Chapter 10
Differential Equations

2

Given: $y'' + 7y' + 10y = 20\cos 4x$

111

If the RHS of a DE is of the form $R_1 \sin ax + R_2 \cos ax$

a trial solution is $A \sin ax + B \cos ax$.

Let the PI be $y_p = A \sin 4x + B \cos 4x$.

Thus $y_p' = 4A \cos 4x - 4B \sin 4x$ and $y_p'' = -16A \sin 4x - 16B \cos 4x$

Substituting in the DE we have

$-16B \cos 4x - 16A \sin 4x - 28B \sin 4x + 28A \cos 4x + 10B \cos 4x + 10A \sin 4x = 20 \cos 4x$

For this to be satisfied for all values of x, the coefficients of $\cos 4x$ must be the same on both sides of the equation; similarly those of $\sin 4x$. Hence we obtain two simultaneous equations:

$$-6B + 28A = 20$$
$$-28B - 6A = 0$$

Solving for A and B yields $A = \frac{28}{41}$ and $B = -\frac{6}{41}$

Hence the PI $y_p =$ ....

112

Before going on to the topic of differential equations you should recapitulate the content of the chapter on complex numbers. Write down the important keywords. Do not spend more than 5 minutes on it! 2

--------------------▷ 3

$$y_p = -\frac{6}{41}\cos 4x + \frac{28}{41}\sin 4x$$ 112

Let us now consider another DE.

The charge Q in a particular electrical circuit is given by the DE

$$\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$$

Obtain a PI of this DE.

The current I in the circuit is $\dot{Q}$. It is the rate of change of the charge with time. Find the current:

$$I = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 117

Explanation and detailed solution wanted --------------------▷ 113

These keywords could be: 3

(i) The imaginary unit $\mathrm{j} = \sqrt{-1}$

(ii) A complex number consists of a real part and an imaginary part; i.e.
$z = x + \mathrm{j}y$
$x =$ real part
$\mathrm{j}y =$ imaginary part; y is a real number.
The modulus $|z| = \sqrt{x^2 + y^2}$ and the argument $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

(iii) Complex numbers can be represented in the Argand plane: x is the real axis and y the imaginary axis.

(iv) The complex number z may be expressed as $z = r\mathrm{e}^{\mathrm{j}\phi}$.

--------------------▷ 4

To obtain a PI for the DE $\dot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$ 113
we assume a solution of the form $Q_{\mathrm{p}} = A\sin 2t + B\cos 2\mathrm{t}$
A and B must be chosen to satisfy the equation.
Differentiating Q we obtain

$$\dot{Q}_{\mathrm{p}} = 2A\cos 2t - 2B\sin 2t$$

and

$$\ddot{Q}_{\mathrm{p}} = -4A\sin 2t - 4B\cos 2t$$

By substituting in the DE and equating sine and cosine terms we have two equations:

....

....

Solution found for A and B --------------------▷ 115

Further explanation wanted --------------------▷ 114

You could have added: 4

(v) The complex conjugate z^* of a complex number $z = x + \mathrm{j}y$ is defined as

$$z^* = x - \mathrm{j}y$$

(vi) Euler's formula:

$$r\mathrm{e}^{\mathrm{j}\phi} = r(\cos\phi + \mathrm{j}\sin\phi)$$

(vii) The exponential form of a complex number can be transformed thus:

$$\mathrm{e}^{(x+\mathrm{j}y)} = \mathrm{e}^x \mathrm{e}^{\mathrm{j}y} = \mathrm{e}^x(\cos y + \mathrm{j}\sin y)$$

Since differential equations require a knowledge of complex numbers you need to be sure that you understand the fundamental operations of complex algebra. If you are in doubt you should return to Chapter 9 in the textbook.

------------------▷ 5

The DE was 114

$$\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$$
$$Q_{\mathrm{p}} = A\sin 2t + B\cos 2t$$
$$\dot{Q}_{\mathrm{p}} = 2A\cos 2t - 2B\sin 2t$$
$$\ddot{Q}_{\mathrm{p}} = -4A\sin 2t - 4B\cos 2t$$

Substituting in the DE yields

$$-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t$$
$$+2A\sin 2t + 2B\cos 2t = 3\sin 2t$$

Equating coefficients of sin $2t$ and cos $2t$ we find

$$\sin 2t(-4A - 4B + 2A) = 3\sin 2t$$
$$\cos 2t(-4B + 4A + 2B) = 0$$

The two equations to determine A and B are

....
....

------------------▷ 115

Differential equations are a branch of the differential and integral calculus. They play a vital role in many aspects of physics and engineering as well as in such topics as economics. 5

Before going any further make sure that you are confident of your ability to differentiate and integrate.

Prerequisites for this chapter are:

(i) differentiation and integration
(ii) algebra and trigonometry
(iii) exponential functions
(iv) complex numbers

-------------------▷ 6

$$-2A - 4B = 3$$
$$+4A - 2B = 0$$

115

Now we solve the equations, obtaining

$$A = -\frac{3}{10}; \ B = -\frac{6}{10}$$

Therefore the PI for the DE is

$$Q_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Remember that we tried a solution of the form

$$Q = A\sin 2t + B\cos 2t$$

-------------------▷ 116

On differentiation:

Test yourself with the following

6

$\frac{d}{dx}(3x^4) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dx}(\cos kx) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dt}(4\sin^2 3t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\frac{d}{dx}(xe^{ax}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 7

$$Q_p = -\frac{3}{10}\sin 2t - \frac{6}{10}\cos 2t$$

116

It follows that for this particular solution the current I is

$$I = \frac{dQ_p}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 117

7

$$\frac{d}{dx}(3x^4) = 3 \times 4x^{(4-1)} = 12x^3$$
$$\frac{d}{dx}(\cos ks) = -k \sin kx$$
$$\frac{d}{dt}(4\sin^2 3t) = 4\frac{d}{dx}(\sin 3t)^2$$
$$= 4 \times 2\sin 3t \cos 3t \times 3 = 24\sin 3t \cos 3t = 12\sin 6t$$

(Note: Apply the function of a function rule twice: put $z = \sin 3t$ and $u = 3t$.) $\frac{d}{dx}(xe^{ax}) = xae^{ax} + e^{ax} = e^{ax}(1 + ax)$ (Note: Use the product rule.)

Differentiate:

$$\frac{d^2}{dx^2}(4e^{7x}) = \text{......................}$$
$$\frac{d^2}{dt^2}(A\cos\omega t + B\sin\omega t) = \text{........................} \quad (A, B, \omega \text{ are constants})$$
$$\frac{d^2}{dt^2}[a\sin(\omega t - \phi)] = \text{..........................} \quad (a, \omega, \phi \text{ are constants})$$

-------------------▷ 8

117

$$\frac{dQ_p}{dt} = I = \frac{6}{5}\sin 2t - \frac{3}{5}\cos 2t$$

We have established a PI for the DE $\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$:

$$Q_p = -\frac{3}{10}\sin 2t - \frac{6}{10}\cos 2t$$

Now we want to obtain the complete solution for the current I.

To do this we have to find the complementary function, i.e. I_c.

The complete solution consists of ....

-------------------▷ 118

$$\frac{d^2}{dx^2}(4e^{7x}) = 4\frac{d}{dx}\left(\frac{d}{dx}e^{7x}\right) = 4\frac{d}{dx}\left(7e^{7x}\right) = 28\frac{d}{dx}\left(e^{7x}\right) = 196e^{7x}$$ 8

$$\begin{aligned}\frac{d^2}{dt^2}(A\cos\omega t + B\sin\omega t) &= \frac{d}{dt}\left[\frac{d}{dt}(A\cos\omega t + B\sin\omega t\right]\\ &= \frac{d}{dt}(-A\omega\sin\omega t + B\omega\cos\omega t)\\ &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\\ &= -\omega^2(A\cos\omega t + B\sin\omega t)\end{aligned}$$

$$\frac{d^2}{dt^2}[a\sin(\omega t - \phi)] = -a\omega^2(\omega t - \phi)$$

If you encountered difficulties you should revise Chapter 5, calculation of differential coefficients, now. This is a prerequisite for the present chapter and your attention must not be diverted by difficulties with elementary calculations.

-------------------▷ 9

The complete solution consists of the complementary function, and the particular integral. Since we have already found the PI we now require the CF. 118

The homogeneous DE is

$$\ddot{Q} + 2\dot{Q} + 2Q = 0$$

The auxiliary equation is

....

whose roots are

$r_1 =$ ...., $r_2 =$ ....

-------------------▷ 119

On integration:

Test yourself with the following integrals: 9

$$\int a e^{bt}\,dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\int \cos(\omega t - \phi)\,dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\int \left(10x^3 + 7x^2 - \frac{5}{x}\right) dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\int (15\cos 7x - 4\sin x)\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

→ 10

119

$$r^2 + 2r + 2 = 0$$
$$r_{1,2} = -\frac{2}{2} \pm \frac{1}{2}\sqrt{2^2 - 4 \times 2}$$

Therefore $r_1 = -1 + \mathrm{j}$, $r_2 = -1 - \mathrm{j}$

Hence Q_c = ...

The complete solution, I = ...

Solution found → 122

Further explanation wanted → 120

10

$$\int a\mathrm{e}^{bt}\,\mathrm{d}t = a\int \mathrm{e}^{bt}\,\mathrm{d}t = \frac{a}{b}\mathrm{e}^{bt} + C$$
$$\int \cos(\omega t - \phi)\mathrm{d}t = \int \cos u \frac{\mathrm{d}u}{\omega} = \frac{1}{\omega}\sin u + C = \frac{1}{\omega}\sin(\omega t - \phi) + C$$
$$\int \left(10x^3 + 7x^2 - \frac{5}{x}\right)\mathrm{d}x = 10\int x^3\,\mathrm{d}x + 7\int x^2\,\mathrm{d}x - 5\int \frac{\mathrm{d}x}{x}$$
$$= \frac{10}{4}x^4 + \frac{7x^3}{3} - 5\ln|x| + C$$
$$\int (15\cos 7x - 4\sin x)\,\mathrm{d}x = 15\int \cos 7x\,\mathrm{d}x - 4\int \sin x\,\mathrm{d}x$$
$$= \frac{15}{7}\sin 7x + 4\cos x + C$$

Do not forget the constant of integration!

If you experienced difficulties you should reread Chapter 6, calculation of integrals. Integration, too, is a prerequisite for the present chapter.

-------------------▷ 11

120

If r_1 and r_2 are the roots of the auxiliary equation then

$$Q_\mathrm{c} = A_1\mathrm{e}^{r_1 t} + A_2\mathrm{e}^{r_2 t}$$
$$= A_1\mathrm{e}^{(-t+\mathrm{j}t)} + A_2\mathrm{e}^{(-t-\mathrm{j}t)} = \mathrm{e}^{-t}(A_1\mathrm{e}^{\mathrm{j}t} + A_2\mathrm{e}^{-\mathrm{j}t})$$

This is a complex quantity. We are interested in its real part:

$$Q_\mathrm{c} = \mathrm{e}^{-t}(C_1\cos t + C_2\sin t)$$

-------------------▷ 121

You should commit to memory the derivatives and integrals of fundamental functions, i.e. 11

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$
$$\frac{d}{dx}\sin x = \cos x \qquad \int \cos x\,dx = \sin x + C$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \int \sin x\,dx = -\cos x + C$$
$$\frac{d}{dx}e^x = e^x \qquad \int e^x = e^x + C$$
$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \int \frac{dx}{x} = \ln|x| + C,\ x \neq 0$$

The derivatives and integrals of other functions will be found in the tables in the textbook.

