Chapter 18
Fourierseries

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Integral 1 is: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos ntdt = \left[\frac{t}{\pi} \cdot \frac{1}{n} \sin nt \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{\pi} \cdot \frac{1}{n} \sin ntdt$

The bracket vanishes because $\sin n\pi = \sin(-n\pi) = 0$
The remaining integral is solved easily:

$$\int_{-\pi}^{\pi} \sin ntdt = \left[-\frac{1}{n} \cos nt \right]_{-\pi}^{\pi}$$

We remember:

$\cos n\pi = \cos(-n\pi)$

Thus: $\int_{-\pi}^{\pi} \sin nt \quad dt = 0$

By this we calculated integral 1. The result is

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \cos ntdt = \ldots\ldots\ldots\ldots\ldots$$

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18.1 Expansion of a Periodic Function into a Fourierseries

1

In this section we show that every periodic function can be expanded into a sum of trigonometric functions. This is important for physics and electrical engineering.

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Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos ntdt = 0$

We had to compute

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{t}{\pi} + 1 \right) \cos ntdt = \frac{1}{\pi} \left[\underbrace{\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt \cdot dt}_{\text{Integral 1}} + \underbrace{\int_{-\pi}^{\pi} \cos nt \cdot dt}_{\text{Integral 2}} \right]$$

Integral 1 vanishes.

Integral 2 can be solved easily and does not pose any difficulty:

$$\int_{-\pi}^{\pi} \cos nt \cdot dt = \Big[\ldots\ldots\ldots\ldots \Big]_{-\pi}^{\pi} = \ldots\ldots\ldots\ldots\ldots\ldots$$

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In the following we will frequently use the addition theorems of trigonometric functions. Your understanding of the following will be much easier if you recapitulate some of the relations treated in chapters 1, 5, and 6 as a preparation. 2

Solve the following integrals:

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = \ldots\ldots\ldots\ldots$$

$$\int_{-\pi}^{+\pi} \cos nx \quad dx = \ldots\ldots\ldots\ldots$$

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$$\int_{-\pi}^{\pi} \cos nt \cdot dt = \left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi} = 0$$ 46

Thus, we obtained: $a_n = 0$

Now we have to calculate the coefficients b_n :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt \cdot dt$$

We insert: $f(t) = \frac{t}{\pi} + 1$

By this we get: $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

We summarize: The integral of a sum of functions is the sum of the integrals of these functions n:

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \ldots\ldots\ldots\ldots\ldots\ldots + \int_{-\pi}^{\pi} \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

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3

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \quad dx = 0$$

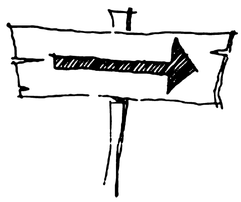
Remember the following relationships.
In case of difficulties you may read again section 1.5.6 (addition formulae) in chapter 1.

$\sin(n+m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$\sin(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

$\cos(n+m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$

$\cos(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$



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$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\frac{t}{\pi}+1)\sin nt \quad dt$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \frac{t}{\pi}\sin ntdt + \int_{-\pi}^{\pi} \sin ntdt\right]$$

Integral 1 Integral 2

We start with integral 1.

Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi}\sin ntdt = \ldots\ldots\ldots$

Solution found -------------------▷ 54

Recapitulation of integration by parts -------------------▷ 48

Detailed solution -------------------▷ 50

$$\sin(n+m)x = \sin nx \cdot \cos mx + \sin mx \cdot \cos nx$$
$$\sin(n-m)x = \sin nx \cdot \cos mx - \sin mx \cdot \cos nx$$
$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

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Using the addition formulae you can derive the following relationships which will be used repeatedly:

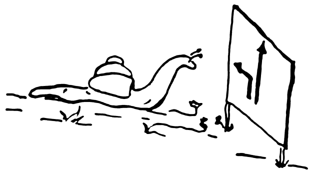
$\sin nx \cdot \cos mx =$ ……………………………...

$\cos nx \cdot \cos mx =$ ……………………………..

$\sin nx \cdot \sin mx =$ ……………………………...

These tasks may seem tedious but they can be solved by you.

Solution found: -------------------▷ 11



Hints wanted -------------------▷ 5

Integration by parts is often useful if products have to be integrated. This method is explained in the textbook section 6.5.4. Perhaps it may help to read that section again. Here is a short recapitulation.

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Given two different functions $u(t)$ and $v(t)$. The limits of integration may be a and b.

Then holds the formula for integration by parts: $\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$

$$\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$$

To calculate Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \sin nt \quad dt$

We let

$u = \frac{t}{\pi}$ $\qquad v' = \sin nt$

From this we obtain

$u' =$ ………… $v =$ ………………….

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Let us start with the first task: 5

$\sin nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

We use the addition formula

$\sin(n+m)x = \sin nx \cdot \cos mx + \sin mx \cdot \cos nx$

$\sin(n-m)x = \sin nx \cdot \cos mx - \sin mx \cdot \cos nx$

If we add both lines we obtain:

$\sin(n+m)x + \sin(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots..$

------------------▷ 6

$u' = \frac{1}{\pi} \qquad v = -\frac{\cos nt}{n}$ 49

In case you are unsure you may verificate by deriving *v*. As we know the formula for integration by parts reads:

$\int_a^b u \cdot v' = [u \cdot v]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u' \cdot v$

Using the functions above we get:

$\int_{-\pi}^{\pi} \frac{t}{\pi} \sin nt \quad dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 50

$\sin(n+m)x + \sin(n-m)x = 2\sin nx \cdot \cos mx$ 6

From this follows directly the result of the first task:

$\sin nx \cdot \cos mx = \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x$

In the same manner you can solve the second task:

$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found: -------------------▷ 8

More hints wanted -------------------▷ 7

$$\int_{-\pi}^{\pi} \frac{t}{\pi}\sin nt\, dt = \left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{+\pi}\frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt$$ 50

First we solve the terms in brackets. This can be done easily, but difficulties may arise if the signs are interchanged. This happens often in calculations.

We remember:

$\cos n\pi = \cos(-n\pi)$

$\cos n\pi = (-1)^n$

$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Answer found -------------------▷ 53

Detailed solution -------------------▷ 51

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The second task:

$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

This time we use the addition formulae in which appear products of cos-functions.

