Chapter 5
Differential Calculus

--------------------▷ 2

$$x = 4y + 3$$
$$y = \frac{x-3}{4} = f^{-1}(x)$$

124

Given: $f(x) = 4x + 3$

Obtain the derivative of the inverse function $f^{-1}(x)$.

Use the rule given in the textbook and check the result by direct computation; differentiating a linear function is easy.

The inverse function reads

$$x = 4y + 3$$
$$y = \frac{x-3}{4}$$
$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 125

5.1 Sequences and Limits

2

Objective: Concepts of sequences of numbers, limit of a sequence of numbers, convergent sequence of numbers, divergent sequence of numbers.

Depending on your previous knowledge, now choose:
Concepts are known, go to

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Concepts are not known, or revision is needed:

READ: 5.1.1 The concept of sequence
5.1.2 Limit of a sequence
Textbook pages 87–90

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125

Applying the rule: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{4}$

Direct differentiation of the inverse function gives $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{4}$ too.

Given: $f(x) = y = x^3$
The inverse function $f^{-1}(x)$ reads $x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Find the derivative:

Step 1: Obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = \ldots\ldots\ldots\ldots\ldots\ldots$

Step 2: Express $\frac{\mathrm{d}x}{\mathrm{d}y}$ as a function of x; $\frac{\mathrm{d}x}{\mathrm{d}y} = \ldots\ldots\ldots\ldots\ldots\ldots$

Step 3: Insert into the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 126

3

The expression
$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$
is called ....
a_n is the ....

------------------▷ 4

126

$$x = y^3; \quad y = \sqrt[3]{x}; \quad \frac{\mathrm{d}x}{\mathrm{d}y} = 3y^2;$$
$$\frac{\mathrm{d}x}{\mathrm{d}y} = 3(\sqrt[3]{x})^2 = 3\sqrt[3]{x^2}; \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3\sqrt[3]{x^2}}$$

You can obtain the derivative of inverse functions as follows. Follow these steps with $f(x) = y = x^2 + 2 \qquad (x \geq 0)$

Step 1: Write down the inverse function in both forms

$x =$ .... $y =$ .... Obtain the derivative $\frac{\mathrm{d}x}{\mathrm{d}y} =$ ....

Step 2: Express the derivative $\frac{\mathrm{d}x}{\mathrm{d}y}$ as a function of x

$\frac{\mathrm{d}x}{\mathrm{d}y} =$ ....

Step 3: Insert into the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} =$ ....

------------------▷ 127

a sequence of numbers
general term

4

State the first five terms of the sequence of numbers, given that

$$a_n = \frac{(-1)^n}{(1+n)^2}$$

....

-------------------▷ 5

Step 1: $x = y^2 + 2 \qquad y = \sqrt{x-2}$

Step 2: $\dfrac{dx}{dy} = 2y = 2\sqrt{x-2}$

Step 3: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x-2}}$

127

The derivative of the inverse function is a subtle concept, but following the given steps is quite simple.

The necessity of this concept will become clearer later on when the derivatives of, for example, trigonometric and exponential functions are known. We suggest you reread the arguments given in the textbook carefully and try to understand fully the geometrical reasoning. It may prove to be wise to return to this section in the textbook where the derivatives of specific inverse functions are discussed.

-------------------▷ 128

$$-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, -\frac{1}{26}, \ldots$$

5

The above sequence of numbers is a
☐ convergent sequence

☐ divergent sequence
Has the sequence a limiting value?
☐ Yes

☐ No

If yes, which one? ....

-------------------▷ 6

5.6 Differentiation of Fundamental Functions (Part 1)

128

Objective: Differentiation of trigonometric functions and inverse trigonometric functions.

READ (and take notes): **5.5.3 Differentiation of fundamental functions:**
1. Trigonometric functions
2. Inverse trigonometric functions
Textbook pages 108–114

-------------------▷ 129

convergent sequence

6

Yes: $\lim\limits_{n\to\infty} \frac{(-1)^n}{1+n^2} = 0$

If the general term of a sequence of numbers tends to a fixed value as n grows beyond all bounds ($n \to \infty$) this value is called ....

Calculate

$$\lim_{n\to\infty} \frac{1}{n+10} = ...$$

--------------------▷ 7

First let us practise differentiating trigonometric functions.

129

Differentiate:

$$y = 3\sin x$$
$$y' = ...$$
$$y = 2\cos x$$
$$y' = ...$$

--------------------▷ 130

the limiting value
0

7

If you are experiencing difficulties with the concepts, then you are advised to study the textbook again.

When doing this, write down the concepts on a separate sheet of paper with short explanations. With the help of keywords try to reproduce the definitions aloud.

------------------▷ 8

$$y' = 3\cos x$$
$$y' = -2\sin x$$

130

Obtain the derivative of the tangent function using the quotient rule.

Remember that $\sin^2 x + \cos^2 x = 1$.

$$y = \tan x = \frac{\sin x}{\cos x}$$
$$y' = \text{............................}$$

------------------▷ 131

Calculate 8

$$\lim_{n\to\infty} \frac{1}{n} = \dots\dots\dots\dots\dots\dots$$

$$\lim_{n\to\infty} \frac{2}{n} + 3 = \dots\dots\dots\dots\dots\dots$$

→ 9

131

$$y' = \frac{1}{\cos^2 x}$$

As a help, if needed, here is the detailed working:

$$y = \frac{\sin x}{\cos x}$$

$$y' = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Another way to represent y' is:

$$y' = 1 + \tan^2 x$$

Did you obtain the correct result?

☐ Yes → 134

☐ No → 132

$$\lim_{n\to\infty} \frac{1}{n} = 0$$
$$\lim_{n\to\infty} \frac{2}{n} + 3 = 3$$

9

Calculate the following three limits:

$$(1)\ \lim_{n\to\infty} \frac{1}{\sqrt{n}} = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$(2)\ \lim_{n\to\infty} \left(3 + \frac{1}{n^2}\right) = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$(3)\ \lim_{n\to\infty} \frac{(-1)^n}{2^n} = \ldots\ldots\ldots\ldots\ldots\ldots$$

- - - - - - - - - -▷ 10

Given:

$$y = \sin x \qquad y' = \cos x$$
$$y = \cos x \qquad y' = -\sin x$$

132

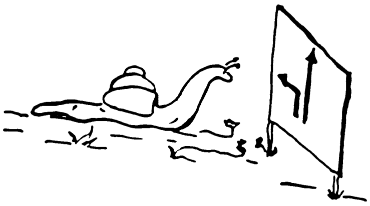
Obtain the derivative of the cotangent function by using the quotient rule.

$$y = \cot x = \frac{\cos x}{\sin x} \qquad (\textit{Hint:}\ \sin^2 x + \cos^2 x = 1)$$
$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

- - - - - - - - - -▷ 133

0
3
0

10



No errors ----> 14

Errors, or further explanation on calculating limits required ----> 11

$$y' = -\frac{1}{\sin^2 x} = -1 - \cot^2 x$$

133

The detailed working was:

$$y = \frac{\cos x}{\sin x}$$

$$y' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

Another way of representing y' is:

$$y' = -\cot^2 x - 1$$

----> 134

11

One method that is successful in many cases for the determination of limits is as follows:

We try to transform the numerator and denominator of the expression so that we obtain integral powers of $\frac{1}{n}$.

The reason is clear; we know that for $\frac{1}{n}$ the limit vanishes because the denominator grows larger and larger. All the terms disappear in the limit. The limiting value of the expression follows from the remainder.

Example:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}$$

By a simple transformation we have

$$\lim_{n\to\infty}\sqrt{\frac{1}{n}}=\sqrt{0}=0$$

Try the following for yourself:

$$\lim_{n\to\infty}\frac{n}{3+n}=\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

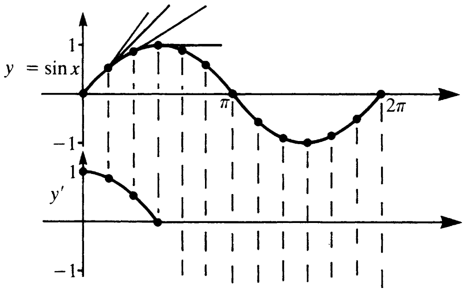
------------------▷ 12

134

Differentiation of trigonometric functions requires knowing that:

$$y=\sin(ax) \qquad y'=a\cos(ax)$$
$$y=\cos(ax) \qquad y'=-a\sin(ax)$$

We should also understand the geometrical meaning. Hence, referring to the figure which shows the function sin x, we draw the slope at four points.



Now plot on the second diagram the values of the slopes of the tangent for each point.

------------------▷ 135

1

12

Let us calculate step by step:

$$\lim_{n\to\infty} \frac{n}{3+n} = ?$$

We transform the expression in order to obtain terms involving $\frac{1}{n}$; thus:

$$\lim_{n\to\infty} \frac{n}{3+n} = \lim_{n\to\infty} \left(\frac{n}{n} \times \frac{1}{\frac{3}{n}+1}\right) = \lim_{n\to\infty} \frac{1}{\frac{3}{n}+1} = \frac{1}{0+1} = 1$$

What is $\lim_{n\to\infty} \frac{n+2}{n+4} = ?$............................

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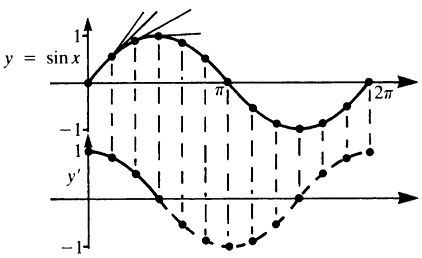
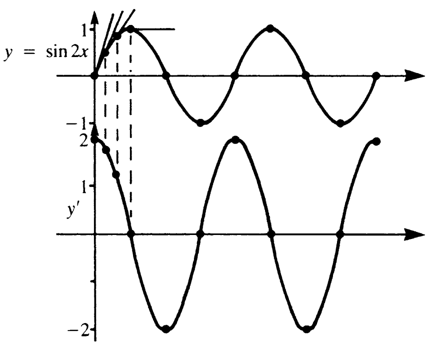


Notice that we obtain the cosine function

135

$$y' = \cos x$$



In a similar way we consider the slope of the function $y = \sin(2x)$. The tangents are steeper. In fact all slopes are doubled. This geometrical approach shows clearly that

$$y' = 2\cos(2x)$$

Differentiate:

$y = 7\sin(cx)$ $\qquad$ $y = \frac{1}{5}\cos(6x)$

$y' =$ .... $\qquad$ $y' =$ ....

-------------------▷ 136

1

13

Another sequence of numbers whose limiting value is 0 is:

$$a_n = \frac{1}{2^n}$$

Here the denominator grows beyond all bounds as $n \to \infty$, and the term vanishes.
Generally:

$$\lim_{n\to\infty} \frac{1}{c^n} = 0, \quad \text{if} \quad c > 1$$

Calculate the limiting value of

$$\lim_{n\to\infty} 2\frac{3^n}{3+3^n} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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$$y' = 7c\cos(cx)$$
$$y' = -\frac{6}{5}\sin(6x)$$

136

All correct

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If you made some mistakes differentiate the following:

$$y_1 = 3\sin\left(\tfrac{1}{3}x\right)$$
$$y_2 = 4\cos(2x)$$

with the aid of the textbook.

Then go to

-------------------▷ 137

2

14

Determine

$$\lim_{n\to\infty}\frac{3n^2-2}{2n+n^2}$$

The result is ....

--------------------▷ 17

I need further help

--------------------▷ 15

$$y_1' = \cos\left(\tfrac{1}{3}x\right)$$
$$y_2' = -8\sin(2x)$$

137

You might still have some difficulties.

If we continue now without overcoming them we will not save time, but create more difficulties for you in the future. We must, therefore, isolate them before going any further.

Obtain the derivatives of the following functions:

(1)	power rule	$y = 4x^3$	$y' = $ ....
(2)	power or quotient rule	$y = \dfrac{1}{2x}$	$y' = $ ....
(3)	power rule	$y = 3x^{-1/2} + x^{1/2}$	$y' = $ ....
(4)	chain rule	$y = 7\sin(ax)$	$y' = $ ....

--------------------▷ 138

The exercise was $\lim\limits_{n\to\infty} \dfrac{3n^2-2}{2n+n^2}$ 15

We try to transform the numerator and denominator so that we obtain integral powers of $\frac{1}{n}$. We factorise out the highest power of n; here it is n^2, so that we have

$$\lim_{n\to\infty} \frac{n^2\left(\ldots\ldots\right)}{n^2\left(\ldots\ldots\right)}$$

Fill in the brackets!

--------------------▷ 16

138

(1) $y' = 3(4x^2)$

(2) $y' = -\dfrac{1}{2x^2}$

(3) $y' = -\frac{3}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}$

(4) $y' = 7a\cos(ax)$

You will find further exercises in the textbook, at the end of each chapter.
You know by now that exercises are useful if they seem difficult to you!

--------------------▷ 139

$$\lim_{n\to\infty} \frac{n^2\left(3-\dfrac{2}{n^2}\right)}{n^2\left(\dfrac{2}{n}+1\right)}$$ 16

If you have had difficulties, convince yourself of the correctness of the expression by carrying out the multiplications.

The expression now becomes:

$$\lim_{n\to\infty} \frac{3-\dfrac{2}{n^2}}{\dfrac{2}{n}+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Further hint: Determine the limiting values for the numerator and denominator separately. You should be able to do this!

-------------------▷ 17

An alternating current is given by the expression 139

$$I = I_0 \sin(\omega t + \phi)$$

I_0, ω and ϕ are constants.

Obtain the derivative with respect to the time t

$$\frac{dI}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 140

3

17

The principle behind the determination of limiting values is often as has just been described. The expression is transformed so that we obtain terms which we know to disappear in the passage to the limit.

