Chapter 1
Dynamics of Open Quantum Systems

Exercise 1.1 (Transformation to the Interaction Picture).
Assuming a time-independent Hamiltonian $H = H_0 + V$, show that the Schrödinger equation in the interaction picture becomes

$$\left|\dot{\tilde{\Psi}}(t)\right\rangle = -\mathrm{i}\boldsymbol{V}(t)\left|\tilde{\Psi}(t)\right\rangle ,$$

where $\boldsymbol{V}(t) = e^{+\mathrm{i}H_0 t} V e^{-\mathrm{i}H_0 t}$ denotes the time-dependent Hamiltonian and $\left|\tilde{\Psi}(t)\right\rangle = e^{+\mathrm{i}H_0 t}\left|\Psi(t)\right\rangle$ the state vector in the interaction picture.

Solution 1.1. We first define the state vector in the interaction picture $\left|\tilde{\Psi}(t)\right\rangle = e^{+\mathrm{i}H_0 t}\left|\Psi(t)\right\rangle$. Taking the time derivative we obtain

$$\begin{aligned}\left|\dot{\tilde{\Psi}}(t)\right\rangle &= \mathrm{i}H_0\left|\tilde{\Psi}(t)\right\rangle - e^{+\mathrm{i}H_0 t}\mathrm{i}(H_0 + V)\left|\Psi(t)\right\rangle = -\mathrm{i}e^{+\mathrm{i}H_0 t}V\left|\Psi(t)\right\rangle \\ &= -\mathrm{i}e^{+\mathrm{i}H_0 t}Ve^{-\mathrm{i}H_0 t}\left|\tilde{\Psi}(t)\right\rangle ,\end{aligned}$$

which solves the exercise. Usually, the transformation into the interaction picture makes sense when H_0 is simple and/or V is small.

Exercise 1.2 (Temporal Dynamics of a two-level system).
Calculate the solution of Eq. (12). What is the stationary state? Show that detailed balance is satisfied.

Solution 1.2. Although the solution can be easily found with a computer algebra system by exponentiating the matrix, we exploit here the conservation of the trace and eliminate $P_1 = 1 - P_0$ leaving just one equation

$$\dot{P}_0 = -(T_{10} + T_{01})P_0 + T_{01} .$$

This equation is solved by the general solution of its homogeneous version and a special solution of the inhomogeneous version. The latter is found by computing the stationary state

$$\bar{P}_0 = \frac{T_{01}}{T_{10} + T_{01}} ,$$

such that the full solution becomes

$$P_0(t) = Ae^{-(T_{10}+T_{01})t} + \frac{T_{01}}{T_{10} + T_{01}} ,$$

where the constant A is fixed by the initial condition

$$P_0^0 = A + \frac{T_{01}}{T_{10} + T_{01}} .$$

Consequently, the solution becomes

$$P_0(t) = P_0^0 e^{-(T_{10}+T_{01})t} + \left[1 - e^{-(T_{10}+T_{01})t}\right] \frac{T_{01}}{T_{10}+T_{01}}\,,$$
$$P_1(t) = (1-P_0^0) e^{-(T_{10}+T_{01})t} + \left[1 - e^{-(T_{10}+T_{01})t}\right] \frac{T_{10}}{T_{10}+T_{01}}\,.$$

We directly see that local detailed balance is satisfied for the stationary state

$$T_{01}\bar{P}_1 = \frac{T_{01}T_{10}}{T_{10}+T_{01}} = T_{10}\bar{P}_0$$

Exercise 1.3 (Reaction-Diffusion Equation).
Along a linear chain of compartments consider the master equation for two species

$$\dot{P}_i = T\left[P_{i-1}(t) + P_{i+1}(t) - 2P_i(t)\right] - \gamma P_i(t)\,,$$
$$\dot{p}_i = \tau\left[p_{i-1}(t) + p_{i+1}(t) - 2p_i(t)\right] + \gamma P_i(t),$$

where $P_i(t)$ may denote the concentration of a molecule that irreversibly reacts with chemicals in the soluble to an inert form characterized by $p_i(t)$. To which partial differential equation does the master equation map?

Solution 1.3. We multiply the first terms in both equations with $\frac{\Delta x^2}{\Delta x^2}$, where Δx is the width of the compartments. Letting the hopping rates T and τ go to infinity and the compartment size go to zero, keeping however their products

$$D_T = \lim_{T\to\infty,\Delta x\to 0} T\Delta x^2\,, \qquad D_\tau = \lim_{\tau\to\infty,\Delta x\to 0} \tau\Delta x^2$$

finite, we obtain by using the discretization of the second derivative

$$f''(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{\Delta x^2}$$

a reaction-diffusion equation

$$\partial_t P = D_T \partial_x^2 P(x,t) - \gamma P(x,t)\,,$$
$$\partial_t p = T_\tau \partial_x^2 p(x,t) + \gamma p(x,t)\,.$$

Here, the diffusion terms are supplemented by a reaction which acts as a sink for species $P(x,t)$ that is converted into species $p(x,t)$.

Exercise 1.4 (Cell culture growth).
Confirm the validity of Eq. (20).

Solution 1.4. One possible solution for this system of equations is by brute force: The probability P_0 is obviously stationary, such that it may be left out from further considerations. The probability $P_1(t)$ follows a simple first order differential equation, which for the initial condition $P_1^0 = 1$ is readily solved by (provided $K > 1$) $P_1(t) = e^{-\alpha t}$.

Inserting this solution into the equation for P_2 yields the differential equation $\dot{P}_2 = -2\alpha P_2 + \alpha e^{-\alpha t}$, which can be solved by variation of constants $P_2(t) = C_2(t)e^{-2\alpha t}$ yielding $P_2(t) = (e^{\alpha t} - 1)e^{-2\alpha t}$.

The equation of $P_3(t)$ is also a simple differential equation with time-dependent coefficients $\dot{P}_3 = -3\alpha P_3 + 2\alpha(e^{\alpha t} - 1)e^{-2\alpha t}$, which is solved with the same method $P_3(t) = C_3(t)e^{-3\alpha t}$. The solution becomes $P_3(t) = (e^{\alpha t} - 1)^2 e^{-3\alpha t}$ and is thus similar to $P_2(t)$.

In fact, the other probabilities can be solved in a similar fashion and they become

$$P_{\ell<K}(t) = e^{-\ell\alpha t}(e^{\alpha t} - 1)^{\ell-1}\,.$$

Finally, the last equation becomes $\dot{P}_K = \alpha(K-1)e^{-(K-1)\alpha t}(e^{\alpha t} - 1)^{K-2}$, which is readily solved by

$$P_K(t) = e^{-(K-1)\alpha t}(e^{\alpha t} - 1)^{K-1}\,.$$

With these solutions for the probabilities, we can directly confirm the solution of the mean occupation. We note that the first probabilities $P_1, \ldots, P_{K-1}$ are independent of the carrying capacity K, which implies for the means of different carrying capacities the recursion formula

$$\begin{aligned}
\langle \ell \rangle_K &= \langle \ell \rangle_{K-1} - (K-1) e^{-(K-2)\alpha t} (e^{\alpha t}-1)^{K-2} + (K-1) e^{-(K-1)\alpha t} (e^{\alpha t}-1)^{K-2} \\
&\quad + K e^{-(K-1)\alpha t} (e^{\alpha t}-1)^{K-1} \\
&= \langle \ell \rangle_{K-1} + e^{-(K-1)\alpha t} (e^{\alpha t}-1)^{K-1} = \langle \ell \rangle_{K-1} + (1-e^{-\alpha t})^{K-1} \,.
\end{aligned}$$

Together with an initial value, e.g. $\langle \ell \rangle_2 = P_1 + 2P_2$ for $K = 2$, this confirms the general solution

$$\langle \ell \rangle = e^{+\alpha t} \left[1 - \left(1 - e^{-\alpha t} \right)^K \right] \,.$$

Alternatively, this can also be obtained via computing the matrix exponential of the rate matrix with a computer algebra system.

Exercise 1.5 (Logistic growth Equation).
Solve Eq. (21).

Solution 1.5. The logistic growth equation is best solved using separation of variables

$$\begin{aligned}
\int\limits_{N_0}^{N(t)} \frac{dN}{N\left(1-\frac{N}{K}\right)} &= \alpha t \\
&= \int\limits_{N_0}^{N(t)} \left[\frac{1}{N} - \frac{1}{n-K} \right] dN = \ln \frac{N(t)}{N_0} - \ln \frac{N(t)-K}{N_0-K} = \ln \frac{N(t)(N_0-K)}{N_0(N(t)-K)} \,.
\end{aligned}$$

This can be solved for $N(t)$

$$N(t) = \frac{N_0 e^{\alpha t}}{1 + \frac{N_0}{K}\left(e^{\alpha t}-1\right)} \,,$$

and we can easily confirm the initial condition $N(0) = N_0$ and the final state $\bar{N} = K$.

Exercise 1.6 (Superposition versus Localized States).
Calculate the density matrix for a statistical mixture in the states $|0\rangle$ and $|1\rangle$ with probability $p_0 = 3/4$ and $p_1 = 1/4$. What is the density matrix for a statistical mixture of the superposition states $|\Psi_a\rangle = \sqrt{3/4}\,|0\rangle + \sqrt{1/4}\,|1\rangle$ and $|\Psi_b\rangle = \sqrt{3/4}\,|0\rangle - \sqrt{1/4}\,|1\rangle$ with probabilities $p_a = p_b = 1/2$.

Solution 1.6. Both statistical mixtures map to the density matrix (in the basis $\{|0\rangle, |1\rangle\}$)

$$\rho = \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix} \,,$$

which demonstrates that these statistical mixtures cannot be distinguished. It is however, of course possible to distinguish between the pure states by suitable measurements.

Exercise 1.7 (Preservation of density matrix properties by unitary evolution).
Show that the von-Neumann equation (29) preserves self-adjointness, trace, and positivity of the density matrix.

Solution 1.7. The von-Neumann-equation has the formal solution

$$\rho(t) = U(t)\rho_0 U^\dagger(t) \,,$$

with the unitary operator $U^\dagger(t) = U^{-1}(t)$ defined in Eq. (5). Since the initial density matrix ρ_0 must be hermitian $\rho_0 = \rho_0^\dagger$, normalized $\mathrm{Tr}\{\rho_0\} = 1$, and positive definite $\langle\Psi|\,\rho\,|\Psi\rangle \geq 0$, all these properties are inherited by $\rho(t)$. For example, considering hermiticity and trace this is simple to show

$$\begin{aligned}
\rho^\dagger(t) = U(t)\rho_0^\dagger U^\dagger(t) = U(t)\rho_0 U^\dagger(t) = \rho(t) \,, \\
\mathrm{Tr}\{\rho(t)\} = \mathrm{Tr}\left\{U(t)\rho_0 U^\dagger(t)\right\} = \mathrm{Tr}\left\{U^\dagger(t) U(t)\rho_0\right\} = \mathrm{Tr}\{\rho_0\} = 1 \,,
\end{aligned}$$

and positivity is of course also preserved since unitary transformations do not change the eigenvalues of a matrix.

Exercise 1.8 (Preservation of density matrix properties by measurement).
Show that the measurement postulate preserves self-adjointness, trace, and positivity of the density matrix.

Solution 1.8. We consider the measurement outcome m, under which the density matrix becomes

$$\rho' = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\left\{M_m^\dagger M_m \rho\right\}} .$$

Most obviously, we have $\mathrm{Tr}\{\rho'\} = 1$, and also hermiticity is preserved

$$(\rho)'^\dagger = \frac{M_m \rho^\dagger M_m^\dagger}{\mathrm{Tr}\left\{\rho^\dagger M_m^\dagger M_m\right\}} = \frac{M_m \rho M_m^\dagger}{\mathrm{Tr}\left\{M_m^\dagger M_m \rho\right\}} = \rho' .$$

To show preservation of positivity, we use that just before the measurement, ρ must be a valid density matrix and thus have a spectral decomposition $\rho = \sum_\alpha P_\alpha |\alpha\rangle \langle\alpha|$ with $P_\alpha \in [0,1]$ and $\sum_\alpha P_\alpha = 1$. Furthermore, we use that the normalization factor is the probability $P_m = \mathrm{Tr}\left\{M_m^\dagger M_m \rho\right\}$ of obtaining measurement outcome m and thus by construction positive. As a side remark, we note that in general the basis within which the density matrix is diagonal and the measurement basis need not coincide. Inserting this, we can write the effect of the measurement as

$$\langle\Psi| \rho' |\Psi\rangle = \sum_\alpha P_\alpha \langle\Psi| M_m |\alpha\rangle \langle\alpha| M_m^\dagger |\Psi\rangle / P_m = \sum_\alpha P_\alpha / P_m |\langle\Psi| M_m |\alpha\rangle|^2 \geq 0 ,$$

which proves positivity of ρ'.

Exercise 1.9 (Tensor Products of Operators).
Let $\boldsymbol{\sigma}$ denote the Pauli matrices, i.e.,

$$\sigma^1 = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ +\mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

Compute the trace of the operator

$$\Sigma = a\mathbf{1} \otimes \mathbf{1} + \sum_{i=1}^{3} \alpha_i \sigma^i \otimes \mathbf{1} + \sum_{j=1}^{3} \beta_j \mathbf{1} \otimes \sigma^j + \sum_{i,j=1}^{3} a_{ij} \sigma^i \otimes \sigma^j .$$

Solution 1.9. We use that the Pauli matrices are traceless $\mathrm{Tr}\left\{\sigma^1\right\} = \mathrm{Tr}\left\{\sigma^2\right\} = \mathrm{Tr}\left\{\sigma^3\right\} = 0$ and that the trace of a simple tensor product can be represented by the product of traces within the subspaces only $\mathrm{Tr}\{A \otimes B\} = \mathrm{Tr}_A\{A\}\mathrm{Tr}_B\{B\}$. Furthermore, the trace of a 2×2 identity matrix is $\mathrm{Tr}\{\mathbf{1}\} = 2$. Then, the task becomes trivial

$$\mathrm{Tr}\{\Sigma\} = a \mathrm{Tr}\{\mathbf{1}\}^2 = 4a .$$

Exercise 1.10 (Partial Trace).
Compute the partial trace $\rho_A = \mathrm{Tr_B}\{\rho_{AB}\}$ of a pure density matrix $\rho_{AB} = |\Psi\rangle \langle\Psi|$ in the bipartite state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \equiv \frac{1}{\sqrt{2}} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) .$$

Show that ρ_A is no longer pure.

Solution 1.10. The full density matrix is given by

$$\begin{aligned} \rho_{AB} &= \frac{1}{2} [|01\rangle \langle 01| + |01\rangle \langle 10| + |10\rangle \langle 01| + |10\rangle \langle 10|] \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \end{aligned}$$

where in the matrix representation we have assumed the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The partial trace over subsystem B is now defined as

$$\begin{aligned}\rho_A &= \mathrm{Tr_B}\{\rho_{AB}\} = \langle 0|_B\, \rho_{AB}\, |0\rangle_B + \langle 1|_B\, \rho_{AB}\, |1\rangle_B \\ &= \frac{1}{2}\left[|0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A\right] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

where we have used the basis $\{|0\rangle_A, |1\rangle_A\}$. This – since $\rho_A^2 \neq \rho_A$ – is obviously a mixed (not pure) state.

Exercise 1.11 (Trace and Hermiticity preservation by Lindblad forms).
Show that the Lindblad form master equation preserves trace and hermiticity of the density matrix.