$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

We add both lines and obtain:

$\cos(n+m)x + \cos(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots$

Thus, the solution is easily obtained:

$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$

------------------▷ 8

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To solve: $\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi}$

We remember: $\cos n\pi = \cos(-n\pi) = (-1)^n$

Inserting the given limits we obtain:

$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = [\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$

------------------▷ 52

$$\cos nx \cdot \cos mx = \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$ 8

In the same manner we can solve the third task:

$\sin nx \cdot \sin mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: Use the addition formulae in which you find products of sin-functions

Solution found -------------------▷ 10

You want more hints: -------------------▷ 9

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \left[\frac{\pi}{\pi}\left(-\frac{\cos n\pi}{n}\right) - \frac{-\pi}{\pi}\left(-\frac{\cos(-n\pi)}{n}\right)\right]$$ 52

We simplify and get now:

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \left[-\frac{\cos n\pi}{n} - \frac{\cos(-n\pi)}{n}\right] = -2\frac{\cos n\pi}{n}$$

Using $\cos n\pi = (-1)^n$ we can write:

$-\cos n\pi = (-1)^{n+1}$

Finally we obtain:

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \ldots\ldots\ldots\ldots$$

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9

To solve: $\sin nx \cdot \sin mx =$…………………………

We use:

$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

We subtract the lines and obtain:

$\cos(n+m)x - \cos(n-m)x =$ ………………………

Thus we obtain the solution:

$simnx \cdot \sin mx =$……………………….

-------------------▷ 10

53

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \frac{2}{n}(-1)^{n+1}$$

Thus our task is to solve:

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \sin ntdt = \frac{2}{n}(-1)^{n+1} - \int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt$$

The remaining integral may be solved easily because it is a fundamental integral

: $\int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt =$ ………………
……………

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$$\cos(n+m)x - \cos(n-m)x = -2\cos nx \cdot \sin mx$$
$$\sin nx \cdot \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x$$

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Through these repetitions we found the relationships we need to get the Fourierseries.

$\sin nx \cdot \cos mx =$ ……………………………...

$\cos nx \cdot \cos mx =$ ……………………………..

$\sin nx \cdot \sin mx =$ ……………………………...

------------------▷ 11

$$\int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt = \left[-\frac{1}{\pi}\frac{\sin n\pi}{n^2}\right]_{-\pi}^{\pi} = 0$$

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The integral vanishes because: $\sin n\pi = \sin(-n\pi) = 0$

Integral 2 vanishes

$$\int_{-\pi}^{\pi} \sin nt \quad dt = \left[\frac{1}{n}(-\cos nt)\right]_{-\pi}^{\pi} = 0$$

Because: $\cos(n\pi) = \cos(-n\pi)$

Finally, we determine b_n to be

$$b_n = \frac{2}{\pi \cdot n}(-1)^{n+1}$$

The sawtooth curve was defined by:

$$f(t) = \left(\frac{t}{\pi} + 1\right) \quad for \quad -\pi < t < \pi$$

The Fourierseries is thus given by

$f(t) =$ ……………………………

Hint: Do not forget : a_0 ..

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11

$$\sin nx \cdot \cos mx = \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x$$
$$\cos nx \cdot \cos mx = \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$
$$\sin nx \cdot \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x$$

Take notes on a separate sheet of paper of these relationships and the following ones:

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = \int_{-\pi}^{+\pi} \cos nx \quad dx = 0$$

Now we have gained all prerequisites to resolve our main objective.
Read carefully and control all transformations parallel to the text

Study **18.1 Expansion of a Periodic Function into a Fourierseries**
Textbook pages 491–496

Then go to

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$$f(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{\pi n} \cdot (-1)^{n+1} \sin nt$$

This Fourierseries differs from the Fourierseries calculated in the textbook only by the value of a_0.
Both curves can be shifted in the y-direction to coincide.

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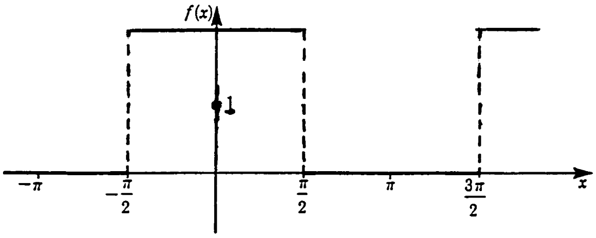
12

You might have had difficulties. This may be because the relationships we repeated beforehand are not familiar enough to you. In this case it is useful to calculate the transformations on a separate sheet and look up the required relationships in the textbook.
With the summary in the textbook on page 494 we can obtain the Fourierseries for the rectangular function $f(x)$, which is defined in the interval $-\pi < x < \pi$.

$$f(x) = \begin{cases} 0 & \textit{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \textit{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \textit{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found: ------------------▷ 21

Help and further explanation ------------------▷ 13

18.3 Expansion of Functions of Period *L*

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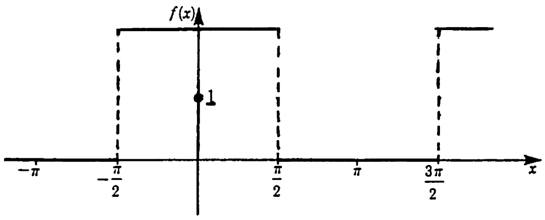
L may be any period depending on the given situation. Reading the section in the textbook you should control all substitutions using a separate sheet.

Study in the textbook: **18.3 Expansion of functions of Period *L***
Textbook page 501–502

------------------▷ 57

Wanted is the Fourierseries for the rectangular function defined by: **13**

$$f(x)=\begin{cases}0 & \text{für} \quad -\pi \leq x \leq -\frac{\pi}{2}\\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2}\\ 0 & \text{für} \quad +\frac{\pi}{2} \leq x \leq +\pi\end{cases}$$



This rectangular function is defined by separate parts or successive separate sections. This implies that the coefficients in the summary (18.2) on page 494 have to be computed separately for each part or section; in other words the integrals must be computed for each part separately.