Terms such as

$$\frac{1}{n},\ \frac{1}{\sqrt{n}},\ c^{-n},\ \frac{1}{2^n},\ \text{and more}$$

can be treated in this way.

You will find more exercises in the textbook. You should do the exercises until you have no difficulty with their solution.

▷ 18

140

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I_0 \omega \cos(\omega t + \phi)$$

All correct ▷ 142

I need further explanation ▷ 141

Test yourself! 18

The expression $a_1, a_2, \ldots, a_n, a_{n+1}$ is called ....

a_n is the ....

$$\lim_{n\to\infty} \frac{n^2+2n+1}{5n^2+1} = \text{......................................}$$

$$\lim_{n\to\infty} \frac{1}{2+2^{-n}} = \text{......................................}$$

Which sequence is divergent?

☐ $a_n = \dfrac{(-1)^n}{n^2}$

☐ $b_n = n(n-1)^n$

-------------------▷ 19

You were asked to differentiate with respect to time t: 141

$$I = I_0 \sin(\omega t + \phi)$$

Let us first rename the variables and constants so that:

$$\begin{aligned} t &= x \quad \omega = a \\ I &= y \quad \phi = c \\ I_0 &= y_0 \end{aligned}$$

Hence $y = y_0 \sin(ax+c)$

Now we apply the chain rule $g = (ax+c)$:

$$y = y_0 \sin g$$

$$\frac{\mathrm{d}y}{\mathrm{d}g} = y_0 \cos g \qquad \frac{\mathrm{d}g}{\mathrm{d}x} = a$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} = y_0 \cos(ax+c)a$$

It follows that with the original notation:

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \omega I_0 \cos(\omega t + \phi)$$

-------------------▷ 142

19

a sequence of numbers
$a_n =$ the general term

$\frac{1}{5}$
$\frac{1}{2}$
a_n : convergent sequence of numbers
b_n : divergent sequence of numbers

If you still have doubts, try the appropriate exercises at the end of Chapter 5 of the textbook.

--------------------▷ 20

142

Differentiation of the inverse trigonometric functions.

$$y = A\sin^{-1}(x) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \cos^{-1}(ax) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \tan^{-1}(ax) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 143

5.2 Limit of a Function, Continuity

20

Objective: Concepts of limit of a function, continuity.

Concepts are known

--------------------▷ 34

Concepts are new:

READ: 5.1.3 Limit of a function
5.1.4 Examples for the practical determination of limits
5.2 Continuity
Textbook pages 91–93

--------------------▷ 21

143

$$y' = \frac{A}{\sqrt{1-x^2}}$$
$$y' = \frac{-a}{\sqrt{1-a^2x^2}}$$
$$y' = \frac{a}{1+a^2x^2}$$

The proof is analogous to that in the textbook.
Use it to obtain the derivative of the inverse tangent function.

$y = \tan^{-1}(x) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

You found the solution

--------------------▷ 146

You need help

--------------------▷ 144

The concept of the limit of a sequence of numbers can be extended to functions. 21

We consider limits of functions for $x \to \infty$.

We also consider limits for $x \to x_0$.

This means that the limits are calculated for a fixed value of x, e.g. $x_0 = 0$.

We indicate this under the lim sign as follows: $\lim\limits_{x \to x_0}$

--------------------▷ 22

To differentiate 144

$$y = \tan^{-1}(x)$$

Step 1: $x = \tan y$

hence

$$\frac{dx}{dy} = \frac{1}{\cos^2 y} = 1 + \tan^2 y$$

Step 2: Express $\frac{dx}{dy}$ as a function of x

$$\frac{dx}{dy} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Step 3: Apply the formula $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 146

If you are in doubt try to express $\cos^2 y$ as a function of x using

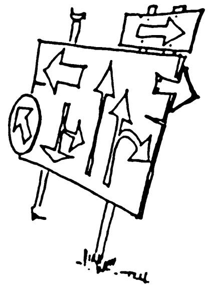
$$x = \frac{\sin y}{\cos y} = \tan y$$

--------------------▷ 145

Given that $f(x) = \frac{1}{x^2}$,
it is required to calculate the limiting value

22

$$\lim_{x \to 2} = \frac{1}{x^2}$$



☐ 0 → 23

☐ $\frac{1}{2}$ → 24

☐ $\frac{1}{4}$ → 25

☐ ∞ → 26

We know that $\frac{dx}{dy} = \frac{1}{\cos^2 y} = 1 + \tan^2 y$

145

Since $x = \tan y$

$$x^2 + 1 = \tan^2 y + 1$$

Inserting into the formula $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ yields

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

→ 146

Wrong, unfortunately! **23**

The limiting value of the function as $x \to 2$ was required; this means the value of the function at $x_0 = 2$.

Your error was probably that you calculated the limiting value for $x \to \infty$.

From now on we have to be very careful and be certain of the value of x_0 the limit is required for. This is clearly stated under the lim sign.

Calculate again:

$$\lim_{x \to 2} \frac{1}{x^2} =$$

☐ $\frac{1}{2}$ --------------------▷ 24

☐ $\frac{1}{4}$ --------------------▷ 25

☐ ∞ --------------------▷ 26

$$\frac{dx}{dy} = 1 + x^2, \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{1 + x^2}$$ **146**

Now for a general revision.

In the following examples the more usual notations have been changed; all of them are simple exercises. If you have any doubts replace the unfamiliar symbols by the known ones, i.e. x and y.

$$U = V^2 \qquad U' = \ldots\ldots\ldots$$

$$E_{\text{kin}} = \frac{m}{2} v^2 \qquad \frac{d}{dv}(E_{\text{kin}}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$s = \frac{g}{2} t^2 \qquad \frac{ds}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$p = \rho h \qquad \frac{dp}{dh} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$A = A_0 \sin(\omega t) \qquad \frac{dA}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$S = A_0 \cos(\omega t) \qquad \frac{ds}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 147

There is an error. 24

It is possible that you have calculated the value of $\lim\limits_{x\to 2}\dfrac{1}{x}$.

But the given function is $f(x) = \dfrac{1}{x^2}$.

Calculate again:

$\lim\limits_{x\to 2}\dfrac{1}{x^2} =$

☐ 0 → 23

☐ $\dfrac{1}{4}$ → 25

☐ ∞ → 26

147

$U' = 2V$

$\dfrac{\mathrm{d}}{\mathrm{d}v}(E_{\mathrm{kin}}) = mv$

$\dfrac{\mathrm{d}s}{\mathrm{d}t} = gt$

$\dfrac{\mathrm{d}p}{\mathrm{d}h} = \rho$

$\dfrac{\mathrm{d}A}{\mathrm{d}t} = A_0\omega\cos(\omega t)$

$\dfrac{\mathrm{d}s}{\mathrm{d}t} = -A_0\omega\sin(\omega t)$

→ 148

Correct! 25

We must always pay attention to the value of x_0 for which the limit of $f(x)$ is required.

-------------------▷ 27

5.7 Differentiation of Fundamental Functions (Part 2)

148

Objective: Differentiation of the exponential function, the logarithmic function and the hyperbolic functions.

READ: 5.5.3 Differentiation of fundamental functions
Exponential and logarithmic functions
Hyperbolic functions
Textbook pages 108–114

-------------------▷ 149

Wrong, unfortunately! 26

This result can only be obtained when calculating the limit for $x \rightarrow 0$. But the required limit was for $x \rightarrow 2$; this means that we wish to calculate the value of the function at $x_0 = 2$.

Try again

$\lim\limits_{x \rightarrow 2} \frac{1}{x^2} =$

☐ 0 → 23

☐ $\frac{1}{2}$ → 24

☐ $\frac{1}{4}$ → 25

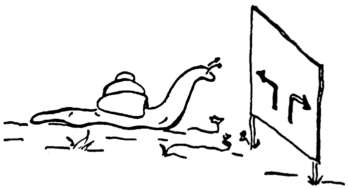
Obtain the derivatives: 149

$$y = 3e^x \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = e^{2x} \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = 2\ln x \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \ln(3x) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$

→ 150

Calculate the following limit: 27

$$\lim_{x\to 0} \frac{x^2+6x}{2x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found 29

Help is required 28

150

$$y' = 3e^x \qquad y' = 2e^{2x}$$

$$y' = \frac{2}{x} \qquad y' = \frac{3}{3x} = \frac{1}{x}$$

The charge of a capacitor is given by

$$Q = Q_0 e^{-\frac{t}{RC}}$$

where R = resistance
C = capacitance
Q = charge

The current is given by $\frac{dQ}{dt} = \dot{Q}$

$$\frac{dQ}{dt} = \ldots\ldots\ldots\ldots\ldots$$

151

Wanted: $\lim\limits_{x\to o} \frac{x^2+6x}{2x}$ 28

In this expression both the numerator and the denominator approach the value zero as $x \to 0$.

This gives the indeterminate expression $\frac{0}{0}$.

We have to try to transform the expression in order to obtain another expression which has a determinate form. One way, in this case, is to factorise out x from the numerator and denominator. The remainder will then be determinate.

$$\lim_{x\to 0} \frac{x^2+6x}{2x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 29

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{Q_0}{RC}\,\mathrm{e}^{-\frac{t}{RC}}$$ 151

Obtain the derivatives of the hyperbolic functions

$y = A\cosh(ax) \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = 3\tanh(2x) \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 152

$$\lim_{x \to 0} \frac{x^2 + 6x}{2x} = 3$$ 29

All correct 31

Detailed explanation required 30

$$y' = Aa \sinh(ax)$$
$$y' = 6(1 - \tanh^2 2x)$$ 152

Obtain the derivative of the hyperbolic cosine function:

$$y = \cosh x = \tfrac{1}{2}(e^x + e^{-x})$$
$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

153

It was required to obtain 30

$$\lim_{x \to 0} \frac{x^2+6x}{2x}$$

In this expression both the numerator and denominator approach zero as $x \to 0$, resulting in an indeterminate form. We therefore factorise out x so that the expression becomes

$$\frac{x^2+6x}{2x} = \frac{x(x+6)}{2x} = \frac{x+6}{2} = \frac{x}{2}+3$$

We know that $\lim\limits_{x \to 0} \frac{x}{2} = 0$,

consequently $\lim\limits_{x \to 0} \frac{x^2+6x}{2x} = \lim\limits_{x \to 0} \left(\frac{x}{2}+3\right) = 3$

--------------------▷ 31

$$y' = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$

153

The derivatives of the inverse hyperbolic functions have a special significance in the integral calculus. We shall not, however, derive them here. But for the reader who has a particular interest we shall prove how to obtain the derivative of the inverse hyperbolic tangent.

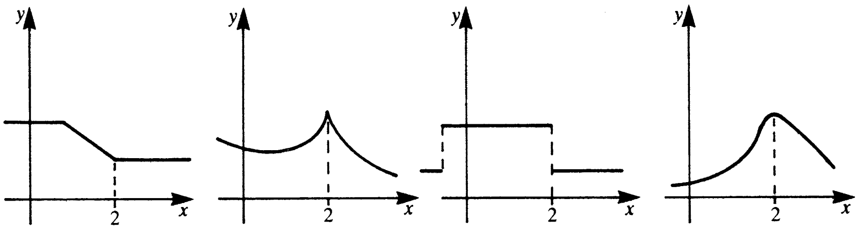
Do you wish to omit the proof?

--------------------▷ 157

Would you like to go through the proof?

--------------------▷ 154

Which of the following functions are not continuous at $x = 2$? 31



☐ A ☐ B ☐ C ☐ D

--------------------▷ 32

The proof follows exactly that of the example in the textbook. 154

To differentiate:

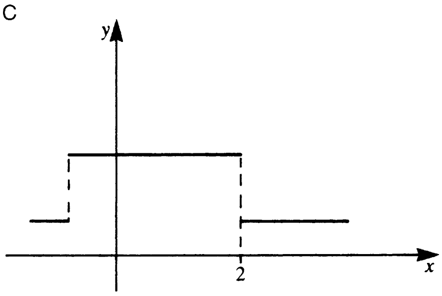
$$y = \tanh^{-1}\ x$$

Step 1: Obtain

$$x = \dots\dots\dots\dots \qquad \frac{dx}{dy} = \dots\dots\dots\dots$$

--------------------▷ 155

C



This function is discontinuous where $x = 2$.

32

At a point of discontinuity the function 'jumps'. The limit of the function when approached from the left is, in the given case, different from the limit when approached from the right.

Is it allowable for a continuous function to possess a sharp point or cusp?

☐ Yes
☐ No

-------------------▷ 33

$$x = \tanh y \quad \frac{dx}{dy} = 1 - \tanh^2 y$$

155

Step 2: Substitute for $\tanh^2 y$. $\frac{dx}{dy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Can you carry on?
Here is a little more help:

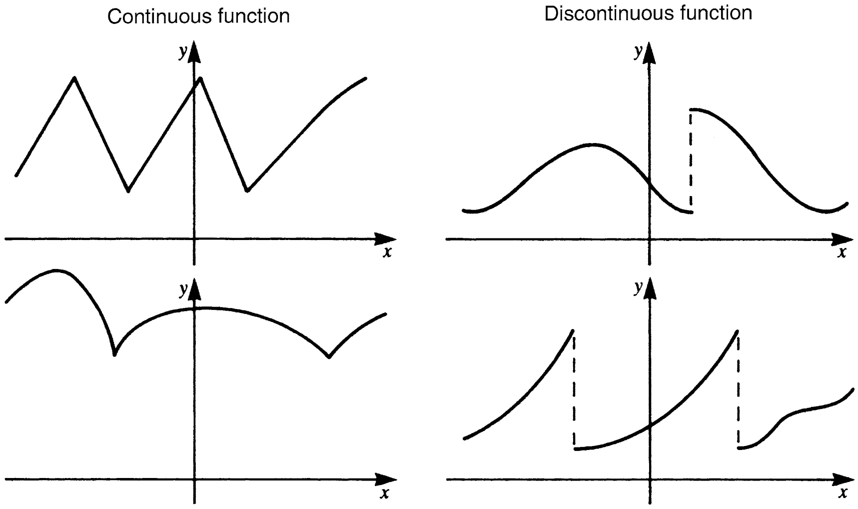
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 156

Yes

33



34

Since $$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

156

and $$\frac{dx}{dy} = 1 - x^2$$

the result is: $$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

157

We now have a short test to see if you have understood the subject of limits. 34

$$\lim_{x\to 0}\frac{x^2+1}{x-1} = \ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to 2}\left(\frac{1}{x}\right) = \ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to 0}\frac{x^2+10x}{2x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to\infty} e^{-x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 35

Now you should be able to solve the exercises in the textbook without too much difficulty. 157

It is important to be able to apply the product rule, the quotient rule and the chain rule with confidence.

