Chapter 2
Vector Algebra II: Scalar and Vector Products

--------------------▷ 2

$\mathbf{a} \cdot \mathbf{b} = 0, \quad \phi = 90°$

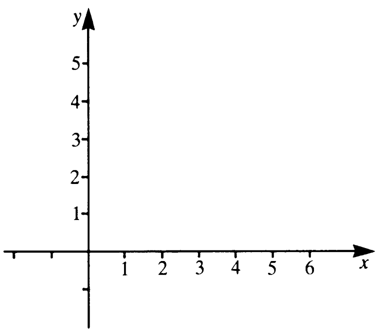
The vectors are perpendicular to each other.

49

Check the result geometrically by completing the diagram

$\mathbf{a} = (4, \quad 1)$

$\mathbf{b} = (-1, \quad 4)$



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A well-known maxim when delivering a lecture is: 2

Say what you are going to say.
Say it.
Say what you have said.

In other words:

— before starting a lecture give a brief explanation of what it is about;
— go through the subject matter in detail;
— at the end summarise the lecture.

By following this maxim it is easier for the audience to learn and retain the content.

The maxim ensures that important facts are repeated .... times.

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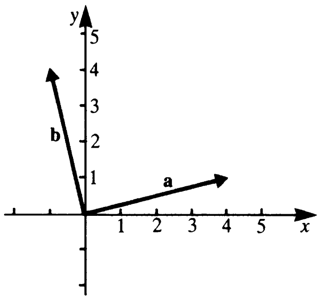

The scalar product vanishes for perpendicular vectors. This relation is frequently used to check whether two vectors are perpendicular to each other. If the components of the vectors are known, we form the scalar product to see if it vanishes.

Given:

$$\begin{aligned} \mathbf{a} &= (a_x, \quad a_y) \\ \mathbf{a_1} &= (-a_x, \quad -a_y) \\ \mathbf{a_2} &= (-a_x, \quad a_y) \\ \mathbf{a_3} &= (a_y, \quad -a_x) \\ \mathbf{a_4} &= (-a_y, \quad a_x) \end{aligned}$$

Which of the vectors are perpendicular to **a**? ....

--------------------▷ 51

three

3

The method is useful for another reason: if the essential facts are given at the beginning and repeated at the end we have a better chance of realising which aspects are fundamental, and this means that we have established priorities.

The maxim is thus also useful for the learner if he applies it to his own way of learning. It is just as useful to recall the essential facts of the last lecture before starting a new one.

Write down, briefly, the essential points of the chapter on Vector Algebra I, including basic formulae and symbols.

Stop working after 5 minutes.

------------------▷ 4

$\mathbf{a}$ and $\mathbf{a_3}$ as well as $\mathbf{a}$ and $\mathbf{a_4}$ are perpendicular.

51

Let $\mathbf{F} = (1\,\mathrm{N}, \quad -1\,\mathrm{N}, \quad 2\,\mathrm{N})$.

Which position vectors are perpendicular to $\mathbf{F}$?

$$\begin{aligned}
\mathbf{s_1} &= (2\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad 1\,\mathrm{m}) \\
\mathbf{s_2} &= (-1\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad 1\,\mathrm{m}) \\
\mathbf{s_3} &= (1\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad -2\,\mathrm{m}) \\
\mathbf{s_4} &= (3\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad -1\,\mathrm{m})
\end{aligned}$$

Perpendicular vectors ....

------------------▷ 52

Your keywords may differ but the following illustrates what you could have written: 4

1 Vectors have magnitude and direction; geometrical representation: directed line segments, e.g. arrows.
2 The addition of vectors is in accordance with the parallelogram or triangle law. For more than two vectors a chain is formed.
3 Subtracting a vector is equivalent to adding the negative of it.
4 Projection of a vector **a** on to a vector **b**:
Drop perpendicular lines from the start and end of vector **a** on to the line of action of **b**.
5 A vector may be expressed in terms of its components. The addition of two vectors **a** and **b** would be

$$\mathbf{a}+\mathbf{b}=(a_x+b_x, \quad a_y+b_y, \quad a_z+b_z)$$

6 The magnitude of a vector in terms of its components is

$$a=\sqrt{a_x^2+a_y^2+a_z^2}$$

7 The unit vector $\mathbf{e}_a$ of a vector **a** has magnitude unity and the direction of **a**.

Thus $$\mathbf{e}_a=\frac{\mathbf{a}}{a}$$

-------------------▷ 5

$\mathbf{s}_2$ and $\mathbf{s}_4$ are perpendicular to **F**. 52

All correct -------------------▷ 54

Additional explanation

The scalar product of two vectors which are perpendicular to each other is zero. We use this fact to check whether two vectors **F** and **s** are perpendicular to each other.

Let $\mathbf{F}=(1\,\mathrm{N}, \quad -1\,\mathrm{N}, \quad 2\,\mathrm{N})$.

It is required to check if $\mathbf{s}=(2\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad 1\,\mathrm{m})$ is perpendicular to **F**.

To check, we know that the scalar product must be zero.

$$\text{Now } \mathbf{F}\cdot\mathbf{s} = 1\,\mathrm{N}\times 2\,\mathrm{m} - 1\,\mathrm{N}\times 1\,\mathrm{m} + 2\,\mathrm{N}\times 1\,\mathrm{m}$$
$$= 3\,\mathrm{Nm}$$

Are **F** and **s** perpendicular to each other?

☐ Yes
☐ No

-------------------▷ 53

It does not matter if your brief résumé does not coincide with the one just given. 5

Your own may be more concise or more lengthy and it may contain other concepts, such as free vectors, bound vectors, multiplication of a vector by a scalar, etc. What matters most is that you are able to summarise the chapter recently studied. This is a useful general rule: before you start something new, recall what you have learnt in the last section.

------------------▷ 6

No. 53

Since $\mathbf{F} \cdot \mathbf{s}$ is not zero the vectors are not perpendicular to each other. If we have to check whether two vectors are perpendicular to each other, we evaluate the scalar product. If $\mathbf{F} \cdot \mathbf{s} = 0$ then the vectors $\mathbf{F}$ and $\mathbf{s}$ are perpendicular to each other, provided $\mathbf{F} \neq 0$ and $\mathbf{s} \neq 0$.

Further exercises can be found at the end of each chapter in the textbook.
If you have difficulties working through the exercises always consult the corresponding section of the textbook.

------------------▷ 54

2.1 The Scalar Product

6

Objective: Concepts of scalar product, inner product, calculation of the scalar product.

In the textbook the scalar product is derived from a mechanics example and then the rule is generalised.

READ: 2.1 Scalar product
2.1.1 Application: Equation of a line and a plane
2.1.2 Special cases
2.1.3 Commutative and distributive laws
Textbook pages 23–27

Then return to the study guide.

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54

Here are a few comments on working independently and in a group.

Group work and working alone are not mutually exclusive methods of study. They complement each other.

Independent study is appropriate when facts must be memorised, when calculations have to be checked, when proofs must be studied and when coherent material must be worked out.

Group work is suitable:

- in the preparation phase, for identifying and analysing problems;
- for the discussion of results and for solving new problems with new methods;
- as a means of checking up on others.

Group work can prepare for and guide private study. Working in a group is particularly fruitful when the work is prepared by the individuals so that all members of the group can take part in the discussion — all being equally competent to do so. Group work cannot replace private study and, similarly, private study cannot replace certain functions of work done in a group.

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The inner or .... product of two vectors can be obtained if the .... and the .... are given. 7

-------------------▷ 8

2.3 The Vector Product; Torque

55

Objective: Concepts of vector product, outer product, torque.

It is shown in the textbook how to calculate the torque for the general case of arbitrary forces and any points of application.

