Chapter 20
Probability Calculus

--------------------▷ 2

Using keywords, write down the definitions for

(a) compound probability

....

....

and (b) statistically independent events

....

....

35

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20.1 Introduction

2

Objective: Concepts of macroscopic and microscopic properties of physical systems.

READ: 20.1 Introduction
Textbook page 519

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The definitions are:

36

(a) Probabilities of the simultaneous occurrence of two (or more) events are called compound events.
(b) Consider two groups of events, A and B. If the occurrence of events in group A is not influenced by the occurrence or non-occurrence of events in group B then the events of group A are statistically independent of the events in group B.

What is the probability that when throwing two dice the number 12 will show?
This means that each die must show the number 6.

$$P = \ldots\ldots$$

Solution --------------------▷ 38

Further explanation required --------------------▷ 37

Without looking in the book name three 3

(a) macroscopic quantities
....
....
....

(b) microscopic quantities
....
....
....

--------------------▷ 4

You appear to be unsure whether you should apply the addition theorem or the compound probability theorem. 37

(1) The addition theorem is valid for questions concerning the occurrence of one event *or* the other. Each event excludes the other; the two are disjoint.
(2) Compound probabilities are valid for questions concerning the occurrence of one event *and* the other.

The problem was: what is the probability, when throwing two dice, that the number 12 will show? This event can only occur if the first die shows a 6 *and* the second die shows a 6.

The probability for this to occur is $P = \ldots\ldots$

--------------------▷ 38

Macroscopic quantities; they describe a total system, e.g. 4

pressure
volume
temperature
electrical and thermal conductivity
magnetism

Microscopic quantities: they describe the properties of individual elements of a system, e.g.

position of a particle
momentum of a particle
velocity of a particle
potential energy of a particle
kinetic energy of a particle

The resistivity of a conductor is a .... quantity.
The vibrational energy of a molecule is a .... quantity.

--------------------▷ 5

$$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$$ 38

Do the following exercise:

A coin is tossed twice. What is the probability that on each occasion it will be a tail?

$$P = \ldots\ldots$$

Solution --------------------▷ 40

Further explanation wanted --------------------▷ 39

macroscopic
microscopic 5

The next section in the textbook contains many new concepts. Divide it into two or three parts and check your understanding after each part using your notes.

By the way, reading without pen and paper is day-dreaming! This should not be new to you since we have repeated it many times.

We do not propose to give you many more instructions about how to study since you should by now have developed a good technique, one which is the most beneficial for your needs.

Remember then to work with the text and have a break when you feel you need one, quite apart from what we may suggest. When you have a break, make it a real break by doing something very different. Finally, time your break and stick to your timing.

--------------------▷ 6

The probability of obtaining a tail when tossing a coin is $\frac{1}{2}$. The probability of getting a tail twice in succession must be less than $\frac{1}{2}$. 39

The compound probability of statistically independent events A and B is

$$P_{\mathrm{AB}} = P_{\mathrm{A}} \times P_{\mathrm{B}}$$

Therefore the probability of two tails in succession is $P_{\mathrm{TT}} = \ldots\ldots$

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20.2 Concept of Probability

6

Objective: Concepts of classical and statistical definition of probability, elementary event, event, evaluation of probabilities.

READ: **20.2.1 Random experiment, outcome space and events**
20.2.2 The classical definition of probability
20.2.3 The statistical definition of probability
Textbook pages 520–523

--------------------▷ 7

$$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$$

40

A box contains 18 balls

5 yellow
4 black
7 green
2 white

If we take out one ball, there are
.... elementary events
.... events with colours

--------------------▷ 41

A student can choose 3 books out of 5 (A, B, C, D, E) arbitrarily. What is the outcome space? 7

....

--------------------▷ 8

---

18 elementary events
4 events with colours 41

The box with 18 balls is made up of:

5 yellow
4 black
7 green
2 white

We now take out 3 balls in succession, and each time we put the ball back in the box. The probability of the compound event

1 black
1 green
1 white is $P(\text{bgw}) =$ ....

The probability of the compound event

1 yellow
1 black
1 green is $P(\text{ybg}) =$ ....

