Chapter 9
Complex Numbers

2

$$z = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{j}\sin\frac{\pi}{4})$$

64

65

First, a little revision of the previous chapter! 2

What are the three most important facts about Taylor's and Maclaurin's series?

(1) ....

(2) ....

(3) ....

------------------▷ 3

You have just converted a complex number from its algebraic form $x + \mathrm{j}y$ into its polar form $r(\cos\phi + \mathrm{j}\sin\phi)$. We now try to formulate an algorithm for this conversion. 65

Conversion of a complex number of the form $z = x + \mathrm{j}y$ into the form $z = r(\cos\phi + \mathrm{j}\sin\phi)$:

Step 1: Conversion relationships:

$$r = \sqrt{x^2 + y^2}$$
$$\tan\phi = \frac{y}{x}$$

Insert the given values into these equations.

Step 2: Obtain the possible values of ϕ using tables or a calculator.

Step 3: Select the value of ϕ which corresponds to the given complex number by considering its position in the complex plane.

Step 4: Substitute the values of r and ϕ in the equation:

$$z = r(\cos\phi + \mathrm{j}\sin\phi)$$

------------------▷ 66

3

(1) A function $f(x)$ can be expanded as a power series of the form $a_0 + a_1 x + a_2 x^2 + \ldots$ (or equivalent).

(2) The coefficients of the power series can be obtained if we know the derivatives of $f(x)$:

$$a_n = \frac{f^{(n)}}{n!}$$

(3) Power series enable us to derive approximations for many functions; this is helpful in calculations.

--------------------▷ 4

66

A further example:

Put

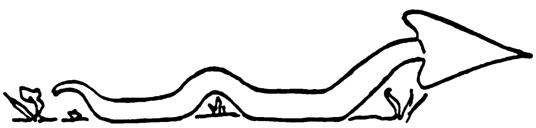
$$z = -1 + \mathrm{j} \text{ in the form}$$

$$z = r(\cos\phi + \mathrm{j}\sin\phi)$$

$$r = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\phi = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$z = \ldots\ldots\ldots\ldots\ldots\ldots$$



--------------------▷ 67

9.1 Definition and Properties of Complex Numbers

4

Objective: Concepts of imaginary numbers, complex numbers, real part, imaginary part; addition, subtraction, multiplication and division of complex numbers.

Complex numbers are helpful in the solution of differential equations, particularly in connection with problems involving oscillations. If you already know about complex numbers then read in the textbook those parts of the topic which are new to you.

READ: 9.1 Definition and properties of complex numbers
Textbook pages 249–252

-------------------▷ 5

$$r = \sqrt{2},\ \phi = \frac{3}{4}\pi$$

$$z = \sqrt{2}(\cos\frac{3}{4}\pi + \mathrm{j}\sin\frac{3}{4}\pi)$$

67

Correct -------------------▷ 69

Wrong, detailed solution -------------------▷ 68

Which of the following numbers are imaginary? 5

☐ j^2 ⟶ 6

☐ $4j$ ⟶ 8

☐ $4+4j$ ⟶ 7

68

$$z = -1 + j$$

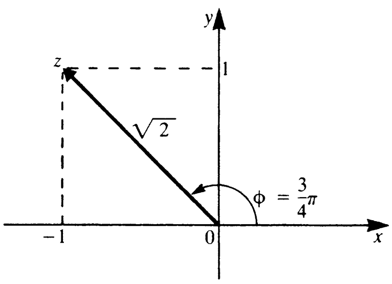
Given: $x = -1$, $y = 1$
Required: r and ϕ

Step 1: $r = \sqrt{x^2 + y^2} = \sqrt{2}$
$\tan\phi = -1$

Step 2: $\phi = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$

Step 3: $P(z)$ lies in the second quadrant, i.e. ϕ lies between $\frac{\pi}{2}$ and π, therefore $\phi = \frac{3}{4}\pi$.

Step 4: $z = \sqrt{2}(\cos\frac{3}{4}\pi + j\sin\frac{3}{4}\pi)$



⟶ 69

Wrong! j is imaginary but $j^2 = -1$ is real. 6

Try again; which of the following numbers is imaginary?

☐ 4j --------------------▷ 8

☐ $4+4j$ --------------------▷ 7

The conversion of a complex number z from the form $x + jy$ into the form $r(\cos\phi + j\sin\phi)$ (and the converse) is not difficult; the only subtlety is the correct determination of the angle ϕ. 69

Now some advice on study techniques for effective revision of study material and some basic results of experiments on memory.

Shortcut --------------------▷ 75

Experiments on memory were first carried out by Ebbinghaus a hundred years ago. First some material is given to be memorised (meaningless syllables, poems, series of numbers, terms, definitions or mathematical statements).

After a certain period of time the degree of retention of the material is investigated. There are different methods:

Unaided recall: The experimental subject must recall what he has remembered without assistance.

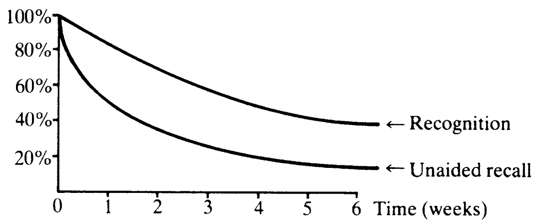
Recognition: Here we measure the percentage of the original material that is recognised.

--------------------▷ 70

Almost correct, but not quite: $4 + 4j$ is a complex number because it consists of a non-zero real part and a non-zero imaginary part. 7

------------------▷ 8

Results of experiments on memory (meaningless syllables) 70



The diagram shows the so-called 'retention curves' which express as a percentage the temporal decrease of the original memory content.

Curves result which are roughly exponentially decreasing. Most facts that one recognises while reading cannot be actively reproduced. Recognition of facts gives one the false impression of possessing a higher degree of knowledge. This can be seen in all situations in which you are required to present facts without any help, and to apply your knowledge. This was mentioned previously in the study guide for Chapter 8.

------------------▷ 71

Correct! 8

Now simplify j^4

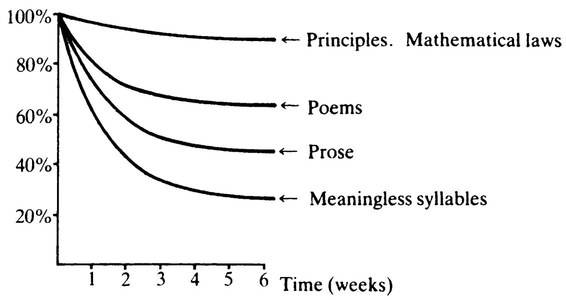
$j^4 = \dots\dots\dots\dots$

Solution found --------------------▷ 10

Explanation wanted --------------------▷ 9

The availability of memory content depends on the structuring of the subject content. Material that is well learned, and understood in its context, remains at your disposal for a longer time. 71

The diagram shows retention curves for different subject matter which has been learned thoroughly. It is clear that it is advantageous always to think of the learned material *in its context*.



--------------------▷ 72

$j^2 = -1$, by definition

Hence $j^4 = j^2 j^2 = (-1)(-1) = 1$

9

Now try

$$j^{12} = \text{..............................}$$

--------------------▷ 10

Effect on retention of one's attitude towards learning:

72

Lewin (1963) reports the following experiment:

A student was asked to read mnemonic material to his fellow students until they were able to reproduce it. Afterwards, the student who had presented the material was invited to present the text unaided. Unlike the other students, he remembered almost nothing.