Let us start with a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ------------------▷ 15

Help and explanation wanted ------------------▷ 14

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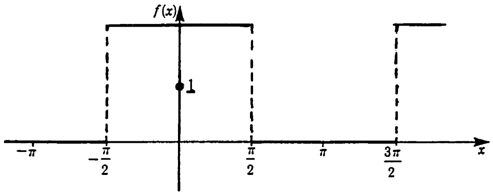
No problems regarding the arbitrary period L ------------------▷ 60

More and detailed explanations wanted ------------------▷ 58

To compute: $a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx =$ 14

We had

$$f(x) = \begin{cases} 0 & \text{für} \quad -\pi \leq x \leq -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \leq x \leq +\pi \end{cases}$$



The integral has to be computed for the whole interval $-\pi \leq x \leq \pi$. For the whole the function is defined in three separate parts. Thus, the whole integral is the sum of three separate integrals for each part or section of the function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} f(x)dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(x)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)dx$$

What remains is for us to insert the function for each part and to compute the integrals:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

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We started with the Fourierseries for the variable *x* and the period 2π . The Fourierseries is summarized in the table on page xxxx of the textbook. Use this table. 58

Now let us substitute *x* by $\frac{2\pi}{T}t$ and *dx* by $\frac{2\pi}{T}dt$.

For $t = T$ we get $x =$…………….

Now try this substation for a_n : $x = \frac{2\pi}{T}t$ and $dx = \frac{2\pi}{T}dt$

We obtain

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 59

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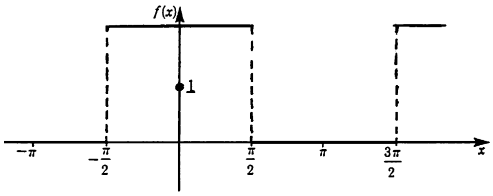
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} 0\cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0\cdot dx = \frac{2}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 2$$

Now we compute partwise $a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nxdx$

The function was defined:

$$f(x) = \begin{cases} 0 & \text{für } -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für } -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für } +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$a_n =$ …………………………..

Solution found -------------------▷ 18

Help and more explanation -------------------▷ 16

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For $t = T$ we get $x = 2\pi$.

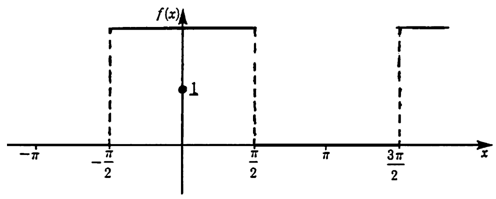
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos\left(\frac{n2\pi}{T}t\right)\cdot\frac{2\pi}{T}dt \quad = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos\left(\frac{n2\pi}{T}t\right)dt$$

-------------------▷ 60

To compute: 16

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx \text{ for}$$

$$f(x) = \begin{cases} 0 & \text{für } -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für } -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für } +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



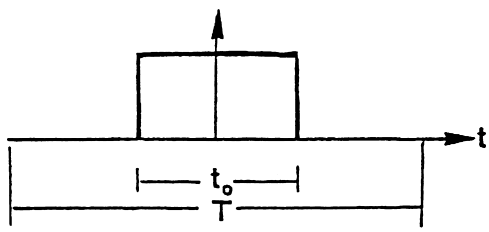
Again we have to compute the integrals for the three parts separately.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx \cdot dx = \ldots\ldots\ldots\ldots\ldots\ldots..$$

------------------▷ 17

Calculate the Fourierseries for a rectangular curve. If the variable is time we have a rectangular waveform. Calculate the rectangular waveform using the variable t instead of x. 60

$$f(t) = \begin{cases} 0 & \text{for } -\frac{T}{2} \le t \le -\frac{t_0}{2} \\ 1 & \text{for } -\frac{t_0}{2} \le t \le \frac{t_0}{2} \\ 0 & \text{for } \frac{t_0}{2} \le t \le \frac{T}{2} \end{cases}$$



Because the function is even $b_n = 0$

We have three sections of the function. For the first and for the third section we have $f(t) = 0$.
Thus, we do not need to calculate the integrals for the first and the third section. It remains to calculate the integral for the section in the middle:

We start with the calculation of a_0.

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$$

$a_0 = \ldots\ldots\ldots\ldots\ldots$

------------------▷ 61

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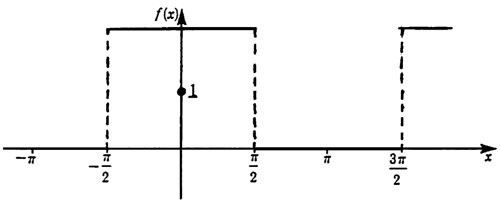
$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx \cdot dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} f(x)\cos nx \cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(x)\cos nx \cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx \cdot dx$$

The function has been

$$f(x) = \begin{cases} 0 & \text{für } -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für } -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für } +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



Inserted we obtain three integrals to solve:

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 18

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$$a_0 = \frac{2}{T}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t)dt = 2\frac{t_0}{T}$$

Now we calculate a_n:

$$a_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t)\cdot\cos\left(\frac{2\pi n}{T}t\right)dt$$

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found -------------------▷ 63

Further explanation wanted -------------------▷ 62

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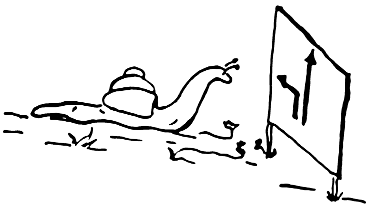
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\frac{\pi}{2}} 0\cdot\cos nx dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cdot\cos nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0\cdot\cos nx dx$$

$$a_n = 0 + \frac{1}{\pi}\cdot\frac{2}{n}\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right] + 0$$

$$a_n = \frac{4}{\pi\cdot n}\sin n\frac{\pi}{2}$$

What remains is the task of calculating b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found -------------------▷ 21

Help and more explanation -------------------▷ 19

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To calculate.

$$a_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t)\cdot\cos\left(\frac{2\pi n}{T}t\right)dt$$

For the section in the middle holds:

$$f(t) = 1 \qquad \text{for} \qquad -\frac{t_0}{2} < t < \frac{t_0}{2}$$

We remember:

$$\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} \cos\left(\frac{2\pi n}{T}t\right)dt = \left[\frac{T}{2\pi n}\cdot\sin\frac{2\pi n\cdot t}{T}\right]_{-\frac{t_0}{2}}^{\frac{t_0}{2}}$$

Now we insert regarding the limits:

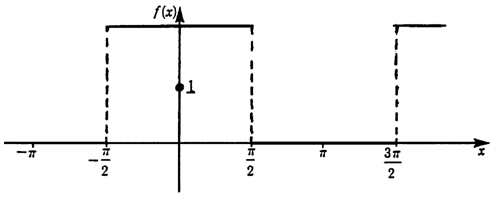
$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 63

Again we must decompose the whole integral into three parts, which have to be solved separately. 19

The function was:

$$f(x)=\begin{cases} 0 & \text{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 20

$$a_n = \frac{2}{T}\cdot\frac{T}{n2\pi}\cdot 2\sin\frac{n\cdot\pi\cdot t_0}{T} = \frac{2}{n\pi}\sin\frac{n\pi t_0}{T}$$ 63

Regarding $a_0 = \frac{2t_0}{T}$ we obtain finally the Fourierseries for the rectangular waveform of duration t_0 and the period T.

