

# *Wave Turbulence*

Sergey Nazarenko



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# *Book ``Wave Turbulence''*

Sergey Nazarenko

Springer, Lecture Notes in Physics  
series, 2011.

# *Course outline*

- ✓ Define Wave Turbulence. Physical examples.
- ✓ Mathematical assumptions and statistical closures.
- ✓ Properties: energy cascades, Kolmogorov-Zakharov spectra, coherent structures & intermittency.
- ✓ ``Cheat sheet" on Wave Turbulence. Shortcuts & tricks to derive ``everything" in WT.
- ✓ Random - coherent transitions. WT life cycle. WT in BEC. Waves & vortices.
- ✓ Summary & Outlook.

# *What is Wave Turbulence?*

- WT is a statistical system of nonlinear waves.
- WT is a phenomenon, not a theory.
- Hence it works (hard).



# *Examples of WT*

- Water Waves: gravity and capillary,
- Waves in rotating and stratified fluids (inertial and internal waves, Rossby waves)
- Plasma waves (ion sound, Langmuir, whistler, drift waves),
- Waves in Bose-Einstein condensates,
- Kelvin waves on quantised vortex filaments,
- MHD turbulence in interstellar turbulence & solar wind,
- Nonlinear optics,
- Solids: phonons, spin waves.

# Weak Turbulence

- Weak Turbulence is a theory that works for weakly nonlinear dispersive waves with **generic** properties.
- Most of wave systems are special, hence Weak Turbulence **fails** most of the time.
- Description of WT needs to be extended to include coherent structures, condensates, finite-box effects, interaction with strong turbulence.

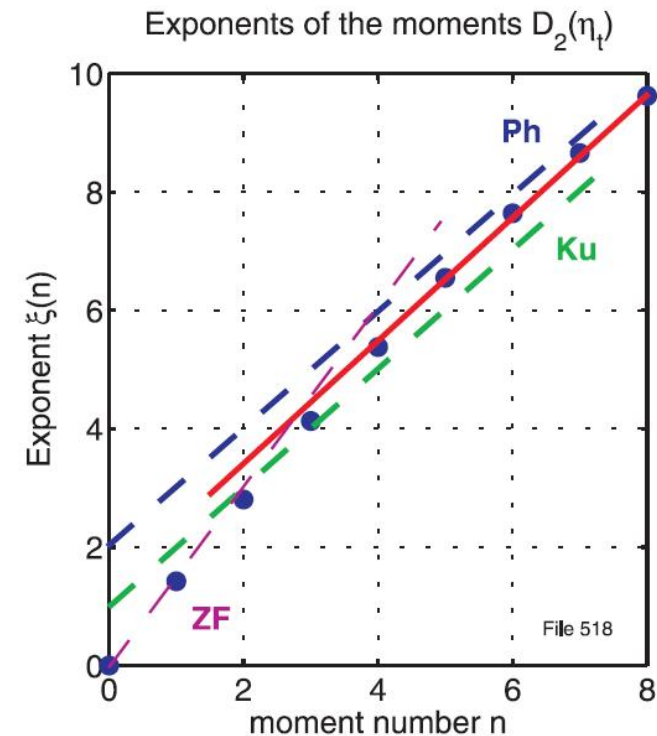
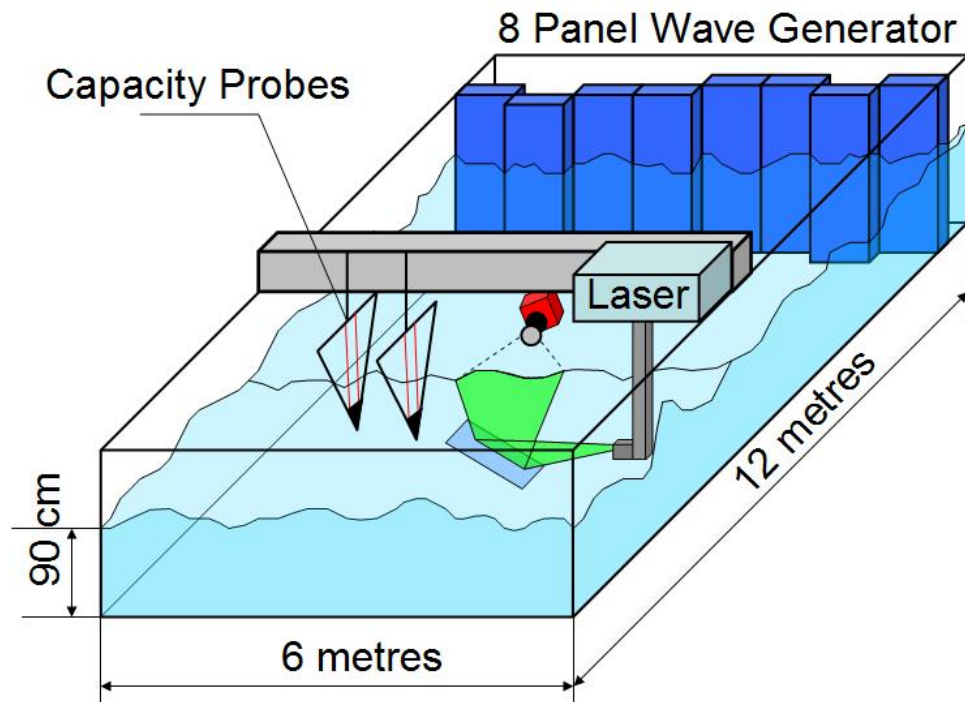


