Chapter 19
Fourier Integrals and Fourier Transforms

$$e^{ja} = j\sin a + \cos a$$
$$e^{ja+b} = e^{ja} \cdot e^{b} = e^{b}(j\sin a + \cos a)$$

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Do you remember the following expressions?

$$e^{j\omega t} = \text{.............................}$$
$$e^{-j\omega t} = \text{.............................}$$
$$e^{j\omega t} + e^{-j\omega t} = \text{.............................}$$

→ 30

19.1 Transition from Fourier series to Fourier integral

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In the preceding chapter it has been shown that any given periodic function can be represented by a sum of discrete sine and cosine functions. It is an important fact in physics and especially in information technology that periodic signals can be seen and produced by the superposition of sine and cosine oscillations or waves.

In this chapter we will show that even a single signal which is limited in duration (and hence non-periodic) can be represented as a sum of sine and cosine functions. The sum will turn into an integral and discrete functions will turn into a continuous distribution of oscillations or waves.

READ: 19.1 Transition from Fourier series to Fourier integral
Textbook pages 509–511

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$$\begin{aligned} e^{j\omega t} &= j \cdot \sin\omega t + \cos\omega t \\ e^{-j\omega t} &= -j \cdot \sin\omega t + \cos\omega t \\ e^{j\omega t} + e^{-j\omega t} &= 2\cos\omega t \end{aligned}$$

In case of difficulties recapitulate chapter 9 "Complex numbers". In the next section complex numbers will be used extensively.

Solve the two integrals using Euler's formula explained in the textbook Sect. 9.3.1

$$\int_{t_1}^{t_2} e^{-j\omega t}\,dt = \text{.............................} + \text{.............................}$$

$$\int_{t_1}^{t_2} e^{j\omega t}\,dt = \text{.............................} + \text{.............................}$$

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3

No difficulties in understanding the transition from the Fourier series to the Fourier integral

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I want further explanations

--------------------▷ 4

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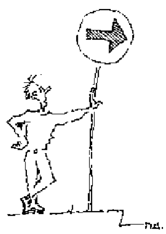
$$\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{-\mathrm{j}\omega}\cdot\left[\mathrm{e}^{-\mathrm{j}\omega t_2} - \mathrm{e}^{-\mathrm{j}\omega t_1}\right] \quad = \frac{1}{\omega}(\sin\omega t_2 - \sin\omega t_1) + \frac{\mathrm{j}}{\omega}(\cos\omega t_2 - \cos\omega t_1)$$

$$\int_{t_{\mathrm{j}1}}^{t_2} \mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{\mathrm{j}\omega}\cdot\left[\mathrm{e}^{\mathrm{j}\omega t_2} - \mathrm{e}^{\mathrm{j}\omega t_1}\right] \quad = \frac{1}{\omega}(\sin\omega t_2 - \sin\omega t_1) + \frac{\mathrm{j}}{\omega}(\cos\omega t_1 - \cos\omega t_2)$$



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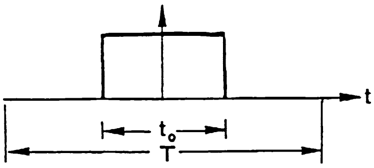
Consider a Fourier series for a rectangular function. The rectangular function is a periodic signal of duration t_0 and period T. 4

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin \frac{n\pi t_0}{T} \cdot \cos \frac{2\pi n t}{T}$$



To obtain a single and isolated signal of duration t the period's duration T has to increase up to infinity. The duration of the period T corresponds to an oscillation whose fundamental harmonic frequency is $\omega_o = \frac{2\pi}{T}$

Hence the n^{th} harmonic frequency is $\omega =$ ....

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19.2.3 Complex Representation of the Fourier Transform 32

19.3 Shift Theorem

Useful hint: calculate the derivations in the textbook separately on a sheet of paper

READ: **19.2.3 Complex Representation of the Fourier Transform**
19.3 Shift Theorem
Textbook pages 514–515

-------------------▷ 33

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$$\omega = n\omega_0 = n\frac{2\pi}{T}$$

This result can be transformed into $T = \frac{n \cdot 2\pi}{n \cdot \omega_o} = \frac{2\pi}{\omega_o}$

The given Fourier series of a rectangular periodic signal was

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin\frac{n\pi t_0}{T} \cdot \cos\frac{2\pi n t}{T}$$

The iteration number n increases from term to term by 1

We denote this by $\Delta n = 1$

We now insert $\Delta n = 1$ to obtain $f(t) =$ ....

