Name: __________________________

PROBABILITY MODULE

An understanding of probability is important to the decision maker. Many decisions must be based on predictions of future events. Inevitably, the prediction of future events has uncertainties and probable errors. An example is population projection, as discussed in Chapter 2 of this text. An understanding of probability concepts helps the decision maker to appreciate the significance of such uncertainties and probable errors. This module serves to introduce or review the fundamental probability concepts, which allows an understanding of what is *information* and *imperfect information*.

This activity module is divided into three sections. The *first* section covers some of the theories of probability. The *second* section covers some rules of counting. Finally, the *third* section builds upon the first and second sections and illustrates with some interesting examples.

By the end of this exercise, the student

a) would be familiar with these concepts: sample space, events, union and intersection of events, empirical or frequency probability, subjective probability, and permutation;

b) would have seen some useful application of these concepts.

THEORY OF PROBABILITY

Definition: *Sample space* refers to all the possible outcomes of a process or experiment. (Sample space is the more general technical term for "equally likely results" used in many textbooks.)

Illustrations:

1) Suppose we define an experiment of tossing two pennies. The set of possible outcomes of tossing the two pennies is (assuming that they do not land on the edge):

		1st Penny	2nd Penny
Possibility	A	head	head
Possibility	B	head	tail
Possibility	C	tail	head
Possibility	D	tail	tail

If we use H for head, and T for tail, and parentheses with two spaces (,); with the 1st space representing the 1st penny, and the 2nd space representing the 2nd penny, the set of all possible outcomes, or in other words, the *sample space* can be summarized:

S = { (H, H), (H, T), (T, H), (T, T) }

Note that we are using S as the symbol for the sample space. Each entry in S is called a *sample point* or *sample element*. Thus (H, H) is called a sample point, likewise for (H, T), (T, H) and (T, T).

A final note, the concept of sample space can only be applied to experiments with well-defined outcomes, and where the set of all possible outcomes can be specified. An example that violates the rule is this: suppose we give questionnaires to all the students on campus to determine their heights. The questionnaire is designed as follows:

A) 4 ft - 5½ ft B) 5 ft - 6½ ft C) 6 ft - 7½ ft

The elements of this sample space overlap each other and thus it is not a set of objects. Note: one of the requirements for a set is that the elements in the set must be distinct. *Sample space* is a *set*.

2) Suppose we define an experiment of tossing one die. The set of all possible outcomes of tossing one die is:

Possible outcome A: the face with one dot is on top
Possible outcome B: the face with two dots is on top
Possible outcome C: the face with three dots is on top
Possible outcome D: the face with four dots is on top
Possible outcome E: the face with five dots is on top
Possible outcome F: the face with six dots is on top or in shorter notation:
S = {1, 2, 3, 4, 5, 6 } each number represents the top face of the die

3) Suppose we define an experiment of tossing three coins, and recording the outcome of the toss of each coin as head or tail. The *sample space* of this experiment has eight *sample points*. Please fill in the missing points:

S = { (H, H, H), (H, H, T), ( ), ( ), ( ), (T, H, T), ( ), (T, T, T) }

4) Suppose we define a process of surveying voting individuals. We ask each voter his/her sex (male or female) and political affiliation (Democrat, Republican, or Other). The *sample space* of this process has six *sample points.* Let the symbols M, F, D, R, and 0 stand for the sex and political affiliation respectively. Please fill in the missing sample points: S = { (M, D), (M, R), ( ), ( ), ( ), ( ) }

5) Suppose we define an experiment of tossing two dice and recording the faces on top. The sample space of this experiment has 36 sample points. In other words, this experiment has 36 possible outcomes. One of the possible outcomes is (1, 1) -1st die shows one and 2nd die shows one. *Please list in the space below the other 35 possible outcomes of this experiment.* There are 36 possible outcomes because the 1st die has six possible outcomes, and the 2nd die has six possible outcomes, and six times six is 36**.**

Definition: Suppose we are interested in *some* of all of the possible outcomes. For example, we may be interested only in one of the possible outcomes, or two of the possible outcomes, or six of the possible outcomes, and so on. These possible are *subsets* of all of the possible outcomes. These subsets of the sample space are called *events* in probability theory. (Event is the technical term for "a favorable result" used in many textbooks.)

6) The sample space of tossing two coins is: S = { (H, H), (H, T), (T, H), (T, T) }

The event, "two tails," is { (T, T) }.