-------------------▷ 12

Since $I_c = \frac{dQ_c}{dt} = \dot{Q}_c$ 121
Differentiating Q_c we have

$$\begin{aligned} I_c = \dot{Q}_c &= -e^{-t}(C_1\cos t + C_2\sin t) + e^{-t}(-C_1\sin t + C_2\cos t) \\ &= e^{-t}[\cos t(C_2 - C_1) + \sin t(-C_1 - C_2) \end{aligned}$$

or $I_c = e^{-t}(A\cos t + B\sin t)$
where $A = C_2 - C_1$, $B = -(C_1 + C_2)$
The complete solution is

$$\begin{aligned} I &= I_c + I_p \\ &= \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

-------------------▷ 122

Finally, before starting on differential equations, do the following exercises: 12

(a) If $y = 24e^{-2t}\cos(3t-5)$
then $\frac{dy}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(b) If $x^2 + 4x - 21 = 0$
the roots are $r_1 = \ldots\ldots\ldots\ldots\ldots\ldots$, $r_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

(c) If $x^2 - x + 2.5 = 0$
the roots are $r_1 = \ldots\ldots\ldots\ldots\ldots$, $r_2 = \ldots\ldots\ldots\ldots\ldots$

(d) Evaluate $\int 24\cos(15t + ß)\,dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 13

$$I = e^{-t}(A\cos t + B\sin t) + \frac{6}{5}\sin 2t - \frac{3}{5}\cos 2t$$ 122

Let us now discuss another difficult DE:
Obtain a PI for the DE

$$\ddot{x} + 4x = 3\cos 2t$$
$$x_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 127

Hints and explanation wanted --------------------▷ 123

13

(a) $\dfrac{dy}{dt} = -24e^{-2t}\,[3\sin(3t-5)+2\cos(3t-5)]$ (product rule)

(b) $r_1 = 3$, $r_2 = -7$

(c) $r_1 = \dfrac{1}{2}+\dfrac{3}{2}j$, $r_2 = \dfrac{1}{2}-\dfrac{3}{2}j$ (quadratic equation, complex numbers)

(d) $\dfrac{24}{15}\sin(15t+\pi)+C$

If you have doubts then you must return to the relevant chapters in the textbook and the study guide before going on to differential equations. You must overcome your difficulties first.

→ 14

123

The auxiliary equation is $r^2+4=0$, i.e. $r=\pm 2j$.

This is the breakdown situation, because of the factor 2 in cos $2t$.

In this case a trial solution is

$$x_p = At\sin 2t + Bt\cos 2t$$

We need the second differential coefficient:

$$\ddot{x}_p = \ldots\ldots\ldots\ldots$$

→ 124

10.1 Differential Equations: Concepts and Classification

14

Objective: Concept of a differential equation, types of differential equation.

READ: 10.1 Concept and classification of differential equations
Textbook pages 275–279

It is especially useful to memorise the somewhat cumbersome classification of differential equations. Take notes.

--------------------▷ 15

$$\ddot{x}_\text{p} = 4A\cos 2t - 4B\sin 2t - 4At\sin 2t - 4Bt\cos 2t$$

124

Substituting in the DE $\ddot{x} + 4x = 3\cos 2t$ for x_p and $\ddot{x}_\text{p}$ leads to
.... $= 3\cos 2t$

--------------------▷ 125

Answer the following questions without any help and check your results using your notes. Which of the following equations are DEs? **15**

(a) $x^n = y^3$
(b) $f(x) = 4x^{-1} + 3$
(c) $f(x) = f'(x)$
(d) $y = y^3$
(e) $y = (y'')^3 + 2xy + 17$
(f) $0 = y' + ay' + by + C$
(g) $y'' = C$

--------------------▷ 16

$$4A\cos 2t - 4B\sin 2t = 3\cos 2t$$ **125**

Solving for A and B yields

$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\quad B = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Thus the PI is

$x_\text{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution --------------------▷ 127

Further hints wanted --------------------▷ 126

The DEs are: (c), (e), (f), (g). 16

Not more than one mistake --------------------▷ 19

More than one mistake --------------------▷ 17

We have $4A\cos 2t - 4B\sin 2t = 3\cos 2t$ 126

Comparing coefficients of sin $2t$ and cos $2t$ we find

$$4A = 3, \text{ hence } A = \ldots\ldots\ldots\ldots$$
$$-4B = 0, \text{ hence } B = \ldots\ldots\ldots\ldots$$

Since we assumed $x_p = At\sin 2t + Bt\cos 2t$ we obtain the PI

$$x_p = \ldots\ldots\ldots\ldots$$

--------------------▷ 127

Using the textbook, which are DEs? 17

(a) $y' + C = y'' + y^3$
(b) $f(x) = x^3 + 2x^2 + 3x + 5$
(c) $y'' = (y')^5 + (y'')^2$
(d) $y^3 = 2xy$
(e) $y'' = y$
(f) $y = y^2$

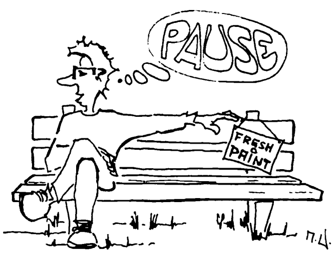
$\dashrightarrow$ 18

$$A = \frac{3}{4};\ B = 0$$ 127

$$x_{\mathrm{p}} = \frac{3}{4} t \sin 2t$$

You deserve a break!

TIME YOURSELF!



$\dashrightarrow$ 128

The DEs are: (a), (c), (e). 18

--------------------▷ 19

The break is over! 128

Before going any further you should recapitulate, if possible without reference to the textbook or the study guide.

If the $f(x)$ in the DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

is of the form

(i) $a + bx + cx^2 + \cdots$
a trial solution for the PI is ....

(ii) $C\mathrm{e}^{bx}$
a trial solution for the PI is ....

(iii) $R_1 \sin ax + R_2 \cos ax$
a trial solution for the PI is ....

--------------------▷ 129

Which of the following are second-order DEs? 19

(a) $(y'')^3 + (y')^4 + y^5 = C$
(b) $y^2 + (y')^2 = x$
(c) $y'' = 0$
(d) $y''' + y'' = 0$

--------------------▷ 20

(i) $y_p = A + Bx + Cx^2 + \cdots$ 129
(ii) $y_p = Ae^{\lambda x}$
(iii) $y_p = A\sin ax + B\cos ax$

Remember that in case (ii), if λ is a root of the auxiliary equation the trial solution should be

$y_p =$ ....

In case (iii), if a is a term of the CF the trial solution must be

$y_p =$ ....

--------------------▷ 130

(a), (c) are second-order DEs. 20

All correct ▷ 23

Mistakes; further explanation wanted ▷ 21

(ii) $y_p = xAe^{\lambda x}$

(iii) $y_p = x(A\sin ax + B\cos ax)$ 130

This section has been quite demanding. But DEs are important for understanding quite a number of basic topics in physics, economics and engineering.

Some remarks on motivation: ▷ 131

Straight on: ▷ 136

A DE is said to be of the second order if the *highest* derivative in the equation is the second derivative. 21

Which of the following are second order DEs?

(a) $y'' + y''' = 0$
(b) $y'' + C = y^3$
(c) $y' = 2xy + y^2$
(d) $y' - y'' = 0$

--------------------▷ 22

In a survey successful students were asked about the reasons for their success in studying. 131

Their report was as follows:

Good studying habits	38%
Interest	25%
Intelligence	15%
Other reasons	22%

Less successful students questioned in the survey gave the following reasons for their failure:

Lack of effort	25%
Lack of interest	35%
Personal problems	8%
Various reasons	32%

Lack of interest, lack of effort, inefficient studying habits are all connected.

There is a reason — a motive — behind every action. Motives also determine the intensity and the course of the learning process.

--------------------▷ 132

Second-order DEs are (b) and (d). 22

--------------------▷ 23

Two students are having an animated conversation about music and about the concerts they have been to recently. They exchange expert opinions on conductors and performances and compare the different interpreters' conceptions of Mozart's piano concertos. At the end they talk about their studies. Both deplore their memory. One reads chemistry, the other biology. 132

Which statement would you feel inclined to agree with?

The students have a selective memory. It is bad at chemistry and biology, but when it comes to music it is exceptionally good.

Both students have a normal memory. They are just more interested in music than in their studies.

--------------------▷ 133

Which of the following DEs are linear? 23

(a) $C_2 y'' + C_1 y' + C_0 y = f(x)$
(b) $x y'' + x^2 y' = y$
(c) $(y'')^2 + y' = y + C$
(d) $y' = y^3$

▷ 24

That both students possess a good memory is probably an accurate claim. 133

Success in learning and motivation are connected.

The students' interest in music is a *primary motivation*. Spending time on music is the result of one's own personal desire. It is a satisfying activity.

Spending one's time on studying doesn't appear to be so much fun. Perhaps it is done only to enable one to earn a living later. In this case we are talking about *secondary motivation*.

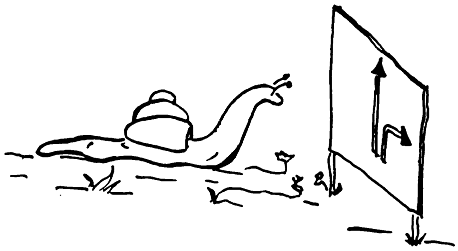
As far as studying is concerned, secondary motivation is the more frequent of the two types. This can, however, be partly transformed into primary motivation when interest is awakened through study success. Many psychological studies have shown that the more time one spends on a subject and the more one understands about it, the more interesting it becomes.

Even studying mathematics can be interesting.

▷ 134

(a), (b).

24



All correct

------------------▷ 27

Mistakes

------------------▷ 25

134

The proportion of primary or secondary motivation in the different fields of study differs from person to person.

The experience of success has a stimulating effect on both types of motivation.

Setting oneself attainable goals, while also keeping a check on oneself, can produce positive results. Quickly attainable goals like, for example, the mastering of easy exercises increase self-confidence and thus indirectly increase the chances of becoming interested.

Sometimes you have to decide between two activities. For example

(a) You must study a chapter.
(b) A friend calls and suggests you go swimming together.

Here, a conflict of motivation presents itself. Both activities are desirable but you can pursue only one.

What would you do?

------------------▷ 135

A DE is linear if its derivatives $(y', y''$, etc.) and the function itself (y) occur to the first power, i.e. to the first degree, and there are no products like yy' etc. 25

Which DEs are linear?

(a) $y' + y'' + y^2 = 0$
(b) $y'' + 3xy + C = 0$
(c) $y' = C + x^2$
(d) $y' + y'' = 2x + 5$

------------------▷ 26

Regardless of the answer you give, one thing can be assumed. For example, let's consider secondary-motivated activities. Here, in any conflict of motivation, the danger of opting for those activities which involve the least effort is far greater. 135

Someone who has to prepare a seminar paper may succumb to the temptation of washing the car, papering the living room or painting the furniture. Thus he avoids making the necessary effort and still retains the feeling of having done something that was necessary.

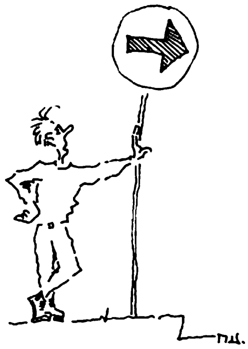
Difficult and unpleasant tasks which subjectively appear daunting can be changed into easier tasks if you break them down into smaller parts. That's what the study guide does for you. But you can do that for yourself, too. A work plan helps to break down difficult tasks into smaller steps.

Note the Chinese proverb:

Every long march begins with a first step.

------------------▷ 136

The linear DEs are: (b), (c), (d). 26



--------------------▷ 27

10.6 Variation of Parameters

136

This section is not compulsory. It is supplementary and may be studied later on.

I would like to skip it.

--------------------▷ 141

I would like to work through it!

Variation of parameters is a systematic but possibly lengthy method to find a PI of a non-homogeneous DE.

READ: 10.3.2 Non-homogeneous linear DE
Method of variation of parameters
Textbook pages 287–292

--------------------▷ 137

Which of the following DEs are homogeneous? 27

(a) $y'' + y + C = 0$
(b) $y'' + y = x^3$
(c) $y'' + 5x = 0$
(d) $y' + y = 0$

--------------------▷ 28

Obtain the PI of the DE 137

$$y'' - 4y = x$$

using the method of variation of parameters.
Use the scheme given in the textbook.