# *Water Waves.*

Random weak waves coexist with  
intermittent strong coherent structures -  
sharp crests, wave breaking



# *Hull wave tank of Lukaschuk et al*

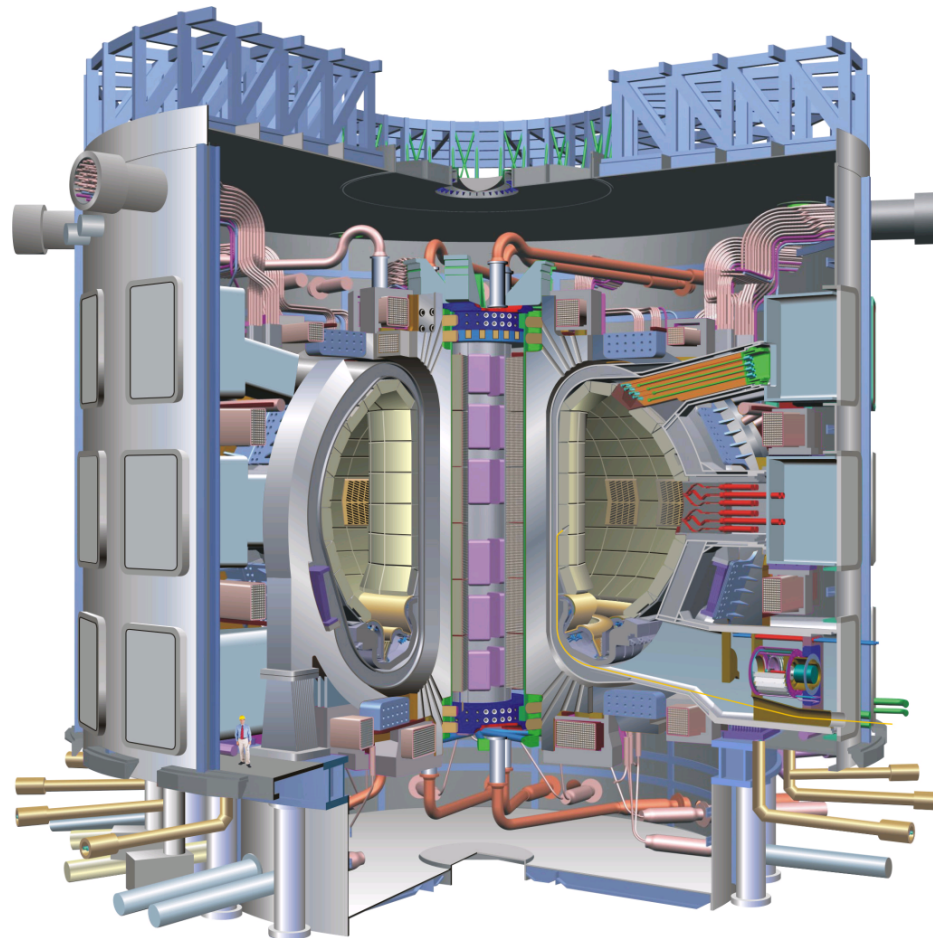


- Low moments - random waves
- High moments - coherent crests



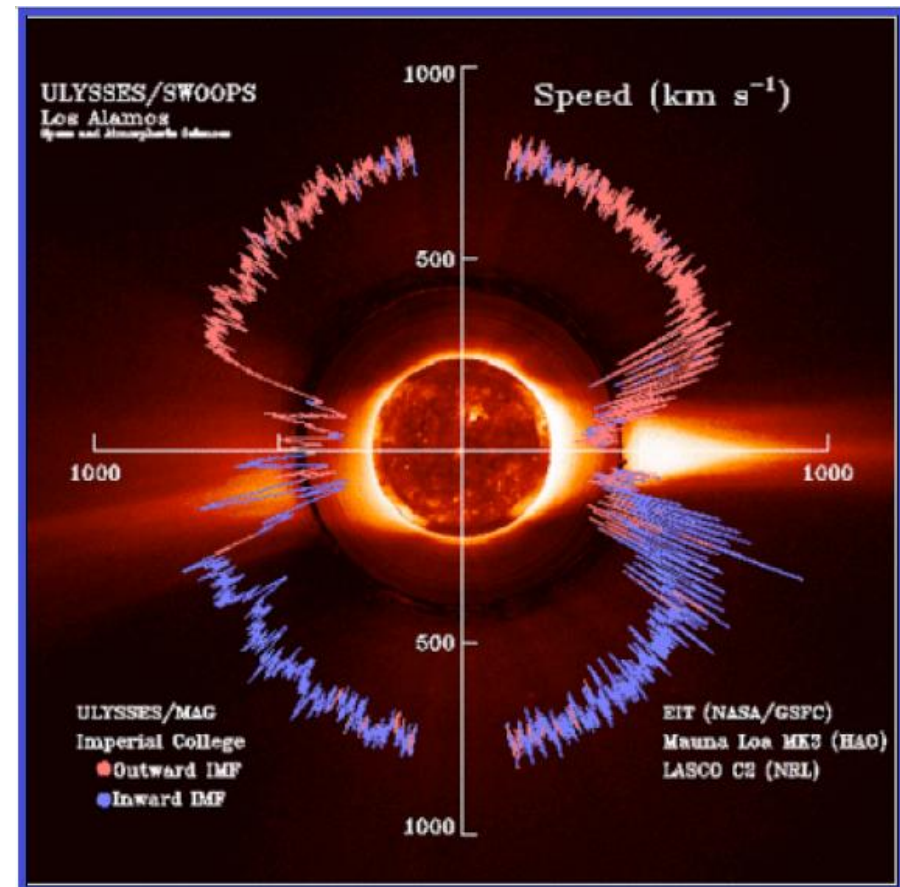
# *Waves in fusion plasmas*

- Drift waves appear naturally due to an instability – they cause anomalous heat and particle loss – major problem for fusion.
- Device size is often important.