The event, "one head" is { (H, T), (T, H) }.

The event, "at least one tail" is { (H, T), (T, H), (T, T) }.

Fill in the missing:

The event, "at least one head," is { (H, H), ( ), ( ) }.
The event, "at least one head or one tail," is { (H, H), ( ), ( ), ( ) }.
The event, "two heads," is { ( ) }.
The event, “one tail,” is { ( ), ( ) }.
The event, "first coin is head," is { (H, H), ( ) }.
The event, "second coin is tail," is { (H, T), ( ) }.

7) Suppose we define an experiment of tossing two dice. The sample space of this experiment has 36 sample points or possible outcomes. (See illustration 5).

The event, "the sum of the spots on the two dice is two," is { (1, 1) }
The event, "the sum of the spots on the two dice is 11," is { (5, 6), (6, 5) }.

Fill in the missing:

The event, "the sum of the spots on the 2 dice is 7," is { (6, 1), (5, 2), ( ), ( ), ( ), (1, 6) }.

The event, “number on the second die is twice the number on the first die," is { (1, 2), (2, 4), ( ) }
The event, "the number on the second die is larger than the number on the first die,” is { (1, 2), (1, 3), ( ), ( ), ( ), (2, 3), ( ), ( ), ( ), ( ), ( ), ( ), ( ), ( ), (5, 6) }.

The event, "the number on the first die is 2," is { (2, 1), ( ), ( ), ( ), ( ), ( ) }.

It is very useful to use set operations in probability theory. The three set operations are *union*, *intersection*, and *complement*.

Illustrations:

8) Suppose we define an experiment of tossing two coins. The sample space for this experiment is: (see illustration 1) S = { (H, H), (H, T), (T, H), (T, T) }

The event, "first coin shows head," is:

A = { (H, H), (H, T) }

The event, "at least one coin shows tail, " is:

B = { (H, T), (T, H), (T, T) }

The event, "at least one coin shows head," is:

C = { (H, H), (H, T), (T, H) }

We thus have so far identified events A, B, and C. The event, "first coin shows head or at least one coin shows tail," is { (H, H), (H, T), (T, H), (T, T) }, which is just A union B, or A U B in set notation. This union process is to combine elements of A and B, and to retain only one of each redundancies, in this case we have two (H, T)'s and we just retain one (H, T).

The event, "first coin shows head and at least one coin shows tail," is { (H, T) }, which is A *intersection* B or $A \cap B$ in set notation. We see immediately this intersection process is to pick out the common elements of A and B, which is (H, T).

The event, “first coin shows tail," is { (T, H), (T, T) },” which is just A complement or $\bar{A}$ or A' in set notation. We see immediately that A complement is the elements in S that are *not* in A.

Fill in the missing:

The event, “first coin shows head or at least one coin shows head," is

{ (H, H), (   ), (   ) },

The event, “first coin shows head and at least one coin shows head," is

{ (H, H), (   ) },

The event, “at least one coin shows tail or at least one coin shows head," is

{ (   ), (   ), (   ), (   ) }

The event, “at least one coin shows tail and at least one coin shows head," is

{ (   ), (   ) }

Now that we have some understanding of sample space and events, we can begin to assign numbers to probabilities of events, There are three ways to assign numbers to probabilities of events, One way is called *a priori* or *theoretical* probabilities. For example, the sample space of tossing two coins is: HH, HT, TH, TT. These four possible outcomes are equally likely to happen. We reasoned that the coins will not land on the edge, or break apart upon landing on the surface, or fly up into space or disappear, etc. Thus the probability of the event "two heads" is ¼, because there is only *one* outcome of two heads, { (H, H) } and there are *four* equally likely outcomes, The probability of the event, "one tail " is ½ because there are *two* outcomes of "one tail," (HT) and (TH) and there are *four* equally likely outcomes. In general, in *a priori* probability:

$$P(E) = n(E) / n(S)$$

where n(E) = number of sample points in event E
n(S) = number of *equally likely sample points* in the sample space S.

The second way of assigning probabilities to an event is called *empirical* or *frequency* probabilities. For example, an inspector examined 1,000 diesel trucks and found five trucks with sub-standard brakes, Thus the probability of a diesel truck with a sub-standard braking system is

$$P(E) = f / n$$

where

n = the number of occurrence of event E
f = the total number of times the process or experiment has been performed or observed

Finally, the third way of assigning probability is called *subjective* (or *Bayesian*) probability. It is the measure of a "knowledgeable" person's belief in the likelihood of the occurrence of some event. For example, an economist may double a price increase in crude petroleum within two years with a probability 0.6; or a marketing vice-president may predict sales of $100 million with a probability of 0.8; or a cancer researcher predicts finding a cure for lung before the year 2000 with a probability of 0.70. All these probabilities are based on the prior knowledge of the expert on the subject.

