Chapter 13
Multiple Integrals; Coordinate Systems

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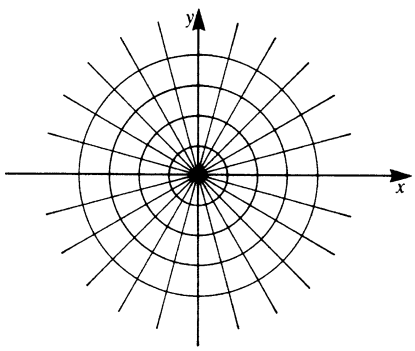
54



In Cartesian coordinates the lines

$x =$ constant and

$y =$ constant

are perpendicular to each other.

Similarly, in polar coordinates the lines

$r =$ constant and

$\phi =$ constant

are perpendicular to each other.

Can you write down the element of area $\mathrm{d}A$ in polar coordinates?

$$\mathrm{d}A = \ldots\ldots\ldots\ldots\ldots\ldots$$

Can you give a drawing of $\mathrm{d}A$?

--------------------▷ 55

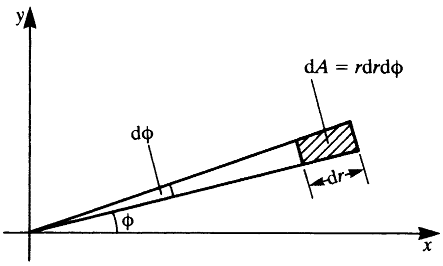
2

Before starting a new topic we should recall the important concepts of the last chapter.
Name at least four of the most important concepts met within Chapter 12.

(1) ....
(2) ....
(3) ....
(4) ....

--------------------▷ 3

55



$$dA = r\,dr\,d\phi$$

The area of a circle, for example, can now be computed easily.

$$A = \int dA$$

The integral can be written as a double integral, bearing in mind that we have two variables r and ϕ. To cover the entire circle r ranges from 0 to R and ϕ from 0 to 2π radians.

Hence, in polar coordinates

$$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 56

(1) function of several variables
(2) partial derivative
(3) total differential
(4) total derivative

3

---

(1) Given the function $f(x, y, z)$, give two of the notations used for the partial derivative with repect to y:
....
....
(2) Given the function $z = x^2 + y^2$, the total differential is:
....

--------------------▷ 4

$$A = \int_0^{2\pi} \int_0^R r \, \mathrm{d}r \, \mathrm{d}\phi$$

56

The limits are constant in this case.

What is the value of the integral?

$$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 57

(1) $\dfrac{\partial f}{\partial y}$, f_y

(2) $\mathrm{d}z = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y$

4

The meaning of the differential is most important.

Complete the following sentence:

The total differential is a measure of the change in the function $z = f(x, y)$ if ....

--------------------▷ 5

$$A = \pi R^2$$

57

The drawing shows the graph of a curve known as the cardioid whose equation is

$$r = 1 + \cos\phi$$

Using the knowledge you have gathered so far calculate the area of the cardioid.

$A =$ ....

--------------------▷ 58

The total differential is a measure of the change in the function $z = f(x, y)$ if x is increased by $\mathrm{d}x$ and y by $\mathrm{d}y$. 5

The total differential can be computed for functions of more than two variables. For example, the temperature T can be a function of three space coordinates. The total differential of the function $T = T(x, y, z)$ is a measure of the change in the temperature T if x, y and z are increased by $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, respectively. Evaluate $\mathrm{d}T$ for $T(x, y, z)$

$$\mathrm{d}T = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 6

$$A = \int_{\phi=0}^{2\pi} \int_{r=0}^{R=1+\cos 2\phi} r\,\mathrm{d}r\,\mathrm{d}\phi = \frac{3}{2}\pi$$ 58

Did you obtain this result?

Yes --------------------▷ 61

No --------------------▷ 59

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}dz$$

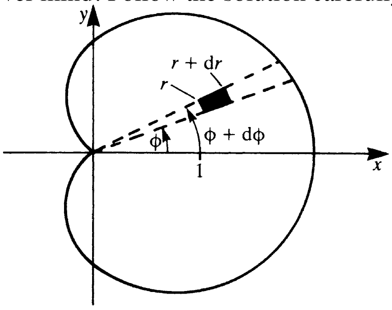
6

The last frames should have told you whether you ought to revise Chapter 12.

7

59

Never mind! Follow the solution carefully:



Consider the graph of the curve $r = 1 + \cos\phi$ and the elemental area shown in the figure.

Its area is $dA = r\,dr\,d\phi$.

To obtain the total area we have to integrate along r and along ϕ. To do this we must determine the limits of integration.

Since r is a function of ϕ we must integrate with respect to r first, whereas the limits for ϕ are constant.

Hence the area is given by

$$A = \int_0^{2\pi}\int_0^{1+\cos\phi} r\,dr\,d\phi$$

60

13.1 Multiple Integrals

7

The concept of multiple integrals generalises the concept of integrals of a single variable to several variables.

READ: 13.1 Multiple integrals
Textbook pages 379–381

--------------------▷ 8

Now we must solve the integral: $A = \int_{\phi=0}^{2\pi} \int_{r=0}^{1+\cos\phi} r \,\mathrm{d}r \,\mathrm{d}\phi$ 60

Evaluating the inner one:

$$A = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{1+\cos\phi} \mathrm{d}\phi = \frac{1}{2}\int_0^{2\pi} (1+\cos\phi)^2 \,\mathrm{d}\phi$$

$$A = \frac{1}{2}\int_0^{2\pi} (1+2\cos\phi+\cos^2\phi) \,\mathrm{d}\phi$$

You should be able to solve this integral with the knowledge acquired in Chapter 6. If necessary consult the table of integrals.

$$A = \frac{1}{2}\left[\phi - 2\sin\phi + \frac{\phi}{2} + \frac{\sin\phi\cos\phi}{2}\right]_0^{2\pi}$$

Hence $A = \frac{3}{2}\pi$.

--------------------▷ 61

The concept of multiple integrals in the textbook is developed following the considerations in Chapter 6. Instead of calculating areas we now calculate volumes. For example, the volume of a cube is the sum of all partial volumes, i.e. 8

$$V = \sum \Delta V_i$$

Each partial volume ΔV_i is the product of the lengths of the edges Δx_i, Δy_i and Δz_i.