Thus, while studying, it is always advantageous to decide what is worth retaining in order to extract or at least to underline it.

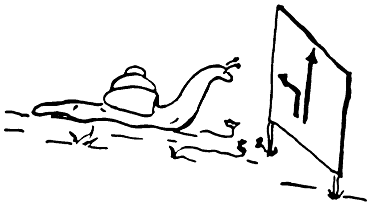
--------------------▷ 73

1 | 10

To simplify powers of j we always use the following procedure: We split the given power of j into a number of factors, each factor being $j^2 = -1$.

$j^n = (j \times j)(j \times j)..... = (-1)(-1)...$; possibly one factor j remains unmatched.

Aside: Such a method leads inevitably to the solution of the problem. It is referred to as an 'algorithm'. If we can find an algorithm then the solution of a particular problem is in our pocket.



Compute j^5

$$j^5 = \text{..............................}$$

-------------------▷ 12

Explanation required

-------------------▷ 11

Revision within the framework of the course: | 73

We have already mentioned revision at the end of a chapter before a break, and revision after a week before starting a new chapter.

These forms of revision should be supplemented by additional systematic revisions after a longer time interval. Therefore you should make out a plan for your revision. This plan may imply that while you are busy working through an advanced chapter you are also revising a chapter you worked on four weeks previously.

-------------------▷ 74

Let us apply the algorithm: 11

We split the powers of j into a number of factors, each factor being $j^2 = -1$.

Example: $j^7 = (j \times j)(j \times j)(j \times j)j = (-1)(-1)(-1)j = -j$

$$j^5 = \ldots\ldots\ldots\ldots\ldots\ldots$$

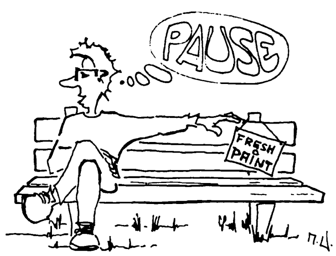
-------------------▷ 12

Each revision should be done in stages: 74

Step 1: You write down, from memory, terms introduced in the chapter. Compare this list with your notes or the textbook and expand it.

Step 2: You try to recall the meaning of the terms without help. Check these against the text and your notes. Incorrectly reproduced meanings, as well as those terms which in the beginning are not remembered, must be learned again.

Step 3: You work out the exercises of the chapter.



These revision techniques *are* useful — but only for those who apply them.
Now it is time for a break.

-------------------▷ 75

$$j^5 = j$$

12

Calculate $\sqrt{-9} = $ ....

--------------------▷ 14

Explanation required

--------------------▷ 13

9.3 Exponential Form of the Complex Number

75

Objective: Concepts of Euler's formula, exponential form of cosine and sine functions.

READ: 9.3 Exponential form of complex numbers
Textbook pages 256–263

--------------------▷ 76

Here is the sequence of operations for $\sqrt{-16}$: 13

$$\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4\mathrm{j}$$

Now try again

$$\sqrt{-9} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 14

Express the complex number z in exponential form: 76

$$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 77

$$\sqrt{-9} = 3\mathrm{j}$$

14

Extracting the root of a negative number always follows the same algorithm.

$$\sqrt{-a^2} = \sqrt{a^2(-1)} = \sqrt{a^2}\sqrt{-1} = a\mathrm{j}$$

Evaluate

$$\sqrt{-b^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sqrt{-9} + \sqrt{-25} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 15

$$z = r\mathrm{e}^{\mathrm{j}\alpha}$$

77

There exists a mathematical relationship between the exponential function $\mathrm{e}^{\mathrm{j}\alpha}$ and the cosine and sine functions. Can you name it?
Can you give the formula?

$$\mathrm{e}^{\mathrm{j}\alpha} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 78

$$bj$$
$$3j + 5j = 8j$$

15

Simplify

$$\sqrt{-18} + \sqrt{-50} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 16

Euler's formula

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

78

The expressions for cos α and sin α in exponential form are

$$\cos\alpha = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sin\alpha = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

These relationships are very important; for example, in the study of oscillations (Chapter 10).

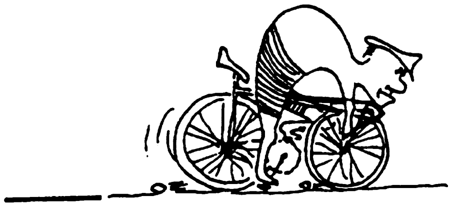
-------------------▷ 79

$$8\sqrt{2}\mathrm{j}$$

16

Correct

17



Error: follow the solution below:

$$\begin{aligned}\sqrt{-18}+\sqrt{-50} &= \sqrt{(-1\times 9\times 2)}+\sqrt{(-1)25\times 2}\\ &= 3\sqrt{2}\mathrm{j}+5\sqrt{2}\mathrm{j}\\ &= 8\sqrt{2}\mathrm{j}\end{aligned}$$

17

$$\cos\alpha = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\alpha}+\mathrm{e}^{-\mathrm{j}\alpha}\right)$$
$$\sin\alpha = \frac{1}{2}\left(\mathrm{e}^{\mathrm{j}\alpha}-\mathrm{e}^{-\mathrm{j}\alpha}\right)$$

79

What is the complex conjugate number of $z = r\mathrm{e}^{\mathrm{j}\alpha}$?

$z^* =$ ....

80

Simplify these expressions 17

(a) $\sqrt{-2}\sqrt{-8} =$ ....

(b) $\dfrac{\sqrt{-6}}{\sqrt{3}} =$ ....

(c) $\dfrac{1}{(-\mathrm{j})^3} =$ ....

--------------------▷ 18

$$z^* = r\mathrm{e}^{-\mathrm{j}\alpha}$$ 80

Correct

--------------------▷ 81

The complex conjugate is obtained, very simply, replacing j by $-$j. You saw this before; it is the definition of the complex conjugate.

--------------------▷ 81

(a) -4
(b) $\sqrt{2}j$
(c) $-j$

18

Correct ⟶ 20

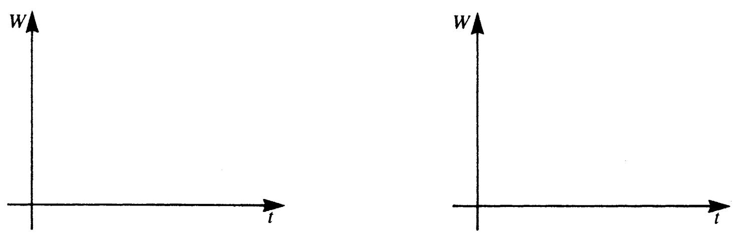
Errors; detailed solution required ⟶ 19

Have a look at the expression

81

$$W(t) = e^{(\sigma t + j\omega t)}$$
$$= e^{\sigma t}(\cos\omega t + j\sin\omega t)$$

What do the graphs of the real part look like when
(a) $\sigma > 0$?
and (b) $\sigma < 0$?