Decide for yourself how many exercises you want to do. You may wish to solve the odd-numbered ones first; if you do not experience any difficulty you may assume that you have understood the subject matter.

------------------▷ 158

35

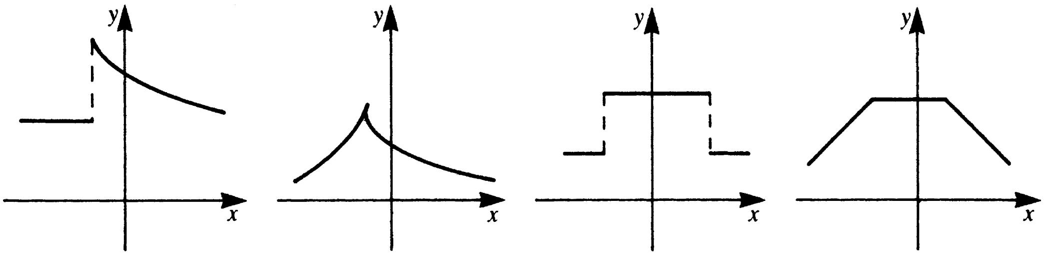
-1

$\frac{1}{2}$

5

0

Which functions are continuous?



☐ A ☐ B ☐ C ☐ D

---------------▷ 36

158

5.8 Higher Derivatives

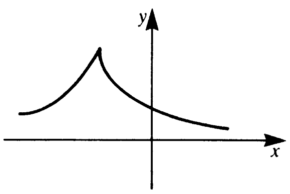
Objective: Generalisation of the concept of differentiation, second derivative, formation of higher derivatives.

READ: 5.6 Higher derivatives
Textbook pages 114–115

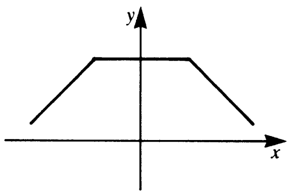
---------------▷ 159

Continuous functions: 36

B



D



Have you made errors or experienced difficulties?

No difficulties, answers correct --------------------▷ 37

Yes, I experienced difficulties. In this case it is advisable to consult the textbook.

READ: 5.1.3 Limit of a function
5.1.4 Examples for the practical determition of limits
5.2 Continuity
Textbook pages 91–93

Then try the exercises in this study guide again. --------------------▷ 21

In the following examples we have changed the notation. In one of them the second derivative is required; in the last the fourth derivative is required. 159

$g(\phi) = a\sin\phi + \tan\phi$ $\quad g'(\phi) = \ldots\ldots\ldots\ldots\ldots\ldots$

$v(u) = u^2 e^u$ $\quad v'(u) = \ldots\ldots\ldots\ldots\ldots\ldots$

$f(x) = \ln x$ $\quad f''(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

$h(x) = x^5 + 2x^2$ $\quad h^{(4)}(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 160

As you progress through this programmed text there are two important aspects concerning working periods and breaks: 37

(i) fixing the end of the break, and
(ii) ....

-------------------▷ 38

$$g'(\phi) = a\cos\phi + \frac{1}{\cos^2\phi}$$
$$v'(u) = e^u(2u + u^2)$$
$$f''(x) = -\frac{1}{x^2}$$
$$h^{(4)}(x) = 120x$$
160

All correct -------------------▷ 163

Errors or difficulties -------------------▷ 161

observing the end of the break 38

In general, the start of a break accords with one's inclination.

The end of a break does not always accord with one's inclination.

--------------------▷ 39

Higher derivatives are calculated as follows: 161

Consider the function $y = \ln x$

The first derivative $y' = \dfrac{1}{x}$

The second derivative is obtained by differentiating once more the first derivative with respect to x. Hence

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x} y'(x) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

Similarly for higher derivatives. They are obtained by differentiating successively. Thus the fourth derivative of

$$\begin{aligned}
h(x) &= x^5 + 2x^2 : \\
h'(x) &= 5x^4 + 4x \\
h''(x) &= 20x^3 + 4 \\
h'''(x) &= 60x^2 \\
h^{(4)}(x) &= 120x
\end{aligned}$$

--------------------▷ 162

Success in learning depends very much on attentiveness and concentration on the subject matter. 39

Concentration depends on many factors:
Tiredness in which case a break or sleep is needed.
Interest in the subject.
Attitude towards study.
Timing of working periods.
Disturbances (noise, interruptions).
Physical and *psychological* state.

The influence of these factors on concentration is obvious and can be demonstrated experimentally, but we can modify their effects within certain limits.
Therefore, let us examine these effects.

-------------------▷ 40

Proficiency in differentiating requires practice. If you are still experiencing difficulties in solving the exercises it is a sign that you have not yet mastered the subject matter. It is therefore necessary for you to practise with more exercises. 162

You will find them at the end of the chapter in the textbook.

-------------------▷ 163

Tiredness can be counteracted by a limited break after a defined working period. 40

Interest in the subject normally increases with progress but decreases with failure. Thus giving correct answers tends to increase your interest.

Timing: For one working period, limit your target; e.g. one section in the textbook. Do not try to take too big a bite at a time.

Do not always regard a *disturbance* as a welcome distraction. The learner who is concentrating on a piece of work has the right to scare off intruders in a friendly but determined way.

------------------▷ 41

There now follows a short recapitulation. 163

Differentiate the function of a function:

$$y(x) = f(g(x)) = \sqrt{2x^3+5}$$

To do this we apply the *chain rule*.

Recall it, please!

$$y = f(g(x)) \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 164

From the foregoing it follows that: 41

Absorbing, digesting and memorising subject matter are very much dependent on attention and concentration.

If you have problems with your concentration, then you must first trace the disturbances which affect your learning process and result in weakness in concentration, then try to discover the causes of them and take action accordingly.

Often a change in the working plan is beneficial.

-------------------▷ 42

$$y' = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} \qquad \text{or} \qquad \frac{\mathrm{d}f}{\mathrm{d}g}g'(x)$$ 164

To differentiate $y(x) = \sqrt{2x^3+5}$

let $g(x) = 2x^3+5$

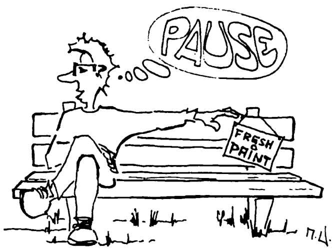
so that $f(g) = \sqrt{g}$

Obtain the derivative

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 165

Now a break is advisable! 42



43

$$y'(x) = \frac{3x^2}{\sqrt{2x^3+5}}$$ 165

All correct 167

Different result, or difficulties 166

5.3 Series

43

Objective: Concepts of series, geometric series, finite series, summation sign.

READ: **5.3 Series**
5.3.1 Geometric series
Textbook pages 94–96

--------------------▷ 44

The function $y = f(g(x)) = \sqrt{2x^3+5}$ is made up of

166

$$g(x) = 2x^3 + 5$$
$$f(g) = \sqrt{g}$$

To use the chain rule we have to multiply $f'(g) = \frac{\mathrm{d}f}{\mathrm{d}g}$ ('outer derivative') with $g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x}$ ('inner derivative').

Consequently we obtain the following derivatives:

$$g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2x^3+5) = 6x^2$$
$$f'(g) = \frac{\mathrm{d}f}{\mathrm{d}g} = \frac{\mathrm{d}}{\mathrm{d}g}(\sqrt{g}) = \frac{1}{2\sqrt{g}}$$

The chain rule gives: $y' = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$

Substituting leads to $y'(x) = \frac{1}{2\sqrt{g}}6x^2 = \frac{3x^2}{\sqrt{2x^3+5}}$

--------------------▷ 167

The infinite series: $1+4+9+16+$ .... is abbreviated to 44

Example of a geometric series: ....
....

-------------------▷ 45

Obtain the following derivatives: 167

$y = (3x^2+2)^2$ $y' =$ ....

$y = a\sin(bx+c)$ $y' =$ ....

$y = \mathrm{e}^{2x^3-4}$ $y' =$ ....

-------------------▷ 168

$\sum\limits_{n=1}^{\infty} n^2$ or, using a different variable, by $\sum\limits_{j=1}^{\infty} j^2$ 45

$$a + aq + aq^2 + \ldots + aq^{r-1} \text{ or}$$

$$1 + x + x^2 + x^3 + \ldots + x^n$$

Given the *sequence* of odd numbers:

$$1,\ 3,\ 5,\ 7, \ldots,\ 19$$

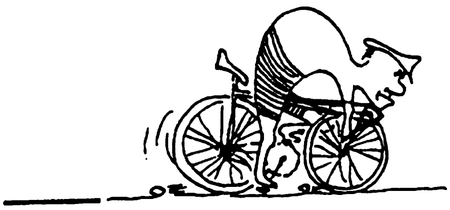
write down the corresponding *series*.

. . . .

--------------------▷ 46

$$y' = 12x(3x^2 + 2)$$
$$y' = ab\cos(bx + c)$$
$$y' = 6x^2 e^{2x^3 - 4}$$

168



--------------------▷ 169

$$1 + 3 + 5 + 7 + \ldots + 19$$

46

Let the sum of this series be s_r, then

$$s_r = 1 + 3 + 5 + 7 + \ldots + 19$$

Express this series using the summation sign!

$$s_r = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Instead of n we could take v, for example, as the variable.

We have to learn to manage expressions with different symbols.

-------------------▷ 47

You will frequently have to use the

169

— product rule
— quotient rule
— chain rule

Now do the exercises in the textbook corresponding to section 5.6.
For the solutions of the exercises use the table of derivatives at the end of chapter 5 in the textbook.

-------------------▷ 170

$$s_r = \sum_{v=0}^{9}(2v+1) = \sum_{v=1}^{10}(2v-1)$$ 47

Completely correct → 51

Error in specifying the bounds → 49

Error in obtaining an expression for the general term → 48

There are two ways of plotting a curve defined by $f(x)$: 170

(1) We tabulate many values of the function. This was explained in Chapter 1. The procedure is laborious and time consuming.

(2) We look for *particular features* of the curve in order to obtain a picture of the *quantitive nature* of the graph.

The second method is most important; it enables us to get a quick picture of the general character of the function. It is usually referred to as curve sketching.

Section 5.7 of the text book deals with the determination of these particular features.

→ 171

First let us consider the general term of a very simple sequence, the sequence of positive even numbers: 48

$$2,\ 4,\ 6,\ 8, \dots,\ 20$$

The factor of the general term of this sequence is

$$a = 2$$

If we use the same variable n as in the textbook we obtain

$$a_n = 2n$$

--------------------▷ 49

5.9 Extreme Values and Points of Inflexion; Curve Sketching 171

READ: **5.7.1 Maximum and minimum values of a function**
5.7.2 Further remarks on points of inflexion (contraflexure)
5.7.3 Curve sketching
Textbook pages 115–123

--------------------▷ 172

The given series was $1+3+5+7+\ldots+19$ 49

The series should be expressed using the summation sign. Let us take v as the variable.

(i) **Solution:** The general term is $a_v = 2v-1$.

$$\begin{aligned} \text{Proof: for } v=1, &\quad a_1 = 1 \\ \text{for } v=2, &\quad a_2 = 3 \\ &\vdots \\ \text{for } v=10, &\quad a_{10} = 19 \end{aligned}$$

In this case the bounds are $v=1$ to $v=10$, hence $s_r = \sum\limits_{v=1}^{10}(2v-1)$

(ii) **Alternative solution:** The general term is $a_v = 2v+1$.

$$\begin{aligned} \text{Proof: for } v=1, &\quad a_1 = 1 \\ \text{for } v=2, &\quad a_2 = 3 \\ &\vdots \\ \text{for } v=10, &\quad a_{10} = 19 \end{aligned}$$

In this case the bounds are $v=0$ to $v=9$, hence $s_r = \sum\limits_{v=0}^{9}(2v+1)$

--------------------▷ 50

---

(1) What do we call the points of intersection of a function $y(x)$ with the x-axis? 172

(2) What do we call the position x_0 for which in the neighborhood of x_0 the following inequality is valid?

$$f(x) > f(x_0)$$

....

--------------------▷ 173

Write with the summation sign: 50

A) $3+7+11+\ldots+31 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
B) $5+5^2+5^3+\ldots+5^{11} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

The solution is given below this time:

$$\text{A)}\ \sum_{v=1}^{8}(4v-1) \qquad \text{or} \qquad \sum_{v=0}^{7}(4v+3)$$

$$\text{B)}\ \sum_{v=1}^{11}5^v$$

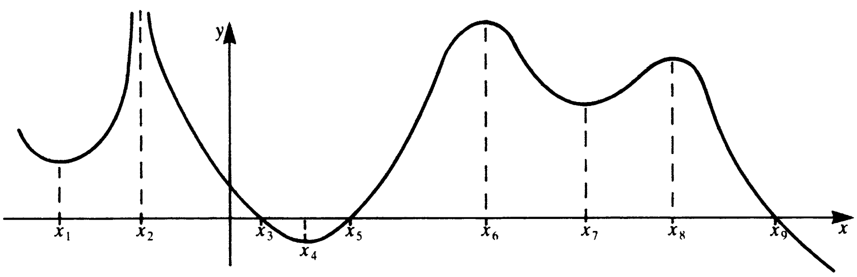
------------------▷ 51

---

(1) Zeros
(2) Local minimum

173

The graph in the figure shows a fairly complicated curve.



