Chapter 15
Sets of Linear Equations; Determinants

2

$$\mathbf{A}^{-1} = \begin{pmatrix} -1 & -4 & -1 & 2 \\ -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & 0 & -1 \end{pmatrix}$$

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We have thus calculated the inverse of $\mathbf{A}$. And your check should result in $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

Now for the solution of the original system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ 8 \\ 64 \end{pmatrix}$$

Write down the augmented matrix $\mathbf{A}|\mathbf{b}$

$$\left(\begin{array}{c|c} \quad & \quad \\ & \\ & \end{array}\right)$$

....

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15.1 Gauss Elimination, Successive Elimination of Variables, Gauss-Jordan Elimination

2

In this chapter several numerical examples are shown in the textbook. It is advisable to try to solve the examples by yourself and to check your solution afterwards. Since the algorithms are explained explicitly you should be able to find the solutions in most cases.

READ: **15.1 Introduction**
15.2.1 Gauss elimination: successive elimination of variables
15.2.2 Gauss-Jordan elimination
Textbook pages 431–434

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$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 16 \\ 1 & 2 & -1 & 5 & 25 \\ 0 & 1 & 0 & 1 & 8 \\ 3 & 5 & -2 & 12 & 64 \end{array}\right)$$

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Solve the system of linear equations by executing the Gauss-Jordan elimination for the augmented matrix.

The steps are exactly the same as in the previous example. The difference is that the matrix **A** has been augmented by **b** instead of **I**. Thus all transformations are applied to **b**. All steps are explained in detail in frames 12 to 15, and in case of difficulties repeat these frames. Result:

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

$x_1 =$ ....
$x_2 =$ ....
$x_3 =$ ....
$x_4 =$ ....

-------------------▷ 19

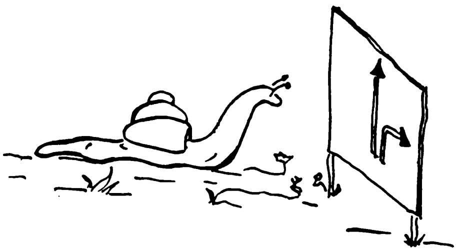
3

Given a system of two linear equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

find the solution using the Gauss-Jordan elimination.

$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_2 = \ldots\ldots\ldots\ldots\ldots\ldots$$



Solution ------------------▷ 7

Detailed calculation ------------------▷ 4

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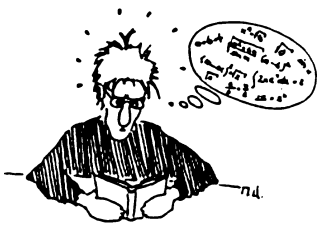
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{aligned} x_1 &= 4 \\ x_2 &= 6 \\ x_3 &= 1 \\ x_4 &= 2 \end{aligned}$$

The execution of the transformations requires attention but they are not difficult in principle. The matrix notation and the execution of all transformations with the augmented matrices helps to avoid errors. It may be useful, too, to write down all transformations explicitly, as has been done in frames 12 to 15.



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Given: 4

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

First step: Dividing the first equation by a_{11} and eliminating x_1 in the second equation yields

$$x_1 + \frac{a_{12}}{a_{11}}x_2 = \frac{b_1}{a_{11}}$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 = b_2 - b_1\frac{a_{21}}{a_{11}}$$

Now execute the second step and write down the result:

$$x_1 + \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 = b_2 - b_1\frac{a_{21}}{a_{11}}$$

Solution ------------------▷ 6

Further explanation ------------------▷ 5

15.3 Existence of Solutions

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In the textbook we again have worked examples at the end of the section. Within the examples matrix notation is used. In case of difficulties it may be advisable to write down the equations explicitly.

READ: 15.2.4 Existence of solutions
Textbook pages 437–439

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Further explanation: Given the result of the first step

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$$x_1 + \frac{a_{12}}{a_{11}} x_2 = \frac{b_1}{a_{11}}$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) x_2 = b_2 - b_1 \frac{a_{21}}{a_{11}}$$

Second step: Elimination of the coefficient of x_2 in the first equation. This step requires division of the second equation by the actual coefficient of x_2 which is bracketed. Then we can eliminate x_2 in the first equation.

$$x_1 + 0 = \text{..........................}$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) x_2 = b_2 - b_1 \frac{a_{21}}{a_{11}}$$

-------------------▷ 6

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(A) If you have to solve 4 non-homogeneous equations and 6 variables:
Then at most .... variables can be determined.
At least .... variables are free to be chosen.