--------------------▷ 42

The outcome space consists of all possible groups of 3 books (denoted by A, B, C, D, E)
$\{$ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE$\}$ 8

Write down the classic definition of probability for the occurrence of an event A:

$$P(A) = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 9

$$P\,(\text{bgw}) = \frac{4}{18} \times \frac{7}{18} \times \frac{2}{18} = \frac{7}{729} \approx 0.01$$
$$P\,(\text{ybg}) = \frac{5}{18} \times \frac{4}{18} \times \frac{7}{18} = \frac{35}{1458} \approx 0.024$$

42

If you got the wrong results for both, then a revision will help.

--------------------▷ 43

$$P(A) = \frac{N_A}{N} = \frac{\text{number of elementary events contained in event } A}{\text{total number of possible elementary events}}$$ 9

The classic definition of probability refers to the following situation:
An experiment has N equally possible outcomes and event A consists of N_A of them.
You will need this definition throughout this chapter.

A box contains six balls:

3 black
2 green
1 yellow

If we take out one ball it is
an .... or ....

-------------------▷ 10

20.5 Permutations and Combinations

43

Objectives: Concepts of permutations, factorials, evaluating the number of permutations, computing factorials, evaluating the number of permutations of elements some of which are identical.

READ: 20.3 Permutations and combinations
20.3.1 Permutations
Textbook pages 527–528

-------------------▷ 44

elementary event or a random experiment 10

A box contains six balls:

3 black
2 green
1 white

One ball is taken. How many elementary events are there?

Solution --------------------▷ 12

Further explanation wanted --------------------▷ 11

Five friends are sitting on a bench in the following order: 44

Alice, Betty, Caroline, Doreen, Evelyn.

This is obviously a possible arrangement of the five, referred to as elements for short by mathematicians.

One possible arrangement is called ....

--------------------▷ 45

We have to distinguish between 'elementary event' and 'event'. 11

Consider 8 balls in a box: 3 black, 2 green and 3 white. Let us place them side by side thus:

Each ball can be chosen. This represents an *elementary event*. Thus we have 8 elementary events. Now let us consider only the colours. Taking out a ball with a distinct colour constitutes an '*event*', i.e. 'black'. Thus we have 3 events.

A box contains 3 black and 3 white balls. One ball is taken. We have .... elementary events.

-------------------▷ 12

---

a permutation 45

Write down all the permutations of the three elements of x, y, z.

-------------------▷ 46

6 elementary events

12

An urn contains 12 balls

6 red
3 green
2 white
1 black

One ball is taken out.
Number of elementary events ....
Number of events if colours are considered ....

--------------------▷ 13

---

xyz yzx zxy
xzy yxz zyx

46

A permutation is a .... arrangement of ....
For three elements there are ...... permutations.

--------------------▷ 47

12 elementary events
4 events

13

A deck of cards consists of 52 cards. They include 4 different kings: clubs, spades, hearts, diamonds. What is the probability of drawing the king of clubs, the deck having been properly shuffled?

$$P(\text{king of clubs}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution ------------------▷ 15

Further explanation wanted ------------------▷ 14

A permutation is a possible arrangement of a number of elements.
For three elements there are 3! = 6 permutations

47

The symbol $N!$ is called ....
and it means, $N! =$ ....

------------------▷ 48

In the deck of 52 cards there is only 1 king of clubs. 14

Number of 'favourable elementary events' = 1

Number of 'possible elementary events' = 52

Hence $P(\text{king of clubs}) = \frac{1}{52}$

i.e. there is one chance in 52 of getting the king of clubs.

Do you know the exact definition of classic probability?

No? Then read the definition once more in the textbook, with pen and paper at hand.

Yes? Then do the following example:

What is the probability of drawing the king of spades?

$$P(\text{king of spades}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 15

$N!$ is called 'factorial N' or 'N factorial'. 48

$$N! = 1 \times 2 \times 3 \times \ldots \times (N-1) \times N$$

Compute

0! = ...

1! = ...

2! = ...

3! = ...

4! = ...

5! = ...

6! = ...

--------------------▷ 49

$\frac{1}{52}$

15

Eight cards are lying on the table face down. We know there are 4 different jacks and 4 different queens.

What is the probability of drawing the queen of hearts?

$$P = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 16

49

$$0! = 1$$
$$1! = 1$$
$$2! = 1 \times 2 = 2$$
$$3! = 1 \times 2 \times 3 = 6$$
$$4! = 1 \times 2 \times 3 \times 4 = 24$$
$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$
$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$$

Let us go back to our five friends Alice, Betty, Caroline, Doreen, Evelyn.