- Drift WT generates zonal jets which form transport barriers (Balk et al 1990) -> H-mode (improved confinement).



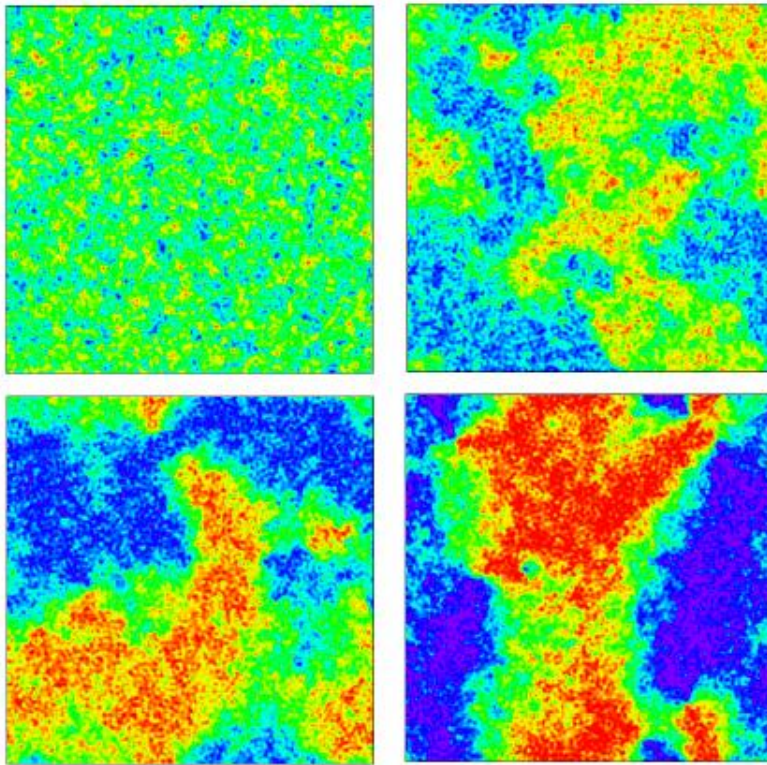
# MHD waves

- MHD turbulence in astrophysics: interstellar medium, solar wind.
- Interaction with non-wave ( $k_{||}=0$ ) component.



# Waves in Bose-Einstein condensates.

- Inverse cascade, - condensation.
- Condensate strongly affects WT



From SN & Onorato 2006.

4-wave WT  $\rightarrow$  strong turbulence  
& vortices  $\rightarrow$  3-wave WT on  
condensate

3D NLS –Onorato &Proment  
2009.