(a) $\sigma > 0$ (b) $\sigma < 0$

⟶ 82

19

(a)
$$\sqrt{-2}\sqrt{-8} = \sqrt{2(-1)}\sqrt{2\times 4(-1)}$$
$$= \sqrt{2}\mathrm{j} \times \sqrt{2} \times 2\mathrm{j}$$
$$= 2 \times 2\mathrm{j}^2 = -4$$

(b)
$$\frac{\sqrt{-6}}{\sqrt{3}} = \frac{\sqrt{3\times 2(-1)}}{\sqrt{3}} = \frac{\sqrt{3\times 2}}{\sqrt{3}}\mathrm{j} = \sqrt{2}\mathrm{j}$$

(c)
$$\frac{1}{(-\mathrm{j})^3} = \frac{1}{(-1)^3\mathrm{j}^3} = \frac{1}{-1(-j)}$$
$$= \frac{1}{\mathrm{j}} = \frac{\mathrm{j}}{\mathrm{j}\times\mathrm{j}} = -\mathrm{j}$$

Alternatively,
$$\frac{1}{(-\mathrm{j})^3} = \frac{(-\mathrm{j})}{(-\mathrm{j})^3(-\mathrm{j})} = -\mathrm{j}\frac{1}{(-1)^4}(-\mathrm{j})^4 = -\mathrm{j}\times 1$$

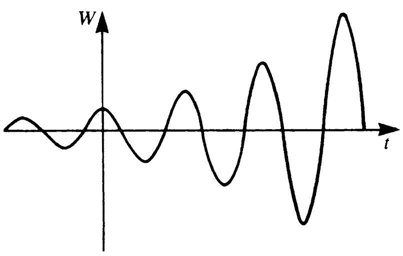
If you still have problems go back to the textbook or ask one of your fellow students.

20

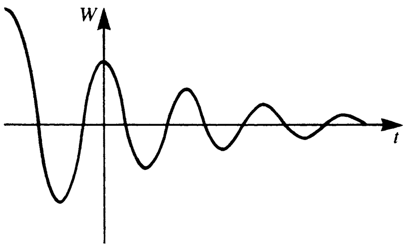
82

The real part is $e^{\sigma t} \cos \omega t$

When $\sigma > 0$ the graph represents an oscillation whose amplitude is increasing with time.



When $\sigma < 0$ the graph represents a damped oscillation.



83

The general form of a complex number is 20

$$z = x + \mathrm{j}y$$

x is called: ....
y is called: ....

--------------------▷ 21

Given: two complex numbers 83

$$z_1 = 2\mathrm{e}^{\mathrm{j}\pi}$$
$$z_2 = 2\mathrm{e}^{\mathrm{j}2\pi/3}$$

$$z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 85

Hints --------------------▷ 84

x is called the real part
y is called the imaginary part

21

Given the complex number

$$z = 3 + 4\mathrm{j}$$

what is the complex conjugate of z?

$$z^* = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 22

To multiply two complex numbers we multiply the moduli (the magnitudes), and add the arguments (the angles).

84

If

$$z_1 = 2\mathrm{e}^{\mathrm{j}\pi}$$
$$z_2 = 1\mathrm{e}^{2\mathrm{j}\pi}$$
$$z_1 z_2 = 2 \times 1\,\mathrm{e}^{\mathrm{j}\pi + \mathrm{j}2\pi} = 2\mathrm{e}^{\mathrm{j}3\pi}$$

Now try

If

$$z_1 = 2\mathrm{e}^{\mathrm{j}\pi}$$
$$z_2 = 2\mathrm{e}^{\mathrm{j}2\pi/3}$$
$$z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 85

$$z^* = 3 - 4j$$

22

The complex conjugate is obtained by replacing j with −j.

23

$$z_1 z_2 = 2 \times 2e^{j(2\pi/3+\pi)} = 4e^{j(5\pi/3)}$$

85

Remember: multiply the moduli and add the arguments.

Given: $z_1 = 2e^{\pi j/3}$ and $z_2 = 2e^{-2\pi j/3}$

Obtain $\dfrac{z_1}{z_2} = $ ....

86

Compute the sum of the following complex numbers: 23

$$z_1 = 1 + \mathrm{j}$$
$$z_2 = -3 - \mathrm{j}$$

$$z_1 + z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 24

$$\frac{z_1}{z_2} = \mathrm{e}^{\pi \mathrm{j}}$$

86

Correct

--------------------▷ 87

Wrong: follow the solution given below.

To divide two complex numbers you divide the moduli (the magnitudes) and subtract the arguments (the angle). Thus

$$\frac{z_1}{z_2} = \frac{2}{2}\mathrm{e}^{(\pi/3+2\pi/3)\mathrm{j}} = \mathrm{e}^{\pi \mathrm{j}}$$

--------------------▷ 87

24

-2

Here is how the solution is arrived at:

To obtain a sum of complex numbers we add the real parts and the imaginary parts separately:

$$z_1 = 1 + \mathrm{j}$$
$$z_2 = -3 - \mathrm{j}$$

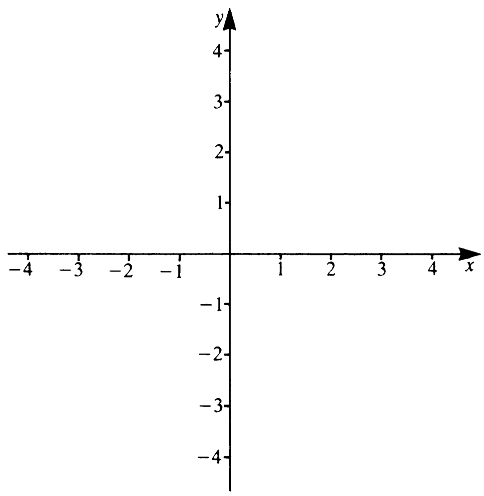
$$z_1 + z_2 = (1-3) + (1-1)\mathrm{j} = -2$$

Now try
$$z_3 = 7 + 3\mathrm{j}$$
$$z_4 = -9 - 3\mathrm{j}$$
$$z_3 + z_4 = \text{..............................}$$

--------------------▷ 25

87

Simplify the expression $4\mathrm{e}^{\pi \mathrm{j}}$ and show its position in the complex plane.



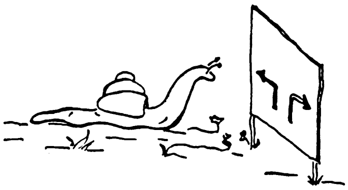
--------------------▷ 88

-2

25

Subtract $z_2 = 1 - j$ from $z_1 = 3 + j$

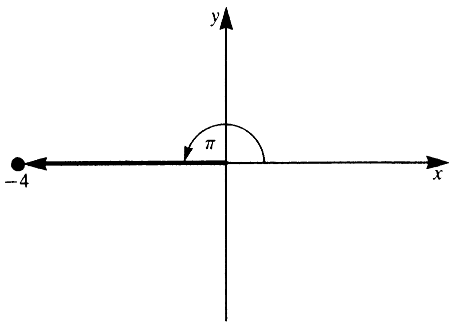
$$z_1 - z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found --------------------▷ 27

Detailed explanation --------------------▷ 26

88



$z = 4e^{\pi j} = -4$

Correct --------------------▷ 90

Wrong --------------------▷ 89

26

The problem was:

$$\text{if } z_1 = 3 + \mathrm{j}$$
$$\text{and } z_2 = 1 - \mathrm{j}$$

then $z_1 - z_2$ = difference of real parts + difference of imaginary parts
$= (3-1) \qquad + (+\mathrm{j} - (-\mathrm{j}))$
$= 2 + 2\mathrm{j}$

Now try again

$$z_3 = 5 + 4\mathrm{j}$$
$$z_4 = 3 + 2\mathrm{j}$$

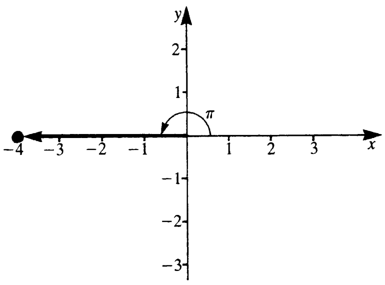
$$z_3 - z_4 = \ldots\ldots\ldots\ldots\ldots\ldots$$

27

89

$z = 4e^{\pi \mathrm{j}}$ has $r = 4$ for magnitude
and $\phi = \pi = 180°$ for the angle.
Hence the point $P(z)$ in the complex plane is obtained from the fact that the coordinates are

$$x = -4 \text{ and } y = 0$$



Since $z = r\cos\alpha + \mathrm{j}r\sin\alpha$
$x = r\cos\alpha = 4\cos 180° = -4$
and $y = r\sin\alpha = 4\sin 180° = 0$

Therefore $z = -4$.