READ: 2.2.1 Torque
2.2.2 Torque as a vector
Textbook pages 30–32

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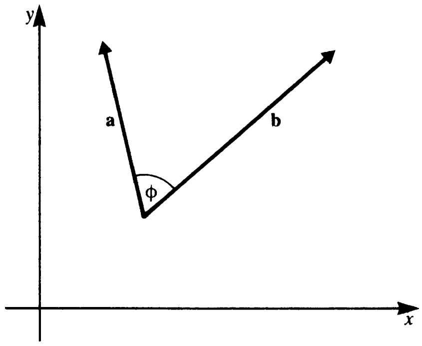
scalar
magnitude, included angle

8



State the formula for the inner product of the vectors **a** and **b**.

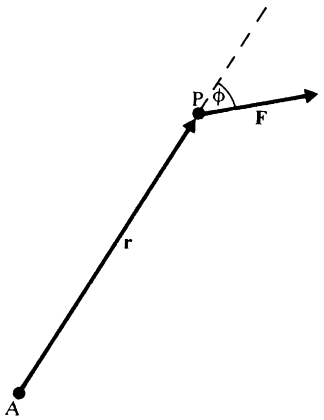
$\mathbf{a} \cdot \mathbf{b} =$ ....

--------------------▷ 9

56

A force **F** is applied to a body at point P causing it to rotate about an axis through A.

In order to calculate the torque we resolve the force into its component perpendicular to the radius vector **r** and its component in the direction of **r**.



Complete the diagram by drawing the components of **F**.
The magnitude of the component perpendicular to **r** is

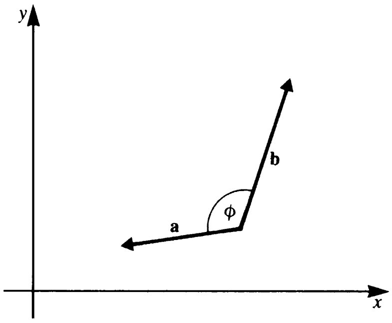
$F_{\mathrm{P}} =$ ....

--------------------▷ 57

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \phi$$

9

The scalar product of two vectors **a** and **b** is equal to the product of the magnitude of vector **a** and the magnitude of the projection of .... on ....



Complete the sketch in such a way that it fits the above statement.

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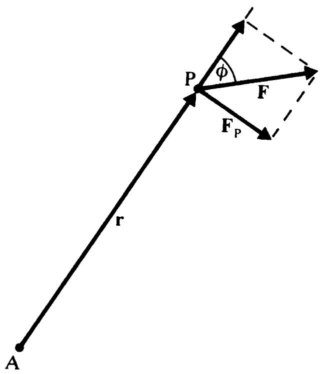
$$F_P = F \sin \phi$$

57



The component of **F** in the direction of **r** does not contribute to the torque; it has no turning effect on the body. Only the component perpendicular to **r** is to be considered.

Hence the magnitude of the torque C is:

$C =$ ....

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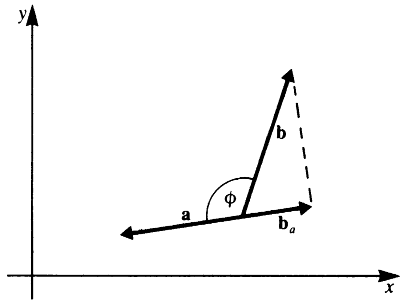
10

The scalar product of two vectors **a** and **b** is equal to the product of the magnitude of vector **a** and the magnitude of the projection of **b** on **a**.



In the above example the sign of the scalar product is:

☐ positive

☐ negative

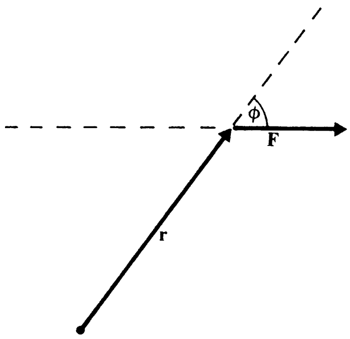
--------------------▷ 11

58

$$C = rF \sin\phi$$

Draw the components of **r** in the sketch, one perpendicular to **F** and one parallel to **F**. The magnitude of the former is:

$r_P = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



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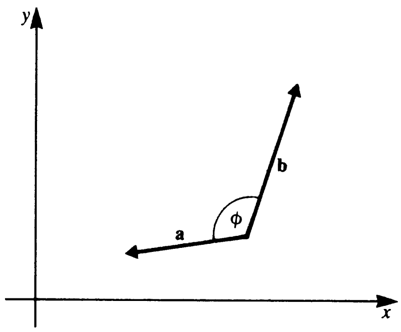
negative: $\cos\phi$ is negative. The projection of **b** on **a** is opposite to **a** in direction. 11

The scalar product of the vectors **a** and **b** is also equal to the product of the magnitude of **b** and the magnitude of the projection of .... on ....



The vectors **a** and **b** are the same as those in the last example. Complete the sketch for this case. The scalar product of the vectors is:

☐ positive
☐ negative

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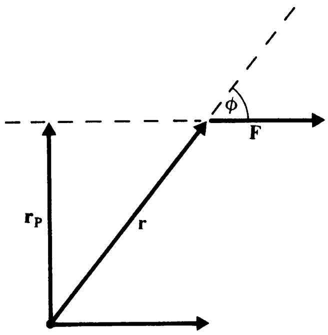
$$r_P = r \sin\phi$$ 59



By this construction the problem is again reduced to the special case of force and radius being perpendicular to each other.

Hence the magnitude of the torque is:

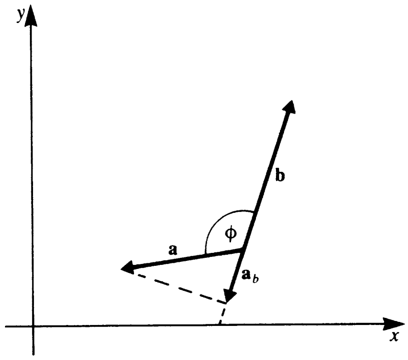
$C =$ ....

-------------------▷ 60

12

The scalar product of the vectors **a** and **b** is equal to the product of the magnitude of **b** and the magnitude of the projection of **a** on **b**.

The scalar product of the vectors is negative.



All correct 16

Further explanation required, or you have made errors 13

60

$$C = rF \sin\phi$$

When calculating torque the vectors cannot be considered as free vectors. We can only shift them along their line of action. A parallel shift of vectors is not permissible in this case.

Torque is

☐ a scalar
☐ a vector

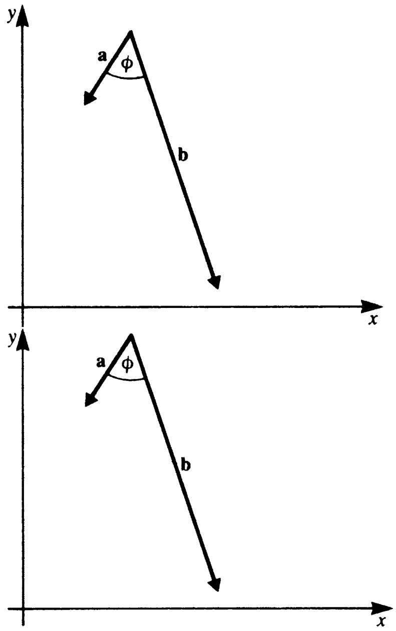
61

13

In order to evaluate the scalar product we need to know the projection of **a** on **b** or of **b** on **a**.

(i) Given **a** and **b** draw the projection of **a** on **b**.
$\mathbf{a}_b$ has a
magnitude = ...

(ii) Draw the projection of **b** on **a**.
$\mathbf{b}_a$ has a
magnitude = ...



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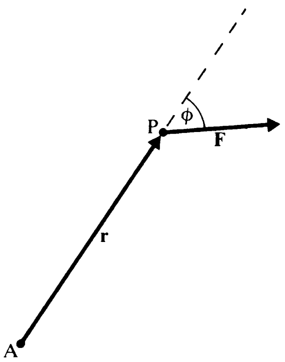
vector

61



Determine the magnitude and direction of the torque defined in the figure.
Magnitude, C = ...
Direction of **C**:

(1) **C** is .... to **r** and **F**.
(2) When **r** is rotated towards **F** in accordance with the right-hand rule, a right-handed screw moves forward in the direction ....