To obtain the actual volume V we proceed to the limit by letting $\Delta V_i \to 0$, thus taking a larger and larger number of partial volumes N. Hence

$$V = \lim_{N\to\infty} \sum \Delta V_i = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0\\ \Delta z\to 0}} \sum \Delta x \Delta y \Delta z$$

so that V = $\int \mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 9

Now let us extend this exercise and obtain the position of the centroid of the cardioid. This is a harder exercise. 61

If A = total area

$\bar{x}$ = position of the centroid, then

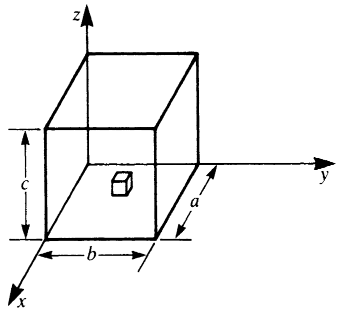
$$A\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 62

$$V = \int\int\int_v \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

9

This is a .... integral. It refers to the calculation of the volume of a cube.
Insert the limits of integration for each variable:



$$V = \int_{z=\dots}^{\dots}\int_{y=\dots}^{\dots}\int_{x=\dots}^{\dots} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

--------------------▷ 10

$$A\bar{x} = \int_A x\,\mathrm{d}A$$

62

Since $x = r\cos\phi$ and $\mathrm{d}A = r\mathrm{d}r\ \mathrm{d}\phi$:

$$A\bar{x} = \int_0^{2\pi}\int_0^{1+\cos\phi} r\cos\phi\, r\,\mathrm{d}r\,\mathrm{d}\phi$$

Now evaluate the integral and determine $\bar{x} =$ ....
After that, determine $\bar{y} =$ .... (Think!)

Solution --------------------▷ 63

Help or explanation wanted --------------------▷ 64

multiple integral

$$V = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

10

Exercises in the practical evaluation of integrals with constant limits will be given in the next section. There are no fundamentally new operations to learn.

--------------------▷ 11

$$\bar{x} = \frac{5}{6}, \quad \bar{y} = 0$$

63

Correct --------------------▷ 66

Wrong, or explanation wanted --------------------▷ 64

13.2 Multiple Integrals with Constant Limits

11

Objective: Rules for the solution of one type of multiple integrals.

READ: 13.2 Multiple integrals with constant limits
13.2.1 Decomposition of a multiple integral into a product of integrals
Textbook pages 381–383

------------------▷ 12

64

Follow the solution carefully. Use a piece of paper and work in parallel; don't just read the explanation, you might miss something:

$$A\bar{x} = \int_0^{2\pi} \int_0^{1+\cos\phi} r^2 \cos\phi \, \mathrm{d}r \, \mathrm{d}\phi$$

We must integrate with respect to r first, since its upper limit is variable.

This yields

$$\begin{aligned} A\bar{x} &= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1+\cos\phi} \cos\phi \, \mathrm{d}\phi \\ &= \frac{1}{3}\int_0^{2\pi} (1+\cos\phi)^3 \cos\phi \, \mathrm{d}\phi \end{aligned}$$

Now we integrate with respect to ϕ, but first we must expand the integrand.

$$A\bar{x} = \frac{1}{3}\int_0^{2\pi} (\cos\phi + 3\cos^2\phi + 3\cos^3\phi + \cos^4\phi) \, \mathrm{d}\phi$$

The result is ....

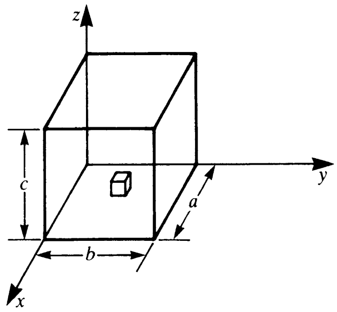
------------------▷ 65

12

The volume of a cube is a triple integral with constant limits of integration, i.e.

$$V = \int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

For the lower limit we explicitly denote the relevant variable (later on this can be dropped when there is no ambiguity) to ensure that no mistakes are made. Integration is carried out for one variable at a time; the other variables are assumed to be constant. The order of integration is immaterial. First integrate with respect to x:



$V =$ ....

13

65

$$A\bar{x} = \frac{1}{3}\left[\sin\phi + 3\left(\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right) + 3\left(\frac{\sin\phi\cos^2\phi}{3} + \frac{2}{3}\sin\phi\right)\right.$$
$$\left. + \frac{1}{4}\sin\phi\cos^3\phi + \frac{3}{4}\left(\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right)\right]_0^{2\pi}$$

Substituting the limits yields

$$A\bar{x} = \frac{5}{4}\pi$$

Since $A = \frac{3}{2}\pi$ it follows that

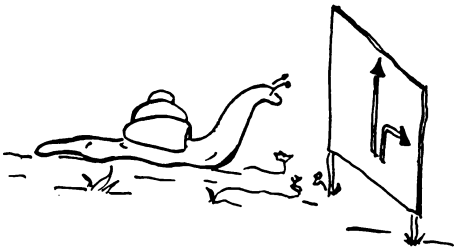
$$\bar{x} = \frac{5}{6}$$

Because of symmetry $\bar{y} = 0$, i.e. the centroid lies on the x-axis.
This was quite a hard exercise in integration!

66

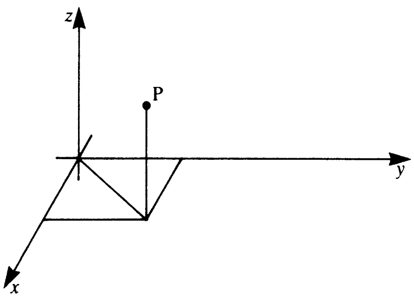
$$V = a \int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\, \mathrm{d}z$$

13



Correct --> 15

Wrong, further explanation needed --> 14



Indicate the position of a general point P whose cartesian coordinates are (x, y, z) in cylindrical coordinates:

$x =$ ....

$y =$ ....

$z =$ ....

66

--> 67

14

You were asked to evaluate the triple integral

$$V = \int_{x=0}^{c} \int_{y=0}^{b} \int_{z=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

with respect to x.