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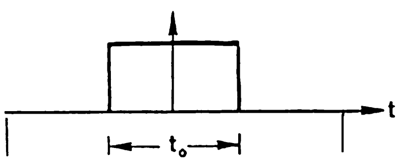
In the textbook the continuous amplitude spectrum has been calculated for a rectangular function of duration t_0. Calculate it on your own.

$F(\omega) = \frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} 1 \cdot e^{-j\omega t}\,dt$ .... Hint: regard the signs

The rectangular function has been defined

$$f(t) = \begin{cases} 0 & \text{for} \quad -\infty < t < -\frac{t_0}{2} \\ 1 & \text{for} \quad -\frac{t_0}{2} < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < \infty \end{cases}$$



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$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin \omega \frac{t_0}{T} \cdot \cos \omega t \cdot \Delta n$$ 6

We obtain a single signal of duration t_0 if we increase the distance between the signals, the period T, to infinity. Then the fundamental harmonic $\omega_0 = \frac{2\pi}{T}$ vanishes.

In the formula above the variable is n. If we want to transform the sum into an integral we must first substitute n by ω and Δn by $\Delta\omega$.

$$\omega = n \cdot \frac{2\pi}{T} \qquad \Delta\omega = \Delta n \cdot \frac{2\pi}{T}$$

$$n = \omega \cdot \frac{T}{2\pi} \qquad \Delta n = \frac{T}{2\pi}\Delta\omega$$

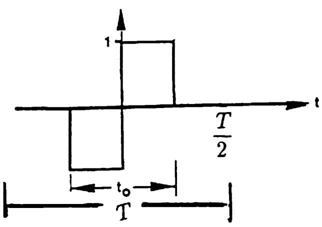
Now insert n and Δn

------------------▷ 7

$$\mathrm{F}(\omega) = \frac{1}{\omega\pi} \sin\left(\omega \frac{t_0}{2}\right)$$ 34

Given the function for the alternating rectangular signal of duration t_0:

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



The signal in complex representation by the Fourier integral

$f(t) =$ ....

The amplitude spectrum is $F(\omega) =$ ....

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$$f(t) = \frac{t_0}{T} + \sum_{\omega=0}^{\omega=\infty} \frac{2 \cdot 2\pi}{\omega \cdot T\pi} \cdot \sin\omega\frac{t_0}{2} \cdot \cos\omega t \cdot \frac{T}{2\pi}\Delta\omega = \frac{t_0}{T} + \sum_{\omega=0}^{\omega=\infty} \frac{2}{\omega\pi} \sin\omega\frac{t_0}{2} \cdot \cos\omega t \cdot \Delta\omega$$

Now we can transform the sum into an integral.

T increases to infinity: $T \to\propto$ $f(t) =$ ....

8

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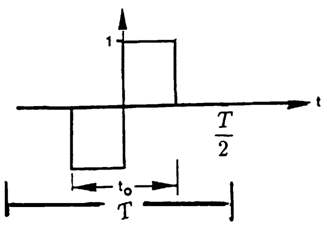
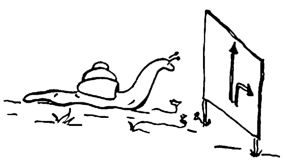
$$f(t) = \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} dt$$
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

Calculate the amplitude function of the alternating rectangular signal of duration t_0

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$





Solution obtained 40

Help and detailed solution 36

$$f(t) = \int_0^\infty \frac{2}{\pi\omega} \cdot \sin\omega\frac{t_0}{2} \cdot \cos\omega t \cdot \mathrm{d}\omega$$ 8

Using $A(\omega) = \frac{2}{\pi\omega} \cdot \sin\omega\frac{t_0}{2}$ we can write the integral as $f(t) =$ ....

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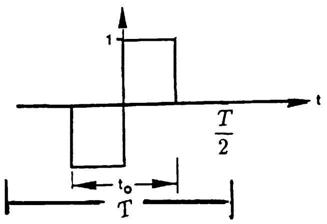
To be obtained: amplitude function of the alternating rectangular signal 36

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot \mathrm{e}^{-\omega t} \mathrm{d}t$$

The alternating rectangular signal has been defined

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We have to calculate the integral of the parts where $f(t) =$ .... is not zero.

Write down the remaining integrals $F(\omega) =$ ....

