Chapter 16
Eigenvalues and Eigenvectors of Real Matrices

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The two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthogonal.

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$$\text{Proof} : \mathbf{r}_1\mathbf{r}_2 = (2, \ -1)\begin{pmatrix}1\\2\end{pmatrix} = 2\times 1 - 1\times 2 = 0$$

More abstract reasoning: The matrix $\mathbf{A}$ under consideration is symmetric. A theorem says that eigenvectors in that case are orthogonal.

$$\text{For } \mathbf{A} = \begin{pmatrix}-1 & -1 & 2\\ -1 & -1 & -2\\ 2 & -2 & -2\end{pmatrix}$$

Prove that $\lambda_1 = -2$ is an eigenvalue of $\mathbf{A}$.
Find the corresponding eigenvector.

I need some help --------------------▷ 17

I can do it --------------------▷ 20

16.1 Eigenvalues and Eigenvectors

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Objective: Concepts of real eigenvalue of a square matrix, eigenvector corresponding to an eigenvalue, characteristic polynomial, geometric significance of eigenvectors, determination of all the real eigenvalues and corresponding eigenvectors of any given 2×2 or 3×3 matrix.

READ: 16 Eigenvalues and eigenvectors of real matrices
Textbook pages 453–461

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In order to verify that $\lambda_1 = -2$ is an eigenvalue one may compute det $(\mathbf{A} - \lambda_1 \mathbf{I})$ which has to vanish.

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix}, \quad (\mathbf{A} + 2\mathbf{I}) = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

Compute its determinant.

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Suppose a square matrix **A** is applied to a vector **r** yielding $\mathbf{r}'$ such that $\mathbf{r}' = \mathbf{A}\mathbf{r} = \lambda\mathbf{r}$
Then **r** is said to be an .... of **A** and λ is an ...., provided $\lambda \neq$ ...... and $\mathbf{r} \neq$ 3

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One can, for example, use Sarrus' rule: 18

$$0+4+4-4-4-0=0$$

The first part of the problem is done. For the second part, finding an eigenvector $\mathbf{r}_1$, we must solve a system of equations. As a matrix equation we must have

$$\mathbf{A}\mathbf{r}_1 = -2\mathbf{r}_1 \text{ or equivalently } (\mathbf{A}+2\mathbf{I})\mathbf{r}_1 = 0$$

Write down the system of equations with $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$.

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eigenvector
eigenvalue
$\lambda \neq 0, \ \mathbf{r} \neq \mathbf{0}$

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Let

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

Can you be certain that a real eigenvalue exists?

Yes --------------------▷ 6

No --------------------▷ 5

$$\begin{aligned} x_1 - y_1 + 2z_1 &= 0 \\ -x_1 + y_1 - 2z_1 &= 0 \\ 2x_1 - 2y_1 \qquad &= 0 \end{aligned}$$

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The determinant of the matrix of coefficients has been shown to vanish. Therefore, one of the equations depends linearly on the others. In fact:

$$\text{first equation} = -\text{second equation}$$

We can thus disregard the second equation. Now compute a particular solution $\mathbf{r}_1$ of the remaining two equations.

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In principle you are right: a 2×2 matrix need not have any real eigenvalue. But take a closer look at the matrix $\mathbf{A}$. It is symmetric and it is not singular. Now, there is a theorem saying that such a symmetric matrix always has the maximum number of real eigenvalues. Thus for a symmetric 2×2 matrix there must be two real eigenvalues. 5

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The determinant of $(\mathbf{A} + 2\mathbf{I})$ vanishes, so $\lambda_1 = -2$ is an eigenvalue of $\mathbf{A}$. The system of linear equations determining a corresponding eigenvector $\mathbf{r}_1$ reduces to 20

$$x_1 = y_1 \quad \text{and} \quad z_1 = 0$$

A particular solution is given by choosing $x_1 = 1$:

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Any other non-zero choice of x_1 leading to another particular solution is equally valid.

Check that $\mathbf{r}_1$ is indeed an eigenvector of

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

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Since $\mathbf{A}$ is symmetric, two real eigenvalues must exist. For an arbitrary 2×2 matrix no real eigenvalue need exist. 6

In order to find the eigenvalues we look for solutions of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ with $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

What is the name of that equation? ....

Write down the equation for the matrix under consideration.

Find its roots. λ_1 = ... λ_2 = ...