Name the x-values of the following salient points:
Zeros ....
Local maxima ....
Local minima ....
Pole ....

------------------▷ 174

Given the following series: 51

$$s_r = 5\times\frac{1}{2}+5\times\frac{1}{4}+5\times\frac{1}{8}+\dots$$
$$= 5\left(\frac{1}{2}\right)+5\left(\frac{1}{2}\right)^2+5\left(\frac{1}{2}\right)^3+\dots$$

Such a series is called: ....

--------------------▷ 52

Zeros: $x_3,\ x_5,\ x_9$ | Local minima: $x_1,\ x_4,\ x_7$ 174

Local maxima: $x_6,\ x_8$ | Pole: x_2

All correct --------------------▷ 180

Zeros wrong --------------------▷ 175

Extreme values wrong --------------------▷ 177

Pole wrong --------------------▷ 179

a geometric series 52

With the aid of the textbook calculate the sum of the series

$$S = 5 + 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \dots$$

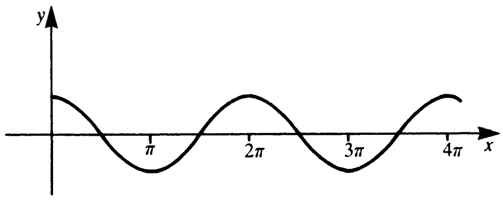
$$S = 5\sum_{v=0}^{\infty}\left(\frac{1}{2}\right)^v$$

$$S = \dots\dots\dots\dots$$

------------------▷ 53

Read once more the definition concerning zero positions in the textbook. Then state the position of all the zero values of the cosine function in the interval 0 to 4π. 175

$y = \cos x$



The positions of the zero values are: ....

------------------▷ 176

$$S = 5 \times \frac{1}{\left(1 - \frac{1}{2}\right)} = 5 \times 2 = 10$$ 53

Did you obtain this result?

☐ Yes --------------------▷ 55

☐ No --------------------▷ 54

$$\frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5ß}{2}, \ \frac{7\pi}{2}$$ 176

Finding the positions where the function is zero is easy: they occur at the points of intersection of the curve with the x-axis.

All correct --------------------▷ 180

Extreme values wrong --------------------▷ 177

Pole wrong --------------------▷ 179

You need some assistance. Here it is. 54

The infinite series

$$a + aq + aq^2 + \ldots = \sum_{v=0}^{\infty} aq^v$$

converges for $|q| < 1$. It then has the value

$$S = a\left(\frac{1}{1-q}\right)$$

In our example the series is

$$5 + 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \ldots$$

It follows that $a = 5$ and $q = \frac{1}{2}$ by comparison. The series converges to the limit

$$S = 5 \times \frac{1}{1 - \frac{1}{2}} = 5 \times 2 = 10$$

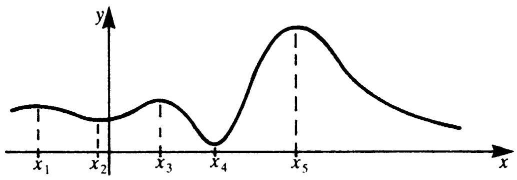
--------------------▷ 55

Identify the *local maxima* of the graph shown below. 177

A local *maximum* occurs at a point where the curve has a *peak*.

Similarly a local *minimum* occurs where the curve has a *trough* like the bottom of a valley.

We call certain maxima and minima local because it is possible that at other points there are maxima (or minima) having greater (or smaller) values. A maximum is not absolute unless it is the highest compared with all other maxima. Similarly for an absolute minimum.



Identify the:

Local maxima . . . .

Local minima . . . .

Absolute maximum . . . .

Absolute minimum . . . .

--------------------▷ 178

In the programmed study guide, we often mention ‘concepts’ and ‘operations’ which we have previously studied in the relevant sections in the textbook. 55

Do you find that after reading a particular section with the impression of understanding the subject you are sometimes unable to completely recall the concepts afterwards?

☐ Yes
--------------------▷ 57

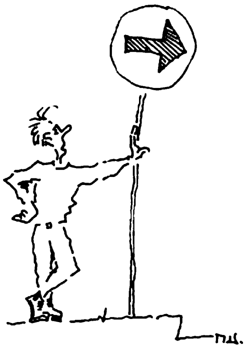
☐ No
--------------------▷ 56

Local maxima: x_1, x_3, x_5 178
Local minima: x_2, x_4
Absolute maximum: x_5
Absolute minimum: x_4

All correct
--------------------▷ 180

Pole wrong
--------------------▷ 179

You are remarkably talented! 56

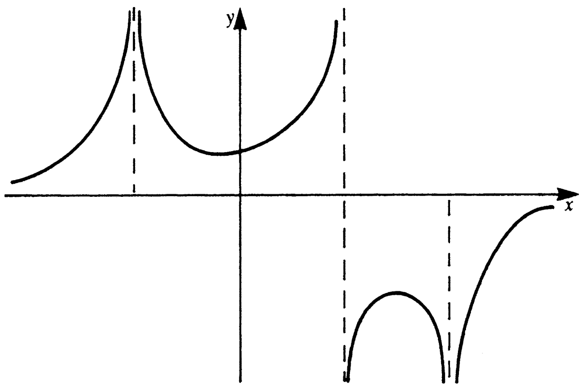


Nevertheless, read on!

57

A pole occurs at a position where the function tends to plus or minus infinity. The function under consideration had one pole position. The function shown here has three pole positions. 179

At one position the function tends to plus infinity from both sides, at a second position it tends to plus infinity on one side and minus infinity on the other side, while at the third position it tends to minus infinity from both sides.



180

Some remarks on reading skills: 57

Intensive reading

No one is able to retain all that he reads. For an average person the rate at which information is perceived is 10 to 20 times greater than the rate at which it is stored in the memory. In other words, we are able to perceive, to read, to hear and to grasp and to understand much more than we memorise. For instance, try to reproduce a lecture which you have understood on a subject of great interest to you!

Everyone is always surprised how little has been retained.

It is the aim of many reading techniques to enable you to .... more of the subject.

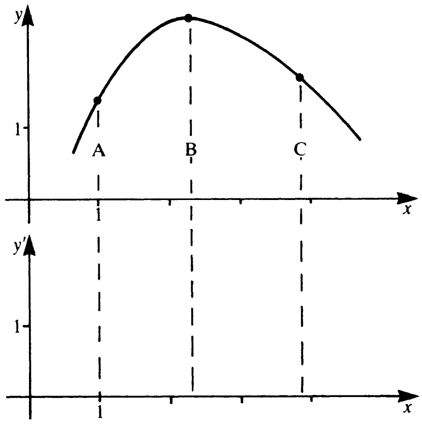
------------------▷ 58

One of the advantages of the differential calculus is that it aids us in finding maxima and minima. 180

The function shown below has one maximum. Draw the tangents to the curve at A, B and C.

Then plot the value of the slope of the tangent at each point.

This is the function $y'(x)$.



------------------▷ 181

It is the aim of many reading techniques to enable you to *memorise* more of the subject. 58

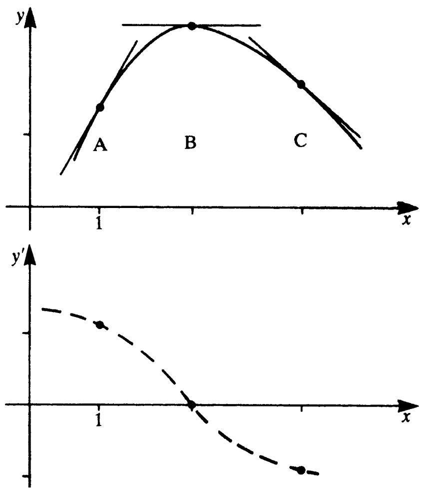
You may have discovered that there is a particular pattern in this programmed study guide. Questions about new concepts are asked and are followed by exercises. The pattern is:

— A new concept is presented.
— This new concept is then written down by you from memory.

The reason for this: What we write down is memorised better.

▷ 59

Leaving aside the question of the scale of the diagram, we observe that the values of the slope on the left of B are positive while those on the right are negative. 181



Plotted in the lower diagram is the curve representing y'.

▷ 182

What we write down is memorised better than what we merely read. 59

Understanding is all-important in mathematics. We understand things better if we know the concepts used in a test or in a lecture. Mathematics, physics and engineering are coherent subjects which require a special technique for their study. What 'coherent' means is best illustrated by an example.

In section 5.1.4 we calculated the following limit:

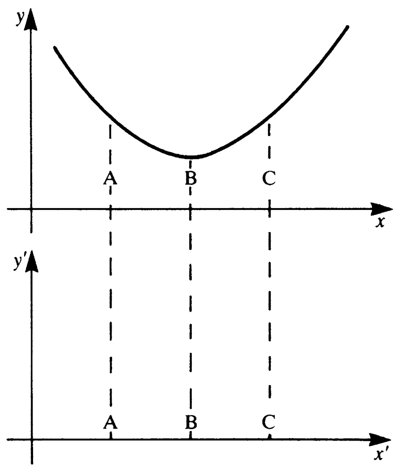
$$\lim_{x \to 0} \frac{\sin x}{x}$$

You can follow the chain of reasoning only if you know the concepts of limit and of sine and cosine. The concept of trigonometric functions can only be understood if you know what a function is. And a function can only be understood if you know, at least, the fundamental operations of arithmetic.

Such a sequence can be extended and the meaning should be immediately clear. We can only understand facts if certain prerequisites are known. Subjects in which there exist many relations with very long chains of prerequisites are called .... subjects.

-------------------▷ 60

Let us now carry out the same procedure in the case of a minimum. 182



Draw on the curve the tangents at A, B and C and hence sketch the curve of y'.

-------------------▷ 183

coherent

60

A person who wants to know where Tokyo is located does not need to know where Paris is, or a person who wants to learn where Tunis is does not need to know how long the River Nile is. These geographical data are not coherent.

The degree of coherence of a subject has an influence on the most appropriate method of study. With a coherent subject we need to study intensively. (At school the teacher took care of these things, but when you study by yourself you have to take over this role to some extent.) Mathematics and physics are coherent subjects.

To learn intensively means, follow the subject matter actively:

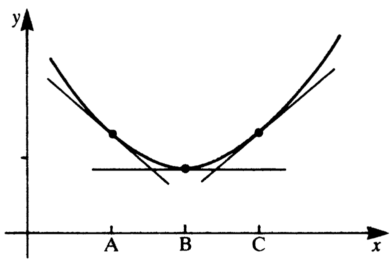
(i) you should not accept anything that you have not understood;
(ii) you recognise, sum up, extract and repeat fundamental concepts and rules.

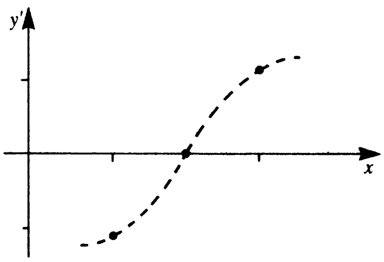
This last point is explained in the next frame.

------------------▷ 61

183

Drawing tangents is not difficult.
Estimating the values of the slopes can only be approximate. What is important is that they are negative to the left of B and positive to the right of B.





We also notice that the slope of the curve for y' is different in this case.
The values of y' increase from left to right.
Recall that for a maximum, y' decreases in value from left to right.

------------------▷ 184

In the textbook new concepts are often written in italics. Definitions and rules are set out. 61

What then is the best way to learn new concepts and definitions?

☐ By careful reading

--------------------▷ 62

☐ By reading again and again until you know them by heart

--------------------▷ 63

☐ By extracting (taking notes) and repeating

--------------------▷ 64

We now know the conditions required to determine the location x_0 of a maximum or a minimum. 184

In both cases, *the tangent is horizontal*.

Mathematically this means: $y'(x_0) =$ ....

--------------------▷ 185

Reading with care is good but it may lead to self-deception. In order to ensure that a new concept has been mastered we need checks. 62

In this study guide such checks are given and you are required to carry them out. Further on we shall talk about a method of study which will help you to carry out these checks yourself.

Here is a hint: mutual questioning and solving problems with fellow students is helpful in carrying out these checks.

Reading with care is not sufficient.

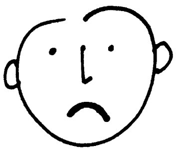
How do we learn new concepts and definitions?

☐ By reading again and again until we know the subject by heart --------------------▷ 63

☐ By extracting (taking notes) and repeating --------------------▷ 64

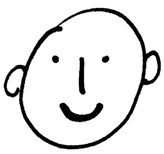
$$y'x_0 = 0$$ 185

For a *maximum*: the value of the derivative decreases from left to right.
Mathematically this means: $y''(x_0) < 0$



max = neg

For a *minimum*: the value of the derivative increases form left to right.
Mathematically this means: $y''(x_0) > 0$



min = pos

If you have not yet understood everything --------------------▷ 186

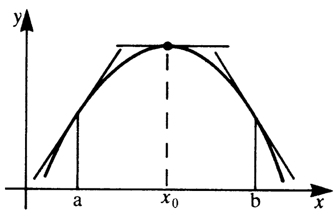
Otherwise --------------------▷ 190

Reading a definition or an explanation of a new concept again and again until you know it word for word is a bit dull. The danger is that you learn the words but not the meaning. 63

An effective procedure is to extract new concepts and definitions. Extracting (taking notes) means writing down key words. The most important part of the text is taken out. By doing so we have to think and digest the content. The notes only need to be such that we can later reconstruct the meaning.

