Chapter 22
Theory of Errors

→ 2

Visualisation activates your brain!
Many facts can be visualised:

42

— All (perhaps not really all) curves in which one parameter varies.
— You can also create mental pictures of vectors, for instance.

Visualisation or the creation of mental pictures can become a useful habit.

→ 43

The theory of errors requires the knowledge gained from Chapter 20. A brief recapitulation is advisable. Name at least three concepts from Chapter 20. 2

....

....

....

-------------------▷ 3

It is often useful to sketch facts on a sheet of paper and then to visualise them by creating a mental picture. 43

These pictures and their connections with the facts are coded and stored by the brain, making it easier to recall the facts when required later.

Using this technique, the same facts are coded within the brain in different ways and, so neurophysiologists say, in different places.

Furthermore, there are connections between these and consequently the same facts can be better retrieved afterwards.

-------------------▷ 44

Discrete probability distribution
Continuous probability distribution
Binomial distribution
Normal distribution
Mean value
Standard deviation

3

A coin is tossed 5 times. What is the probability $P(5;3)$ that a head will appear 3 times? (Head and tail are equally probable events.)

$$P(5;3) = \ldots\ldots$$

Hint: you can solve this exercise with the help of the binomial distribution.

--------------------▷ 4

You should be able to answer the next questions without the assistance of the textbook.
For a normal distribution:

44

(1) ......% of all test data are to be expected within the interval $\mu \pm \sigma$, and
(2) ......% of all test data are to be expected within the interval $\mu \pm 2\sigma$.

--------------------▷ 45

$$P(5;3) = \binom{5}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 = 0.3125$$

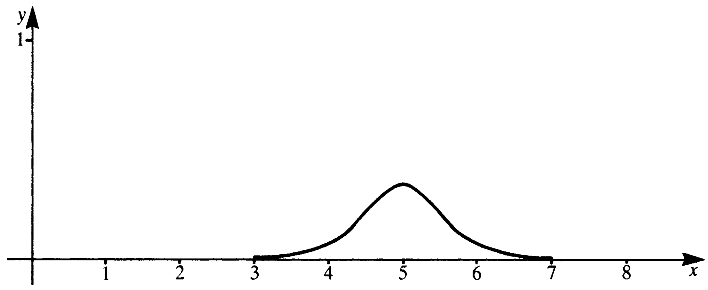
4

Given the probability density function of the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Its graph is shown below for $\sigma = 1$ and $\mu = 5$.

Sketch the graph of this function for $\sigma = \frac{1}{2}$ and $\mu = 5$ in the same coordinate system.



--------------------▷ 5

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(1) 68% (2) 95%

45

If we assume that the dispersion of test data is normally distributed about the mean value then it can be proved that:

The mean values of a series of readings are also normally distributed. The standard deviation of the mean values is given by

$$\sigma_M = \frac{\sigma}{\sqrt{n}}$$

The standard deviation of the mean value leads to the evaluation of the confidence intervals.

The diameter of a wire has been measured:

$$d = (0.1420 \pm 0.00065)\,\mathrm{mm}$$

Within what limits will the diameter lie with a probability of 95%?

Upper limit: ....

Lower limit: ....

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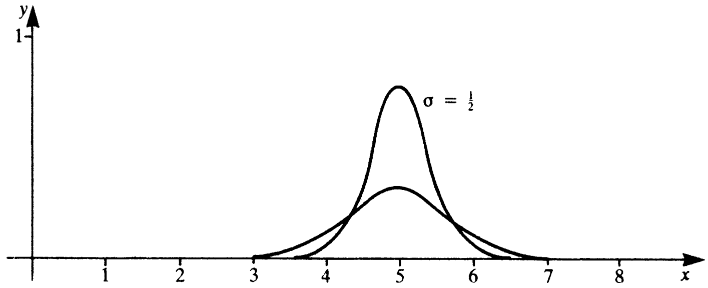


5

This exercise should have made it clear that the dispersion of the normal distribution is fixed by the parameter ....
and the mean value is given by ......