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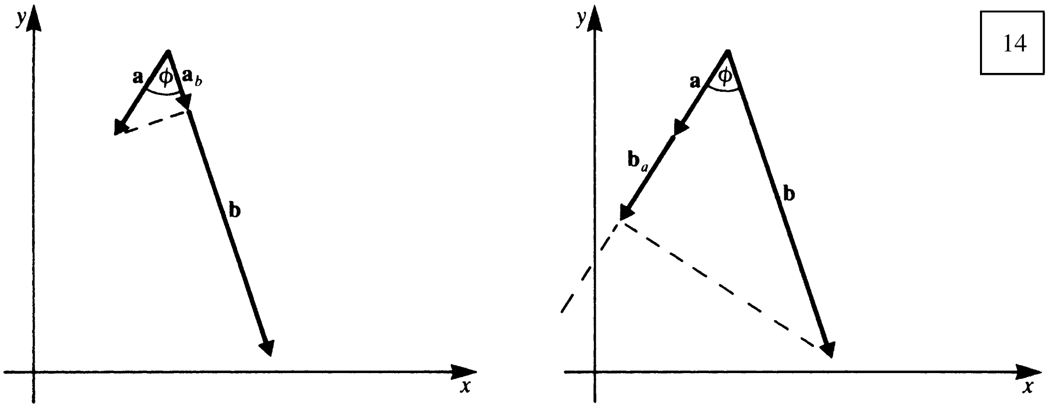


$\mathbf{a}_b = a \cos \phi$ $\mathbf{b}_a = b \cos \phi$

14

The length of the projection of a vector depends on the included angle.

If the included angle is greater than 90° or $\pi/2$ the projected vector is opposite to the vector on which it is projected since the cosine is negative.

------------------▷ 15

$C = rF \sin \phi$ (1) Perpendicular
(2) perpendicularly into the plane of the page.

62

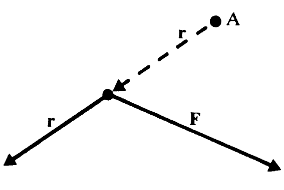
The right-hand screw rule is perhaps a little difficult to formulate clearly; it is easier to demonstrate. We proceed as follows:

(1) We shift **r** and **F** along their line of action to a common point.
(2) **r** is turned in the shortest way so that it coincides with **F**.
(3) The rotation takes place in the way a right-handed screw would move.

Anyone who has had occasion to use a screwdriver will have developed a feel for right-handed screw threads; we know from experience that such a screw would move forward and we use this fact to define the direction of **C**.

The torque in the new figure on the right
$\mathbf{C} = \mathbf{r} \wedge \mathbf{F}$
is perpendicular to the page of this text and points

☐ upwards
☐ downwards



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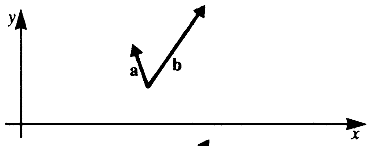
15

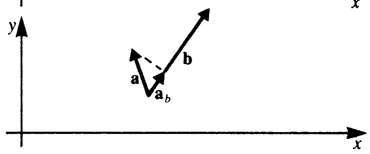
Evaluation of the scalar product.

Step 1: Choose **b** as reference vector.

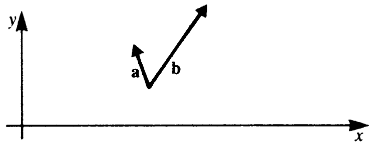
Step 2: Project **a** on the reference vector.

Step 3: The scalar product is obtained by multiplying the reference vector, **b**, by the component of the other vector, $\mathbf{a}_b$, on the reference vector.



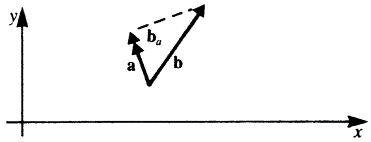


The reference vector can also be **a**. Complete the diagram with **a** as reference.



This is the solution.

We have the line of action of **a** as reference.



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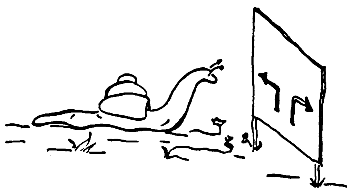
63

The vector **C** points upwards.



Further explanation required -------------------▷ 64

I want to go on -------------------▷ 65

Different symbols are used in the literature for the scalar product. Pick out three correct notations from the following: 16

☐ $\mathbf{a} \cdot \mathbf{b}$

☐ $\mathbf{a}, \quad \mathbf{b}$

☐ $<\mathbf{a}, \quad \mathbf{b}>$

☐ $\mathbf{a} \times \mathbf{b}$

☐ $(\mathbf{a}, \quad \mathbf{b})$

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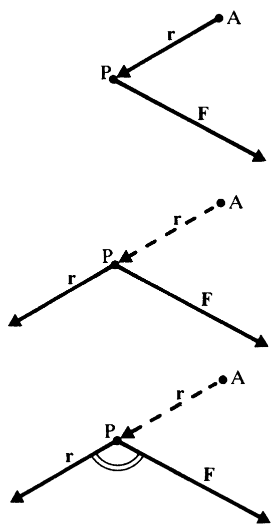


We proceed one step at a time. 64

Given: the shaft axis A of a body, the force **F** and the point of application **P**.

Step 1: Shift **r** along its line of action so that **r** and **F** meet at a common point, **P**.

Step 2: A right-handed screw at **P** operating from below the page would rotate **r** towards **F** and the screw would move out of the paper towards the reader; this is the direction of the torque vector **C**.

------------------▷ 65

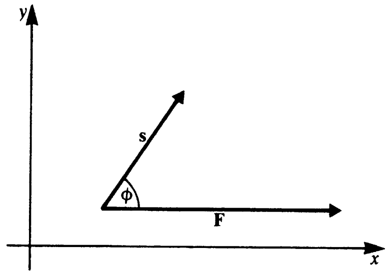
$$\mathbf{a} \cdot \mathbf{b}$$
$$(\mathbf{a}, \mathbf{b})$$
$$<\mathbf{a}, \quad \mathbf{b}>$$

17

A body is moved a distance **s** by a force **F** applied to it.

Calculate the work done by the force, given:

$$F = 6\,\text{N}$$
$$s = 2\,\text{m}$$
$$\phi = \pi/3 \text{ radians}$$



Distance is measured in metres (m), force in newtons (N).
The work done is $\mathbf{F} \cdot \mathbf{s} =$ ....

------------------▷ 18

2.4 Definition of the Vector Product

65

Objective: Concepts of outer product, vector product, null vector, calculation of the vector product (magnitude and direction).

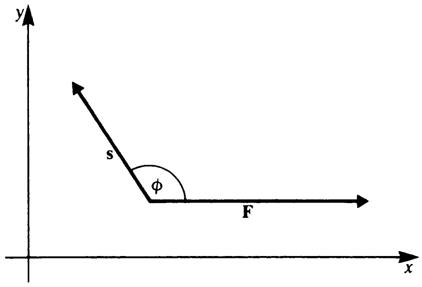
READ: 2.2.3 Definition of the vector product
2.2.4 Special cases
2.2.5 Anti-commutative law for vector products
Textbook pages 32–34

------------------▷ 66

$$\mathbf{F} \cdot \mathbf{s} = 6 \times 2 \cos \frac{\pi}{3} = 6\,\text{N m}$$

18

If $\mathbf{F} = 6\,\text{N}$
$\mathbf{s} = 2\,\text{m}$
$\phi = 120°$
then $\mathbf{F} \cdot \mathbf{s} =$ ....



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The need to calculate a torque leads to the need to define a new product of vectors.