-------------------▷ 37

$$f(t) = \int_0^{\infty} A(\omega) \cos \omega t \cdot \mathrm{d}\omega$$ 9

Now we proceed to the next section

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$$F(\omega) = \frac{1}{2\pi} \int_{-\frac{t_0}{2}}^{0} (-1) \cdot \mathrm{e}^{-\mathrm{j}\omega t} \cdot \mathrm{d}t + \frac{1}{2\pi} \int_0^{\frac{t_0}{2}} \mathrm{e}^{-\mathrm{j}\omega t} \cdot \mathrm{d}t$$ 37

You can solve these integrals above regarding the limits

$F(\omega) =$ ....

Solution --------------------▷ 40

Further help and detailed solution --------------------▷ 38

19.2 Fourier transforms

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READ: 19.2.1 Fourier cosine transform
19.2.2 Fourier sine transform
Textbook pages 511–513

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38

To solve

$$F(\omega) = \frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{0}(-1)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\cdot \mathrm{d}t + \frac{1}{2\pi}\int_{0}^{\frac{t_0}{2}} \mathrm{e}^{-\mathrm{j}\omega t}\cdot \mathrm{d}t$$

Remember

$$\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t = \frac{1}{-\mathrm{j}\omega}\cdot\left[\mathrm{e}^{-\mathrm{j}\omega t_2} - \mathrm{e}^{-\mathrm{j}\omega t_1}\right]$$

Using the last form you should be able to solve the integrals.

You can solve these integrals above regarding the limits

$F(\omega) =$ ....

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The formulae to obtain the amplitude spectrum $A(\omega)$ and $B(\omega)$ are given in the textbook. They are verified but not derived. 11

Verify that $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cdot\cos\omega t\,\mathrm{d}t$ equals the $A(\omega)$ which we obtained in the preceding section from the transition from the sum to the Fourier integral: $A(\omega) = \frac{2}{\pi\omega}\cdot\sin\omega\frac{t_0}{2}$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cdot\cos\omega t\cdot\mathrm{d}t = \int \text{..............................} = [\text{..............................}]$$

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39

$$F(\omega) = \frac{1}{2\pi}\left(\frac{1}{-\mathrm{j}\omega}\right)\cdot\left(\left[(-1)\cdot\mathrm{e}^{-\mathrm{j}\omega t}\right]_{-\frac{t_0}{2}}^{0} + \left[\mathrm{e}^{-\mathrm{j}\omega t}\right]_{0}^{\frac{t_0}{2}}\right)$$

We insert the limits to obtain

$$F(\omega) = \frac{1}{2\pi(-\mathrm{j}\omega)}\cdot\left[\text{..............................}\right]$$

Now go back to frame 39 at the lowest part of the page

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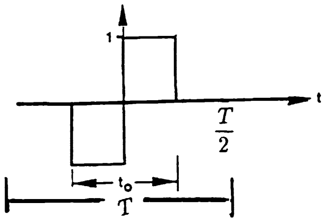
$$A(\omega) = \frac{1}{\pi} \int_{-\frac{t_0}{2}}^{+\frac{t_2}{0}} \cos \omega t \cdot \mathrm{d}t = \frac{1}{\pi \omega} [\sin \omega t]_{-\frac{t_0}{2}}^{+\frac{t_0}{2}} = \frac{2}{\pi \omega} \cdot \sin \omega \frac{t_0}{2}$$

We will also verify the formula for the amplitude spectrum of the Fourier sine transform. First we calculate the Fourier sequence of a periodic alternating rectangular wave form of duration t_0 and period T

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



The function is ☐ even ☐ odd. Hence the .... vanish

-------------------▷ 13

40

$$F(\omega) = \frac{1}{2\pi(-\mathrm{j}\omega)} \cdot \left[-1 + \mathrm{e}^{\mathrm{j}\omega \frac{t_0}{2}} + \mathrm{e}^{-\mathrm{j}\omega \frac{t_0}{2}} - 1 \right]$$

We carry out the sum and rearrange

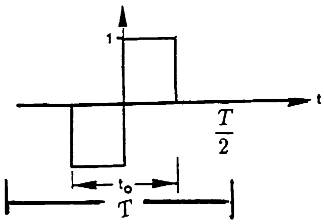
$F(\omega) =$ ....

