Chapter 11
Laplace Transforms

-------------------▷ 1

Now you decide how to proceed. 74

Difficulties with example 2 -------------------▷ 75

Difficulties with example 3 -------------------▷ 78

Difficulties with example 4 -------------------▷ 84

Straight on, no difficulties so far -------------------▷ 86

Laplace transforms 1

Objective: In the first section the definition of Laplace transforms will be explained and developed. It will prove useful, to copy all definitions and results into your notebook when going through this and subsequent sections. Then they will be readily accessible to you and you will not have to take recourse to the textbook for each and every single computation.

READ **11.1 Introduction**
11.2 The Laplace transform definition
Textbook pages 321–322

When done -------------------▷ 2

Example 2 deals with: $y''+5y'+4y=0$ initial conditions: $t=0$ $y_0=0$ $y'_0=3$ 75

Finding a solution always requires following these three steps:
Step 1: Apply the Laplace transform. Insert the initial conditions.
Step 2: Simplify the equation and solve for $\bar{y}(s)$ or in another notation $F(s)$.
Step 3: Apply the transform.
If needed, read again the first part of section 11.4 in the textbook.
. The following shows all details for example 2 from the textbook.
Difficulties may arise with regard to rewriting $\bar{y}(s)=\dfrac{3}{s^2+5s+4}$

Determine the roots in the denominator by solving the quadratic equation. That will enable us to represent the expression $\bar{y}(s)=\dfrac{3}{s^2+5s+4}$ as a product of linear factors.

$s_1=$....................
$s_2=$..................

-------------------▷ 76

Write down the definition of the Laplace transform: 2

$\mathcal{L}[f(t)] =$ ………………………….

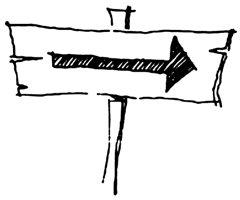
Being well versed in the notation greatly eases not only studying the textbook but to an even greater extent eases later practice.

Write down the same definition using different notations

$\mathcal{L}[y(t)] =$ …………..

$\mathcal{L}[f(x)] =$ …………..

$\mathcal{L}[y(x)] =$ …………..



------------------▷ 3

$s_1 = -4$ $\quad$ $s_2 = -1$ 76

Therefore, the denominator can be written as a product of linear factors: $s^2 + 5s + 4 = (s+4)\cdot(s+1)$

Thus, the transform reads: $\bar{y}(s) = \dfrac{3}{(s+4)\cdot(s+1)}$

For this expression of $\bar{y}(s)$ the inverse transform can be found using the table: $y =$ ....

So we are done.
If, however, you prefer to use a more elementary approach, you may first split the expression for $\bar{y}(s)$ into partial fractions and then find the inverse transforms for elementary expressions of the type $\dfrac{1}{(s-a)}$, which are given by e^{at}, of course. For the similar case of $\bar{y}(s) = \dfrac{5s+11}{s(s+2)\cdot(s+4)}$ this has been demonstrated in frame 71, which you may like to refer to.

------------------▷ 77

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)\cdot e^{-s\cdot t}dt = \bar{f} \qquad \text{or } F(s)$$

$$\mathcal{L}\left[y(t)\right] = \int_0^\infty y(t)\cdot e^{-s\cdot t}dt = \bar{y} \qquad \text{or } Y(s)$$

$$\mathcal{L}\left[f(x)\right] = \int_0^\infty f(x)\cdot e^{-s\cdot x}dx = \bar{f} \qquad \text{or } F(s)$$

$$\mathcal{L}\left[y(x)\right] = \int_0^\infty y(x)\cdot e^{-s\cdot x}dx = \bar{y} \qquad \text{or } Y(s)$$

3

Complete the definition

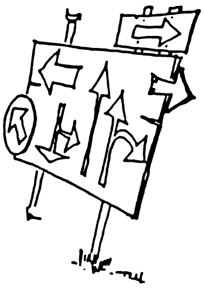
$\mathcal{L}^{-1}\left[F(s)\right] =$…………………………..

-------------------▷ 4

$y = e^{-t} - e^{-4t}$

77

Again, please make a choice



Difficulties with example 3 -------------------▷ 78

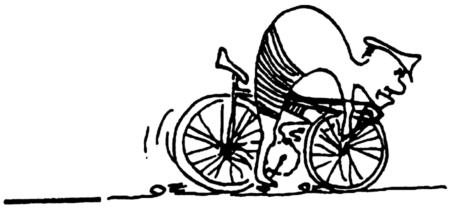
Difficulties with example 4 -------------------▷ 84

Straight on -------------------▷ 86

The symbol $\mathcal{L}^{-1}[F(s)]$ stands for the inverse Laplace transform. 4

$\mathcal{L}^{-1}\ f(s) = F(t)$ or $F(x)$ in case we use the variable x.

Before going through examples that will show the extreme usefulness of Laplace transforms, you will have to stand a certain number of dry spells, while you take in the rules for the transforms and copy them into your notebook.



------------------▷ 5

Example 3 78

Solve the following differential equation: $y''+8y'+17y=0$
given the initial conditions: $t=0 \quad y_0=0 \quad y'_0=3$

Step 1: Apply the Laplace transform to the differential equation according to the rules:

……………………..…………………..= 0

------------------▷ 79

11.1 Laplace transform of standard functions and general theorems

5

Objective: In this section you will acquaint yourself with the main facts on how to transform basic functions like e^{at}, $\sin\omega t$, $C \cdot t$, etc. Funnily enough, t^n will have to wait for a while.
Please follow all arguments, and copy all calculations and results into your notebook.
Since this section is quite long we suggest you study it in parts. In this section we will use the following notation: few denote the original function by $y(t)$ or $f(x)$.

Study **11.3 Laplace transform of standard functions**
Theorem I: The shift Theorem
Textbook pages 322–324

When done

-------------------▷ 6

$\bar{y}(s) \cdot s^2 - y_0 - y'_0 - sy_0 + 8\bar{y}(s) \cdot s - 8y_0 + 17\bar{y}(s) = 0$

79

Step 2: Insert the initial conditions $(t = 0, y_0 = 0, y'_0 = 2)$ and solve for $\bar{y}(s)$.

$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 80

Given the constant function $y(t)$
Try to derive its Laplace transform $\bar{y}(s)$ on your own.
$\bar{y}(s)$ = …………………………

6

Solution found ------------------▷ 8

Help needed ------------------▷ 7

$\bar{y}(s) = \frac{3}{s^2 + 8s + 17}$

80

In the table we find the inverse transform for the following function:

$\bar{y}(s) = \frac{1}{(s-a)^2 + \omega^2}$ $\qquad y = \frac{1}{\omega} \cdot e^{at} \cdot \sin \omega t$

Thus, we face the task to rearrange the given denominator so that the inverse transform according to the rule above can be applied. We succeed by completing the square for $s^2 + 8s$ and obtain

$\bar{y}(s) = \frac{3}{\left((s+4)^2 + 1\right)}$

So, using the notation from the rule, $a = -4$ and $\omega^2 = 1$

$y(t) =$ ....

------------------▷ 81

Given the constant function $y(t) = C$ 7
Wanted: Laplace transform $\bar{y}(s)$.
You have to solve

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot y(t) \cdot dt$$

For $y(t) = C$ we insert C into the integral:

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot C \cdot dt$$

Factor the constant C, solve the integral, and insert the limits of integration:

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot C \cdot dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 8

$y = 3 \cdot e^{-4t} \cdot \sin t$ 81

We could have transformed the function $\bar{y}(s)$ differently. This is shown in example 2 in the textbook.

After determining the roots a and b of the denominator, we obtain the following expression:

$$\bar{y}(s) = \frac{3}{(s-a) \cdot (s-b)}$$

Show me this transformation also ------------------▷ 82

Difficulties with this example ------------------▷ 84

Ready to go on ------------------▷ 86

$$\bar{y}(s) = C \cdot \int_0^\infty e^{-st} dt$$ 8

$$\bar{y}(s) = C\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{C}{s}$$

Likewise we can determine the Laplace transform of an exponential function.

Given: $y(t) = e^{at}$

To do: Compute $\bar{y}$:

$\bar{y} =$……………………..........

Computation successfully done -------------------▷ 10

Help needed -------------------▷ 9

Given: $\bar{y}(s) = \frac{3}{s^2 + 8s + 17}$ 82

Determine the roots of the denominator (solve the quadratic equation):

$s_1 =$ ....

$s_2 =$ ....

-------------------▷ 83

Given: $y(t) = e^{at}$ 9
Wanted: Laplace transform $\vec{y}$.

Again, you must compute the Laplace transform by evaluating an integral.

$$\mathcal{L}[y(t)] = \vec{y} = \int_0^\infty e^{-st} \cdot y(t) \cdot dt$$

Insert $y(t)$ as given:

$$\mathcal{L}[y(t)] = \vec{y}(s) = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

By factoring t in the exponent, we get an integral, which we have already solved frequently.
Lastly, we insert the limits of integration.

$\mathcal{L}[y(t)] = \vec{y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

In case of remaining difficulties consult the textbook.