They wish to sit on a bench in a park.

In how many ways can they sit on the bench, i.e. how many permutations are there of five different elements?

There are .... permutations.

------------------▷ 50

$\frac{1}{8}$

16

Steps required to compute the classic probability:
Step 1: Determine the number of favourable elementary events N_A.
Step 2: Determine the number of all possible elementary events N.

Then $P(A) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

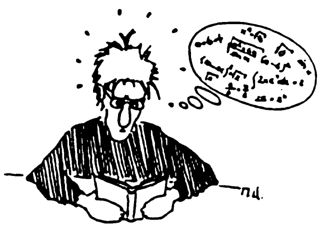
-------------------▷ 17

120
The number of permutations of five different elements is 5!

50

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

How many different arrangements are possible with 5 equal elements? ....



-------------------▷ 51

$$P(A) = \frac{N_A}{N}$$

17

In a drawer there are 10 shirts. Three shirts have a button missing. In the morning we hurriedly grab a shirt.

What is the probability of taking a shirt with all its buttons?

$$P = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 18

Exactly ONE,
since, if we have *AAAAA*, interchanging always results in *AAAAA*.

51

How many different arrangements are there with five elements *aabbc*?

Arrangements which result in interchanging an element a with another a, or b with another b, are considered to be equal.

Hence there are .... different arrangements.

Solution --------------------▷ 53

Further explanation wanted --------------------▷ 52

$$P = \frac{7}{10}$$ 18

It helps to wear a necktie, anyway!

Solve the following problem:
When throwing a die what is the probability that the side with 3 spots will appear?

$P =$ .... ------------------▷ 19

The total number of arrangements of 5 distinct elements is given by the number of permutations of 5 elements: 5! = 120. 52

In our case among the 120 permutations there exist some that differ only in that a pair of elements *a* or *b* has been interchanged. Thus the number of different arrangements is smaller.

Read section 20.3.1 once more and then try again.
How many different permutations are there with the 5 elements *aabbc*? ......

------------------▷ 53

$$P = \frac{1}{6}$$ 19

If you are still having difficulties then further exercises will be helpful. Do you wish to do some more exercises?

No, I want to carry on

--------------------▷ 21

Yes; see below

(1) A deck of cards has 26 red cards and 26 black cards. What is the probability of drawing a black card? ....

(2) When throwing a die what is the probability of obtaining a prime number? ....

--------------------▷ 20

$$\frac{5!}{2!2!} = 30$$ 53

--------------------▷ 54

(1) $P = \frac{26}{52} = \frac{1}{2}$ 20

(2) $P = \frac{3}{6} = \frac{1}{2}$ (prime numbers are 2, 3 and 5)

Hopefully, both your answers were correct!

--------------------▷ 21

20.6 Combinations

54

Objective: Concepts of combinations, binomial coefficient, determination of the number of combinations and evaluation of binomial coefficients.

READ: 20.3.2 Combinations
Textbook pages 528–530

--------------------▷ 55

A particular experiment has been carried out 530 times. The result A has been obtained 50 times. The quantity 21

$h_A = \frac{50}{530}$ is called ....

For a very large N the ratio becomes the ....

If you cannot answer straight away refer back to your notes where you should have a list of keywords and their meanings. If that does not help go back to the textbook.

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How is the binomial coefficient defined? 55

$${}_nC_k = \binom{n}{k} =$$

....

--------------------▷ 56

relative frequency
statistical probability

22

Which probability can be determined experimentally?
.... probability

To relieve the boredom of a car journey a girl starts counting the numbers of a particular model of car, e.g. the Escort, coming in the opposite direction. She found that out of 144 oncoming cars 8 were Escorts. The relative frequency of Escorts is

$$h_{\text{Escort}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 23

$$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

56

Calculate:

(a) $\binom{3}{2} = \ldots\ldots\ldots\ldots$ (b) $\binom{5}{3} = \ldots\ldots\ldots\ldots$

(c) $\binom{5}{5} = \ldots\ldots\ldots\ldots$ (d) $\binom{4}{1} = \ldots\ldots\ldots\ldots$

Solutions
--------------------▷ 59

Explanation wanted
--------------------▷ 57

statistical probability 23

$$h_{\text{Escort}} = \frac{8}{144} = \frac{1}{18}$$

If you have been concentrating hard for some time you should have a few minutes' rest.