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$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & +0 \\ -1 & -1 & -0 \\ 2 & -2 & -0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = -2\mathbf{r}_1$$

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Do you feel in a position to compute the characteristic polynomial of $\mathbf{A}$?

Yes: Splendid! Please do it.

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No: Reread section 16.2 in the textbook. Then try to compute it.

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characteristic equation of **A**

$$\det\begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda)-4 = \lambda^2 - 7\lambda + 6 = 0$$

Roots: $\lambda_1 = 1,\ \lambda_2 = 6$

There are two eigenvalues of **A**. In order to find corresponding eigenvectors, one must solve two systems of homogeneous linear equations.

Do you remember how these arise? Yes:

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No: Read on!

For the sake of clarity, let us consider the first eigenvalue $\lambda_1 = 1$. For a corresponding eigenvector $\mathbf{r}_1$ the following matrix equation must hold:

$$\mathbf{A}\mathbf{r}_1 = \lambda_1 \mathbf{r}_1$$

Equivalently: $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{r}_1 = \mathbf{0}$

Write this down explicitly, using $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$: ....

Recall that $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

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The characteristic polynomial of **A** reads

$$\lambda^3 + 4\lambda^2 - 4\lambda - 16 = 0$$

Splitting off the linear factor $(\lambda - (-2))$ yields:

$$\lambda^3 + 4\lambda^2 - 4\lambda - 16 = (\lambda + 2)(\dots\dots\dots\dots\dots\dots)$$

The roots of the quadratic term are

$$\lambda_2 = \dots\dots\dots\dots$$
$$\lambda_3 = \dots\dots\dots\dots$$

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$$\begin{pmatrix} 2-1 & 2 \\ 2 & 5-1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ 8

Multiplying out yields the system of two homogeneous linear equations for the first eigenvector:

$$x_1 + 2y_1 = 0$$
$$2x_1 + 4y_1 = 0$$

A solution of these equations gives the components of an eigenvector.

How can you be certain that a non-zero solution exists?

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$$(\lambda + 2)(\lambda^2 + 2\lambda - 8)$$
$$\lambda_2 = 2, \quad \lambda_3 = -4$$ 23

An eigenvector corresponding to $\lambda_2 = 2$ is asserted to be

$$\mathbf{r}_2^* = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}$$

Check if this assertion is true.

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Given: $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$, $\lambda_1 = 1$, $\lambda_2 = 6$. 9

In order to find an eigenvector for $\lambda_1 = 1$ the following system of equations must be solved non-trivially:

$$\begin{aligned} x_1 + 2y_1 &= 0 \\ 2x_1 + 4y_1 &= 0 \end{aligned}$$

A non-zero solution must exist, since the determinant of the coefficients, which is det $(\mathbf{A} - \lambda_1 \mathbf{I})$, vanishes.

Find a solution to the system of equations given above.

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$\mathbf{r}_2^*$ is *not* an eigenvector of $\mathbf{A}$: 24

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 24 \end{pmatrix} \neq \lambda_2 \mathbf{r}_2^*$$

Find an eigenvector corresponding to $\lambda_2 = 2$.

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We have $x_1 = -2y_1$. Choosing $x_1 = 2$, a particular solution reads $x_1 = 2, \quad y_1 = -1$. 10
Written differently:

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(Other choices for x_1 leading to other particular solutions are equally valid.)

This vector $\mathbf{r}_1$ must satisfy the equation

$$\mathbf{A}\mathbf{r}_1 = \lambda_1 \mathbf{r}_1 \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

or, equivalently,

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{r}_1 = \mathbf{0},$$

Check this numerically.

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To solve: $(\mathbf{A} - 2\mathbf{I})\mathbf{r}_2 = 0$. 25

$$\begin{aligned} -3x_2 - y_2 + 2z_2 &= 0 \\ -x_2 - 3y_2 - 2z_2 &= 0 \\ 2x_2 - 2y_2 - 4z_2 &= 0 \end{aligned}$$

Thus: $x_2 = -y_2$ and $z_2 = x_2$.

A particular solution is obtained by letting $x_2 = 1$:

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Find an eigenvector corresponding to $\lambda_3 = -4$.

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$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2-2 \\ 4-4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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The other eigenvalue of the matrix was found to be $\lambda_2 = 6$. Write down the corresponding matrix equation for an eigenvector $\mathbf{r}_2$.