--------------------▷ 10

$s_1 = -4 + i$ 83
$s_2 = -4 - i$

Thus $\bar{y}(s) = \dfrac{3}{(s+4+i)\cdot(s+4-i)}$

So we have a type of expression that we know how to handle.

$$\bar{y}(s) = \frac{3}{(s-a)\cdot(s-b)}$$

$$y = \frac{1}{2i}\cdot\left(e^{-(4-i)t} - e^{-(4+i)t}\right)\cdot 3$$

Using Euler's formula we obtain the result already known
$y = 3 \cdot e^{-4t} \cdot \sin t$

Difficulties with example 4 --------------------▷ 84

Ready to go straight on --------------------▷ 86

$$\bar{y} = \int_0^\infty e^{-(s-a)} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{(s-a)}$$ **10**

The Laplace transforms of trigonometric functions are obtained by a direct approach, too.
Let us start with the sine function:

$y(t) = \sin \omega t$

$\mathcal{L}[\sin \omega t] = \ldots\ldots\ldots\ldots\ldots$

Hint: Use Euler's formula to express $\sin \omega t$ by exponential functions.

Solution found -------------------▷ 13

Help needed -------------------▷ 11

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Example 4 asks us to solve: $y'' + 6y = t$
with initial conditions: $t = 0 \quad y_0 = 0 \quad y'_0 = 1$

Applying the Laplace transform and rearranging according to the known procedures results in

$$\bar{y}(s) = \frac{1}{s^2+6} + \frac{1}{s^2(s^2+6)}$$

Only the inverse transformations may pose a problem. But in the table we do find an inverse transform for both expressions. Putting $\omega^2 = 6$ we arrive at

$$y(t) = \frac{1}{\sqrt{6}} \sin \sqrt{6} \cdot t + \frac{1}{6\sqrt{6}} \left(\sqrt{6} \cdot t - \sin \sqrt{6} \cdot t \right)$$

Simplify $y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 85

The Laplace transform of the sine function $y(t) = \sin \omega t$ is to be found. 11
We already know the transformation of the exponential function
$\mathcal{L}\left[e^{at}\right] = \bar{y}(s) = \frac{1}{s-a}$

Hint: You should have taken a note of this result before, in order to quickly have access to it.

We recall Euler's formula

$$e^{i\omega t} = i \sin \omega t + \cos \omega t$$

$$e^{-i\omega t} = -i \sin \omega t + \cos \omega t$$

So, $\sin \omega t$ can be expressed as a difference of exponential functions.

$e^{i\omega t} - e^{-i\omega t} = \ldots\ldots\ldots\ldots$

------------------▷ 12

$$y(t) = \frac{1}{6}\left(t + \frac{5}{\sqrt{6}} \sin \sqrt{6}t\right)$$ 85

Go on ------------------▷ 86

$e^{i\omega t} - e^{-i\omega t} = 2i \sin \omega t$ 12

This implies $\sin \omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

Since we know the transform of an exponential function, the problem is almost solved. A little bit of calculation is all that remains.

We know:

$\mathcal{L}\left[e^{at}\right] = \bar{y}(s) = \frac{1}{s-a}$

By inserting we obtain:

$\mathcal{L}[\sin \omega t] = y(s) = \frac{1}{2i}\left[\frac{1}{s-i\omega} - \frac{1}{s+i\omega}\right]$

Using the common denominator for the fractions we get:

$\vec{y}(s)$=…………………

------------------▷ 13

You should now be in a position to solve the following DE. 86

$y''-6y'+8y = 4$ Initial conditions: $t = 0$ $y_0 = 0$ $y'_0 = 0$

Step 1: Obtain the Laplace transform of the differential equation

………………………………=………………………….

------------------▷ 87

$$\mathcal{L}[\sin\omega t] = F(s) = \frac{1}{2i}\left[\frac{s+i\omega - s + i\omega}{s^2+\omega^2}\right]$$ 13

Or simply

$$\mathcal{L}[\sin\omega t] = \bar{y}(s) = \left[\frac{\omega}{s^2+\omega^2}\right]$$

The Laplace transform of the cosine function can be arrived at in quite a similar way.
Try to solve the problem on your own.

$\mathcal{L}[\cos\omega t] = \bar{y}(s) =$ ....

Solution found -------------------▷ 18

Help needed -------------------▷ 14

$$s^2\bar{y}(s) - sy_0 - y'_0 - 6s\bar{y}(s) + 6y_0 + 8\bar{y}(s) = \frac{4}{s}$$ 87

Please note: The constant 4 on the right side of the original DE needed to be transformed also.

Step 2: Insert the initial conditions $(t = 0, y_0 = 0, y'_0 = 0)$ and solve for $\bar{y}(s)$:

$\bar{y}(s) =$ ....

-------------------▷ 88

To be found: $\mathcal{L}[\cos\omega t]$. 14

Again, we start by expressing $\cos\omega t$ as a sum or a difference of exponential functions using Euler's Formula.

Recall:

$e^{i\omega t} =$ ……………………………………..

$e^{-i\omega t} =$ …………………………………….

Therefore, the cosine function can alternatively be expressed as

$\cos\omega t =$ ……………………………………

-------------------▷ 15

$0\ \bar{y}(s) = \dfrac{4}{s(s^2 - 6s + 8)}$ 88

Decompose into partial fractions in order to find expressions that can easily be inverse transformed.

$\bar{y}(s) =$ ....

Solution successfully found -------------------▷ 96

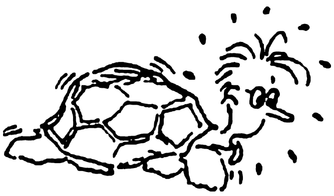
Help and explanation -------------------▷ 89

$$e^{i\omega t} = i\sin\omega t + \cos\omega t$$
$$e^{-i\omega t} = -i\sin\omega t + \cos\omega t$$
$$\cos\omega t = \frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

15

Now we can determine:

$\mathcal{L}[\cos\omega t] = \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



Solution found -------------------▷ 18

One more hint needed? -------------------▷ 16

We must rewrite

89

$$\bar{y}(s) = \frac{4}{s\left(s^2 - 6s + 8\right)}$$

The bracketed expression in the denominator can be expressed as a product:

$$\left(s^2 - 6s + 8\right) = (s-a)\cdot(s-b)$$

Hint: In this special case you can either guess the correct values or, more generally, you must solve for the roots of the quadratic expression.

$$\bar{y}(s) = \frac{4}{s(s\ldots\ldots\ldots\ldots)\cdot(s\ldots\ldots\ldots\ldots)}$$

-------------------▷ 90

The task is to determine the Laplace transform of $f(t) = \cos\omega t$. 16

$$\mathcal{L}[\cos\omega t] = \int_0^\infty e^{-st}\cos\omega t\, dt$$

We know already

$$\mathcal{L}[e^{at}] = \frac{1}{s-a}$$

We also know

$$\cos\omega t = \frac{1}{2}\left[e^{i\omega t} + \mathrm{e}^{-i\omega t}\right]$$

By combining both results we obtain

$$\mathcal{L}[\cos\omega t] = \mathcal{L}\left[\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

------------------▷ 17

$$\bar{y}(s) = \frac{4}{s(s-2)\cdot(s-4)}$$ 90

Now the fraction can be split into partial fractions. In a later step the inverse transform will be applied.

$$\frac{4}{s(s-2)\cdot(s-4)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-4)}$$

$$\frac{4}{s(s-2)\cdot(s-4)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ------------------▷ 95

Help ------------------▷ 91

$$\mathcal{L}[\cos \omega t] = \mathcal{L}\left[\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] = \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right]$$ 17

After rearranging and using the common denominator we finally end up with:

$\mathcal{L}[\cos \omega t] = \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

-------------------▷ 18

Our approach for the decomposition into partial fractions is: 91

$$\frac{4}{s(s-2)\cdot(s-4)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-4)}$$

We must determine *A*, *B*, and *C*.

By using the common denominator of the partial fractions we obtain

$$\frac{4}{s(s-2)\cdot(s-4)} = \frac{(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)}{s(s-2)\cdot(s-4)}$$

-------------------▷ 92

$$\mathcal{L}[\cos \omega t] = \bar{y}(s) = \frac{s}{s^2 + \omega^2}$$ 18

Compute the Laplace transform

$\bar{y}(s) = \mathcal{L}[5 \cdot \sin 4t] =$ ………………

------------------▷ 19

$$\frac{4}{s(s-2)\cdot(s-4)} = \frac{A\cdot(s-2)\cdot(s-4) + B\cdot s(s-4) + C\cdot s(s-2)}{s(s-2)\cdot(s-4)}$$ 92

Both numerators must be equal.
Expand the numerator on the right side and collect terms according to the different powers of s:

$4 =$ ....

------------------▷ 93

$\bar{y}(s) = \frac{5 \cdot 4}{s^2 + 16}$ 19

Compute the Laplace transform $\bar{y}(s)$ for $y(t) = 5 \cdot \cos 4t$

$\bar{y}(s)$ ....

------------------▷ 20

$4 = s^2[A + B + C] + s[-6A - 4B - 2C] + 8A$ 93

The equation can only be satisfied, if the right hand side is independent of both s and s^2. This amounts to both brackets being equal to 0.
From this we obtain the following equations to determine the unknown values of A, B, and C.