-------------------▷ 24

Take one more look at the definition of the binomial coefficient 57

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (2 \times 1)} = \frac{6 \times 5}{2} = 15$

Compute:

(a) $\binom{6}{5} =$ ....

(b) $\binom{6}{1} =$ ....

(c) $\binom{6}{2} =$ ....

-------------------▷ 58

20.3 General Properties of Probabilities

24

Objective: Concepts of normalisation condition, addition theorem, certain event, impossible event.

READ: 20.2.4 General properties of probabilities
Textbook pages 523–525

------------------▷ 25

58

(a) $\binom{6}{5} = \frac{6!}{5!(6-5)!} = \frac{6!}{5!\,1!} = 6$

(b) $\binom{6}{1} = \frac{6!}{1!(6-1)!} = 6$

(c) $\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15$

Now compute $\binom{3}{2} =$ .... $\binom{5}{3} =$ ....

$\binom{5}{5} =$ .... $\binom{4}{1} =$ ....

------------------▷ 59

A box contains 9 white balls. What is the probability of taking out 1 white ball? 25

$P = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution ------------------▷ 28

Further explanation required ------------------▷ 26

$\binom{3}{2} = 3; \quad \binom{5}{3} = 10; \quad \binom{5}{5} = 1; \quad \binom{4}{1} = 4$ 59

A group of 3 elements is to be formed out of 5 different elements.
How many ways to form a group are there? .... ------------------▷ 60

The box contains white balls only. If we take out a ball it can only be a white one. This is a *certain event*. A certain event has the probability 1. 26

There are two extreme cases:
A certainty has the probability $P = \ldots\ldots\ldots$
An impossible event has the probability $P = \ldots\ldots\ldots$

→ 27

There are $\binom{5}{3}$ different ways to form a group of three out of 5 different elements: 60

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$

A club has 20 members. A committee is to be formed consisting of 5 members.
How many possibilities are there of forming such a committee?

→ 61

$$P_{\text{certain}} = 1$$
$$P_{\text{impossible}} = 0$$

27

If a box contains a number of green balls and if you take one the probability that you get a green one is

$$P_{\text{green}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 28

There are $\binom{20}{5}$ possibilities to form a committee of 5 members out of 20 members; i.e. it is a combination of 20 elements, taking 5 at a time.

61

Hence $$\binom{20}{5} = \frac{20!}{5!\,15!} = \frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \approx 15{,}000$$

We have come to the end of Chapter 20. But before leaving it altogether would you like to solve one final problem, on clairvoyance?

Yes

------------------▷ 62

No

------------------▷ 68

$$P = 1$$

28

In this case we have a certain event, which has the probability 1.

A box contains 3 white balls and 3 black balls. The probability of taking out a green ball is

$$P_{\text{green}} = \ldots\ldots\ldots$$

--------------------▷ 29

62

Here is the problem:

A researcher investigates whether some people are clairvoyant. For this purpose he sets 500 people the task of guessing the result of an experiment.

A coin is tossed 10 times behind a screen. Each person has to guess whether the outcomes are heads or tails.

It is assumed that people who make one mistake only in their forecasts are gifted with clairvoyance.

Do you think that such an experiment is suitable in order to discover whether a person is clairvoyant?

☐ Can you answer the question?

--------------------▷ 66

☐ Do you want a hint?

--------------------▷ 63

$$P_{\text{green}} = 0$$

29

An impossible event has the probability 0.

Write down in keywords the meaning of the normalisation condition:
....
....

--------------------▷ 30

63

The problem can be solved with the help of probability calculus.

First hint: Determine the probability that no one amongst the 500 people satisfies the condition.