90

$$2+2j$$

27

Multiplication of complex numbers.
Given:

$$z_1 = 3+5j$$
$$z_2 = 2+4j$$

$$z_1 z_2 = \text{..............................}$$

Solution → 29

Explanation wanted → 28

Given $z = 4e^{\pi j}$
Calculate $z^3 =$ ....

90

In case of difficulty refer to the textbook. → 91

28

The multiplication of two complex numbers is carried out in a very similar way to the multiplication of two binomial expressions. We must keep in mind that $j^2 = -1$.

Look at this new example:

$$\begin{aligned} z_1 &= (2 + j) \\ z_2 &= (1 - 2j) \\ z_1 z_2 &= (2 + j)\,(1 - 2j) \end{aligned}$$

Follow the arrows, first the top ones and then the bottom ones, and obtain

$$\begin{aligned} z_1 z_2 &= 2 - 4j + j + 2 \\ &= 4 - 3j \end{aligned}$$

Now try the original problem again:

$$\begin{aligned} z_1 &= 3 + 5j \\ z_2 &= 2 + 4j \\ z_1 z_2 &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

--------------------▷ 29

91

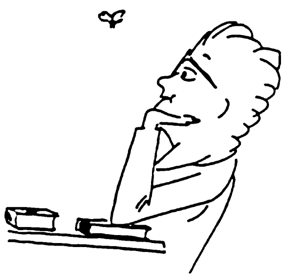
$$z^3 = 64e^{3\pi j} = -64$$

Here is the detailed solution:

$$z = 4e^{\pi j}, \quad z^3 = 4^3 e^{3\pi j} = 64 e^{3\pi j} = 64(-1)$$

Extract the square root of

$$\begin{aligned} z &= 4e^{\pi j} \\ \sqrt{z} &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$



--------------------▷ 92

$$-14+22j$$

29

Now for this problem:
given

$$z_1 = 1 + j$$
$$z_2 = 2 + 3j$$
$$z_3 = 1 - 4j$$

obtain the product $z_1 z_2 z_3 =$ ....

--------------------▷ 30

$$z = 2e^{\pi j/2}$$

92

Here is the detailed solution:

To extract the root of a complex number we find the root of the magnitude and divide the angle by the radical index. Thus:

$$\sqrt{z} = \sqrt{4}e^{j\pi/2} = 2e^{\pi j/2}$$

Given a complex number in the form

$$z = 1 + j$$

express it in the exponential form $z = re^{j\alpha}$

$$z = \text{...........................}$$

--------------------▷ 93

$$z_1 z_2 z_3 = 19 + 9\mathrm{j}$$

30

Correct ---> 32

Wrong, or explanation wanted ---> 31

$$z = \sqrt{2}\mathrm{e}^{\pi\mathrm{j}/4}$$

93

Correct ---> 94

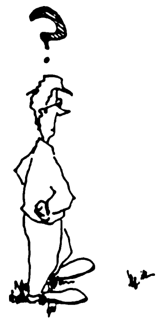
Wrong: follow the solution step by step:

Step 1: $x = 1$, $y = 1$
Step 2: $r = \sqrt{x^2 + y^2} = \sqrt{2}$
$\tan\alpha = \dfrac{y}{x} = 1$
Step 3: $\tan\alpha = 1$ gives $\alpha = \dfrac{\pi}{4}$ or $\dfrac{5}{4}\pi$
Step 4: $\mathrm{P}(z)$ lies in the first quadrant, hence $\alpha = \dfrac{\pi}{4}$
Step 5: The complex number in exponential form is
$z = \sqrt{2}\mathrm{e}^{\pi\mathrm{j}/4}$

---> 94

Since we have more than two factors we multiply in stages: 31

$$\begin{aligned} z_1 z_2 z_3 &= (1+\mathrm{j})(2+3\mathrm{j})(1-4\mathrm{j}) \\ &= (2+3\mathrm{j}+2\mathrm{j}+3\mathrm{j}^2)(1-4\mathrm{j}) \\ &= (-1+5\mathrm{j})(1-4\mathrm{j}) \\ &= -1+4\mathrm{j}+5\mathrm{j}-20\mathrm{j}^2 \\ &= 19+9\mathrm{j} \end{aligned}$$



--------------------▷ 32

Consider $z(\alpha) = r\mathrm{e}^{\mathrm{j}\alpha}$ 94
as a function of the angle α, as indicated by $z(\alpha)$.
What is the period of $z(\alpha)$?.....................

--------------------▷ 95

Obtain the product $z_1 z_1{}^*$,
given $z_1 = 4 + 2\mathrm{j}$.
z^* is called ....
$z_1^* =$ ....
$z_1 z_1^* =$ 32

33

$$2\pi$$ 95

$z(\alpha)$ has a period of 2π which means that

$$z(\alpha) = z(\alpha + 2\pi) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi)}$$

or
$$z(\alpha) = z(\alpha + 2\pi + 2\pi + \ldots) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi+2\pi+\ldots)}$$

and generally
$$z(\alpha) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi k)}$$

where $k = 1, 2, 3, \ldots$

96

z^* is called the complex conjugate of z 33

$$z^* = 4 - 2j$$

$$z_1 z_1^* = 20$$

Note: The product of a complex number and its complex conjugate is always a real number.

Correct --------------------▷ 35

Wrong; explanation required --------------------▷ 34

You must be exhausted! 96

You deserve a break!

Before you start your break recapitulate the most important aspects of what you have just learned, i.e. concepts, definitions, formulae.

Resume your work at the end of your timed break.

--------------------▷ 97

Follow the solution: 34

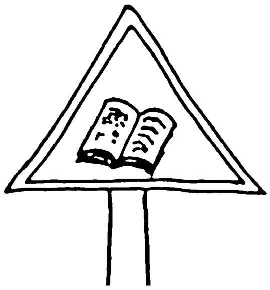
Required: $z_1 z_1^*$; given: $z_1 = 4 + 2\mathrm{j}$.

First we need $z_1^* = 4 - 2\mathrm{j}$, the complex conjugate.

The next step should be easy:

$$\begin{aligned} z_1 z_1^* &= (4+2\mathrm{j})(4-2\mathrm{j}) \\ &= 16 - 8\mathrm{j} + 8\mathrm{j} + 4\mathrm{j}^2 \\ &= 16 - 4(-1) = 16 + 4 = 20 \end{aligned}$$

A real number!



Remember that the product of a complex number and its conjugate is real.

------------------▷ 35

9.4 Complex Numbers Expressed in Polar Form

97

These operations are equivalent to those using the exponential form. Both forms are used in practice.

