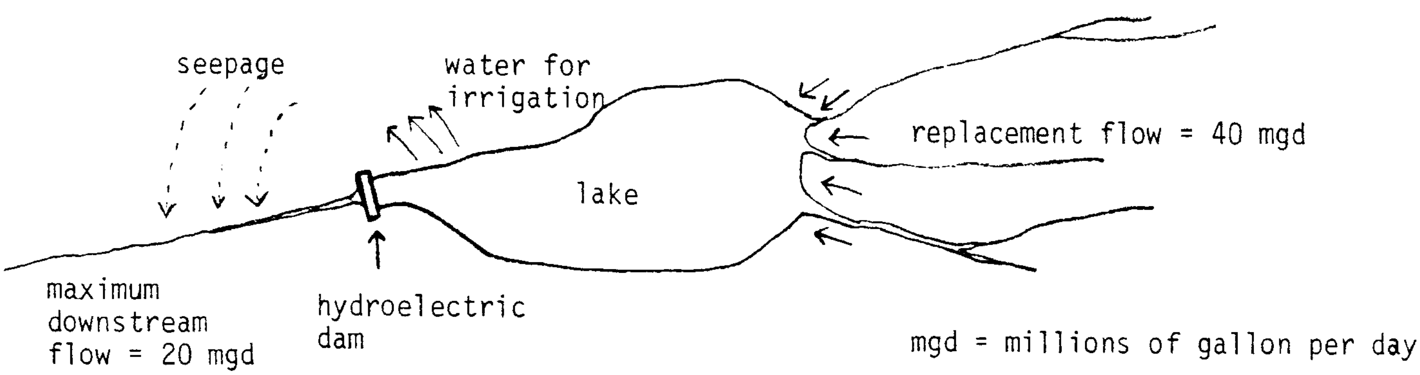
Constraint equations:

220 X_1	+	( )( )	+	( )( )	$\le$	( )	(Money constraint)
X_1	+	( )	+	( )	$\le$	40	(Pilot constraint)
( )	+	( )	+	( )	$\le$	( )	(Maintenance facility constraint)

$X_1 \ge 0$, $X_2 \ge 0$, $X_3 \ge 0$.

Illustration 6)

PROBLEM:



A hydroelectric dam was constructed for two purposes: 1) electricity generation and 2) irrigation. The average flow from rain, runoffs, and streams is 40 mgd. The maximum downstream flow is 20 mgd. There would be erosion and flood if downstream flow is greater than 20 mgd. The generating efficiency of the dam is one million watt-hours per one million gallons per day. Also, it was determined that 1/3 of the water used for irrigation seeps through the ground and contributes to the downstream flow. (The seepage varies with the season but for simplicity sake we assume 1/3. The hydroelectric authority has been authorized to charge $60 per one million gallons of irrigation water and $70 per one million watt-hours of electricity. Assuming unlimited demands for electricity and irrigation, how should the hydroelectric authority allocate the water for revenue?

FORMULATION:

Activities:
X_1 = water allocated, in millions of gallon, for irrigation
X_2 = water allocated, in millions of gallon, for electricity generation.

Objective function:
Maximize revenue Z = ( ) ( )+( )X_2

Constraint equations:

( ) + ( )	≤ 40	(Inflow constraint: the use must be equal or less than replacement.)
( ) ( ) + X_2	≤ 20	(Downstream flow constraint: the flow must be equal or less than maximum allowed downstream flow.)

$X_1 \geq 0$, $X_2 \geq 0$.

Illustration 7)

PROBLEM: The Precision Glass Co. produces six glass products. The production schedule indicates that the next month's production should be:

- 200 pieces of coated glass #1
- 200 pieces of coated glass #2
- 150 pieces of coated glass #3
- 50 pieces of lens #1
- 75 pieces of lens #2
- 25 pieces of lens #3

The actual monthly production is not to fall short of the planned production by more than 10% for each glass product.

Production times for each product:

- 0.012 unit operating hour per coated glass #1
- 0.014 unit operating hour per coated glass #2
- 0.018 unit operating hour per coated glass #3
- 0.040 unit operating hour per lens #1
- 0.045 unit operating hour per lens #2
- 0.060 unit operating hour per lens #3

Because of maintenance requirements of the production machines, the production time for each month is 1,000 unit operating hours.

The packing operations requirement:

- 0.010 unit packing hour per coated glass #1
- 0.015 unit packing hour per coated glass #2
- 0.018 unit packing hour per coated glass #3
- 0.025 unit packing hour per lens #1
- 0.035 unit packing hour per lens #2
- 0.040 unit packing hour per lens #3

The packing can be done on two units: unit #1 for the coated glasses and unit #2 for the lenses. Because of maintenance requirements, unit #1has 500 unit hours available per month, and unit #2 has 900 unit hours available per month.

The total approximate cost per glass product for the glass company is:

- $1.20 per coated glass #1
- $2.00 per coated glass #2
- $2.50 per coated glass #3
- $3.00 per lens #1
- $5.00 per lens #2
- $6.00 per lens #3

What is the optimal production?

FORMULATION:

Activities:
X_1= the number of coated glass #1 produced
X_2= the number of coated glass #2 produced
X_3= the number of coated glass #3 produced
X_4= the number of lens #1 produced
X_5= the number of lens #2 produced
X_6= the number of lens #3 produced

Objective function:
Minimize cost $Z = 1.20\, X_1 + 2.00\, X_2 + (\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad)$

Constraint equations:
Actual production must be within 10% of the planned production constraint.
$200 - X_1 \leq (0.10)\,(200)$
$200 - X_2 \leq (0.10)\,(200)$
$(\quad) - (\quad) \leq (\quad)\,(\quad)$
$(\quad) - (\quad) \leq (\quad)\,(\quad)$
$(\quad) - (\quad) \leq (\quad)\,(\quad)$

$25 - X_6 \leq (0.10)(25)$

Production time constraint:
$0.012\,X_1 + (\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad) \leq 1{,}000$

Packing time constraints:
$0.010\,X_1 + (\quad)(\quad) + (\quad)(\quad) \leq 500$ (for unit 1)
$(\quad)(\quad) + (\quad)(\quad) + (\quad)(\quad) \leq 900$ (for unit 2)

$X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0, X_5 \geq 0, X_6 \geq 0$

Illustration 8)

A dietician must make up a diet consisting of meat, bread and spinach, at the minimum cost.

Minimum daily requirements of nutrients are:
protein 100 units
carbohydrate 50 units
iron 15 units

The nutrient compositions, in units per pound, of the three food items are:

	Protein	Carbohydrate	Iron
Meat	40	8	5
Bread	10	35	2
Spinach	5	6	20

The costs of the three food items are:
meat $ 1.65 / lb.
bread $ 0.70 / lb.
spinach $ 0.60 / lb.

Model this problem. (Please write out the linear programming formulation.)