RULES OF COUNTING

Now that you have some understanding of basic definitions in probability, this section will help you to learn some basic rules of counting or calculating the number of possible outcomes of a process or experiment. These rules or formulas are short cuts. Obviously, the most fundamental counting rule is to list all the possible outcomes and count them individually; but this process is tedious and time consuming and could involve large numbers.

Rule (I)

If a process or experiment consists of k separate steps, and the first step can occur in n_1 different ways, the second step in n_2 different ways, and n_k the kth step in different ways, then the number of different possible outcomes is $n_1 \times n_2 \times n_3 \ldots \times n_k$

Illustrations

9) Suppose we define an experiment of tossing a coin three times and each time recording whether head or tail. This experiment has three steps:

1st step: a coin is tossed with the result of head or tail, thus the first step has two

possible outcomes.
2nd step: the coin is tossed and again this step has two possible outcomes.
3rd step: the coin is tossed and again this step has two possible outcomes.

The number of different possible outcomes for this experiment is:

$$2 \times 2 \times 2 = 8$$

Using the hand notation introduced in the first section of this module, the possible outcomes are: { (HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT) }

The first entry in represents the first toss, the second entry represents second toss and the third entry represents the third (Note that this experiment is slightly different from illustration 3)

10) Suppose we define an experiment of drawing individually five cards, without replacing each, from a full deck of playing cards, What is the number of different possible outcomes?

This experiment has five steps:

1st step: a card is drawn from the deck, this step has 52 possible outcomes, and there remain 51 cards,

2nd step: a card is drawn from the deck; this step has 51 possible outcomes; there remain 50 cards,

3rd step: a card is drawn from the deck; this step has 50 possible outcomes; there remain 49 cards.

4th step: a card is drawn from the deck; this step has 49 possible outcomes; there remain 48 cards.

5th step: a card is drawn from the deck; this step has 48 possible outcomes; there remain 47 cards,

The number of different possible outcomes is: 52 x 51 x 50 x 49 x 48 = 311,875,200

11) A student can select five courses from a total offering of 30 courses with no time conflicts. Please calculate the number of different possible course loads for the student.

1st selection has 30 possible choices.

2nd selection has 29 possible choices.

3rd selection has (   ) possible choices.

4th selection has (   ) possible choices.

5th selection has (   ) possible choices.

The no. of possible different course loads is: 30 x 29 x (   ) x (   ) x (   ) = (   ).

12) Suppose we define an experiment of tossing a die three times. Please calculate the number of different possible outcomes of this experiment.

1st toss has 6 possible outcomes.

2nd toss has ( ) possible outcomes.

3rd toss has ( ) possible outcomes.

The number of different possible outcomes is:

6 x ( ) x ( ) = 216

13) Suppose we define an experiment of tossing two dice three times, Please calculate the number of different possible outcomes of this experiment,

1st toss has 36 possible outcomes (See illustration 5 for this result).

2nd toss has ( ) possible outcomes.

3rd toss has ( ) possible outcomes.

The number of different possible outcomes is:

36 x ( ) x ( ) = 46,656

Rule (2)

Permutation rules. Permutation refers to an ordered arrangement of objects. For example, if we select three students from a group of students, each different possible arrangement of the three students is a permutation,

Rule (2a)

The formula for determining the number of possible permutations of r objects selected from a set of n objects is: $_nP_r = n! / (n - r)!$

Illustrations

14) n! is read "n-factorial."

6! means 6 x 5 x 4 x 3 x 2 x 1

10! means 10 x 9 x 8 x 7 x 6 x 5 x 4 x 3 x 2 x 1

15) If we select three students from a of 10 students, the number of possible arrangements is:

$$\begin{aligned} {}_{10}P_3 &= 10! / (10 - 3)! \\ &= 10! / 7! \\ &= (10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) / (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) \\ &= 720 \end{aligned}$$

16) There are eight possible river crossings but we have money only for four bridges. What is the number of possible arrangements of the four bridges? Please fill in the missing.

$$_8P_{(\quad)} = 8! / (\quad)! = (\quad)$$

Rule (2b)

The formula for determining the number of possible permutations of r objects selected from a set of n objects' if *repetitions of the r objects are allowed* (once the object is put back into the n objects) is $_nP^r = n^r$.