$f(t)$=……………………………..

------------------▷ 64

20

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} 0 \cdot \sin nx \cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2 \cdot \sin nx \cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0 \cdot \sin nx \cdot dx = \frac{1}{\pi} 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx \cdot dx$$

This time we do not expect more difficulties:

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx dx = \ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 21

64

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi t_0}{T} \cdot \cos \frac{n2\pi t}{T}$$

If you would like one more exercise -------------------▷ 65

If you prefer to go on -------------------▷ 70

21

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin nx dx = \frac{2}{\pi}\left[-\frac{\cos n\frac{\pi}{2}}{n} + \frac{\cos n\frac{\pi}{2}}{n}\right] = 0$$

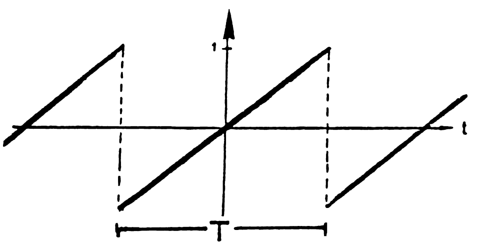
Thus our problem is solved. We obtained the Fourierseries for the rectangular function.

The series contains with exception of $a_0 = 1$ only cos-functions, the amplitudes of which decrease if n increases. Furthermore, the coefficients b vanish for n = even.

------------------▷ 22

65

The example of the sawtooth waveform has been treated exhaustively But up to now only for the period 2π . Now we will calculate an example for the period T.



The sawtooth waveform is defined for the period T by:

$f(t)$=…………..

Hint: for $t = \frac{T}{2}$ holds: $f(t) = 1$.

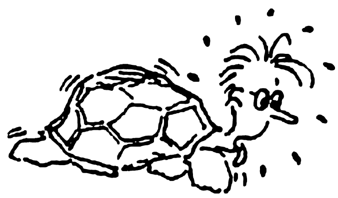
------------------▷ 66

18.2 Examples of Fourierseries

22

In the following section we do not expect fundamental difficulties. But within the calculations writing errors or computing errors are likely to occur. Thus, you have to concentrate on the accuracy of your calculations. Do not forget to control all transformations using a separate sheet of paper.

Study **18.2 Examples of Fourierseries**
Textbook pages 496–501



After your study go to ------------------▷ 23

$f(t) = \frac{2}{T} \cdot t$ for $-\frac{T}{2} < t < \frac{T}{2}$

66

We state: $f(t)$ is odd. Then all $a_n = 0$. We have to calculate only the coefficients b_n.

The integral can be solved by integration by parts:

$b_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 68

Help and explanations wanted ------------------▷ 67

The rectangular function treated in the last section of the study guide beginning with frame 12 is a 23

- ☐ Even function
- ☐ Odd function

------------------▷ 24

To calculate: 67

$$b_n = \left(\frac{2}{T}\right)^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} t \sin\left(\frac{n2\pi}{T}t\right) dt$$

We remember integration by parts: $: \int u \cdot v' = [u \cdot v] - \int u' \cdot v$

If $u = t$ and $v' = \sin\left(\frac{n2\pi}{T}t\right)dt$ we obtain:

$$u' = 1 \text{ and } v = -\cos\left(\frac{n2\pi}{T}t\right) \cdot \frac{T}{2\pi n}$$

You have to insert this into the integral shown above and calculate with a lot of patience:

$$b_n = \left(\frac{2}{T}\right)^2 \left[\left[\ldots\ldots\ldots\ldots \right]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} \ldots\ldots\ldots\ldots \right]$$

------------------▷ 68

Even function

24

For even functions we have:

☐ $a_n = 0$

☐ $b_n = 0$

For even functions the Fourierseries is composed of …………… functions.

------------------▷ 25

68

$$b_n = \left(\frac{2}{T}\right)^2 \left[\left[t\left(-\cos\left(\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right)\right]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} 1\cdot\left(-\cos\left(\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right)\right]$$

The remaining integral is a fundamental integral and may be solved easily.

Because of $\sin\left(\frac{n2\pi}{T}\cdot\frac{T}{2}\right) = \sin n2\pi = 0$ the integral vanishes.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{n2\pi}{T}t\right) = \frac{T}{n2\pi}\left[\sin\left(\frac{n2\pi}{T}t\right)\right]_{-\frac{T}{2}}^{\frac{T}{2}} = 0$$

Thus we obtain:

$b_n = \left(\frac{2}{T}\right)^2$ ………………………..

Hint: $-\cos n\pi = (-1)^{n+1}$

------------------▷ 69

For even functions $b_n = 0$ holds. 25

The Fourierseries is composed of cos-functions.

In the examples in the textbook the variable *x* might have been substituted by the variable *t* which denotes time. In physics and engineering we mostly deal with time. So we will use the variable *t* in the next example.

Here follows another exercise regarding a rectangular function

------------------▷ 26

69

$$b_n = \left(\frac{2}{T}\right)^2 \left[2\left(\frac{T}{2}\right)^2 \cdot \frac{1}{n \cdot \pi} (-1)^{n+1} \right]$$

$$b_n = \frac{2}{n\pi}(-1)^{n+1}$$

The Fourierseries of the sawtooth waveform with the period *T* reads:

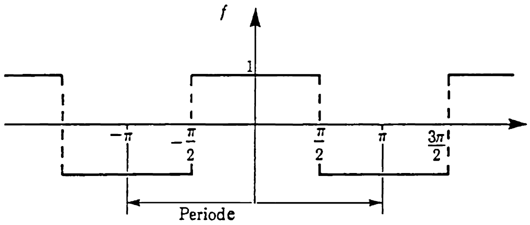
$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot (-1)^{n+1} \cdot \sin\left(\frac{n2\pi}{T} t\right)$$

This is in agreement with the result on page 497 of the textbook.