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 140

Further explanation and detailed solution --------------------▷ 138

(d) is homogeneous

28

Correct ----------> 31

Wrong ----------> 29

Given the DE: $y'' - 4y = x$

138

Two independent solutions of the CF are $y_1 = e^{-2x}$ and $y_2 = e^{2x}$
and the derivatives are $y_1' = -2e^{-2x}$ and $y_2' = 2e^{2x}$
The particular integral is assumed to be of the form $y_p = V_1 e^{-2x} + V_2 e^{+2x}$.
Substituting in equations 3 and 6 of section 10.3.2 of the textbook yields the relations:

$$V_1' e^{-2x} + V_2' e^{2x} = 0 \quad [1]$$

$$-2V_1' e^{-2x} + 2V_2' e^{2x} = x \quad [2]$$

Solve [1] for V_1' and find $V_1' = -V_2' e^{4x}$
Substitute in [2] and obtain

$$V_2' = \frac{x e^{-2x}}{4} \quad [3]$$

and
$$V_1' = -\frac{x e^{2x}}{4} \quad [4]$$

----------> 139

Read the definition in the textbook. Which of the following DEs are homogeneous? Check your answer using the textbook. 29

(a) $y'' + x = C$
(b) $xy = 0$
(c) $xy' = x$
(d) $y'' + y' = 2xy^2$

-------------------▷ 30

The functions V_1 and V_2 are obtained by integration of equations 3 and 4. 139

Hence

$$V_2(x) = \frac{1}{4}\int x\mathrm{e}^{-2}\,\mathrm{dx} \qquad (\text{let } u = x,\ \ v' = \mathrm{e}^{-2x})$$
$$V_1(x) = -\frac{1}{4}\int x\mathrm{e}^{2x}\,\mathrm{d}x \qquad (\text{let } u = x,\ \ v' = \mathrm{e}^{2x})$$

The solutions are

$$V_2(x) = \frac{\mathrm{e}^{-2x}}{16}(-2x-1)$$
$$V_1(x) = -\frac{\mathrm{e}^{2x}}{16}(2x-1)$$

Remembering that the PI is given by

$$y_\mathrm{p} = V_1(x)y_1 + V_2(x)y_2$$

then by substituting y_1 and y_2 we have

$$y_\mathrm{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 140

(b) and (d) are homogeneous

30

31

$$y_{\mathrm{p}} = -\frac{1}{4}x$$

140

You should verify that $y_{\mathrm{p}} = -\dfrac{1}{4}x$ is a particular integral of $y'' - 4y = x$.

You should always satisfy yourself of the correctness of your solution.

141

In section 10.1 it was shown that the general solution of a DE contains arbitrary constants. How many such constants are there in the general solution of a second-order DE? 31

Solution --------------------▷ 33

Further explanation wanted --------------------▷ 32

10.7 Boundary Value Problems

141

Objective: Evaluation of the arbitrary constants given certain boundary conditions.

READ: 10.4 Boundary value problems
Textbook pages 293–295

--------------------▷ 142

The number of arbitrary constants in a DE is easily memorised: the number corresponds to the number of integrations necessary to obtain a solution. In the case of an nth order DE we have to integrate n times and, therefore, there will be n arbitrary constants. 32

How many arbitrary constants are there in the general solution of a second-order DE? ....

--------------------▷ 33

The DE $y' - 4y = 0$ has the general solution 142

$y = C\mathrm{e}^{4x}$

Obtain the particular solution of the DE given that when $x = \frac{1}{4}$, $y = 2\mathrm{e}$.

$y =$ ....

--------------------▷ 143

Two constants

33

How many boundary conditions are required to obtain the values of the constants in a second order DE? ....

If the values of the constants are fixed we have a special solution called ....

Answers found ---------> 35

Further explanation wanted ---------> 34

$$y = 2e^{4x}$$

143

Correct ---------> 145

Wrong, or explanation wanted ---------> 144

Since an nth order DE contains n arbitrary constants it follows that these n constants can be determined if we specify n boundary conditions, i.e. one condition for each constant. 34

How many boundary conditions are required to obtain the values of the constants in a second-order DE? ....

Find in the textbook the name of the special solution if the constants are determined according to boundary conditions: ....

------------------▷ 35

To obtain the particular solution of the DE $y' - 4y = 0$, we have to insert into the general solution $y = C\mathrm{e}^{4x}$ the boundary conditions given. 144

In this case the boundary conditions are those when $x = \frac{1}{4}$, $4y = 2\mathrm{e}$. This means the curve of the solution must contain the point $\left(\frac{1}{4},\quad 2\mathrm{e}\right)$. Substituting in the general solution we have

$$2\mathrm{e} = C\mathrm{e}^{4\times 1/4} = C\mathrm{e}, \quad \text{hence } C = 2,$$

and the particular solution is $y = 2\mathrm{e}^{4x}$.

------------------▷ 145

two boundary conditions
particular solution, or particular integral

35

Let us recapitulate:

The order of a DE is given by the order of the highest derivative in the equation.

A DE is linear if y and the derivatives of y are of the first power and no products of y and its derivatives occur.

The general solution of a DE of nth order contains n arbitrary constants. They can be determined if n additional conditions are given which are called boundary conditions.

If the constants in the general solution are determined according to certain boundary conditions we call this special solution the *particular solution* or *particular integral*.

------------------▷ 36

The DE $\dot{v}(t) = -g$ has the general solution

145

$$v(t) = -gt + C$$

Obtain the value of the constant if $t = 0$ when $v(t) = v_0$.

$$C = \text{..........................}$$

------------------▷ 146

10.2 Preliminary Remarks

36

In this section it is shown that the general solution of a non-homogeneous linear DE consists of two parts: a general solution of the homogeneous DE and a particular solution of the non-homogeneous DE.

READ: 10.2 Preliminary remarks
Textbook pages 279–280

37

146

$$C = v_0$$

Hence $v(t) = -gt + v_0$.

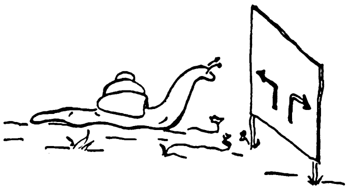
147

Given the non-homogeneous DE **37**

$$y'' = x$$

obtain the general solution

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found --------------------▷ **41**

Further explanation wanted --------------------▷ **38**

Now to boundary value problems for second-order DEs. **147**

We know that we need two boundary conditions.

Obtain the particular solution of the DE

$$y'' - 3y' + \frac{9}{4}y = 0$$

given that (a) $x = \frac{2}{3}$ when $y = 3\mathrm{e}$

and (b) $x = \frac{2}{3}$ when $y' = \frac{15}{2}\mathrm{e}$

The general solution is

$$y = C_1\mathrm{e}^{\frac{3}{2}x} + C_2 x\mathrm{e}^{\frac{3}{2}x}$$

$$C_1 = \ldots\ldots\ldots\ldots, \quad C_2 = \ldots\ldots\ldots\ldots$$

Particular solution: $y = \ldots\ldots\ldots\ldots$ --------------------▷ **148**

$y'' = x$ is a non-homogeneous DE
$y'' = 0$ is the homogeneous DE

38

The general solution can be found in three steps, as is outlined in the textbook.

Step 1: Find the complementary function y_c of the homogeneous DE.
Note: complementary function means the general solution of the homogeneous DE.

In this case

$$y'' = 0$$
$$y_c = \dots\dots\dots\dots\dots\dots$$

→ 39

$$C_1 = 1,\ C_2 = 3$$
$$y = e^{3x/2} + 3xe^{3x/2}$$

148

Correct → 150

Wrong, or further explanation wanted → 149

39

Verification:

$$y_c = C_1 x + C_2$$
$$y_c{}' = C_1$$
$$y_c{}'' = 0$$

Step 2: Find a particular integral y_p of the non-homogeneous DE.
Note: particular integral means a solution of the non-homogeneous DE.

In this case

$$y'' = x$$
$$y_p = \text{..........................}$$

--------------------▷ 40

149

We have two boundary conditions:

$$y\left(\frac{2}{3}\right) = 3\mathrm{e}$$
$$y'\left(\frac{2}{3}\right) = \frac{15}{2}\mathrm{e}$$

The general solution is:

$$y = C_1\mathrm{e}^{3x/2} + C_2 x\mathrm{e}^{3x/2}$$

The derivative is

$$y' = \frac{3}{2}C_1\mathrm{e}^{3x/2} + C_2\mathrm{e}^{3x/2} + \frac{3}{2}C_2 x\mathrm{e}^{3x/2}$$

Substituting the boundary conditions yields

$$3\mathrm{e} = C_1\mathrm{e} + \frac{2}{3}\mathrm{e}C_2 \quad \text{and} \quad \frac{15}{2}\mathrm{e} = \left(\frac{3}{2}C_1 + C_2(1+1)\right)$$

from which we get

$$C_1 = 1 \text{ and } C_2 = 3$$

--------------------▷ 150

$$y_{\mathrm{p}} = \frac{x^3}{6}$$ 40

Verification:

$$y_{\mathrm{p}}'' = \frac{3 \times 2 \times x}{6} = x$$

Step 3: The general solution of the non-homogeneous DE is the sum of the complementary function y_{c} and a particular integral y_{p}.

In our case

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 41

The general solution of the DE $\ddot{x} = -g$ (motion of a freely falling body) is 150

$$x = -\frac{g}{2}t^2 + C_1 t + C_2$$

Obtain the values of the constants such that the equation satisfies the boundary conditions:
when $t = 0$, $x = 0$ and $\dot{x} = v_0$

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 151

$$y = \frac{x^3}{6} + C_1 x + C_2$$ 41

Given the non-homogeneous DE

$$2y' = x + 1$$

Write down the homogeneous DE

....

The general solution of the homogeneous DE is called

....

Obtain it (step 1)

$$y_c = \text{...........................}$$

--------------------▷ 42

$$x = \frac{g}{2}t^2 + v_0 t$$ 151

Solution: First condition: $x(0) = C_2 = 0$
Therefore $C_2 = 0$

Second condition: $\dot{x}(0) = C_1 = v_0$
Therefore $C_1 = v_0$

Further examples of using boundary conditions will be given in the following section on applications.

--------------------▷ 152

$$2y' = 0$$

42

the complementary function y_c:

$$y_c = C_1 \text{ (since } y_c' = 0)$$

Given: $$2y' = x + 1$$

Obtain a solution of the non-homogeneous DE (step 2). It is called ....

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 43

10.8 Application to Problems in Physics and Engineering

152

Objective: Solution of typical DEs encountered in practice.

This section shows typical applications of DEs in science. We will use the symbols generally used in practical applications.

READ: 10.5 Some applications of differential equations
Textbook pages 295–304

-------------------▷ 153

a particular integral 43

$$y_p = \frac{x^2}{4} + \frac{x}{2}$$

Given: $2y' = x + 1$

Complementary function $y_c = C_1$, particular integral $y_p = \dfrac{x^2}{4} + \dfrac{x}{2}$

Now obtain the general solution (step 3)

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 44

Growth and decay functions. 153

We frequently encounter in practice DEs whose solutions are of the form

$$y = \alpha e^{\beta t} \quad \text{or} \quad y = \alpha e^{-\beta t}$$

The first one shows that the quantity y increases exponentially with time whilst the second one shows that y decreases exponentially with time. They represent processes of 'growth' and 'decay' respectively, e.g. the growth of viruses and the decay of a radioactive substance. Other examples are to be found in electrical networks and in oscillations. The time t is not always the independent variable.

Develop the DE of a bacterial culture for which the rate of growth $\dfrac{d}{dt}N(t)$ is proportional to the actual number $N(t)$ of bacteria present. Call α the constant of proportionality.

....