66

We call this product .... product
or .... product

To distinguish this product from the inner or scalar product we need new symbols.
Two such symbols are:

....
or ....

--------------------▷ 67

$$\mathbf{F}\cdot\mathbf{s} = -6\,\mathrm{N\,m}$$

19

All correct --------------------▷ 23

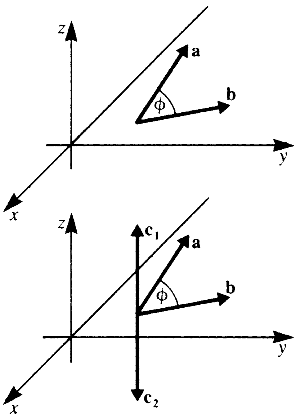
Units not clear, explanation required --------------------▷ 20

Concept of mechanical work not clear, explanation required --------------------▷ 22

Vector product or outer product

If $\mathbf{A}$ and $\mathbf{B}$ are two vectors,
the vector product is written $\mathbf{A}\times\mathbf{B}$ or $\mathbf{A}\wedge\mathbf{B}$.

67



Given: two vectors $\mathbf{a}$ and $\mathbf{b}$ in the $x-y$ plane, and ϕ the angle between them. If $\mathbf{c}$ is their vector product then

$\mathbf{c} = \mathbf{a}\wedge\mathbf{b}$ has the following properties:

Magnitude $c =$ ....

It is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Its direction follows the right-hand rule and is:

☐ $\mathbf{c_1}$

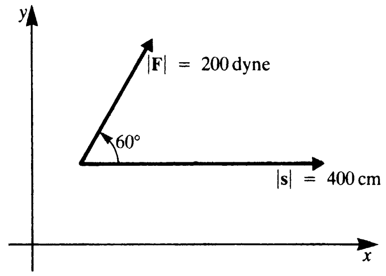
☐ $\mathbf{c_2}$

--------------------▷ 68

Most quantities encountered in physics and engineering need to be given magnitudes and units. For vectors we need direction as well. Vectors are represented by arrows and operations can be performed with these arrows. Each arrow carries with it a unit, and when carrying out calculations we have to consider the units. 20

Examples:

	(SI unit)
force	newton (N)
velocity	m/s
displacement	m
electrical field intensity	V/m



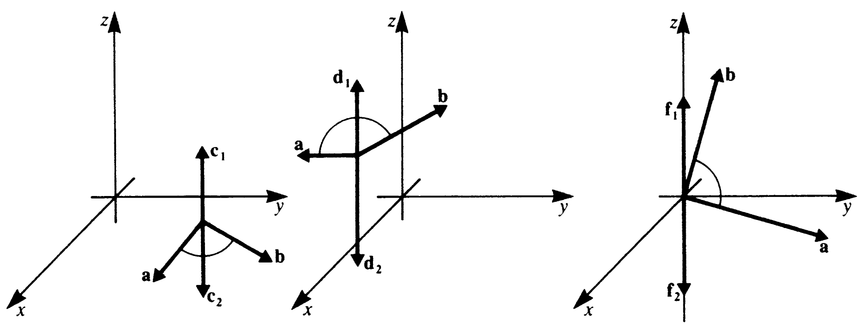
A force applied to a body has a magnitude of 200 dyn and a displacement of 400 cm at an angle of 60° to the line of action of the force. Since cos $60° = 0.5$ the work done (W) is:

$$W = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 21

$$|\mathbf{c}| = |\mathbf{a} \wedge \mathbf{b}| = ab\ \sin\phi, \quad \mathbf{c_2}$$ 68

Indicate the direction of the vector product $\mathbf{a} \wedge \mathbf{b}$ for the cases below. $\mathbf{a}$ and $\mathbf{b}$ lie in the $x-y$ plane.



☐ $\mathbf{c_1}$
☐ $\mathbf{c_2}$

☐ $\mathbf{d_1}$
☐ $\mathbf{d_2}$

☐ $\mathbf{f_1}$
☐ $\mathbf{f_2}$

-------------------▷ 69

$$W = 40\,000\,\text{dyn cm} = 0.004\,\text{N m}$$

21

In this case it was important not to forget the units!

Concept of mechanical work is not clear, explanation required

--------------------▷ 22

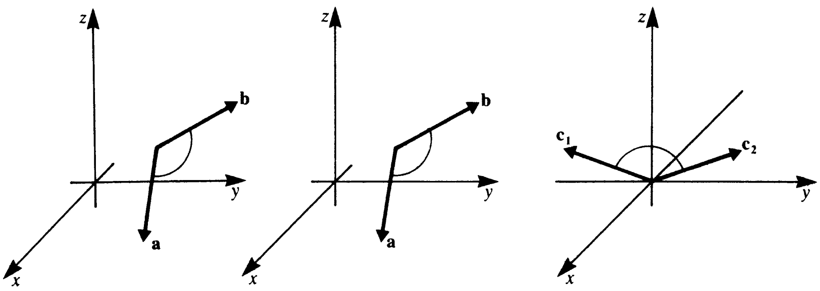
No difficulties

--------------------▷ 23

$\mathbf{c_1}$, $\mathbf{d_2}$, $\mathbf{f_1}$

69

Insert in the sketches below the directions of the vector products. **a** and **b** lie in the $x-y$ plane and $\mathbf{c_1}$ and $\mathbf{c_2}$ in the $y-z$ plane.



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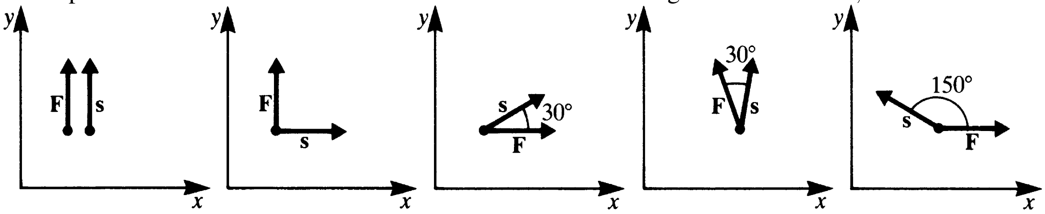
22

The mechanical work done by a force is

— the product of that force and the component of the displacement; or
— the product of the component of the force in the direction of the displacement and the displacement.

Work is considered to be positive if the displacement is in the same direction as the force and negative if these directions are opposite.

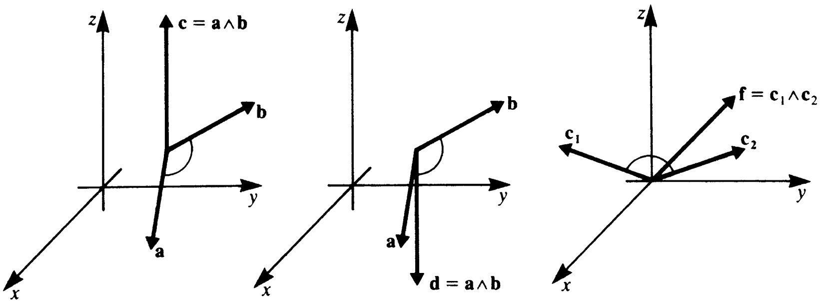
Thus mechanical work is obtained by forming the scalar product of the force and the displacement of the point of action. Calculate the work done **F** for the following cases if $F = 1\,\mathrm{N},\ s = 1\,\mathrm{m}$.



$W_1 =$ .... $W_2 =$ .... $W_3 =$ .... $W_4 =$ .... $W_5 =$ ....

--------------------▷ 23

70



All correct: right-hand screw rule fully understood --------------------▷ 75

Errors, further explanation required --------------------▷ 71

$W_1 = 1\,\text{N m}, \quad W_2 = 0, \quad W_3 = 0.87\,\text{N m}, \quad W_4 = 0.87\,\text{N m}, \quad W_5 = -0.87\,\text{N m}$

23

The inner or scalar product is an arithmetical operation which we have introduced by means of an example from mechanics.

But the concept of a scalar product is not restricted to applications in mechanics.