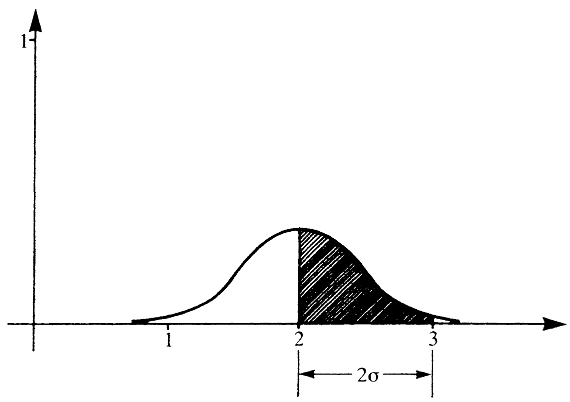
------------------▷ 6

Upper limit: 0.1433 mm
Lower limit: 0.1407 mm

46

In case of difficulty read section 22.5 again.

For a normal distribution the shaded area contains ......% of all test data.



------------------▷ 47

standard deviation σ
μ

6

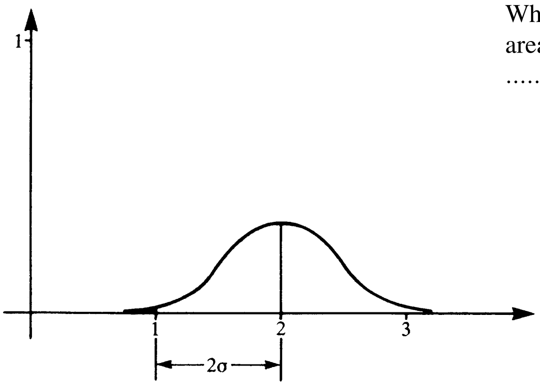
And now let us go on to the theory of errors.

--------------------▷ 7

47.5%

47

Explanation: The Gaussian distribution is symmetrical about the mean value μ. Since 95% of all data lie in the range $\mu - 2\sigma$ and $\mu + 2\sigma$ then half will lie inside the interval, i.e. 47.5%.



What percentage of the test data lies in the shaded area?
......% of all test data.

--------------------▷ 48

22.1 Purpose of the Theory of Errors, Mean Value, Variance

7

Objective: Concepts of systematic errors, random errors, variance, standard deviation, random sample, parent population, calculation of the standard deviation and variance for a series of readings.

READ: 22.1 Purpose of the theory of errors
22.2 Mean value and variance
Textbook pages 549–552

Suggestion: It will be beneficial to do the exercises in 22.1 and 22.2 in parallel with your studies.

------------------▷ 8

2.5%
Explanation: 5% of all test data lie outside the 2σ-interval (2 standard deviations), hence the area shown represents 2.5% of all test data.

48

------------------▷ 49

All measurements are subject to random errors. In addition, systematic (constant) errors can also occur. 8

A tape measure is used to obtain the length of a room.

(a) A tape measure is elongated by heavy use and its true length is now 1004 mm instead of 1000 mm. The error in measurement is ....
(b) A measurement is carried out using a flexible steel tape which has to be set several times. The datum lines are marked on the floor with a pencil. Wrong setting will result in .... errors.
(c) At the wall the steel tape has to be bent, making it difficult to read; this will result in .... errors.

------------------▷ 9

22.4 Error Propagation Law

49

Objective: Concept of error propagation law; calculation of the standard deviation of combined independent measurements.

Many physical quantities are computed from two or more measured quantities, e.g. $g = f(x, \ y, \ z)$. Therefore, the mean error of a quantity g depends on the error in the measurements of the individual quantities $x, \ y, \ z$, etc...

READ: 22.6 Law of error propagation
22.7 Weighted average
Textbook pages 558–560

------------------▷ 50

(a) systematic or constant
(b) random
(c) random

9

Correct --------------------▷ 13

Help required --------------------▷ 10

The Gaussian error propagation law states how the standard deviation of quantities that are not directly measurable can be determined by a combination of directly measurable quantities.

50

The formula in the textbook has not been derived. The application of this formula depends on your knowledge of partial derivatives which were treated in Chapter 12.