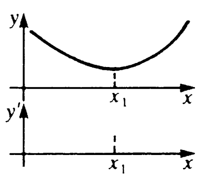
------------------▷ 64

We repeat once more the chain of reasoning: 186



(1) At the point x_0 the tangent to the curve $y(x)$ is horizontal, i.e. the slope is zero. This is expressed mathematically by $y' = 0$
(2) In the interval (a, x_0) the slope of the curve $y(x)$ is positive. Thus in this interval $y' > 0$, but the slope of the tangent decreases from left to right.
(3) In the interval (x_0, b) the slope of the curve $y(x)$ is negative. Thus in this interval $y' < 0$, and the slope of the tangent decreases from left to right.

Sketch the curve for $y'(x)$ in the neighbourhood of a minimum x_1.

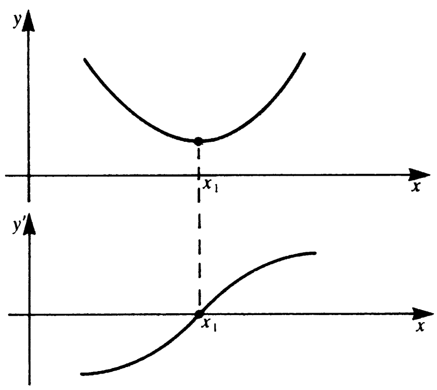


------------------▷ 187

Yes, good. In fact, extracting is the most effective method of memorizing something new. Extracting means to take out of the text the most salient parts. These are mostly new concepts, rules and definitions as well as short explanations. These extracts only need to be so detailed that we can later reconstruct the meaning. 64

-------------------▷ 65

187



Look at the neighbourhood of x_1 in the diagram above.

Which of the following is true?

☐ $y''(x) < 0$

☐ $y''(x) > 0$

-------------------▷ 188

65

Active learning is more effective than passive learning.

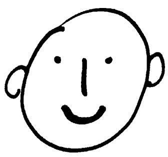
On the next page we describe a psychological experiment to illustrate the effect of active learning.

66

If you are not interested in the experiment and its results, proceed to 67

188

$y'' > 0$



To determine whether a curve has a maximum or a minimum we need to remember two things:

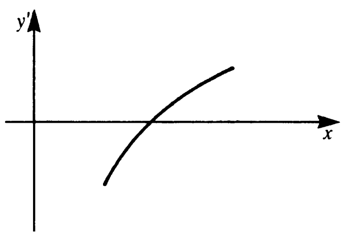
(1) The slope at an extreme value is zero: $y' = 0$.
(2) Whether there is a maximum or a minimum can be determined by investigating the nature of the slope.

You should remember that for a maximum the slope is positive before and negative after the point where the maximum occurs.

This implies that the slope is getting smaller, i.e. $y'' < 0$.

For a minimum the reverse is true.

The figure shows the slope y' of a curve.



The curve y has a

☐ minimum

☐ maximum

189

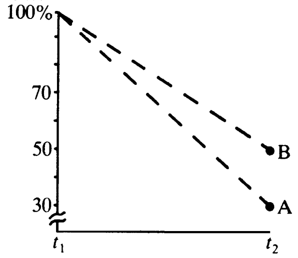
Two groups of students A and B write at time t_1 a dictation of foreign words.

66

Group A is given back the dictation with the errors underlined in red. They then have to look up the words they got wrong in order to carry out the corrections. If they have made mistakes in the punctuation they have to look up the rules.

Group B is given back the corrected dictation where the correct way of writing and punctuation has been inserted.

After four weeks (time t_2) they are both given a second dictation. The results are shown in the figure below.



Ordinate: relative number of total errors.

At time t_1 it is the same for both groups.

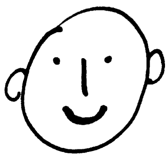
The decrease in the relative number of errors in group A is attributed to the more active way of learning.

------------------▷ 67

---

minimum

189



Remember: the mouth represents the curve, the smiling expression represents the positive value of y''.

------------------▷ 190

We not only have to know new concepts and rules, we also need to know how to apply them. 67

By taking notes we learn

☐ to apply the subject matter --------------------▷ 68

☐ to memorize the subject matter --------------------▷ 69

To determine characteristic points of a curve the following conditions apply: 190

(1) Zero values: $y = 0$
(2) Local maxima: $y' = 0, \quad y'' < 0$
(3) Local minima: $y' = 0, \quad y'' > 0$

For what value x_E has the function $y = x^2$ an extreme value?

The curve is a parabola.
To obtain the required value:

(1) Find the derivative

$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(2) Solve the equation
$y' = 0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\quad x_E = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 191

This is not entirely correct. 68

By means of extracts it is easier to memorise new concepts. With notes we have the possibility of reproducing the subject matter later on. In doing so we do not necessarily learn how to apply these new concepts.

We can only apply those concepts and rules that we know. For the time being then we only deal with the first step, i.e. getting to know.

-------------------▷ 69

$y' = 2x$ 191
$0 = 2x$
hence $x_E = 0$

Is there a maximum or a minimum at that point?

$y'' =$ ....

☐ $y'' > 0$
☐ $y'' < 0$

☐ Minimum
☐ Maximum

-------------------▷ 192

Extracting makes it easier to memorise and retain. 69

If you want to apply new concepts, definitions and rules in a *different* context then you must have them firmly fixed in your memory.

For this purpose extracts are helpful. The method of making extracts is not difficult and like all important methods it is very simple.

Extracting information is an active form of learning.

Now, as an exercise, extract the most salient aspects of section 5.3 in the textbook.

------------------▷ 70

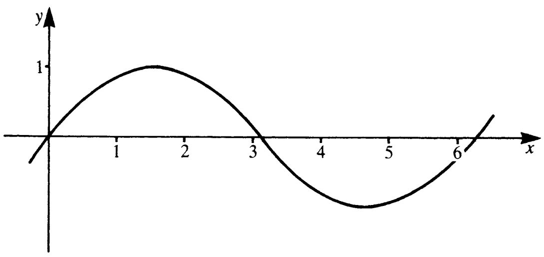
$y'' = 2 > 0$
Minimum 192

For what values of x in the range $0 \leq x \leq 2\pi$ is the sine function $y = \sin x$ zero?

....

Obtain the derivative.

For what values of x in the interval $0 \leq x \leq 2\pi$ has the function
$y = \sin x$
maxima ....
minima ....



------------------▷ 193

Have you written down the salient aspects? 70

☐ No, I did not have a piece of paper handy → 71

☐ No, I already know the salient aspects → 72

☐ Yes → 73

In the interval $0 \leq x \leq 2\pi$: 193

$y = 0$ at $x = 0$

at $x = \pi$

and at $x = 2\pi$

Maximum at $x = \frac{\pi}{2}$

Minimum at $x = \frac{3\pi}{2}$

The steps required to obtain the position of a maximum or minimum are:

Step 1: $y' = \cos x$
Step 2: $y' = 0$ for $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$
Step 3: $y'' = -1$ for $x = \frac{\pi}{2}$, hence a maximum
$y'' = +1$ for $x = \frac{3\pi}{2}$, hence a minimum

→ 194

No comments 71

--------------------▷ 73

It is often important to determine the extreme values of a function. 194

Whether there is a minimum or a maximum often follows from the nature of the problem, and for this we do not need to apply the second condition.

The steps are:

— obtain the derivative
— set the derivative = 0, and solve the equation $y' = 0$.

Here is an example for you.

$$y = x^3 + x^2$$

$$y' = \text{.......................................}$$

Extreme values occur at

$$x_{E1} = \text{...................................}$$
$$x_{E2} = \text{...................................}$$

--------------------▷ 196

If you require help

--------------------▷ 195

Well, if you already know the salient aspects of the work the exercise is pointless. You should start to make extracts as soon as new subject matter is presented to you. Please do it! 72

→ 73

Given: $y = x^3 + x^2$ 195

Required: extreme values

For the extreme values: $y'(x_E) = 0$

Step 1: Derive y'

$$y' = 3x^2 + 2x$$

Step 2: $y' = 0$

$$0 = 3x^2 + 2x$$

Solve this equation for the values of x.

We can factorise out x to obtain

$0 = x(3x+2)$

You should be able to solve this equation for x.

$x_{E1} = \ldots\ldots\ldots\ldots\ldots\ldots$

$x_{E2} = \ldots\ldots\ldots\ldots\ldots\ldots$

→ 196

You could have written down: 73

Relation between sequence and series;
leading term, last term;
infinite sequence/finite sequence;
geometric series.

We shall use this technique frequently.
From now on you should make a short extract from each text studied intensively.
These extracts can be collected in some kind of order and kept in a binder, folder or filing cabinet.

------------------▷ 74

$$x_{E1} = 0$$
$$x_{E2} = -\frac{2}{3}$$

196

Obtain the value of x for which the function is zero and where there is an extreme value:

$$y = -x^2 + 2x$$

Zero ....
Extreme values ....

------------------▷ 200

If you need help with the zero

------------------▷ 197

If you need help with the extreme values

------------------▷ 198

You have found the solution

------------------▷ 200

5.4 Differentiation of a Function

74

Objective: Concepts of slope of a curve, difference quotient, differential quotient, derivative, differential.

When learning from the textbook use a scribbling pad to perform calculations. Check the transformations.

A chain of reasoning is better understood and retained if an active part is taken, e.g. if you take notes. It may be tiresome but it is well worth while.

READ: 5.4 Differentiation of a function
Textbook pages 96–102

------------------▷ 75

197

To obtain the values of x for which the function is zero we set $y = 0$.

Hence if $y = -x^2 + 2x$
then $y = 0$ leads to

$$0 = -x^2 + 2x \quad or \quad x^2 - 2x = 0$$

$$x(x-2) = 0$$

hence $x_1 = 0$
$x_2 = 2$

Now calculate the extreme values of the function

$$y = -x^2 + 2x$$

Extreme values ....

------------------▷ 200

If you need help

------------------▷ 198

Have you made an extract of section 5.4 and taken notes? 75

☐ Yes --------------------▷ 76

☐ No --------------------▷ 78

Given: $y = -x^2 + 2x$
Required: Extreme values x_E 198

Step 1: Obtain the derivative. That is the slope of a tangent, y' : ....
Step 2: Set $y' =$ .... $= 0$

and solve for x.

$= -2x + 2$

giving $x_E =$ ....

--------------------▷ 199

76

Splendid. You have applied one of the important techniques of study.
What are the following symbols called?

$$\frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$dy = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------> 77

199

$$x_E = 1$$

Now obtain the extreme values of the function

$$y = 2x^2 + 4x$$

Obtain the derivative y'.
Set $y' = 0$.
Show that the function has an extreme value at

$$x_E = -1$$

Check your solution by yourself.

--------> 200

77

$\frac{\Delta y}{\Delta x}$ = difference quotient

$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = differential coefficient

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = differential coefficient

$\mathrm{d}x$ = differential of the independent variable

$\mathrm{d}y$ = differential of the dependent variable

$f'(x)$ = derivative of the function $f(x)$

With the aid of your notes it should have been possible to write down the concepts.

------------------▷ 80

200

The function $y = -x^2 + 2x$
is zero at $x_1 = 0$ and $x_2 = 2$;
it has an extreme value at $x_E = 1$

As the last example of this sequence, sketch the function

$$y(x) = \frac{2}{\mathrm{e}^{-(x+1)} + 1}$$

First establish a few salient points!

Solution found ------------------▷ 203

Hints on zeros, poles and asymptotes ------------------▷ 201

Hints on extreme values and points of inflexion ------------------▷ 202

It is a pity that you did not make extracts. But if you already knew the content of the section very well then you were right not to do so. 78

We make extracts if we want to learn new facts. This method of working is highly recommended.

Now try, from memory, to name the following symbols:

$$\frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 79

Zeros: $y(x_z) = 0$ 201

The numerator does not vanish (and the denominator is always positive). Thus there are no zeros.

Poles: The denominator does not vanish. Thus there are no poles.

Asymptotes: We examine the behaviour of the function as

$x \to +\infty$ and $x \to -\infty$

$$\lim_{x \to \infty} y(x) = \lim_{x \to \infty} \frac{2}{\mathrm{e}^{-(x+1)} + 1} = 2$$

Thus the asymptote for $x \to \infty$ is $y = 2$.

The other asymptote is found in the same way: for $x \to -\infty$ the asymptote is $y = 0$.

Full solution established --------------------▷ 203

Hints on extreme values and points of inflexion --------------------▷ 202

$\frac{\Delta y}{\Delta x}$ = difference quotient **79**

$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = differential coefficient

$\frac{dy}{dx}$ = differential coefficient

dx = differential of the independent variable

dy = differential of the dependent variable

$f'(x)$ = derivative of the function $f(x)$

All correct ▷ 81

Errors ▷ 80

Extreme values: $y'(x_E) = 0$ **202**

$$y'(x) = \frac{2e^{-(x+1)}}{(e^{-(x+1)}+1)^2}$$

The numerator never vanishes. Thus the function has no extreme values.
Points of inflexion: $y''(x_i) = 0$

$$y''(x) = \frac{-2e^{-(x+1)}(e^{-(x+1)}+1) + 4e^{-(x+1)}e^{-(x+1)}}{(e^{-(x+1)}+1)^3}$$

After rearranging we get

$$y''(x) = \frac{2e^{-(x+1)}[2e^{-(x+1)} - e^{-(x+1)} - 1]}{(e^{-(x+1)}+1)^3}$$

The bracket in the numerator must be zero for

$$y''(x_i) = 0$$

$0 = e^{-(x_i+1)} - 1$
Since $e^{-(x_i+1)} = 1 = e^0$
Thus the function has a point of inflexion at $x_i = -1$

▷ 203

While taking notes you memorise new concepts more easily. At the same time you are forced to distinguish between the essential and the non-essential. Finally, you will have in your notes all the important keywords. 80

If you take notes and collect them together they will help you to recall the subject matter more quickly before an examination.