The rule is:

Except for x, all variables are treated as constants. We place in brackets what is considered constant, thus

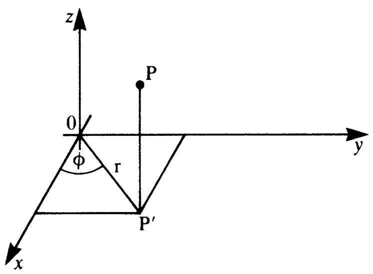
$$V = \left[\int_{z=0}^{c} \int_{y=0}^{b}\right] \int_{x=0}^{a} \mathrm{d}x\,[\mathrm{d}y\,\mathrm{d}z]\,.$$

Now rearrange the equation so that all quantities considered constant for the time being are bracketed.

$$V = \int_{x=0}^{a} \ldots\ldots\ldots\ldots \left[\ldots\ldots\ldots\ldots\right]$$

-------------------▷ 15

67



$x = r\cos\phi$
$y = r\sin\phi$
$z = z$

What are the cylindrical coordinates of the point $P(-1, 1, 3)$?

$r = \ldots\ldots\ldots\ldots$
$\tan\phi = \ldots\ldots\ldots\ldots$
$z = \ldots\ldots\ldots\ldots$

-------------------▷ 68

$$V = \int_{x=0}^{a} \mathrm{d}x \left[\int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\, \mathrm{d}z \right]$$

15

The integral with respect to x can now be evaluated. The brackets are untouched during this operation.

$$V = \int_{x=0}^{a} \mathrm{d}x \left[\int \int \mathrm{d}y\, \mathrm{d}z \right] = (a-0) \left[\int \int \mathrm{d}y\, \mathrm{d}z \right]$$

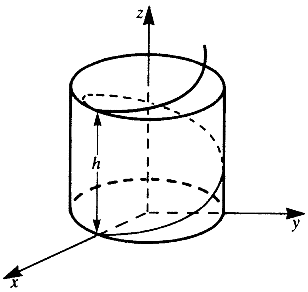
Now we can do away with the brackets and evaluate

$$V = a \int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\, \mathrm{d}z$$

------------------▷ 16

$r = \sqrt{2}$, $\tan \phi = -1$, $z = 3$

68



The figure shows a sketch of a helix of radius R and pitch h.

Can you write down the equation for it?

....

....

------------------▷ 69

Evaluate the double integral 16

$$V = a \int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y \, \mathrm{d}z$$

with respect to y:

$$V = \text{..............................}$$

--------------------▷ 17

$$r = R, \; z = \frac{h}{2\pi}\phi$$ 69

Wrong, or explanation wanted --------------------▷ 70

Correct --------------------▷ 71

$$V = ab \int_0^c \mathrm{d}z$$

17

Now evaluate the final integral.

$V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Errors or difficulties

--------------------▷ 18

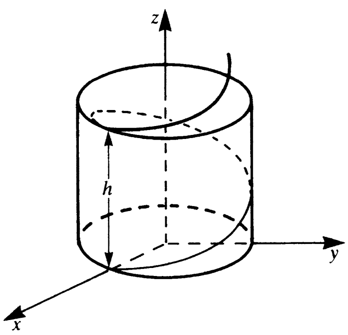
I want to go on

--------------------▷ 21

The helix is such that the distance r from the z-axis (central axis) is constant and equal to its radius R. 70

The projection of the helix on the $x - y$ plane is a circle for which we know the equation. The height z depends on the angle ϕ. In one revolution the height increases by an amount h (known as the pitch of the helix). It follows that

$$z = \frac{h}{2\pi}\phi.$$



--------------------▷ 71

While integrating with respect to y we assume that all other variables are regarded as constants. You already know of something similar, namely partial differentiation. There we differentiated with respect to one variable at a time, regarding all others as constants. 18

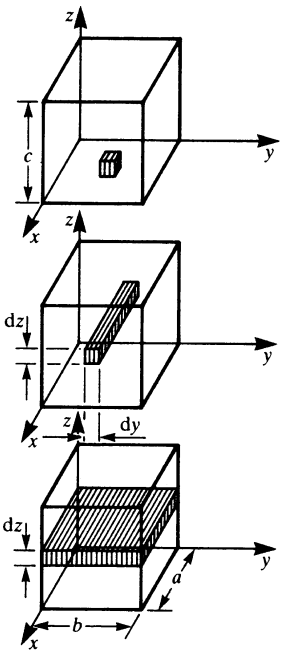
Perhaps it will become clearer in the next frame when we consider the geometric meaning.

--------------------▷ 19

Now derive the elemental volume in cylindrical coordinates by yourself. Check it with the textbook. 71

$$dV = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 72



The volume integral 19

$$\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

had to be evaluated.

The geometric meaning of integration with respect to x is that elemental volumes are added together in the x-direction forming a column having a base equal to $\mathrm{d}y\ \mathrm{d}z$. This column is now the new integrand

$$\int_{z=0}^{c}\int_{y=0}^{b} a\,\mathrm{d}y\,\mathrm{d}z$$

The geometric meaning of integration with respect to y is that all the elemental columns are added together in the y-direction. A slice is then generated having a base equal to ab and thickness $\mathrm{d}z$.

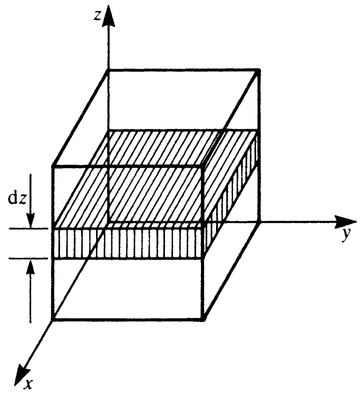
Hence $\int_{z=0}^{c} ab\,\mathrm{d}z$ is left to be evaluated.

------------------▷ 20

13.5 Spherical Coordinates 72

READ: 13.4.3 Spherical coordinates
Textbook pages 393–396

------------------▷ 73



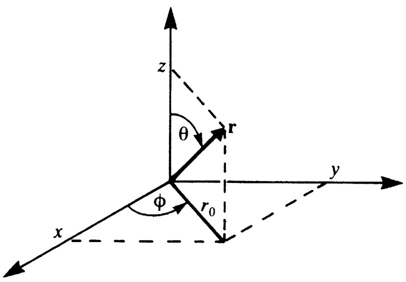
If we integrate with respect to z we are effectively adding up all the slices in the z-direction. The result is a cube of volume

20

$$V = abc$$

Let us recapitulate. The first integration corresponds to adding up the elemental volumes in one direction, giving rise to a column. The second integration is a summation of all the columns in the second direction, giving rise to a slice. The third integration is a summation in the third direction of all the slices; with this final step we have generated a cube and obtained its volume.