$4 = 8A$
$0 = A + B + C$
$0 = -6A - 4B - 2C$

Determine A, B, and C

$A =$ ....
$B =$ ....
$C =$ ....

Solution found ------------------▷ 95

One last hint ------------------▷ 94

$$\bar{y}(s) = \frac{5 \cdot s}{s^2 + 16}$$ 20

Given a linear function $y(t) = C \cdot t$

To be found the Laplace transform: $\mathcal{L}[C \cdot t] = \bar{y}(s)$

$\bar{y}(s) =$……………………

Solution found -------------------▷ 22

Hint needed -------------------▷ 21

The first equation implies $A = \frac{1}{2}$ 94

By inserting into the other equations

$$A + B + C = 0 \qquad \text{and} \qquad -6A - 4B - 2C = 0$$

we obtain

$$\frac{1}{2} + B + C = 0 \qquad \text{and} \qquad -3 - 4B - 2C = 0$$

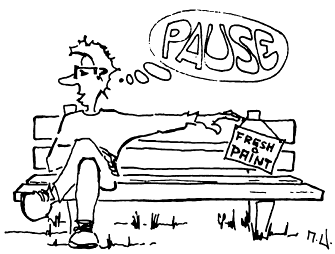
We rearrange and add

$$2B + 2C = -1$$
$$-4B - 2C = +3$$

This implies

$B =$ ....

$C =$ ....



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How to determine the Laplace transform of a linear function is described in detail in the textbook. Please recap section 11.3 and follow all computations meticulously. If necessary, also look up the technique of integrating by parts. 21

In parallel to the procedure shown in the textbook compute

$\mathcal{L}[C \cdot t] = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 22

$A = \frac{1}{2}$ $\qquad B = -1$ $\qquad C = \frac{1}{2}$ 95

From this we obtain the desired form of the Laplace transform

$\ldots \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 96

$\mathcal{L}[C \cdot t] = \bar{y} = \dfrac{C}{s^2}$ 22

Given $y(t) = 5 \cdot t$

Compute $\mathcal{L}[y(t)] = \bar{y}$ ………………………………….

------------------▷ 23

$\bar{y}(s) = \dfrac{1}{2s} - \dfrac{1}{(s-2)} + \dfrac{1}{2}\dfrac{1}{(s-4)}$ 96

Step 3 requires us to obtain the inverse transform using the given table.

$y(t) =$ ....

------------------▷ 97

$\bar{y}(s) = \frac{5}{s^2}$ 23

Horizontal shift: If a function is shifted to the right by a units, its transform is

$\mathcal{L}[y(t-a)] = e^{-as} \cdot \bar{y}(s) \quad \int_0^\infty y(t-a) \cdot e^{-s \cdot t} dt =$ ……………………………….

Hint: Substitute u for t-a and evaluate the integral with respect to u

------------------▷ 24

$y(t) = \frac{1}{2} + \frac{1}{2} e^{4t} - e^{2t}$ 97

Let us turn to one last exercise which will be well known to you from physics -- the equation of motion in the gravitational field near the earth's surface. As is customary in physics, we denote time by t and the derivative with respect to time by $\dot{y}$. The direction of y is up. Then the acceleration is directed down, i.e. negative.

$\ddot{y} = -g$

Laplace transform
……………………………….

Solve for $\mathcal{L}(s)$
$\mathcal{L}(s) =$ ....

Inverse transform
$y(t) =$ …………………………….

------------------▷ 98

$$\mathcal{L}[y(t-a)] = e^{-as} \cdot \bar{y}(s) \text{ because } \int_0^\infty y(t-a) \cdot e^{-s \cdot t} dt = \int_{-a}^\infty y(u) \cdot e^{-s \cdot (u+a)} du = e^{-as} \cdot \int_{-a}^\infty y(u) \cdot e^{-s \cdot u} du$$ 24

Let us now start with the linear function $y(t) = 5t$, which is represented by a straight line through the origin.

We already know its transform to be $\bar{y} = \frac{5}{s^2}$

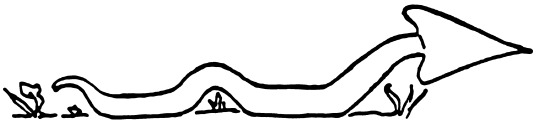
Let us now look at the slightly more general linear function $y(t) = 5t - 15$

As can easily be seen, it is a straight line which is shifted to the right by 3 units, since

$y(t) = 5(t-3) = 5t - 15$

Compute the transform of this function using the result on horizontal shifting:

$\mathcal{L}[5(t-3)] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



------------------▷ 25

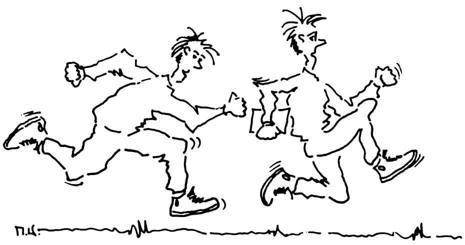
98

$$s^2 \cdot \mathcal{L}(s) - sy_0 - \dot{y}_0 = \frac{-g}{s}$$

$$\mathcal{L}(s) = \frac{-g}{s^3} + \frac{\dot{y}_0}{s^2} + \frac{y_0}{s}$$

$$y(t) = \frac{-g}{2} \cdot t^2 + \dot{y}_0 \cdot t + y_0$$

This is the famous and well-known equation for a freely falling object.



------------------▷ 99

$\mathcal{L}[5(t-3)] = e^{-3s} \cdot \frac{5}{s^2}$ 25

Using the same argument we are in a position to transform any linear function.

Given $y(t) = a \cdot t - b$

Step 1: Rearrange the formula, so that you can apply the Laplace transform

$y(t) =$ ………………………………

------------------▷ 26

11.4 Solving simultaneous differential equations with constant coefficients 99

Read **11.5 Solving simultaneous differential equations with constant coefficients**
Textbook pages 330–331

------------------▷ 100

Step 1: The straight line is shifted to the right by $\frac{b}{a}$ units. 26

Step 2: The Laplace transform is $\bar{y}(s) = \frac{a}{s^2} \cdot e^{-\frac{b}{a} \cdot f}$.

Example: Given the straight line $y(t) = 5t - 50$

Rewrite the function so that the Laplace transform can be obtained

$y(t) =$ …………………

Now compute the Laplace transform

$\bar{y} =$ ………………….

------------------▹ 27

I have understood the principles of finding solutions and 100

have followed all examples in the textbook ------------------▹ 138

Explanations for example 1 in the textbook ------------------▹ 101

Explanations for example 2 in the textbook ------------------▹ 121

27

$y(t) = 5t - 50 = 5(t - 10)$

$\dot{y}(s) = \frac{5}{s^2} \cdot e^{-10s}$

Look up the shift theorem in the textbook.

Given a function $y(t)$ and its transform $\bar{y}(s)$

We then know $\bar{y}(s + a) =$ ………………….

------------------▷ 28

101

The first example in the textbook was the following system of equations:

$3\dot{x} + 2x + \dot{y} = 1$ for $t = 0 : x_0 = y_0 = 0$

$\dot{x} + 4\dot{y} + 3y = 0$

Step 1: Apply the Laplace transform to both equations, observe the initial conditions, and simplify all expressions. Consult the textbook, if necessary.

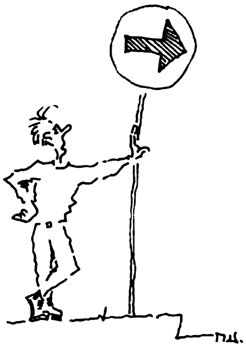
……………………………………………………………………………………………………

……………………………………………………………………………………………………

------------------▷ 102

$\bar{y}(s+a) = \mathcal{L}\left[e^{-at} \cdot y(t)\right]$ 28

Proof from the textbook understood -------------------▷ 30



Additional explanations for the proof -------------------▷ 29

$$(3s+2)\ \mathcal{L}\ [x] + s\ \mathcal{L}\ [y] = \frac{1}{s}$$
$$+s \cdot \ \mathcal{L}\ [x] + (4s+3)\ \mathcal{L}\ [y] = 0$$
102

Step 2: Now the system of equations must be solved for $\mathcal{L}\ [x]$ and $\mathcal{L}\ [y]$, consecutively. We start with $\mathcal{L}\ [x]$ and eliminate $\mathcal{L}\ [y]$. For this purpose let us multiply the first equation by $(4s+3)$ and the second equation by $(-s)$. That leads to

$$(3s+2)\mathcal{L}[x] \cdot (4s+3) + s(4s+3)\mathcal{L}[y] = \frac{4s+3}{s}$$
$$-s^2\,\mathcal{L}[x] \qquad -s(4s+3)\mathcal{L}[y] = 0$$

Add the equations to obtain

…………………………..=………………………..

-------------------▷ 103

29

To be shown: $y(s+a) = \mathcal{L}\left[e^{-at} \cdot y(t)\right]$

We must determine the Laplace transform of the function
$e^{-at} \cdot y(t)$

Assuming that the transform of $y(t)$ is known to be $\bar{y}(s)$,
we recollect the definition of the Laplace transform

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \int_0^\infty e^{-at} \cdot e^{-st} \cdot y(t)dt$$

The integrand can be simplified as follows

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \int_0^\infty e^{-(s+a)t} \cdot y(t)dt$$

The integral evaluates to $\bar{y}(s+a)$, which is assumed to be known.