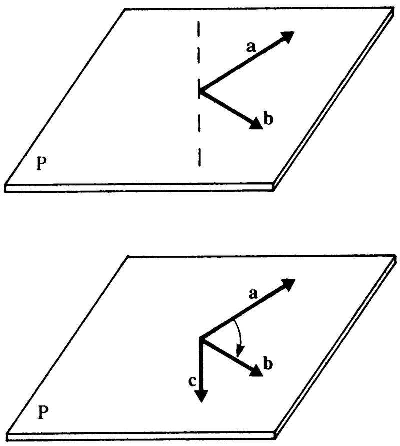
Given $a = 2$
$b = 1$
$\phi = 135°$
Calculate $\mathbf{a} \cdot \mathbf{b} =$ ....

ϕ		$\cos\phi$	$\sin\phi$
$0 = 0.00$	$0°$	1	0
$\frac{\pi}{6} = 0.52$	$30°$	0.866	0.500
$\frac{\pi}{4} = 0.79$	$45°$	0.707	0.707
$\frac{\pi}{3} = 1.05$	$60°$	0.500	0.866
$\frac{\pi}{2} = 1.57$	$90°$	0	1

------------------▷ 24

71

The result of the vector product $\mathbf{a} \wedge \mathbf{b}$ is a new vector $\mathbf{c}$.$\mathbf{a}$ and $\mathbf{b}$ define a plane P as shown in the figure. $\mathbf{c}$ is perpendicular to that plane and its direction is defined by the right-handed screw rule.



Steps required:

1: Turn the first vector, **a** in this case, towards the second vector **b** until they coincide. Take the shortest route. The direction of the vector product $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$ is the same as the displacement of a right-handed screw.
2: To determine the sense of the vector **c** we imagine the way a right-handed corkscrew would advance as we turn it.

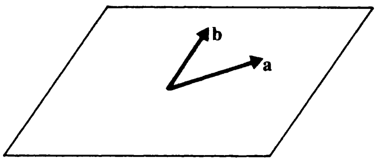
------------------▷ 72

$$\mathbf{a} \cdot \mathbf{b} = 2 \times 1 \cos 135^\circ = 2 \times 1(-0.707) = -1.414$$

24

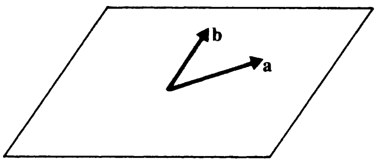
Correct result 27

Wrong result, explanation wanted 25



Insert the direction of the vector product $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$. The order of the vectors is important. To obtain the direction of $\mathbf{c}$ we always start with the first vector, i.e. $\mathbf{a}$ towards $\mathbf{b}$. Thus the direction of $\mathbf{c}$ depends on the order.

72



Insert now the direction of

$$\mathbf{d} = \mathbf{b} \wedge \mathbf{a}$$

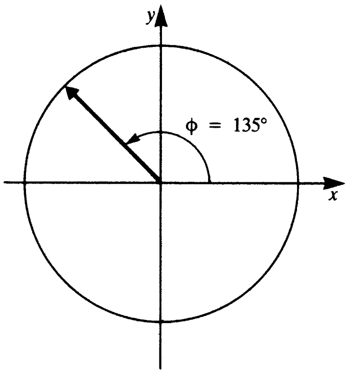
Note the change in the order.

73

The scalar product is sometimes negative since the cosine of a particular angle ϕ can be negative. By using the unit circle you can easily determine the right sign. Let us consider an example: The unit circle is drawn and the angle $\phi = 135°$ is marked. 25

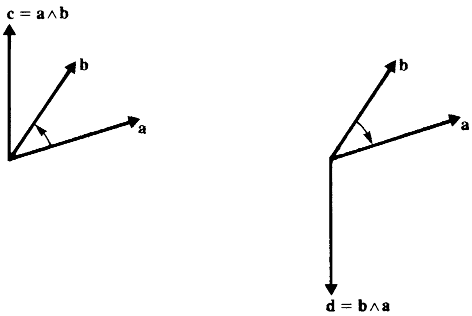
☐ $\cos 135° = 0.707$

☐ $\cos 135° = -0.707$



------------------▷ 26

73



The direction of the vector product depends on the order in which we take the vectors. This is quite different from the scalar product where

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

For the vector product, however,

$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 74

$$\cos 135^\circ = -0.707$$

26

Observing the sign of the cosine, try again to solve:

$a = 2$
$b = 1$
$\phi = 135^\circ \quad \cos\phi = -0.707$

$\mathbf{a}\cdot\mathbf{b} =$ ....

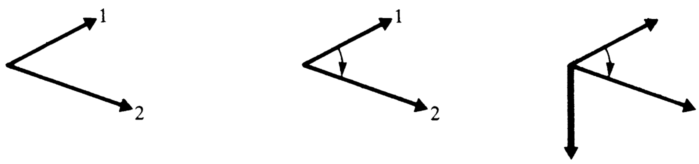
27

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a}$$

74

To obtain the direction of the vector product proceed as follows:

(1) Consider the first vector.
(2) Rotate the first vector towards the second in the shortest way.
(3) Imagine this rotation to take place as if you were turning a right-handed screw. The advance of this screw defines the direction of the vector product.



75

$$\mathbf{a}\cdot\mathbf{b} = 2\times 1\times(-0.707) = -1.414$$

27

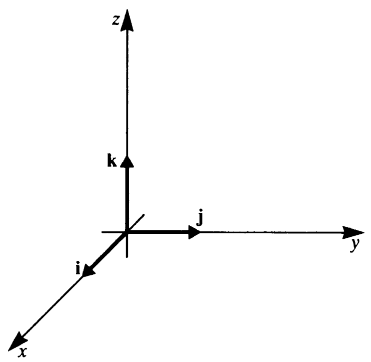
Given: $a = 2,\quad b = 4$.

Calculate the scalar products for the following included angles.

$\alpha = 45°$
$\mathbf{a}\cdot\mathbf{b} =$ ....
$\alpha = \frac{\pi}{2}$
$\mathbf{a}\cdot\mathbf{b} =$ ....
$\alpha = 120°$
$\mathbf{a}\cdot\mathbf{b} =$ ....

ϕ		$\cos\phi$	$\sin\phi$
$0 = 0.00$	$0°$	1	0
$\frac{\pi}{6} = 0.52$	$30°$	0.87	0.5
$\frac{\pi}{4} = 0.79$	$45°$	0.71	0.71
$\frac{\pi}{3} = 1.05$	$60°$	0.50	0.87
$\frac{\pi}{2} = 1.57$	$90°$	0	1

------------------▷ 28



75

The diagram shows a three-dimensional coordinate system with the unit vectors **i**, **j** and **k** on the axes.

Determine the magnitude of the vector product

$|\mathbf{i}\wedge\mathbf{j}| =$ ....

The vector product

$\mathbf{i}\wedge\mathbf{j} =$ ....

------------------▷ 76

$\alpha = 45°\mathbf{a} \cdot \mathbf{b} = 4 \times 2 \times 0.71 = 5.68$
$\alpha = \frac{\pi}{2}\mathbf{a} \cdot \mathbf{b} = 4 \times 2 \times 0 = 0$
$\alpha = 120°\mathbf{a} \cdot \mathbf{b} = 4 \times 2 \times (-0.5) = -4$

28

The following special cases are worth memorising:

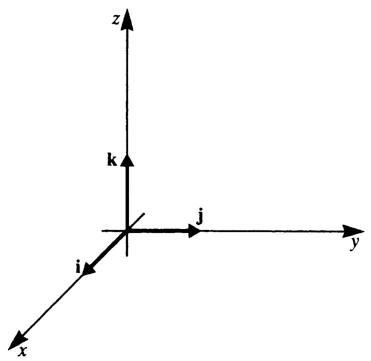
— The scalar product of parallel vectors is equal to the product of their magnitudes.
— The scalar product of perpendicular vectors is zero.