------------------▷ 21



Express the cartesian coordinates in terms of the spherical ones, referring only to the figure.

73

$x =$ ....

$y =$ ....

$z =$ ....

------------------▷ 74

$$V = abc$$

21

Given the double integral

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2 \,\mathrm{d}x\,\mathrm{d}y$$

to evaluate the inner integral we rearrange and place the inner integral in brackets. Carry out this step; which of the following is correct?

☐ $\int_{y=0}^{2}\left[\int_{x=0}^{1} x^2 \,\mathrm{d}x\right]\mathrm{d}y$ ------------------▷ 22

☐ $\int_{x=0}^{1}\left[\int_{y=0}^{2} x^2 \,\mathrm{d}x\right]\mathrm{d}y$ ------------------▷ 23

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

74

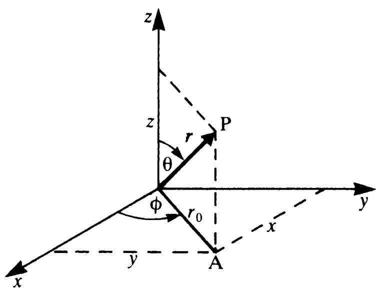
Correct ------------------▷ 77

Wrong ------------------▷ 75

Correct! 22

You have realised that to integrate with respect to x we must first place the integrals in the right order.
Evaluate the integral.

▷ 25



Let's have a look at the diagram. 75

First we calculate the projection of r (the length of the position vector from O to P) on to the $x - y$ plane:

$$r_0 = r \sin\theta.$$

Remaining in the $x - y$ plane, we have by simple trigonometry

$$x = r_0 \cos\phi = r \sin\theta \cos\phi$$
$$y = r_0 \sin\phi = r \sin\theta \sin\phi$$

Now we consider the triangle OAP and get $z = r\cos\theta$.

Thus the transformation from spherical to cartesian coordinates is

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

▷ 76

You made a mistake! 23

If we integrate with respect to a particular variable we have to make sure that we take the limits appropriate to that variable.

The integral was

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2 \, dx \, dy$$

If we integrate with respect to x we have to place the limits in such a way that we can evaluate the inner integral first. Enter the appropriate limits

$$\int \left[\int x^2 \, dx \right] dy$$

-------------------▷ 24

There is no particular need to memorise the expression for the element of volume dV in the case of spherical coordinates. What is important at this stage is: 76

(1) that you understand the derivation, and
(2) that you know where to find it when required, either in a table or in the textbook.

-------------------▷ 77

$$\int_{y=0}^{2}\left[\int_{x=0}^{1} x^2\,\mathrm{d}x\right]\mathrm{d}y$$

24

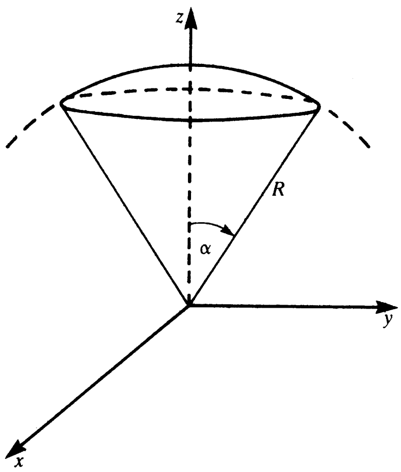
Now you should be able to solve the inner integral and substitute the correct limits.

$$\int_{y=0}^{2}\left[\dots\dots\dots\dots\dots\dots\right]\mathrm{d}y$$

--------------------▷ 25

Let us try an example using spherical coordinates.

77



The figure shows a spherical sector of radius R, angle α.

We require its volume.

The elemental volume $\mathrm{d}V$ is

$$\mathrm{d}V = \dots\dots\dots\dots$$

--------------------▷ 78

$$\int_{y=0}^{2}\left[\frac{1}{3}\right]\mathrm{d}y = \frac{2}{3}$$ 25

Note:

(1) A multiple integral with constant limits can be reduced to the successive evaluation of definite integrals.
(2) When evaluating an integral with respect to a particular variable we have to ensure that the limits inserted belong to that variable.

-------------------▷ 26

$$\mathrm{d}V = r^2 \sin\theta \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$$ 78

The volume is given by the integral. Insert the limits:

$$V = \int_{\cdots\cdots}^{\cdots\cdots}\int_{\cdots\cdots}^{\cdots\cdots}\int_{\cdots\cdots}^{\cdots\cdots} r^2 \sin\theta\ \mathrm{d}r\ \mathrm{d}\theta\ \mathrm{d}\phi$$

-------------------▷ 79

It is sometimes possible to decompose an integrand into a product of functions if each function depends on a single variable. In this case integration is particularly easy. 26

Which of the following integrands can be expressed as a product of independent functions?

(A) $\int_{x=0}^{2}\int_{y=1}^{2}\frac{x^2}{y^2}\,dx\,dy$

(B) $\int_{x=0}^{2}\int_{y=1}^{2}x\left(x+\frac{1}{y^2}\right)dx\,dy$

☐ A → 27

☐ B → 28

☐ A and B → 29

$$V=\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{R}r^2\sin\theta\,dr\,d\theta\,d\phi$$ 79

Note that all the limits of integration are constant.