A further hint is required --------------------▷ 64

I can carry on --------------------▷ 66

The normalisation condition is:
The sum of the probabilities is 1 when referring to all the events of a defined sample space. 30

What is the probability of drawing out the jack of clubs *or* the king of diamonds *or* the queen of spades in a pack of 52 cards?

$$P = \ldots\ldots\ldots\ldots$$

-------------------▷ 31

Second hint: In order to determine the probability that no one amongst the 500 people satisfies the condition we have to determine the probability that a particular person will be correct at least 9 times. 64

Use the classic definition of probability

$$P = \ldots\ldots\ldots\ldots$$

I still have difficulties -------------------▷ 65

I want to check my solution -------------------▷ 66

$$P = \frac{3}{52}$$

31

An urn contains 12 balls:

6 red
4 white
1 green
1 black

The probability of drawing either a white or a green ball is

$$P(\text{white or green}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 33

Further explanation wanted --------------------▷ 32

For a particular person the number of possible forecasts is $2 \times 2 \ldots 2\times = 2^{10} = 1024$. Number of favourable outcomes $= 10 + 1 = 11$.

65

Reasons:

A favourable outcome exists if there is at most one mistake in 10 forecasts. This mistake can occur in the first, second, ..., tenth number. There are 10 cases $= \binom{10}{1}$. A favourable case also exists if there is no mistake. This one case $= \binom{10}{0}$.

It therefore follows that $P = \frac{11}{2^{10}} \approx 10^{-2}$

The probability that one person will make more than one mistake is then $(1 - P) = (1 - 0.01) = 0.99$. The probability that all 500 people make more than one mistake is $(1 - 0.01)^{500} = 0.006$. This means that it is most likely that at least one person satisfies, by chance $(1 - 0.006) = 0.994$, the condition of '1 error at best'.

--------------------▷ 66

Consider this example. 32

Given: 6 lottery tickets with 2 first prizes, 2 consolation prizes and 2 blanks.

Probability of a first prize $P\,(\mathrm{F}) = \frac{2}{6} = \frac{1}{3}$

Probability of a consolation prize $P\,(\mathrm{C}) = \frac{2}{6} = \frac{1}{3}$

Probability of a first prize *or* a consolation prize is $P\,(\mathrm{F}) + P\,(\mathrm{C}) = \frac{2}{3}$

This was an example of the application of the *addition theorem*.

An urn contains 12 balls:

6 red
4 white
1 green
1 black

The probability of drawing either a white or a green ball is

$$P\,(\text{white or green}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 33

The experiment is not suitable for testing the hypothesis. 66

According to the random law it is almost certain that at least one person in the sample will satisfy the condition.

The probability that at least one person satisfies by chance the given condition is 0.994.

To develop the problem a little further, let us assume that the researcher investigates a particular person. He starts with the assumption that at least 8 correct scores are sufficient in order to prove that a person has clairvoyant abilities. Is this assumption justified?

(Assume that the random probability of the occurrence of such an event has to be smaller than 0.01, i.e. smaller than 1%.)

-------------------▷ 67

$$P\,(\text{white or green}) = \frac{5}{12}$$ 33

This was one of the applications of the addition theorem. It is applicable if the required probability is for one of a number of events, the events being independent.

-------------------▷ 34

He is also mistaken in this case. 67

There are $\binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 56$ favourable events (8 or 9 or 10 correct scores).

Hence $P\,(x \geq 8) = \dfrac{56}{2^{10}} \approx 0.055$.

Thus the random probability is greater than 0.01.

-------------------▷ 68

20.4 Probability of Compound Events Which are Statistically Independent

34

Objective: Concepts of statistically independent events, compound probability, evaluation of compound probability of statistically independent events.

READ: 20.2.5 Probability of statistically independent events; compound probability
Textbook pages 525–527

------------------▷ 35

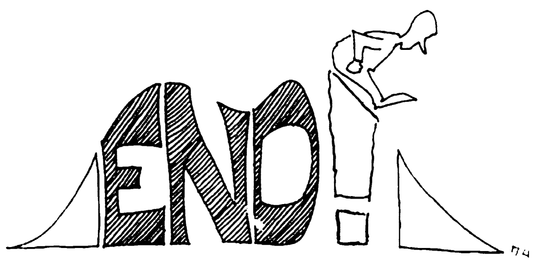
Please continue on page 1
(bottom half)

For the solution of this problem we have used

68

— the classic definition of probability
— the addition theorem of probability
— the compound probability for independent events
— binomial coefficients

In the next chapter you will encounter probability distributions, which will enable you to deal with the solution of similar problems.



OF CHAPTER 19