Solution 1.11. We want to show that the evolution equation

$$\dot{\rho} = -\mathrm{i}\,[H,\rho] + \sum_{\alpha,\beta=1}^{N^2-1} \gamma_{\alpha\beta}\left(A_\alpha \rho A_\beta^\dagger - \frac{1}{2}\left\{A_\beta^\dagger A_\alpha, \rho\right\}\right),$$

preserves initial hermiticity and trace of ρ_0. Hermiticity is obviously preserved if

$$\begin{aligned}\dot{\rho}^\dagger &= +\mathrm{i}(\rho H - H\rho) + \sum_{\alpha,\beta=1}^{N^2-1} \gamma_{\alpha\beta}^*\left(A_\beta \rho A_\alpha^\dagger - \frac{1}{2}\left\{A_\alpha^\dagger A_\beta, \rho\right\}\right) \\ &= -\mathrm{i}\,[H,\rho] + \sum_{\alpha,\beta=1}^{N^2-1} \gamma_{\beta\alpha}\left(A_\beta \rho A_\alpha^\dagger - \frac{1}{2}\left\{A_\alpha^\dagger A_\beta, \rho\right\}\right) \\ &= -\mathrm{i}\,[H,\rho] + \sum_{\alpha,\beta=1}^{N^2-1} \gamma_{\alpha\beta}\left(A_\alpha \rho A_\beta^\dagger - \frac{1}{2}\left\{A_\beta^\dagger A_\alpha, \rho\right\}\right) = \dot{\rho}\end{aligned}$$

holds. Here, we have used hermiticity of $\rho = \rho^\dagger$ and $H = H^\dagger$ and $\gamma_{\beta\alpha}^* = \gamma_{\alpha\beta}$. In the last equation, we have simply renamed $\alpha \leftrightarrow \beta$. It is also straightforward to show that the trace is conserved

$$\mathrm{Tr}\left\{\dot{\rho}\right\} = -\mathrm{i}\mathrm{Tr}\left\{H\rho - \rho H\right\} + \sum_{\alpha,\beta=1}^{N^2-1} \gamma_{\alpha\beta}\mathrm{Tr}\left\{A_\alpha \rho A_\beta^\dagger - \frac{1}{2}\left\{A_\beta^\dagger A_\alpha, \rho\right\}\right\} = 0\,,$$

where one only has to use $\mathrm{Tr}\left\{AB\right\} = \mathrm{Tr}\left\{BA\right\}$.

Exercise 1.12 (Shift invariance).
Show the invariance of the diagonal representation of a Lindblad form master equation (46) with respect to the transformation (47).

Solution 1.12. To show that the simplest form of a Lindblad equation

$$\dot{\rho} = -\mathrm{i}\,[H,\rho] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\left\{L_\alpha^\dagger L_\alpha, \rho\right\}\right).$$

is invariant with respect to the transformation

$$\begin{aligned}L_\alpha &\to L'_\alpha = L_\alpha + a_\alpha \mathbf{1} \\ H &\to H' = H + \frac{1}{2\mathrm{i}}\sum_\alpha \left(a_\alpha^* L_\alpha - a_\alpha L_\alpha^\dagger\right) + b\mathbf{1}\,,\end{aligned}$$

we simply insert these

$$
\begin{aligned}
\dot{\rho} &= -\mathrm{i}\left[H + \frac{1}{2\mathrm{i}}\sum_\alpha \left(a_\alpha^* L_\alpha - a_\alpha L_\alpha^\dagger\right) + b, \rho\right] + \sum_\alpha \left(\left[L_\alpha + a_\alpha\right] \rho \left[L_\alpha^\dagger + a_\alpha^*\right] - \frac{1}{2}\left\{\left[L_\alpha^\dagger + a_\alpha^*\right]\left[L_\alpha + a_\alpha\right], \rho\right\}\right) \\
&= -\mathrm{i}\left[H, \rho\right] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\left\{L_\alpha^\dagger L_\alpha, \rho\right\}\right) \\
&\quad -\frac{1}{2}\sum_\alpha \left(a_\alpha^* L_\alpha - a_\alpha L_\alpha^\dagger\right)\rho + \frac{1}{2}\rho\sum_\alpha \left(a_\alpha^* L_\alpha - a_\alpha L_\alpha^\dagger\right) + \sum_\alpha \left[\frac{a_\alpha}{2}\rho L_\alpha^\dagger + \frac{a_\alpha^*}{2} L_\alpha \rho - \frac{a_\alpha}{2} L_\alpha^\dagger \rho - \frac{a_\alpha^*}{2}\rho L_\alpha\right] \\
&= -\mathrm{i}\left[H, \rho\right] + \sum_\alpha \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\left\{L_\alpha^\dagger L_\alpha, \rho\right\}\right).
\end{aligned}
$$

This invariance enables one to gauge the energy of the Hamiltonian b and to choose e.g. traceless Lindblad operators L_α. for all systems. Such transformations will not affect the solution to the Lindblad equation.

Exercise 1.13 (Hermiticity of effective Hamiltonian).
Show that the effective Hamiltonian $\boldsymbol{H}_{\mathbf{eff}}(t) = \mathrm{i}\dot{U}(t)U^\dagger(t)$ *is hermitian.*

Solution 1.13. To show this we only need the unitarity of U. Taking the time derivative of $U(t)U^\dagger(t) = \mathbf{1}$ we obtain the relation

$$
\dot{U}(t)U^\dagger(t) + U(t)\dot{U}^\dagger(t) = \mathbf{0}\,.
$$

This can be exploited to obtain for the effective Hamiltonian

$$
\boldsymbol{H}_{\mathbf{eff}}{}^\dagger(t) = -\mathrm{i}U(t)\dot{U}^\dagger(t) = +\mathrm{i}\dot{U}(t)U^\dagger(t) = \boldsymbol{H}_{\mathbf{eff}}(t)\,.
$$

Its hermiticity and occurrence in the commutator with the density matrix motivates the terminology effective Hamiltonian.

Exercise 1.14 (Moments).
Calculate the expectation value of the number operator $n = a^\dagger a$ *and its square* $n^2 = a^\dagger a a^\dagger a$ *in the stationary state of the master equation (56).*

Solution 1.14. We use that the stationary solution of Eq. (56) is just the thermal state $\bar{\rho} = e^{-\beta H}\mathrm{Tr}\left\{e^{-\beta H}\right\}$ with $H = \Omega a^\dagger a$. First, we calculate the proper normalization of the stationary density matrix in the Fock state basis of the harmonic oscillator eigenstates

$$
Z = \mathrm{Tr}\left\{e^{-\beta\Omega a^\dagger a}\right\} = \sum_{n=0}^\infty e^{-\beta\Omega n} = \frac{1}{1 - e^{-\beta\Omega}}\,,
$$

where we have used the geometric series. To calculate the expectation value of n, we have to evaluate

$$
\langle n\rangle = \sum_{n=0}^\infty n\frac{e^{-\beta\Omega n}}{Z} = -\frac{1}{Z}\sum_{n=0}^\infty \frac{\partial}{\partial(\beta\Omega)} e^{-\beta\Omega n} = -\left(1 - e^{-\beta\Omega}\right)\frac{\partial}{\partial(\beta\Omega)}\frac{1}{1 - e^{-\beta\Omega}} = \frac{1}{e^{\beta\Omega} - 1}\,.
$$

This is nothing but the Bose distribution at frequency Ω with the reservoir temperature β, i.e., the system occupation equilibrates with the reservoir. Similarly, we evaluate the expectation value of n^2

$$
\left\langle n^2\right\rangle = \sum_{n=0}^\infty n^2\frac{e^{-\beta\Omega n}}{Z} = \frac{1}{Z}\sum_{n=0}^\infty \frac{\partial^2}{\partial(\beta\Omega)^2} e^{-\beta\Omega n} = \left(1 - e^{-\beta\Omega}\right)\frac{\partial^2}{\partial(\beta\Omega)^2}\frac{1}{1 - e^{-\beta\Omega}} = \frac{e^{\beta\Omega} + 1}{\left(e^{\beta\Omega} - 1\right)^2}\,.
$$

Exercise 1.15 (Coherent state).
Using the driven cavity master equation, show that the stationary expectation value of the cavity occupation fulfils

$$
\lim_{t\to\infty}\left\langle a^\dagger a\right\rangle = \frac{|P|^2}{\gamma^2 + 4(\Omega - \omega)^2}
$$

Solution 1.15. We first map the driven cavity master equation (in the comoving frame) to a set of equations for operator expectation values by using $\left\langle \dot{A} \right\rangle = \mathrm{Tr}\{A\dot{\rho}\} = \mathrm{Tr}\{A\mathcal{L}\rho\}$. Starting with $\left\langle a^\dagger a \right\rangle$ we see that it couples to two further operators a and $a^\dagger$, for which we also seek to find the dynamics. Altogether, the dynamics is closes with just three operators

$$\begin{aligned}
\frac{d}{dt}\left\langle a^\dagger a \right\rangle &= +\mathrm{i}\frac{P}{2}\left\langle a \right\rangle - \mathrm{i}\frac{P^*}{2}\left\langle a^\dagger \right\rangle - \gamma\left\langle a^\dagger a \right\rangle ,\\
\frac{d}{dt}\left\langle a \right\rangle &= \left[\frac{\gamma}{2} - \mathrm{i}(\Omega-\omega)\right]\left\langle a \right\rangle - \mathrm{i}\frac{P^*}{2} ,\\
\frac{d}{dt}\left\langle a^\dagger \right\rangle &= \left[\frac{\gamma}{2} + \mathrm{i}(\Omega-\omega)\right]\left\langle a^\dagger \right\rangle + \frac{P}{2} .
\end{aligned}$$

The stationary state of this equation is given by

$$\lim_{t\to\infty}\left\langle a^\dagger a \right\rangle = \frac{|P|^2}{\gamma^2 + 4(\Omega-\omega)^2} ,$$

which is independent on the frame.

Exercise 1.16 (Preservation of Positivity).
Show that the super-operator in Eq. (70) preserves positivity of the density matrix provided that initial positivity ($-1/4 \le \left|\rho_{ge}^0\right|^2 - \rho_{gg}^0\rho_{ee}^0 \le 0$) is given.

Solution 1.16. Due to its simple structure, we may readily obtain the solution of Eq. (70)

$$\begin{aligned}
\rho_{gg} &= \rho_{gg}^0 + \rho_{ee}^0\left(1 - e^{-\gamma t}\right) , \qquad \rho_{ee} = e^{-\gamma t}\rho_{ee}^0\\
\rho_{ge} &= e^{(-\gamma/2+2\mathrm{i}\Omega)t}\rho_{ge}^0 , \qquad \rho_{eg} = e^{(-\gamma/2-2\mathrm{i}\Omega)t}\rho_{eg}^0 ,
\end{aligned}$$

where we have to keep in mind that the initial density matrix must be hermitian ($\rho_{eg}^0 = (\rho_{ge}^0)^*$ and $\rho_{gg}, \rho_{ee} \in \mathbb{R}$), normalized ($\rho_{ee}^0 + \rho_{gg}^0 = 1$) and positive. The last condition implies that both eigenvalues

$$\lambda_\pm = \frac{1}{2}\left(\rho_{ee} + \rho_{gg} \pm \sqrt{4\rho_{eg}\rho_{ge} + (\rho_{ee} - \rho_{gg})^2}\right)$$

of the initial density matrix must be positive. In particular, this requires that the argument of the root

$$A = 4\left|\rho_{eg}\right|^2 + (\rho_{ee} - \rho_{gg})^2 = 1 + 4\left|\rho_{eg}\right|^2 - 4\rho_{ee}\rho_{gg}$$

must be positive and upper-bounded by one. In particular, this implies the hint given in the exercise. To preserve positivity at all times, it is required that the above eigenvalues must be positive throughout. Inserting the solutions we obtain

$$\begin{aligned}
A(t) &= 4\left|\rho_{eg}\right|^2 + (\rho_{ee} - \rho_{gg})^2 = e^{-\gamma t}4\left|\rho_{eg}^0\right|^2 + \left[\rho_{ee}^0 - \rho_{gg}^0 - 2\rho_{ee}^0\left(1 - e^{-\gamma t}\right)\right]^2\\
&= \left(e^{-\gamma t} - 1\right)4\left|\rho_{eg}^0\right|^2 + 4\left|\rho_{eg}^0\right|^2 + \left(\rho_{ee}^0 - \rho_{gg}^0\right)^2 - 4\left(\rho_{ee}^0 - \rho_{gg}^0\right)\rho_{ee}^0\left(1 - e^{-\gamma t}\right) + 4(\rho_{ee}^0)^2\left(1 - e^{-\gamma t}\right)^2\\
&= A(0) - 4\left|\rho_{eg}^0\right|^2\left(1 - e^{-\gamma t}\right) + 4\left(1 - e^{-\gamma t}\right)\left(\rho_{ee}^0\rho_{gg}^0 - e^{-\gamma t}(\rho_{ee}^0)^2\right)\\
&\le A(0) - 4\left|\rho_{eg}^0\right|^2\left(1 - e^{-\gamma t}\right) + 4\left(1 - e^{-\gamma t}\right)\rho_{ee}^0\rho_{gg}^0 = A(0) + 4\left(1 - e^{-\gamma t}\right)\left(-\left|\rho_{eg}^0\right|^2 + \rho_{ee}^0\rho_{gg}^0\right)\\
&\le A(0) \le 1 .
\end{aligned}$$

The first line obviously shows that $A(t)$ is positive, and its upper bound follows from the upper bound of the initial condition. Therefore, despite the seemingly unrelated evolution of populations and coherences the parameters entering these equations ensure that the solution density matrix is positive.

Exercise 1.17 (Expectation values from superoperators).
Show that for a Liouvillian super-operator connecting N populations (diagonal entries) with M coherences (off-diagonal entries) acting on the density matrix $\rho(t) = (P_1, \ldots, P_N, C_1, \ldots, C_M)^T$, the trace in the expectation value of an operator can be mapped to the matrix element

$$\langle A(t)\rangle = (\underbrace{1,\ldots,1}_{N\times},\underbrace{0,\ldots,0}_{M\times})\cdot\mathcal{A}\cdot\rho(t)\,,$$

where the matrix $\mathcal{A}$ is the super-operator corresponding to multiplication with A from the left.

Solution 1.17. The expectation value of an operator is simply given by

$$\langle A\rangle = \mathrm{Tr}\left\{\rho(t)A\right\} = \mathrm{Tr}\left\{A\rho(t)\right\}\,.$$

Since the trace is just the sum over the diagonal entries of a matrix, in superoperator notation this corresponds to a sum of the entries corresponding to populations. When $\mathcal{A}\rho\hat{=}A\rho$, these are obtained by multiplying with the described row vector.

Similarly, one could define the superoperator corresponding to multiplication from the right via $\bar{\mathcal{A}}\rho\hat{=}\rho A$, and – despite in general $\bar{\mathcal{A}}\neq\mathcal{A}$ the trace formula would be the same

$$\langle A(t)\rangle = (\underbrace{1,\ldots,1}_{N\times},\underbrace{0,\ldots,0}_{M\times})\cdot\mathcal{A}\cdot\rho(t) = (\underbrace{1,\ldots,1}_{N\times},\underbrace{0,\ldots,0}_{M\times})\cdot\bar{\mathcal{A}}\cdot\rho(t)\,.$$

Chapter 2
Microscopic Derivation

Exercise 2.1 (Jordan-Wigner transform).
Show that for fermions distributed on N sites, the decomposition

$$c_i = \underbrace{\sigma^z \otimes \ldots \otimes \sigma^z}_{i-1} \otimes \sigma^- \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{N-i}$$

preserves the fermionic anti-commutation relations

$$\{c_i, c_j\} = \mathbf{0} = \left\{c_i^\dagger, c_j^\dagger\right\}, \qquad \left\{c_i, c_j^\dagger\right\} = \delta_{ij}\mathbf{1}\,.$$

Show also that the fermionic Fock space basis $c_i^\dagger c_i \left|n_1, \ldots, n_N\right\rangle = n_i \left|n_1, \ldots, n_N\right\rangle$ obeys $\sigma_i^z \left|n_1, \ldots, n_N\right\rangle = (-1)^{n_i+1} \left|n_1, \ldots, n_N\right\rangle$.