------------------▷ 70

Given the rectangular function: 26

$$f(t)=\begin{cases}-1 & \textit{für} \quad -\pi \le t \le -\dfrac{\pi}{2}\\ 1 & \textit{für} \quad -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}\\ -1 & \textit{für} \quad \dfrac{\pi}{2} \le t \le \pi\end{cases}$$



Shift the rectangular function to obtain an odd function: Is there only one solution?

$$f(t)=\begin{cases} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{cases}$$

------------------▷ 27

18.4 Fourier Spectrum

70

We remember a notation which we use frequently: $\sin\dfrac{n2\pi}{T}t = \sin\omega t$, and $\omega = \dfrac{n2\pi}{T}$

Study **18.4 Fourier Spectrum**
Textbook page 502–503

Go to ------------------▷ 71

There exist two solutions that are of equal value: **27**

Solution1: $f(t)=\begin{cases} -1 & \textit{für} \quad -\pi < t < 0 \\ +1 & \textit{für} \quad 0 < t < \pi \end{cases}$

Solution 2: $f(t)=\begin{cases} +1 & \textit{für} \quad -\pi < t < 0 \\ -1 & \textit{für} \quad 0 < t < \pi \end{cases}$

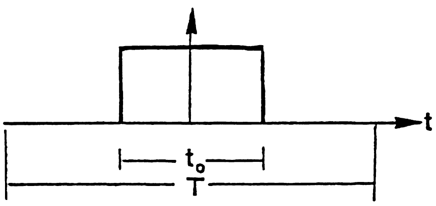
Compute the Fourierseries for the rectangular function of solution 1:

$f(t)=$ …………………………

Solution found -------------------▷ 33

Help wanted -------------------▷ 28

You have already calculated the Fourierseries for the rectangular waveform. From the result you can construct the Fourier spectrum. **71**



We just got: $f(t)=\frac{t_0}{T}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi t_0}{T}\cdot\cos\frac{n2\pi t}{T}$

Now calculate the coefficients a_n for $n=1$ up to $n=8$, for $T=2$ and $t_0=1$

$a_1=$ ……… $a_2=$ ………

$a_3=$ ……… $a_2=$ ………

$a_5=$ ……… $a_6=$ ………

$a_7=$ ……… $a_8=$ ………

Solution found -------------------▷ 73

Help wanted -------------------▷ 72

Given the rectangular waveform: 28

$$f(t) = \begin{cases} -1 & \textit{für} \quad -\pi < t < 0 \\ +1 & \textit{für} \quad 0 < t < \pi \end{cases}$$

It is a ……… function. In this case which coefficients vanish? ……

…………

------------------▷ 29

In our case the coefficients a_n are given by: 72

$$a_n = \frac{2}{n\pi} \cdot \sin \frac{n\pi t_0}{T}$$

We transform and insert the values: $T = 2$, $t_0 = 1$ to obtain:

$$a_n = \frac{2}{n\pi} \cdot \sin\left(n \cdot \frac{\pi}{2}\right)$$

Thus, we get rounded to two decimal places:

$a_1 = \ldots\ldots$	$a_2 = \ldots\ldots$
$a_3 = \ldots\ldots$	$a_2 = \ldots\ldots$
$a_5 = \ldots\ldots$	$a_6 = \ldots\ldots$
$a_7 = \ldots\ldots$	$a_8 = \ldots\ldots$

------------------▷ 73

29

Odd function

The coefficients a_n vanish.

Given again

$$f(t) = \begin{cases} -1 & \textit{für} \quad -\pi < t < 0 \\ +1 & \textit{für} \quad 0 < t < \pi \end{cases}$$

Please show that a_0 vanishes as well.

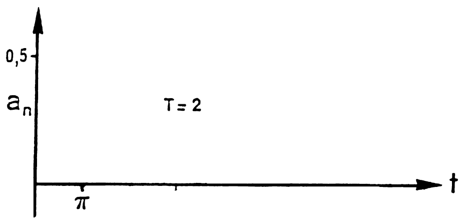
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt == \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 30

73

$a_1 = \ 0{,}64$ $\qquad a_2 = 0$
$a_3 = -0{,}21$ $\qquad a_2 = 0$
$a_5 = \ 0{,}13$ $\qquad a_6 = 0$
$a_7 = -0{,}09$ $\qquad a_8 = 0$

Sketch the Fourier spectrum of the rectangular function:



Hint:: $\omega_0 = \dfrac{2\pi}{T}, \quad \omega_n = \dfrac{n2\pi}{T} = n \cdot \pi$

------------------▷ 74

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot dt + \frac{1}{\pi}\int_{0}^{\pi}(1)\cdot dt = -1+1 = 0$$ 30

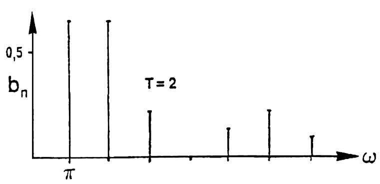
Compute now the coefficients b_n for the given rectangular waveform.

$$f(t) = \begin{cases} -1 & für \quad -\pi < t < 0 \\ +1 & für \quad 0 < t < \pi \end{cases}$$

Split the integral below into two segments.:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot \sin nt \quad dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 31



74

Again we regard the Fourierseries of the rectangular waveform:

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\frac{n\pi t_0}{T}\cdot\cos\frac{n2\pi t}{T}$$

Calculate the coefficients a_n given different values: $T = 8$ and $t_0 = 1$. Calculate again rounding to two decimal places:

$a_1 = \ldots\ldots$ $a_2 = \ldots\ldots$
$a_3 = \ldots\ldots$ $a_2 = \ldots\ldots$
$a_5 = \ldots\ldots$ $a_6 = \ldots\ldots$
$a_7 = \ldots\ldots$ $a_8 = \ldots\ldots$

I want further help ------------------▷ 75

Solution ------------------▷ 76

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot \sin nt \quad dt + \frac{1}{\pi}\int_{0}^{\pi}\sin nt \cdot dt$$ 31

Solve the integrals regarding the given limits:………………………….