--------------------▷ 154

$$y = y_c + y_P = \frac{x^2}{4} + \frac{x}{2} + C_1$$ 44

Note: To determine C_1 we need a supplementary (boundary) condition.
Suppose $y(0) = 1$
We obtain

$$y(0) = 1 = 0 + 0 + C_1$$

Thus $C_1 = 1$, in this case.

--------------------▷ 45

$$\frac{\mathrm{d}}{\mathrm{d}t} N(t) = \alpha N(t)$$ 154

or

$$\dot{N}(t) = \alpha N(t)$$

Now solve the equation $\dot{N}(t) = \alpha N(t)$

$$N(t) = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 155

10.3 General Solution of First- and Second-Order DEs with Constant Coefficients

45

The following section is somewhat lengthy. You would be well advised to divide it into two or three parts, revising at the end of each part.

Take notes and follow the calculations separately.

READ: 10.3 General solution of first- and second-order DEs with constant coefficients
10.3.1 Homogeneous linear DE
Textbook pages 281–286

46

$$N(t) = C\mathrm{e}^{\alpha t}$$

155

Explanation of the solution:

$$\dot{N}(t) - \alpha N(t) = 0$$

This is a homogeneous equation of the first order with constant coefficients.

The auxiliary equation is $r - \alpha = 0$, root $r = \alpha$.

Hence the solution is

$$N = C\mathrm{e}^{\alpha t}$$

It may also be noted that in this case it is possible to separate the variables:

$$\frac{\mathrm{d}N}{N} = \alpha \mathrm{d}t$$

Integrating yields

$$\ln N = \alpha t + C$$

Solving for N we find $N = C\mathrm{e}^{\alpha t}$

156

What is the auxiliary equation of this DE?

46

$$y'' + 2y' - y = 0$$

(Use the exponential solution $y = C\mathrm{e}^{rx}$.)

Answer found --------------------▷ 48

Further explanation wanted --------------------▷ 47

Given: $N(t) = C\mathrm{e}^{\alpha t}$

156

If at $t = 0$, $N = 100$ bacteria are present, obtain the particular solution of the DE.

$$N(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: With $N = 100$ when $t = 0$ we have a boundary condition. Note that $\mathrm{e}^0 = 1$.

--------------------▷ 157

Let us discuss an example: 47

Given: $y'' - y = 0$

If $y = C\mathrm{e}^{rx}$ then $y' = Cr\mathrm{e}^{rx}$ and $y'' = Cr^2\mathrm{e}^{rx}$.

Substituting in the DE yields

$$(r^2 - 1)C\mathrm{e}^{rx} = 0$$

The term in the brackets must be zero, since $C\mathrm{e}^{rx} \neq 0$.

Thus the auxiliary equation reads $r^2 \quad - 1 = 0$.

Using $y = C\mathrm{e}^{rx}$, obtain the auxiliary equation of the DE

$$y'' + 2y' - y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 48

$$N(t) = 100\mathrm{e}^{\alpha t}$$ 157

Radioactive decay.

The decay of a sample of radium has been investigated.

If N is the total number of nuclei present in the sample at time t and $\mathrm{d}N$ is the number decaying in time $\mathrm{d}t$ then

$$\mathrm{d}N = -N\lambda\mathrm{d}t$$

or

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

which may be written as $\dot{N} + \lambda N = 0$, a first order DE with constant coefficients.

If N_0 is the number of nuclei when $t_0 = 0$, obtain the particular solution of the DE.

$$N = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 158

$$r^2 + 2r - 1 = 0$$

48

The auxiliary equation of the DE

$$3y'' + 2y' - 2y = 0$$

is ....

The roots of the auxiliary equation are:

$$r_1 = \text{...........................}, \quad r_2 = \text{...........................}$$

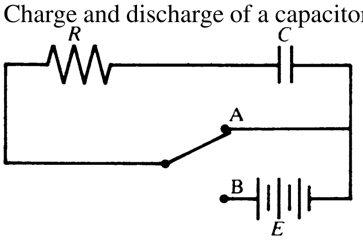
Answers found --------------------▷ 50

Difficulties --------------------▷ 49

$$N = N_0 e^{-\lambda t}$$

158

Charge and discharge of a capacitor in an electrical circuit.



The figure shows a simple circuit consisting of a capacitor of C farads, a resistor of R ohms, two switches A and B and a battery having a constant voltage E.

Case (a): Initially the capacitor is not charged, i.e. the boundary condition is that when $t = 0$, $Q = 0$. (Q is the charge.)

When switch B is closed and A open a current flows in the circuit and the charge in the capacitor increases. The DE for the charge as a function of time is

$$R\dot{Q} + \frac{1}{C}Q = E$$

Solve the equation: i.e. obtain a particular solution

$$Q = \text{................................}$$

--------------------▷ 159

Go back to section 10.3.1 in the textbook and study it more thoroughly, then try again. 49

Auxiliary equation of $3y'' + 2y' - 2y = 0$ (use $y = Ce^{rx}$): ....

Obtain the roots:

$$r_1 = \text{...........................},\ \ r_2 = \text{...........................}$$

Perhaps you ought to revise quadratic equations? (These were treated in Chapter 1.)

--------------------▷ 50

$$Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$$ 159

Correct --------------------▷ 162

Wrong, or detailed solution wanted --------------------▷ 160

$$3r^2 + 2r - 2 = 0$$
$$r_1 = \frac{1}{3}\left(-1 + \sqrt{7}\right)$$
$$r_2 = \frac{1}{3}\left(-1 - \sqrt{7}\right)$$

50

What is the general solution of the DE

$$3y'' + 2y' - 2y = 0?$$
$$y = \text{.................................}$$

Solution --------------------▷ 53

Further explanation wanted --------------------▷ 51

The DE is $R\dot{Q} + \frac{1}{C}Q = E$.

160

We will solve this equation by using the method we have developed.

But it may be added that the method of separating the variables and direct integration is equally convenient in this case.

The solution will consist of two functions, namely the CF and the PI. The CF is the solution of

$$R\dot{Q} + \frac{1}{C}Q = 0$$

The CF is $Q_c =$ ....

Hint: If you rewrite the equation using the substitutions $R = a_1$; $\frac{1}{C} = a_0$; $Q = y$ you will find the problem quite easy.

--------------------▷ 161

Let's do an easier example. 51

Find the general solution of the DE

$$4y'' - y = 0$$

by following the scheme:

Step 1: Exponential solution: $y = Ce^{rx}$

Step 2: Establish the auxiliary equation: $ar^2 + br + c = 0$

....

Step 3: Find the roots of the auxiliary equation: $r_1 =$ ...., $r_2 =$ ....

Step 4: General solution of the DE: $y =$ ....

Note: Since the DE is of the second order we find two roots and thus two exponential solutions. The general solution is the sum of both.

--------------------▷ 52

$$Q_c = Ae^{-\frac{t}{RC}}$$ 161

$\left(\text{You will have noted that the auxiliary equation is } r + \frac{1}{RC} = 0. \text{ The root is } r = -\frac{1}{RC}\right)$

The PI of $R\dot{Q} + \frac{1}{C}Q = E$

is $Q_p =$ ....

--------------------▷ 162

52

Auxiliary equation: $4r^2 - 1 = 0$
Roots of the auxiliary equations: $r_1 = 0.5$, $r_2 = -0.5$
General solution of the DE: $y(x) = C_1 e^{0.5x} + C_2 e^{-0.5x}$
Note: the general solution contains two arbitrary constants.

Now try again, following the same scheme.

Given:

$$3y'' + 2y' - 2y = 0$$
$$y = Ce^{rx}$$

Auxiliary equation $3r^2 + 2r - 2 = 0$
The roots of the quadratic equation are

$$r_1 = \frac{1}{3}(-1 + \sqrt{7}),\ r_2 = \frac{1}{3}(-1 - \sqrt{7})$$

Since both roots are real and distinct, the general solution is given by

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Remember the two arbitrary constants; the DE is of the second order!

--------------------▷ 53

$$Q_{\mathrm{p}} = EC$$

162

Explanation:
Let $Q_{\mathrm{p}} = B$, a constant, then $\dot{Q}_{\mathrm{p}} = 0$.
It follows that $0 + \frac{1}{C}B = E$.

Therefore $B = EC$ is a particular integral.
Now find the general solution using $Q_{\mathrm{c}} = Ae^{-\frac{t}{RC}}$; $\quad Q_{\mathrm{p}} = EC$:

$$Q = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 163

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

or $$y = C_1 e^{1/3(-1+\sqrt{7})x} + C_2 e^{1/3(-1-\sqrt{7})x}$$

53

Given the DE: $16y'' - 8y' + 26y = 0$
Auxiliary equation: $16r^2 - 8r + 26 = 0$

The roots are $$r_1 = \frac{1}{4} + \frac{5}{4}\mathrm{j}$$

and $$r_2 = \frac{1}{4} - \frac{5}{4}\mathrm{j}, \ (\mathrm{j} = \sqrt{-1})$$

The real form of the solution of the DE is $y =$ ....
Note: In case of difficulties solve the example step by step, using the textbook.

--------------------▷ 54

$$Q = Ae^{-\frac{t}{RC}} + CE$$

163

Now use the boundary condition ($t = 0$, $Q = 0$) to calculate A and hence the particular solution.

$A =$ ....
$Q =$ ....

--------------------▷ 164

$$y = e^{\frac{1}{4}x}\left(C_1 \cos\frac{5}{4}x + C_2 \sin\frac{5}{4}x\right)$$ 54

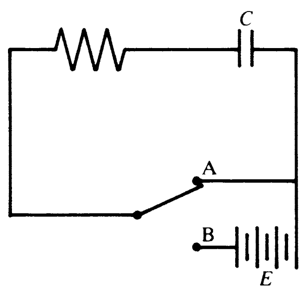
What is the general real-valued solution of this DE?

$$3y'' + 5y' + 4y = 0$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 55

$$A = -EC$$
$$Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$$ 164

(Note that the charge Q tends to the value EC exponentially as $t \to \infty$.)



Case (b): Referring to the diagram, if the charge is Q_0 when $t = 0$ and switch A is closed (B open) a current will flow in the circuit and the capacitor will discharge.

The DE in this case is

$$R\dot{Q} + \frac{1}{C}Q = 0.$$

Obtain the particular solution

$$Q = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 165

$$y = \mathrm{e}^{-\frac{5}{6}x}\left(C_1 \cos\frac{\sqrt{23}}{6}x + C_2 \sin\frac{\sqrt{23}}{6}x\right)$$

55

Hint: The roots of the auxiliary equation
$3r^2 + 5r + 4 = 0$ are

$$r_1 = -\frac{5}{6} + \frac{1}{6}\sqrt{23}\mathrm{j}, \qquad r_2 = -\frac{5}{6} - \frac{1}{6}\sqrt{23}\mathrm{j}$$

Find the real-valued solution of the DE

$$y'' + 2y' + 5y = 0$$

$y = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 56

$$Q = Q_0 \mathrm{e}^{-\frac{t}{RC}}$$

165

When $t = RC$ by what percentage of its original value has the charge dropped?

-------------------▷ 166

$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$ 56

Detailed solution:
Auxiliary equation: $r^2 + 2r + 5 = 0$
The roots are $r_1 = -1 + 2j$, $r_2 = -1 - 2j$
According to the formula in the textbook the real-valued solution is

$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

Obtain the general solution of the DE

$$\frac{3}{2}y'' + \frac{1}{2}y' + \frac{1}{24}y = 0$$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 57

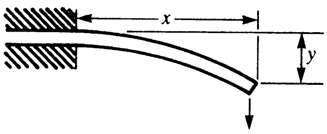
63% 166

(Explanation: When $t = RC$, $Q_1 = Q_0 e^{-1} \approx 0.37\,Q_0$. Hence the drop in the charge is $Q_0 - Q_1 = Q_0 - 0.37\,Q_0 = Q_0(1 - 0.37) = 0.63\,Q_0$.)

In your studies you will come across problems of this kind; the product *RC* is known as the 'time constant'.

Now for quite an interesting application.