Solution 2.1. Without loss of generality (we have $\{c_i, c_j\} = \{c_j, c_i\}$ and $c_i^2 = 0$ is trivially fulfilled due to $(\sigma^-)^2 = \mathbf{0}$) we consider the case $i < j$

$$\begin{aligned}
\{c_i, c_j\} &= \underbrace{(\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2}_{(i-1)\times} \otimes \underbrace{\sigma^-\sigma^z}_{i} \otimes \underbrace{\sigma^z \otimes \ldots \otimes \sigma^z}_{(i+1)\ldots(j-1)} \otimes \underbrace{\sigma^-}_{j} \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(N-j)\times} \\
&\quad + (\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2 \otimes \sigma^z\sigma^- \otimes \sigma^z \otimes \ldots \otimes \sigma^z \otimes \sigma^- \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
&= \mathbf{0}\,,
\end{aligned}$$

where we have used $(\sigma^z)^2 = \mathbf{1}$ and $\sigma^-\sigma^z = -\sigma^z\sigma^-$. From the adjoint equation we automatically have $\{c_i^\dagger, c_j^\dagger\} = \mathbf{0}$. To show the last commutation relation we consider first $i < j$ ($i > j$ need not be considered due to the symmetry)

$$\begin{aligned}
\left\{c_i, c_j^\dagger\right\} &= \underbrace{(\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2}_{(i-1)\times} \otimes \underbrace{\sigma^-\sigma^z}_{i} \otimes \underbrace{\sigma^z \otimes \ldots \otimes \sigma^z}_{(i+1)\ldots(j-1)} \otimes \underbrace{\sigma^+}_{j} \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(N-j)\times} \\
&\quad + (\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2 \otimes \sigma^z\sigma^- \otimes \sigma^z \otimes \ldots \otimes \sigma^z \otimes \sigma^+ \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} \\
&= \mathbf{0}
\end{aligned}$$

due to the same reasons as before. For $i = j$ we have

$$\begin{aligned}
\left\{c_i, c_i^\dagger\right\} &= \underbrace{(\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2}_{(i-1)\times} \otimes \underbrace{\sigma^-\sigma^+}_{i} \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(N-i)\times} \\
&\quad + \underbrace{(\sigma^z)^2 \otimes \ldots \otimes (\sigma^z)^2}_{(i-1)\times} \otimes \underbrace{\sigma^+\sigma^-}_{i} \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(N-i)\times} \\
&= \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(i-1)\times} \otimes \left[\frac{1}{2}\left(\mathbf{1} - \sigma^z\right) + \frac{1}{2}\left(\mathbf{1} + \sigma^z\right)\right] \otimes \underbrace{\mathbf{1} \otimes \ldots \otimes \mathbf{1}}_{(N-i)\times} \\
&= \mathbf{1}\,,
\end{aligned}$$

which proves in combination that $\{c_i, c_j^\dagger\} = \mathbf{1}\delta_{ij}$. The second part can be answered via the mapping $c_i^\dagger c_i \left|n_1, \ldots, n_N\right\rangle = \frac{1}{2}\left[\mathbf{1} + \sigma_i^z\right]\left|n_1, \ldots, n_N\right\rangle = n_i \left|n_1, \ldots, n_N\right\rangle$, which can be solved for

$$\sigma_i^z \left|n_1, \ldots, n_N\right\rangle = (2n_i - 1)\left|n_1, \ldots, n_N\right\rangle = (-1)^{n_i+1}\left|n_1, \ldots, n_N\right\rangle ,$$

since n_i can only take values 0 and 1.

Exercise 2.2 (Transforming the coupling operators).
Given an interaction Hamiltonian $\mathcal{H}_\mathrm{I} = \sum_\alpha A_\alpha \otimes B_\alpha$ where $\langle B_\alpha\rangle \neq 0$, show that there exists a simple transformation $B_\alpha \to B'_\alpha$ and $\mathcal{H}_\mathrm{S} \to \mathcal{H}'_\mathrm{S}$ which obeys $\langle B'_\alpha\rangle = 0$. Find B'_α and $\mathcal{H}'_\mathrm{S}$.

Solution 2.2. We note that the total Hamiltonian is invariant when we simultaneously transform

$$B_\alpha \to B'_\alpha = B_\alpha + g_\alpha \mathbf{1} , \qquad \mathcal{H}_\mathrm{S} \to \mathcal{H}'_\mathrm{S} = \mathcal{H}_\mathrm{S} - \sum_\alpha g_\alpha A_\alpha ,$$

such that $\mathcal{H}_\mathrm{S} + \sum_\alpha A_\alpha \otimes B_\alpha = \mathcal{H}'_\mathrm{S} + \sum_\alpha A_\alpha \otimes B'_\alpha$ holds. To obtain vanishing lineary expectation values of the bath coupling operators $\langle B'_\alpha\rangle = 0$, we therefore set $g_\alpha = -\langle B_\alpha\rangle$, such that the new reservoir coupling operators and system Hamiltonian read

$$B'_\alpha = B_\alpha - \langle B_\alpha\rangle \mathbf{1} , \qquad \mathcal{H}'_\mathrm{S} = \mathcal{H}_\mathrm{S} + \sum_\alpha \langle B_\alpha\rangle A_\alpha .$$

Hermiticity of the interaction Hamiltonian implies that also $\mathcal{H}'_\mathrm{S}$ will be hermitian. This transformation has the interesting consequence that the interaction may change the pointer-basis from the eigenbasis of $\mathcal{H}_\mathrm{S}$ to the eigenbasis of $\mathcal{H}'_\mathrm{S}$.

Exercise 2.3 (Properties of Correlation functions).
Show that when $[\mathcal{H}_\mathrm{B}, \bar\rho_\mathrm{B}] = 0$ (which is e.g. the case in thermal equilibrium), the correlation functions in Eq. (86) only depend on the difference of their time arguments

$$C_{\alpha\beta}(t_1, t_2) = C_{\alpha\beta}(t_1 - t_2, 0) .$$

Solution 2.3. We use the definition of the correlation function and the fact that the thermal equilibrium state commutes with the reservoir Hamiltonian

$$\begin{aligned} C_{\alpha\beta}(t_1, t_2) &= \mathrm{Tr_B}\left\{e^{+\mathrm{i}\mathcal{H}_\mathrm{B}t_1} B_\alpha e^{-\mathrm{i}\mathcal{H}_\mathrm{B}t_1} e^{+\mathrm{i}\mathcal{H}_\mathrm{B}t_2} B_\beta e^{-\mathrm{i}\mathcal{H}_\mathrm{B}t_2}\bar\rho_\mathrm{B}\right\} = \mathrm{Tr_B}\left\{e^{+\mathrm{i}\mathcal{H}_\mathrm{B}t_1} B_\alpha e^{-\mathrm{i}\mathcal{H}_\mathrm{B}(t_1-t_2)} B_\beta \bar\rho_\mathrm{B} e^{-\mathrm{i}\mathcal{H}_\mathrm{B}t_2}\right\} \\ &= \mathrm{Tr_B}\left\{e^{+\mathrm{i}\mathcal{H}_\mathrm{B}(t_1-t_2)} B_\alpha e^{-\mathrm{i}\mathcal{H}_\mathrm{B}(t_1-t_2)} B_\beta \bar\rho_\mathrm{B}\right\} = C_{\alpha\beta}(t_1 - t_2, 0) \equiv C_{\alpha\beta}(t_1 - t_2) . \end{aligned}$$

In practice, this dependence on a single variable enables one to compute Fourier transforms of the reservoir correlation functions easily.

Exercise 2.4 (KMS condition).
Show the validity of the KMS condition for a thermal bath with $\bar\rho_\mathrm{B} = \frac{e^{-\beta\mathcal{H}_\mathrm{B}}}{\mathrm{Tr}\left\{e^{-\beta\mathcal{H}_\mathrm{B}}\right\}}$.

Solution 2.4. For a thermal state the bath correlation functions (single time argument) read

$$C_{\alpha\beta}(\tau) = \mathrm{Tr}\left\{e^{+\mathrm{i}\mathcal{H}_\mathrm{B}\tau} B_\alpha e^{-\mathrm{i}\mathcal{H}_\mathrm{B}\tau} B_\beta \frac{e^{-\beta\mathcal{H}_\mathrm{B}}}{Z_B}\right\} ,$$

where $Z_B = \mathrm{Tr}\left\{e^{-\beta\mathcal{H}_\mathrm{B}}\right\}$ simply provides a normalization and where we have also assumed that B_α and B_β are coupling operators belonging to the same bath. We find

$$\begin{aligned} C_{\beta\alpha}(-\tau - \mathrm{i}\beta) &= \mathrm{Tr}\left\{e^{-\mathrm{i}\mathcal{H}_\mathrm{B}(\tau+\mathrm{i}\beta)} B_\beta e^{\mathrm{i}\mathcal{H}_\mathrm{B}(\tau+\mathrm{i}\beta)} B_\alpha \frac{e^{-\beta\mathcal{H}_\mathrm{B}}}{Z_B}\right\} \\ &= \mathrm{Tr}\left\{e^{\mathrm{i}\mathcal{H}_\mathrm{B}(\tau+\mathrm{i}\beta)} B_\alpha \frac{e^{-\beta\mathcal{H}_\mathrm{B}}}{Z_B} e^{-\mathrm{i}\mathcal{H}_\mathrm{B}(\tau+\mathrm{i}\beta)} B_\beta\right\} \\ &= \mathrm{Tr}\left\{e^{\mathrm{i}\mathcal{H}_\mathrm{B}\tau} B_\alpha e^{-\mathrm{i}\mathcal{H}_\mathrm{B}\tau} B_\beta \frac{e^{\mathrm{i}^2\beta\mathcal{H}_\mathrm{B}}}{Z_B}\right\} = C_{\alpha\beta}(\tau) . \end{aligned}$$

This also further transforms to a corresponding symmetry of the Fourier transforms of the reservoir correlation functions.

Chapter 3
Exactly solvable Models

Exercise 3.1 (Interaction Picture).
Show that Eq. (139) arises in the interaction picture.

Solution 3.1. In the interaction picture, all operators transform according to $\boldsymbol{O}(t) = e^{+\mathrm{i}(\mathcal{H}_\mathrm{S}+\mathcal{H}_\mathrm{B})t} O e^{-\mathrm{i}(\mathcal{H}_\mathrm{S}+\mathcal{H}_\mathrm{B})t}$. Since $[\mathcal{H}_\mathrm{S}, \mathcal{H}_\mathrm{B}] = 0$ by construction, one can separate the exponentials e.g. $e^{\mathrm{i}(\mathcal{H}_\mathrm{S}+\mathcal{H}_\mathrm{B})t} = e^{\mathrm{i}\mathcal{H}_\mathrm{S} t} e^{\mathrm{i}\mathcal{H}_\mathrm{B} t}$, such that system and bath operators can be separately transformed. Furthermore, since $[\mathcal{H}_\mathrm{S}, \sigma^z] = 0$, the system coupling operator remains invariant

$$\boldsymbol{\sigma^z}(t) = e^{+\mathrm{i}\mathcal{H}_\mathrm{S} t} \sigma^z e^{-\mathrm{i}\mathcal{H}_\mathrm{S} t} = \sigma^z \,.$$

This is different for the bosonic operators. We first compute their time derivative

$$\frac{d}{dt}\boldsymbol{b_k}(t) = e^{+\mathrm{i}\mathcal{H}_\mathrm{B} t} \left(+\mathrm{i}\mathcal{H}_\mathrm{B} b_k - \mathrm{i} b_k \mathcal{H}_\mathrm{B}\right) e^{-\mathrm{i}\mathcal{H}_\mathrm{B} t} = \mathrm{i} e^{+\mathrm{i}\mathcal{H}_\mathrm{B} t} [\mathcal{H}_\mathrm{B}, b_k] e^{-\mathrm{i}\mathcal{H}_\mathrm{B} t} = -\mathrm{i}\omega_k e^{+\mathrm{i}\mathcal{H}_\mathrm{B} t} b_k e^{-\mathrm{i}HBt} = -\mathrm{i}\omega_k \boldsymbol{b_k}(t) \,.$$

This differential operator equation is readily solvable, and by hermitian conjugation we can directly conclude the interaction picture evolution of both operators

$$\boldsymbol{b_k}(t) = e^{-\mathrm{i}\omega_k t} b_k \,, \qquad \boldsymbol{b_k^\dagger}(t) = e^{+\mathrm{i}\omega_k t} b_k^\dagger \,.$$

Inserting these solutions into the interaction Hamiltonian yields Eq. (139).

Exercise 3.2 (BCH formula).
Show the generalization from Eq. (141) to Eq. (142).

Solution 3.2. We sucessively use Eq. (141) to separate one operator at a time and furthermore exploit that the exponential of the commutator commutes with all other operators

$$\begin{aligned}
e^{A_1+\ldots+A_n} &= e^{A_1+\ldots+A_{n-1}} e^{A_n} e^{-[A_1+\ldots+A_{n-1},A_n]/2} \\
&= e^{A_1+\ldots+A_{n-2}} e^{A_{n-1}} e^{A_n} e^{-[A_1+\ldots+A_{n-2},A_{n-1}]/2} e^{-[A_1+\ldots+A_{n-1},A_n]/2} \\
&\vdots \\
&= e^{A_1} \times \ldots \times e^{A_n} e^{-[A_1,A_2]/2} e^{-[A_1+A_2,A_3]/2} \times \ldots \times e^{-[A_1+\ldots+A_{n-2},A_{n-1}]/2} e^{-[A_1+\ldots+A_{n-1},A_n]/2} \,.
\end{aligned}$$

Then, we use that the commutator terms in the exponentials do all mutually commute, such that they can be combined into a single exponential operator

$$\begin{aligned}
e^{A_1+\ldots+A_n} &= e^{A_1} \times \ldots \times e^{A_n} e^{-[A_1,A_2]/2-[A_1+A_2,A_3]/2-\ldots-[A_1+\ldots+A_{n-2},A_{n-1}]/2-[A_1+\ldots+A_{n-1},A_n]/2} \\
&= e^{A_1} \times \ldots \times e^{A_n} e^{\sum_{i<j}[A_i,A_j]/2} \,.
\end{aligned}$$

Exercise 3.3 (Matrix Exponentials).
Show that for a unit vector $|\boldsymbol{n}| = 1$ and a vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ one has

$$e^{(\boldsymbol{n}\cdot\boldsymbol{\sigma})\otimes A} = \mathbf{1} \otimes \cosh(A) + (\boldsymbol{n}\cdot\boldsymbol{\sigma}) \otimes \sinh(A) \,.$$

Solution 3.3. We first show that – using well-known properties of the Pauli matrices

$$
\begin{aligned}
(\boldsymbol{n}\cdot\boldsymbol{\sigma})^2 &= (n_x\sigma^x + n_y\sigma^y + n_z\sigma^z)^2 = \left(n_x^2+n_y^2+n_z^2\right)\mathbf{1} + n_x n_y\{\sigma^x,\sigma^y\} + n_x n_z\{\sigma^x,\sigma^z\} + n_y n_z\{\sigma^y,\sigma^z\} \\
&= \mathbf{1}\,.
\end{aligned}
$$

This implies for even and odd powers

$$
(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k} = \mathbf{1}\,, \qquad (\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k+1} = (\boldsymbol{n}\cdot\boldsymbol{\sigma})\;.
$$