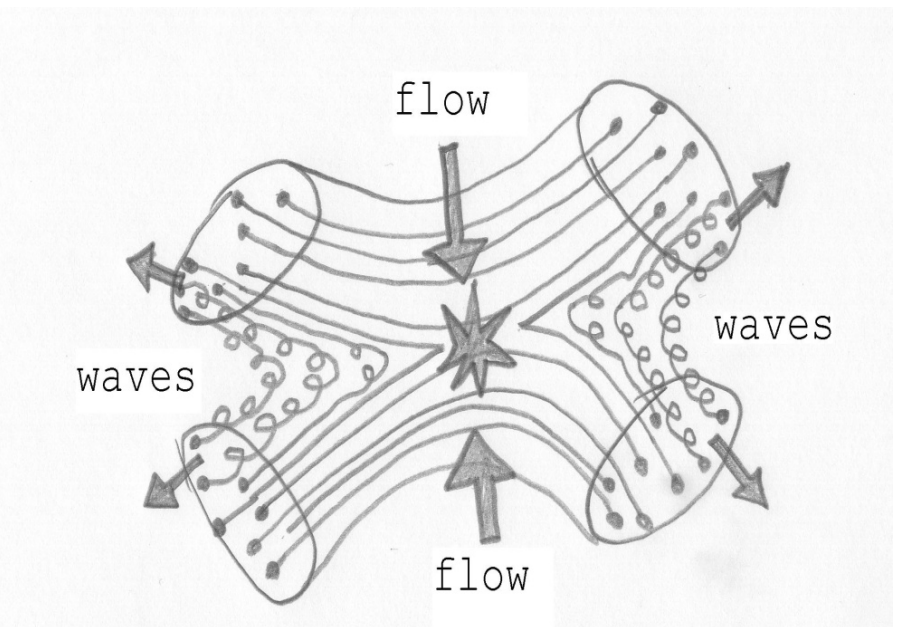
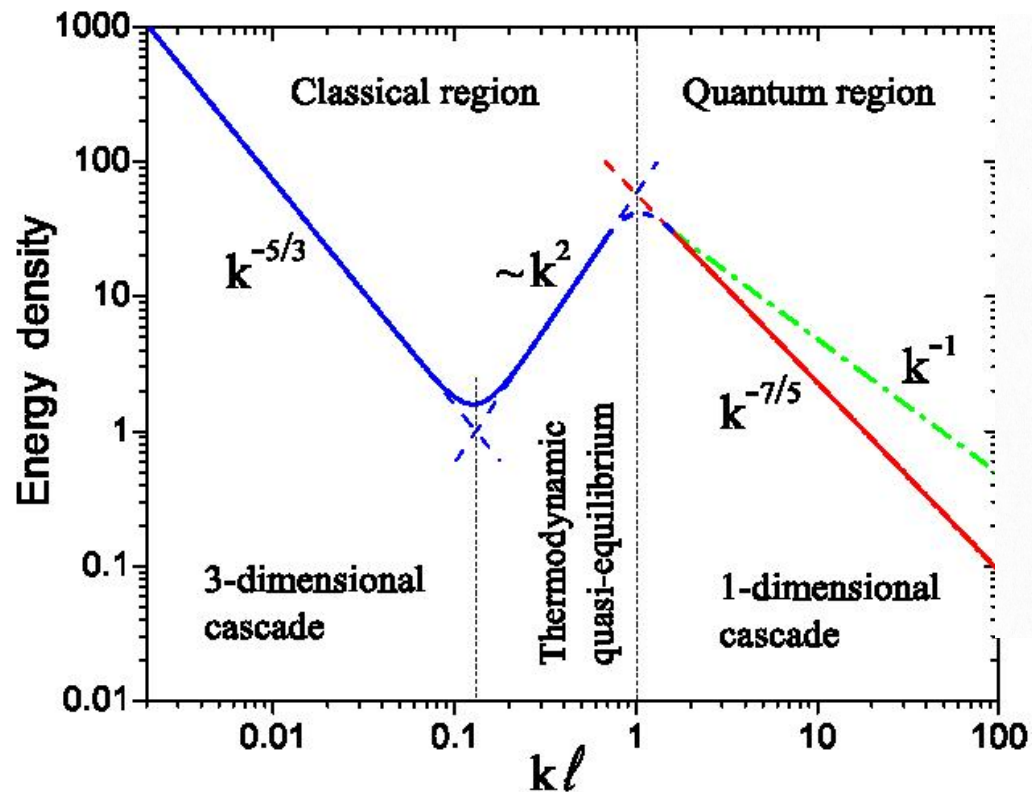
$Re[\psi(x, y)]$  at different times:  $t = 2500, t = 5000, t = 7500, t = 10000$ .



# Quantum Turbulence

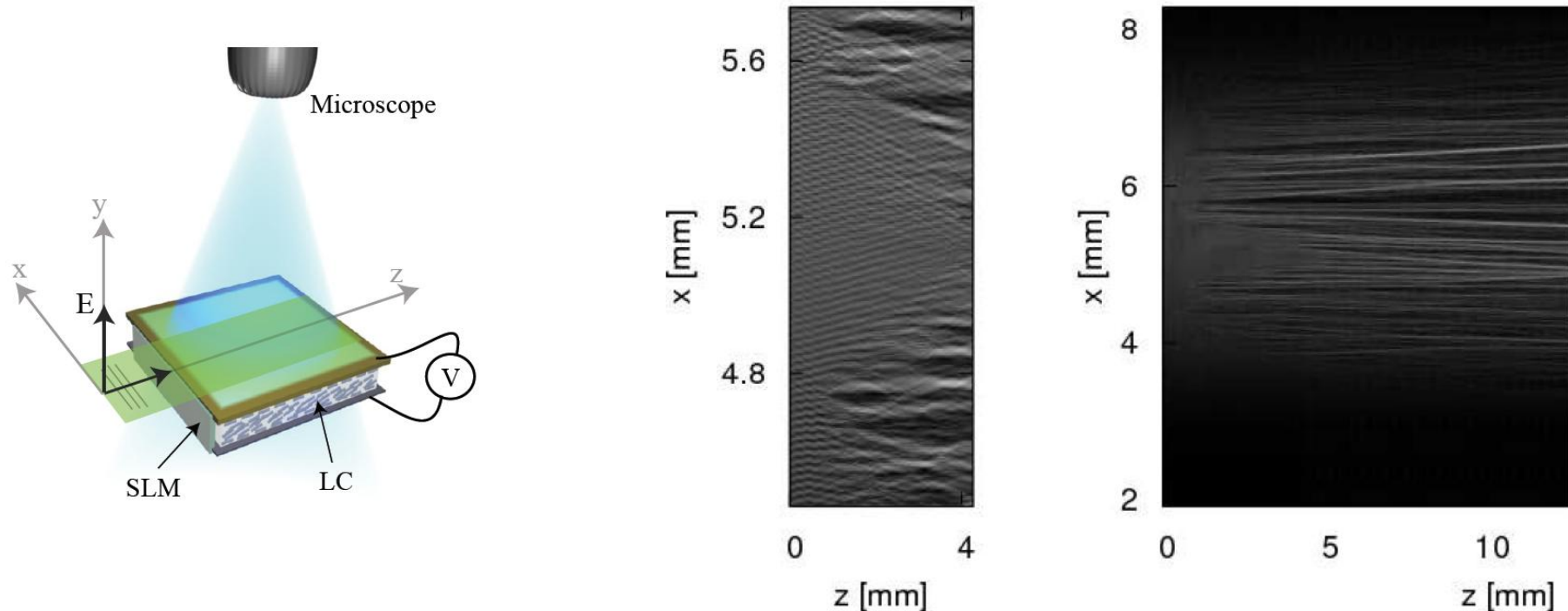
Kelvin waves on quantised vortex filaments (Vinen & Barenghi talks).