READ: 9.4 Operations with complex numbers expressed in polar form
Textbook pages 263–267

------------------▷ 98

Multiply $(2-3\mathrm{j})$ by a suitable factor in order to obtain a real number. 35

$(2-3\mathrm{j})(\ldots\ldots\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ a real number.

Hint: From the previous frames you should know that the product of conjugate complex numbers yields a real number. Hence the complex conjugate of $(2-3\mathrm{j})$ is $(2+3\mathrm{j})$; the rest is straightforward. You have done it before.

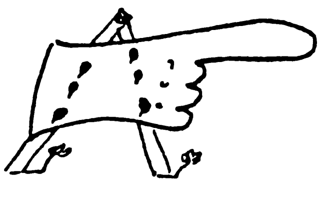
-------------------▷ 36

Multiplication of two complex numbers expressed in polar form. 98

Given: $z_1 = r_1(\cos\phi_1 + \mathrm{j}\sin\phi_1)$

and $z_2 = r_2(\cos\phi_2 + \mathrm{j}\sin\phi_2)$

then $z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



If you have difficulties go back to the textbook and solve this example while consulting section 9.4.1

-------------------▷ 99

$$(2-3\mathrm{j})(2+3\mathrm{j})=13$$

36

Given: $z_1 = 27+\sqrt{3}\,\mathrm{j}$ and $z_2 = 3\sqrt{3}$

What is z_1 divided by z_2?

$$\frac{z_1}{z_2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 37

$$z_1 z_2 = r_1 r_2[\cos(\phi_1+\phi_2)+\mathrm{j}\sin(\phi_1+\phi_2)]$$

99

Now for a numerical example.

Calculate $z = z_1 z_2$ if

$$z_1 = 4(\cos 30^\circ + \mathrm{j}\sin 30^\circ)$$
$$z_2 = 5(\cos 55^\circ + \mathrm{j}\sin 55^\circ)$$
$$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 100

$$\frac{z_1}{z_2} = \frac{27+\sqrt{3}\mathrm{j}}{3\sqrt{3}} = 3\sqrt{3} + \frac{1}{3}\mathrm{j}$$

37

Correct

--------------------▷ 38

Wrong? Here is the solution:

$$\frac{27+\sqrt{3}\mathrm{j}}{3\sqrt{3}} = \frac{27}{3\sqrt{3}} + \frac{\sqrt{3}\mathrm{j}}{3\sqrt{3}} = \frac{9}{\sqrt{3}} + \frac{1}{3}\mathrm{j}$$
$$= \frac{9\sqrt{3}}{3} + \frac{1}{3}\mathrm{j} = 3\sqrt{3} + \frac{1}{3}\mathrm{j}$$

The result is obtained by dividing the real part and the imaginary part by the real number, separately.

--------------------▷ 38

$$z = 20(\cos 85^\circ + \mathrm{j}\sin 85^\circ)$$

100

Correct

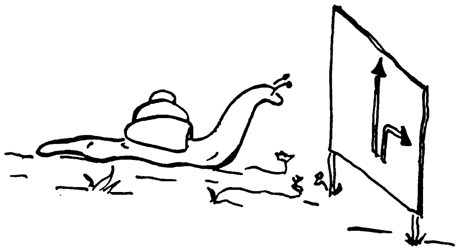
--------------------▷ 102

Wrong

--------------------▷ 101

Have a look at this: 38

$$\frac{(4-\sqrt{3}\mathrm{j})}{2\mathrm{j}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found --------------------▷ 40

Detailed solution wanted --------------------▷ 39

To multiply two complex numbers expressed in polar form we multiply their moduli and add their arguments. 101

Since $z_1 = 4(\cos 30° + \mathrm{j}\sin 30°) = r_1(\cos\phi_1 + \mathrm{j}\sin\phi_1)$
and $z_2 = 5(\cos 55° + \mathrm{j}\sin 55°) = r_2(\cos\phi_2 + \mathrm{j}\sin\phi_2)$

Then

$$z_1 z_2 == r(\cos\phi + \mathrm{j}\sin\phi)$$

where $r = r_1 r_2 = 4 \times 5 = 20$
and $\phi = \phi_1 + \phi_2 = 30° + 55° = 85°$

--------------------▷ 102

We transform the fraction to get a real denominator: 39

$$\frac{4-\sqrt{3}j}{2j} = \frac{4-\sqrt{3}j}{2j}\frac{j}{j}$$
$$= \frac{4j-\sqrt{3}j^2}{2j^2} = \frac{4j+\sqrt{3}}{-2} = \text{..............................}$$

------------------▷ 40

Write down the general expression for the quotient of two complex numbers, given 102

$$z_1 = r_1(\cos\phi_1 + j\sin\phi_1)$$

and $$z_2 = r_2(\cos\phi_2 + j\sin\phi_2)$$

$$z_3 = \frac{z_1}{z_2} = \text{..............................}\ldots = r_3\,(\cos\phi_3 + j\sin\phi_3)$$

Where $r_3 =$ ....
and $\phi_3 =$ ....

In case of difficulty consult the textbook

------------------▷ 103

$$-\frac{1}{2}\sqrt{3}-2\text{j}$$ 40

Given: $z_1 = 8+7\text{j}$ and $z_2 = 3+4\text{j}$
Obtain:

$$\frac{z_1}{z_2} = \text{..............................}$$

Solution --------------------▷ 42

Explanation --------------------▷ 41

$$z_3 = \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\phi_1-\phi_2)+\text{j}\sin(\phi_1-\phi_2)\right]$$ 103

hence $$r_3 = \frac{r_1}{r_2},\ \phi_3 = \phi_1-\phi_2$$

Let's have a numerical example.
Given: $$z_1 = 6(\cos 72^\circ + \text{j}\sin 72^\circ)$$

$$z_2 = 3(\cos 20^\circ + \text{j}\sin 20^\circ).$$

Obtain: $$z_3 = \frac{z_1}{z_2} = \text{...............................}$$

--------------------▷ 104

To divide two complex numbers we first convert the denominator to a real number. This is achieved by multiplying the denominator by its conjugate, and of course the numerator is multiplied by the number also. 41

We then have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{z_1 \times z_2^*}{z_2 \times z_2^*} = \frac{(8+7j)}{(3+4j)}\frac{(3-4j)}{(3-4j)} = \frac{(8+7j)(3-4j)}{9+16}\\ &= \frac{24-28j^2-32j+21j}{25}\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

--------------------▷ 42

$$z = \frac{z_1}{z_2} = 2(\cos 52^\circ + j\sin 52^\circ)$$ 104

Correct

--------------------▷ 105

Mistake:

To divide two complex numbers expressed in polar form we divide their moduli and subtract their arguments.

Now do it!

--------------------▷ 105

$$2.08 - 0.44j$$ 42

Now it is time for a short break.

→ 43

Let us now move on to the problem of raising a complex number to a power n. 105

Can you write the result down now without referring to any notes?

$$z^n = [r(\cos\phi + j\sin\phi)]^n = \text{............................}$$

Write down the rule in you own words.

→ 106

9.2 Graphical Representation of Complex Numbers

43

Objective: Concepts of the Argand diagram, modulus and argument of a complex number, polar form of a complex number.