Thus, it is proved that $\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \bar{y}(s+a)$

------------------▷ 30

103

$$\mathcal{L}[x] \cdot \left[(3s+2) \cdot (4s+3) - s^2\right] = \frac{4s+3}{s}$$

Now we must rearrange in order to obtain expressions that are amenable to the inverse transforms that are shown on page 327 of the textbook. Let us start with the expression in brackets on the left and expand the product:

$$\mathcal{L}[x] \cdot \left(12s^2 + 9s + 8s + 6 - s^2\right) = \mathcal{L}[x \cdot] \cdot \left(11s^2 + 17s + 6\right)$$

In order to obtain an expression of the type $A(s+a) \cdot (s+b)$, we factor the constant 11 and solve the quadratic equation. Thus we obtain

$$\mathcal{L}[x] \cdot 11 \cdot (\ldots\ldots\ldots\ldots\ldots\ldots) \cdot (\ldots\ldots\ldots\ldots\ldots\ldots) = \frac{4s+3}{s}$$

------------------▷ 105

Further help

------------------▷ 104

Given $y(t) = 3 \cdot \sin 2t \cdot e^{-4t}$ **30**
Find the transform $\bar{y}(s)$

$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 33

Help needed ------------------▷ 31

The task is to solve $\mathcal{L}[x] \cdot (11s^2 + 17s + 6) = \frac{4s+3}{s}$ **104**

By factoring 11 we obtain

$$\mathcal{L}\ [x] \cdot 11 \cdot \left(s^2 + \frac{17}{11}s + \frac{6}{11}\right) = \frac{4s+3}{s}$$

Let us now solve the quadratic equation

$$\left(s^2 + \frac{17}{11}s + \frac{6}{11}\right) = 0$$

$$s_1 = -\frac{17}{2 \cdot 11} + \sqrt{\frac{17^2}{(2 \cdot 11)^2} - \frac{6}{11}} = -\frac{17}{2 \cdot 11} + \sqrt{\frac{289 - 264}{(2 \cdot 11)^2}} = -\frac{17}{2 \cdot 11} + \sqrt{\frac{25}{(2 \cdot 11)^2}}$$

$$= -\frac{17}{2 \cdot 11} + \frac{5}{2 \cdot 11} = -\frac{12}{2 \cdot 11} = -\frac{6}{11}$$

$$s_2 = -\frac{17}{2 \cdot 11} - \frac{5}{2 \cdot 11} = -\frac{22}{2 \cdot 11} = -1$$

Thus we can express

$$\mathcal{L}[x] \cdot 11 \cdot \left(s^2 + \frac{17}{11}s + \frac{6}{11}\right) = \mathcal{L}[x] \cdot 11 \cdot (\ldots\ldots\ldots\ldots\ldots) \cdot (\ldots\ldots\ldots\ldots\ldots) = \frac{4s+3}{s}$$

------------------▷ 105

31

Given $y(t) = 3\sin 2t \cdot e^{-4t}$
Because of the damping factor e^{-4t} we can use the shift theorem.

We start by computing the transform of

$\mathcal{L}[3\sin 2t] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: This result was already obtained. Possibly you may wish to consult your notes or the textbook.
..

$\mathcal{L}[3\sin 2t] = \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$..

------------------▷ 32

105

$$\mathcal{L}[x]\cdot 11\cdot(s+1)\cdot\left(s+\frac{6}{11}\right) = \mathcal{L}[x]\cdot(s+1)\cdot(11s+6) = \frac{4s+3}{s}$$

Solving for $\mathcal{L}[x]$ this yields:

$$\mathcal{L}[x] = \frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{1}{11}\cdot\frac{(4s+3)}{s(s+1)\cdot\left(s+\frac{6}{11}\right)}$$

The fraction can be split into three partial fractions, whose denominators are given by the three linear factors. After performing a slightly tedious calculation, which is prone to errors by miscalculation, we obtain:
$\mathcal{L}[x] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$..

Calculation happily done ------------------▷ 112

Help and explanation ------------------▷ 106

Please note: The calculation could also be performed for the equivalent expression

$$\mathcal{L}[x] = \frac{(4s+3)}{11\cdot\left(s(s+1)\cdot\left(s+\frac{6}{11}\right)\right)}$$

The result, of course, would be unchanged.

$$\mathcal{L}[3\sin 2t] = \bar{y}(s) = \frac{3 \cdot 2}{s^2 + 4}$$ 32

However, we must determine $\mathcal{L}[\bar{y}(t)] = \mathcal{L}\left[3\sin 2t \cdot e^{-4t}\right]$

We now use the shift theorem

$$\mathcal{L}\left[e^{-at} \cdot 3\sin 2t\right] = \bar{y}(s+a)$$

$$\mathcal{L}\left[e^{-4t} \cdot 3\sin 2t\right] = \text{..............................}$$

------------------▷ 33

The following fraction must be decomposed: $\dfrac{4s+3}{s(s+1)\cdot(11s+6)}$ 106

We expect to decompose into partial fractions with the three linear factors as the denominators:

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$$

Now let us add the partial fractions using the common denominator, in order to gain equations for determining A, B, and C.

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{\text{.........................}}{s(s+1)\cdot(11s+6)}$$

------------------▷ 107

$$\mathcal{L}\left[e^{-4t}\cdot 3\sin 2t\right]=\frac{3\cdot 2}{(s+4)^2+4}$$ 33

Next task:

Given: $\bar{y}(t)=e^{-2t}\cdot\cos\pi t$

Wanted: $\mathcal{L}\left[e^{-2t}\cdot\cos\pi t\right]=$ ....

Solution found -------------------▷ 35

Help needed -------------------▷ 34

$$\frac{4s+3}{s(s+1)\cdot(11s+6)}=\frac{A\cdot(s+1)\cdot(11s+6)+B\cdot s(11s+6)+C\cdot s(s+1)}{s(s+1)\cdot(11s+6)}$$ 107

We multiply the expressions in the numerator and rearrange according to the powers of *s:*

$$\frac{4s+3}{s(s+1)\cdot(11+6)}=\frac{s^2(\ldots\ldots\ldots\ldots\ldots\ldots)+s\cdot(\ldots\ldots\ldots\ldots\ldots\ldots)+\ldots\ldots\ldots}{s(s+1)\cdot(11s+6)}$$

-------------------▷ 108

We use the same procedure as in the preceding exercise.
First, determine $\mathcal{L}[\cos \pi t]$ and, second, apply the shift theorem. 34

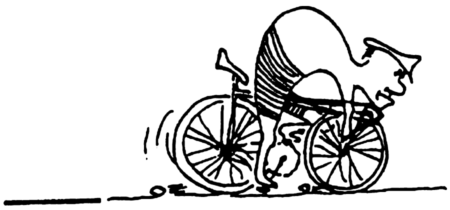
$$\mathcal{L}[\cos \pi t] = \frac{s}{s^2 + \pi^2}$$

The shift theorem tells us

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \bar{y}(s+a)$$

Thus

$$\mathcal{L}\left[e^{-2t} \cdot \cos \pi t\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



------------------▷ 35

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{s^2(11A+11B+C)+s(17A+6B+C)+6A}{s(s+1)\cdot(11s+6)}$$ 108

Comparing coefficients we obtain three equations for A, B, and C.

constant terms: $3 = 6A$
for s: $4 = 17A + 6B + C$
for s^2: $0 = 11A + 11B + C$

From this we easily determine

$A = \ldots\ldots\ldots\ldots$

$B = \ldots\ldots\ldots\ldots$

$C = \ldots\ldots\ldots\ldots$

Solution ------------------▷ 111

Explicit calculation ------------------▷ 109

$$\mathcal{L}\left[e^{-2t} \cdot \cos \pi t\right] = \frac{s+2}{(s+2)^2 + \pi^2}$$ 35

Understanding and applying the linearity theorem should be quite obvious.
Obtain the Laplace transform of

$y(t) = 2\cos 2\pi t + \sin 2\pi t$:

$\mathcal{L}[2\cos 2\pi t + \sin 2\pi t] =$ ....

Hint: All transforms have already been introduced. If necessary consult your notes or the textbook.

-------------------▷ 36

We obtained three equations containing three variables 109

1) $3 = 6A$

2) $4 = 17A + 6B + C$

3) $0 = 11A + 11B + C$

Equation 1): $A = \frac{1}{2}$

Equation 2) minus Equation 3): $4 = -\frac{11}{2} - 11B - C + \frac{17}{2} + 6B + C$

$4 = 3 - 5B$

Hence $B =$ ....

Insert B into equation 3): $0 = \frac{11}{2} - \frac{11}{5} + C$

$C = \frac{22-55}{10}$

$C =$ ....

-------------------▷ 110

$$\mathcal{L}[2\cos 2\pi t + \sin 2\pi t] = \frac{2\cdot s}{s^2 + (2\pi)^2} + \frac{2\pi}{s^2 + (2\pi)^2} = \frac{2s + 2\pi}{s^2 + (2\pi)^2}$$ 36

------------------▷ 37

$A = \frac{1}{2}$ $\quad B = -\frac{1}{5}$ $\quad C = -\frac{33}{10}$ 110

Thus, the decomposition into partial fractions is complete:

$$\mathcal{L}[x] = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$$

$\mathcal{L}[x] =$ ……………………….