Interaction with hydro eddies (vortex bundles) is important.

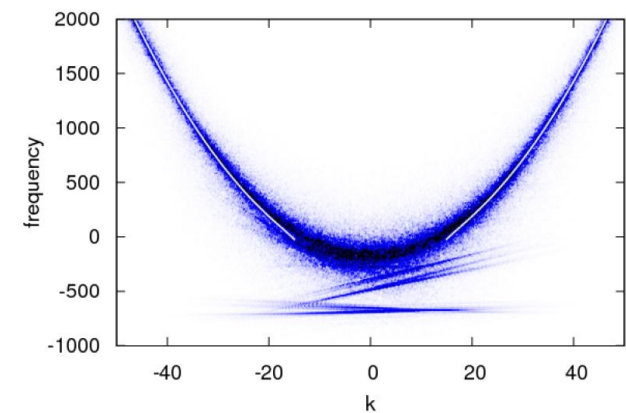
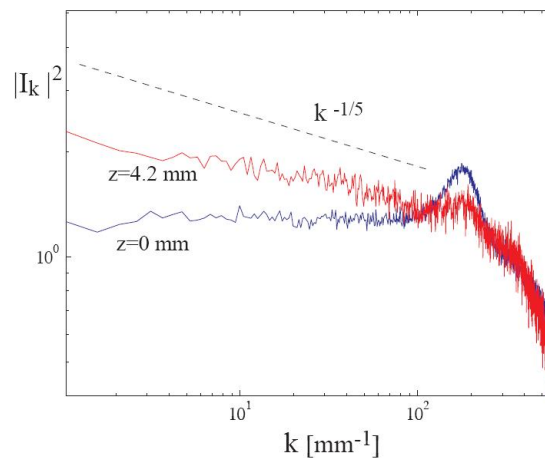


From Lvov et al 2007

# 1D Optical Turbulence: inverse cascade & condensation into solitons



Bortolozzo et al  
2008



# *WT extensions.*

- Condensates, finite size effects, strong-weak transition.
- Generalized WT beyond spectrum to include effect of coherent structures onto random waves.