The following DE occurred in a problem concerned with the deflection y of a rod.



$$y'' + 0.5y + 2.5\cos 0.8x = 0$$

Obtain the particular solution. The boundary conditions are

$$x = 0, \qquad y = 0, \qquad y' = 0.$$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution ------------------▷ 177

Further explanation wanted ------------------▷ 167

$$y = C_1 e^{-\frac{1}{6}x} + C_2 x e^{-\frac{1}{6}x}$$
$$= e^{-\frac{1}{6}x}(C_1 + C_2 x)$$

57

Correct → 59

Wrong, or detailed solution wanted → 58

Rewriting the DE we have

167

$$y' + 0.5y = -2.5\cos 0.8x$$

The homogeneous equation is
....

→ 168

Given: $\frac{3}{2}y'' + \frac{1}{2}y' + \frac{1}{24}y = 0$ 58

Auxiliary equation:

$$\frac{3}{2}r^2 + \frac{1}{2}r + \frac{1}{24} = 0$$

Roots: $r_1 = r_2 = -\frac{1}{6}$ i.e. equal roots.

The solution is:

$$y = C_1 e^{-\frac{1}{6}x} + C_2 x e^{-\frac{1}{6}x}$$

With the help of the scheme in the textbook find the general solution of the DE

$$y'' - 2y' + y = 0$$
$$y = \text{..........................}$$

Check your solution by yourself; you must obtain

$$y = C_1 e^x + C_2 x e^x = e^x (C_1 + x C_2)$$

--------------------▷ 59

$$y'' + 0.5y = 0$$ 168

The auxiliary equation is ....

The roots of the auxiliary equations are

$$r_1 = \text{..........................}, \quad r_2 = \text{..........................}$$

Thus the complementary function is

$$y_c = \text{.................................}$$

--------------------▷ 169

The most important aim of this section is to learn how to solve homogeneous DEs using the method of exponential solution. 59

This method will enable you to solve many DEs encountered in physics and engineering.

One more exercise!

Find the general solution of the first order DE

$$2y' = 3y$$

$$y = \text{................................}$$

→ 60

The auxiliary equation is $r^2 = -0.5$. 169

The roots are $r = \pm\sqrt{-0.5} = \pm 0.707\text{j}$.

CF: $y_c = C_1 e^{0.707\text{j}x} + C_2 e^{-0.707\text{j}x}$

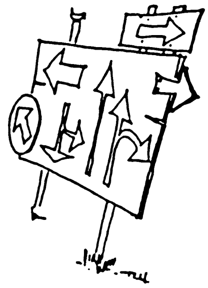
or $y_c = C \sin(0.707x + \phi)$ (the real part)

For the PI of the DE $y'' + 0.5y = -2.5\cos 0.8x$

let $y_p = \text{...........................}$

→ 170

$$y = C\,e^{\frac{3}{2}x}$$ 60



Correct --------> 64

Wrong, or further explanation wanted --------> 61

$$y_p = A \sin 0.8x + B \cos 0.8x$$ 170

The derivatives are

$$y_p' = \ldots\ldots\ldots\ldots$$
$$y_p'' = \ldots\ldots\ldots\ldots$$

--------> 171

The DE is

$$a_1 y' + a_0 y = 0$$ 61

Its auxiliary equation is

$$a_1 r_1 + a_0 = 0$$

whose root is $r_1 =$ ....

--------------------▷ 62

$$y'_{\mathrm{p}} = 0.8A\cos 0.8x - 0.8B\sin 0.8x$$
$$y''_{\mathrm{p}} = -0.64A\sin 0.8x - 0.64B\cos 0.8x$$ 171

Substitute in the DE

$$y'' + 0.5y = -2.5\cos 0.8x$$

and obtain

$$-0.64A\sin 0.8x - 0.64B\cos 0.8x$$
$$+0.5A\sin 0.8x + 0.5B\cos 0.8x = -2.5\cos 0.8x$$

Solve for A and B:

$a =$ ...., $B =$ ....

Solution --------------------▷ 173

Further explanation wanted --------------------▷ 172

$$r_1 = -\frac{a_0}{a_1}$$

62

The solution of the homogeneous first order DE is therefore

$$y = C\mathrm{e}^{r_1 x} = C\mathrm{e}^{-\frac{a_0}{a_1}x}$$

Now we calculate the solution of the DE $2y' = 3y$.

Rewriting the equation we have

$$2y' - 3y = 0$$

Hence the auxiliary equation is

$$2r_1 - 3 = 0$$

Therefore $r_1 = \dfrac{3}{2}$

and the solution is

$$y = C\mathrm{e}^{\frac{3}{2}x}$$

Find the solution of the DE: $\dot{N}(t) = -\lambda N(t)$

$$N(t) = \text{..............................}$$

-------------------▷ 63

Given $-0.64A\sin 0.8x - 0.64B\cos 0.8x + 0.5A\sin 0.8x + 0.5B\cos 0.8x = -2.5\cos 0.8x$

172

To solve for A and B equate the coefficients of the sine and cosine terms on both sides of the equation. You should obtain

$-0.64A + 0.5A = 0$; therefore $A =$ ....

$-0.64B + 0.5B = -2.5$; therefore $B =$ ....

-------------------▷ 173

$$N(t) = C\,\mathrm{e}^{-\lambda t}$$

63

Note: this is a function of the variable t.

64

$$A = 0, \qquad B = \frac{2.5}{0.14} = 17.86$$

173

The general solution of the DE
$y'' + 0.5y' + 2.5\cos 0.8x = 0$ (using $y_c = C\sin(0.707x + \phi)$
and $y_p = 17.86\cos 0.8x$)
is $y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

174

We finally summarize the procedure for the solution of homogeneous, linear first- and second-order DEs with constant coefficients. The general form of such a DE is 64

$$a_2 y'' + a_1 y' + a_0 y = 0$$

The solution is carried out in three steps:

Step 1: Establish the auxiliary equation; this means

(i) replacing y'' by r^2
(ii) replacing y' by r
(iii) replacing y by 1

Step 2: Determine the roots r_1 and r_2 of the auxiliary equation.

-------------------▷ 65

$$y = y_c + y_p = C\sin(0.707x + \phi) + 17.86\cos 0.8x$$ 174

To calculate C and ϕ, the two arbitrary constants, substitute the boundary conditions in the general equation. The boundary conditions are:
when $x = 0$, y and y' are both zero

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(i) When $x = 0$, $y = 0$; hence $0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
(ii) When $x = 0$, $y' = 0$; hence $0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 175

65

Step 3: The general solution of the DE depends on the nature of the roots r_1 and r_2. There are three possible cases:

(a) r_1 and r_2 are real and unequal, i.e. $r_1 \neq r_2$
(b) r_1 and r_2 are real and equal, i.e. $r_1 = r_2$
(c) r_1 and r_2 are complex, i.e. $r_1 = a + jb$; $r_2 = a - jb$

The corresponding solutions are:

(a) $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$
(b) $y = C_1 e^{r_1 x} + C_2 x e^{r_1 x} = e_1^r x(C_1 + C_2 x)$
(c) $y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

Such a procedure can be used to solve any problem of a given type. It is called an algorithm. Algorithms are used extensively in computer programming.

--------------------▷ 66

175

$$y' = 0.707C\cos(0.707x + \phi) - 17.86 \times 0.8\sin 0.8x$$

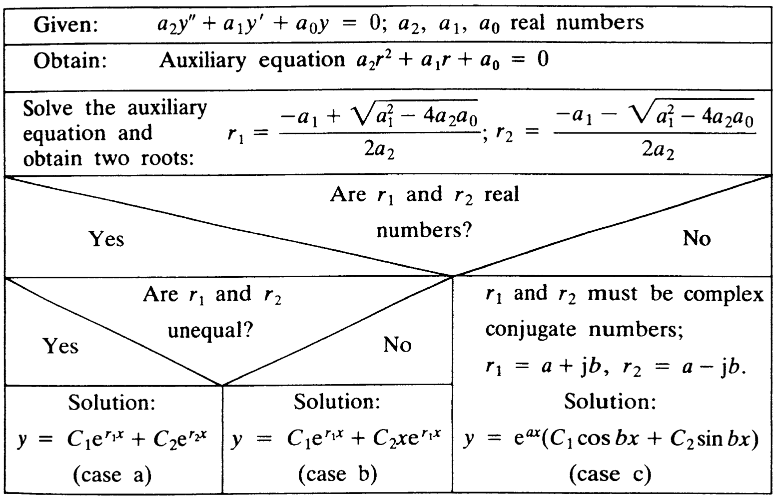
(i) $0 = C\sin\phi + 17.86$
(ii) $0 = 0.707C\cos\phi$

Solve for C and ϕ

$C =$ ...., $\phi =$ ....

--------------------▷ 176

The algorithm for the solution of homogeneous second-order linear DEs with constant coefficients can be depicted graphically. (We here use a technique called Nassi-Shneiderman diagram.) 66



------------------▷ 67

$$\phi = \frac{\pi}{2}, \; C = -17.86$$ 176

Explanation: From (ii) $C \cos\phi = 0$, from (i) we know $C \neq 0$. Hence $\cos\phi = 0$. Therefore $\phi = \dfrac{\pi}{2}$ is a possible solution.

Substitution in (i) yields: $C = -17.86$

The particular solution is

$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 177

10.4 Solution of the Non-Homogeneous Second-Order DE with Constant Coefficients

67

Objective: General and particular solution of the non-homogeneous linear DE with constant coefficients.

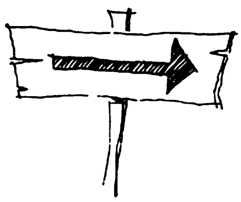
READ: 10.3.2 Non-homogeneous linear DE
Textbook pages 287–292

-------------------▷ 68

$$y = 17.86(\cos 0.8x - \sin(0.707x + \frac{\pi}{2}))$$

177

Have a 10 minute break to relax and collect your thoughts before you continue with the last example of applications. Time yourself!



-------------------▷ 178

Given a non-homogeneous DE of the form 68

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Let y_c be the general solution of the accompanying homogeneous DE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

y_c is also called the complementary function, or CF for short.

Let a solution of the non-homogeneous DE be y_p.

y_p is also called a particular integral, or PI for short.

What is the general solution of the non-homogeneous DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)?$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 69

The following DE arose while the performance of an anti-vibration mounting was being investigated: 178

$$\ddot{x} + 60\dot{x} + 3600x = 150\sin 65t$$

x is the displacement, t is the time.

The boundary conditions are:

when $t = 0$, $x = 0$ and $\dot{x} = 0$

Solve the equation.

Make sure that you carry out the solution step by step: above all, don't use short-cuts because they do not save time in the long run, and you are more likely to make mistakes. Follow a logical sequence by drawing up an algorithm, if you wish, and following it. Switch on your calculator, you will need it.

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution -------------------▷ 183

Detailed solution wanted -------------------▷ 179

$$y = y_c + y_p$$ 69

The general solution of the non-homogeneous DE is the sum of the general solution of the homogeneous DE and a particular solution of the non-homogeneous DE.

This rule holds true in general for linear non-homogeneous DEs of any order, but we shall only consider first- and second-order DEs now.

The CF (complementary function) of the DE

$$y'' + 3y' = x + \frac{1}{3}$$

is $y_c = C_1 + C_2 e^{-3x}$.

A PI (particular integral) of that DE is

$$y_p = \frac{x^2}{6}$$

What is the general solution of $y'' + 3y' = x + \frac{1}{3}$?

$y = \ldots\ldots\ldots\ldots\ldots\ldots$ --------------------▷ 70

Here is the detailed solution to check against your own: 179

Step 1: The DE and boundary conditions: $\ddot{x} + 60\dot{x} + 3600x = 150\sin 65t$

$x = 0,\ \ \dot{x} = 0$ when $t = 0$

Step 2: Solution of the homogeneous DE: $\ddot{x} + 60\dot{x} + 3600x = 0$

The auxiliary equation is: $r^2 + 60r + 3600 = 0$

Its roots are $r_{1,2} = -\frac{60}{2} \pm \frac{1}{2}\sqrt{60^2 - 4 \times 3600} = -30 \pm \sqrt{3} \times 30\mathrm{j}$

i.e. $r_1 = -30 + 51.96\mathrm{j}$, $\quad r_2 = -30 - 51.96\mathrm{j}$.