READ: 9.2 Graphical representation of complex numbers
Textbook page 252–255

--------------------▷ 44

$$z^n = [r(\cos\phi + \mathrm{j}\sin\phi)]^n = r^n(\cos n\phi + \mathrm{j}\sin n\phi)$$

106

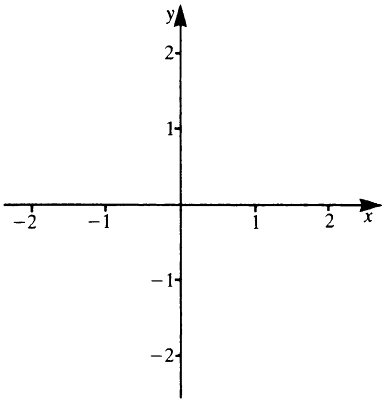
Rule: To raise a complexnumber to a power n we raise the modulus to that power and multiply the argument by that power.

If you had any difficulty you should read section 9.4.2 of the textbook.

If we set $r = 1$, we obtain De Moivre's theorem, namely

$$(\cos\phi + \mathrm{j}\sin\phi)^n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 107



Indicate on the diagram the position of the point $P(z)$ which belongs to the complex number 44

$$z = 1 - 2j$$

-------------------▷ 45

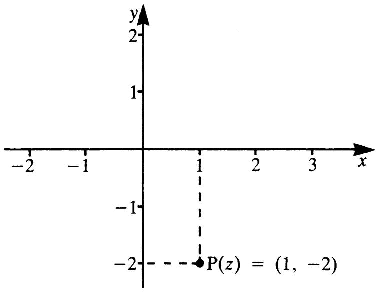
$$(\cos\phi + j\sin\phi)^n = \cos n\phi + j\sin n\phi$$ 107

Example:

$$z = (\cos 20^\circ + j\sin 20^\circ)^6$$

then $z =$ ....

-------------------▷ 108



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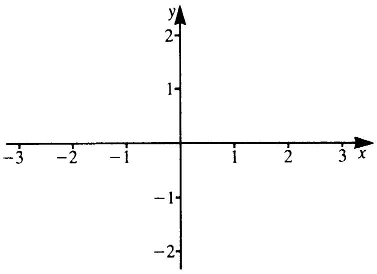


Indicate on the diagram the following complex numbers

$$z_1 = 1 + j$$
$$z_2 = -2 + j$$
$$z_3 = -2j$$

--------------------▷ 46

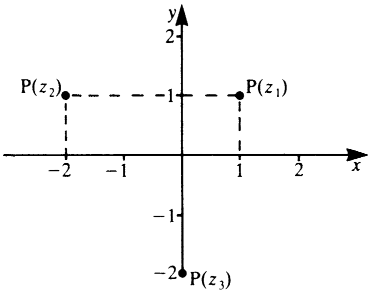
$$z = \cos 120^\circ + j \sin 120^\circ$$

108

Calculate:

$$z = \left(1 - j\sqrt{3}\right)^5 = \text{..........................}$$

--------------------▷ 109



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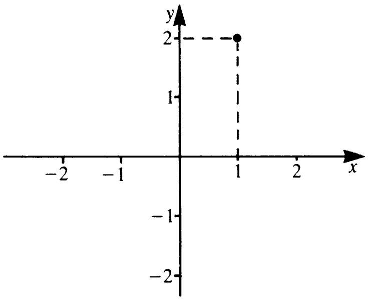


What are the real part x and the imaginary part y of the complex number z shown in the diagram?

$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 47

$$z = 32\,(\cos 60^\circ + \mathrm{j}\sin 60^\circ) = 16 + 16\sqrt{3}\,\mathrm{j}$$

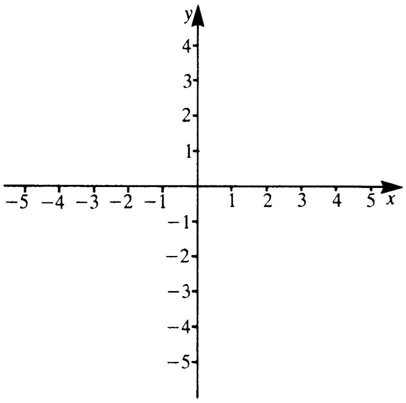
109

Correct --------------------▷ 111

Wrong, or detailed solution wanted --------------------▷ 110

$z = 1 + 2j$

47



Obtain the sum of $z_1 = 5 + 2j$ and $z_2 = -3 - 3j$ graphically. Remember the rules of geometric addition and subtraction of vectors. What is the new complex number z?

$z =$ ....

------------------▷ 48

The problem is to calculate $z = \left(1 - j\sqrt{3}\right)^5$

110

To solve it we must first convert it into a polar form, since the rules for raising a complex number to a power are based on their polar form. We have $z = x + jy$; therefore $x = 1$, $y = -\sqrt{3}$.

In polar form: $r = \sqrt{1+3} = 2$

$\tan\phi = -\sqrt{3} = 300°$ (fourth quadrant)

The given complex number in polar form is

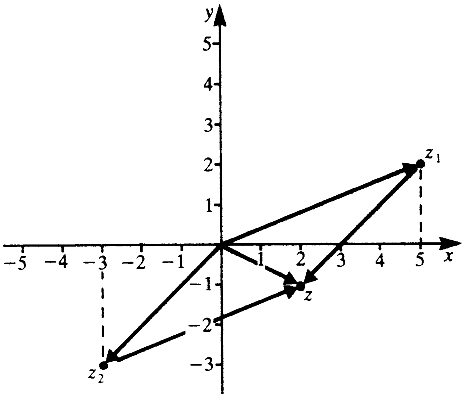
$z = 2(\cos 300° + j\sin 300°)$

When raised to the fifth power

$$\begin{aligned} z^5 &= 2^5(\cos(5 \times 300°) + j\sin(5 \times 300°)) \\ &= 32(\cos 60° + j\sin 60°) \\ &= 32\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 16 + 16\sqrt{3}j \end{aligned}$$

------------------▷ 111

48



$z = 2 - j$

Correct → 50

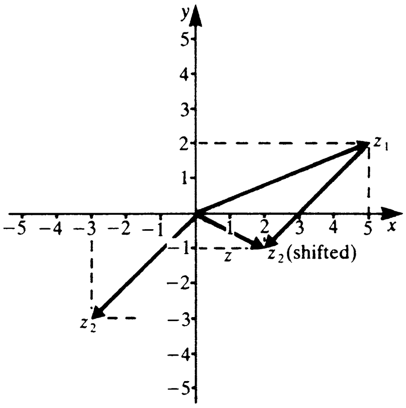
Wrong, I have difficulties → 49

De Moivre's theorem holds true for positive, negative and fractional powers.

Give the general formula for extracting the n roots of a complex number: 111

$$\sqrt[n]{x+jy} = \text{..............................}$$

In case of difficulty look it up in your textbook (sections 9.4.2 and 9.4.3). → 112



49

Given: $z_1 = 5 + 2j$
$z_2 = -3 - 3j$
What is $z = z_1 + z_2$ graphically?

The solution is obtained by the following steps: We regard the complex numbers as vectors. We obtain a sum by adding $\mathbf{z_1}$ and $\mathbf{z_2}$ geometrically. Note that the addition of vectors means adding the components. Now the x-components are the real parts and the y-components are the imaginary parts.

Note: We could have obtained the same result by shifting vector $\mathbf{z_1}$ instead of $\mathbf{z_2}$. Check it for yourself!