Now evaluate the volume:

$V = \ldots\ldots\ldots\ldots$

→ 80

Correct! 27

Now decompose the integrand into the product of independent functions.

$$\int_{x=0}^{2}\int_{y=1}^{2}\frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 31

$$V = \frac{2}{3}\pi R^3(1-\cos\alpha)$$ 80

Correct --------------------▷ 82

Wrong --------------------▷ 81

Wrong; the integrand was $x\left(x+\frac{1}{y^2}\right)$. The bracket contains both x and y. Try again! Which of the following functions can be decomposed into the product of functions which depend on one variable only? 28

☐ $f_1 = (x+2y)y$

☐ $f_2 = (x+x^2)(y+y^2)$

☐ $f_3 = \sin x \cos y$

☐ $f_4 = (\sin x + \sin y)y$

-------------------▷ 30

The integral was $V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{R} r^2 \sin\theta \, dr\, d\theta\, d\phi$. 81

The expression consists of simple functions, namely r^2 and $\sin\theta$. Let us integrate with respect to r first.

$$V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\left[\frac{r^3}{3}\right]_0^R \sin\theta\, d\theta\, d\phi$$

Remember that while you integrate with respect to r, θ and ϕ are treated as constants.

$$V = \frac{R^3}{3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha} \sin\theta\, d\theta\, d\phi$$

We can now integrate with respect to θ; this gives

$$V = \frac{R^3}{3}\int_{\phi=0}^{2\pi}\Big[-\cos\theta\Big]_0^{\alpha} d\phi = \frac{R^3}{3}(1-\cos\alpha)\int_{\phi=0}^{2\pi} d\phi$$

Finally, we integrate with respect to ϕ. We find

$$V = \frac{2}{3}\pi R^3(1-\cos\alpha)$$

-------------------▷ 82

29

Wrong, unfortunately. You are right with the first integrand.

$\frac{x^2}{y^2}$ is a product of two functions each of which depends on one variable only.

But you are wrong with the second integrand: $x\left(x+\frac{1}{y^2}\right)$. The bracket contains both x and y and therefore depends on two variables.

Which of the following functions can be decomposed into the product of functions which depend on one variable only?

☐ $f_1 = (x+2y)y$

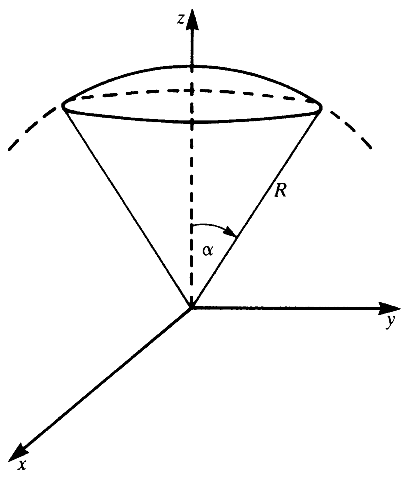
☐ $f_2 = (x+x^2)(y+y^2)$

☐ $f_3 = \sin x \cos y$

☐ $f_4 = (\sin x + \sin y)y$

--------------------▷ 30

82



What is the position $\bar{z}$ of the centroid of the spherical sector?

$$\bar{z} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 83

Explanation wanted

--------------------▷ 84

$$f_2 = (x+x^2)(y+y^2) = g(x)h(y)$$
$$f_3 = (\sin x)(\cos y) = g(x)h(y)$$

30

Decompose the integral into a product:

$$\int_{x=0}^{2}\int_{y=1}^{2}\frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 31

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha)$$

83

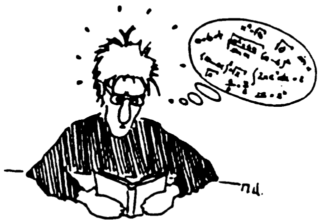
Correct --------------------▷ 91

Wrong, or detailed solution wanted --------------------▷ 84

$$\left(\int_0^2 x^2\,\mathrm{d}x\right)\left(\int_1^2 \frac{1}{y^2}\,\mathrm{d}y\right)$$

31

Now evaluate this integral!



-------------------▷ 32

By definition, the z-component of the position of the centroid is

84

$$V\bar{z} = \int_V z\,\mathrm{d}V$$

Now we establish the integral using spherical coordinates. First find the limits and express $\mathrm{d}V$:

$$\int_V z\,\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 85

$$\left(\int_{x=0}^{2} x^2\,\mathrm{d}x\right)\left(\int_{y=1}^{2}\frac{1}{y^2}\,\mathrm{d}y\right)=\frac{4}{3}$$ 32

Correct → 34

Wrong, or further explanation wanted → 33

$$\int_V z\,\mathrm{d}V=\int_0^{2\pi}\int_0^{\alpha}\int_0^{R} z\,r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$ 85

We must express z in terms of r, θ and ϕ before evaluating the integral:

$z =$ ....

→ 86

$$\text{Let}\quad I = \left(\int_{x=0}^{2} x^2\,\mathrm{d}x\right)\left(\int_{y=1}^{2} \frac{\mathrm{d}y}{y^2}\right)$$ 33

Since the functions and hence the integrals are independent we can solve them separately

(1) $I_1 = \int_0^2 x^2\,\mathrm{d}x = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$

(2) $I_2 = \int_1^2 \frac{\mathrm{d}y}{y^2} = \left[-\frac{1}{y}\right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$

Hence $I = I_1 I_2 = \frac{8}{3} \times \frac{1}{2} = \frac{4}{3}$

--------------------▷ 34

$$z = r\cos\theta$$ 86

If you didn't get this result look up the formula in the textbook (Table 13.1).

Substituting for z in the expression we have:

$$\int_V z\,\mathrm{d}V = \text{...........................}$$

--------------------▷ 87

Evaluate the following integral, either by successive integration or by decomposing the integrand into a product. 34

$$A = \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/2} \sin\phi \cos\theta \, \mathrm{d}\phi \, \mathrm{d}\theta = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 35

$$\int_V z \, \mathrm{d}V = \int_0^{2\pi} \int_0^{\alpha} \int_0^{R} r^3 \sin\theta \cos\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$$ 87

Now evaluate the integral.

$$V\bar{z} = \int_V z \, \mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 88

$$A = 2$$

35

Further exercises are given in the textbook.

Exercises are most useful if you are not quite 100 per cent correct. → 36

$$\frac{1}{4}\pi R^4 \sin^2 \alpha$$

88

Correct → 90

Wrong, or further explanation wanted → 89

13.3 Multiple Integrals with Variable Limits

36

In general, multiple integrals have variable limits of integration, in which case the order of integration is important.
Objective: Evaluation of multiple integrals with variable limits.