Do you know how to obtain the partial derivatives of a function?

Yes --------------------▷ 54

No --------------------▷ 51

Let us make clear once more the difference between systematic (constant) and random errors: 10

Systematic errors arise as a result of, for example, inaccurate calibration or inherent errors in measuring instruments or faulty measuring techniques.

Examples: If the diameter of a hose is measured using a pair of callipers then the hose will be slightly deformed each time because of the pressure exerted. The result is an error in the measured diameter. If a tape measure is elongated from heavy use then the measurements will be too small.

Systematic or constant errors usually falsify the measurement in one direction only giving either too high or two low a value.

The characteristic of random errors is that they are subject to unpredictable statistical fluctuations. Hence the results will turn out to be too high on one occasion and too low on another.

-------------------▷ 11

Partial derivatives are a prerequisite for this section. They are explained in Chapter 12. 51

You should revise or study Chapter 12 now.

-------------------▷ 52

11

What types of error will occur in the following situations?

(a) Unless an instrument with a pointer is read full face on a parallax error results. It is a .... error.
(b) The divisions of the scale of an ammeter are wide (thick) so that a reading has to be estimated. For the same position of the pointer different people will give different readings. The errors are ....
(c) The temperature of a very small mass of liquid is measured using a mercury thermometer. The thermometer absorbs heat from the liquid, hence its temperature decreases; the error is ....
(d) Scales not positioned horizontally give rise to a .... error.
(e) When weighing very small objects on a precision balance a draught will upset the reading because the pan will not always settle in the same position, giving rise to .... errors.

------------------▷ 12

52

(1) Given: $f(x,\ y) = ax^2y + \frac{3}{4}xy^3$

Obtain

$$\frac{\partial f}{\partial y} = f_y = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial x} = f_x = \ldots\ldots\ldots\ldots\ldots\ldots$$

(2) Given: another function with different variables: $f(D,\ G) = \dfrac{G}{\frac{4}{3}\left(\frac{D}{2}\right)^2}$

Obtain

$$\frac{\partial f}{\partial G} = f_G = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial D} = f_D = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 53

(a) systematic
(b) random
(c) systematic
(d) systematic
(e) random

12

--> 13

---

(1) $f_y = ax^2 + \frac{9}{4}xy^2, \;\; f_x = 2axy + \frac{3}{4}y^3$

(2) $f_G = \dfrac{3}{D^2} \quad f_D = \dfrac{-6G}{D^3}$

53

If you didn't get these answers then you ought to read the relevant section in Chapter 12 again, and do the accompanying exercises.

--> 54

The volume of a necklace with a pendant is determined by the overflow method. A container is filled with a liquid and the necklace is immersed in it. The liquid overflows into a groove and is collected in a measuring cylinder. 13

The test is repeated 10 times. We want to calculate the mean value, variance and standard deviation for this series of measurements.

Data:

Volume (unit of measurement: $10^3\,\mathrm{mm}^3$)

2.4, 2.7, 2.6, 2.5, 2.4, 2.6, 2.7, 2.6, 2.8, 2.7

First calculate the mean value

$$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 14

Two electrical resistors R_1 and R_2 have been measured several times and the following resistance values were obtained: 54

$$R_1 = (150 \pm 0.9)\,\text{ohms}(\Omega)$$
$$R_2 = (220 \pm 1.1)\,\text{ohms}(\Omega)$$

If these two resistors are connected in parallel what is the value and the standard deviation of the resultant resistance R?

The resultant of total resistance for a parallel connection is given by the equation:

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

hence

$$R = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 55

$$\bar{x} = 2.6 \times 10^3\,\text{mm}^3$$

14

To calculate the variance and standard deviation we can draw up a table and compute the deviations of the measurements about the mean and the squares of these deviations.

-------------------▷ 15

$R = 89.19\Omega$ to 2 d.p.

55

We now apply the law of error propagation to calculate the mean error in R.