------------------▷ 50

112

$$\sqrt[n]{x + jy} = \sqrt[n]{r}\left[\cos\left(\frac{\phi}{n} + \frac{360° k}{n}\right) + j\sin\left(\frac{\phi}{n} + \frac{360° k}{n}\right)\right]$$

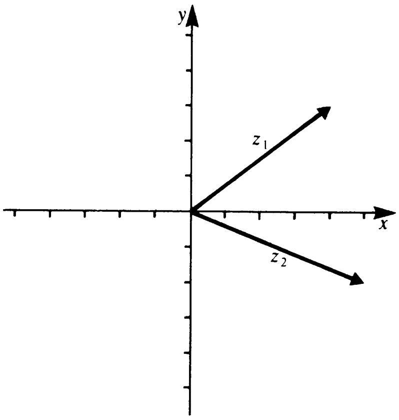
$k = 0, \pm 1, \pm 2, \pm 3$

k is a whole number in the formula.

What does it account for?

k accounts for the fact that ....
....

------------------▷ 113



Obtain graphically the difference

50

$$\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$$

given that

$$\mathbf{z}_1 = 4 + 3\mathrm{j}$$

and

$$\mathbf{z}_2 = 5 - 2\mathrm{j}$$

Hence $\mathbf{z} =$ ....

------------------▷ 51

---

k accounts for the fact that we obtain n roots by considering the periodicity of the trigonometric functions.

113

A numerical example should clarify the matter.

Obtain the three solutions of

$$z = \sqrt[3]{\cos 135° + \mathrm{j}\sin 135°}$$

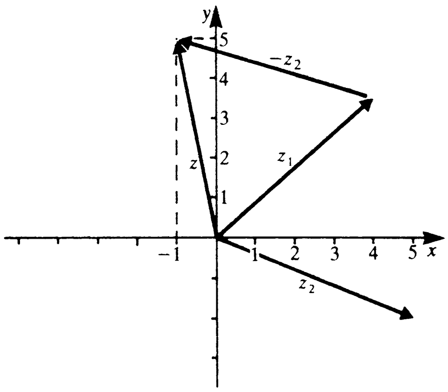
$z_1 =$ ....

$z_2 =$ ....

$z_3 =$ ....

------------------▷ 114

51



$\mathbf{z} = -1 + 5j$

Correct 53

Wrong 54

114

$z_1 = 0.707 + 0.707j$
$z_2 = -0.966 + 0.259j$
$z_3 = 0.259 - 0.966j$ to 3 decimal places

Correct: splendid! 116

Wrong: too bad, don't despair 115

You might find it useful to revise basic facts on vectors in a plane.
Read Chapter 3 of the textbook again, especially section 3.3. 52

------------------▷ 53

To calculate 115

$$z = \sqrt[3]{\cos 135^\circ + \mathrm{j}\sin 135^\circ}$$

we apply the general rule:

$$\begin{aligned}\sqrt[n]{x+\mathrm{j}y} &= \sqrt[n]{r(\cos\phi + \mathrm{j}\sin\phi)}\\ &= \sqrt[n]{r}\left[\cos\left(\frac{\phi}{n} + \frac{360^\circ k}{n}\right) + \mathrm{j}\sin\left(\frac{\phi}{n} + \frac{360^\circ k}{n}\right)\right]\end{aligned}$$

To get three roots we take $k = 0, 1, 2$.

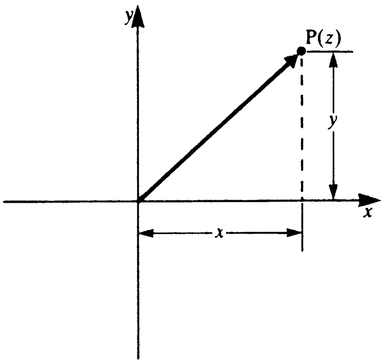
Applying the formula to our example gives

$$z = \sqrt[3]{1}\,[\cos(45^\circ + 120^\circ k) + \mathrm{j}\sin(45^\circ + 120^\circ k)]$$

and the roots are

$$\begin{aligned}k = 0, z_1 &= \cos 45^\circ + \mathrm{j}\sin 45^\circ = 0.707 + 0.707\mathrm{j}\\ k = 1, z_2 &= \cos 165^\circ + \mathrm{j}\sin 165^\circ = -0.966 + 0.259\mathrm{j}\\ k = 1, z_3 &= \cos 285^\circ + \mathrm{j}\sin 285^\circ = 0.259 - 0.966\mathrm{j}\end{aligned}$$

------------------▷ 116



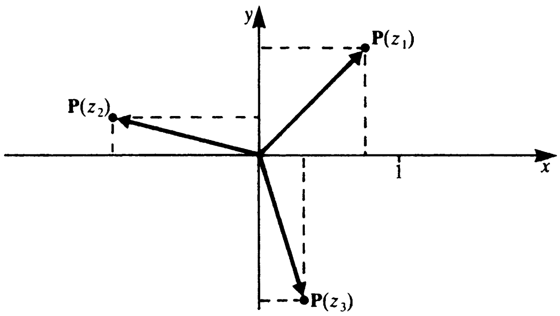
Given the complex number $z = x + \mathrm{j}y$ as shown. 53

The point $\mathrm{P}(z)$ can also be fixed using polar coordinates r and ϕ.

Fill in r and ϕ.

------------------▷ 54

The result of the previous example can be displayed on the Argand diagram as shown 116



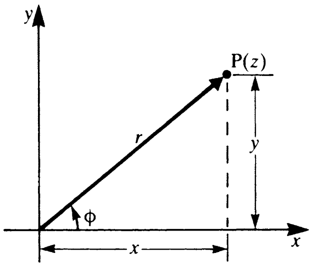
You should note that the argument increases by $\dfrac{360°}{3} = 120°$.

The three roots represented by the three vectors $\mathbf{P}(z_1)$, $\mathbf{P}(z_2)$ and $\mathbf{P}(z_3)$ form an equilateral triangle and their tips lie on a circle of radius 1.

What is the principal root of

$$\sqrt[n]{x + \mathrm{j}y} = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 117



54

Let z be given in the forms (a) $z = x + \mathrm{j}y$
and (b) $z = r(\cos\phi + \mathrm{j}\sin\phi)$

What are the relationships between x and y, and r and ϕ?

$x = \ldots\ldots\ldots\ldots\ldots\ldots$ $y = \ldots\ldots\ldots\ldots\ldots\ldots$

Can you answer from memory?

☐ Yes --------------------▷ 56

☐ No --------------------▷ 55

$$\sqrt[n]{x + \mathrm{j}y} = \sqrt[n]{r}\left(\cos\frac{\phi}{n} + \mathrm{j}\sin\frac{\phi}{n}\right)$$

117

One more example:

Obtain the three roots of unity i.e. $\sqrt[3]{1}$.

$z_1 = \ldots\ldots\ldots\ldots\ldots\ldots$
$z_2 = \ldots\ldots\ldots\ldots\ldots\ldots$
$z_3 = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution --------------------▷ 119

Hints and detailed solution --------------------▷ 118

Look it up in the textbook in section 9.2.2 or in the table of formulae. 55

Now express x and y in terms of r and ϕ.