# *Nonlinear wave equation*

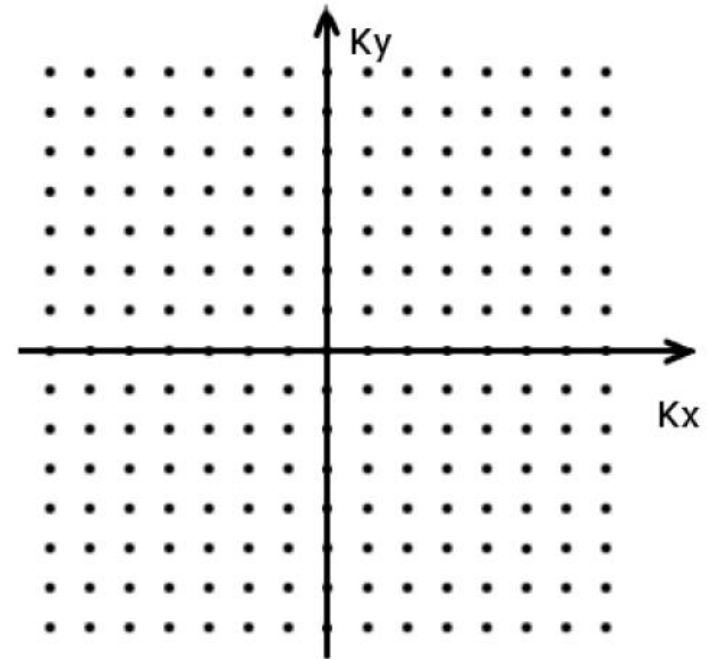
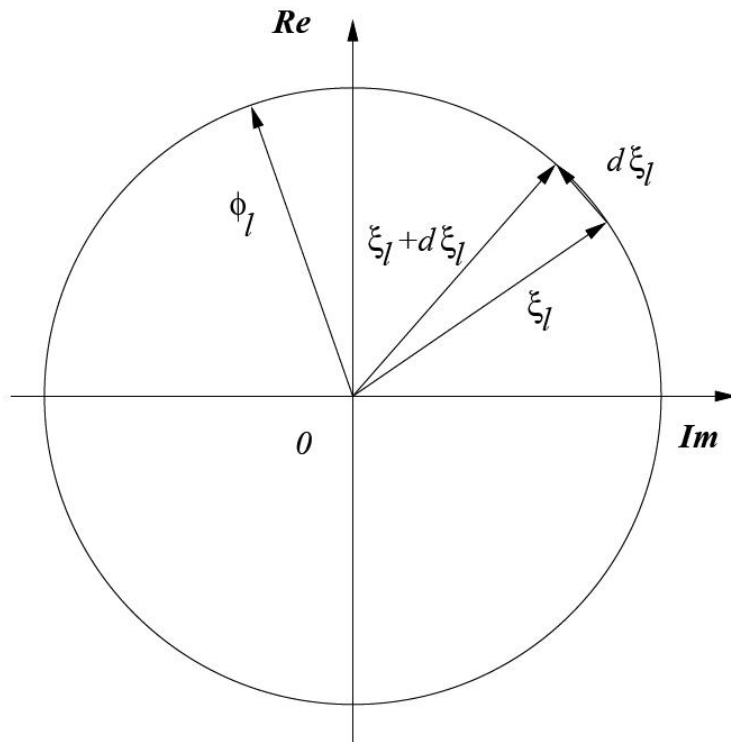
$$i\dot{\psi} + \Delta \psi \pm |\psi|^2 \psi = 0,$$

$$\psi(\mathbf{x}, t) \in \mathbb{C}, \mathbf{x} \in \mathbb{R}^d \text{ (} d = 2 \text{ or } 3 \text{)}$$

Let us place the system in a periodic box,  $\mathbb{T}^d$   
in Fourier space

$$i\dot{\hat{\psi}}(\mathbf{k}) - k^2 \hat{\psi}(\mathbf{k}) \pm \sum_{\mathbf{k}_1 + \mathbf{k} = \mathbf{k}_2 + \mathbf{k}_3} \hat{\psi}^*(\mathbf{k}_1) \hat{\psi}(\mathbf{k}_2) \hat{\psi}(\mathbf{k}_3) = 0.$$

# Set of wave modes



$$\hat{\psi}_l = \hat{\psi}(\mathbf{k}_l, t) = \sqrt{J_l} \phi_l,$$

$J_l \in \mathbb{R}^+$  is the intensity of the mode  $\mathbf{l}$ ,  
 $\phi_l \in \mathbb{S}^1$  is the phase factor, i.e.  $\phi_l = e^{i\varphi_l}$ ,  
 set of all  $J_l$  and  $\phi_l$  with  $\mathbf{l}$  such that  $\mathbf{k}_l \in B_N$ :  
 $\{J_l, \phi_l; \mathbf{k}_l \in B_N\}$  or for short:  $\{J, \phi\}$ .

# *Amplitudes and phases*

$$\hat{\psi}_l = \hat{\psi}(\mathbf{k}_l, t) = \sqrt{J_l} \phi_l ,$$

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# *N-mode joint PDF*

The probability of finding  $J_l$  inside  $(s_l, s_l + ds_l) \subset \mathbb{R}^+$  and finding  $\phi_l$  in the sector  $(\xi_l, \xi_l + d\xi_l) \subset \mathbb{S}^1$  is given by the joint PDF  $\mathcal{P}^{(N)}\{s, \xi\}$

$$\mathcal{P}^{(N)}\{s, \xi\} \prod_{\mathbf{k}_l \in B_N} ds_l |d\xi_l|.$$

For a function  $f = f\{J, \phi\}$ , its mean value

$$\langle f\{J, \phi\} \rangle = \left( \prod_{\mathbf{k}_l \in B_N} \int_0^\infty ds_l \oint_{\mathbb{S}^1} |d\xi_l| \right) \mathcal{P}^{(N)}\{s, \xi\} f\{s, \xi\}$$

# *RP and RPA systems*

**Definition.** A random phase field (RP): all  $\phi$  are i.r.v.'s such that uniformly distributed on  $\mathbb{S}^1$ :

$$\mathcal{P}^{(N)}\{s, \xi\} = \frac{1}{(2\pi)^N} \mathcal{P}^{(N,a)}\{s\}.$$

**Definition.** Random Phase and Amplitude fields (RPA)

1. All amplitudes and all phases are i.r.v.
2. All phases are uniformly distributed on  $\mathbb{S}^1$ :

$$\mathcal{P}^{(N)}\{s, \xi\} = \frac{1}{(2\pi)^N} \prod_{j \in B_N} \mathcal{P}_j^{(a)}(s_j)$$