The CF is $x_c = C\,e^{-30t}\cos(51.96t - \phi)$.

C and ϕ are two arbitrary constants.

--------------------▷ 180

$$y = C_1 + C_2 e^{-3x} + \frac{x^2}{6}$$ 70

If the given DE is

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

let the complementary function be denoted by y_c, and let a particular integral be denoted by y_p.

The general solution is

$$y = \text{..............................}$$

----------------▷ 71

Step 3: Obtain the PI 180

$$\text{Let } x_p = A \sin 65t + B \cos 65t$$
$$\dot{x}_p = 65A \cos 65t - 65B \sin 65t$$
$$\ddot{x}_p = -65^2 A \sin 65t - 65^2 B \cos 65t$$

We substitute the expressions x_p, $\dot{x}_p$ and $\ddot{x}_p$ in the DE and get

$$\begin{aligned}
&- 4225A \sin 65t - 4225B \cos 65t \\
&+ 65 \times 60A \cos 65t - 65 \times 60B \sin 65t \\
&+ 3600A \sin 65t + 3600B \cos 65t = 150 \sin 65t
\end{aligned}$$

To calculate the constants A and B equate the coefficients of sin $65t$ and cos $65t$ on both sides of the equation.

This gives (1) ....

(2) ....

----------------▷ 181

$$y = y_c + y_p$$ 71

Given: $y'' + 23y' + 15y = 6$
Can you guess a PI (particular integral) of the DE?

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 74

Further explanation wanted --------------------▷ 72

---

(1) $-625A - 3900B = 150$
(2) $-625B + 3900A = 0$

181

Determination of A and B:

$$A = -0.006$$
$$B = -0.0375$$

Hence the PI is $x_p = \ldots\ldots\ldots\ldots\ldots$ --------------------▷ 182

Let us try a simpler problem. Consider the DE **72**

$$y'' + y = 2$$

Required: a PI (particular integral).
Try these functions:

$$y = 1$$
$$y = 2$$
$$y = x$$
$$y = 2x$$

Which one fits the DE?

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ **73**

$$x_{\mathrm{p}} = -0.006 \sin 65t - 0.0375 \cos 65t$$ **182**

Step 4: The general solution is $x = x_{\mathrm{c}} + x_{\mathrm{p}}$; i.e.

$$x = C\mathrm{e}^{-30t} \cos(51.96t - \phi) - 0.006 \sin 65t - 0.0375 \cos 65t$$

Step 5: Boundary conditions: At $t = 0$, $x = 0$, $\dot{x} = 0$.

This gives for the first condition $0 = C \cos \phi - 0.0375$. [1]

Differentiating the general solution yields

$$\dot{x} = -51.96C\mathrm{e}^{-30t} \sin(51.96t - \phi) - 30C\mathrm{e}^{-30t} \cos(51.96t - \phi)$$
$$- 0.39 \cos 65t + 2.4375 \sin 65t$$

This gives for the second condition $\dot{x} = 0$, $t = 0$.

$$0 = 51.96C \sin \phi - 30C \cos \phi - 0.39 \qquad [2]$$

(Remember $\sin(-\phi) = -\sin \phi$, $\cos(-\phi) = \cos \phi$.)
Solving equations 1 and 2 we find $C = 0.0475$, $\phi = 0.66$ radians.

Step 6: Write down the particular solution
$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ **183**

$$y_p = 2$$ 73

To guess a particular solution may be quite subtle a problem. Only experience will help in many cases, but sometimes it is quite easy. Given the DE

$$y'' + 23y' + 15y = 6$$

Try

$$y = 6$$
$$y = 6 + x$$
$$y = \frac{2}{5}$$
$$y = \frac{6}{15} + 23x$$
$$y_p = \dots\dots\dots\dots\dots\dots$$

--------------------▷ 74

$$x = 0.0475e^{-30t}\cos(51.96t - 0.66) - 0.006\sin 65t - 0.0375\cos 65t$$ 183

Straight on

--------------------▷ 186

You may need a break now. But reflect for a moment on the aims of the study guide. There are two fundamental aims.

First aim: To impart mathematical knowledge for application to physics and engineering problems.
Second aim: To promote your study skills and your ability to use written texts.

The first aim need not be discussed. The second aim is worth commenting on. Study skills and the ability to use written texts are important for you.

The advanced student relies even more than the beginner on his or her ability to use written sources of information.

--------------------▷ 184

$$y_p = \frac{2}{5}$$ 74

There are special cases of non-homogeneous DEs where the variables can be separated.

Solve the DE

$$y'' = a \qquad \text{(a constant)}$$

Integrating twice yields

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 76

Detailed solution --------------------▷ 75

The promotion of your study skills by the study guide is important because once you have finished with the guide you will have to control your learning process by yourself. Thus you should understand the control techniques recommended by the study guide. 184

The study guide tries to build up confidence in your own ability by means of examples of varying difficulty and by giving you hints or part solutions where necessary, or by directing you back to sections in the textbook.

You therefore adopt the habit of reading the textbook carefully and of controlling your progress.

After each section you are asked simple questions about the new concepts, and even their names. Then you are encouraged to learn how to test yourself, beginning with learning by rote and continuing with active problem solving.

--------------------▷ 185

Detailed solution of the DE: 75

$$y'' = a \quad (a \text{ is a constant})$$
$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = a$$

In this case we can integrate directly.

The first step is to integrate once with respect to x; this yields

$$y' = ax + C_1$$

Integrating once more yields

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 76

Some special techniques have been outlined and practised by the guide: 185

— how to break great learning tasks into bits which can be handled more easily,
— how to use different reading techniques; intensive reading combined with note taking versus selective or orientational reading,
— how to make the best use of breaks,
— how to control learning progress,
— how to make learning more active by using self-testing techniques.

-------------------▷ 186

$$y(x) = \frac{ax^2}{2} + C_1 x + C_2$$ 76

This example illustrates a special case of a DE whose solution is obtained by direct integration because the variables can be separated.

A practical example is that of a body thrown vertically upwards, if we neglect air resistance.

The DE governing the motion has been mentioned at the beginning of Chapter 10 in the textbook.

$\ddot{y} = -g$ (g is acceleration due to gravity.)

(The dot notation refers to differentiation with respect to the time t.)

Solve this equation on your own:

$$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 77

10.9 General Linear First-Order DEs

186

The following sections offer in a more concise fashion some additional methods of solving certain types of DE. For beginners it may be advisable to skip the rest of this chapter during a first course and to return to it when the need arises.

Objective: Determination of the general solution of *any* linear first-order differential equation (i.e. the coefficients are not necessarily constants).

READ: 10.6 General linear first-order DEs
Textbook pages 304–308

------------------▷ 187

$$y(t) = -g\frac{t^2}{2} + C_1 t + C_2$$ 77

The subject matter we have been discussing required a fair amount of concentration, perhaps more than usual. If you study a certain amount each week, or within a given period, and recapitulate at the end of each section you will progress more rapidly than if you do not follow a definite study plan.

Before closing the study guide you should recall what you have just learnt. Write down the most important aspects of the subject matter and compare them with the textbook.

------------------▷ 78

We are now going to illustrate the straightforward method of solving a linear first-order differential equation using the integrating factor. In case of difficulties consult the textbook. 187

Given: $xy' + y - x^2 = 0$
First identify the coefficients:

$$p(x) = \ldots\ldots\ldots\ldots\ldots\ldots, \quad q(x) = \ldots\ldots\ldots\ldots\ldots\ldots, \quad f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 188

End of the first part of differential equations. 78

→ 79

$$p(x) = x, \quad q(x) = 1, \quad f(x) = x^2$$ 188

Write down the formula for the integrating factor in terms of $p(x)$, $q(x)$ and $f(x)$:

$$I(x) = \text{...........................}$$

Now let us solve the differential equation under consideration:

$$xy' + y - x^2 = 0$$

We follow the steps described in section 10.6.2.

Step 1: Determine the integrating factor $I(x) =$ ....

I need some help → 189

Solution → 190

10.5 Solution by Substitution or by Trial

79

In this section we shall explain techniques to obtain or to guess particular solutions for frequent types of non-homogeneous DEs of the linear form.

READ: 10.3.2 Non-homogeneous linear DE
Solution by substitution or by trial
Textbook pages 287–292

--------------------▷ 80

189

The general formula for the integrating factor $I(x)$ is:

$$I(x) = \mathrm{e}^{\int \frac{q(x)}{p(x)}\,\mathrm{d}x}$$

It must be computed for $p(x) = x, \quad q(x) = 1$.

Compute $\int \frac{q(x)}{p(x)}\,\mathrm{d}x =$ ...., $\quad I(x) =$ ....

--------------------▷ 190

The homogeneous linear DE of the second order can be solved algorithmically. On the other hand, certain types of non-homogeneous DEs are best solved by trial. 80

In the following frames we are concerned with finding the particular integral (PI) by trial, i.e. by assuming a function of the same type as $f(x)$ in the DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Type 1: If $f(x)$ is a polynomial of degree n, i.e.

$$f(x) = a + bx + cx^2 + \cdots$$

then a trial solution for the PI is

$$y_\mathrm{p} = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 81

$$I(x) = \mathrm{e}^{\int \frac{q(x)}{p(x)}\,\mathrm{d}x}$$

$$I(x) = \mathrm{e}^{\int \frac{1}{x}\,\mathrm{d}x} = \mathrm{e}^{\ln|x|} = |x|$$ 190

(As an aside, the constant of integration is of no significance for our purposes.)

Step 2: The following must be evaluated:

$$y(x) = \frac{1}{I(x)} \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x$$

We know $p(x) = x, \quad q(x) = 1, \quad f(x) = x^2, \quad I(x) = |x|.$

$\frac{1}{I(x)} = \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 191

$$y_p = A + Bx + Cx^2 + \cdots$$ 81

Obtain a PI of the DE

$$y'' - 5y' + 6y = x^2$$
$$y_p = \text{..........................}$$

Solution found ⊳ 87

Further explanation and detailed solution wanted ⊳ 82

$$y(x) = \frac{1}{I(x)} \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x = \frac{1}{x}\left(\frac{x^3}{3} + C\right)$$
$$= \frac{x^2}{3} + \frac{C}{x}$$ 191

Let us tackle another first-order linear differential equation by the same method:

$$(x+1)y' + y = (x+1)^2$$

Preliminary step:

$$p(x) = \text{.....................}, \quad q(x) = \text{.....................}, \quad f(x) = \text{.....................}$$

Step 1:

$$\int \frac{q(x)}{p(x)}\,\mathrm{d}x = \text{.....................}, \quad I(x) = \text{.....................}$$

Step 2:

$$y(x) = \text{..........................}$$

⊳ 192

For the PI of $y'' - 5y' + 6y = x^2$ **82**

we can try

$y_p = A + Bx + Cx^2$ as a solution,

i.e. a polynomial of the second degree.

Note: No intermediate power of x can be omitted, even if the right-hand side of the DE does not contain all powers.

Hence $y_p' = $ ....

$y_p'' = $ ....

--------------------▷ 83

$$p(x) = x + 1, \quad q(x) = 1, \quad f(x) = (x+1)^2$$ **192**

$$\int \frac{q(x)}{p(x)} \mathrm{d}x = \ln|x+1|, \quad I(x) = |x+1|$$

$$y(x) = \frac{1}{x+1} \int (x+1)^2 \mathrm{d}x = \frac{1}{3}(x+1)^2 + \frac{C}{x+1}$$

Can every linear first-order differential equation be solved using the integrating factor?