------------------▷ 111

11.2 Laplace transform of derivatives

37

Objective: Obtain a good command of how to use two very useful theorems and, as a by-product, transform old friends like t^n.
This is the last section preparing you for highly useful applications of the Laplace transforms. Once again, we advise you to take notes of all results, rules, and theorems; this will aid your memory.

Read the second part of Section 11.3

Theorem II: Transforms of products *t y(t)*
Theorem III: Linearity
Theorem IV: Transforms of derivatives
Textbook pages 324–327

When done

------------------▷ 38

$$\mathcal{L}[x] = \frac{1}{2s} - \frac{1}{5(s+1)} - \frac{33}{10}\cdot\frac{1}{(11s+6)}$$, or, equivalently $$\mathcal{L}[x] = \frac{1}{2s} - \frac{1}{5}\cdot\frac{1}{(s+1)} - \frac{3}{10}\cdot\frac{1}{\left(s+\frac{6}{11}\right)}$$

111

Step 3:

Now we are in a comfortable position to do the inverse transform using the table on page 333 of the textbook. We obtain

$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 112

38

In this chapter we use the following abbreviations for denoting the values of the original function $y(t)$ or $f(t)$ and its derivatives at $t=0$:

$y(0)=$ ………………

$y'(0)=$ ……………...

$y''(0)=$ ……………..

Equivalently we use the notation

$f(0)=$ ……………..

$f'(0)=$ …………….

$f''(0)=$ ……………

-------------------▷ 39

112

$x=\frac{1}{2}-\frac{1}{5}\cdot e^{-t}-\frac{3}{10}\cdot e^{-\frac{6}{11}t}$

Now we must find y. In the given system of equations we eliminate $\mathcal{L}[x]$.
We have

$$\begin{aligned}(3s+2)\ \mathcal{L}[x] \quad &+ s\ \mathcal{L}[y]=\frac{1}{s}\\ +\ s\mathcal{L}[x] \quad &+(4s+3)\ \mathcal{L}[y]=0\end{aligned}$$

A convenient way to eliminate $\mathcal{L}[x]$ is to multiply the first equation by $-s$ and the second one by $(3s+2)$, and then add the resulting equations.
That yields: ……………………..=…………………….

and ……………………..=…………………….

-------------------▷ 113

$$y(0) = y_0 \qquad f(0) = f_0$$

$$y'(0) = y'_0 \qquad f'(0) = f'_0$$

$$y''(0) = y''_0 \qquad f''(0) = f''_0$$

39

The advantage of this shorthand notation will become apparent when we turn to solving differential equations.

Given: $y(t) = t \cdot e^{-at}$

To be found: $\mathcal{L}[y(t)] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found -------------------▷ 43

Help needed -------------------▷ 40

$$-s(3s+2)\cdot \mathcal{L}[x] \qquad\qquad -s^2 \cdot \mathcal{L}\ [y] = -1$$
$$s(3s+2)\cdot \mathcal{L}[x] + (4s+3)\cdot(3s+2)\cdot \mathcal{L}[y] = 0$$

113

We now add and obtain

$$\mathcal{L}[y]\cdot\left[(4s+3)\cdot(3s+2) - s^2\right] = -1$$

Multiplying and collecting powers this leads to

$$\mathcal{L}[y]\cdot\left(11s^2 + 17s + 6\right) = -1$$

The quadratic expression has already been factored in step 104 with the roots
$s_1 = -\frac{6}{11}, \quad s_2 = -1$

Therefore $\mathcal{L}[y] = \dfrac{-1}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$

-------------------▷ 114

In the textbook you find that the Laplace transform of a function $t \cdot y(t)$ is given by a derivative: **40**

$\mathcal{L}[t \cdot y(t)] = -\frac{d}{ds}[\bar{y}(s)]$ if we denote $\mathcal{L}[y(t)]$ by $\bar{y}(s)$

In our case we already know the Laplace transform $\bar{y}(s)$ for $y(t) = e^{-at} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

If necessary, consult your notes or the textbook pages 332 and 334.

-------------------▷ 41

$$\mathcal{L}[y] = \frac{-1}{(s+1)\cdot(11s+6)}$$ **114**

We decompose the fraction into two partial fractions and obtain

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found -------------------▷ 118

Help and detailed computation -------------------▷ 115

The Laplace transform of $y(t) = e^{-at}$ is given by $\bar{y}(s) = \frac{1}{s+a}$ **41**

In order to determine the Laplace transform of $t \cdot y(t)$ the theorem tells us to find the derivative of $\bar{y}(s)$ with respect to s, and change the sign:

$$\frac{d}{ds}[\bar{y}(s)] = -\mathcal{L}[t \cdot y(t)]$$

In our case $\bar{y}(s) = \frac{1}{s+a}$

Thus, we get $\frac{d}{ds}[\bar{y}(s)] = \ldots\ldots\ldots\ldots\ldots\ldots..$

-------------------▷ 42

Hint (redundant, we *do* hope): $\frac{1}{s+a} = (s+a)^{-1}$

To decompose into partial fractions: $\frac{-1}{(s+1)\cdot(11s+6)}$ **115**

Assumption:

$$\frac{-1}{(s+1)\cdot(11s+6)} = \frac{A}{(s+1)} + \frac{B}{(11s+6)}$$

We use the common denominator

$$\frac{-1}{(s+1)\cdot(11s+6)} = \frac{A(11s+6)+B(s+1)}{(s+1)\cdot(11s+6)}$$

After multiplying and ordering according to powers of s we obtain

$$\frac{-1}{(s+1)\cdot(11s+6)} = \frac{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}{(s+1)\cdot(11s+6)}$$

-------------------▷ 116

$$\frac{d}{ds}[\bar{y}(s)] = \frac{-1}{(s+a)^2}$$

42

So we now know $\frac{d}{ds}[\bar{y}(s)] = -\frac{1}{(s+a)^2} = -\mathcal{L}\left[t \cdot e^{-at}\right]$

Thus, we have identified the Laplace transform for $t \cdot e^{-at}$:

$\mathcal{L}\left[t \cdot e^{-at}\right] =$.............................

------------------▷ 43

$$\frac{-1}{(s+1)\cdot(11s+6)} = \frac{s[11A+B]+6A+B}{(s+1)\cdot(11s+6)}$$

116

This leads us to the equations for determining A and B

$-1 = 6A + B$
$0 = 11A + B$

From this we obtain

$A =$............................ $B =$............................

------------------▷ 117

$\mathcal{L}\left[t \cdot e^{-at}\right] = \frac{1}{(s-a)^2}$ **43**

Let us turn to another exercise. This time we use the notation $f(t)$ and for the Laplace transform $F(s)$

Given: $f(t) = t^3$
To be found: $F(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 44

$A = \frac{1}{5}$ $B = -\frac{11}{5}$ **117**

Thus we obtain

$\mathcal{L}\,[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 118

$f(t) = t^3 \qquad F(s) = \frac{3!}{s^4} = \frac{6}{s^4}$ **44**

Easy, wasn't it? $\mathcal{L}[t \cdot t^2] = -\frac{d}{ds}\mathrm{L}[t^2] = \frac{d^2}{(ds)^2}\mathrm{L}[t] = \frac{d^2}{(ds)^2}\frac{1}{s^2}$

Next task

-------------------▷ 45

$\mathcal{L}[y] = \frac{1}{5(s+1)} - \frac{11}{5}\frac{1}{(11s+6)}$, or, equivalently $\mathcal{L}[y] = \frac{1}{5(s+1)} - \frac{1}{5}\frac{1}{\left(s + \frac{6}{11}\right)}$ **118**

Step 3: Inverse transform

$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$

-------------------▷ 119

Now this exercise will be slightly more demanding: 45

Given: $f(t) = 3\sin^2(2t)$

Wanted: $F(s) = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution of the somewhat tricky task found ------------------▷ 51

Help needed ------------------▷ 46

$$y(t) = \frac{1}{5}\cdot e^{-t} - \frac{1}{5}\cdot e^{-\frac{6}{11}t} = \frac{1}{5}\left(e^{-t} - e^{-\frac{6}{11}t}\right)$$ 119

Most difficulties in algebraic operations arise by oversight or slips of the pen which afterwards can hardly be discovered.

------------------▷ 120

46

The transform of $\sin^2(2t)$ can neither be found in the table nor, as yet, in our notes.

Therefore, let us try to rewrite the given expression $\sin^2(2t)$ so that we can find the Laplace transform with our given arsenal. We will use Pythagoras' theorem and the addition formulae for trigonometric functions so that the squared expression is eliminated.

$\sin^2(2t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution of this intermediate step found ------------------▷ 50

Further hints needed ------------------▷ 47

120

No difficulties with example 2 from the textbook ------------------▷ 138

Show me a detailed walk through example 2 from the textbook ------------------▷ 121

47

Recall Pythagoras' theorem $\cos^2 t + \sin^2 t = 1$ and
the addition formula $\cos(u+v) = \cos u \cdot \cos v - \sin u \cdot \sin v$

Now, in a seemingly round about way, let us first compute

$\cos(t+t) = \cos 2t =$……………………..