Remember $\frac{1}{\mathrm{j}} = \frac{i}{\mathrm{j}^2} = -\mathrm{j} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$

-------------------▷ 41

The function is odd. Hence $a_n = 0$ 13

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We calculate the b_n. Because we have two branches we have to split the integration.

Obtain the two integrals. b_n ....

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$$F(\omega) = \frac{1}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right] \cdot (-\mathrm{j})$$
$$= \frac{1}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right] \cdot \mathrm{e}^{-\frac{i\pi}{2}}$$

Using the amplitude function we obtain the continuous amplitude spectrum

$A(\omega) =$ ....

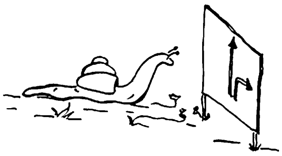
The phase spectrum in our case is

$\phi(\omega) =$ ....

--------------------▷ 42

$$b_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{0}(-1)\cdot\sin\frac{n2\pi}{T}t\,\mathrm{d}t + \frac{2}{T}\int_{0}^{\frac{t_0}{2}} 1\cdot\sin\frac{n2\pi}{T}t\cdot\mathrm{d}t$$ 14

The integral above can be resolved. You have to regard limits and may rearrange b_n ....



Integrals solved --------------------▷ 17

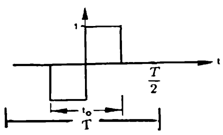
Hints and step-by-step calculation --------------------▷ 15

$$A(\omega) = \frac{1}{\pi\omega}\cdot\left[1-\cos\omega\frac{t_0}{2}\right] \qquad \text{Phasespectrum}: \varphi(\omega) = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$$ 42

Exercise for shift theorem.

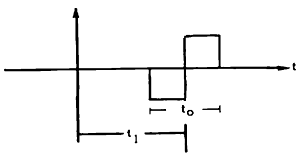
Given the alternating rectangular signal.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < \frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < \frac{T}{2} \end{cases}$$



We shift the Signal by the time t_1.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2}+t_1 < t < -\frac{t_0}{2}+t_1 \\ -1 & \text{for} \quad -\frac{t_0}{2}+t_1 < t < t_1 \\ +1 & \text{for} \quad t_1 < t < \frac{t_0}{2}+t_1 \\ 0 & \text{for} \quad \frac{t_0}{2}+t_1 < t < \frac{T}{2}+t_1 \end{cases}$$



Calculate $F(\omega) =$ .... --------------------▷ 46

Help and detailed solution --------------------▷ 43

To calculate $b_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{0}(-1)\cdot\sin\frac{n2\pi}{T}t\cdot dt + \frac{2}{T}\int_{0}^{\frac{t_0}{2}}1\cdot\sin\frac{n2\pi}{T}t\cdot dt$ 15

We remember $\int_{t_1}^{t_2}\sin\frac{n2\pi}{T}t\cdot dt = \frac{T}{n2\pi}\left[-\cos\frac{n2\pi}{T}t\right]_{t_1}^{t_2}$

We calculate the integrals and obtain

$$b_n = \frac{2}{T}\left[(-1)\cdot\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{-\frac{t_0}{2}}^{0} + \frac{2}{T}\left[\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{0}^{\frac{t_0}{2}}$$

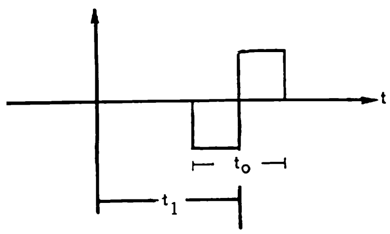
We rearrange and regard the limits to obtain b_n ....

Solution found ------------------▷ 17

Hint and help ------------------▷ 16

The given signal 43

$$f(t) = \begin{cases} 0 & \text{for } -\frac{T}{2}+t_1 < t < -\frac{t_0}{2}+t_1 \\ -1 & \text{for } -\frac{t_0}{2}+t_1 < t < t_1 \\ +1 & \text{for } t_1 < t < \frac{t_0}{2}+t_1 \\ 0 & \text{for } \frac{t_0}{2}+t_1 < t < \frac{T}{2}+t_1 \end{cases}$$



The amplitude function was

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot e^{-j\omega t}\cdot dt$$

We write down the integral for the parts where $f(t)$ is not equal to zero

$F(\omega) =$ .... + ... ------------------▷ 44

Given $b_n = \frac{2}{T}\left[(-1)\cdot\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{-\frac{t_0}{2}}^{0} + \frac{2}{T}\left[\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{0}^{\frac{t_0}{2}}$ 16

We isolate the term $\frac{T}{n2\pi}$ and insert the limits. Because of $\cos(0) = 1$ we obtain

$$b_n = \frac{2}{T}\cdot\frac{T}{n2\pi}\left[\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)+\left(-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)+1\right)\right]$$

Now we rearrange and summarize b_n ....