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

The expression for R corresponds to $g = f(x,\ y)$ in the textbook. We then obtain

$$\sigma_{MR} = \dots\dots\dots\dots\dots\dots$$

I can solve the exercise

-------------------▷ 59

I need advice

-------------------▷ 56

Complete the table: 15

Measured values x_i [$10^3\,\text{mm}^3$]	Deviation from the mean $(x-\bar{x})$ [$10^3\,\text{mm}^3$]	$(x-\bar{x})^2$ [$10^6\,\text{mm}^6$]
2.4		
2.7		
2.6		
2.5		
2.4		
2.6		
2.7		
2.6		
2.8		
2.7		

It is advisable to repeat the calculations. Use your calculator.

--------------------▷ 16

The resistance R corresponds to the quantity g in the textbook, with R_1 corresponding to x and R_2 to y. 56

The function is

$$g = f(x, \; y) = \frac{xy}{x+y}$$

or

$$R = f(R_1, \; R_2) = \frac{R_1 R_2}{R_1 + R_2}$$

The formula for the law of propagation of the error is

$$\sigma_{Mg} = \sqrt{(f_x \sigma_x)^2 + (f_y \sigma_y)^2}$$

Hence in the case of the resistances

$$\sigma_{MR} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 57

Here is the completed table. We shall use it again. 16

x_i $[10^3\,\text{mm}^3]$	$(x_i - \bar{x})$ $[10^3\,\text{mm}^3]$	$(x_i - \bar{x})^2$ $[10^6\,\text{mm}^6]$
2.4	−0.2	0.04
2.7	0.1	0.01
2.6	0.0	0.00
2.5	−0.1	0.01
2.4	−0.2	0.04
2.6	0.0	0.00
2.7	0.1	0.01
2.6	0.0	0.00
2.8	0.2	0.04
2.7	0.1	0.01

--------------------▷ 17

$$\sigma_{MR} = \sqrt{\left(\frac{\partial f}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial f}{\partial R_2}\right)^2 \sigma_{R_2}^2}$$ 57

We know $\sigma_{R1} = 0.9\Omega$ and $\sigma_{R2} = 1.1\Omega$.

We must obtain the partial derivatives:

$$\frac{\partial f}{\partial R_1} = \frac{\partial}{\partial R_1}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \frac{R_2^2}{(R_1 + R_2)^2} = \ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial R_2} = \frac{\partial}{\partial R_2}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \frac{R_1^2}{(R_1 + R_2)^2} = \ldots\ldots\ldots\ldots$$

Now insert the numerical values of R_1 and R_2.

--------------------▷ 58

We must be careful when calculating the variance because we have to distinguish between variance of the random sample and the estimated variance for the parent population. 17

Variance of the random sample: $s^2 =$ ....

Estimated variance of the parent population: $\sigma^2 =$ ....

------------------▷ 18

58

$$\frac{\partial f}{\partial R_1} = \frac{(220)^2}{(150+220)^2} = 0.35\Omega$$

$$\frac{\partial f}{\partial R_2} = \frac{(150)^2}{(150+220)^2} = 0.16\Omega$$

We now recall the general equation:

$$\sigma_{MR} = \sqrt{\left(\frac{\partial f}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial f}{\partial R_2}\right)^2 \sigma_{R_2}^2}$$

Substituting numerical values yields:

$$\sigma_{MR} = \text{..............................}$$

and $R =$ ....

------------------▷ 59

$$s^2 = \frac{0.16 \times 10^6}{10} = 0.016 \times 10^6\,\text{mm}^6$$
$$\sigma^2 = \frac{0.16 \times 10^6}{9} = 0.018 \times 10^6\,\text{mm}^6$$

18

Finally, calculate the best estimate of the standard deviation of the test data $\sigma =$ ....

-------------------▷ 19

$$\sigma_{MR} = 0.36\,\Omega$$

Hence $$R = (89.19 \pm 0.36)\,\Omega$$

59

What is the total resistance R and the standard deviation if the resistors $R_1 = (150 \pm 0.9)\,\Omega$ and $R_2 = (220 \pm 1.1)\,\Omega$ are connected in series?

$$R = R_1 + R_2$$
$$R = \text{.............}$$
$$\sigma_{MR} = \text{.............}$$

-------------------▷ 60

$$\sigma = 0.13 \times 10^3\,\text{mm}^3$$

19

Try to formulate in your own words (keywords) the significance of the standard deviation.