(B) Suppose you have 4 homogeneous linear equations. State the form of the solution.

Trivial solution: ....
If a non-trivial solution exists it ....

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$$x_1 + 0 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} \times \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$0 + x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

6

The first equation can be rearranged to a form similar to the second one, giving:

$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

------------------▷ 7

(A) At most 4 variables can be determined.
(B) At least 2 variables are free to be chosen.

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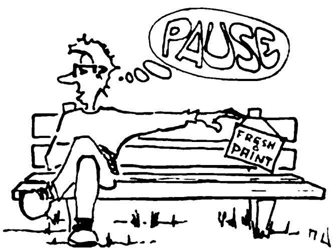
Trivial solution: $x_j = 0, \quad j = 1, 2, 3, 4.$
If a non-trivial solution exists it is not unique and has at least one variable free to be chosen.

In practical applications it is useful first to check whether solutions exist and whether they are unique.
By applying Gauss-Jordan elimination the solution clearly shows its structure.



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$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

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In case of difficulties repeat the transformations, beginning with frame 4.

Now solve the numerical example

$$x_1 + x_2 = \frac{7}{10}$$
$$2x_1 + 5x_2 = 2$$
$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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15.4 Determinants

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Objective: Concept of a determinant, competence to evaluate the determinants of 2×2 and 3×3 matrices.

READ: **15.3.1 Preliminary remarks on determinants**
15.3.2 Definition and properties of an *n*-row determinant
Textbook pages 440–446

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$$x_1 = \frac{1}{2}$$
$$x_2 = \frac{1}{5}$$

8

The explicit formula for systems of two linear equations for two unknowns is the only formula derived in detail. The formulae for other systems are formidable. In practice one would not use explicit formulae; rather the algorithm for the solution would be followed. You should, therefore, understand the logical basis of the elimination procedure.

--------------------▷ 9

Given: the determinant of the matrix which we used in frames 11 to 15

24

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix}$$

Determine the minor of a_{12} : $\begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix}$ ....

Determine the cofactor A_{12} : $\begin{vmatrix} \quad & \quad & \quad \\ & & \\ & & \end{vmatrix}$ ....

In case of difficulties refer to the textbook.

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15.2 Matrix Notation of Systems of Equations and Calculation of the Inverse Matrix

9

Solving systems of linear equations is facilitated by the matrix notation. We will use Gauss-Jordan elimination.

Also the calculation of the inverse matrix is shown to be possible using Gauss-Jordan elimination.

READ: 15.2.3 Matrix notation of sets of equations and determination of the inverse matrix
Textbook pages 434–437

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minor of $a_{12} = \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

cofactor $A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

Evaluate the cofactor: $A_{12} = - \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

Try both methods, Sarrus' rule and evaluating using cofactors.

$A_{12} = \ldots\ldots\ldots\ldots\ldots$

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Given a system of linear algebraic equations **Ax = b** 10

$$\begin{aligned} x_1 + x_2 + 0 + 3x_4 &= 16 \\ x_1 + 2x_2 - x_3 + 5x_4 &= 25 \\ 0 + x_2 + 0 + x_4 &= 8 \\ 3x_1 + 5x_2 - 2x_3 + 12x_4 &= 64 \end{aligned}$$

The augmented matrix **A** | **I** is a .... matrix.
Write down the augmented matrix **A** | **I**

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

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Using cofactors, expanding for the first row: 26

$$-\begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix} = [1 \times (-2) - (-1)(-3)] = 1$$

Sarrus' rule

$$-\begin{pmatrix} 1 & -1 & 5 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & -2 & 12 & 3 & -2 \end{pmatrix} = -(-3 - (-2)) = 1$$

--------------------▷ 27

$\mathbf{A} \mid \mathbf{I}$ is a 4×8 matrix 11

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & -2 & 12 & 0 & 0 & 0 & 1 \end{array}\right)$$

Evaluate the inverse $\mathbf{A}^{-1}$. To do this transform the matrix so that the first part is a unit matrix. Then the second part will be $\mathbf{A}^{-1}$.