These can be separately written in the exponential

$$
\begin{aligned}
e^{(\boldsymbol{n}\cdot\boldsymbol{\sigma})\otimes A} &= \sum_{k=0}^{\infty}\frac{(\boldsymbol{n}\cdot\boldsymbol{\sigma})^k\otimes A^k}{k!} = \sum_{k=0}^{\infty}\frac{(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k}\otimes A^{2k}}{(2k)!} + \sum_{k=0}^{\infty}\frac{(\boldsymbol{n}\cdot\boldsymbol{\sigma})^{2k+1}\otimes A^{2k+1}}{(2k+1)!} \\
&= \mathbf{1}\otimes\sum_{k=0}^{\infty}\frac{A^{2k}}{(2k)!} + (\boldsymbol{n}\cdot\boldsymbol{\sigma})\sum_{k=0}^{\infty}\frac{A^{2k+1}}{(2k+1)!} = \mathbf{1}\otimes\cosh(A) + (\boldsymbol{n}\cdot\boldsymbol{\sigma})\sinh(A)\,.
\end{aligned}
$$

Exercise 3.4 (Weak Coupling Limit).
Show that Eq. (177) reduces in the weak-coupling limit to Eq. (178 by using a representation of the Dirac-Delta distribution

$$
\delta(x) = \lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}\,.
$$

Solution 3.4. In the wide-band limit $\Gamma_\alpha(\omega)\to\Gamma_\alpha$, the stationary occupation of the central dot can be written as ($\Gamma = \sum_\alpha \Gamma_\alpha$)

$$
\begin{aligned}
\bar{n} &= \sum_\alpha \int d\omega \Gamma_\alpha f_\alpha(\omega)\frac{2}{\pi}\frac{1}{\Gamma^2+4(\omega-\epsilon)^2} = \sum_\alpha \frac{\Gamma_\alpha}{\Gamma}\int f_\alpha(\omega)\frac{1}{\pi}\frac{\Gamma/2}{(\Gamma/2)^2+(\omega-\epsilon)^2}d\omega \\
&\to \sum_\alpha \frac{\Gamma_\alpha}{\Gamma}\int f_\alpha(\omega)\delta(\omega-\epsilon)d\omega = \sum_\alpha\frac{\Gamma_\alpha}{\Gamma}f_\alpha(\epsilon)\,,
\end{aligned}
$$

where we have assumed that while $\Gamma\to 0$, the ration Γ_α/Γ remains finite. For two terminals $\alpha\in\{L,R\}$, this directly corresponds to Eq. (178). In any case one can see that the occupation of the dot is a convex combination of the lead occupations at the dot energy ϵ: For coupling only to a single lead, the dot occupation equilibrates perfectly in the weak-coupling limit.

Exercise 3.5 (Weak-Coupling Limit).
Show that Eq. (186) follows from Eq. (185) when $\Gamma\to 0$.

Solution 3.5. Similar to the previous exercise we may employ a representation of the Dirac-Delta function. Assuming that while $\Gamma=\Gamma_L+\Gamma_R\to 0$, the prefactor $\Gamma_L\Gamma_R/(\Gamma_L+\Gamma_R)$ remains finite, we use

$$
\delta(\omega-\epsilon) = \lim_{\Gamma\to 0}\frac{\Gamma/2}{(\omega-\epsilon)^2+(\Gamma/2)^2}\,,
$$

which directly converts Eq. (185) to Eq. (186).

Chapter 4
Technical Tools

Exercise 4.1 (Single Resonant Level).
Calculate the matrix exponential of the Liouvillian super-operator for a single resonant level tunnel-coupled to a single junction

$$\mathcal{L} = \begin{pmatrix} -\Gamma f & +\Gamma(1-f) \\ +\Gamma f & -\Gamma(1-f) \end{pmatrix}$$

when the dot level is much lower than the Fermi edge ($f \to 1$) and when it is much larger than the Fermi edge $f \to 0$.

Solution 4.1. Though it is straightforward to exponentiate the Liouvillian $\mathcal{L}$ also for finite f, we consider for brevity only the special cases $f \to 0$ and $f \to 1$, where the Liovillian becomes

$$\mathcal{L}_0 = \Gamma \begin{pmatrix} 0 & +1 \\ 0 & -1 \end{pmatrix}, \qquad \mathcal{L}_1 = \Gamma \begin{pmatrix} -1 & 0 \\ +1 & 0 \end{pmatrix},$$

and we can directly confirm the relations $\mathcal{L}_0^2 = -\Gamma\mathcal{L}_0$ and $\mathcal{L}_1^2 = -\Gamma\mathcal{L}_1$, which by induction generalize to

$$\mathcal{L}_0^{n\geq 1} = (-1)^{n-1}\mathcal{L}_0\,, \qquad \mathcal{L}_1^{n\geq 1} = (-1)^{n-1}\mathcal{L}_1\,,$$

which can be inserted in the matrix exponential

$$e^{\mathcal{L}_{0/1}} = \mathbf{1} + \sum_{n=1}^{\infty} \frac{\mathcal{L}_{0/1}^n}{n!} = \mathbf{1} + \frac{\mathcal{L}_{0/1}}{-\Gamma} \sum_{n=1}^{\infty} \frac{(-\Gamma)^n}{n!} = \mathbf{1} - \frac{\mathcal{L}_{0/1}}{\Gamma}\left(e^{-\Gamma} - 1\right),$$

and in particular we obtain

$$e^{\mathcal{L}_0} = \begin{pmatrix} 1 & 1 - e^{-\Gamma} \\ 0 & e^{-\Gamma} \end{pmatrix}, \qquad e^{\mathcal{L}_1} = \begin{pmatrix} e^{-\Gamma} & 0 \\ 1 - e^{-\Gamma} & 1 \end{pmatrix}.$$

We note here that $\Gamma = \gamma t$ must be dimensionless. In the large-time limit $\Gamma \to \infty$, these propagators simply map any initial state to the empty or filled state, respectively.

Exercise 4.2 (EOM for the harmonic oscillator).
Calculate the expectation value of $a + a^\dagger$ for a cavity in a vacuum bath

$$\dot\rho = -\mathrm{i}\,[H, \rho] + a\rho a^\dagger - \frac{1}{2}\left\{a^\dagger a, \rho\right\}.$$

Solution 4.2. The vaccuum master equation reads explicitly

$$\dot\rho = -\mathrm{i}\left[\Omega a^\dagger a, \rho\right] + \gamma\left[a\rho a^\dagger - \frac{1}{2}a^\dagger a\rho - \frac{1}{2}\rho a^\dagger a\right].$$

It is straightforward to use Eq. (190) to calculate the equations of motion for the displacement $x = \sqrt{\frac{1}{2m\Omega}}(a + a^\dagger)$ and momentum $p = -\sqrt{m\Omega}\mathrm{i}(a - a^\dagger)$ of the harmonic oscillator. Fortunately, these already form a closed set

$$
\begin{aligned}
\frac{d}{dt}\left\langle a + a^\dagger \right\rangle &= -\frac{\gamma}{2}\left\langle a + a^\dagger \right\rangle - \Omega \left\langle \mathrm{i}\left(a - a^\dagger\right)\right\rangle \\
\frac{d}{dt}\left\langle \mathrm{i}(a - a^\dagger)\right\rangle &= \Omega \left\langle a + a^\dagger \right\rangle - \frac{\gamma}{2}\left\langle \mathrm{i}\left(a - a^\dagger\right)\right\rangle .
\end{aligned}
$$

However, it is even simpler to calculate the evolution of expectation values of the creation and annihilation operators

$$
\frac{d}{dt}\left\langle a(t)\right\rangle = \left(-\frac{\gamma}{2} - \mathrm{i}\Omega\right)\left\langle a(t)\right\rangle , \qquad \frac{d}{dt}\left\langle a^\dagger(t)\right\rangle = \left(-\frac{\gamma}{2} + \mathrm{i}\Omega\right)\left\langle a^\dagger(t)\right\rangle .
$$

From the evident solution of these equation, we may also construct the evolution of the displacement

$$
\left\langle a(t) + a^\dagger(t)\right\rangle = e^{-\gamma t/2}\cos(\Omega t)\left\langle a + a^\dagger\right\rangle - e^{-\gamma t/2}\sin(\Omega t)\left\langle \mathrm{i}(a - a^\dagger)\right\rangle ,
$$

i.e., position expectation values are simply damped to the minimum, as one would expect for a vacuum reservoir. This also holds for the momentum.

Exercise 4.3 (Order of the RK scheme).
Acting with the Liouville super-operator performs the time-derivative of the density matrix. Show that the presented scheme (199) is of fourth order in Δt, i.e., that

$$
\rho_{n+1} = \left[\mathbf{1} + \mathcal{L}\Delta t + \mathcal{L}^2\frac{\Delta t^2}{2!} + \mathcal{L}^3\frac{\Delta t^3}{3!} + \mathcal{L}^4\frac{\Delta t^4}{4!}\right]\rho_n + \mathcal{O}\{\Delta t\}^5 .
$$

Solution 4.3. We may recursively plug in the recipe of the Runge-Kutta scheme

$$
\begin{aligned}
\rho_{n+1} &= \rho_n + \frac{1}{6}\sigma_{n,1} + \frac{1}{3}\sigma_{n,2} + \frac{1}{3}\sigma_{n,3} + \frac{1}{6}\sigma_{n,4} \\
&= \rho_n + \frac{1}{6}\Delta t\mathcal{L}\rho_n + \frac{1}{3}\Delta t\mathcal{L}\left(\rho_n + \frac{1}{2}\Delta t\mathcal{L}\rho_n\right) + \frac{1}{3}\Delta t\mathcal{L}\left[\rho_n + \frac{1}{2}\Delta t\mathcal{L}\left(\rho_n + \frac{1}{2}\Delta t\mathcal{L}\rho_n\right)\right] \\
&\quad + \frac{1}{6}\Delta t\mathcal{L}\left\{\rho_n + \Delta t\mathcal{L}\left[\rho_n + \frac{1}{2}\Delta t\mathcal{L}\left(\rho_n + \frac{1}{2}\Delta t\mathcal{L}\rho_n\right)\right]\right\} \\
&= \left[\mathbf{1} + (\Delta t)\mathcal{L} + \frac{(\Delta t)^2}{2}\mathcal{L}^2 + \frac{(\Delta t)^3}{6}\mathcal{L}^3 + \frac{(\Delta t)^4}{24}\mathcal{L}^4\right]\rho_n ,
\end{aligned}
$$

which corresponds to the fourth order expansion of $\rho(t + \Delta t)$ around t, since the k-th derivative of $\rho(t)$ at time t is numerically approximated by $\mathcal{L}^k\rho(t)$.

Exercise 4.4 (Norm for continuous evolution).
Calculate the norm of the state vector $\langle\Psi(t)|\Psi(t)\rangle$ from Eq. (202).

Solution 4.4. The state vector for nonlinear evolution

$$
|\Psi(t)\rangle = \frac{e^{-\mathrm{i}Mt}|\Psi_0\rangle}{\langle\Psi_0| e^{+\mathrm{i}M^\dagger t}e^{-\mathrm{i}Mt}|\Psi_0\rangle^{1/2}}
$$

with $M = H - \frac{\mathrm{i}}{2}\sum_\alpha \gamma_\alpha L_\alpha^\dagger L_\alpha$ is obviously normalized, which follows from

$$
\left(e^{-\mathrm{i}Mt}\right)^\dagger = e^{+\mathrm{i}M^\dagger t} .
$$

Therefore, we have $\langle\Psi(t)|\Psi(t)\rangle = 1$. Any loss in the norm of the state vector would result in false jump probabilities in the stochastic Schrödinger equation, since these are determined from $\langle\Psi(t)| L_\alpha^\dagger L_\alpha |\Psi(t)\rangle$.

Exercise 4.5 (additivity of rates).
Show that for an interaction Hamiltonian of the form $\mathcal{H}_\mathrm{I} = \sum_\alpha A_\alpha \otimes B_\alpha = \sum_a \sum_\nu A_{a\nu} \otimes B_{a\nu}$ where ν labels the reservoir and where $\langle B_{a\nu}\rangle = 0$ holds, different reservoirs do not interfere, such that the rates can be calculated additively

$$C_{\alpha\beta}(\tau) = C_{a\nu,b\mu}(\tau) = \langle \boldsymbol{B_{a\nu}}(\tau) B_{b\nu} \rangle \, \delta_{\mu\nu} \, .$$

Solution 4.5. This claim of course only holds for reservoirs that remain independent, i.e., where the reservoir density matrix can be written as a tensor product

$$\bar{\rho}_{\rm B} = \bigotimes_{\nu} \bar{\rho}_{\rm B}^{(\nu)} \, ,$$

and where $\bar{\rho}_{\rm B}^{(\nu)}$ denotes the density matrix of reservoir ν. Since these must be separately normalized $\mathrm{Tr}_\nu\{\bar{\rho}_{\rm B}^{(\nu)}\} = 1$, the trace in the reservoir correlation functions can be simplified to

$$\begin{aligned}
C_{\alpha\beta}(\tau) &= \mathrm{Tr}_{\rm B}\left\{\boldsymbol{B_\alpha}(\tau) B_\beta \bar{\rho}_{\rm B}\right\} = \mathrm{Tr}_{\rm B}\left\{\boldsymbol{B_{a\nu}}(\tau) \boldsymbol{B_{b\mu}} \bigotimes_{\sigma} \bar{\rho}_{\rm B}^{(\sigma)}\right\} \\
&= \begin{cases} \mathrm{Tr}_\nu\{\boldsymbol{B_{a\nu}}(\tau)\bar{\rho}_{\rm B}^{(\nu)}\}\mathrm{Tr}_\mu\{\boldsymbol{B_{b\mu}}\bar{\rho}_{\rm B}^{(\mu)}\} & : \nu \neq \mu \\ \mathrm{Tr}_\nu\{\boldsymbol{B_{a\nu}}(\tau)\boldsymbol{B_{b\nu}}\bar{\rho}_{\rm B}^{(\nu)}\} & : \nu = \mu \end{cases} \\
&= \langle \boldsymbol{B_{a\nu}}(\tau) B_{b\nu} \rangle \, \delta_{\mu\nu} \, ,
\end{aligned}$$

where we have used that $\langle B_{a\nu} \rangle = 0$ (which can always be achieved by suitable transformations of both system and interaction Hamiltonians).

Exercise 4.6 (Logarithmic Sum Inequality).
Show that for non-negative a_i and b_i

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq a \ln \frac{a}{b}$$

with $a = \sum_i a_i$ and $b = \sum_i b_i$.

Solution 4.6. Defining the function $f(x) = x \ln(x)$ we can write

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} = \sum_i b_i f\left(\frac{a_i}{b_i}\right) = b \sum_i \frac{b_i}{b} f\left(\frac{a_i}{b_i}\right) \, ,$$

and we note that $f(x)$ is a convex function. This implies that we can apply Jensen's inequality (we have $\sum_i b_i/b = 1$ and $b_i/b \geq 0$): For a convex function $f(x)$, a convex combination of the functions values at a set of points $f_i = f(x_i)$ is larger than the function value when evaluated at the convex combination built from the set of points $\bar{x} = \sum_i \alpha_i x_i$. This yields

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq b f\left(\sum_i \frac{a_i}{b}\right) = b f\left(\frac{a}{b}\right) = a \ln\left(\frac{a}{b}\right) \, .$$

Exercise 4.7 (SET Onsager relations).
Confirm the validity of Eq. (237).