....

....

....

--------------------▷ 20

$$R = 370\,\Omega$$
$$\sigma_{MR} = 1.42\,\Omega$$
$$R = (370 \pm 1.42)\,\Omega$$

60

Correct --------------------▷ 62

Wrong --------------------▷ 61

You could have written: 20

The individual values are dispersed about the mean value; the standard deviation is a measure of the dispersion of the values from the mean. The smaller the value of the standard deviation the closer the measured values are to the mean, and vice versa.

68% of the test values have a deviation less than $\pm\sigma$ (1 standard deviation) and about 32% have a deviation greater than $\pm\sigma$. These values are valid for random errors which are normally distributed as explained in section 22.5.

Calculation of the mean value and standard deviation is a routine task but we have to pay attention to the units used. It is advisable to use some standard method for carrying out these calculations.

------------------▷ 21

Here is the solution: 61

$$R_1 = (150 \pm 0.9)\,\Omega, \quad R_2 = (220 \pm 1.1)\,\Omega$$

$$R = R_1 + R_2 = 370\,\Omega$$

$$\frac{\partial R}{\partial R_1} = 1, \quad \frac{\partial R}{\partial R_2} = 1$$

$$\sigma_{MR} = \sqrt{\left(\frac{\partial R}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial R}{\partial R_2}\right)^2 \sigma_{R_2}^2} = \sqrt{1 \times 0.9^2 + 1 \times 1.1^2}$$

$$= \sqrt{(0.81 + 1.21)\,\Omega^2} = \sqrt{2.02\,\Omega^2}$$

$$\sigma_{MR} = 1.42\,\Omega$$

$$R = (370 \pm 1.42)\,\Omega$$

------------------▷ 62

The diameter of a wire has been measured 5 times. The results are given in the table: 21

d_i $[10^{-2}\text{mm}]$	$(d_i - \bar{d})$	$(d_i - \bar{d})$
4		
3		
4		
5		
6		

Calculate the mean value and estimate the standard deviation

$\bar{d} =$ ....

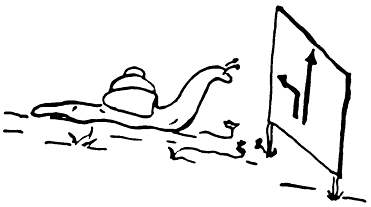
$\sigma =$ ....

-------------------▷ 22

Interpretation of the error propagation law, in words: 62

If we determine a quantity from several individual quantities then the quality of the final result is determined by the quality of the individual values.

Do you think you would benefit from another exercise?



☐ No -------------------▷ 65

☐ Yes -------------------▷ 63

$$\bar{d} = 4.4 \times 10^{-2}\,\text{mm}$$
$$\sigma = 1.14 \times 10^{-2}\,\text{mm}$$

22

You will find the details of the computation in the table:

d_i in mm	$(d_i - \bar{d})$ in mm	$(d_i - \bar{d})$ in mm^2
4×10^{-2}	-0.4×10^{-2}	0.16×10^{-4}
3×10^{-2}	-1.4×10^{-2}	1.96×10^{-4}
4×10^{-2}	-0.4×10^{-2}	0.16×10^{-4}
5×10^{-2}	0.6×10^{-2}	0.36×10^{-4}
6×10^{-2}	1.6×10^{-2}	2.56×10^{-4}
sum: 22×10^{-2}	0	5.20×10^{-4}

$$\bar{d} = \frac{22 \times 10^{-2}\,\text{mm}}{5} = 4.4 \times 10^{-2}\,\text{mm}$$
$$\sigma^2 = \frac{1}{(5-1)} \times 5.20 \times 10^{-4} = 1.30 \times 10^{-4}\,\text{mm}^2$$
$$\sigma = 1.14 \times 10^{-2}\,\text{mm}$$

-------------------▷ 23

Example:
The sides of a cube have been measured; they are

63

$$x = (22 \pm 0.1)\,\text{mm}$$
$$y = (16 \pm 0.08)\,\text{mm}$$
$$z = (10 \pm 0.08)\,\text{mm}$$

Calculate the volume $V = xyz$ of the cube and the standard deviation σ_M.