It follows that:
If the scalar product of two vectors is zero, then the vectors are .... to each other, unless one (or both) of the vectors are ....

------------------▷ 29

$|\mathbf{i} \wedge \mathbf{j}| = 1$
$\mathbf{i} \wedge \mathbf{j} = \mathbf{k}$

76



Write down the vector products of the following unit vectors:

$\mathbf{j} \wedge \mathbf{i} =$ ....
$\mathbf{i} \wedge \mathbf{k} =$ ....
$\mathbf{j} \wedge \mathbf{k} =$ ....

------------------▷ 77

perpendicular
zero

29

The scalar product of a vector with itself implies parallelism:

$$\mathbf{a}\cdot\mathbf{a} = a^2$$

Given: $c = 3$, $a = 3$.

If $\mathbf{c}\cdot\mathbf{a} = 9$, the included angle $\phi =$ ....

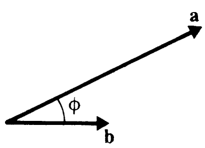
If $\mathbf{c}\cdot\mathbf{a} = 0$, the included angle $\phi =$ ....

Obtain $\mathbf{c}\cdot\mathbf{c} =$ ....

--------------------▷ 30

$$\mathbf{j}\wedge\mathbf{i} = -\mathbf{k} \quad \text{(opposite of } \mathbf{i}\wedge\mathbf{j}\text{)}$$
$$\mathbf{i}\wedge\mathbf{k} = -\mathbf{j}$$
$$\mathbf{j}\wedge\mathbf{k} = \mathbf{i}$$

77



Given: $a = 4$

$b = 2$

$\phi = \frac{\pi}{6}$

$|\mathbf{a}\wedge\mathbf{b}| =$ ....

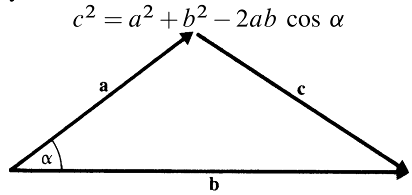
--------------------▷ 78

$$\phi = 0$$
$$\phi = 90^\circ$$
$$\mathbf{c} \cdot \mathbf{c} = 9$$

30

Try to prove the cosine rule by yourself:

$$c^2 = a^2 + b^2 - 2ab \cos \alpha$$



Successful --------------------▷ 32

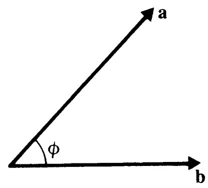
Hint required --------------------▷ 31

$$|\mathbf{a} \wedge \mathbf{b}| = ab \sin \phi = 4 \times 2 \times \sin 30^\circ = 4 \times 2 \times 0.5 = 4$$

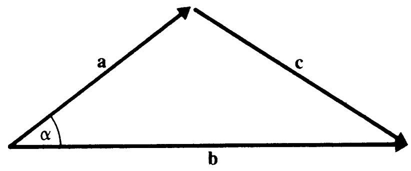
78

The magnitude of the vector product has a geometrical meaning; it is the area of a certain plane surface.

Sketch the area determined by $\mathbf{a} \wedge \mathbf{b}$.



--------------------▷ 79



We express **c** in terms of **a** and **b** 31

$$\mathbf{b} = \mathbf{a} + \mathbf{c}$$

hence $\mathbf{c} = \mathbf{b} - \mathbf{a}$.

We now form

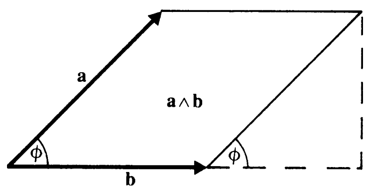
$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}),$$

hence $c^2 =$ ....

Check your result with the help of the textbook.

--------------------▷ 32

79



We had to know that $|\mathbf{a} \wedge \mathbf{b}| = ab\ \sin\ \phi$.
We can learn it in two ways:

(1) By learning the geometrical meaning, i.e. that $\mathbf{a} \wedge \mathbf{b}$ represents the surface area of a parallelogram. Hence with the help of a diagram we can reconstruct the formula.
(2) By memorising the formula.

The first method forces us to reason, and hence understand the problem, and is far superior to the second one.

--------------------▷ 80

2.2 The Scalar Product in Terms of the Components of the Vectors

32

Objective: To calculate the scalar product of two vectors when their components are known.

In the textbook it is shown that the calculations of the scalar product can be simplified if the components of the vectors are known.

READ: 2.1.4 Scalar product in terms of the components of the vectors
Textbook pages 27–29

--------------------▷ 33

80

The vector product (or outer product) has magnitude

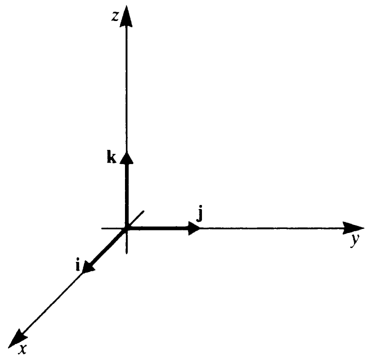
$$|\mathbf{a} \wedge \mathbf{b}| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The scalar product (or inner product) has magnitude

$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Try to derive both formulae geometrically.

--------------------▷ 81



The unit vectors along the Cartesian axes are shown. Obtain: 33

$\mathbf{i}\cdot\mathbf{i} =$ ......
$\mathbf{i}\cdot\mathbf{j} =$ ......
$\mathbf{i}\cdot\mathbf{k} =$ ......
$\mathbf{j}\cdot\mathbf{i} =$ ......
$\mathbf{j}\cdot\mathbf{j} =$ ......
$\mathbf{j}\cdot\mathbf{k} =$ ......
$\mathbf{k}\cdot\mathbf{i} =$ ......
$\mathbf{k}\cdot\mathbf{j} =$ ......
$\mathbf{k}\cdot\mathbf{k} =$ ......

------------------▷ 34

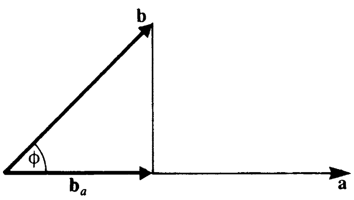
$$|\mathbf{a}\wedge\mathbf{b}| = ab\ \sin\phi$$
$$\mathbf{a}\cdot\mathbf{b} = ab\ \cos\phi$$

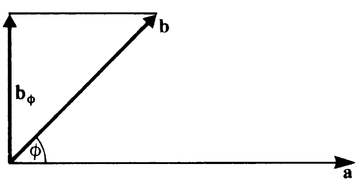
81

The two diagrams below also help us to calculate the scalar and vector products. For **a** and **b**, the magnitude of the:

scalar product = **a** times the projection of **b** on to **a**;
vector product = **a** times the projection of **b** on to the perpendicular to **a**.

$\mathbf{a}\cdot\mathbf{b} =$ .... $|\mathbf{a}\wedge\mathbf{b}| =$ ....





------------------▷ 82

34

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}=1 & \mathbf{j}\cdot\mathbf{i}=0 & \mathbf{k}\cdot\mathbf{i}=0 \\ \mathbf{i}\cdot\mathbf{j}=0 & \mathbf{j}\cdot\mathbf{j}=1 & \mathbf{k}\cdot\mathbf{j}=0 \\ \mathbf{i}\cdot\mathbf{k}=0 & \mathbf{j}\cdot\mathbf{k}=0 & \mathbf{k}\cdot\mathbf{k}=1 \end{array}$$

All correct --------------------▷ 39

Errors or difficulties, or more examples wanted --------------------▷ 35

82

$$\mathbf{a}\cdot\mathbf{b}=ab\ \cos\ \phi$$

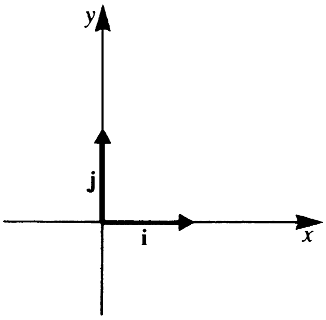
$$|\mathbf{a}\wedge\mathbf{b}|=ab\ \sin\ \phi$$

Using the definitions you should be able to answer the following questions:

$\mathbf{a}\wedge\mathbf{a}=$ ....?