------------------▷ 48

121

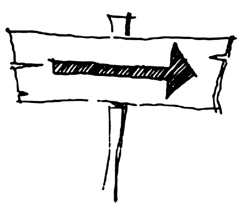
Example 2: The following system of equations is to be solved

$\ddot{x} + 2x - \dot{y} = 1$ for $t = 0$: $x_0 = 1$, $\dot{x}_0 = y_0 = \dot{y}_0 = 0$
$\dot{x} + \ddot{y} + 2y = 0$

Step 1: Applying the Laplace transform and inserting the initial conditions results in:

…………………………………=………………………………

…………………………………=……………………………….



------------------▷ 122

$\cos 2t = \cos^2 t - \sin^2 t$ 48

Because of $\cos^2 t = 1 - \sin^2 t$, we can eliminate $\cos^2 t$ from the expression above.

$\cos 2t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 49

$$(s^2+2)\ \mathcal{L}[x] \qquad -5\ \mathcal{L}[y] = \frac{1}{s} + s$$

$$s\,\mathcal{L}[x] + (s^2+2)\ \mathcal{L}[y] = x_0 = 1$$

122

Step 2: Solve for $\mathcal{L}[x]$ by eliminating $\mathcal{L}[y]$.
For this we multiply the first equation by (s^2+2) and the second by s. Then by adding $\mathcal{L}\ [y]$ is eliminated.

We obtain

$\mathcal{L}[x] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

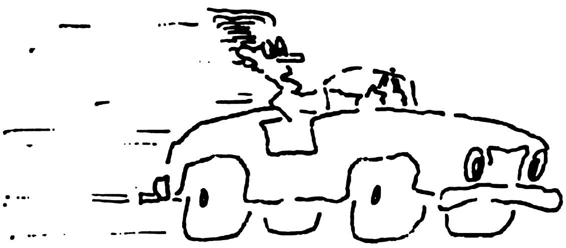
------------------▷ 123

$\cos 2t = 1 - 2\sin^2 t$

49

So we are in a position to rewrite $\sin^2 t$ as follows

$\sin^2 t =$…………………….…



------------------▷ 50

$\mathcal{L}[x] \cdot \left[(s^2+2)^2 + s^2 \right] = \frac{s^2+2}{s} + s(s^2+2) + s$

123

We rearrange the right hand side and use the common denominator s. On the left side we expand the square:

$\mathcal{L}[x] \cdot [\ldots\ldots\ldots\ldots\ldots] = \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{s}$

------------------▷ 124

$\sin^2 t = \frac{1}{2}[1-\cos 2t]$ 50

The original task was to obtain the Laplace transform of the function $f(t) = 3\sin^2(2t)$.
Using the above result we get:

$f(t) = 3 \cdot \frac{1}{2}(1-\cos 4t)$

This expression is now accessible to transformation: $\mathcal{L}\left[\frac{3}{2}(1-\cos 4t)\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 51

$\mathcal{L}[x] \cdot \left[s^4 + 5s^2 + 4\right] = \dfrac{s^4 + 4s^2 + 2}{s}$ 124

This leads to

$\mathcal{L}[x] = \dfrac{s^4 + 4s^2 + 2}{s\left(s^4 + 5s^2 + 4\right)}$

We wish to decompose into partial fractions, which will yield expressions that can then be inversely transformed. For this we must expand the bracketed expression into a product of the type $\left(s^2 + a\right) \cdot \left(s^2 + b\right)$.

For finding *a* and *b* we solve the equation $s^4 + 5s^2 + 4$ for s^2. With regard to s^2 it is just a quadratic equation. The zeroes are not difficult to find:

$s_1^{\,2} = \ldots\ldots\ldots$

$s_2^{\,2} = \ldots\ldots\ldots$

Hence

$s^4 + 5s^2 + 4 = \left(s^2 - \ldots\ldots\ldots\ldots\right) \cdot \left(s^2 - \ldots\ldots\ldots\right)$

Solution found ------------------▷ 126

Show me the solution of the quadratic equation ------------------▷ 125

51

$$\mathcal{L}\left[\frac{3}{2}(1-\cos 4t)\right] = \frac{3}{2}\cdot\frac{1}{s} - \frac{3}{2}\cdot\frac{s}{s^2+4^2}$$

$$= \frac{3}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right] = \frac{3}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$$

$$= \frac{3\cdot 8}{s(s^2+16)} = \frac{24}{s(s^2+16)}$$

------------------▷ 52

125

To be solved for s^2: $s^4 + 5s^2 + 4 = 0$

We solve the quadratic equation:

$$s^2 = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} \quad = -\frac{5}{2} \pm \sqrt{\frac{9}{4}} = -\frac{5}{2} \pm \frac{3}{2}$$

$s_1{}^2 = \ldots\ldots\ldots\ldots$

$s_2{}^2 = \ldots\ldots\ldots\ldots$

$s^4 + 5s^2 + 4 = (s^2 + \ldots\ldots\ldots\ldots)\cdot(s^2 + \ldots\ldots\ldots\ldots)$

------------------▷ 126

Let us do two more exercises using the table of inverse transformations. 52

Given the Laplace transform $F(s)=\frac{4}{s(s^2+4)}$

Find the inverse transform $f(t)=$ ……………………

Solution found -------------------▷ 54

Help needed -------------------▷ 53

$s_1{}^2=-1 \qquad s_2{}^2=-4$ 126

$s^4+5s^2+4=(s^2+1)\cdot(s^2+4)$

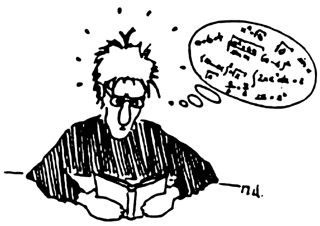
Now the Laplace transform can be expressed as $\mathcal{L}[x]=\frac{s^4+4s^2+2}{s(s^2+1)\cdot(s^2+4)}$

Next, we decompose into partial fractions:

$$\frac{s^4+4s^2+2}{s(s^2+1)\cdot(s^2+4)}=\frac{A}{s}+\frac{B_1+B_2s}{(s^2+1)}+\frac{C_1+C_2s}{(s^2+4)}$$

Express the right side as a single fraction with the original common denominator

$$\frac{s^4+4s^2+2}{s(s^2+1)\cdot(s^2+4)}=\frac{\dots\dots\dots\dots\dots\dots}{s(s^2+1)\cdot(s^2+4)}$$



-------------------▷ 127

We need to find the inverse transform of: $F(s)=\frac{4}{s(s^2+4)}$ 53

In the table you find that

given $F(s)=\frac{1}{s(s^2+\omega^2)}$ the inverse transform is $f(t)=\frac{1}{\omega^2}(1-\cos\omega t)$

Constant factors stay unchanged. If you identify $\omega^2=4$ and $\omega=2$, then you obtain:

$f(t)=$ ....

-------------------▷ 54

$$\frac{s^4+4s^2+2}{s(s^2+1)\cdot(s^2+4)}=\frac{A(s^4+5s^2+4)+B_1\cdot s(s^2+4)+B_2 s^2(s^2+4)+C_1 s(s^2+1)+C_2 s^2(s^2+1)}{s(s^2+1)\cdot(s^2+4)}$$ 127

Clear the fractions. On the right side expand all products and order them according to powers of s.

$s^4+4s^2+2=$ ....

-------------------▷ 128

$f(t) = \frac{4}{4}(1 - \cos 2t) = 1 - \cos 2t$ **54**

One last task.

Given $F(s) = \frac{2}{(s^2 - 4s + 3)}$

Then the inverse transform is $f(t)$=……………………..

Solution found -------------------▷ 57

Help -------------------▷ 55

$s^4 + 4s^2 + 2 = s^4(A + B_2 + C_2) + s^3(B_1 + C_1) + s^2(5A + 4B_2 + C_2) + s(4B_1 + C_1) + 4A$ **128**

Since the coefficients of like powers of s must be equal, we obtain five equations for determining A, B_1, B_2, C_1, and C_2.

Solving for each unknown, consecutively, we get:

A =…….. B_1 =…… B_2 =…… C_1 =….... C_2 =…….

Solution found -------------------▷ 130

Help and explicit calculation -------------------▷ 129

We must transform

$$F(s) = \frac{2}{\left(s^2 - 4s + 3\right)}$$

55

In the table you can find

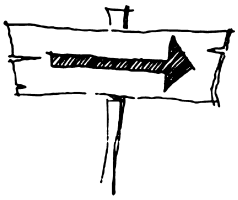
$F(s) = \frac{1}{(s-a)\cdot(s-b)}$ which leads to the inverse transform $f(t) = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$

In order to factor the denominator let us find the roots of the quadratic expression:

$s_1 = ...$ $s_2 = ...$

Thus, the denominator can be written as a product:

$s^2 - 4s + 3 = (...........................)\cdot(................................)$



------------------▷ 56

Compare coefficients for like powers of s

129

$$s^4 + 4s^2 + 2 = s^4\left(A + B_2 + C_2\right) + s^3\left(B_1 + C_1\right) + s^2\left(5A + 4B_2 + C_1\right) + s\left(4B_1 + C_1\right) + 4A$$

Coefficients of s^4: $1 = \left(A + B_2 + C_2\right)$
Coefficients of s^3: $0 = B_1 + C_1$
Coefficients of s^2: $4 = \left(5A + 4B_2 + C\right)$
Coefficients of s^1: $0 = 4B_1 + C_1$
Coefficients of s^0: $2 = 4A$

Solve this set of linear equations, starting with A from the last equation.