☐ Yes --------------------▷ 193

☐ No --------------------▷ 195

$$y_p' = B + 2Cx$$
$$y_p'' = 2C$$

83

Substituting in the DE $y'' - 5y' + 6y = x^2$ yields:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = x^2$$

--------------------▷ 84

You are too optimistic! Try to solve:

193

$$\ln|x|\,y' + \frac{1}{x}\,y = \frac{1}{(\ln|x|)^2}$$

$p(x) = \ldots\ldots\ldots\ldots\ldots\ldots$, $q(x) = \ldots\ldots\ldots\ldots\ldots\ldots$, $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$,

$$\int \frac{q(x)}{p(x)}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots, \quad I(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(Hint\colon \int \frac{dx}{x\ln|x|} = \int \frac{(\ln|x|)'}{\ln|x|}\,dx\right)$$

--------------------▷ 194

$$2C - 5B - 10Cx + 6A + 6Bx + 6Cx^2 = x^2$$ 84

To find the values of A, B and C that will satisfy the DE equate the coefficients of x^2, x and the constant terms.

Hence $C =$ ...., $B =$ ...., $A =$ ....

Solution found: --------------------▷ 86

Further explanation needed: --------------------▷ 85

$$p(x) = \ln|x|, \quad q(x) = \frac{1}{x}, \quad f(x) = \frac{1}{(\ln|x|)^2}$$ 194

$$\int \frac{q(x)}{P(x)}\mathrm{d}x = \int \frac{\mathrm{d}x}{x\ln|x|} = \ln(\ln|x|),\ I(x) = \ln|x|$$

We have succeeded in completing the first step. But we encounter difficulties during the second step. We must evaluate:

$$y(x) = \frac{1}{I(x)}\int \frac{I(x)}{p(x)} f(x)\mathrm{d}x = \frac{1}{\ln|x|}\int \frac{\mathrm{d}x}{(\ln|x|)^2}$$

But this integral cannot be solved at all by elementary methods! --------------------▷ 195

Given: $2C - 5B - 10Cx + 6A + 6Bx + 6Cx^2 = x^2$ 85

Equating coefficients of x^2: $6C = 1$,

therefore $C = \frac{1}{6}$

Equating coefficients of x: $-10C + 6B = 0$ (since there is no x on the RHS)

therefore $B = \frac{10}{6} \times \frac{1}{6} = \frac{5}{18}$

Constant term: $2C - 5B + 6A = 0$ (since there is no constant term)
Solving for A yields

$$A = \frac{1}{6}(5B - 2C) = \frac{1}{6}\left(\frac{25}{18} - \frac{1}{3}\right) = \frac{19}{108}$$

Thus we have $A =$ ...., $B =$ ...., $C =$ ....

------------------▷ 86

You are right. *In practice*, not all linear first-order differential equations can be solved by this method. The reason is that there is no guarantee that the necessary integrations can be performed. A solution does exist, but it may not be soluble otherwise than by numerical means. 195

Here is one last linear first-order differential equation for you to solve:

$$(\sin x)y' + (\cos x)y = \cos^2 x$$

Work through the necessary steps and write down the solution.

$$y(x) = \text{..........................}$$

Solution ------------------▷ 197

I need some help! ------------------▷ 196

$$C = \frac{1}{6}; \quad B = \frac{5}{18}; \quad A = \frac{19}{108}$$ 86

Inserting into the trial solution

$$y_p = A + Bx + Cx^2 \text{ yields}$$
$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 87

Given: $(\sin x)y' + (\cos x)y = \cos^2 x$ 196

Determine: $p(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $q(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compute: $\int \frac{q(x)}{p(x)} \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $\int \frac{I(x)}{p(x)} f(x) \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(*Hints*: You can look up both integrals in the table of the textbook at the end of Chapter 6. Recall: $\frac{\cos x}{\sin x} = \cot x$.)

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 197

$$y_p = \frac{19}{108} + \frac{5}{18}x + \frac{1}{6}x^2$$

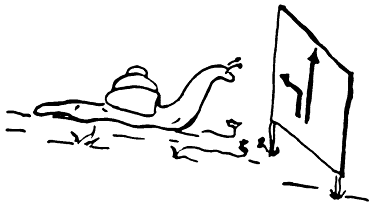
87

or

$$y_p = \frac{1}{108}(18x^2 + 30x + 19)$$

Next problem:
Obtain a PI of the DE

$$y'' + 4y\prime + 5y = 3x - 2$$
$$y_p = \text{.........................}$$



Solution found --------------------▷ 92

In difficulty; further explanation wanted --------------------▷ 88

$$p(x) = \sin x, \quad q(x) = +\cos x, \quad f(x) = \cos^2 x$$

197

$$\int \frac{q(x)}{p(x)}\mathrm{d}x = +\ln|\sin x|, \quad \int \frac{I(x)}{p(x)} f(x)\mathrm{d}x = \frac{x}{2} + \frac{\sin x \cos x}{2} + C$$

$$y(x) = \frac{x}{2\sin x} + \frac{\cos x}{2} + \frac{C}{\sin x}$$

Further examples of first-order linear differential equations are given in the exercises in the textbook.

--------------------▷ 198

$f(x) = 3x - 2$ is a polynomial function. 88

What should your trial solution be, according to the textbook?

$y_p = $ ....

--------------------▷ 89

10.10 Some Remarks on General First-Order DEs 198

Bernoulli type DEs are new but you have encountered the separation of variables before. It is a method which is quite straightforward — provided it can be applied. This is the case if the DE can be rearranged in such a way that one variable only is found on the RHS and the other on the LHS.

READ: 10.7.1 Bernoulli's equations
10.7.2 Separation of variables
Textbook pages 308–310

--------------------▷ 199

$$y_p = A + Bx$$ 89

Note: If you try Bx alone as a solution it is not the general polynomial of the first degree.

Hence $y_p' =$ ....

$y_p'' =$ ....

--------------------▷ 90

Which of the following differential equations can be written in the form of a Bernoulli differential equation? 199

(1) $y' + xy = (\cos x)y$
(2) $xy' + y = x^2y^2$
(3) $y' + x^2y = xe^y$

--------------------▷ 200

$$y_\text{p}' = B;\ y_\text{p}'' = 0$$

90

Substitute in the DE $y'' + 4y' + 5y = 3x - 2$

.... $= 3x - 2$

Equate coefficients; hence

$$A = \text{.......................}, \quad B = \text{.......................}$$

Finally, the PI is

$$y_\text{p} = \text{........................}$$

PI found --------------------▷ 92

Further hints --------------------▷ 91

200

Only equation (2) is of the Bernoulli type. Equation (1) looks like a Bernoulli equation, but it is in fact linear:

$$y' + (x - \cos x)y = 0$$

The substitution $u = y^{1-n}$ would not work in that case ($n = 1$), and, in fact, it is not necessary.

Now try solving differential equation (2) written in normal form:

$$y' + \frac{y}{x} = xy^2$$

$n = $ ...., $u = $ ...., $u' = $ ....

Substitute into the given differential equation to obtain an equation for $u(x)$: ....

--------------------▷ 201

Substituting in the DE yields **91**

$$4B + 5A + 5Bx = 3x - 2$$

Now we equate the coefficients of x and the constant terms:

$$B = \frac{3}{5} \quad (\text{since } 5Bx = 3x)$$
$$A = -\frac{22}{25} \quad (\text{since } 4B + 5A = -2)$$

Hence the PI is

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 92

$$n = 2, \quad u = y^{-1}. \quad \text{Thus } uy^2 = y.$$ **201**
$$u' = -y'y^{-2}. \quad \text{Thus } -u'y^2 = y'.$$

Differential equation for $u(x)$:

$$u' - \frac{1}{x}u = -x$$

Solve the differential equation above for u: $u(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Resubstitute: $y(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 202

$$y_p = \frac{1}{25}(15x - 22)$$ 92

Obtain a PI of the DE

$$y''' - y'' - 6y = x^2 - 3x - 2 \quad (\textit{Note}:\ f(x) \text{ is a polynomial again.})$$

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 97

Further explanation wanted --------------------▷ 93

$$u = x\int \frac{1}{x}(-x)\,dx = x(C - x) = Cx - x^2$$ 202

$$y = \frac{1}{Cx - x^2}$$

In the general Bernoulli equation are fractional values for n admissible?

☐ Yes

☐ No --------------------▷ 203

Given: $y''' - y'' - 6y = x^2 - 3x - 2$ 93

The fact that it is a third-order linear DE does not require any change of method in evaluating the PI, so find the trial solution first:

$$y_\text{p} = \text{...........................}$$

-------------------▷ 94

Yes, indeed! 203

Reduce the following differential equation to a linear one:

$$y' + 2xy = x\sqrt[4]{y}$$
$$n = \text{...........................}, \quad u(x) = \text{...........................}$$

Now the linear differential equations reads: $\frac{4}{3}u' + \text{................} = \text{.................}$

-------------------▷ 204

$$y_p = A + Bx + Cx^2$$ 94

Differentiate three times successively

$$y'_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y''_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y'''_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 95

$$n = \frac{1}{4}$$
$$u = y^{3/4}$$
$$\frac{4}{3}u' + 2xu = x$$ 204

Solve this linear first order equation using the method described in 10.6.2 (integrating factor):

$$u(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Then resubstitute to find $y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 205

$$y_p' = 2Cx + B\,,\quad y_p'' = 2C\,,\quad y_P''' = 0$$ 95

We substitute in the DE: $y''' - y'' - 6y = x^2 - 3x - 2$ and get

$$0 - 2C - 6A - 6Bx - 6Cx^2 = x^2 - 3x - 2$$

Equate coefficients:
for x^2: ....
for x: ....
Constant term: ....

--------------------▷ 96

$$u(x) = \frac{1}{2} + C\mathrm{e}^{-\frac{3}{4}x^2}$$ 205

$$y = \sqrt[3]{\left(\frac{1}{2} + C\mathrm{e}^{-\frac{3}{4}x^2}\right)^4}$$

Now on to the last Bernoulli differential equation of this chapter! Solve:

$$y' + 2xy + xy^4 = 0$$

$n =$ ....
$u(y) =$ ....
$y =$ ....

--------------------▷ 206

$-6C = 1$;	therefore $C =$	96
$-6B = -3$;	therefore $B =$	
$-2C - 6A = -2$;	therefore $A =$	

The trial solution was $y_p = A + Bx + Cx^2$
The PI is $y_p =$ ....

--------------------▷ 97

206

$$n = 4, \quad u = y^{-3}, \quad u' - 6xu = 3x$$

$$u(x) = -\frac{1}{2} + Ce^{3x^2}, \; y = \frac{1}{\sqrt[3]{Ce^{3x^2} - \frac{1}{2}}}$$

Let us now turn to another technique. Solve:

$$(1 + x^2)\, y' - xy^2 = 0$$

I need some help! --------------------▷ 207

Solution --------------------▷ 208

$$y_p = \frac{7}{18} + \frac{1}{2}x - \frac{1}{6}x^2$$ 97

Now we tackle the second type of function $f(x)$.
Given the DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

If $f(x) = C e^{\lambda x}$ what would your trial solution be for the PI?

$$y_p = \text{..........................}$$

------------------▷ 98

Try to solve the differential equation by separation of the variables: 207

$$(1 + x^2)\, y' - x y^2 = 0$$

Rewrite it in the following form with the variables separated:

$$p(y)\mathrm{d}y = -q(x)\,\mathrm{d}x$$

Then integrate to solve the differential equation.

------------------▷ 208

Trial solution: $y_p = A\,e^{\lambda x}$

98

Since this function is to be a particular integral of the DE we have to find the values of A that will satisfy the equations.