Solution 4.7. The matter current from source to drain through the SET reads

$$I_M = \frac{\Gamma_S \Gamma_D}{\Gamma_S + \Gamma_D} \left[f_S(\epsilon) - f_D(\epsilon)\right] \, ,$$

where $f_\alpha(\omega)$ denotes the Fermi function. The heat current from source to drain is in linear response simply given by $\dot{Q} = (\epsilon - \mu) I_M$. To obtain an expression valid for the linear response regime, we expand the Fermi functions

$$f_S(\epsilon) = \frac{1}{e^{\left(\beta - \frac{\Delta\beta}{2}\right)\left[\epsilon - \left(\mu + \frac{\Delta\mu}{2}\right)\right]} + 1} \, , \qquad f_D(\epsilon) = \frac{1}{e^{\left(\beta + \frac{\Delta\beta}{2}\right)\left[\epsilon - \left(\mu - \frac{\Delta\mu}{2}\right)\right]} + 1}$$

for small $\Delta\beta$ and small $\Delta\mu$

$$f_S(\epsilon) - f_D(\epsilon) \approx \Delta\beta(\epsilon-\mu)\frac{e^{\beta(\epsilon-\mu)}}{(e^{\beta\epsilon}+e^{\beta\mu})^2} + \beta\Delta\mu\frac{e^{\beta(\epsilon-\mu)}}{(e^{\beta\epsilon}+e^{\beta\mu})^2} = f(1-f)\,.$$

Therefore, we deduce for the matter current and the heat current in linear response

$$\begin{aligned} I_M &\approx \frac{\Gamma_S\Gamma_D}{\Gamma_S+\Gamma_D} f(1-f)\left[(\epsilon-\mu)\Delta\beta + \beta\Delta\mu\right]\,,\\ \dot{Q} &\approx \frac{\Gamma_S\Gamma_D}{\Gamma_S+\Gamma_D} f(1-f)\left[(\epsilon-\mu)^2\Delta\beta + (\epsilon-\mu)\beta\Delta\mu\right]\,, \end{aligned}$$

which when written in matrix form directly corresponds to Eq. (237).

Exercise 4.8 (Cumulant-generating function).
Calculate the long-term cumulant-generating function for current through the SET

$$\mathcal{L}(\chi) = \begin{pmatrix} -\Gamma_L f_L - \Gamma_R f_R & +\Gamma_L(1-f_L) + \Gamma_R(1-f_R)e^{+\mathrm{i}\chi} \\ +\Gamma_L f_L + \Gamma_R f_R e^{-\mathrm{i}\chi} & -\Gamma_L(1-f_L) - \Gamma_R(1-f_R) \end{pmatrix}\,.$$

What are the first two cumulants for the current, i.e., current $I = \frac{d}{dt}\left\langle\left\langle n\right\rangle\right\rangle$ *and noise* $S = \frac{d}{dt}\left\langle\left\langle n^2\right\rangle\right\rangle = \frac{d}{dt}\left(\left\langle n^2\right\rangle - \left\langle n\right\rangle^2\right)$*?*

Solution 4.8. The two eigenvalues of the Liouvillian become

$$\lambda_\pm(\chi) = \frac{1}{2}\left(-\Gamma_L - \Gamma_R \pm \sqrt{\Gamma_L^2 + \Gamma_R^2 + 2(1-2f_L)(1-2f_R)\Gamma_L\Gamma_R + 4\Gamma_L\Gamma_R f_L(1-f_R)e^{+\mathrm{i}\chi} + 4\Gamma_L\Gamma_R(1-f_L)f_R e^{-\mathrm{i}\chi}}\right)\,,$$

and we note that $\lambda_-(0) = -\Gamma_L - \Gamma_R$ and $\lambda_+(0) = 0$. Therefore, the long-term CGF is given by $\mathcal{C}(\chi,t) = \lambda_+(\chi)t$. The first cumulant is obtained by performing the derivative with respect to the counting field, which after some algebra yields

$$I = (-\mathrm{i})\partial_\chi\lambda_+(\chi)\big|_{\chi=0} = \frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}(f_L - f_R)\,.$$

We note that when $f_L > f_R$, the current is always positive, which simply expresses the fact that the current normally flows from the lead with large chemical potential towards the lead with low chemical potential. Similarly, the noise becomes

$$\begin{aligned} S &= (-\mathrm{i})^2\partial_\chi^2\lambda_+(\chi)\big|_{\chi=0} \\ &= \frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^3}\left[(f_L+f_R-2f_Lf_R)\left(\Gamma_L^2+\Gamma_R^2\right) + 2\Gamma_L\Gamma_R\left(f_L(1-f_L)+f_R(1-f_R)\right)\right]\,, \end{aligned}$$

which we note is positive throughout: The tunneling rates are positive by construction $\Gamma_\alpha > 0$, and since the Fermi functions obey $f_\alpha \in (0,1)$, we conclude $f_\alpha(1-f_\alpha) > 0$ and $f_L+f_R-2f_Lf_R > f_L^2+f_R^2-2f_Lf_R = (f_L-f_R)^2 \geq 0$. Since they originate from the same cumulant-generating function, current and noise are not unrelated. A popular symmetry is the Johnson-Nyquist relation, which holds for equal temperatures $\beta = \beta_L = \beta_R$ left and right $S\big|_{V=0} = \frac{2}{\beta}\frac{dI}{dV}\big|_{V=0}$.

Exercise 4.9 (Fluctuation Theorem).
Find the fluctuation theorem, i.e., a symmetry in the cumulant-generating function, for the SET

$$\mathcal{L}(\chi) = \begin{pmatrix} -\Gamma_L f_L - \Gamma_R f_R & +\Gamma_L(1-f_L) + \Gamma_R(1-f_R)e^{+\mathrm{i}\chi} \\ +\Gamma_L f_L + \Gamma_R f_R e^{-\mathrm{i}\chi} & -\Gamma_L(1-f_L) - \Gamma_R(1-f_R) \end{pmatrix}\,.$$

Solution 4.9. We can either deduce this symmetry of the cumulant-generating function from the characteristic polynomial of the Liouvillian

$$\mathcal{D}(\chi) = |\mathcal{L}(\chi) - \lambda\mathbf{1}| = \lambda^2 + \lambda(\Gamma_L+\Gamma_R) + (f_L+f_R-2f_Lf_R)\Gamma_L\Gamma_R - \Gamma_L\Gamma_R f_L(1-f_R)e^{+\mathrm{i}\chi} - \Gamma_L\Gamma_R(1-f_L)f_R e^{-\mathrm{i}\chi}$$

or directly from the dominating eigenvalue

$$\lambda(\chi) = \frac{1}{2}\left(-\Gamma_L - \Gamma_R + \sqrt{\Gamma_L^2 + \Gamma_R^2 + 2(1-2f_L)(1-2f_R)\Gamma_L\Gamma_R + 4\Gamma_L\Gamma_R f_L(1-f_R)e^{+\mathrm{i}\chi} + 4\Gamma_L\Gamma_R(1-f_L)f_R e^{-\mathrm{i}\chi}}\right) .$$

In both expressions we note that the counting-field-dependence is completely contained in the function

$$f(\chi) = \Gamma_L\Gamma_R\left[f_L(1-f_R)e^{+\mathrm{i}\chi} + (1-f_L)f_R e^{-\mathrm{i}\chi}\right] ,$$

from which we can deduce that the tunneling rates will not enter the symmetry, if existent. We therefore just have to find a symmetry of $f(\chi)$, which is straightforward to see

$$f(-\chi) = f\left(+\chi + \mathrm{i}\ln\frac{f_L(1-f_R)}{(1-f_L)f_R}\right) .$$

For the counting statistics of transferred particles this maps for the probabilities $P_n(t)$ of transferring n particles from left to right after time t to the fluctuation theorem

$$\lim_{t\to\infty}\frac{P_{+n}(t)}{P_{-n}(t)} = e^{n\ln\frac{f_L(1-f_R)}{(1-f_L)f_R}} = e^{n[(\beta_R-\beta_L)\epsilon+\beta_L\mu_L-\beta_R\mu_R]} ,$$

which further simplifies e.g. for equal temperatures.

Chapter 5
Composite Non-Equilibrium Environments

Exercise 5.1 (DQD bath correlation functions).
Calculate the Fourier transforms (293) of the bath correlation functions for the double quantum dot.

Solution 5.1. The bath correlation functions can easily be linked to their Fourier transforms by introducing the tunneling rates $\Gamma_\alpha(\omega) = 2\pi \sum_k |t_{k\alpha}|^2 \delta(\omega - \epsilon_{k\alpha})$, such that we have with $\left\langle c_{k\alpha}^\dagger c_{k'\alpha} \right\rangle = \delta_{kk'} f_\alpha(\epsilon_{k\alpha})$ and $\left\langle c_{k\alpha} c_{k'\alpha}^\dagger \right\rangle = \delta_{kk'} \left[1 - f_\alpha(\epsilon_{k\alpha})\right]$

$$\begin{aligned} C_{12}(\tau) &= \sum_{kk'} t_{kL} t_{k'L}^* e^{+\mathrm{i}\epsilon_{kL}\tau} \left\langle c_{kL}^\dagger c_{kL} \right\rangle = \sum_k |t_{kL}|^2 e^{+\mathrm{i}\epsilon_{kL}\tau} f_L(\epsilon_{kL}) = \frac{1}{2\pi} \int \Gamma_L(\omega) f_L(\omega) e^{+\mathrm{i}\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int \Gamma_L(-\omega) f_L(-\omega) e^{-\mathrm{i}\omega\tau} d\omega \,. \end{aligned}$$

Similar calculations are performed for $C_{21}(\tau)$

$$C_{21}(\tau) = \sum_k |t_{kL}|^2 e^{-\mathrm{i}\epsilon_{kL}\tau} \left[1 - f_L(\epsilon_{kL})\right] = \frac{1}{2\pi} \int \Gamma_L(\omega) \left[1 - f_L(\omega)\right] e^{-\mathrm{i}\omega\tau} d\omega \,.$$

We may therefore directly read off the Fourier transforms ($C_{34}(\tau)$ and $C_{43}(\tau)$ follow by replacing $L \to R$)

$$\begin{aligned} \gamma_{12}(\omega) &= \Gamma_L(-\omega) f_L(-\omega) \,, \qquad & \gamma_{21}(\omega) &= \Gamma_L(\omega) \left[1 - f_L(\omega)\right] \,, \\ \gamma_{34}(\omega) &= \Gamma_R(-\omega) f_R(-\omega) \,, \qquad & \gamma_{43}(\omega) &= \Gamma_R(\omega) \left[1 - f_R(\omega)\right] \,. \end{aligned}$$

We note that for Lorentzian tunneling rates $\Gamma_\alpha(\omega) = \Gamma_\alpha \frac{\delta_\alpha^2}{\omega^2 + \delta_\alpha^2}$ it is also possible to represent the bath correlation functions explicitly in terms of hypergeometric functions, the above way of obtaining their Fourier transforms is however usually much more convenient.

Exercise 5.2 (Stationary Current).
Calculate the stationary currents corresponding to rate matrices Eq. (303) and Eq. (304).

Solution 5.2. In the Coulomb-blockade and high-bias limit of Eq. (303), it is implicitly assumed that the bias – though very large – does not allow to populate both dots simultaneously, such that we have $\bar{P}_2 = 0$. Its is straightforward to insert the corresponding counting fields for all jumps crossing the right terminal

$$\mathcal{L}(\chi) = \frac{1}{2} \begin{pmatrix} -2\Gamma_L & \Gamma_R e^{+\mathrm{i}\chi} & \Gamma_R e^{+\mathrm{i}\chi} \\ \Gamma_L & -\Gamma_R & 0 \\ \Gamma_L & 0 & -\Gamma_R \end{pmatrix} ,$$

and we note that transport is unidirectional from left to right (as should be the case in high bias limits). The stationary probabilities read

$$\bar{P}_0 = \frac{\Gamma_R}{2\Gamma_L + \Gamma_R} \,, \qquad \bar{P}_- = \frac{\Gamma_L}{2\Gamma_L + \Gamma_R} \,, \qquad \bar{P}_+ = \frac{\Gamma_L}{2\Gamma_L + \Gamma_R} \,,$$

and can be used to evaluate the current via

$$I=(-\mathrm{i})\mathrm{Tr}\left\{\mathcal{L}'(0)\bar{\rho}\right\}=\frac{\Gamma_L\Gamma_R}{2\Gamma_L+\Gamma_R}\,.$$

In contrast to the infinite bias current through the SET, we note that the above expression is no longer symmetric under exchanging $L \leftrightarrow R$. This is a consequence of the level hybridization: The energy eigenstates are not symmetrically coupled to left and right leads.

In the infinite bias limit, we can no longer neglect the population of the highest level. Inserting the counting fields, we obtain the Liouvillian

$$\mathcal{L}(\chi)=\frac{1}{2}\begin{pmatrix}-2\Gamma_L & \Gamma_R e^{+\mathrm{i}\chi} & \Gamma_R e^{+\mathrm{i}\chi} & 0\\ \Gamma_L & -\Gamma_L-\Gamma_R & 0 & \Gamma_R e^{+\mathrm{i}\chi}\\ \Gamma_L & 0 & -\Gamma_L-\Gamma_R & \Gamma_R e^{+\mathrm{i}\chi}\\ 0 & \Gamma_L & \Gamma_L & -2\Gamma_R\end{pmatrix},$$

which again corresponds to unidirectional transport. The stationary occupations become

$$P_0=\frac{\Gamma_R^2}{(\Gamma_L+\Gamma_R)^2}\,,\qquad P_-=\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}\,,\qquad P_+=\frac{\Gamma_L\Gamma_R}{(\Gamma_L+\Gamma_R)^2}\,,\qquad P_2=\frac{\Gamma_L^2}{(\Gamma_L+\Gamma_R)^2}\,,$$

and can be used to calculate the current

$$I=\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\,,$$

which is the same as the infinite bias current through a normal SET.

Exercise 5.3 (Nonequilibrium Stationary State).
Show that the stationary state of Eq. (303) cannot be written as a grand-canonical equilibrium state by disproving the equations $\bar{\rho}_{--}/\bar{\rho}_{00}=e^{-\beta(E_--E_0-\mu)}$, $\bar{\rho}_{++}/\bar{\rho}_{00}=e^{-\beta(E_+-E_0-\mu)}$ *and* $\bar{\rho}_{++}/\bar{\rho}_{--}=e^{-\beta(E_+-E_-)}$

Solution 5.3. A thermal equilibrium stationary state would fulfill (compare the solution of the last exercise)

$$\begin{aligned}\frac{\bar{P}_-}{\bar{P}_0}&=e^{-\beta(E_--E_0-\mu)}=\frac{\Gamma_L}{\Gamma_R}\,,\\ \frac{\bar{P}_+}{\bar{P}_0}&=e^{-\beta(E_+-E_0-\mu)}=\frac{\Gamma_L}{\Gamma_R}\,,\\ \frac{\bar{P}_+}{\bar{P}_-}&=e^{-\beta(E_+-E_-)}=1\,.\end{aligned}$$

The last equation would e.g. require $\beta=0$, which is in clear conflict with the first two equations. Therefore, the stationary state is clearly a nonequilibrium one (unless in some specific cases, e.g. $\Gamma_L=\Gamma_R$).

Exercise 5.4 (Transition Rates).
Derive the Fourier transforms of the reservoir correlation functions and confirm the rates in the Liouvillian (311).