$A = ...$ $B_1 = ...$ $B_2 = ...$ $C_1 = ...$ $C_2 = ...$

------------------▷ 130

$s_1 = 3 \qquad s_2 = 1$ 56

So we rewrite the quadratic expression as a product of linear factors:

$s^2 - 4s + 3 = (s-3)\cdot(s-1)$

Given

$F(s) = \frac{1}{(s-a)\cdot(s-b)}$ the inverse transform is $f(t) = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$

Given

$F(s) = \frac{2}{(s-3)\cdot(s-1)}$ the inverse transform is $f(t)=$……………..……………….

------------------▷ 57

$A = \frac{1}{2} \qquad B_1 = 0 \qquad B_2 = \frac{1}{3} \qquad C_1 = 0 \qquad C_2 = \frac{1}{6}$ 130

Thus $\mathcal{L}[x] = \frac{s^4 + 4s^2 + 2}{s(s^4 + 5s^2 + 4)}$ can be written as a sum of three fractions

$\mathcal{L}[x] =$………+……………+……………….

------------------▷ 131

$f(t) = \frac{2}{2}\left[e^{3t} - e^{t}\right]$ 57

More exercises and their solutions can be found in the textbook.

Please keep in mind: Easy exercises are fun. But the important thing is to tackle the hard ones.

------------------▷ 58

$\mathcal{L}[x] = \frac{1}{2s} + \frac{1}{3}\frac{s}{(s^2+1)} + \frac{1}{6}\frac{s}{(s^2+4)}$ 131

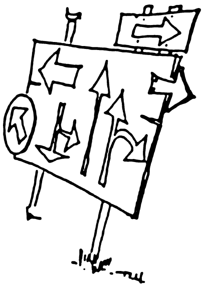
Step 3: In order to determine the inverse transform we use the table in the textbook on page 332 and 333 and obtain
$x(t) =$ …………………………..

------------------▷ 132

58

In the next section we will finally encounter applications. Certain algebraic transformations of fractions, i.e. the decomposition into partial fractions, will prove especially useful.

This technique was already dealt with in the textbook, in a previous section on integration by partial fractions.
If you do not feel familiar enough with it, go back to section 6.5.7 in the textbook pages 170–174 and also to frames 117–138 in the accompanying chapter in the study guide.



After studying the textbook go to ------------------▷ 61

Partial fractions well known ------------------▷ 59

132

$x(t) = \frac{1}{2} + \frac{1}{3}\cos t + \frac{1}{6}\cos 2t$

So the first part of example 2 is done. It still remains to determine $\mathcal{L}[y]$ and $y(t)$.
Recall that the transformed system of equations was:

$$\left(s^2+2\right)\ \mathcal{L}[x] - s\ \mathcal{L}[y] = \frac{1}{s} + s$$
$$s\ \mathcal{L}[x] + \left(s^2+2\right)\ \mathcal{L}[y] = 1$$

In order to eliminate $\mathcal{L}[x]$, multiply the first equation by $(-s)$ and the second by $\left(s^2+2\right)$ and add them. That leads to

$\mathcal{L}[y] =$ ………………………..

------------------▷ 133

Just to make sure, split the following expressions into partial fractions: 59
1.) Roots real and unequal:

$$f_1(x) = \frac{5x+11}{x^2+6x+8}$$

$f_1(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Roots real and repeated:

$$f_2(x) = \frac{1}{x-3x^2+4} = \frac{1}{(x+1)\cdot(x-2)^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Roots real and complex

$$f_3(x) = \frac{2x^2-13x+20}{x(x^2-4x+5)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 60

$$\mathcal{L}[y]\cdot\left((s^2+2)^2+s^2\right)=1$$ 133

We expand the square and solve for $\mathcal{L}[y]$:

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 134

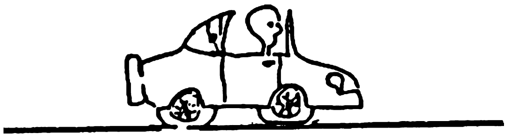
$$f_1(x)=\frac{5x+11}{x^2+6x+8}=\frac{5x+11}{(x+2)\cdot(x+4)}=\frac{1}{2(x+2)}+\frac{9}{2(x+4)}$$ 60

$$f_2(x)=\frac{1}{9(x+1)}+\frac{1}{3(x-2)}-\frac{1}{9(x-2)^2}$$

$$f_3(x)=\frac{4}{x}+\frac{-2x+3}{x^2+4x+5}$$

…………………………………………………………………………

If you succeeded, you know the basic rules for finding partial fractions and you may proceed to the next section.



------------------▷ 61

If, however, you encountered difficulties, it makes sense to return to the textbook and study the relevant pages 170–174, work through the relevant frames in the study guide and do some examples. You will be happier knowing the basics well when entering subsequent sections.

Afterwards

------------------▷ 61

$$\mathcal{L}[y]=\frac{1}{\left(s^4+5s^2+4\right)}$$ 134

The bracketed expression has already been factored in the preceding frames 125–127 resulting in $\left(s^2+1\right)\cdot\left(s^2+4\right)$. Next on the agenda is the decomposition into partial fractions:

$$\frac{1}{\left(s^2+1\right)\cdot\left(s^2+4\right)}=\frac{A}{s^2+1}+\frac{B}{s^2+4}$$

$$\frac{1}{\left(s^2+1\right)\cdot\left(s^2+4\right)}=\frac{A\left(s^2+4\right)+B\left(s^2+1\right)}{\left(s^2+1\right)\cdot\left(s^2+4\right)}=\frac{s^2(A+B)+4A+B}{\left(s^2+1\right)\cdot\left(s^2+4\right)}$$

From this we deduce in the usual way

$$A+B=0$$
$$4A+B=1$$

$A=$…………… $B=$……………

------------------▷ 135

11.3 Solution of linear differential equations with constant coefficients

61

After having successfully gone through all preliminary steps, which might have been somewhat depletive in places, you now encounter applications and you will reap the benefits. Follow all the examples arduously and if necessary, consult the table on page 275.
You will learn to transform a set of linear differential equations into a set of algebraic equations and after solving these to use the inverse transform, thus obtaining a solution of the original DEs.
.

READ **11.4 Solution of linear differential equations with constant coefficients**
Textbook pages 328–329

When done

-------------------▷ 62

$A = \frac{1}{3}$ $B = -\frac{1}{3}$

135

By inserting we obtain $\mathcal{L}[y] = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$

Step 3: Find the inverse transform using the table on page 333 of the textbook:

$y(t) =$ …………………………….

-------------------▷ 136

62

All examples meticulously done and understood ------------------▷ 86

Hints and detailed calculations of the examples ------------------▷ 63

$y(t) = \frac{1}{3}\sin t - \frac{1}{6}\sin 2t$ 136

Just one more exercise. The solution should not be difficult for you after working through the preceding examples.
The solution follows the now customary scheme consisting of three steps:
- Laplace Transformation
- Solve for $\mathcal{L}[x]$ and $\mathcal{L}[y]$ and rearrange into a convenient expression for applying the inverse transform.
- Perform the inverse transformation

------------------▷ 137

One slight irritation may arise from the notation used in this section:
the function to be determined is denoted by $y(t)$ and, consequently, its derivatives by $y'(t)$ and $y''(t)$. This is in parallel to the notation $f(t)$, $f'(t)$, and $f''(t)$ that was used in some previous sections. 63

Difficulties with the first example -------------------▷ 64

Difficulties with the other example -------------------▷ 74

Given two simultaneous differential equations with constant coefficients: 137

$$4\dot{x} - \dot{y} + x = 1 \qquad \text{for } t = 0: \quad x_0 = y_0 = 0$$
$$4\dot{x} - 4\dot{y} - y = 1$$

$x(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found -------------------▷ 142

Stepwise solution showing intermediate results -------------------▷ 138

64

The first example deals with the differential equation $y'+4y = e^{-2t}$
The initial conditions are given as $t = 0$, $y_0 = 5$

In a first step we perform the Laplace transform of the differential equation and recall the rules of transformation

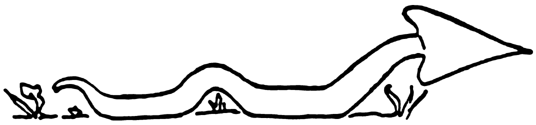
$$\mathcal{L}[y'] = s \cdot \bar{y}(s) - y_0$$

$$\mathcal{L}[y] = \bar{y}(s)$$

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a}$$

The transform of the given differential equation is:…………………………………..

Please, check your results of Laplace transforming each member of the given equation using the tables on page 332 and 333.



------------------▷ 65

138

Given

$$4\dot{x} - \dot{y} + x = 1 \qquad \text{for } t = 0: \; x_0 = y_0 = 0$$
$$4\dot{x} - 4\dot{y} - y = 1$$

Step 1: Determine the Laplace transforms and insert the initial conditions:

………………………………………….

………………………………………….