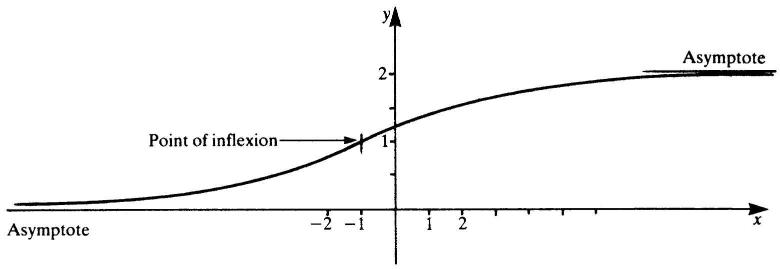
READ again and take notes **5.4 Differentiation of a function**
Textbook pages 96–102

-------------------▷ 81

There are no zeros and no poles. 203

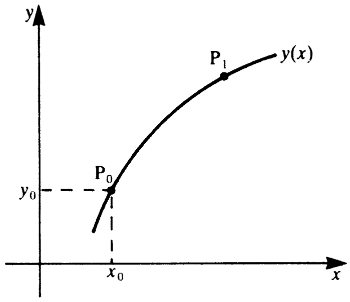
Asymptotes $x \to \infty \quad y = 2$

Point of inflexion $x_i = -1 \quad x \to -\infty \quad y = 0$



If you have mastered the subject you do not need more exercises at the moment. If you had difficulties some more exercises should be useful to get more practice. In any case, have a look at some exercises in one or two weeks' time. As you already know, further exercises are given in the textbook at the end of Chapter 5.

-------------------▷ 204



The derivative of a function $y(x)$ at the point x_0 has a geometrical significance. Pick out the *wrong* sentence! 81

(a) The derivative $y'(x_0)$ indicates the slope of the secant through $P_0(x_0, y_0)$ and an arbitrary point P_1 on the curve $y(x)$

-------------------▷ 82

(b) The derivative $y'(x_0)$ indicates the slope of the tangent to the curve $y(x)$ at the point $P_0(x_0, y_0)$

-------------------▷ 83

(c) The derivative indicates the slope of the curve $y(x)$ at the point $P_0(x_0, y_0)$

-------------------▷ 85

5.10 Applications of the Differential Calculus

204

Section 5.8 in the textbook is intended to give you an impression of possible applications. As there are many applications for the physicist and engineer, it is impossible to present an exhaustive summary. Some of the examples might be studied intensively in later periods when you are facing advanced problems. In any case this section may be useful when you want to review the concept of curvature and the l'Hôpital's rule.

READ: 5.8 Applications of differential calculus
5.8.1 Extreme values
5.8.2 Increments
5.8.3 Curvature
5.8.4 Determination of limits by differentiation: l'Hôpital's rule
Textbook pages 123–129

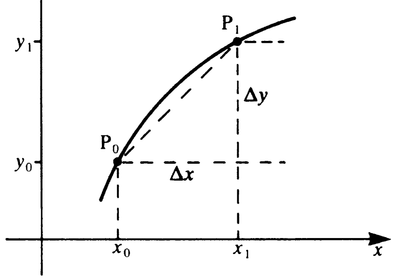
-------------------▷ 205

You are right

82

The slope of a secant through the point P_0 and a point P_1 is not given by the derivative but by the difference quotient

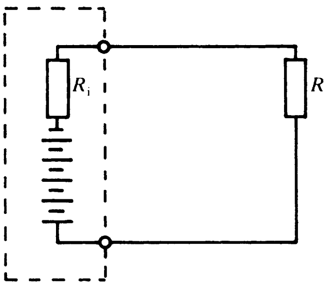
$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$



-------------------▷ 86

We start with a maximum problem in electricity. Consider a battery or an amplifier. Its voltage is V_0, its internal resistance is R_i.

205



The problem is to find the value for R at which the electrical power consumed by R is a maximum.

Want to solve the problem without help -------------------▷ 210

Want hints -------------------▷ 206

Wrong, unfortunately

83

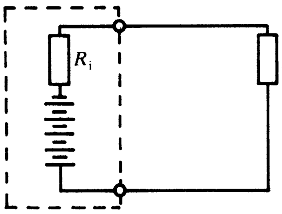
You have not yet entirely understood the geometrical meaning of the derivative of a function.
Have a look again at sections 5.4.2 and 5.4.3 in the textbook and clarify the following statements:

1. The derivative $f'(x)$ of a function indicates the slope of:
   (a) the chord
   (b) the tangent
2. The expression $\frac{\Delta y}{\Delta x}$ is represented by the slope of:
   (a) the chord
   (b) the tangent

------------------▷ 84

The circuit is shown again:

206



The box on the left represents the battery or amplifier. If there is a current I the voltage applied to R is V_0 diminished by the voltage applied to R_i. Thus the voltage applied to R is

$$V_R = V_0 - IR_i$$

The electrical power consumed by R is then

$$L = V_R I$$

Now we want the maximum of L depending on R.

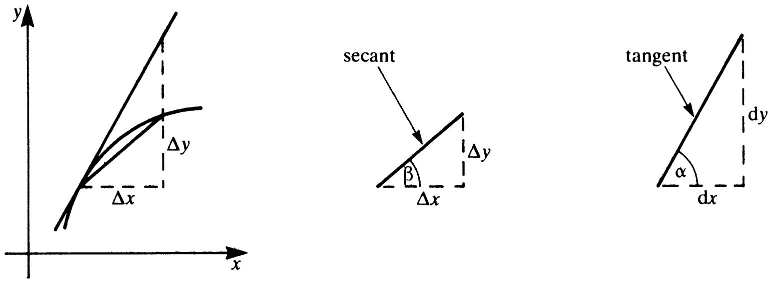
Want to solve the problem on my own ------------------▷ 210

More help is needed ------------------▷ 207

$f'(x)$ = slope of the tangent 84

$\dfrac{\Delta y}{\Delta x}$ = slope of the chord

If you have still made mistakes, then here is some advice.
Have a look at the following sketches.
On the right are drawn the secant and the tangent.
The angles α and β are the measures of the corresponding slopes.



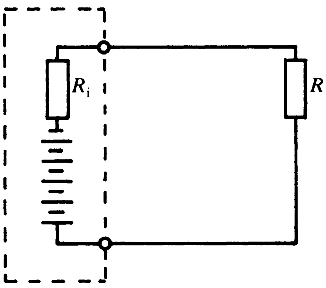
→ 86

The voltage applied to R is $V_R = V_0 - R_i I$ 207
The power consumed by R is $L = V_R I$
The current I is given by Ohm's law: $I = \dfrac{V_0}{R_i + R}$
We express L in terms of R:

$$L = (V_0 - R_i I) I$$

$$L = \left(V_0 - R_i \frac{V_0}{R_i + R}\right) \frac{V_0}{R_i + R}$$

$$L = \frac{R V_0^2}{(R_i + R)^2}$$



We look for the maximum of L with respect to the variable R.

Want to solve the problem on my own → 210

Want further hints → 208

Wrong, unfortunately

85

The question asked for the *wrong* statement about the geometrical significance of the derivative $f'(x)$ of a function at a point x_0. You picked out the *right* answer: the derivative does indeed indicate the slope of the curve at the point $P_0(x_0, \ y_0)$.

Try again; go back to ------------------▷ 81

Wanted: the maximum of L

208

$$L = \frac{RV_0^2}{(R_\mathrm{i}+R)^2}$$

Let us change the variables:

$$L = y$$
$$R = x$$
$$y = \frac{V_0^2\,x}{(R_\mathrm{i}+x)^2}$$

The necessary condition for y being maximal is $y' = 0$.

I can solve the problem now ------------------▷ 210

Further hints wanted ------------------▷ 209

In the textbook we describe the concept of instantaneous velocity. 86

We have to distinguish very clearly between instantaneous velocity and average velocity. In daily life we often distinguish between them very poorly.

The reading on the speedometer of a car shows the .... velocity.

When we speak of average travelling speed we mean the .... velocity, as a rule.

-------------------▷ 87

The given equation reads 209

$$y = \frac{V_0^2 x}{(R_i + x)^2}$$

We differentiate:

$$y' = \frac{V_0^2 (R_i + x)^2 - V_0 x 2(R_i + x)}{(R_i + x)^4} = V_0^2 \frac{R_i - x}{(R_i + x)^3}$$

An extreme value is given if $y' = 0$.

$$0 = \frac{R_i - x}{(R_i + x)^3} \quad \text{i.e. } x = R_i$$

Since we substituted $R = x$ we now have the solution:

$$R = R_i$$

-------------------▷ 210

Speedometer reading: instantaneous velocity
Average travelling speed: average velocity 87

As an aside, let us mention that Newton discovered the differential calculus when investigating velocities and motions (1665–1676, theory of fluxions). At about the same time Leibniz developed the same calculus when investigating mathematical problems (1673–1676).

For derivatives with respect to time Newton used the dot above the variable:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{s}$$

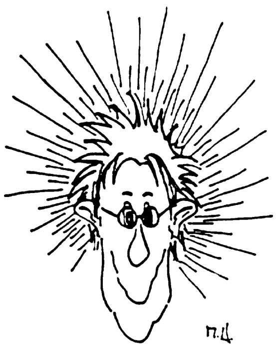
The passage to the limit $\mathrm{d}t \to 0$ is one of the fundamental mathematical abstractions in science. We are not able to measure arbitrarily small times. This abstraction is, however, confirmed by the conclusions.

------------------▷ 88

$$R = R_{\mathrm{i}}$$ 210

From the context it is obvious that this is a maximum and not a minimum. But you may prove it for yourself by determining the second derivative.

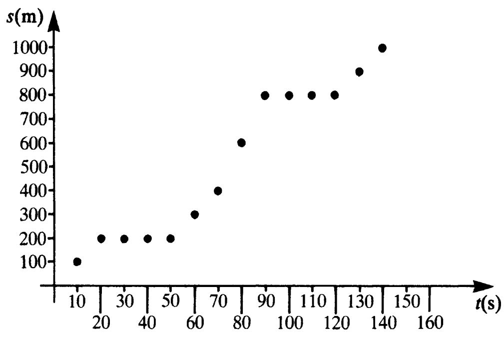
The maximum electrical power that can be supplied by a battery is therefore reached if the resistance in the external circuit is the same as the internal resistance. In this case half of the electric power is consumed within the battery or amplifier.



Note also that the result is independent of V_0.

------------------▷ 211

A car is being driven along a straight main road with many traffic lights. The position of the car is measured at intervals of 10 s and plotted on a graph. The abscissa represents the time and the ordinate the distance covered. 88



The car has stopped ☐ once
☐ twice
☐ three times
☐ four times

The duration of each stop at the traffic lights is .... seconds.

--------------------▷ 89

Now let us tackle a problem on errors. 211

To measure the depth of a well we drop a stone. Since we can hear it hitting the water we measure the time of the fall.

The time is 2 ± 0.2 s.

What is the corresponding error in the depth?

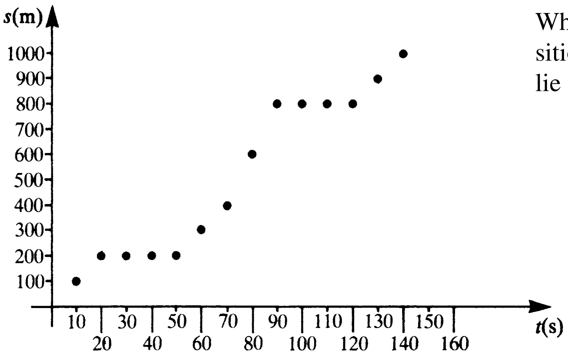
What is the relative error?

(The time of travel of the acoustic signal may be neglected; the approximate value of $g = 9.81$ m/s.)

I want to solve the problem on my own --------------------▷ 213

Further hints --------------------▷ 212

twice
about 30 seconds 89



When the car is at a stop time elapses but its position remains constant. The corresponding points lie on a horizontal line.

Draw a curve through the points on the distance–time graph.

The average speed of the car in the time interval from 60 to 70s after the start of the journey is ....

-------------------▷ 90

The time of fall is $2 \pm 0.2\,\text{s}$. Thus the relative error in measurement is 10%. 212

The depth is

$$s = \frac{g}{2}t^2 = \frac{9.81}{2}\,\text{m/s}^2(2\,\text{s})^2$$
$$s = 19.62\text{m}$$

The error

$$\Delta s = \frac{ds}{dt}\Delta t = gt\Delta t$$
$$\Delta s = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and the relative error

$$\frac{\Delta s}{s} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

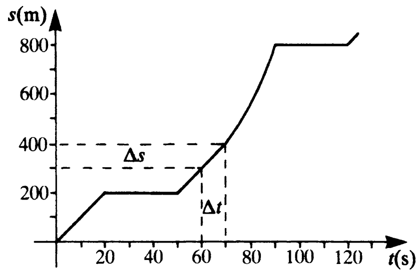
-------------------▷ 213

Average speed = $\frac{\Delta s}{\Delta t} = 10\,\mathrm{m/s}$

90

Newton found that the concept of the difference quotient was not sufficient in order to describe the instantaneous velocity, i.e. the velocity at any given instant t. In order to overcome this problem he was led to the concept of the differential coefficient.

The average speed is obtained by dividing the distance covered by the time taken. It is geometrically identical to the determination of the slope of a secant of a curve. The determination of the instantaneous velocity is geometrically identical to the determination of the slope of the tangent to a curve.