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$$F(\omega) = \frac{1}{2\pi}\int_{-\frac{t_0}{2}+t_1}^{t_1}(-1)\cdot e^{-j\omega t}\cdot dt + \frac{1}{2\pi}\int_{t_1}^{\frac{t_0}{2}+t_1} e^{-j\omega t}\cdot dt$$ 44

We solve both integrals

$F(\omega) = \frac{1}{2\pi}\left[\text{.............................}\right] + \frac{1}{2\pi}\left[\text{.............................}\right]$

- - - - - - - - - - - - - - - - - - -▷ 45

$$b_n = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{T}\cdot t_0\right)\right]$$ 17

Thus the Fourier series is

$f(t) =$ ....

--------------------▷ 18

$$F(\omega) = \frac{1}{2\pi}\left[\frac{-1}{-\mathrm{j}\omega}\cdot \mathrm{e}^{-\mathrm{j}\omega t}\right]_{-\frac{t_0}{2}+t_1}^{t_1} + \frac{1}{2\pi}\left[\frac{1}{-\mathrm{j}\omega}\cdot \mathrm{e}^{-\mathrm{j}\omega t}\right]_{t_1}^{\frac{t_0}{2}+t_1}$$ 45

Now your task is to insert the limits and to rearrange the remaining terms.

Hint: the term $\mathrm{e}^{-\mathrm{j}\omega t_1}$ can be isolated and put before the brackets

$F(\omega) =$ ....

--------------------▷ 46

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\left(\frac{n2\pi}{T} \cdot \frac{t_0}{2}\right)\right) \cdot \left(\sin \frac{n2\pi}{T} t\right)$$ 18

In the textbook, the Fourier coefficients are shown for $t_0 = 1$ and $T = 2t_0$, $T = 4t_0$ and $T = 8t_0$.

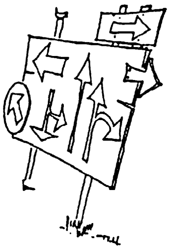
It is a useful exercise to calculate the b_n at least for one case numerically.

Calculation of b_n for $t_0 = 1$ and $T = 2$ -------------------▷ 19

I will skip this exercise -------------------▷ 21

$$F(\omega) = \frac{1}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right] e^{-j\omega t_1 - j\frac{\pi}{2}}$$ 46

Main result: The continuous amplitude spectrum does not depend on a shift of the signal by the time t_1. Only the phase spectrum will be affected.



Remarks concerning the notation of Fourier transforms which are not handled consistently -------------------▷ 47

If you want to skip these remarks -------------------▷ 50

Given the Fourier series $f(t)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)\cdot\left(\frac{n2\pi}{T}t\right)$ 19

To calculate $b_n=\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{T}\cdot t_0\right)\right]$

We insert $t_0=1$ and $T=2$

Fill in the calculated values.

Use a pocket calculator.

n	$\frac{2}{n\pi}$	$\cos n\frac{\pi}{2}$	b_n
1			
2			
3			
4			
5			
6			
7			
8			

--------------------▷ 20

In a note in the textbook we said that the notation of Fourier transform is not handled consistently. This may disturb you as the reader if you come across different notations in the literature such as: 47

a) $f(t)=\int_{-\infty}^{+\infty}F(\omega)\cdot e^{+j\omega t}d\omega \qquad F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}f(t)\cdot e^{-j\omega t}dt$

b) $f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}F(\omega)\cdot e^{+j\omega t}d\omega \qquad F(\omega)=\int_{-\infty}^{+\infty}f(t)\cdot e^{-j\omega t}dt$

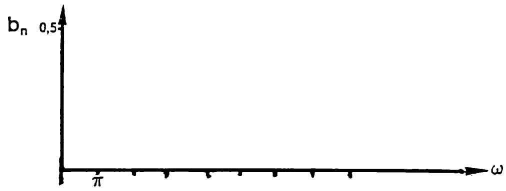
c) $f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}F(\omega)\cdot e^{+j\omega t}d\omega \qquad F(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f(t)\cdot e^{-j\omega t}dt$