------------------▷ 139

$s \cdot \bar{y}(s) - y_0 + 4\bar{y}(s) = \frac{1}{s+2}$ initial condition $y_0 = 5$ **65**

As a second step we solve the equation for $\bar{y}(s)$ and insert $y_0 = 5$.

$\bar{y}(s)=$………………………………

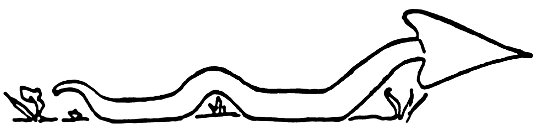
------------------▷ 66

$4s\,\mathcal{L}[x] - s\ \mathcal{L}[y] + \ \mathcal{L}[x] = \frac{1}{s}$ **139**

$4s\,\mathcal{L}[x] - 4s\,\mathcal{L}[y] - \ \mathcal{L}[y] = \frac{1}{s}$

Step 2: Solve for $\mathcal{L}[x]$:

$\mathcal{L}[x]=$…………………………



------------------▷ 140

$$\bar{y}(s)=\frac{5}{(s+4)}+\frac{1}{(s+2)\cdot(s+4)}$$ 66

We are now in a position to perform the third step, finding the inverse transform. It is obtained by consecutively transforming both expressions, yielding y_1 and y_2. The complete inverse transform y then is given by the sum $y_1 + y_2$.
Consult the table on page 332 and 333 in the textbook and identify the inverse transforms of

$\bar{y}_1(s)=\frac{5}{s+4}$ $\qquad$ $y_1 =$ ....

$\bar{y}_2(s)=\frac{1}{(s+2)\cdot(s+4)}$ $\qquad$ $y_2 =$ ....

Solution found -------------------▷ 69

Help needed -------------------▷ 67

$$\mathcal{L}[x]=\frac{3s+1}{s\cdot 12\left(s+\frac{1}{6}\right)\cdot\left(s+\frac{1}{2}\right)}$$ 140

Decompose into partial fractions in order to obtain expressions which are easily inverse transformed:

$\mathcal{L}[x]=$..........................

-------------------▷ 141

In the table on page 333 you see the inverse transforms for general classes of functions. **67**

Rather than $y_1(s) = \frac{5}{s+4}$ the table shows $\bar{y}(s) = \frac{1}{s-a}$, its inverse transform being e^{at}.

Since the constant factor 5 remains unchanged by the transform. If we identify $a = -4$,

we can easily find the inverse transform for $\frac{5}{s+4}$ to be $y_1 =$……………………….

------------------▷ 68

141

$$\mathcal{L}[x] = \frac{1}{s} - \frac{3}{4} \cdot \frac{1}{\left(s + \frac{1}{6}\right)} - \frac{1}{4} \cdot \frac{1}{\left(s + \frac{1}{2}\right)}$$

Step 3: The inverse transformation results in

$x(t) =$………………………………

------------------▷ 142

$y_1 = 5 \cdot e^{-4t}$ 68

It now remains to determine the inverse transform y_2 for $\bar{y}_2(s) = \frac{1}{(s+2)\cdot(s+4)}$

From the table we know that for $\bar{y}(s) = \frac{1}{(s-a)\cdot(s-b)}$ the inverse transform is $y = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)$

Inserting $a = -2$ and $b = -4$ we get:

$y_2 =$ ....

-------------------▷ 69

$x(t) = 1 - \frac{3}{4} e^{-\frac{t}{6}} - \frac{1}{4} \cdot e^{-\frac{t}{2}}$ 142

Now we must determine $\mathcal{L}[y]$ and y.
The Laplace transform has already been performed in frame 139 with the following result:

$$4s\,\mathcal{L}[x] - s\ \mathcal{L}[y] + \ \mathcal{L}[x] = \frac{1}{s}$$
$$4s\,\mathcal{L}[x] - 4s\,\mathcal{L}[y] - \ \mathcal{L}[y] = \frac{1}{s}$$

Now compute

$y(t) =$ …………………………

Solution happily found -------------------▷ 147

Stepwise solution showing intermediate results -------------------▷ 143

$y_2 = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]$ 69

We already know: $y_1 = 5 \cdot e^{-4t}$

Now combine both expressions to obtain the complete inverse transform:

$y = y_1 + y_2 = ...$

-------------------▷ 70

The following must be solved for $\mathcal{L}[y]$ 143

$$4s\ \mathcal{L}[x] - s\,\mathcal{L}[y] + \ \mathcal{L}\ [x] = \frac{1}{s}$$
$$4s\,\mathcal{L}[x] - 4s\,\mathcal{L}[y] - \ \mathcal{L}[y] = \frac{1}{s}$$

By eliminating $\mathcal{L}[x]$ in the usual fashion we obtain

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 144

$$y = 5 \cdot e^{-4t} + \frac{1}{2}\left[e^{-2t} - e^{-4t}\right] = \frac{1}{2}e^{-2t} + \frac{9}{2}e^{-4t}$$ 70

In the textbook the inverse transform is arrived at in a slightly different way.

Show me how the expression from the textbook is arrived at ------------------▷ 71

Continue with the next exercise ------------------▷ 74

$$\mathcal{L}[y] = \frac{-1}{s \cdot 12\left(s + \frac{1}{6}\right) \cdot \left(s + \frac{1}{2}\right)}$$ 144

Decomposition into partial fractions produces expressions that are easily inversely transformed:

$$\mathcal{L}[y] = \frac{-1}{s \cdot 12\left(s + \frac{1}{6}\right) \cdot \left(s + \frac{1}{2}\right)} = \frac{A}{12s} + \frac{B}{\left(s + \frac{1}{6}\right)} + \frac{C}{\left(s + \frac{1}{2}\right)}$$

Determine the unknown values

A =…………….. B =…………….. C =……………

------------------▷ 145

Given $\bar{y}(s)=\frac{5}{s+4}+\frac{1}{(s+2)\cdot(s+4)}$. Use the common denominator to get $\bar{y}(s)=\frac{5s+11}{(s+2)\cdot(s+4)}$ 71

In order to arrive at possibly simpler expressions let us decompose into partial fractions:

$$\frac{5s+11}{(s+2)\cdot(s+4)}=\frac{A}{s+2}+\frac{B}{s+4}$$

We must determine A and B. First express everything with the common denominator:

$$\frac{5s+11}{(s+2)\cdot(s+4)}=\frac{As+4A+Bs+2B}{(s+2)\cdot(s+4)}$$

Clearing the fractions leads to: $5s+11=(A+B)\cdot s+4A+2B$

Determine A and B by comparing coefficients (equality holds for all different values of s).

$$5=A+B$$
$$11=4A+2B$$

$A=$ .... $B=$ ....

-------------------▷ 72

If decomposing into partial fractions proved difficult for you, go back to the textbook again to follow the solution of the exercise above.

$A=-12$ $B=\frac{3}{2}$ $C=-\frac{1}{2}$ 145

Thus

$$\mathcal{L}[y]=\frac{-1}{s\cdot 12\left(s+\frac{1}{6}\right)\cdot\left(s+\frac{1}{2}\right)}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 146

$A = \frac{1}{2}$ $\qquad B = \frac{9}{2}$ 72

Thus, the Laplace transform becomes:

$$\bar{y}(s) = \frac{5s+11}{(s+2)\cdot(s+4)} = \frac{1}{2(s+2)} + \frac{9}{2(s+4)}$$

Again, we quickly obtain the inverse transformation using the table in the textbook on page 332 and 333.

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 73

$$\mathcal{L}[y] = -\frac{1}{s} + \frac{3}{2}\frac{1}{\left(s+\frac{1}{6}\right)} - \frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)}$$ 146

Now we can determine the inverse transform with a little help from the table, as nobody can keep all transforms in memory. The result is:

$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 147

$$y = \frac{1}{2} \cdot e^{-2t} + \frac{9}{2} e^{-4t}$$ 73

This coincides with the result previously obtained in this study guide.

There usually exist various different ways to rearrange the expression $\bar{y}(s)$ or $F(s)$. Experience, combined with some educated guessing, will guide you to finding expressions which are amenable to inverse transformations. One suitable approach often is the decomposition into partial fractions, which was dealt with in the textbook.

------------------▷ 74

Please continue on page 1
(bottom half)

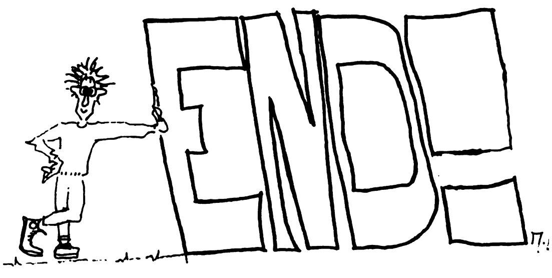
$$y(t) = -1 + \frac{3}{2} \cdot e^{-\frac{t}{6}} - \frac{1}{2} e^{-\frac{t}{2}}$$ 147

Thus, we have determined the complete solution of the system of equations as follows

$$x(t) = 1 - \frac{1}{2}\left(e^{-\frac{t}{6}} + e^{-\frac{t}{2}}\right)$$

$$y(t) = -1 + e^{-\frac{t}{6}} - e^{-\frac{t}{2}}$$

You have successfully completed this somewhat demanding chapter, and you may be proud of your stamina!



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