-------------------▷ 32

The coefficients a_n are given by: 75

$$a_n = \frac{2}{n\pi}\cdot \sin\left(\frac{n\cdot\pi}{T}\cdot t_0\right)$$

But this time the value of T has changed. It is now $T = 8$. The period has increased significantly.
Insert and calculate again rounding to two decimal places starting with:

$\sin\frac{\pi}{8} = ...$ $\sin\frac{2\pi}{4} = ...$

$\sin\frac{3\pi}{8} = ...$ $\sin\frac{4\pi}{8} = ...$

$\sin\frac{5\pi}{8} = ...$ $\sin\frac{6\pi}{8} = ...$

$\sin\frac{7\pi}{8} = ...$ $\sin\frac{8\pi}{8} = ...$

Using these values we obtain finally:

$a_1 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_3 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_5 = \ldots\ldots$ $a_6 = \ldots\ldots$

$a_7 = \ldots\ldots$ $a_8 = \ldots\ldots$

-------------------▷ 76

$$b_n = \frac{1}{\pi}\left[(-1)\cdot\left(-\frac{\cos 0}{n}\right)-(-1)\cdot\left(-\frac{\cos(-n\pi)}{n}\right)\right]+\frac{1}{\pi}\left[\left(-\frac{\cos n\pi}{n}\right)-\left(-\frac{\cos 0}{n}\right)\right]$$ 32

$$= \frac{1}{n\pi}\left[1-\cos(-n\pi)+1-\cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[1-\cos n\pi\right]$$

Now you can obtain: $f(t) = \sum_{n=1}^{\infty}$ ……………………

------------------▷ 33

$\sin\frac{\pi}{8} = 0{,}38$ $\sin\frac{2\pi}{8} = 0{,}71$ 76

$\sin\frac{3\pi}{8} = 0{,}92$ $\sin\frac{4\pi}{8} = 1$

$\sin\frac{5\pi}{8} = 0{,}92$ $\sin\frac{6\pi}{8} = 0{,}71$

$\sin\frac{7\pi}{8} = 0{,}38$ $\sin\frac{8\pi}{8} = 0$

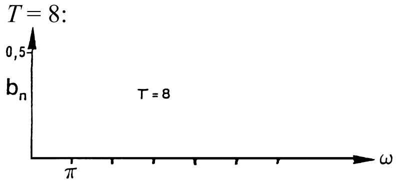
We obtain finally: $a_1 = 0{,}24$ $a_2 = 0{,}23$

$a_3 = 0{,}20$ $a_2 = 0{,}16$

$a_5 = 0{,}12$ $a_6 = 0{,}08$

$a_7 = 0{,}03$ $a_8 = 0$

Using these values you can sketch the Fourier spectrum of the new rectangular waveform with period $T = 8$:



Remember: $\omega_n = n\cdot\omega_0 = n\cdot\frac{2\pi}{8} = n\cdot\frac{\pi}{4}$

------------------▷ 77

33

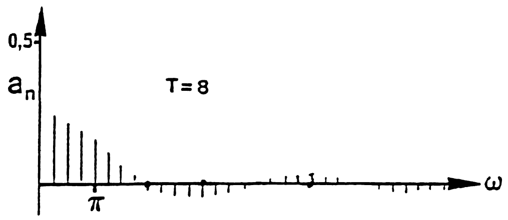
$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - \cos n\pi] \cdot \sin nx$$

Write down the first three terms of the series. Do not forget $\cos n\pi = (-1)^n$

$$f(t) \approx \frac{2}{\pi}[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

------------------▷ 34

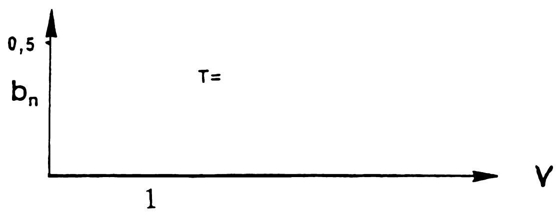
77



In theory we use most frequently the angular frequency ω. But you can as well use the frequency z $\nu = \frac{1}{T}$.

$\omega = 2\pi\nu \qquad \nu = \frac{\omega}{2\pi}$

Sketch the Fourier spectrum for the frequency:

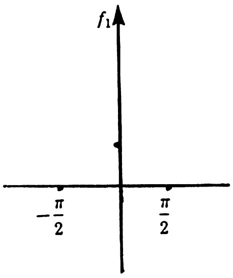


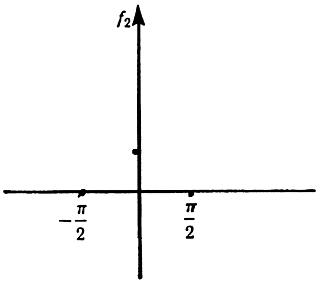
------------------▷ 78

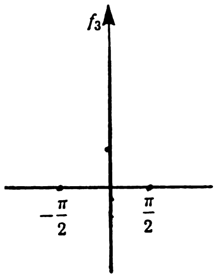
$$f(t) \approx \frac{2}{\pi}\left[\frac{2}{1}\sin t + \frac{2}{3}\sin 3t + \frac{2}{5}\sin 5t\right]$$

34

Sketch each of these three terms:



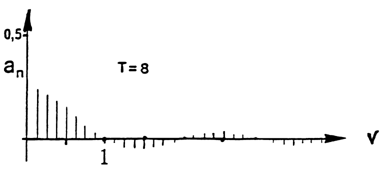




First term second term third term

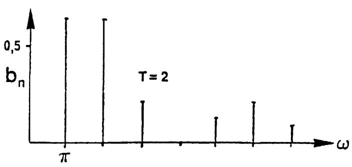
------------------▷ 35

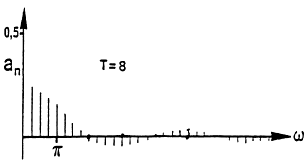
78



The Fourier spectrum is the same. Only the calibration of the *x*-axis has changed.

Now compare the Fourier spectrum of the same rectangular waveform for $t_0 = 1$ and two different periods $T = 2$ and $T = 8$:



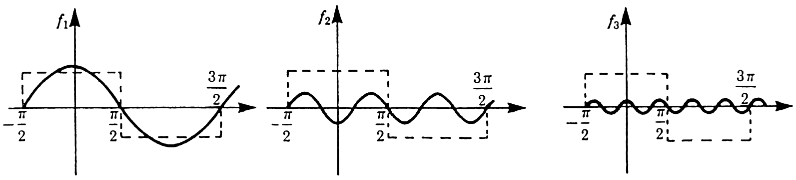


If the period *T* increases the distances between the components of the series decrease.