$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 56

To obtain the solution we proceed in the usual way, and don't be put off by the fact that our number is a real one! 118

In this case $x = 1$, $y = 0$
hence $r = 1$, $\tan\phi = 0°$, $\phi = 0°$

Applying the formula yields

$$\sqrt[3]{1} = \cos\left(\frac{0°}{3} + \frac{360°k}{3}\right) + \mathrm{j}\sin\left(\frac{0°}{3} + \frac{360°k}{3}\right)$$

$$= \cos 120°k + \mathrm{j}\sin 120°k$$

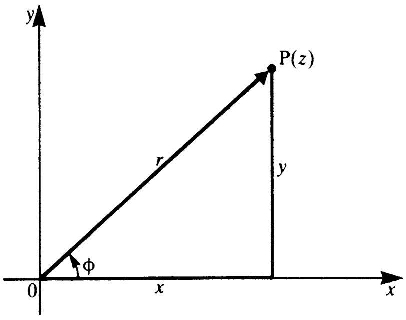
The roots are:

$k = 0$: $z_1 = \cos 0° + \mathrm{j}\sin 0° = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (Principal value)
$k = 1$: $z_2 = \cos 120° + \mathrm{j}\sin 120° = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$k = 2$: $z_3 = \cos 240° + \mathrm{j}\sin 240° = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 119

$$x = r\cos\phi$$
$$y = r\sin\phi$$

56



Express r and ϕ in terms of x and y.
$r = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$\phi = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 57

$$z_1 = 1$$
$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{j} \qquad z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{j}$$

119

Now at the end of this chapter, some remarks on problem solving.

Problem solving requires

— knowledge acquired from previous chapters and sections

— the application of simple problem-solving methods

Here is a problem:

A satellite travels in a circular path round the Earth at a constant speed and at an altitude of 1700 km, measured from the Earth's surface.

The time needed to complete the orbit is $T = 2\,\mathrm{h}$.

The radius of the Earth is 6400 km.

For how long can an observer see this satellite above the horizon?

Before you attempt the problem, please read the general comments on the following pages.

------------------▷ 120

$$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

57

Given the complex number

$$z = a^2 + (b+c)\mathrm{j}$$

a, b and c are real. Express z in polar form:

$r =$ ....

$\phi =$ ....

------------------▷ 58

Solving a problem can be attempted in the following way:

120

Step 1: Analysis of the situation:
The given facts are transferred either into a drawing or into formulae and relations. For our problem, you are advised to make a drawing and to consider carefully which quantities are known.

Step 2: Analysis of the aim:
You attempt to formulate exactly which quantities you would like to know.

Step 3: Solution of the problem:
You try to eliminate the discrepancy between the values you know and those you don't know. An important stage in the process is to look for intermediate values which can be calculated from the known ones, and with the help of which you can deduce the values sought.

Step-by-step calculation of the solution ------------------▷ 121

The solution — in case you need no help ------------------▷ 124

$$r = \sqrt{a^4 + (b+c)^2}$$
$$\phi = \tan^{-1}\left(\frac{b+c}{a^2}\right) \text{ or } \tan\phi = \frac{b+c}{a^2}$$ 58

Correct -------------------▷ 60

Wrong; detailed explanation -------------------▷ 59

121

1. Analysis of the situation:
Very often a sketch reduces the given information to the essential data. Draw a section through the part of the satellite path visible to the observer. Joining of the visible part of the satellite's path with the centre of the Earth forms the angle α.
2. Analysis of the aim:
We are looking for the time T_s, which the satellite needs to pass through the section of the satellite path visible to an observer.
3. Solution of the problem:
We can find intermediate values like travel speed, v and angle α of the known values, from which we can calculate the values we are looking for.

If this is already sufficient advice for you -------------------▷ 124

Now try to draw out a sketch -------------------▷ 122

Given: $z = a^2 + (b+c)\text{j}$ 59

Compare with the standard form:

$$z = x + \text{j}y$$

It follows that $x = a^2$ and $y = b + c$;

therefore
$$r = \sqrt{x^2 + y^2} = \sqrt{a^4 + (b+c)^2}$$

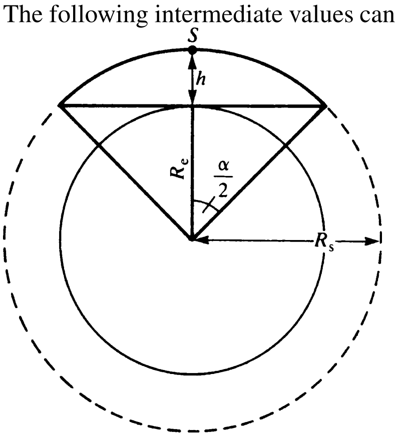
and
$$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{b+c}{a^2}\right)$$

or
$$\tan\phi = \frac{b+c}{a^2}$$

-------------------▷ 60

In the sketch the visible part of the satellite path is S. 122

The following intermediate values can be directly determined:



Speed of travel: from the time needed for orbit and the radius of the satellite's path. The radius of the course is made up of the height above the Earth plus the Earth's radius.

The length of the visible part of the satellite's course: from the radius of the satellite's course and angle α.

Angle α: from the Earth's radius and the radius of the satellite's path.

The problem is not completely explained here.

Detailed solution -------------------▷ 123

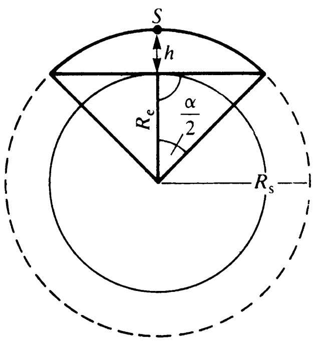
Solution -------------------▷ 124

Given: $z = 1 + \mathrm{j}$
Express z in the polar form $r(\cos\phi + \mathrm{j}\sin\phi)$ 60

☐ Can you do it straight away? --------------------▷ 64

☐ Do you need help? --------------------▷ 61

Given values: 123



Radius of Earth $R_e = 6400\,\text{km}$

Altitude of satellite $h = 1700\,\text{km}$

Time of orbit $T = 2\,\text{h} = 120\,\text{min}$

Intermediate values:

Speed of satellite: $v = \dfrac{2\pi\, R_s}{T}$

Length of visible part: $S = \dfrac{\alpha}{360^\circ} 2\pi R_s$ since $\dfrac{\alpha}{360^\circ} = \dfrac{S}{2\pi R_s}$

Angle α: $\cos\dfrac{\alpha}{2} = \dfrac{R_e}{R_s} = \dfrac{6400}{8100} = 0.7901$

$$\frac{\alpha}{2} = 38^\circ \quad \alpha = 76^\circ$$

Time of observation: $T_s = \dfrac{\alpha}{360^\circ} T$ or $T_s = \dfrac{S}{v}$

$T_s = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 124

We want the expression $z = r(\cos\phi + j\sin\phi)$ 61

Since $z = 1 + j$

then $x = 1$ and $y = 1$

Hence $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

and $\tan\phi = \frac{y}{x} = \frac{1}{1} = 1$

$\tan\phi = 1$ occurs for values of $\phi = \frac{\pi}{4}$ and $\frac{5}{4}\pi$

We must determine to which quadrant our complex number belongs.

$\phi = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 62

Time of observation: $T_s = 0.422\,\text{h} = 25$ min 124

--------------------▷ 125

$$\phi = \frac{\pi}{4}$$

62

Correct?

Good; write down the complex number in polar form, knowing that

$$r = \sqrt{2},\ \phi = \frac{\pi}{4}$$

$$z = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 64

Wrong, explanation wanted

--------------------▷ 63

125

n.u. of Chapter 9

To determine the correct value for the angle you should refer to the position of z in the complex plane. In our example, since $z = 1 + 1j$, i.e. $x = +1$ and $y = +1$, it follows that it lies in the first quadrant. 63

Therefore $0 < \phi < \frac{\pi}{2}$; in our case $\phi = \frac{\pi}{4}$.

Hence $z = \sqrt{2}(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4})$

64

Please continue on page 1
(bottom half)