Solution 5.4. In the interaction Hamiltonian, we can identify 8 reservoir coupling operators

$$\begin{aligned}B_1&=\sum_k t_{kLA}c_{kLA}^\dagger\,, && B_2=B_1^\dagger\,,\\ B_3&=\sum_k t_{kLB}c_{kLB}^\dagger\,, && B_4=B_3^\dagger\,,\\ B_5&=\sum_k t_{kRA}c_{kRA}^\dagger\,, && B_6=B_5^\dagger\,,\\ B_7&=\sum_k t_{kRB}c_{kRB}^\dagger\,, && B_8=B_7^\dagger\,,\end{aligned}$$

and the non-vanishing correlation functions can be straightforwardly derived. We just perform the explicit calculation for the first two

$$
\begin{aligned}
C_{12}(\tau) &= \sum_{kk'} t_{kLA} t^*_{k'LA} \left\langle c^\dagger_{kLA} c_{k'LA} \right\rangle e^{+\mathrm{i}\epsilon_{kLA}t} = \sum_k |t_{kLA}|^2 f_{LA}(\epsilon_{kLA}) e^{+\mathrm{i}\epsilon_{kLA}t} = \frac{1}{2\pi}\int \Gamma_{LA}(\omega) f_{LA}(\omega) e^{+\mathrm{i}\omega t} d\omega \\
&= \frac{1}{2\pi}\int \Gamma_{LA}(-\omega) f_{LA}(-\omega) e^{-\mathrm{i}\omega t} d\omega \,, \\
C_{21}(\tau) &= \sum_{kk'} t^*_{kLA} t_{k'LA} \left\langle c_{kLA} c^\dagger_{k'LA} \right\rangle e^{-\mathrm{i}\epsilon_{kLA}t} = \sum_k |t_{kLA}|^2 [1 - f_{LA}(\epsilon_{kLA})] e^{-\mathrm{i}\epsilon_{kLA}t} \\
&= \frac{1}{2\pi}\int \Gamma_{LA}(\omega)[1 - f_{LA}(\omega)] e^{-\mathrm{i}\omega t} d\omega \,,
\end{aligned}
$$

where we can directly infer the Fourier transforms of the reservoir correlation functions. The remaining correlation functions can be calculated in an absolutely similar fashion, and we obtain

$$
\begin{aligned}
\gamma_{12}(\omega) &= \Gamma_{LA}(-\omega) f_{LA}(-\omega) \,, & \gamma_{21}(\omega) &= \Gamma_{LA}(+\omega)[1 - f_{LA}(+\omega)] \,, \\
\gamma_{34}(\omega) &= \Gamma_{LB}(-\omega) f_{LB}(-\omega) \,, & \gamma_{43}(\omega) &= \Gamma_{LB}(+\omega)[1 - f_{LB}(+\omega)] \,, \\
\gamma_{56}(\omega) &= \Gamma_{RA}(-\omega) f_{RA}(-\omega) \,, & \gamma_{65}(\omega) &= \Gamma_{RA}(+\omega)[1 - f_{RA}(+\omega)] \,, \\
\gamma_{78}(\omega) &= \Gamma_{RB}(-\omega) f_{RB}(-\omega) \,, & \gamma_{87}(\omega) &= \Gamma_{RB}(+\omega)[1 - f_{RB}(+\omega)] \,.
\end{aligned}
$$

According to Eq. (118), these Fourier transforms enter the transition rates from energy eigenstate b to eigenstate a via

$$
\begin{aligned}
\gamma_{ab} = \gamma_{ab,ab} &= \sum_{\alpha\beta} \gamma_{\alpha\beta}(E_b - E_a) \langle a| A_\beta |b\rangle \langle a| A^\dagger_\alpha |b\rangle^* \\
&= \gamma_{12}(E_b - E_a) |\langle a| A_2 \langle b||^2 + \gamma_{21}(E_b - E_a) |\langle a| A_1 |b\rangle|^2 \\
&\quad + \gamma_{34}(E_b - E_a) |\langle a| A_4 \langle b||^2 + \gamma_{43}(E_b - E_a) |\langle a| A_3 |b\rangle|^2 \\
&\quad + \gamma_{56}(E_b - E_a) |\langle a| A_6 \langle b||^2 + \gamma_{65}(E_b - E_a) |\langle a| A_5 |b\rangle|^2 \\
&\quad + \gamma_{78}(E_b - E_a) |\langle a| A_8 \langle b||^2 + \gamma_{87}(E_b - E_a) |\langle a| A_7 |b\rangle|^2 \,,
\end{aligned}
$$

where we have exploited that e.g. $A_2 = A_1^\dagger$. Noting that the system coupling operators are given by

$$
\begin{aligned}
A_1 &= d_A \,, & A_2 &= d_A^\dagger \,, & A_3 &= d_B \,, & A_4 &= d_B^\dagger \,, \\
A_5 &= d_A \,, & A_6 &= d_A^\dagger \,, & A_7 &= d_B \,, & A_8 &= d_B^\dagger \,,
\end{aligned}
$$

we can immediately conclude that only single particle transition rates are non-vanishing. For jumps out of the system these become

$$
\begin{aligned}
\gamma_{00,01} &= \gamma_{43}(\epsilon_B) + \gamma_{87}(\epsilon_B) = \Gamma_{LB}(\epsilon_B)[1 - f_{LB}(\epsilon_B)] + \Gamma_{RB}(\epsilon_B)[1 - f_{RB}(\epsilon_B)] \,, \\
\gamma_{00,10} &= \gamma_{21}(\epsilon_A) + \gamma_{65}(\epsilon_A) = \Gamma_{LA}(\epsilon_A)[1 - f_{LA}(\epsilon_A)] + \Gamma_{RA}(\epsilon_A)[1 - f_{RA}(\epsilon_A)] \,, \\
\gamma_{01,11} &= \gamma_{21}(\epsilon_A + U) + \gamma_{65}(\epsilon_A + U) = \Gamma_{LA}(\epsilon_A + U)[1 - f_{LA}(\epsilon_A + U)] + \Gamma_{RA}(\epsilon_A + U)[1 - f_{RA}(\epsilon_A + U)] \,, \\
\gamma_{10,11} &= \gamma_{43}(\epsilon_B + U) + \gamma_{87}(\epsilon_B + U) = \Gamma_{LB}(\epsilon_B + U)[1 - f_{LB}(\epsilon_B + U)] + \Gamma_{RB}(\epsilon_B + U)[1 - f_{RB}(\epsilon_B + U)] \,,
\end{aligned}
$$

and for jumps into the system the rates read

$$
\begin{aligned}
\gamma_{01,00} &= \gamma_{34}(-\epsilon_B) + \gamma_{78}(-\epsilon_B) = \Gamma_{LB}(\epsilon_B) f_{LB}(\epsilon_B) + \Gamma_{RB}(\epsilon_B) f_{RB}(\epsilon_B) \,, \\
\gamma_{10,00} &= \gamma_{12}(-\epsilon_A) + \gamma_{56}(-\epsilon_A) = \Gamma_{LA}(\epsilon_A) f_{LA}(\epsilon_A) + \Gamma_{RA}(\epsilon_A) f_{RA}(\epsilon_A) \,, \\
\gamma_{11,01} &= \gamma_{12}(-\epsilon_A - U) + \gamma_{56}(-\epsilon_A - U) = \Gamma_{LA}(\epsilon_A + U) f_{LA}(\epsilon_A + U) + \Gamma_{RA}(\epsilon_A + U) f_{RA}(\epsilon_A + U) \,, \\
\gamma_{11,10} &= \gamma_{34}(-\epsilon_B - U) + \gamma_{78}(-\epsilon_B - U) = \Gamma_{LB}(\epsilon_B + U) f_{LB}(\epsilon_B + U) + \Gamma_{RB}(\epsilon_B + U) f_{RB}(\epsilon_B + U) \,.
\end{aligned}
$$

One can immediately see that local detailed balance is satisfied and that these exactly correspond to the rates in Eq. (311).

Exercise 5.5 (reduced affinity).
Confirm the validity of the reduced affinity in Eq. (323).

Solution 5.5. In principle, any two by two rate matrix will obey a fluctuation theorem. We had the effective Liouvillian

$$\mathcal{L}(\chi) = \begin{pmatrix} -\mathcal{L}_{10}^{(L)} - \mathcal{L}_{10}^{(R)} & +\mathcal{L}_{01}^{(L)} + \mathcal{L}_{01}^{(R)} e^{+\mathrm{i}\chi} \\ +\mathcal{L}_{10}^{(L)} + \mathcal{L}_{10}^{(R)} e^{-\mathrm{i}\chi} & -\mathcal{L}_{01}^{(L)} - \mathcal{L}_{01}^{(R)} \end{pmatrix},$$

which no longer obeys local detailed balance. The characteristic polynomial becomes

$$\mathcal{D}(\chi) = \left(\lambda + \mathcal{L}_{01}^{(L)} + \mathcal{L}_{01}^{(R)}\right)\left(\lambda + \mathcal{L}_{10}^{(L)} + \mathcal{L}_{10}^{(R)}\right) - \mathcal{L}_{01}^{(L)}\mathcal{L}_{10}^{(L)} - \mathcal{L}_{01}^{(R)}\mathcal{L}_{10}^{(R)}$$
$$-\mathcal{L}_{10}^{(L)}\mathcal{L}_{01}^{(R)} e^{+\mathrm{i}\chi} - \mathcal{L}_{01}^{(L)}\mathcal{L}_{10}^{(R)} e^{-\mathrm{i}\chi},$$

and we see that the counting-field dependence is only encoded in the last two terms $f(\chi) = \mathcal{L}_{10}^{(L)}\mathcal{L}_{01}^{(R)} e^{+\mathrm{i}\chi} + \mathcal{L}_{01}^{(L)}\mathcal{L}_{10}^{(R)} e^{-\mathrm{i}\chi}$ (a similar result also holds for the eigenvalues). For any function $g(\chi) = a e^{+\mathrm{i}\chi} + b e^{-\mathrm{i}\chi}$ we can directly confirm the symmetry $g(-\chi) = g(+\chi + \mathrm{i}\ln(a/b))$, such that we obtain the symmetry

$$f(-\chi) = f\left(+\chi + \mathrm{i}\ln\frac{\mathcal{L}_{10}^{(L)}\mathcal{L}_{01}^{(R)}}{\mathcal{L}_{01}^{(L)}\mathcal{L}_{10}^{(R)}}\right).$$

This transfers to the characteristic polynomial and all eigenvalues, and thereby also to the long-term cumulant-generating function. The corresponding affinity is thus the one of Eq. (323) and also shows up in the corresponding fluctuation theorem of channel A – provided the time-scale separation limit is well approached. This affinity is therefore also a measurable quantity.

Exercise 5.6 (Correlation functions for the QPC).
Show the validity of Eqs. (333).

Solution 5.6 (Correlation functions for the QPC). The second bath correlation function reads

$$\begin{aligned} C_{21}(\tau) &= \sum_{kk'\ell\ell'} t_{kk'}^* t_{\ell\ell'} e^{+\mathrm{i}(\varepsilon_{kL}-\varepsilon_{k'R})\tau} \left\langle \gamma_{k'R}\gamma_{kL}^\dagger \gamma_{\ell L}\gamma_{\ell' R}^\dagger \right\rangle \\ &= \sum_{kk'} \left|t_{kk'}\right|^2 e^{+\mathrm{i}(\varepsilon_{kL}-\varepsilon_{k'R})\tau} f_L(\varepsilon_{kL})[1 - f_R(\varepsilon_{k'R})] \\ &= \frac{1}{2\pi}\int d\omega \int d\omega' T(\omega,\omega') e^{+\mathrm{i}(\omega-\omega')\tau} f_L(\omega)[1 - f_R(\omega')], \end{aligned}$$

such that its Fourier transform reads

$$\begin{aligned} \gamma_{21}(\Omega) &= \int C_{21}(\tau) e^{+\mathrm{i}\Omega\tau} d\tau \\ &= \int d\omega \int d\omega' T(\omega,\omega')\delta(\Omega + \omega - \omega') f_L(\omega)[1 - f_R(\omega')] \\ &= T_0 \int d\omega f_L(\omega)[1 - f_R(\omega + \Omega)], \end{aligned}$$

where we have assumed the wideband limit $T(\omega,\omega') \approx T_0$. We note that this limit should at least hold in the region where $f_L(\omega)[1 - f_R(\omega + \Omega)] > 0$.

Exercise 5.7 (Normalization terms).
Compute the remaining rates

$$\sum_m \gamma_{(0,m)(0,m),(0,n)(0,n)}, \qquad \text{and} \qquad \sum_m \gamma_{(1,m)(1,m),(1,n)(1,n)}$$

explicitly.

Solution 5.7. The structure of the coupling operators $A_1 = (\mathbf{1} + \tilde{\tau} d^\dagger d) \otimes B^\dagger$ and $A_2 = (\mathbf{1} + \tilde{\tau}^* d^\dagger d) \otimes B$ demonstrates that in the summation only the terms with $m = n \pm 1$ remain (we omit the identical double indices for brevity)

$$\begin{aligned}\sum_m \gamma_{(0m),(0n)} &= \gamma_{(0,n-1),(0n)} + \gamma_{(0,n+1),(0n)} \\ &= \gamma_{12}(0) \langle 0,n-1| A_2 |0,n\rangle \langle 0,n-1| A_1^\dagger |0,n\rangle^* \\ &\quad +\gamma_{21}(0) \langle 0,n+1| A_1 |0,n\rangle \langle 0,n+1| A_2^\dagger |0,n\rangle^* \\ &= \gamma_{12}(0) + \gamma_{21}(0)\,.\end{aligned}$$

Analogously, we obtain for the occupied state

$$\begin{aligned}\sum_m \gamma_{(1m),(1n)} &= \gamma_{(1,n-1),(1n)} + \gamma_{(1,n+1),(1n)} \\ &= \gamma_{12}(0) \langle 1,n-1| A_2 |1,n\rangle \langle 1,n-1| A_1^\dagger |1,n\rangle^* \\ &\quad +\gamma_{21}(0) \langle 1,n+1| A_1 |1,n\rangle \langle 1,n+1| A_2^\dagger |1,n\rangle^* \\ &= [\gamma_{12}(0) + \gamma_{21}(0)] \left|1+\tilde{\tau}\right|^2 .\end{aligned}$$

Exercise 5.8 (QPC current).
Show that the stationary state of the SET is unaffected by the additional QPC dissipator and calculate the stationary current through the QPC for Liouvillian (347).

Solution 5.8. The stationary state of the Liouvillian is found by $\mathcal{L}(0,0)\bar{\rho} = 0$, and since $\mathcal{L}(0,0) = \mathcal{L}_{\rm SET}(0)$, it is simply given by the stationary state of the SET Liouvillian

$$\bar{\rho}_0 = (1-\bar{f})\,, \qquad \bar{\rho}_1 = \bar{f}$$

with the average Fermi function $\bar{f} = (\Gamma_L f_L + \Gamma_R f_R)/(\Gamma_L + \Gamma_R)$. The current through the SET is therefore not influenced by the QPC. This is different for the stationary QPC current, which can be calculated via

$$\begin{aligned} I &= (-{\rm i}){\rm Tr}\left\{\mathcal{L}'_{\rm QPC}(0)\bar{\rho}\right\} \\ &= (-{\rm i}){\rm i}\left(\gamma_{21}-\gamma_{12}\right)(1,1)\begin{pmatrix} 1 & 0 \\ 0 & \left|1+\tilde{\tau}\right|^2\end{pmatrix}\begin{pmatrix} 1-\bar{f} \\ \bar{f}\end{pmatrix} \\ &= (\gamma_{21}-\gamma_{12})\left[1-\bar{f}+\left|1+\tau\right|^2\bar{f}\right] \\ &= I_0(1-\bar{f}) + I_1\bar{f}\,.\end{aligned}$$

The QPC current is therefore just the weighted average current for empty (I_0) or occupied (I_1) SET dot, respectively.

Exercise 5.9 (Independent Fluctation Theorems).
Confirm the validity of Eq. (352).