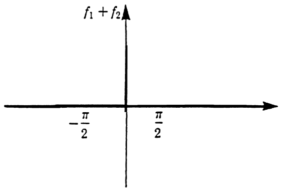
More exercises wanted ------------------▷ 79

Terminate this chapter ------------------▷ 86

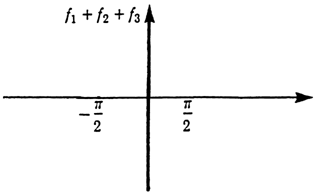
35



Try to sketch the sum of the first two terms $(f_1 + f_2)$



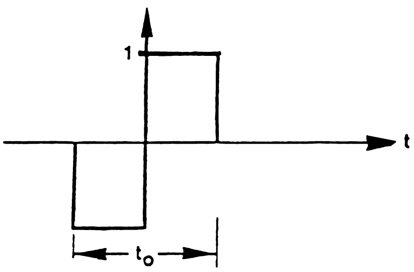
Now try to add the third term f_3.



-------------------▷ 36

Given an alternating rectangular waveform with a period t_0

79



$$f(t) = \left\{ \begin{array}{lll} -1 & \text{für} & -\frac{t_0}{2} < t < 0 \\ +1 & \text{für} & 0 < t < \frac{t_0}{2} \end{array} \right\}$$

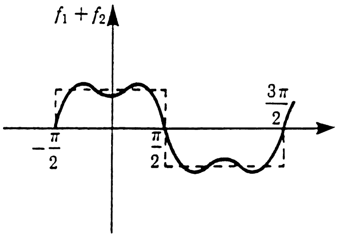
This function is ☐ even

☐ odd

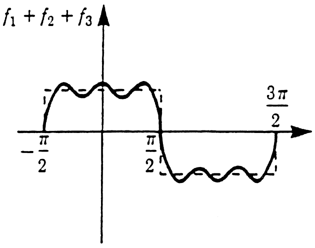
Thus the coefficients: ………………………. vanish.

-------------------▷ 80

Superposition of the first two terms: 36



Superposition of the first three terms:

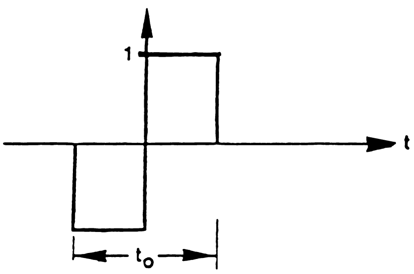


We observe the stepwise formation of the rectangular waveform.

------------------▷ 37

The function is odd. 80

Thus the coefficients a_n vanish.



Now calculate the remaining coefficients

$b_n = \dots\dots\dots\dots$

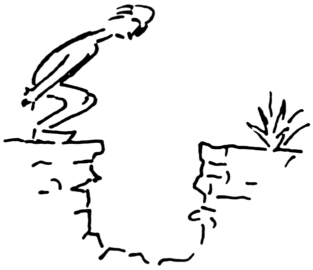
If you are unsure consult the formulae in the textbook.

Solution found ------------------▷ 83

Help needed ------------------▷ 81

If you followed the calculations in the textbook and in this study guide without difficulties you may skip the next exercise.

37



------------------▷ 56

If you felt you had difficulties you should do the next exercise because all transformations will be explained in detail.

------------------▷ 38

81

To calculate: $b_n = \frac{2}{t_0} \int\limits_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t) \sin\left(\frac{n2\pi}{t_0} \cdot t\right) dt$

We divide into two parts because $f(t)$ is defined for two parts:

$$f(t) = \begin{Bmatrix} -1 & \textit{für} & -\frac{t_0}{2} & < t < 0 \\ +1 & \textit{für} & 0 & < t < \frac{t_0}{2} \end{Bmatrix}$$

Thus we obtain:

$$b_n = \frac{2}{t_0} \int\limits_{-\frac{t_0}{2}}^{0} (-1) \sin\left(\frac{n2\pi}{t_0} t\right) dt + \frac{2}{t_0} \int\limits_{0}^{\frac{t_0}{2}} \sin\left(\frac{n2\pi}{t_0} t\right) dt$$

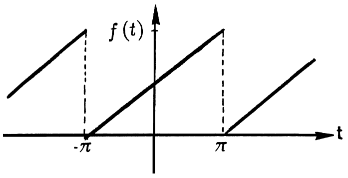
It remains to solve the integrals and to regard the limits of integration.

$b_n = \ldots\ldots\ldots\ldots\ldots\ldots..$

Solution found ------------------▷ 83

Stepwise solution ------------------▷ 82

In the following we will calculate an example in detail.
Given a sawtooth-like function shown below. 38



In this case we do not need to progress stepwise. The function is defined for the entire period by only one function.

$$f(t)=\left(\frac{t}{\pi}+1\right) \quad \textit{für} \quad -\pi<t<\pi$$

Calculation of a_0:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt=\frac{1}{\pi}\int_{-\pi}^{+\pi}\left(\frac{t}{\pi}+1\right)dt$$

There are only simple integrals to solve.

$$a_0=\frac{1}{\pi}\Big[\ldots\ldots\ldots\ldots\ldots\Big]_{-\pi}^{\pi}$$

------------------▷ 39

To calculate: 82

$$b_n=\frac{2}{t_0}\int_{-\frac{t_0}{2}}^{0}(-1)\sin\left(\frac{n2\pi}{t_0}t\right)dt+\frac{2}{t_0}\int_{0}^{\frac{t_0}{2}}\sin\left(\frac{n2\pi}{t_0}t\right)dt$$

Hint:

$$\int_{-\frac{t_0}{2}}^{0}\sin\left(\frac{n2\pi}{t_0}t\right)dt=\left[-\frac{t_0}{n2\pi}\cdot\cos\left(\frac{n2\pi}{t_0}t\right)\right]_{-\frac{t_0}{2}}^{0}$$

If you insert you will obtain:

$$b_n=\frac{2}{t_0}\left[(-1)\cdot\frac{t_0}{n2\pi}\left(-\cos\frac{n2\pi}{t_0}\cdot t\right)\right]_{-\frac{t_0}{2}}^{0}+\frac{2}{t_0}\left[\frac{t_0}{n2\pi}\left(-\cos\frac{n2\pi}{t_0}\cdot t\right)\right]_{0}^{\frac{t_0}{2}}$$

Now insert the limits, simplify, and obtain:

$b_n=$ …………………………..