READ: 13.3 Multiple integrals with variable limits
Textbook pages 384–388

37

89

Perhaps your difficulty is evaluating the integral

$$\int_0^{\alpha} \sin\theta \cos\theta \, d\theta?$$

It is an integral which we can evaluate using the substitution method.
Let

$$u = \sin\theta$$

$$du = \cos\theta \, d\theta$$

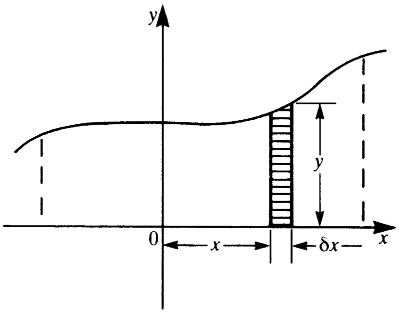
hence

$$\int u \, du = \tfrac{1}{2}u^2 = \tfrac{1}{2}\sin^2\theta$$

Now verify that

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{R} r^3 \sin\theta \cos\theta \, dr \, d\theta \, d\phi = \frac{1}{4}\pi R^4 \sin^2\alpha$$

90



37

One aspect of this section is theoretically very interesting: the evaluation of an area leads to a double integral. The area problem was discussed in Chapter 6. It now turns out to be a special case where one integration has already been carried out without out being aware of it. We considered strips in the y-direction of width dx and height y; this is, in fact, the result of first integrating along the y-direction.

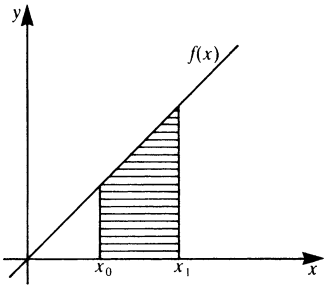
------------------▷ 38

90

Now complete the solution:

$\bar{z} =$ ....

------------------▷ 91



Let us investigate the practical aspects of the evaluation of multiple integrals. **38**

Example: Area under the curve $y = x$ in the interval $x_0 \le x \le x_1$.

The area integral

$$A = \int \mathrm{d}A$$

becomes in terms of cartesian coordinates the integral

$$A = \int\int \mathrm{d}x\,\mathrm{d}y$$

Insert the limits for both variables:

$$A = \int_{y=\ldots}^{\ldots}\int_{x=\ldots}^{\ldots} \mathrm{d}x\,\mathrm{d}y$$

------------------▷ 39

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha)$$ **91**

Well done!

You now have two expressions for a spherical sector:

(a) its volume, $V = \frac{2}{3}\pi R^3(1-\cos\alpha)$, and

(b) the position of its centroid, $\bar{z} = \frac{3}{8}R(1+\cos\alpha)$.

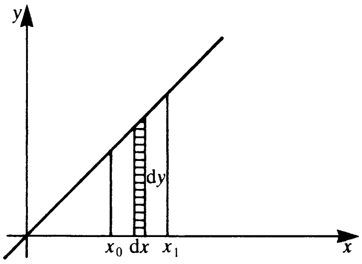
As a final exercise, what are the volume and the centroid of a hemisphere?

$$V = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\bar{z} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution ------------------▷ 93

Explanation wanted ------------------▷ 92

$$A = \int_{y=0}^{f(x)} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y = \int_{y=0}^{x} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y$$ 39



The limits of integration for x are obvious, but the limits for y are from 0 to the value $y = f(x)$. The functional value in this case is $y = x$.

In this example the order of integration is no longer arbitrary. The following rule must be observed:

We take for the inner integral the one whose limits of integration contain variables, and for the outer integral the one that has constant limits. Rearrange the given integrals and write the inner integral in brackets:

$A = \int_{y=0}^{x} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y$ $\qquad$ $A =$ ....

-------------------▷ 40

In the case of a hemisphere the angle α in the expression is equal to 90°, hence $\cos 90^\circ = 0$. The expressions derived in this example are true for all values of α. 92

Now evaluate

$$V = \frac{2}{3}\pi R^3(1-\cos\alpha) = \text{.............................}$$

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha) = \text{..................................}$$

-------------------▷ 93

$$A = \int_{x=x_0}^{x_1} \left[\int_{y=0}^{x} \mathrm{d}y \right] \mathrm{d}x$$ 40

Given the following integral:

$$I = \int_{u=a}^{v} \int_{v=1}^{2} uv \, \mathrm{d}u \, \mathrm{d}v$$

Rearrange the integral in such a way that the bracketed one is the first one to evaluate.

$$I = \int \left[\int \ldots\ldots\ldots\ldots\ldots\ldots \right]$$

--------------------▷ 41

$$V = \frac{2}{3}\pi R^2$$
$$\bar{z} = \frac{3}{8}R$$ 93

Straight on: --------------------▷ 95

Didactical remarks:
Keeping a check on oneself is best done in the following way:

First phase: Attempt the calculation alone with as little help as possible. Compare your solution with the correct one given.

Second phase: If the result is correct this success can have a positive effect on study motivation.
If the result is incorrect you should set out to look for
(a) careless mistakes or slips,
(b) systematic mistakes.

Third phase: If you find a systematic mistake, jot it down on a piece of paper and on finishing the lesson eliminate its cause. This normally means repeating the relevant section of a textbook, along with the accompanying exercises.

--------------------▷ 94

$$I = \int_{v=1}^{2} \left[\int_{u=a}^{v} uv \, \mathrm{d}u \right] \mathrm{d}v$$ 41

We have to integrate with respect to u first since its upper limit is a function of the variable v.

Rearrange the following integral so that it can be evaluated in the correct order.

$$I = \int_{x=0}^{y^2} \int_{y=0}^{z} \int_{z=0}^{1} xyz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$I = \int \left[\int \left[\int \cdots\cdots\cdots \right] \right]$$

--------------------▷ 42

The correction of systematic mistakes is cumbersome but it is the most effective way of improving one's competence. Therefore it is not wrong to deduce that we learn particularly effectively from our mistakes — assuming, of course, that we have *identified*, *analysed*, and *eliminated* the cause of the mistake. 94

--------------------▷ 95

$$I = \int_{z=0}^{1} \left[\int_{y=0}^{z} \left[\int_{x=0}^{y^2} xyz\, \mathrm{d}x \right] \mathrm{d}y \right] \mathrm{d}z$$ 42

Some careful rearranging was necessary in this case.

Now solve the integral!

$I = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution -------------------▷ 46

Explanation and/or help is needed -------------------▷ 43

13.6 Application: Moment of Inertia of A Solid

95

The evaluation of areas and volumes has been used to introduce the concept of multiple integrals. As a further example the evaluation of moments of inertia will be shown.