Find the PI of the DE

$$y'' + 5\,y' - 14y = 2e^x$$
$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found → 100

Explanation wanted → 99

208

$$(1+x^2)y' = xy^2$$
$$\frac{dy}{y^2} = \frac{x}{1+x^2}\,dx$$

Integration:
$$-\frac{1}{y} = \frac{\ln(x^2+1)}{2} + C_1$$
$$y = \frac{-2}{\ln(C\,(x^2+1))}$$

Here is another differential equation which you should be able to solve by separating the variables:

$$y' + xy' + y = 1$$
$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

→ 209

99

Given the DE: $y'' + 5y' - 14y = 2e^x$
We assume a solution of the exponential type, i.e. $y_p = Ae^x$.
We need to find the value of the constant A that will satisfy the DE

$$y'' + 5y' - 14y = 2e^x$$

If $y_p = A e^x$

then $y_p' = A e^x$ and $y_p'' = A e^x$

Substituting in the DE yields

$$A e^x + 5Ae^x - 14Ae^x = 2e^x$$

Dividing by e^x we have $A(1+5-14) = 2$

Therefore $A = -\frac{1}{4}$

The PI is $y_p =$ ....

-------------------▷ 100

209

$$y(x) = \frac{C}{x+1} + 1$$

(Note that the LHS of the given DE $= [(x+1)y]'$.)

This is the end of your work with Chapter 10 for the time being. The succeeding sections of the textbook require some knowledge of partial derivatives which are treated in Chapter 12. Having worked through Chapter 12 (Functions of several variables, partial derivatives) you should return to Chapter 10 on differential equations.

Then read: **10.7.3 Exact equations**
10.7.4 The integrating factor general case
10.8 Simultaneous DEs
Textbook pages 309–318
10.9 Higher order DEs
10.10 Some advice on intractable DEs

We shall not give exercises for sections 10.7.3 and 10.7.4 in the programmed study guide. By now you know well how to proceed on your own. Work through each section in the textbook and try to solve at least one problem posed in the corresponding exercises at the end of Chapter 10. In case of difficulties when solving differential equations try to copy exactly the procedure shown in the examples in the textbook.

-------------------▷ 210

$$y_p = -\frac{1}{4}e^x$$

100

Find the PI of the DE

$$2y'' + 7y' - 15y = 3e^{2x}$$
$$y_p = \text{..........................}$$

Solution found --------------------▷ 102

Explanation wanted --------------------▷ 101

210

Try to solve the following simultaneous differential equations using the first method discussed in section 10.8; t is the independent variable.

Solve for x:

$$2\dot{x} + 3x - y = 0 \qquad [1]$$
$$3\dot{y} + 10x - 4y = 0 \qquad [2]$$

First differentiate equation 1:

$$\text{..........................} = 0 \qquad [3]$$

--------------------▷ 211

Given: $2y'' + 7y' - 15y = 3e^{2x}$ **101**

The DE is similar to the last one except that there is an e^{2x} instead of an e^x.

Hence for the PI we assume a solution

$$y_p = Ae^{2x}$$

Thus $y_p' = 2Ae^{2x}$ and $y_p'' = 2 \times 2Ae^{2x} = 4Ae^{2x}$

Substituting in the DE we find

$$(2 \times 4 + 7 \times 2(-15))Ae^{2x} = 3e^{2x}$$

Dividing by e^{2x} yields $(8 + 14 - 15)A = 3$

Therefore $A = \dfrac{3}{7}$

The PI is: $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

----------▷ **102**

$$2\ddot{x} + 3\dot{x} - \dot{y} = 0 \qquad [3]$$ **211**

Given were:

$$2\dot{x} + 3x - y = 0 \qquad [1]$$
$$3\dot{y} + 10x - 4y = 0 \qquad [2]$$

Solve for y from [1] and for $\dot{y}$ using [3]:

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\dot{y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substitute for y and $\dot{y}$ in [2]. You should then obtain a DE for x.

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

----------▷ **212**

$$y_p = \frac{3}{7}e^{2x}$$

102

The PI of the DE

$$a_2 y'' + a_1 y' + a_0 y = ae^{\lambda x}$$

is obtained by trying a solution

$$y_p = Ae^{\lambda x}$$

This we have seen in the textbook and in the last two examples.
Can this method fail?

☐ Yes --------------------▷ 104

☐ No --------------------▷ 103

$$y = 2\dot{x} + 3x$$
$$\dot{y} = 2\ddot{x} + 3\dot{x}$$
$$6\ddot{x} + \dot{x} - 2x = 0$$

212

Now we must solve the differential equation $6\ddot{x} + \dot{x} - 2x = 0$.
The auxiliary equation is ....
Its roots are $r_1 =$ ...., $r_2 =$ ....
The solution for x is

$x =$ ....

Substituting in the equation $y = 2\dot{x} + 3x$
we get the solution:

$y =$ ....

--------------------▷ 213

You are not right. 103

The method fails if λ is a root r of the auxiliary equation.

The auxiliary equation of the homogeneous DE is

$$a_2r^2 + a_1r + a_0 = 0$$

If λ is a root of the equation it follows that

$$a_2\lambda^2 + a_1\lambda + a_0 = 0$$

hence with substitution $y_\mathrm{p} = A\mathrm{e}^{\lambda x}$ we find that

$$(a_2\lambda^2 + a_1\lambda + a_0)A = a$$

i.e. $A = \dfrac{a}{a_2\lambda^2 + a_1\lambda + a_0} = \dfrac{a}{0}$ which is not defined.

We must, therefore, find a new trial solution.

--------------------▷ 104

213

$$6r^2 + r - 2 = 0$$

$$r_1 = \frac{1}{2},\quad r_2 = -\frac{2}{3}$$

$$x = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{-\frac{2}{3}t}$$

$$y = 4A\mathrm{e}^{\frac{1}{2}t} + \frac{5}{3}B\mathrm{e}^{-\frac{2}{3}t}$$

All correct --------------------▷ 215

I need help and explanation --------------------▷ 214

Yes is the right answer. The method may fail. 104

If in the DE

$$a_2 y'' + a_1 y' + a_0 = a\mathrm{e}^{\lambda x}$$

λ is a root of the auxiliary equation, what would be a trial solution for the PI? (Consult the textbook, if necessary.)

$$y_\mathrm{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 105

Go carefully through example 1 in section 10.8. To solve for x and y as functions of t from the equations we eliminate y first in order to obtain a differential equation in $x = x(t)$ only. Although the two equations are of the first order, the elimination process leads to a second order DE which we can solve by the exponential method we studied previously. Hence we get a solution for x. To obtain a solution for y we substitute for x and $\dot{x}$ in equation [1]: 214

$$2\dot{x} + 3y - y = 0 \qquad [1]$$

$$x = A\mathrm{e}^{\frac{1}{2}t} + B\mathrm{e}^{-\frac{2}{3}t}$$
$$\dot{x} = \frac{1}{2}A\mathrm{e}^{\frac{1}{2}t} - \frac{2}{3}B\mathrm{e}^{-\frac{2}{3}t}$$

The solution is

$$y = 4A\mathrm{e}^{\frac{1}{2}t} + \frac{5}{3}B\mathrm{e}^{-\frac{2}{3}t}$$

--------------------▷ 215

$$y_p = Axe^{\lambda x}$$ 105

Find the PI of the DE

$$y'' + 2y' - 3y = 4e^x$$
$$y_p = \text{.........................}$$

Solution found --------------------▷ 107

Explanation wanted --------------------▷ 106

As our final example: 215
Solve

$$\dot{x} = 2x + 3y$$
$$\dot{y} = 2x + y$$

by the second method of section 10.8.

Let $x = \text{.........................}$, $y = \text{.........................}$
$\dot{x} = \text{.........................}$, $\dot{y} = \text{.........................}$

--------------------▷ 216

The DE $y'' + 2y' - 3y = 4e^x$ 106
has the auxiliary equation $r^2 + 2r - 3 = 0$ whose roots are $r_1 = 1$, $r_2 = -3$.
The CF is $y_c = C_1 e^x + C_2 e^{-3x}$
The first term is the same as the RHS of the DE except for the constant, i.e. $4e^x$ is a part of the CF. We must try
$y_p = Axe^x$ for the PI.
Thus $y'_p = Axe^x + Ae^x$
and $y''_p = Axe^x + 2Ae^x$
Substituting in the DE we have

$$Ae^x(x + 2 + 2x + 2 - 3x) = 4e^x$$

Dividing by e^x and solving for A yields $A = 1$.
Therefore the PI is $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 107

$$x = ae^{rt}, \quad y = be^{rt}$$
$$\dot{x} = rae^{rt}, \quad \dot{y} = rbe^{rt}$$

216

Substitute in the DE:

....

....

-------------------▷ 217

$$y_p = xe^x$$ 107

The following DE occurs frequently in the study of forced oscillation with damping:

$$\ddot{x} + \xi\omega_n\dot{x} + \omega_n^2 x = F\mathrm{e}^{\mathrm{j}\omega t}$$

Find the PI and the value of its amplitude.
(Note the imaginary unit j. ξ, ω_n, ω are constants, t is the time.)

$x_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Amplitude $= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution ------------------▷ 110

Explanation wanted ------------------▷ 108

$$(r-2)a - 3b = 0, \qquad \text{i.e. } (r-2)a = 3b$$
$$-2a + (r-1)b = 0, \qquad \text{i.e. } (r-1)b = 2a$$ 217

Eliminate a and b : $(r-2)(r-1)ab = 6ab$, i.e. $(r-2)(r-1) - 6 = 0$
Then solve for r:

$r_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $r_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 218

The DE

$$\ddot{x} + \xi\omega_n \dot{x} + \omega_n^2 x = F e^{j\omega t}$$ 108

is similar to the type we discussed at the beginning of this sequence, except that the independent variable is the time, hence the reason for using the dot notation.

$f(x) = Fe^{j\omega t}$ is similar to ae^{bx} with $a = F$ and $bx = j\omega t$.

For the PI let

$$x_p = Ae^{j\omega t}$$

Thus

$$\dot{x}_p = j\omega A e^{j\omega t}$$

and $\ddot{x}_p = -\omega^2 A e^{j\omega t}$, since $j^2 = -1$

.

Substituting in the DE yields

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = Fe^{j\omega t}$$

------------------▷ 109

$$r_1 = -1, \quad r_2 = 4$$ 218

The solutions are

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 219

$$Ae^{j\omega t}(-\omega^2 + j\xi\omega_n\omega + \omega_n^2) = Fe^{j\omega t}$$ 109

Rearranging yields

$$A = \frac{F}{\omega_n^2 - \omega^2 + j\xi\omega_n\omega}$$

Thus $x_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The fraction is the complex amplitude.

We remember that the modulus of a complex number

$$z = a + jb \text{ is } |z| = \sqrt{a^2 + b^2}$$

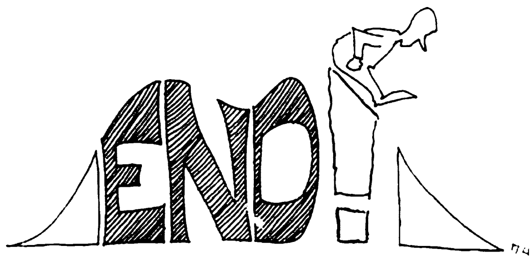
Obtain the modulus of the denominator:

$$\text{Amplitude} = \frac{|F|}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

→ 110

$$x = a_1e^{-t} + b_1e^{4t}$$
$$y = a_2e^{-t} + b_2e^{4t}$$ 219

This chapter on differential equations has been demanding. But if you worked through it carefully you will have gained enough experience to solve many of the DEs which you will encounter later on.



END OF CHAPTER 10

$$x_p = \frac{1}{\omega_n^2 - \omega^2 + j\xi\omega_n\omega} F e^{j\omega t}$$ 110

$$\text{Amplitude} = \frac{|F|}{\sqrt{(\omega_n^2 - \omega^2)^2 + \xi^2\omega_n^2\omega^2}}$$

In case of difficulty ▷ 108

Finally, we look at the third type of non-homogeneous DE.
Obtain the PI of the DE

$$y'' + 7y' + 10y = 20\cos 4x \qquad y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ▷ 112

Detailed solution wanted ▷ 111

Please continue on page 1
(bottom half)