------------------▷ 83

$$a_0 = \frac{1}{\pi}\left[\frac{t^2}{2\pi} + t\right]_{-\pi}^{\pi} = \frac{1}{\pi}[\pi - (-\pi)] = 2$$

39

The sawtooth shown here can be transformed into the sawtooth curve treated in the textbook. To obtain this you may shift the function in the direction of the y-axis.

Go to --------------------▷ 40

$$b_n = \frac{1}{n\pi}\left[1 - \cos\left(\frac{n2\pi}{t_0}\cdot\frac{t_0}{2}\right) + 1 - \cos\left(\frac{n2\pi}{t_0}\cdot\frac{t_0}{2}\right)\right] = \frac{2}{n\pi}[1 - \cos n\pi]$$

83

Since $\cos n\pi = (-1)^n$ we obtain:

$$b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

Now you can calculate:

$b_1 = \ldots\ldots\ldots\ldots$ $b_2 = \ldots\ldots\ldots\ldots$

$b_3 = \ldots\ldots\ldots\ldots$ $b_4 = \ldots\ldots\ldots\ldots$

$b_5 = \ldots\ldots\ldots\ldots$ $b_6 = \ldots\ldots\ldots\ldots$

$b_7 = \ldots\ldots\ldots\ldots$ $b_8 = \ldots\ldots\ldots\ldots$

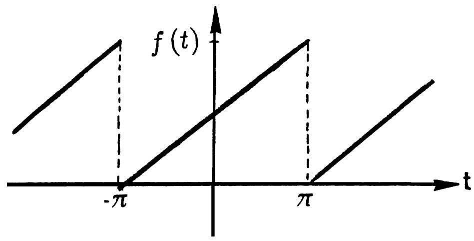
--------------------▷ 84

Calculation of a_n

40

Given the sawtooth curve:

$f(t) = \left(\frac{t}{\pi} + 1\right) \quad for \quad -\pi < t < \pi$



To compute:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\cos(nt)dt$$

We insert $f(t)$ into the integral and remember: The integral of a sum is the sum of the integrals.

Go to

-------------------▷ 41

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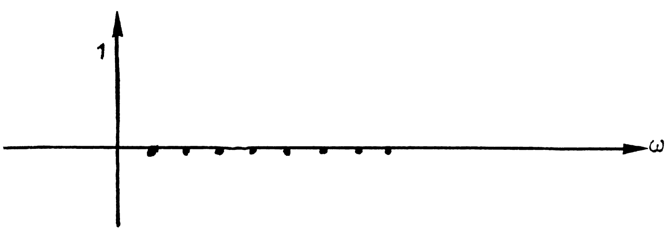
$b_1 = 1{,}3$	$b_2 = 0$
$b_3 = 0{,}42$	$b_4 = 0$
$b_5 = 0{,}25$	$b_6 = 0$
$b_7 = 0{,}18$	$b_8 = 0$

Sketch the Fourier spectrum for $t_0 = 1$.

Remember: $\omega_n = n \cdot \omega_0 = n \cdot \frac{2\pi}{t_0}$



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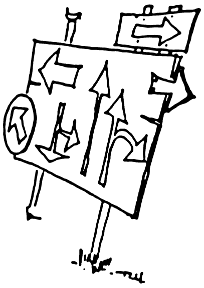
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{t}{\pi}+1\right)\cos(nt)dt = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\frac{t}{\pi}\cos nt \quad dt + \int_{-\pi}^{\pi}\cos nt \quad dt\right]$$

Integral 1 Integral 2

Both integrals in the brackets have to be solved. We start with integral 1. We can solve it using integration by parts

$$\int_{-\pi}^{\pi}\frac{t}{\pi}\cos nt dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found ▷ 45

Repetition of integration by parts ▷ 42

Detailed solution ▷ 44

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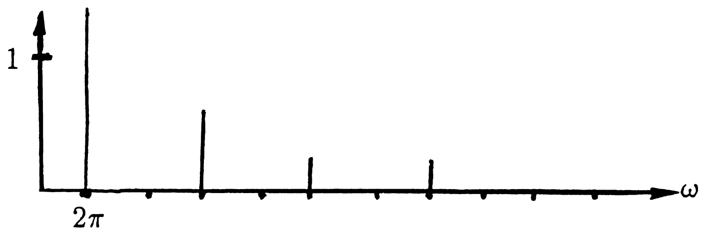


▷ 86

Integration by parts is often useful when products have to be integrated. This method is explained in the textbook section 6.5.4. Perhaps it may help to read that section again. Here is a short recapitulation. 42
Given two different functions $u(t)$ and $v(t)$. The limits of integration may be a and b. Then:

$$\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$$

I
In our case we have:

Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt dt$

We let : $u = \frac{t}{\pi} \quad v' = \cos nt$

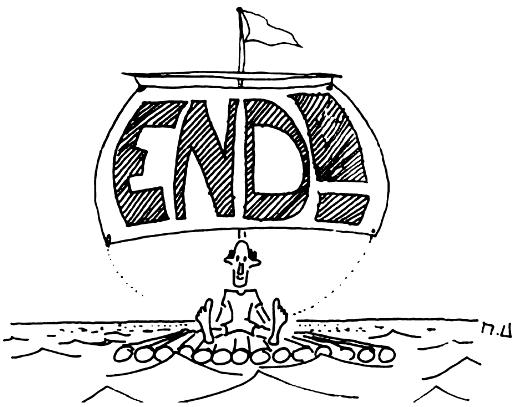
Thus we obtain:

$u' = \ldots\ldots\ldots\ldots\ldots\ldots$

$v = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 43

You have successfully finished the chapter Fourierseries. During this chapter you might have had some trouble with mere calculations and regarding limits of integration and regarding signs. But you overcame all difficulties. You should enjoy your success. Congratulations 86



End of chapter 18

$u' = \frac{1}{\pi} \qquad v = \frac{1}{n}\sin nt$ 43

We had from before:

$u = \frac{t}{\pi} \qquad v' = \cos nt$

If you are not sure you may verify this by differentiating. Having done this we insert our results into the formula of the integration by parts.

$$\int_{-\pi}^{\pi} u \cdot v' = [u \cdot v]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u' \cdot v$$

Thus the first integral can be solved:

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt \quad dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

-------------------▷ 44

Please continue on page 1
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