Solution -------------------▷ 16

Detailed solution with further explanation -------------------▷ 12

The properties of determinants are useful if the determinants of a large matrix have to be evaluated since the effort of calculation can be considerably reduced. 27

Given:

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix}$$

Try to use property 5 (adding multiples of a row to another) to simplify the task of evaluation.

$$\det \mathbf{A} = \begin{vmatrix} & & & \\ & & & \\ & & & \\ & & & \end{vmatrix}$$

.....

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12

Consider the given matrix.
First step: Elimination of the elements in the first column beneath a_{11}.

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & -2 & 12 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \text{Row 2 : subract row 1} \\ \text{Row 3 : } a_{13} \text{ is already zero} \\ \text{Row 4 : substract } 3\times \text{ row 1} \end{array}$$

Execute the subtractions for the whole rows:

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

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There are several simplifications possible. In this case one could do the following:

$$\begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix} \begin{array}{l} (\text{row 2: subtract row 1}) \\ (\text{row 4: subtract } 3\times\text{row 1}) \end{array} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -2 & 3 \end{vmatrix}$$

Incidentally, this simplification equals the first step of the Gauss-Jordan elimination (see frame 12).
Now only the cofactor A_{11} has to be evaluated.
Thus det **A** =

....

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The changed elements are circled 13

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 & -3 & 0 & 0 & 1 \end{array}\right)$$

Second step: Elimination of the elements beneath and above a_{22}:
Row 1: subtract row 2
Row 3: subtract row 2
Row 4: subtract $2 \times$ row 2

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

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$$\det \mathbf{A} = a_{11} \times A_{11} = 1 \times \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -2 & 3 \end{vmatrix} = -1$$ 29

We end the evaluation of determinants now since it is sufficient to know the principles.

30

The changed elements are circled 14

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right)$$

Third step: Elimination of the elements beneath and above a_{33}:
Row 1: subtract row 3
Row 2: add row 3
Row 4: a_{34} is already zero

$$\left(\begin{array}{c|c} & \\ & \\ & \\ & \end{array}\right)$$

....

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15.5 Rank of a Determinant Applications of Determinants 30

You should know the concept of rank of a matrix and its determinant, since the structure of solutions of systems of linear equations is determined by this rank. This is shown in connection with Cramer's rule which is introduced as an application of determinants.

READ: 15.3.3 Rank of a determinant and rank of a matrix
15.3.4 Applications of determinants
Textbook pages 446–451

Follow the examples given in the textbook.

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The changed elements are circled 15

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right)$$

Fourth step: Division of row 4 by $a_{44} = -1$ and elimination of the elements above a_{44}.
After division by a_{44}:
Row 1: subtract $2 \times$ row 4
Row 2: subtract row 4
Row 3: add row 4
Result:

$$\left(\begin{array}{c|c} \qquad\qquad & \qquad\qquad \\ \\ \\ \end{array}\right)$$

....

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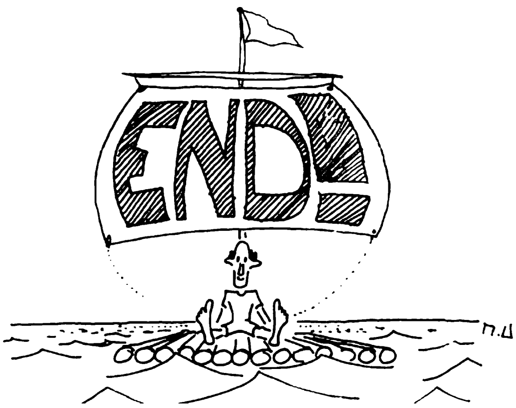
In section 15.3.4 three examples have been given for the application of Cramer's rule. We will not discuss any more examples for this rule in the study guide, since for practical applications we suggest the use of the Gauss or the Gauss-Jordan elimination to solve systems of linear equations. Doing this, the rank of the coefficients matrix will be obtained automatically, too. 31

END OF CHAPTER 15



$$\mathbf{I}|\mathbf{A}^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -4 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & -1 \end{array}\right)$$

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Now write down $\mathbf{A}^{-1}$, which we obtained by this transformation of $\mathbf{A}|\mathbf{I}$ into $\mathbf{I}|\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1} = \left(\quad \right)$$

....

Check your result by calculating
$\mathbf{A}^{-1}\mathbf{A}$ and $\mathbf{AA}^{-1}$

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