Solution 5.9. We may consider the characteristic polynomial of the full Liouvillian

$$\mathcal{L}(\chi,\xi) = \begin{pmatrix} -\Gamma_L f_L - \Gamma_R f_R + g(\xi) & +\Gamma_L(1-f_L) + \Gamma_R(1-f_R)e^{+{\rm i}\chi} \\ +\Gamma_L f_L + \Gamma_R f_R e^{-{\rm i}\chi} & -\Gamma_L(1-f_L) - \Gamma_R(1-f_R) + \left|1+\tilde{\tau}\right|^2 g(\xi)\end{pmatrix},$$

where

$$g(\xi) = +\gamma_{21}\left(e^{+{\rm i}\xi}-1\right) + \gamma_{12}\left(e^{-{\rm i}\xi}-1\right)$$

which becomes

$$\begin{aligned}\mathcal{D}(\chi,\xi) &= (\lambda + \Gamma_L f_L + \Gamma_R f_R - g(\xi))\left(\lambda + \Gamma_L(1-f_L) + \Gamma_R(1-f_R) - \left|1+\tilde{\tau}\right|^2 g(\xi)\right) \\ &\quad -\Gamma_L^2 f_L(1-f_L) - \Gamma_R^2 f_R(1-f_R) - \Gamma_L\Gamma_R f(\chi)\end{aligned}$$

with the function $f(\chi)$ being defined as

$$f(\chi) = f_L(1-f_R)e^{+{\rm i}\chi} + (1-f_L)f_R e^{-{\rm i}\chi}\,.$$

These functions obey the symmetries

$$g(-\xi) = g\left(+\xi + \mathrm{i}\ln\frac{\gamma_{21}}{\gamma_{12}}\right), \qquad f(-\chi) = f\left(+\chi + \mathrm{i}\ln\frac{f_L(1-f_R)}{(1-f_L)f_R}\right),$$

which transfer to the characteristic polynomial $\mathcal{D}(\chi,\xi)$ and thereby to the eigenvalues of the Liouvillian and thus eventually to the cumulant-generating function. The separate fluctuation theorems therefore generally read

$$\lim_{t\to\infty}\frac{P_{+n}^{\mathrm{SET}}}{P_{-n}^{\mathrm{SET}}} = e^{n\ln\frac{f_L(1-f_R)}{(1-f_L)f_R}}, \qquad \lim_{t\to\infty}\frac{P_{+n}^{\mathrm{QPC}}}{P_{-n}^{\mathrm{QPC}}} = e^{n\ln\frac{\gamma_{21}}{\gamma_{12}}}.$$

When the SET leads are characterized by inverse temparature β_{SET} and the QPC leads by inverse temperature β_{QPC}, these fluctuation theorems simplify further

$$\ln\frac{f_L(1-f_R)}{(1-f_L)f_R} \to \beta_{\mathrm{SET}}(\mu_L^{\mathrm{SET}} - \mu_R^{\mathrm{SET}}) = \beta_{\mathrm{SET}}V_{\mathrm{SET}}, \qquad \ln\frac{\gamma_{21}}{\gamma_{12}} \to \beta_{\mathrm{QPC}}V_{\mathrm{QPC}},$$

and do thus correspond to Eq. (352).

Exercise 5.10 (Diagonalization of a single-qubit Hamiltonian).
Calculate eigenvalues and eigenvectors of the system Hamiltonian.

Solution 5.10. The matrix to be diagonalized reads

$$\mathcal{H}_{\mathrm{S}} = \begin{pmatrix} \epsilon+\varDelta & T \\ T & \epsilon-\varDelta \end{pmatrix}.$$

Eigenvalues can be obtained from the roots of the characteristic polynomial

$$|\mathcal{H}_{\mathrm{S}} - \lambda\mathbf{1}| = (\epsilon+\varDelta-\lambda)(\epsilon-\varDelta-\lambda) - T^2 = 0$$

and are given by $\lambda_\pm = \epsilon \pm \sqrt{\varDelta^2+T^2}$. Furthermore, it is straightforward to prove the eigenvalue equations $\mathcal{H}_{\mathrm{S}}\left|\pm\right\rangle = \lambda_\pm\left|\pm\right\rangle$ when

$$\left|\pm\right\rangle = \frac{1}{\sqrt{T^2 + \left(\varDelta \pm \sqrt{\varDelta^2+T^2}\right)^2}}\begin{pmatrix} \varDelta \pm \sqrt{\varDelta^2+T^2} \\ T \end{pmatrix}$$

Note that normalization and orthogonality of the eigenvectors are also clearly visible.

Exercise 5.11 (Qubit Dissipation).
Show the validity of Eqs. (367).

Solution 5.11. The coupling operator was given as

$$A = \frac{\tilde{\tau}_A}{2}(\mathbf{1}+\sigma^z) + \frac{\tilde{\tau}_B}{2}(\mathbf{1}-\sigma^z) = \begin{pmatrix} \tilde{\tau}_A & 0 \\ 0 & \tilde{\tau}_B \end{pmatrix}.$$

Therefore, we first calculate its matrix element in the energy eigenbasis (compare previous solution)

$$\begin{aligned}
\left\langle -\right| A \left|+\right\rangle &= \frac{1}{\sqrt{T^2+\left(\varDelta+\sqrt{\varDelta^2+T^2}\right)^2}}\frac{1}{\sqrt{T^2+\left(\varDelta-\sqrt{\varDelta^2+T^2}\right)^2}} \times \\
&\quad \times \left(\varDelta - \sqrt{\varDelta^2+T^2}, T\right)\begin{pmatrix} \tilde{\tau}_A & 0 \\ 0 & \tilde{\tau}_B \end{pmatrix}\begin{pmatrix} \varDelta+\sqrt{\varDelta^2+T^2} \\ T \end{pmatrix} \\
&= \frac{1}{\sqrt{4T^4+4\varDelta^2T^2}}\left[\tilde{\tau}_A(-T^2) + \tilde{\tau}_B T^2\right] = \frac{(\tilde{\tau}_B-\tilde{\tau}_A)T}{2\sqrt{\varDelta^2+T^2}}.
\end{aligned}$$

This evidently implies for the dissipative rate constants

$$\begin{aligned}
\gamma_{-+,-+} &= \gamma(+2\sqrt{\varDelta^2+T^2})\frac{T^2}{4(\varDelta^2+T^2)}(\tilde{\tau}_A-\tilde{\tau}_B)^2, \\
\gamma_{+-,+-} &= \gamma(-2\sqrt{\varDelta^2+T^2})\frac{T^2}{4(\varDelta^2+T^2)}(\tilde{\tau}_A-\tilde{\tau}_B)^2
\end{aligned}$$

as stated in Eqs. (367).

Exercise 5.12 (Strongly monitored qubit).
Calculate the stationary qubit state for the QPC held at infinite bias $V \to \pm\infty$.

Solution 5.12. Due to trace conservation, the stationary state of the qubit is completely defined by the ratio

$$\frac{\bar{\rho}_{++}}{\bar{\rho}_{--}} = \frac{\gamma(-2\sqrt{\Delta^2+T^2})}{\gamma(+2\sqrt{\Delta^2+T^2})}\,,$$

with the Fourier transform of the bath correlation function being given as

$$\gamma(\Omega) = \frac{\Omega+V}{1-e^{-\beta(\Omega+V)}} + \frac{\Omega-V}{1-e^{-\beta(\Omega-V)}}\,.$$

Asymptotically, this function behaves for large voltages as

$$\gamma(\Omega) \stackrel{V\to+\infty}{\to} \Omega+V\,, \qquad \gamma(\Omega) \stackrel{V\to-\infty}{\to} \Omega-V\,,$$

which implies for the stationary state

$$\lim_{V\to+\infty}\frac{\bar{\rho}_{++}}{\bar{\rho}_{--}} = \lim_{V\to+\infty}\frac{V-2\sqrt{\Delta^2+T^2}}{V+2\sqrt{\Delta^2+T^2}} = \frac{1}{2}\,, \qquad \lim_{V\to-\infty}\frac{\bar{\rho}_{++}}{\bar{\rho}_{--}} = \lim_{V\to-\infty}\frac{-V-2\sqrt{\Delta^2+T^2}}{-V+2\sqrt{\Delta^2+T^2}} = \frac{1}{2}\,.$$

Thus, for large voltages, the qubit decoheres into a balanced statistical mixture of the two energy eigenstates with $\bar{\rho} = \frac{1}{2}\left|-\right\rangle\left\langle-\right| + \frac{1}{2}\left|+\right\rangle\left\langle+\right|$.

Exercise 5.13 (Correlation function at infinite bias).
Confirm the validity of Eq. (387) in the infinite bias limit of the SET.

Solution 5.13. At infinite bias, the shift κ reduces to

$$\begin{aligned}\kappa &= \frac{1}{2\pi}\int\frac{\eta_L}{(\epsilon-\omega)^2+\eta^2/4}d\omega = \mathrm{i}\mathrm{Res}\frac{\eta_L}{(\epsilon-\omega)^2+\eta^2/4}\Big|_{\omega=\epsilon+\mathrm{i}\eta/2}\\ &= \frac{\eta_L}{\eta_L+\eta_R}\,,\end{aligned}$$

where we have closed the integral contour in the upper complex half plane and used the residue theorem.

Similarly, the Fourier transform of the correlation function becomes

$$\gamma(\Omega) \to \frac{\eta_L\eta_R}{2\pi}\int\frac{d\omega}{[(\epsilon-\omega)^2+\eta^2/4]\,[(\epsilon-\omega-\Omega)^2+\eta^2/4]}\,,$$

where $\eta = \eta_L+\eta_R$. The remaining integral can be solved analytically, for example using the theorem of residues. The poles are situated at $\omega_{1\pm} = \epsilon \pm \mathrm{i}\eta/2$ and $\omega_{2\pm} = \epsilon - \Omega \pm \mathrm{i}\eta/2$. Then, we obtain e.g. by closing the integral in the upper half plane

$$\begin{aligned}\gamma(\Omega) &= \eta_L\eta_R\mathrm{i}\left[\mathrm{Res}\frac{1}{[(\epsilon-\omega)^2+\eta^2/4]\,[(\epsilon-\omega-\Omega)^2+\eta^2/4]}\Big|_{\omega=\omega_{1p}} + \mathrm{Res}\frac{1}{[(\epsilon-\omega)^2+\eta^2/4]\,[(\epsilon-\omega-\Omega)^2+\eta^2/4]}\Big|_{\omega=\omega_{2p}}\right]\\ &= \frac{\eta_L\eta_R}{\eta}\frac{2}{\Omega^2+\eta^2} = \frac{\eta_L\eta_R}{\eta_L+\eta_R}\frac{2}{(\eta_L+\eta_R)^2+\Omega^2}\,.\end{aligned}$$

Exercise 5.14 (Polaron transformation).
Show the validity of Eqs. (423).

Solution 5.14. We use the nested-commutator expansion for operators X and Y

$$e^{+X}Ye^{-X} = \sum_{m=0}^{\infty}\frac{1}{m!}\left[X,Y\right]_m$$

with $\left[X,Y\right]_m = \left[X,\left[X,Y\right]_{m-1}\right]$ and $\left[X,Y\right]_0 = Y$. For the particular transformation, it is straightforward to show by induction that

$$[d^\dagger dA, d]_n = (-A)^n d\,.$$

This implies

$$UdU^\dagger = e^{+d^\dagger dA} d e^{-d^\dagger dA} = \sum_{m=0}^{\infty} \frac{1}{m!}\left[d^\dagger dA, d\right]_m = d \sum_{m=0}^{\infty} \frac{1}{m!}(-A)^m = de^{-A}\,.$$

By hermitian conjugation we also conclude that

$$Ud^\dagger U^\dagger = d^\dagger e^{+A}\,.$$

For the remaining transformations we use the nested-commutator expansion again

$$Ua_q U^\dagger = e^{d^\dagger dA} a_q e^{-d^\dagger dA} = \sum_{m=0}^{\infty} \frac{1}{m!}\left[d^\dagger dA, a_q\right]_m\,,$$

where we now need to specify $A = \sum_q \left(\frac{h_q^*}{\omega_q} a_q^\dagger - \frac{h_q}{\omega_q} a_q\right)$ From

$$\left[d^\dagger dA, a_q\right] = d^\dagger d \frac{h_q^*}{\omega_q}\left[a_q^\dagger, a_q\right] = -d^\dagger d \frac{h_q^*}{\omega_q}$$

we conclude that all higher commutators vanish, such that only the two lowest terms $m \in \{0, 1\}$ have to be kept

$$Ua_q U^\dagger = a_q - \frac{h_q^*}{\omega_q} d^\dagger d\,.$$

By hermitian conjugation we also obtain

$$Ua_q^\dagger U^\dagger = a_q^\dagger - \frac{h_q}{\omega_q} d^\dagger d\,.$$

Exercise 5.15 (KMS condition).
Show that the phonon correlation function (442) obeys the KMS condition $C(\tau) = C(-\tau - \mathrm{i}\beta_{\mathrm{ph}})$

Solution 5.15. The phonon correlation function is given by

$$C(\tau) = \exp\left\{\sum_q \left[e^{-\mathrm{i}\omega_q\tau}(1 + n_B^q) + e^{+\mathrm{i}\omega_q\tau} n_B^q - (1 + 2n_B^q)\right]\right\}$$

with the bosonic occupation (we abbreviate $\beta_{\mathrm{ph}} = \beta$)

$$n_B^q = \frac{1}{e^{\beta\omega_q} - 1}\,.$$

With the relation

$$\frac{n_B^q}{1 + n_B^q} = e^{-\beta\omega_q}$$

one obtains

$$\begin{aligned} C(-\tau - \mathrm{i}\beta) &= \exp\left\{\sum_q \left[e^{+\mathrm{i}\omega_q\tau} e^{-\beta\omega_q}(1 + n_B^q) + e^{-\mathrm{i}\omega_q\tau} e^{+\beta\omega_q} n_B^q - (1 + 2n_B^q)\right]\right\} \\ &= \exp\left\{\sum_q \left[e^{+\mathrm{i}\omega_q\tau} n_B^q + e^{-\mathrm{i}\omega_q\tau}(1 + n_B^q) - (1 + 2n_B^q)\right]\right\} = C(+\tau)\,. \end{aligned}$$

This symmetry relation also implies for the Fourier transform

$$\gamma(\omega) = \int C(\tau) e^{+\mathrm{i}\omega\tau} d\tau$$

a relation of the form

$$\gamma(-\omega) = \gamma(+\omega) e^{-\beta\omega} \, .$$

Exercise 5.16 (KMS condition).
Show the validity of relation (448).