$\mathbf{a}\cdot\mathbf{a}=$ ....?

--------------------▷ 83



Let us look at the problem in the $x-y$ plane. The unit vectors have magnitude 1 and are directed along the axes as shown. 35

$\mathbf{i} \cdot \mathbf{i}$ is the scalar product of the unit vector with itself; both have the same direction.

Result: $$\mathbf{i} \cdot \mathbf{i} = 1 \times 1 \times \cos 0° = 1$$

$\mathbf{i} \cdot \mathbf{j}$: The vectors are perpendicular to each other; therefore their scalar product is zero.

$$\mathbf{i} \cdot \mathbf{j} = 1 \times 1 \times \cos 90° = 0$$

Now try the following for yourself:

$\mathbf{j} \cdot \mathbf{j} =$ ....

$\mathbf{j} \cdot \mathbf{i} =$ ....

-------------------▷ 36

$$\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$$
$$\mathbf{a} \cdot \mathbf{a} = a^2$$

83

It is time to have a break!

Before the break try to recapitulate briefly what you have learned. Write down the important concepts.

Now fix the duration of the break.

During the break it is important that you do something totally different.

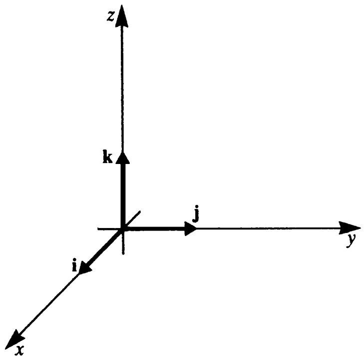
Go and make yourself a cup of coffee or tea, do some exercises (physical ones!) or go for a short walk. If you prefer, go and play your piano or your guitar or put a record on your stereo.

Do something different during your break! Read on for an explanation.

-------------------▷ 84

36

$$\mathbf{j} \cdot \mathbf{j} = 1$$
$$\mathbf{j} \cdot \mathbf{i} = 0$$



Now consider again the three-dimensional case and evaluate the scalar products of the following unit vectors:

$\mathbf{k} \cdot \mathbf{i} =$ ....

$\mathbf{k} \cdot \mathbf{j} =$ ....

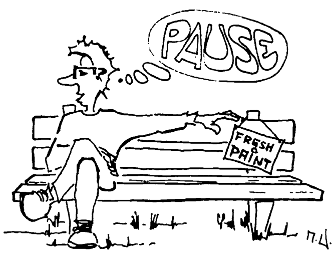
$\mathbf{k} \cdot \mathbf{k} =$ ....

-------------------▷ 37

84

Psychologists have demonstrated that learning is impaired if similar subjects are studied at the same time. For example, a foreign correspondent is studying Spanish and Italian at the same time. She believes that the similarity between the two languages will promote the learning process. Unfortunately she is mistaken; she notices that when studying Spanish, Italian words keep coming into her mind and vice versa. This makes her feel unsure because she does not know if the words that come to her mind belong to Spanish or Italian. This phenomenon is called *interference*. Such interference obstructs the learning process, and increases the learning time and affects one's confidence.

Interference is prevented by doing something totally different during your break, something which has nothing to do with mathematics.
So have a break now and fix its duration.



End of your break: ....

-------------------▷ 85

37

$\mathbf{k} \cdot \mathbf{i} = 0$, the vectors are perpendicular to each other
$\mathbf{k} \cdot \mathbf{j} = 0$, also perpendicular to each other
$\mathbf{k} \cdot \mathbf{k} = 1$, the vectors have the same direction

---

If

$$\mathbf{a} = (1, \quad 4)$$
$$\mathbf{b} = (3, \quad 1)$$
$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 38

85

It is much easier to fix the duration of the break than to keep to it.
Look at your watch and compare the time that you fixed with the time now.
Are they in agreement?
If yes: splendid!
If not: it is not too serious.
However, the difference between intentions and actions should not be allowed to accumulate.

--------------------▷ 86

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 3 + 4 \cdot 1 = 7$$

38

We end the explanations, but if you still have difficulties read section 2.1.4 once more and/or ask someone for help.

-------------------▷ 39

2.5 Resultant of Several Torques Applied to a Body Components of the Vector Product

86

The first section is intended to show you an application of the vector product.

The second shows the calculation of the vector product if the vectors are represented by their components.

READ: 2.2.6 Components of the vector product
Textbook pages 34–35

Write things down for yourself as you read. If you are not familiar with determinants follow the example in the textbook by applying equation (2.11b).

-------------------▷ 87

If

$$\mathbf{a} = (a_x, \quad a_y, \quad a_z)$$
$$\mathbf{c} = (c_x, \quad c_y, \quad c_z)$$
$$\mathbf{a} \cdot \mathbf{c} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

39

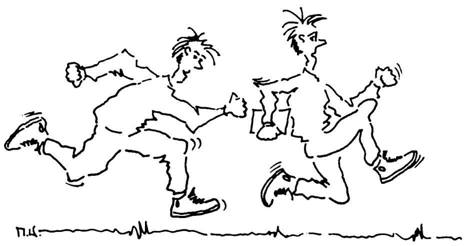
--------------------▷ 40

With the help of the textbook or from your own notes evaluate the vector product $\mathbf{a} \wedge \mathbf{b}$ for

87

$$\mathbf{a} = (2, \quad 1, \quad 1) \quad \text{and} \quad \mathbf{b} = (-1, \quad 2, \quad 1):$$

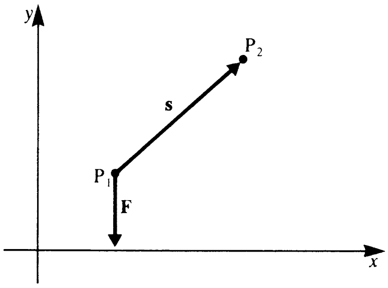
$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



--------------------▷ 88

$$\mathbf{a} \cdot \mathbf{c} = a_x c_x + a_y c_y + a_z c_z$$

40



$\mathbf{F} = (0, \quad -5\,\text{N})$

$\mathbf{s} = (3\,\text{m}, \quad 3\,\text{m}) = \overrightarrow{\text{P}_1\text{P}_2}$

The displacement is given by the vector **s**.

Calculate the work done in a displacement from P_1 to P_2.

$W = \mathbf{F} \cdot \mathbf{s} =$ ....

-------------------▷ 41

$$\mathbf{a} \wedge \mathbf{b} = (1 \times 1 - 1 \times 2)\mathbf{i} + (-1 \ -2)\mathbf{j} + (4+1)\mathbf{k} = -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

88

Note: We have used equation (2.11a) of the textbook.

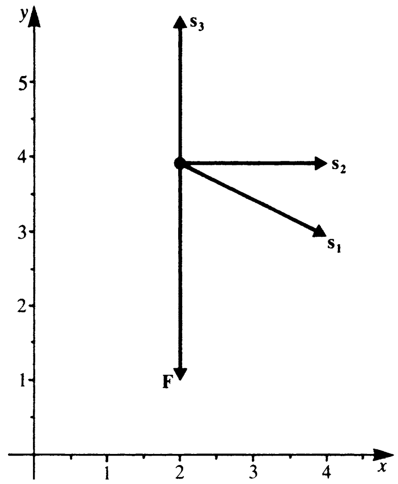
Write the answer in the shorthand way of expressing a vector, i.e.