The difference quotient indicates the slope of the ....

The differential coefficient indicates the slope of the ....

------------------▷ 91

The error $\Delta s = \pm g t \Delta t \approx \pm 4\mathrm{m}$

213

The relative error $\frac{\Delta s}{s} = 2\frac{\Delta t}{t} \approx 20\%$

Now we turn to an example on curvature.

Let us calculate the radius of curvature of a circle. The result is already known. The radius of curvature of a circle is the radius of the circle. Nevertheless this problem is an interesting one because it provides a check for the formula derived in the textbook.

Equation of a circle: $x^2 + y^2 = R_0^2$

$$y = \sqrt{R_0^2 - x^2}$$

First write down the formula for the radius of curvature. In case of doubt use the textbook.

$$R = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 214

secant
tangent 91

Above all it is important for you to have understood the fundamental idea which led to the solution of the tangent problem discussed in section 5.4.

I have understood the fundamental idea --------------------▷ 96

I have not understood everything, I need additional explanation --------------------▷ 92

$$R = \frac{[1+(y')^2]^{3/2}}{y''}$$ 214

Calculate R with

$$y = \sqrt{R_0^2 - x^2}$$

$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ --------------------▷ 217

Want further hints --------------------▷ 215

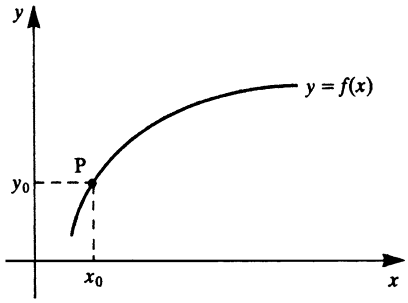
The solution of the tangent problem is now explained in a different way. 92

Problem: The slope of a curve at a point is to be determined.

Coordinates of the point P = $(x_0,\ y_0)$

Alternatively: $P = (x_0,\ f(x_0))$

The y value is calculated in accordance with the functional relation.



-------------------▷ 93

Given: 215

$$R = \frac{[1+(y')^2]^{3/2}}{y''}$$
$$y = \sqrt{R_0^2 - x^2}$$

First find the derivatives

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y'' = \ldots\ldots\ldots\ldots\ldots\ldots$$

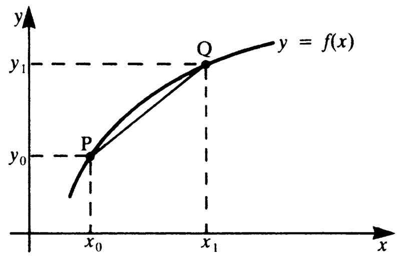
Derivatives found -------------------▷ 217

Further hints wanted -------------------▷ 216

93

We draw a secant by joining P to some other point Q.
Q has coordinates $Q = (x_1, y_1)$
Alternatively: $Q = (x_1, f(x_1))$
We call Δx the difference in the x-values, i.e. $\Delta x = x_1 - x_0$.
Are you now able to express the coordinates of the point Q in terms of x_0 and Δx?

$$Q = (\ldots\ldots\ldots\ldots\ldots\ldots,\quad f(\ldots\ldots\ldots\ldots\ldots\ldots))$$



------------------▷ 94

216

$$y' = -\frac{x}{\sqrt{R_0^2 - x^2}}$$

$$y'' = -\frac{-R_0^2}{(R^2 - x^2)^{3/2}}$$

These derivatives are to be inserted into the formula

$$R = \frac{[1 = (y')^2]^{3/2}}{y''}$$

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 217

$$Q = (x_0 + \Delta x, f(x_0 + \Delta x))$$

94

We are now in a position to state the slope of the secant.

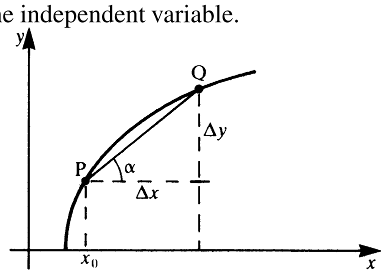
Δy is yet to be defined:

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Then the slope of the secant is $\tan \alpha = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

This is a fundamental equation. The numerator expresses the difference between the functional values. The denominator expresses the difference in the independent variable.

As $\Delta x \to 0$ Q is moved closer and closer to P, and we go from the slope of the secant to that of the tangent.



On the graph draw the tangent to the curve at P.

--------------------▷ 95

$$R = -R_0$$

217

The sign indicates that the curve is concave downwards.

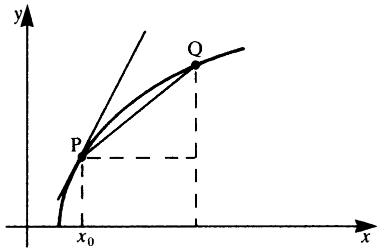
The radius of a circle is the radius of curvature.

R is a constant in this case.

What is the radius of curvature of the parabola $y = x^2$?

$$R = \text{.................................}$$

--------------------▷ 218



95

Now we want to move from the slope of the secant to that of the tangent.

We have to carry out the move to the limit. This is the reason why we discussed the problem of finding limits at the beginning of this chapter. The problem in essence consists of transforming the expression for the slope of the secant in such a way that the limit can be seen to exist. This is done in the textbook for a simple case, namely $f(x) = x^2$.

For many functions the fundamental idea of the proof is always the same; we evaluate the expression $f(x+\Delta x)$ and subtract from it the expression $f(x)$ and obtain Δy. According to the problem, we have to undertake transformations that will enable us to move to the limit. We shall explain this process in greater detail further on. The important thing now is to understand that the tangent problem requires the determination of the limit as $\Delta x \to 0$.

-------------------▷ 96

$$R = \frac{(1+4x^2)^{3/2}}{2}$$

218

In this case R is not a constant. The radius R of curvature grows with increasing magnitude of x. For $x = 0$ we have $R = \frac{1}{2}$. The curvature $\frac{1}{R}$ decreases with increasing magnitude of x.

-------------------▷ 219

The differential dy of the function $y = f(x)$ is defined as 96

dy = ...

--------------------▷ 97

5.11 Further Methods for Calculating Differential Coefficients

219

This section covers slightly more advanced methods. They are useful if you have to deal with complicated functions. For the first pass through the textbook you may skip this section but you should return to it before having completed differential equations (Chapter 10).

READ: 5.9 Further methods for calculating differential coefficients
Textbook pages 129–130

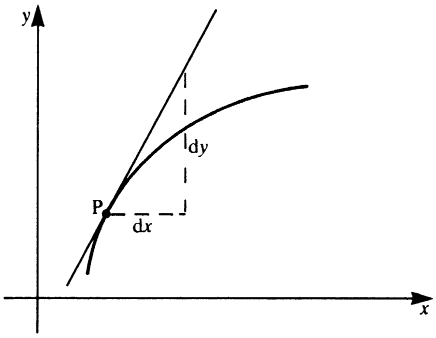
--------------------▷ 220

$$dy = f'(x)dx$$

97

The slope of the tangent is considered as the ratio of small differentials. If the slope of the tangent is determined it is possible to choose a dx to obtain a corresponding dy for the tangent by the equation

$$dy = f'(x)dx$$



--------------------▷ 98

Given: $2x^2 + 3y^3 = 27$.

This is an implicit function.

Calculate y' using one of the methods shown in the textbook.

220

$y' =$ ....

--------------------▷ 223

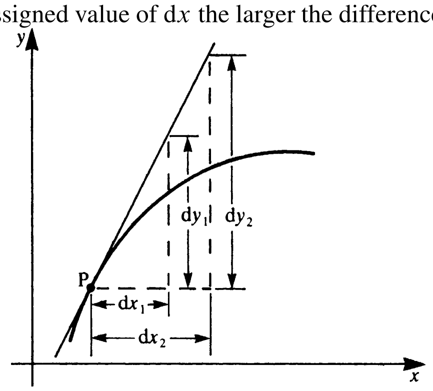
Hints wanted

--------------------▷ 221

98

The differential coefficient $f'(x) = \frac{dy}{dx}$ measures the slope of the tangent to the curve at P.
The quotient $\frac{dy}{dx}$ is independent of the value assigned to dx.

If we assign a value to dx and the slope of the tangent f' is known then we can calculate the value of the differential dy. These differentials are always measured to the tangent. The larger the assigned value of dx the larger the difference between the tangent and the curve.



$$f'(x) = \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2}$$

-------------------▷ 99

221

We want the derivative y' of the *implicit* function

$$2x^2 + 3y^3 = 27$$

We may use the method for differentiating implicit functions or we may use logarithmic differentiation. We start with the first method:

$$\frac{d}{dx}(x^2) = 2x$$
$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = \text{..........................}$$

-------------------▷ 223

Further hints wanted -------------------▷ 222

Let $y = x^2$; then $y' = 2x$ 99

Express $\mathrm{d}y$ in terms of y' and $\mathrm{d}x$:

$$\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 100

Given: $2x^2 + 3y^3 = 27$ 222

We differentiate all terms with respect to x and solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(2x^2 1) + \frac{\mathrm{d}}{\mathrm{d}x}(3y^3) = \frac{\mathrm{d}}{\mathrm{d}x}(27)$$

$$4x + 9y^2\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

Now solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 223

$$y(x) = x^2, \quad y' = \frac{dy}{dx} = 2x$$
$$dy = y'dx$$
$$dy = 2xdx$$

100

Does your result agree with the one above?

Yes → 103

No, or further explanation required → 101

$$\frac{dy}{dx} = \frac{-4x}{9y^2}$$

223

Now try logarithmic differentiation for the same function. It is more complicated in this case.

$$2x^2 + 3y^3 = 27$$

$$2x^2 = 27 - 3y^3$$

$$\frac{dy}{dx} = \dots\dots\dots\dots$$

→ 225

Hints needed → 224

Let us work out another example: 101

$$y = 3x^4$$

We form the difference quotient $\Delta y = f(x+\Delta x) - f(x)$

$$\Delta y = 3(x+\Delta x)^4 - 3x^4$$

or $\Delta y = 3(4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4)$
Dividing by Δx gives

$$\frac{\Delta y}{\Delta x} = 3(4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3)$$

As $\Delta x \to 0$ all the terms vanish except for the first one.
Hence

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 \text{ and } \mathrm{d}y = 12x^3\mathrm{d}x$$

------------------▷ 102

Given: $2x^2 = 27 - 3y^3$ 224
We take logarithms:

$$\ln 2 + 2\ln x = \ln(27 - 3y^3)$$

We differentiate with respect to x

$$\frac{2}{x} = \frac{-3 \times 3y^2}{27 - 3y^3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Inserting $2x^2$ for $(27 - 3y^3)$ from the first equation yields

$$\frac{2}{x} = \frac{-9y^2}{2x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{...........................}$$

------------------▷ 225

Further example: $y = 2x^2 + 2$ 102

We form the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \frac{2(x+\Delta x)^2 + 2 - (2x^2+2)}{\Delta x}$$
$$= \frac{4x\Delta x + 2(\Delta x)^2}{\Delta x} = 4x + 2\Delta x$$

As $\Delta x \to 0$ it follows that

$$\lim_{\Delta x \to 0}(4x + 2\Delta x) = 4x$$

In order to obtain the difference Δy in the value of the function we have to calculate f for the position x and the position $x + \Delta x$ and form the difference. To obtain the difference quotient we have to divide Δy by Δx. Subsequently we carry out the limiting process.

As Δx approaches zero some terms are usually negligible compared with other terms.

------------------▷ 103

$$\frac{dy}{dx} = -\frac{4x}{9y^2}$$ 225

Given the function:

$$y = (x+1)^3(x+2)$$

There are several different methods for solving this problem, but it is easiest to use logarithmic differentiation.

$y' =$ ....

------------------▷ 228

Explanation wanted

------------------▷ 226

Try to obtain the differential coefficient of the function $y = x^3$. Follow the procedure shown in the previous examples. 103

Note that if $\Delta x \to 0$, then $(\Delta x)^2$ and $(\Delta x)^3 \to 0$ also.

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots$$

→ 104

226

$$y = (x+1)^3(x+2)$$

We take logs on both sides:

$$\ln y = 3\ln(x+1) + \ln(x+2)$$

Now we differentiate with respect to x and solve for $\frac{dy}{dx}$.

Then we substitute back $(x+1)^3(x+2) = y$

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots$$

→ 228

Detailed calculation → 227

If $y = x^3$ then $\frac{dy}{dx} = 3x^2$ 104

Now it is time to have a BREAK

... and we know how to do it properly.

→ 105

227

$$y = (x+1)^3(x+2)$$

Taking logarithms

$$\ln y = 3\ln(x+1) + \ln(x+2)$$

Differentiating with respect to x yields:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x+1} + \frac{1}{x+2}$$

Solving for $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\frac{3(x+2)+(x+1)}{(x+1)(x+2)}$$

Substituting back y yields

$$\frac{dy}{dx} = \text{............................}$$

→ 228

5.5 Calculating Differential Coefficients

105

Objective: Application of the rules of differentiation, forming the derivatives of simple functions.

Mastery of the rules of differentiation is an important tool for the scientist and the engineer.

Read on for a further hint.

------------------▷ 106

$$y' = (4x+7)(x+1)^2$$

228

This should have been easy. You can check the answer by applying the rule for differentiating a product.

Here is a very different function:

$$y = \frac{xe^{2x}}{(x+3)^2}$$

$$y' = \text{............................}$$

------------------▷ 229

Learning technique: intensive reading, extracting (taking notes). 106

If we study a text intensively we must do so actively, e.g. by writing down proofs.

New concepts, definitions and symbols should be extracted.

It is recommended that you collect these extracts and keep them in a ring binder or suitable folder. Later we can use them as a basis for recalling the relevant subject matter.