*RPA does not mean Gaussianity*

# *Evolution of joint PDF*

Taking initial field to be weak and RP:

$$\dot{\mathcal{P}} = - \int \frac{\delta F_j}{\delta s_j} dk_j,$$

$F$  is a flux of probability in the  $s$ -space,

$$F_k = 4\pi \int |W_{12}^{k3}|^2 \delta_{12}^{k3} \delta(\omega_{12}^{k3}) s_1 s_2 s_3 \left[ \frac{\delta}{\delta s_k} + \frac{\delta}{\delta s_3} - \frac{\delta}{\delta s_1} - \frac{\delta}{\delta s_2} \right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$



# *From N-mode to 1-mode stats*

- N-mode equation can be used for validation of RPA.
- For obtaining a closure for the 1-mode PDF.

# *Evolution of 1-mode PDF*

$$\dot{\mathcal{P}}_k + \frac{\partial}{\partial s_k} \mathcal{F}_k = 0,$$

$\mathcal{F}_k$  is the probability flux:

$$\mathcal{F}_k = -s_k \left( \gamma_k \mathcal{P}_k + \eta_k \frac{\partial}{\partial s_k} \mathcal{P}_k \right),$$

$$\eta_k = 4\pi \int |W_{12}^{k3}|^2 \delta_{12}^{k3} \delta(\omega_{12}^{k3}) n_1 n_2 n_3 d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3,$$

$$\gamma_k = 4\pi \int |W_{12}^{k3}|^2 \delta_{12}^{k3} \delta(\omega_{12}^{k3}) \left[ n_3(n_1 + n_2) - n_1 n_2 \right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

# *Evolution of the spectrum*

$$n_k = \langle J_k \rangle = \int_0^\infty s_k \mathcal{P} ds_k,$$

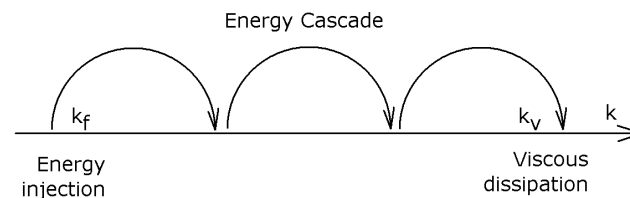
Kinetic equation (Hasselmann 1962):

$$\dot{n}_k = 4\pi \int |W_{12}^{k3}|^2 \delta_{12}^{k3} \delta(\omega_{12}^{k3}) n_1 n_2 n_3 n_k \left[ \frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2} \right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$

# Kolmogorov-Zakharov state

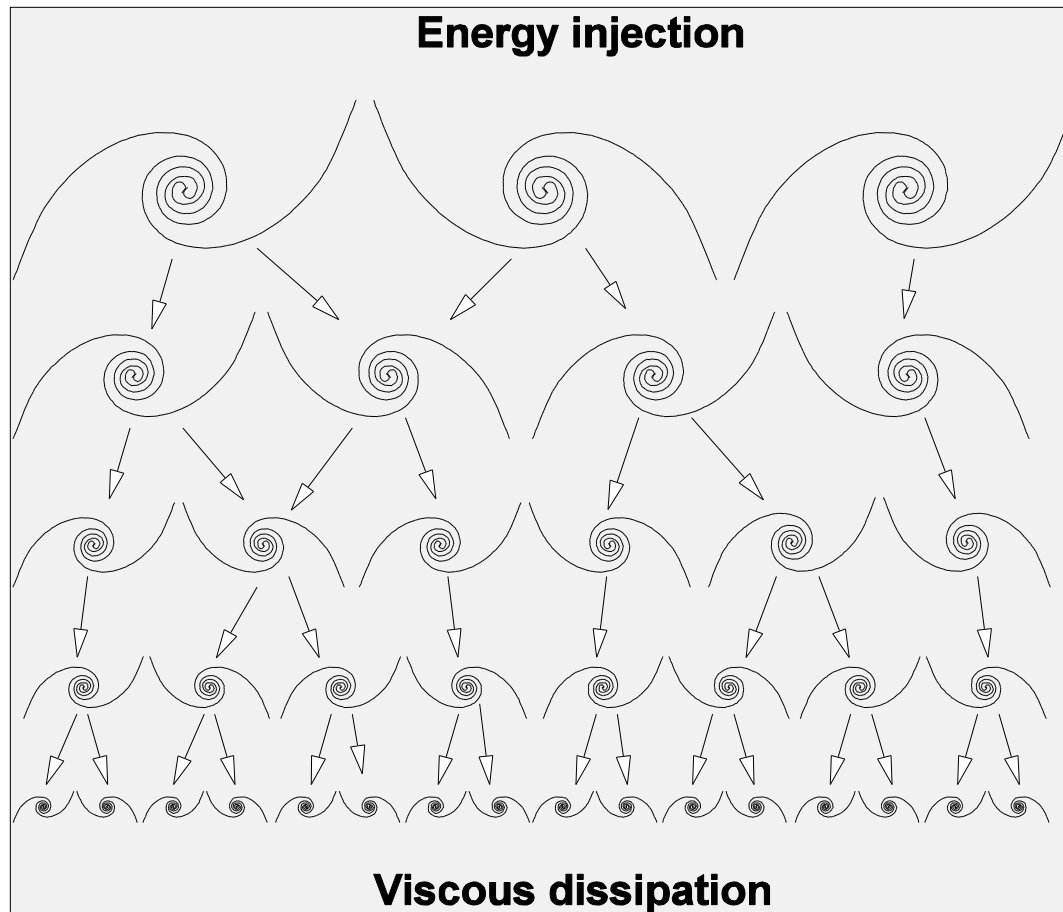
- Steady state spectrum corresponding to energy cascade (Zakharov 1965)