The differences refer to the constant factor $\frac{1}{2\pi}$

--------------------▷ 48

n	$\frac{2}{n\pi}$	$\cos n\frac{\pi}{2}$	b_n
1	0,64	0	0,64
2	0,32	-1	0,64
3	0,22	0	0,22
4	0,16	$+1$	0
5	0,13	0	0,13
6	0,11	-1	0,22
7	0,09	0	0,09
8	0,08	$+1$	0

Draw the amplitude spectrum for $t_0 = 1$ and $T = 2$ 20



------------------▷ 21

The different notations are: 48

$a)$ $f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t}\, d\omega \qquad F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t}\, dt$

$b)$ $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t}\, d\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t}\, dt$

$c)$ $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t}\, d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t}\, dt$

To see that all notations are equivalent you may insert the second integral into the first. In the case a) of our notation we obtain

$$a) \quad f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \quad = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt \cdot e^{j\omega t} \cdot d\omega$$

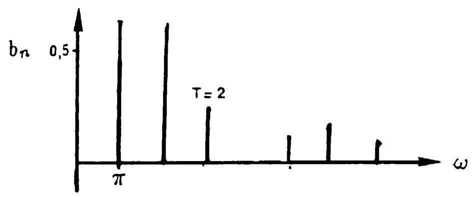
Repeat this procedure for b) and c)

$b)$ $f(t) =$ ....

$c)$ $f(t) =$ ....

Solution ------------------▷ 50

Help ------------------▷ 49



21

The b_n for different values of t_0 and T can be calculated in the same way.

In the textbook the results for $T = 4$ and $T = 8$ are shown.

-------------------▷ 22

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Given

a) $f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

c) $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

We consider b). We insert $F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$ into the first integral and obtain

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}$ ....$\cdot e^{j\omega t} \cdot d\omega$

Now do the same with c)

c) $f(t) =$ ....

-------------------▷ 50

The Fourier series for the periodic alternating rectangular wave form of duration t_0 and period T was $f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)\cdot\sin\left(\frac{n2\pi}{T}t\right)$ 22

Now follows the transition to the Fourier integral.

The sum increases step by step by $\Delta n = 1$. By the transition from the sum to the integral we must regard the relation between Δn and $\Delta\omega$ since $\frac{n2\pi}{T} = \omega$ and $n = \omega\frac{T}{2\pi}$ we get $\Delta n = \Delta\omega\frac{T}{2\pi}$ and $\Delta\omega = \Delta n\cdot\frac{2\pi}{T}$.

Substitute in the sum above T by ω $f(t) = \frac{2}{\pi}\sum_{n=1}^{n=\infty}$ ....

--------------------▷ 23

---

b) = c) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)\cdot e^{-j\omega t}\cdot dt\cdot e^{j\omega t}\cdot d\omega$ 50

Result: all three notations result in the same expression if we perform the Fourier transform and the reverse transform.

These remarks may help you to accept the different notations. Important for you is to use one of these notations consequently and exclusively.

--------------------▷ 51

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{n=\infty} \frac{2\pi}{\omega T} \left(1 - \cos\omega \frac{t_0}{2}\right) \cdot \sin\omega t \cdot \Delta\omega \frac{T}{2\pi}$$ 23

Now we can reduce and perform the limiting process.

$$f(t) = \frac{2}{\pi} \int_0^{\infty} \frac{1}{\omega} \left(1 - \cos\omega \frac{t_0}{2}\right) \cdot \sin\omega t \cdot \mathrm{d}\omega$$

The continuous amplitude spectrum will be $B(\omega) =$ ....

▷ 24

19.4 Discrete Fourier transform, sampling theorem

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19.5 Fourier transform of the Gaussian function

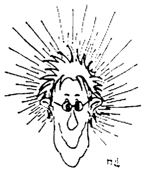
READ: 19.4 Discrete Fourier transform, sampling theorem
19.5 Fourier transform of the Gaussian function
Textbook pages 516–517

▷ 52

$$B(\omega) = \frac{2}{\pi \cdot \omega} \cdot \left(1 - \cos\omega \frac{t_0}{2}\right)$$ 24

We can also obtain the continuous amplitude spectrum with the formula for the sine transform given in the textbook.