$\mathbf{a} \wedge \mathbf{b} = (........, \ ...., \ ....)$

-------------------▷ 89

$$W = 0 \times 3 + (-5) \times 3 = -15\,\text{N m}$$

41



Given: $\mathbf{F} = (0, \quad -5\,\text{N})$.
Calculate the work done if:

$\mathbf{s_1} = (2\,\text{m}, \quad -1\,\text{m})$
$\mathbf{s_2} = (2\,\text{m}, \quad 0\,\text{m})$
$\mathbf{s_3} = (0\,\text{m}, \quad 2\,\text{m})$

$W_1 =$ ....
$W_2 =$ ....
$W_3 =$ ....

--------------------▷ 42

$$\mathbf{a} \wedge \mathbf{b} = (-1, \quad -3, \quad 5)$$

89

A body rotates about the x-axis with an angular velocity $\omega = (\omega, \quad 0, \quad 0)$.
Calculate the velocity of a point $P = (1, \quad 1, \quad 0)$.
Hint: the velocity $\mathbf{v} = \omega \wedge \mathbf{r}$
$\mathbf{r}$ is the position vector of P.
$\mathbf{v} = (\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots)$

--------------------▷ 90

$$W_1 = 5\,\text{N m}, \quad W_2 = 0, \quad W_3 = -10\,\text{N m}$$ 42

All correct --------------------▷ 47

Errors, or further examples wanted --------------------▷ 43

$$\mathbf{v} = (0, \quad 0, \quad \omega)$$ 90

The following are exercises on the whole chapter.

Write down the formulae for the vector product

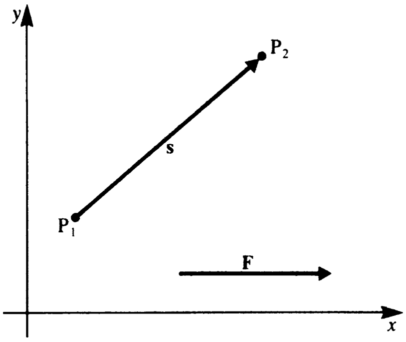
$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and for the scalar product

$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

given the magnitude of the vectors and the included angle ϕ. --------------------▷ 91

43



Consider a force $\mathbf{F}$ such that

$$\mathbf{F} = (F_x, \quad F_y) = (200\,\mathrm{N}, \quad 0)$$

The force is applied to a body and moves it from P_1 to P_2.

The displacement $\mathbf{s}$ has components s_x and s_y.

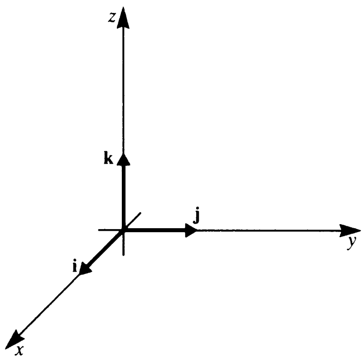
$$\mathbf{s} = (s_x, \quad s_y) = (2\,\mathrm{km}, \quad 2\,\mathrm{km})$$

$W = \mathbf{F} \cdot \mathbf{s} =$ ....

------------------▷ 44

91

$$\mathbf{a} \wedge \mathbf{b} = ab \sin \phi$$
$$\mathbf{a} \cdot \mathbf{b} = ab \cos \phi$$



Write down the vector products of the unit vectors:

$\mathbf{i} \wedge \mathbf{j} =$ ....

$\mathbf{i} \wedge \mathbf{k} =$ ....

$\mathbf{i} \wedge \mathbf{i} =$ ....

and the scalar products

$\mathbf{i} \cdot \mathbf{j} =$ ....

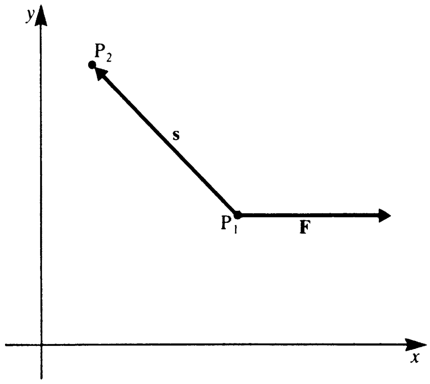
$\mathbf{i} \cdot \mathbf{k} =$ ....

$\mathbf{i} \cdot \mathbf{i} =$ ....

------------------▷ 92

$$\begin{aligned}\mathbf{F}\cdot\mathbf{s} &= (F_x s_x + F_y s_y)\\ &= (200\,\mathrm{N}\times 2\,\mathrm{km} + 0\,\mathrm{N}\times 2\,\mathrm{km})\\ &= 400\,\mathrm{N\,km} = 400\,000\,\mathrm{N\,m} = 4\times 10^5\,\mathrm{N\,m}\end{aligned}$$

44



We now consider a different displacement **s** for the same force.

$$\mathbf{F} = (200\,\mathrm{N}, \quad 0)$$
$$\mathbf{s} = (-2\,\mathrm{km}, \quad 2\,\mathrm{km})$$
$$\mathbf{F}\cdot\mathbf{s} = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 45

$$\mathbf{k}, \quad -\mathbf{j}, \quad \mathbf{0}$$
$$0, \quad 0, \quad 1$$

92

Which maxim for the delivery of a lecture was cited at the beginning of this chapter?

....

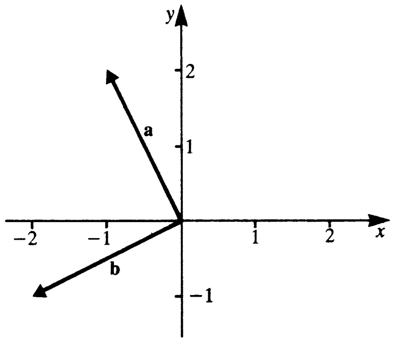
....

....

-------------------▷ 93

$$\mathbf{F} \cdot \mathbf{s} = -400\,\text{N Km} = -400\,000\,\text{N m} = -4 \times 10^5\,\text{N m}$$

45



Given: $\mathbf{a} = (-1, \quad 2)$
$\mathbf{b} = (-2, \quad -1)$
Obtain: $\mathbf{a} \cdot \mathbf{b} =$ ....
The vectors are .... to each other.

-------------------▷ 46

Say what you are going to say, say it, say what you have said.

93

The vector $\mathbf{a} = (2, \quad 3, \quad 1)$ has magnitude $a =$ ....

-------------------▷ 94

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2 = 0$$

The vectors are perpendicular to each other.

46

Write down the scalar product of the following vectors:

$$\mathbf{F} = (F_x, \quad F_y, \quad F_z)$$
$$\mathbf{s} = (s_x, \quad s_y, \quad s_z)$$
$$\mathbf{F} \cdot \mathbf{s} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Check your answer yourself using the textbook

--------------------▷ 47

$a = \sqrt{14} = 3.74$ (to two decimal places)

94

A body moves from P_0 to P_1 under the action of a force $\mathbf{F} = (5\,\mathrm{N}, \quad 0)$. The change in its position is $\mathbf{s} = (3\,\mathrm{m}, \quad 0)$.

The work $W = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 95

Obtain the magnitude of the vector 47

$$\mathbf{c} = (3, \quad 2, \quad -2)$$

$c =$ ....

--------------------▷ 48

$$W = 15\,\mathrm{N\,m}$$ 95

To solve such problems it is important to consider the direction in which the displacement takes place.
Extreme cases occur whenever:

(1) displacement and force have the same or opposite directions, or
(2) displacement and force are perpendicular to each other.

It is advisable to draw sketches.
Some final remarks:
Planning your work is important; it helps you to learn efficiently and to save time and energy.
You can fix times and dates and monitor them.
A notebook and pen help!

$$c = \sqrt{(3^2 + 2^2 + (-2)^2} = \sqrt{17} = 4.12 \text{ to 2 d.p.}$$

48

Calculate the scalar product of the two vectors

$\mathbf{a} = (4, \quad 1)$

$\mathbf{b} = (-1, \quad 4)$

$\mathbf{a} \cdot \mathbf{b} =$ ....

What is the value of the angle between **a** and **b**?

$\phi =$ ....

-------------------▷ 49

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