READ: 13.5 Application: Moments of inertia of a solid
Textbook pages 397–400

-------------------▷ 96

43

We rearrange the integral into a form suitable for evaluation. We have to choose as the inner integrals those whose limits are variable. The outermost integral must have constant limits.

In our example we have

$$I = \int_{z=0}^{1} \int_{y=0}^{z} \int_{x=0}^{y^2} xyz\,\mathrm{d}x \quad \mathrm{d}y \quad \mathrm{d}z$$

Inner integral

Intermediate integral

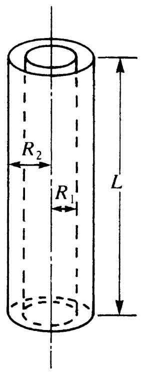
Outer integral with constant limits

For the time being evaluate the inner integral

$$I = \int\int \left[\int_{x=0}^{y^2} xyz\,\mathrm{d}x\right] \mathrm{d}y\,\mathrm{d}z$$

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 44



96

We wish to find the moment of inertia of the hollow cylinder of length L having an inner radius R_1 and an outer radius R_2.

The constant density of the material is ρ. The moment of inertia with respect to the central axis is defined as

$$I = \int_V r^2\,\mathrm{d}m$$

r is the distance from the axis and $\mathrm{d}m$ an element of mass.

$$\mathrm{d}m = \rho\,\mathrm{d}V$$

Hence

$$I = \rho \int r^2\,\mathrm{d}V$$

The problem can be solved most easily using .... coordinates with

$$\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 97

$$I = \int_{z=0}^{1}\int_{y=0}^{z} yz\,\frac{y^4}{2}\,\mathrm{d}y\,\mathrm{d}z$$

44

Remember that, while integrating with respect to x, the variables y and z are considered constant. The integral of x is $\frac{x^2}{2}$ substituting the limits $x = 0$ and $x = y^2$ gives $\frac{y^4}{2}$.

Now evaluate the next one (it was the intermediate one):

$$I = \int_{z=0}^{1}\left[\ldots\ldots\ldots\ldots\right]\mathrm{d}z$$

------------------▷ 45

---

cylindrical coordinates

97

$$\mathrm{d}V = r\,\mathrm{d}\phi\,\mathrm{d}r\,\mathrm{d}z$$

With these coordinates the integral becomes

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Don't forget the limits!

------------------▷ 98

$$I = \int_{z=0}^{1} \frac{z^6}{12} z \,\mathrm{d}z = \int_0^1 \frac{z^7}{12} \mathrm{d}z$$ 45

Finally, evaluate this last integral (which was the outer one):

$I = \ldots\ldots\ldots\ldots$

--------------------▷ 46

$$I = \rho \int_0^{2\pi} \int_0^L \int_{R_1}^{R_2} r^3 \,\mathrm{d}r \,\mathrm{d}z \,\mathrm{d}\phi = \frac{\pi}{2} \rho L \left(R_2^4 - R_1^4\right)$$ 98

Correct --------------------▷ 100

Wrong, or detailed explanation wanted --------------------▷ 99

$$I = \frac{1}{12 \times 8} = \frac{1}{96}$$ 46

In future it should be reasonably obvious which integral is to be the innermost and which is to be the outermost. The intermediate integrals should fall into place.

Have a break!

--------------------▷ 47

In this example the limits are all constant, $r = R_1$ to R_2, $z = 0$ to L and $\phi = 0$ to 2π. 99

Therefore:

$$\begin{aligned} I &= \rho \int r^2 \, \mathrm{d}V \\ &= \rho \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \int_{r=R_1}^{R_2} r^3 \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\phi \end{aligned}$$

Since all the limits are constant the order of integration is not important. Furthermore, this integral may be decomposed into a product of three integrals.

$$\begin{aligned} I &= \rho \int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{z=0}^{L} \mathrm{d}z \int_{r=R_1}^{R_2} r^3 \, \mathrm{d}r \\ &= \frac{\rho}{2} \pi \, L \, ({R_2}^4 - {R_1}^4) \end{aligned}$$

--------------------▷ 100

13.4 Coordinates: Polar, Cylindrical, Spherical

47

Objective: Concepts of polar, cylindrical and spherical coordinates, transformation of coordinates, applications to problems.

We have already met polar coordinates; cylindrical and spherical coordinates are new. The choice depends on the nature of the problem; it should be based on the ease with which the particular problem can be solved.

You know that working parallel with the textbook, pencil in hand, makes sense!

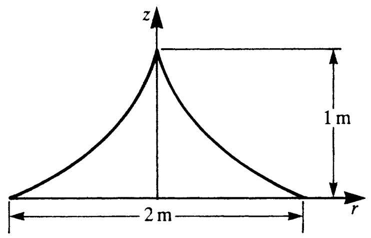
READ: 13.4 Coordinate systems
13.4.1 Polar coordinates
13.4.2 Cylindrical coordinates
Textbook pages 383–393

------------------▷ 48

Here is a little problem for you to solve. It is the last one of this chapter.

100

A stone cutter has just completed the solid shape, shown in the figure, out of sandstone. As you can see, it is a shape with axial symmetry. The density of sandstone is between 2400 and $2700\,\mathrm{kg/m^3}$. The stone cutter has a van with a maximum capacity of 2.5 t. Can he use it to transport his workpiece?



You have to consider:

(1) the volume of the body,
(2) the weight of the body, remembering that the density is not known exactly, only as a range of values.

The equation of the boundary which defines the shape is

$$z = 1 - 1.5r + 0.5r^2$$

Can you do the problem?

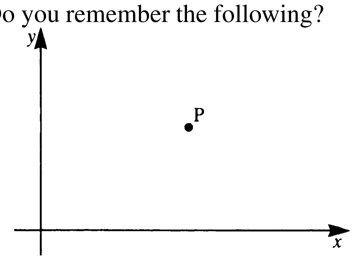
Without help

------------------▷ 104

With help

------------------▷ 101

Do you remember the following?



A point P is to be described in polar coordinates.

Name the two variables:

.... and ....

and place them on the drawing.