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \cdot \sin \omega t \cdot \mathrm{d}t \qquad B(\omega) = \text{.............................}$$



Solution found --------------------▷ 28

Hint and step-by-step solution --------------------▷ 25

The following items are aimed to give a general orientation. The sampling theorem is fundamental for modern information technology which deals with the transformation of analog signals into discrete signals and the reconstruction of analog signals from discrete sample values. 52

A full reconstruction of a function from its sampling values is only possible if the sampling frequency captures the highest frequency in the amplitude spectrum.

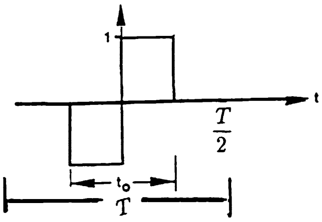
The sampling frequency has to bee the highest frequency of the amplitud spectrum.

--------------------▷ 53

To solve: $B(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\sin\omega t \cdot \mathrm{d}t$ 25

The function is defined in parts.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We calculate the integral for the two parts which do not vanish

$$B(\omega) = \text{............................} + \text{............................}$$

--------------------▷ 26

The sampling frequency has to be *twice* the highest frequency of the amplitude spectrum. 53

Given the bell-shaped Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{a}{2}t^2}$

It corresponds to the likewise bell-shaped amplitude function $F(\omega) = \frac{1}{\sqrt{a}} \cdot \mathrm{e}^{-\frac{\omega^2}{2a}}$

Both functions have a maximum for $t =$ .... and $\omega =$ ....

If t and ω increase, both exponential terms decrease.

Both terms are $\frac{1}{\mathrm{e}}$ if the exponent is -1: $\mathrm{e}^{-\frac{\omega^2}{2a}} = \mathrm{e}^{-\frac{a}{2}t^2} = \frac{1}{e}$

The exponents .... = ... $= -1$

Exponents found
--------------------▷ 55

Further explanation
--------------------▷ 54

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$$B(\omega) = \frac{1}{\pi} \int_{-\frac{t_0}{2}}^{0} (-1) \sin \omega t \cdot \mathrm{d}t + \int_{0}^{\frac{t_0}{2}} \sin \omega t \cdot \mathrm{d}t$$

We solve the integrals and obtain

$$B(\omega) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 27

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Given Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{a}{2}t^2}$.

Its amplitude spectrum $F(\omega) = \frac{1}{\sqrt{a}} \cdot \mathrm{e}^{-\frac{\omega^2}{2a}}$.

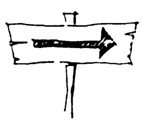
Both functions have its maximum at $\mathrm{e}^0 = 1$

Hence for the maximum holds $t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ and $\omega = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

If t and ω increase the absolute values of the exponents increase as well. Since the exponents are negative the values of the functions decrease up to $\frac{1}{\mathrm{e}}$ if the exponents are -1

In this case $\mathrm{e}^{-\frac{\omega^2}{2a}} = \mathrm{e}^{-\frac{a}{2}t^2} = \mathrm{e}^{-1}$.

Give the exponents $\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots = -1$



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$$B(\omega)=\frac{1}{\pi}\left[(-1)\cdot(-\cos\omega t)\cdot\frac{1}{\omega}\right]_{-\frac{t_0}{2}}^{0}+\frac{1}{\pi}\left[\left(-\cos\omega t\cdot\frac{1}{\omega}\right)\right]_{0}^{\frac{t_0}{2}}$$

Now we regard the limits to obtain

$$B(\omega)=\text{............................}$$

28

55

Maxima for $t=0$ and $\omega=0$.

Decreases to $\frac{1}{e}$ if $-\frac{\omega^2}{2a}=-\frac{a}{2}t^2=-1$.

By this we get for the decrease to $\frac{1}{\mathrm{e}}$

$\frac{a}{2}t^2=1$ and $t=\sqrt{\frac{2}{a}}$

$\frac{\omega^2}{2a}=1$ and $\omega=\sqrt{2a}$

We repeat: If the parameter a is large

We get a narrow signal in the time domain and a wide amplitude spectrum in the frequency domain.

56

$$B(\omega) = \frac{2}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right]$$ 28

Thus we have again verified that by the limiting process $T \to \infty$ and $n \to \infty$ we obtain the same continuous amplitude spectrum as has been given in the textbook.

In the next section we will use complex numbers.

Please remember:

$$e^{ja} = \text{............................}$$
$$e^{ja+b} = \text{............................}$$

------------------▷ 29

Please continue on page 1
(bottom half)

You have successfully reached the end of this chapter. You made an important step forward. 56