$$n_k = Ak^v$$



Like in classical turbulence – Richardson cascade,  
Kolmogorov spectrum. **This is why WT belongs to turbulence.**  
Turbulence is a non-equilibrium statistical system whose state is determined by a flux through phase space (and not by a thermodynamic potential like  $T$  etc).

# *Classical turbulence in fluids*



Richardson cascade.  
Kolmogorov spectrum

$$E(k) = C \varepsilon^{2/3} k^{-5/3}$$

# *Steady states for 1-mode PDF*

$$-s(\gamma P + \eta \partial_s P) = F = \text{const.}$$

in the steady state  $\gamma/\eta = n$

$$P = P_{hom} + P_{part}$$

where

$$P_{hom} = \text{const} \exp(-s/n) \quad \begin{array}{l} F=0: \\ \text{Gaussian} \end{array}$$

*Finite F*

At the tail of the PDF,  $s \gg n_k$ ,

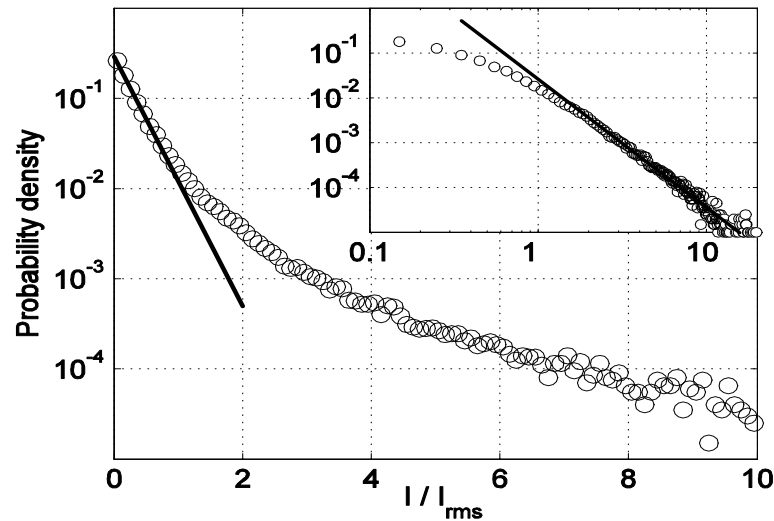
$$P_{part} = -\frac{F}{s\gamma} - \frac{\eta F}{(\gamma s)^2} + \dots$$



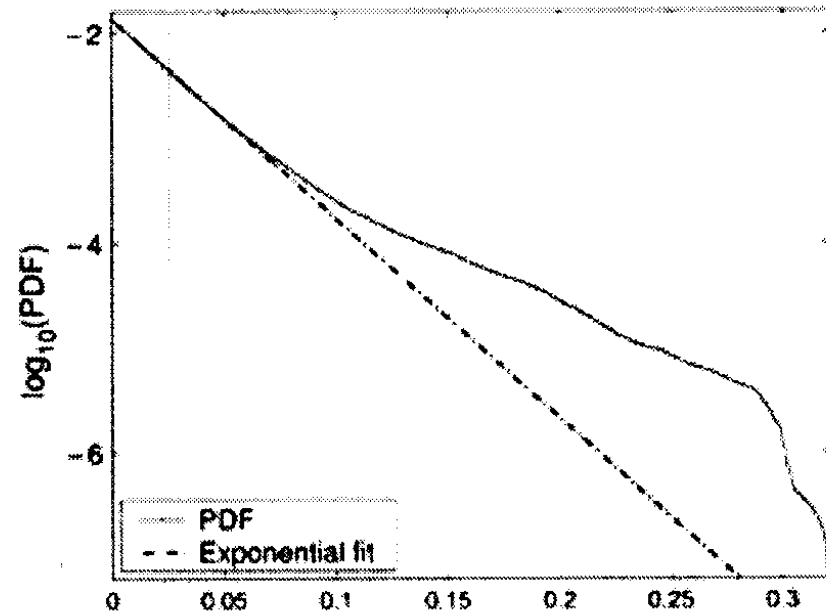
# *Fat tail*

- $P = -F/s$
- Needs cutoff: wave-breaking – no waves stronger than a critical value
- Flux  $F$  in  $s$  is negative – what does it mean?

# 1-mode PDF in water waves



Wavetank experiment  
Denissenko et al 2007



DNS, Lvov et al 2005

Wavebreaking cutoff at  **$J=J_{ph}$**

# Wave Turbulence life cycle