A first recall should take place before each .... --------------------▷ 107

229

$$y' = \frac{e^{2x}(2x^2+5x+3)}{(x+3)^3}$$

Correct --------------------▷ 231

Wrong, detailed solution --------------------▷ 230

break

107

Abstracts and summaries are not exercises in calligraphy but they have to be readable.

A further hint: in order to write extracts clearly (take notes) it is recommended that you write concepts on the left and underline them. Place explanations on the right, using keywords.

Thus you can use your notes more efficiently as a learning aid, to memorise concepts and symbols and to test your knowledge.

First run through: Cover concepts and symbols.
From your explanation you should be able to recall the concepts.
Second run through: Cover explanations.
You should be able to recall the explanations by heart according to the concepts.

It is a sensible alternative idea to write down the concepts on the front side of a card and the explanations in keywords on the back. That way we build up a useful system for learning and revising.

------------------▷ 108

230

The function to differentiate is $y = \dfrac{x e^{2x}}{(x+3)^2}$

Step 1: Take logs on both sides: $\ln y = \ln x + 2x - 2\ln(x+3)$

Step 2: Differentiate with respect to x, remembering that

$$\frac{d}{dx}\ln y = \frac{d}{dy}(\ln y)\frac{dy}{dx}$$

$$\frac{d}{dx}\ln y = \frac{1}{y}y' = \frac{1}{x} + 2 - \frac{2}{x+3}$$

Step 3: Multiply through by y so that $y' = \dfrac{x e^{2x}}{(x+3)^2}\left(\dfrac{1}{x} + 2 - \dfrac{2}{x+3}\right)$

Step 4: Simplify by placing the terms in the brackets under a common denominator $x(x+3)$ and obtain

$$y' = \frac{e^{2x}}{(x+3)^3}(2x^2 + 5x + 3)$$

------------------▷ 231

108

READ (and take notes): **5.5 Calculating differential coefficients**
5.5.1 Derivatives of power functions; constant factors
5.5.2 Rules for differentiation
Textbook pages 102–108

-------------------▷ 109

231

5.12 Parametric Functions and their Derivatives

In the first part of this section the concept of a parametric function will be explained using well-known examples from physics. This concept is very powerful.

In the second part of the section we shall deal with the derivatives of parametric functions.

READ: **5.10 Parametric functions and their derivatives**
5.10.1 Parametric form of an equation
Textbook pages 131–136

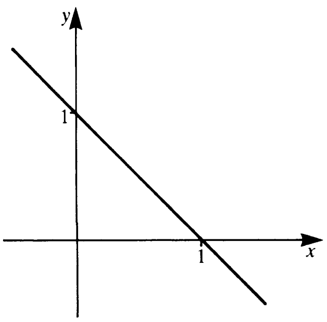
-------------------▷ 232

In sections 5.5.1 and 5.5.2 the following rules are dealt with: 109

Differentiation of

1. ....
2. ....
3. ....
4. ....
5. ....
6. ....
7. ....

-------------------▷ 110



Give the parametric form of the straight line in the figure. Denote the parameter by λ: 232

$x = $ ....

$y = $ ....

-------------------▷ 233

Differentiation of 110

1. power function
2. constant factor
3. sum (sum rule)
4. product (product rule)
5. quotient (quotient rule)
6. function of a function (chain rule)
7. inverse function

Did you recall all the rules?
You should have been able to do so with the help of your notes.

------------------▷ 111

$$x = \lambda$$
$$y = 1 - \lambda$$

233

Eliminate the parameter and get back to the usual form of the equation of this straight line.

$y =$ ....

------------------▷ 234

111

The technique of differentiation requires practice.
For this we need:

(a) knowledge of the general rules for differentiation;
(b) knowledge of the derivatives of simple functions.

For the time being we shall consider power functions.
Differentiate the following power functions:

1. $y = 5$ $\quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
2. $y = x^n$ $\quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
3. $y = \frac{1}{x^n}$ $\quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
4. $y = \sqrt{x}$ $\quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
5. $y = x^{-5/3}$ $\quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 112

234

$$y = 1 - x$$

Correct --------------------▷ 236

Explanation wanted --------------------▷ 235

112

1. $y' = 0$
2. $y' = nx^{n-1}$
3. $y' = (-n)x^{-(n+1)} = -\dfrac{n}{x^{n+1}}$
4. $y' = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$
5. $y' = \left(-\frac{5}{3}\right) x^{-8/3} = \dfrac{-5}{3\sqrt[3]{x^8}}$

☐ All correct --------------------▷ 116

☐ Error in exercise 1 --------------------▷ 113

☐ Errors in exercises 2–5 --------------------▷ 114

235

We start with the parametric form

$$x = \lambda$$
$$y = 1 - \lambda$$

We want to eliminate the parameter in order to obtain one single equation. In this case this is quite easy.
First express λ in terms of x:

$$\lambda = x$$

Second, insert λ in the second equation:

$$y = 1 - \lambda = 1 - x$$

--------------------▷ 236

The derivative of a constant is zero. 113

We have to distinguish between an additive constant and a constant in a product.

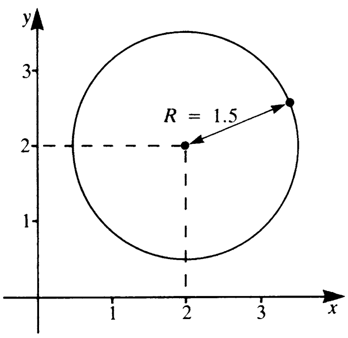
Examples:

$$y = a \qquad y' = 0$$
$$y = ax \qquad y' = a$$
$$y = cx^2 \qquad y' = \ldots\ldots\ldots$$
$$y = c + x^2 \qquad y' = \ldots\ldots\ldots$$

Check your answers with the help of the textbook.

Errors in exercises 2–5 ------------------▷ 114

Exercises 2–5 carried out correctly ------------------▷ 116



A point moves on the circle. 236

Give the parametric equation of the curve.

Denote the parameter by ϕ.

$$x = \ldots\ldots\ldots\ldots\ldots$$

$$y = \ldots\ldots\ldots\ldots\ldots$$

------------------▷ 237

All the exercises were concerned with differentiating a power function 114

$$y = x^n$$

n can be an arbitrary number, it does not have to be integral. The derivative is

$$y' = nx^{n-1}$$

You should know this rule by heart.

Calculate the derivatives of:

$y = x^2$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots$

$y = \sqrt[3]{x}$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots$

$y = x^{-2}$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 115

$$x = 2 + 1.5\cos\phi$$
$$y = 2 + 1.5\sin\phi$$ 237

Correct --------------------▷ 239

Explanation wanted --------------------▷ 238

115

$$y' = 2x$$

$$y' = \tfrac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$y' = -2x^{-3} = -\frac{2}{x^3}$$

0 ______

Write down the general expression for the derivative of

$$y = x^n \qquad y' = \text{............................}$$

Check your answer with the help of the textbook.

--------------------▷ 116

238

The example is similar to one given in the textbook.

The difference is that the center of the circle has the coordinates (2, 2).

The equations are obtained by adding the coordinates of the center to those of the circle centered at the origin.

$$x = 2 + 1.5\cos\phi$$

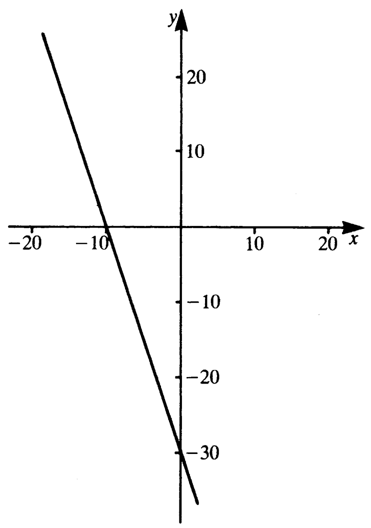
$$y = 2 + 1.5\sin\phi$$

--------------------▷ 239

We now go a step further and differentiate functions composed of power functions. 116

Using your extracts:
If $y = 3x^2 + 2x + 4$
find $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 117



Give parametric equations of the straight line in the diagram. 239

Denote the parameter by t.

$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 240

$$y' = 6x + 2$$

117

We used: sum rule

Differentiate

$$y = 2(x^2+1)\sqrt{x}$$
$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 119

Further help required

--------------------▷ 118

240

$$x = -10 + 10t$$
$$y = -30t$$

Other equations are possible. The following equations are also valid:

$$x = 5t$$
$$y = -30 - 15t$$

In case of doubt or difficulty consult the textbook.

The vector **a** can be chosen arbitrarily; thus different equations may result. But all give the same equation in the usual form if one eliminates the parameter. Eliminate the parameter in both pairs of equations above and show that in both cases the result is

$$y = -3x - 30$$

--------------------▷ 241

To differentiate

$$y = 2(x^2+1)\sqrt{x}$$

118

We note that it is the product of two functions, namely $2(x^2+1)$ and $\sqrt{x}$. We apply the product rule.

You may substitute

$$2(x^2+1) = u(x)$$
$$\sqrt{x} = v(x)$$

Then obtain $u'(x)$ and $v'(x)$ and apply the product rule.

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 119

5.13 Derivatives of Parametric Functions

241

Try to calculate the examples in the second part of this section without using the textbook.

READ: 5.10.2 Derivatives of parametric functions
Textbook pages 136–142

--------------------▷ 242

$$y' = 4x\sqrt{x} + 2(x^2+1)\frac{1}{2}\frac{1}{\sqrt{x}} = \frac{5x^2+1}{\sqrt{x}}$$ 119

We used: product rule, sum rule

Differentiate

$$y = \frac{x}{x^2+1}$$
$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 120

A point moves on a straight line. Its movement is given by the equations 242

$$x = -10\,\text{cm} + 10\,(\text{cm/s})t$$
$$y = -30\,(\text{cm/s})t$$

Give the magnitude of the velocity:

$$v = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: Obtain the components of the velocity first.

-------------------▷ 243

$$y' = \frac{1-x^2}{(x^2+1)^2}$$ 120

We used: quotient rule

When differentiating, be patient, concentrate and take it steadily; this way you will make fewer mistakes in lengthy exercises.

$$y = \frac{(x+1)\sqrt{x}}{(x-1)}$$

$$y' = \text{............................}$$

--------------------▷ 121

$$v = 31.6\,\text{cm/s} = \sqrt{100+900}\,\text{cm/s}$$ 243

A point moves on a helix. The position vector is given by

$$\mathbf{r}(t) = (R\cos\omega t,\ \ R\sin\omega t,\ \ \frac{t}{2\pi})$$

Evaluate the

velocity $\mathbf{v}(t) =$ ....
acceleration $\mathbf{a}(t) =$ ....

--------------------▷ 244

$$y' = \frac{(x-1)\left[\sqrt{x} + (x+1)\frac{1}{2\sqrt{x}}\right] - (x+1)\sqrt{x}}{(x-1)^2}$$ 121

Rearranging yields:

$$y' = \frac{x^2 - 4x - 1}{2\sqrt{x}(x-1)^2}$$

We now apply the chain rule.

Let us differentiate the function

$$y = (x^2 + 2)^5$$

We could multiply out the expression and apply the rule for power functions $y = x^n$. The chain rule makes the work easier. Substitute $x^2 + 2 = g(x)$ so that

$$y = (x^2 + 2)^5 = (g(x))^5$$

then $y' = \ldots\ldots\ldots\ldots\ldots$

-------------------▷ 123

If you require assistance

-------------------▷ 122

$$\mathbf{v}(t) = (-R\omega \sin \omega t, \quad R\omega \cos \omega t, \frac{1}{2\pi})$$
$$\mathbf{a}(t) = (-R\omega^2 \cos \omega t, \quad -R\omega^2 \sin \omega t, \quad 0)$$ 244

Now give the magnitude of velocity and acceleration.

$$|\mathbf{v}(t)| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$|\mathbf{a}(t)| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 245

This example is just like the one in the textbook in the section concerning the chain rule. 122

Look at the textbook, study the example and substitute so that:

$$y = (x^2 + 2)^5 \text{ and with } g = (x^2 + 2)$$

$$y = g^5$$

then $\dfrac{dy}{dg} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and $\dfrac{dg}{dx} = g' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

By the chain rule

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 123

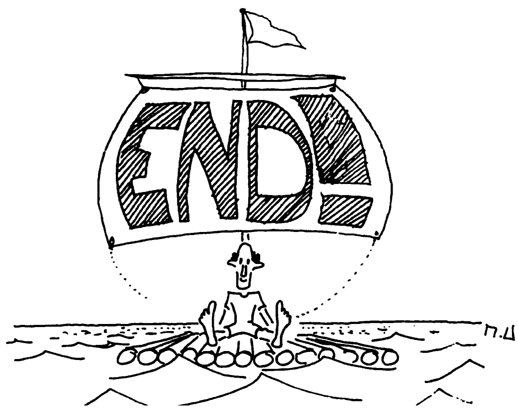
245

$$v = \sqrt{R^2\omega^2 + \left(\frac{1}{2\pi}\right)^2}$$

$$a = R\omega^2$$

This chapter has turned out to be rather lengthy. The reason is that the differential calculus is of fundamental importance for many areas of applied mathematics.

But before you have your break, which you do deserve now without doubt, do recapitulate the most salient points of Chapter 5.



$$y' = 5(x^2+2)^4 2x = 10x(x^2+2)^4$$

123

Derivative of the inverse function.
Given: a function $f(x)$

$$y = 4x+3$$

First obtain the inverse function $f^{-1}(x)$:
Write it down in two ways.

$x =$ ....

$y =$ ....

In case of difficulty consult your textbook, Chapter 1, section 1.4.
The concept of inverse functions is fundamental for the next exercises.

--------------------▷ 124

Please continue on page 1
(bottom half)