Solution 5.16. The Fourier transform of the correlation function could – when specified on $\boldsymbol{n_\alpha}$ phonons created – be written as a product of electronic and phononic contributions

$$\gamma^\alpha_{12,+\boldsymbol{n_\alpha}}(\omega) = \gamma^\alpha_{12,\mathrm{el}}(\omega - \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}) C_{\mathrm{ph}}^{\boldsymbol{n_\alpha}} \, , \qquad \gamma^\alpha_{21,+\boldsymbol{n_\alpha}}(\omega) = \gamma^\alpha_{21,\mathrm{el}}(\omega - \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}) C_{\mathrm{ph}}^{\boldsymbol{n_\alpha}} \, ,$$

where the electronic and phononic contributions, respectively, read

$$\begin{aligned}
\gamma^\alpha_{12,\mathrm{el}}(\omega) = \varGamma_\alpha(-\omega) f_\alpha(-\omega) \, , \qquad \gamma^\alpha_{21,\mathrm{el}}(\omega) = \varGamma_\alpha(+\omega)[1 - f_\alpha(+\omega)] \, , \\
C_{\mathrm{ph}}^{\boldsymbol{n_\alpha}} = \prod_{q=1}^{Q} \left[e^{-\frac{|h_q|^2}{\omega_q^2}(1+2n_B^q)} \left(\frac{1+n_B^q}{n_B^q} \right)^{\frac{n_q}{2}} \mathcal{J}_{n_q} \left(2 \frac{|h_q|^2}{\omega_q^2} \sqrt{n_B^q(1+n_B^q)} \right) \right] \, .
\end{aligned}$$

The Fermi function obey

$$f_\alpha(+\omega) = e^{-\beta_\alpha(\omega-\mu_\alpha)} [1 - f_\alpha(+\omega)] \, ,$$

which implies for the electronic contributions

$$\gamma^\alpha_{12,\mathrm{el}}(-\omega) = \gamma^\alpha_{21,\mathrm{el}}(+\omega) e^{-\beta_\alpha(\omega-\mu_\alpha)} \, .$$

Furthermore, it is easy to see that

$$\begin{aligned}
C_{\mathrm{ph}}^{-\boldsymbol{n_\alpha}} &= \prod_{q=1}^{Q} \left[e^{-\frac{|h_q|^2}{\omega_q^2}(1+2n_B^q)} \left(\frac{1+n_B^q}{n_B^q} \right)^{-\frac{n_q}{2}} \mathcal{J}_{-n_q} \left(2 \frac{|h_q|^2}{\omega_q^2} \sqrt{n_B^q(1+n_B^q)} \right) \right] \\
&= \prod_{q=1}^{Q} \left[e^{-\frac{|h_q|^2}{\omega_q^2}(1+2n_B^q)} \left(\frac{n_B^q}{1+n_B^q} \right)^{\frac{n_q}{2}} \mathcal{J}_{+n_q} \left(2 \frac{|h_q|^2}{\omega_q^2} \sqrt{n_B^q(1+n_B^q)} \right) \right] \\
&= \prod_{q=1}^{Q} \left[e^{-\frac{|h_q|^2}{\omega_q^2}(1+2n_B^q)} \left(\frac{e^{-\beta_{\mathrm{ph}}\omega_q}(1+n_B^q)}{e^{+\beta_{\mathrm{ph}}\omega_q} n_B^q} \right)^{\frac{n_q}{2}} \mathcal{J}_{+n_q} \left(2 \frac{|h_q|^2}{\omega_q^2} \sqrt{n_B^q(1+n_B^q)} \right) \right] \\
&= \prod_{q=1}^{Q} \left[e^{-\frac{|h_q|^2}{\omega_q^2}(1+2n_B^q)} e^{-\beta_{\mathrm{ph}} n_q \omega_q} \left(\frac{1+n_B^q}{n_B^q} \right)^{\frac{n_q}{2}} \mathcal{J}_{+n_q} \left(2 \frac{|h_q|^2}{\omega_q^2} \sqrt{n_B^q(1+n_B^q)} \right) \right] \\
&= e^{-\beta_{\mathrm{ph}} \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}} C_{\mathrm{ph}}^{+\boldsymbol{n_\alpha}} \, .
\end{aligned}$$

Altogether, one obtains

$$\begin{aligned}
\gamma^\alpha_{12,+\boldsymbol{n_\alpha}}(-\omega) &= \gamma^\alpha_{12,\mathrm{el}}(-\omega - \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}) C_{\mathrm{ph}}^{\boldsymbol{n_\alpha}} \\
&= \gamma^\alpha_{21,\mathrm{el}}(+\omega + \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}) e^{-\beta_\alpha(\omega + \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega} - \mu_\alpha)} e^{+\beta_{\mathrm{ph}} \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}} C_{\mathrm{ph}}^{-\boldsymbol{n_\alpha}} \\
&= e^{-\beta_\alpha(\omega + \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega} - \mu_\alpha)} e^{+\beta_{\mathrm{ph}} \boldsymbol{n_\alpha} \cdot \boldsymbol{\omega}} \gamma^\alpha_{21,-\boldsymbol{n_\alpha}}(+\omega) \, ,
\end{aligned}$$

which exactly corresponds to Eq. (448).

Chapter 7
Controlled Systems

Exercise 7.1 (Cumulants).
Show that the cumulants of the probability distribution $P_n(t)$ are given by

$$\langle\langle n^k \rangle\rangle = \left[\gamma + (-1)^k \bar{\gamma}\right] t \,,$$

and can thus be understood as two counter-propagating Poissonian distributions.

Solution 7.1. The moment-generating function is given by $\mathcal{M}(\chi, t) = \exp\left\{\left[\gamma(e^{+\mathrm{i}\chi} - 1) + \bar{\gamma}(e^{-\mathrm{i}\chi} - 1)\right] t\right\}$. The cumulant-generating function is given by its its logarithm

$$\mathcal{C}(\chi, t) = \left[\gamma(e^{+\mathrm{i}\chi} - 1) + \bar{\gamma}(e^{-\mathrm{i}\chi} - 1)\right] t \,,$$

such that for our simple example it is always (not only in the long-term limit) linear in the time t. This implies for the cumulants

$$\begin{aligned}
\langle\langle n^k \rangle\rangle &= (-\mathrm{i})^k \partial_\chi^k \, \mathcal{C}(\chi, t)\big|_{\chi=0} = (-\mathrm{i})^k (+\mathrm{i})^k \gamma t + (-\mathrm{i})^k (-\mathrm{i})^k \bar{\gamma} t \\
&= \gamma t + (-1)^k \bar{\gamma} t \,,
\end{aligned}$$

which should be shown. It is already visible at the cumulant-generating function that it is the sum of two Poissonian cumulant-generating functions. Generally, cumulants of independent stochastic processes are additive, which can therefore be seen already at this level.

Exercise 7.2 (Poissonian limit).
Show that a Poissonian distribution arises in the unidirectional transport limit.

Solution 7.2. Taking the direct limit of the Bessel functions is tedious, therefore we start from the moment-generating function in the unidirectional transport limit ($\bar{\gamma} = 0$)

$$P(\chi, t) = \exp\left\{\gamma t(e^{+\mathrm{i}\chi} - 1)\right\} = \sum_n P_n(t) e^{+\mathrm{i}n\chi},$$

which is also the Fourier transform of the probability distribution $P_n(t)$. The inverse Fourier transform yields

$$\begin{aligned}
P_n(t) &= \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{\gamma t(e^{+\mathrm{i}\chi} - 1)} e^{-\mathrm{i}n\chi} d\chi = e^{-\gamma t} \sum_{a=0}^{\infty} \frac{(\gamma t)^a}{a!} \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{+\mathrm{i}a\chi} e^{-\mathrm{i}n\chi} d\chi \\
&= \begin{cases} e^{-\gamma t} \frac{(\gamma t)^n}{n!} & : n \geq 0 \\ 0 & : \text{ else} \end{cases} \,,
\end{aligned}$$

which is the sought-after Poissonian limit.

Exercise 7.3 (Probability conservation).
Show that the above introduced propagator $\mathcal{W}(\Delta t)$preserves the sum of all probabilities, i.e., that $\sum_n \rho_n(t + \Delta t) = \sum_n \rho_n(t)$.

Solution 7.3. The conservation of the total probability results from the normalization of the Poisson distribution in the propagator

$$\mathcal{W}_{nm}(\Delta t) = \begin{cases} e^{-\gamma \Delta t} \frac{(\gamma \Delta t)^{n-m}}{(n-m)!} & : n \geq m \\ 0 & : \text{ else} \end{cases}$$

which can easily be extended to feedback-dependent protocols. This leads to

$$\sum_n \rho_n(t+\Delta t) = \sum_n \sum_m \mathcal{W}_{nm}(\Delta t)\rho_m(t) = \sum_m \left[\sum_n \mathcal{W}_{nm}(\Delta t)\right]\rho_m(t) = \sum_m \rho_m(t)\,,$$

which is also valid for the feedback-dependent propagator, where we simply have $\gamma \to \gamma_m$.

Exercise 7.4 (Effective Feedback Propagator).
Show the validity of Eq. (503).

Solution 7.4. The matrix elements of the effective propagator read

$$\left(\mathcal{W}^{(n)}(t,\Delta t)\mathcal{P}^{(n)}\right)_{ab} = \sum_i \mathcal{W}_{ai}^{(n)}(t,\Delta t)\mathcal{P}_{ib}^{(n)} = \sum_i \mathcal{W}_{ai}^{(n)}(t,\Delta t)\delta_{i,n}\delta_{b,n} = \mathcal{W}_{a,n}^{(n)}(t,\Delta t)\delta_{b,n}\,,$$

which shows that the n-th column of the propagator has to be conditioned on on result n.

Exercise 7.5 (Variance evolution without feedback).
Show that without feedback $\gamma_m(t) = \gamma$, the variance during the iteration will for arbitrary distributions always increase as $\left(\left\langle n^2\right\rangle_{t+\Delta t} - \left\langle n\right\rangle_{t+\Delta t}^2\right) - \left(\left\langle n^2\right\rangle_t - \left\langle n\right\rangle_t^2\right) = \gamma\Delta t$.

Solution 7.5. We had for the evolution of the variance $C_2(t) = \left\langle n^2\right\rangle_t - \left\langle n\right\rangle_t^2$ the expression

$$C_2(t+\Delta t) - C_2(t) = \Delta t^2\left[\left\langle\gamma_n^2\right\rangle_t - \left\langle\gamma_n\right\rangle_t^2\right] + \Delta t\left[\left\langle\gamma_n\right\rangle_t + 2\left\langle n\gamma_n\right\rangle_t - 2\left\langle n\right\rangle_t\left\langle\gamma_n\right\rangle_t\right]\,.$$

Without feedback, we simply have $\left\langle\gamma_n^\alpha\right\rangle = \gamma^\alpha$ and $\left\langle n\gamma_n\right\rangle = \gamma\left\langle n\right\rangle$, such that we obtain

$$C_2(t+\Delta t) - C_2(t) = \gamma\Delta t\,,$$

which is always positive.

Exercise 7.6 (Variance evolution of a localized distribution).
Show that for arbitrary feedback, the variance of a localized distribution $\rho_m(t) = \delta_{m\bar{m}}$ will always increase unless $\gamma_{\bar{m}} = 0$.

Solution 7.6. For a localized distribution $\rho_m(t) = \delta_{m,\bar{m}}$, the expressions in the variance evolution do collapse

$$C_2(t+\Delta t) - C_2(t) = \Delta t^2\left[\gamma_{\bar{m}}^2 - \gamma_{\bar{m}}^2\right] + \Delta t\left[\gamma_{\bar{m}} + 2\bar{m}\gamma_{\bar{m}} - 2\bar{m}\gamma_{\bar{m}}\right] = \gamma_{\bar{m}}\Delta t\,.$$

Therefore, the variance will in this case always increase, and shortly after the distribution will no longer be localized.

Exercise 7.7 (Second Cumulant for joint distributions).
Show the validity of equation (527).

Solution 7.7. Since the separate distributions were independent (no feedback), we had for the probabilities of counting n particles during a pump cycle (of length T)

$$P_n(T) = \sum_{n_1,n_2}\delta_{n,n_1+n_2}P_{n_1}(T/2)P_{n_2}(T/2)\,,$$

where $P_{n_1}(T/2)$ denoted the probability to have n_1 particles tunneled to the right junction in the first half cycle and $P_{n_2}(T/2)$ the probability to have n_2 particles tunneled in the second half cycle, respectively. The second moment can be evaluated as (omitting the cycle time arguments)

$$\begin{aligned}\langle n^2\rangle &= \sum_n n^2 P_n = \sum_n \sum_{n_1,n_2} n^2 \delta_{n,n_1+n_2} P_{n_1} P_{n_2} = \sum_{n_1,n_2} (n_1+n_2)^2 P_{n_1} P_{n_2}\\ &= \langle n_1^2\rangle + 2\langle n_1\rangle\langle n_2\rangle + \langle n_2^2\rangle\,.\end{aligned}$$

For the second cumulant this implies (with $\langle n\rangle = \langle n_1\rangle + \langle n_2\rangle$)

$$\begin{aligned}\langle\langle n^2\rangle\rangle &= \langle n^2\rangle - \langle n\rangle^2 = \langle n_1^2\rangle + 2\langle n_1\rangle\langle n_2\rangle + \langle n_2^2\rangle - \langle n_1\rangle^2 - 2\langle n_1\rangle\langle n_2\rangle - \langle n_2\rangle^2\\ &= \langle n_1^2\rangle - \langle n_1\rangle^2 + \langle n_2^2\rangle - \langle n_2\rangle^2\,,\end{aligned}$$

where we find that the cumulants of independent processes are additive.

Exercise 7.8 (Measurement superoperators).
Show the correspondence between M_i and $\mathcal{M}_i$ in the above equations.

Solution 7.8. In matrix representation, the (in our case hermitian) measurement operators read

$$M_1 = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\,,\qquad M_2 = \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\,.$$

Hence, their action on an arbitrary density matrix yields

$$\begin{aligned}M_1\rho M_1^\dagger &= \frac{1}{4}\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{pmatrix}\begin{pmatrix}1&1\\1&1\end{pmatrix}\\ &= \frac{\rho_{00}+\rho_{11}+\rho_{01}+\rho_{10}}{4}\begin{pmatrix}1&1\\1&1\end{pmatrix}\,,\\ M_2\rho M_2^\dagger &= \frac{1}{4}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\begin{pmatrix}\rho_{00}&\rho_{01}\\\rho_{10}&\rho_{11}\end{pmatrix}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\\ &= \frac{\rho_{00}+\rho_{11}-\rho_{01}-\rho_{10}}{4}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\,,\end{aligned}$$

which enables one to conclude the action of the corresponding superoperators in the basis $\{\rho_{00},\rho_{11},\rho_{01},\rho_{10}\}$ as

$$\begin{aligned}\mathcal{P}_1 &= \frac{1}{4}\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}\,,\\ \mathcal{P}_2 &= \frac{1}{4}\begin{pmatrix}1&1&-1&-1\\1&1&-1&-1\\-1&-1&1&1\\-1&-1&1&1\end{pmatrix}\,.\end{aligned}$$

Exercise 7.9 (Repeated measurements).
Show the validity of the above equation.

Solution 7.9. It is helpful to note first the block structure of the effective propagator without feedback ($\alpha = 0$) leading to a decoupling of populations and coherences. Without decoherence ($\gamma_0 = 0$), we have

$$e^{\mathcal{L}_A \Delta t}(\mathcal{P}_1+\mathcal{P}_2) = \frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&e^{-\mathrm{i}\Omega\Delta t}&e^{-\mathrm{i}\Omega\Delta t}\\0&0&e^{+\mathrm{i}\Omega\Delta t}&e^{+\mathrm{i}\Omega\Delta t}\end{pmatrix}\,,$$

such that we can treat each block separately. The upper two-by-two block corresponds to a projector, such that we immediately find

$$\left[\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\right]^n = \frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}\,.$$

For the lower two-by-two block

$$M(\Omega\Delta t) = \frac{1}{2}\begin{pmatrix} e^{-i\Omega\Delta t} & e^{-i\Omega\Delta t} \\ e^{+i\Omega\Delta t} & e^{+i\Omega\Delta t} \end{pmatrix}$$

we obtain for two applications

$$M^2(\Omega\Delta t) = \frac{1}{4}\begin{pmatrix} 1 + e^{-2i\Omega\Delta t} & 1 + e^{-2i\Omega\Delta t} \\ 1 + e^{+2i\Omega\Delta t} & 1 + e^{+2i\Omega\Delta t} \end{pmatrix} = \cos(\Omega\Delta t)M(\Omega\Delta t)\,,$$

such that we can easily generalize $M^n(\Omega\Delta t) = \cos^{n-1}(\Omega\Delta t)M(\Omega\Delta t)$.