Illustrations

17) Suppose we draw a card from a deck of 52 playing cards. The card is then returned to the deck and the deck is shuffled. We do this four times, drawing a total of four cards, How many different permutations of the four cards are possible?

We are drawing four objects from 52 objects, with replacement for each object.

` $${}_{52}P^{4} = 52^{4} = 7{,}311{,}616$$

Suppose we have the letters { a, b, c }. We pick one letter. With repetition (or replacement) allowed, we pick a second letter. The number of permutations in this process is:

$${}_{3}P^{2} = 3^{2} = 9$$

The nine permutations are: (please fill in the blanks)

aa	(b)	cc
ab	( )	( )
( )	( )	( )

Applications

Let us examine a classic probability problem: In a room of N people, what is the probability that at least two persons have the same day of the year as birthdays? This section of the module will help you to answer this question. Please fill in the blanks.

Illustrations

18)

Suppose we have a room with five people. We assume 365 days in the year. There are 365 possible days for the first person's birthday. There are 365 possible days for the second person's birthday.

There are ( ) possible days for the 3rd person's birthday,

There are ( ) possible days for the 4th person's birthday.

There are ( ) possible days for the 5th person's birthday.

We are selecting five birthdays (because there are five persons) from 365 days, The number of permutations of five objects (the birthdays) selected from 365 objects (the days in a year) with *repetition* (or replacement) is:

$${}_{365}P^{5} = (\qquad)^{5} = 6.48 \times 10^{12}$$

This result can be arrived at by using Rule 1 of the second section of this module. There are five separate steps in the process, with 365 different ways in each step.

Illustrations

19)

If no two persons have the same birthday, then:

There are 365 possible days for the person's birthday.

There are 364 possible days for the 2nd person's birthday.

There are 363 possible days for the 3rd person's birthday.

There are (   ) possible days for the 4th person's birthday.

There are (   ) possible days for the 5th person's birthday.

The number of permutations of five objects (the five birthdays) from 365 objects (the days in the year) *without repetition* (because we want no two persons to have the same birthday) is: ${}_{365}P_5 = (\quad)! / 360! = 6.30 \times 10^{12}$

or simply,

$$365 \times 364 \times 363 \times (\quad) \times (\quad) = 6.30 \times 10^{12}$$

The experiment we have defined is:

We chose five days from 365 days, *with repetition allowed.* The number of elements in this sample space, n(S), is (see the first section of this module)

$${}_{365}P^5 = 6.48 \times 10^{12}$$

In particular, we are interested in the event no two persons have the same birthday. The number of elements in this event, is:

$${}_{365}P_5 = 6.30 \times 10^{12}$$

The probability of this event, P(E), is:

$$P(E) = n(E) / n(S) = (6.30 \times 10^{12}) / (6.48 \times 10^{12})$$

The above equation translated into words is "the probability of no two persons in a group of five persons have the same birthday is 0.972. In probability, the sum of an event and not that event is always one. For example, in tossing a coin, the probability of a head is ½, and the probability of not a head is ½; or in tossing a die, the probability of a "one" is 1/6 and the probability of not one showing is 5/6.

From the previous discussion, we immediately conclude that the probability of at least two persons in a group of five persons have the same birthday is:

$$1 - 0.972 = 0.028$$

In summary,

$$0.028 = 1 - {}_{365}P_5 / {}_{365}P^5$$

Illustrations

20)

Additional illustrations of the "birthday problem:"

a) The probability of at least two persons with the same birthday in a group of 10 persons:

$$1 - {}_{365}P_{10} / {}_{365}P^{10} = 1 - 0.883 = 0.117$$

b) The probability of least two persons with the same birthday in a group of 20 persons:

$$1 - {}_{(\ \ )}P_{(\ \ )} / {}_{(\ \ )}P^{(\ \ )} = 1 - 0.589 = 0.411$$

c) The probability of least two persons with the same birthday in a group of 25 persons:

$$1 - {}_{(\ \ )}P_{(\ \ )} / {}_{(\ \ )}P^{(\ \ )} = 1 - 0.431 = 0.569$$

Note that with a group of 25 persons, the probability is greater than 0.5 that two persons have the same birthday!