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

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$\mathbf{r}_3$ must be orthogonal to both $\mathbf{r}_1$ and $\mathbf{r}_2$, since $\mathbf{A}$ is symmetric. Therefore, $x_3 = -y_3$ and $z_3 = -2x_3$. A solution reads: 26

$$\mathbf{r}_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

(a) What is the maximum number of real eigenvalues an $n \times n$ matrix can have?
(b) Are there matrices which do not possess the maximum number of real eigenvalues?
(c) Give an example if $n = 2$.

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$$\mathbf{A}\mathbf{r}_2 = 6\mathbf{r}_2 \qquad \text{i.e.} \quad (\mathbf{A} - 6\mathbf{I})\mathbf{r}_2 = \mathbf{0}$$

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$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write down the two homogeneous linear equations embodied by the last matrix equation. Find a particular non-trivial solution, i.e. an eigenvector $\mathbf{r}_2$ to λ_2.

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(a) n
(b) Yes
(c) Any rotation of the plane about an angle $\alpha \neq 0$ or π does not possess any real eigenvalue.

So far we have dealt exclusively with symmetric matrices. Now let us consider a matrix which is non-symmetric:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

Find its eigenvalues and eigenvectors.

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$$\begin{aligned} -4x_2 + 2y_2 &= 0 \\ 2x_2 - y_2 &= 0 \end{aligned}$$ 13

If we choose $y_2 = 2$ then $x_2 = 1$ and

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(Other choices of y_2 leading to other particular solutions are equally valid.)

Verify that $\mathbf{r}_2$ is an eigenvector corresponding to $\lambda_2 = 6$.

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(a) Characteristic equation: $\lambda^2 - \lambda - 6 = 0$ 28
(b) Eigenvalues: $\lambda_1 = 3,\ \ \lambda_2 = -2$
(c) Eigenvectors: $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Draw $\mathbf{r}_1,\ \mathbf{r}_2,\ \mathbf{r}'_1$ and $\mathbf{r}'_2$ in a diagram.
Are $\mathbf{r}_1$ and $\mathbf{r}_2$ orthogonal?

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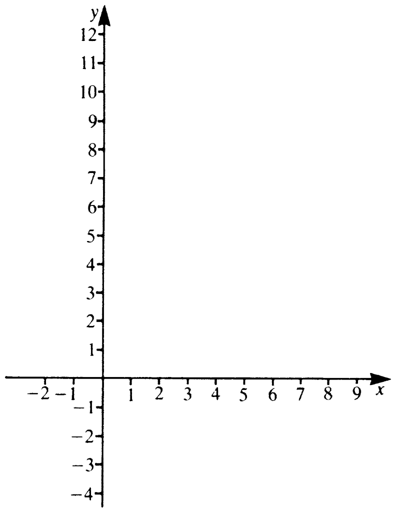
$$\mathbf{A}\mathbf{r}_2 = 6\mathbf{r}_2:$$
$$\mathbf{r}'_2 = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+4 \\ 2+10 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \mathbf{6}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Draw $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}'_1$ and $\mathbf{r}'_2$.



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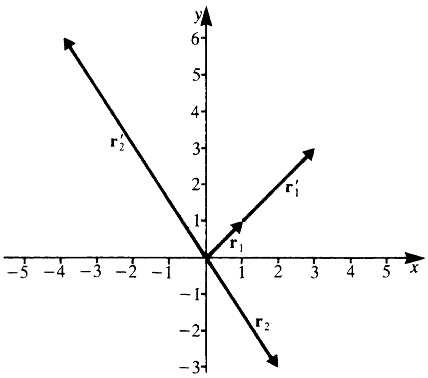
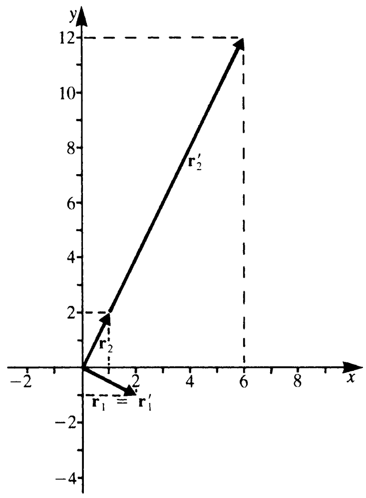


They are not orthogonal:

$$\mathbf{r}_1\mathbf{r}_2 = (1,\ 1)\begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2-3 \neq 0.$$

Well done!
You have worked well, and now you deserve a break.
For further study you should solve the problems given in the textbook.

END OF CHAPTER 16



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What do you notice about the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$? Prove your claim. 16

Please continue on page 1
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