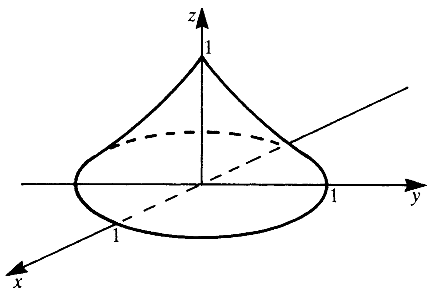
48

-------------------▷ 49

First we need the volume of the solid. The cross-section is shown in the figure and the equation of the boundary line is

101

$$z = 1 - 1.5r + 0.5r^2$$



We have to decide which coordinate system to use. Since it is a solid of revolution, there is axial symmetry. Cylindrical coordinates seem the most appropriate.

The volume is then given by

$$V = \int\int\int r \,\mathrm{d}\phi \,\mathrm{d}r \,\mathrm{d}z$$

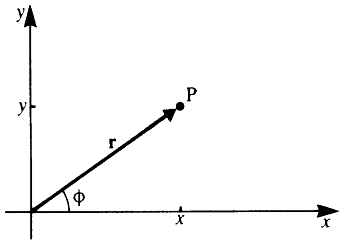
Write down the expression for the volume with limits:

$$V = \int_{\dots}^{\dots} \int_{\dots}^{\dots} \int_{\dots}^{\dots} \dots\dots\dots\dots\dots\dots$$

-------------------▷ 103

Explanation wanted

-------------------▷ 102



Length of the position vector **r**:r.
Angle with the x-axis: ϕ

49

Write down the transformation equations

$$x = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$r = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\tan\phi = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 50

Now for the limits:

102

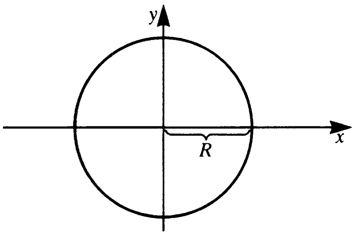
ϕ goes from 0 to 2π
r goes from 0 to 1, as given
z must be given a variable upper limit since it is a function of r, hence
z goes from 0 to $1 - 1.5r + 0.5r^2$
Hence the expression for the volume becomes

$$V = \int_{\phi=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{1-1.5r+0.5r^2} r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\phi$$

------------------▷ 103

$x = r\cos\phi$
$y = r\sin\phi$
$r = \sqrt{x^2 + y^2}$
$\tan\phi = \dfrac{y}{x}$

50



What is the equation of the central circle of radius R in polar coordinates?

....

--------------------▷ 51

$$V = \int_{\phi=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{1-1.5r+0.5r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi$$ 103

Now evaluate the integral above.

$V =$ ....

Solution --------------------▷ 105

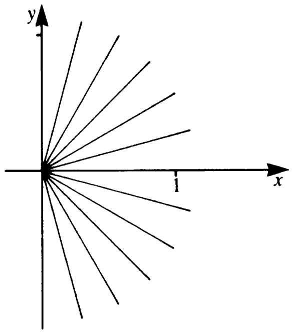
Explanation wanted --------------------▷ 104

$$r = R$$

51

Note: In polar coordinates the equation of the central circle is just as simple as the equation of a straight line parallel to the x-axis (i.e. $y = a$) in Cartesian coordinates.

Sketch the function $r = \cos\phi$ by calculating the value of r for each value of ϕ and mark it on the sketch.



------------------▷ 52

$$V = \int_{\phi=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-1.5r+0.5r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi$$

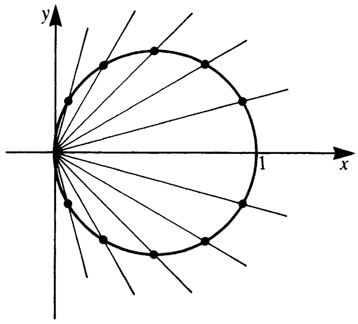
104

The integral with variable limits is the integral with respect to z. It must be done first:

$$V = \int_0^{2\pi}\int_0^1 r\left(1 - 1.5r + 0.5r^2\right)\mathrm{d}r\,\mathrm{d}\phi = 2\pi\int_0^1 (r - 1.5r^2 + 0.5r^3)\,\mathrm{d}r\,\mathrm{d}\phi$$

$$V = \frac{\pi}{4}\,\mathrm{m}^3$$

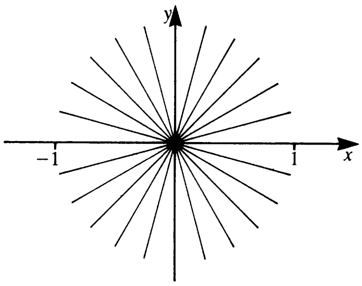
------------------▷ 105



52

Sketch the function $r^2 = \cos 2\phi$.

Note: The function is defined for positive and negative values of ϕ



--------------------▷ 53

$$V = \frac{\pi}{4}\,\mathrm{m}^3$$

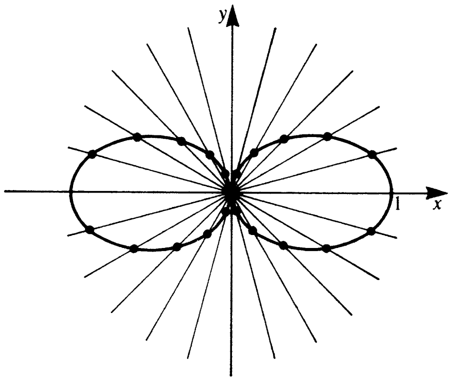
105

What is the mass of the solid to be considered? ρ lies between 2400 and $2700\,\mathrm{kg/m^3}$.

$M = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 106

53



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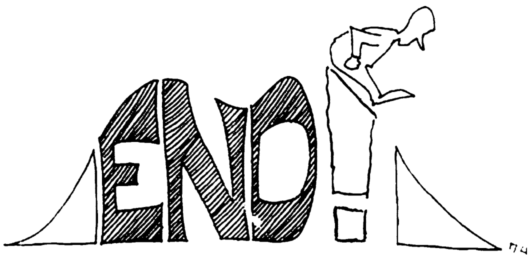
$$M = 2.12\,\mathrm{t}$$

$$M = \text{Volume} \times \text{Density}$$

To be on the safe side it is wiser to take the highest value for the density, hence

$$M = \frac{\pi}{4} \times 2700 = 2120\,\mathrm{kg} = 2.12\,\mathrm{t}$$

This is less than 2.5 tons, and so the van can transport the solid.



END OF CHAPTER 13