$V = \ldots\ldots\ldots\ldots\ldots\ldots$

$\sigma_M = \ldots\ldots\ldots\ldots\ldots\ldots$

Final result, $V = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 64

If you experienced difficulties with the solution of the last example you should read section 22.2 in the textbook again. 23

It is important that you understand the following:

(1) A series of readings is a random sample of all possible test data.
(2) A series of readings has a mean value, a variance and a standard deviation.
(3) The parent population of all test data also has a mean value, a variance and a standard deviation. These values are estimated on the basis of the values obtained for the random sample.

For all calculations we use the formulae on page 541 in the textbook.

If you have not yet done exercise 22.2 on page 541 you should do so now!

-------------------▷ 24

64

$$
\begin{aligned}
V &= 3520\,\mathrm{mm}^3\\
\sigma_{MV} &= 36.9\,\mathrm{mm}^3\\
V &= (3520 \pm 36.9)\,\mathrm{mm}^3
\end{aligned}
$$

$$
\begin{aligned}
x &= (22 \pm 0.1)\,\mathrm{mm},\ y = (16 \pm 0.08)\,\mathrm{mm},\ z = (10 \pm 0.08)\,\mathrm{mm}\\
V &= xyz = 3520\mathrm{mm}^3\\
\frac{\partial V}{\partial x} &= \frac{\partial}{\partial x}(xyz) = yz = 160\,\mathrm{mm}^2\\
\frac{\partial V}{\partial y} &= xz = 220\,\mathrm{mm}^2,\ \frac{\partial V}{\partial z} = xy = 352\,\mathrm{mm}^2\\
\sigma_{MV} &= \sqrt{160^2\mathrm{mm}^4 \times 0.1^2\mathrm{mm}^2 + 220^2\,\mathrm{mm}^4 \times 0.08^2\,\mathrm{mm}^2 + 352^2\,\mathrm{mm}^4 \times 0.08^2\,\mathrm{mm}^2}\\
&= \sqrt{256\,\mathrm{mm}^6 + 4.84 \times 64\,\mathrm{mm}^6 + 12.39 \times 64\,\mathrm{mm}^6}\\
&= \sqrt{(256 + 310 + 793)\,\mathrm{mm}^6} = \sqrt{1359\,\mathrm{mm}^6}\\
\sigma_{MV} &= 36.9\,\mathrm{mm}^3\\
V &= (3520 \pm 36.9)\,\mathrm{mm}^3
\end{aligned}
$$

-------------------▷ 65

22.2 Mean Value and Variance of Continuous Distributions Errors of the Mean Value

24

Objective: Concepts of deviation of the mean value, sampling error of the mean value, confidence intervals, evaluation of the standard deviation of the mean value.

Note: The short section 'Mean value and variance of continuous distributions' is an extension of the concepts derived for discrete test data. The most important section in the theory of errors is the 'error in mean value'. It determines the reliability of the mean value of the test data.

READ: 22.3 Mean value and variance of continuous distributions
22.4 Error in mean value
Textbook pages 554–557

-------------------▷ 25

22.5 Curve Fitting: Method of Least Squares, Regression Line

65

Objective: Concept of curve fitting, regression line.

We have seen that the mean value of a series of readings is more reliable than the individual measurements and that the sum of the squares of the deviations or errors is a minimum. In this section this fundamental idea is used to fit a curve through a series of data points.

However, we illustrate the technique with a straight line only as the best fit to a series of data points.

READ: 22.8 Curve fitting: method of least squares, regression line
22.9 Correlation and correlation coefficient
Textbook pages 561–567

-------------------▷ 66

Our goal is to determine the error of the mean value of a series of measurements. 25

We use a different notation for the standard deviation of the mean value; it is σ_M (as opposed to σ, which is used for a sample). We should become familiar with this distinction. In other texts these concepts are often used synonymously.

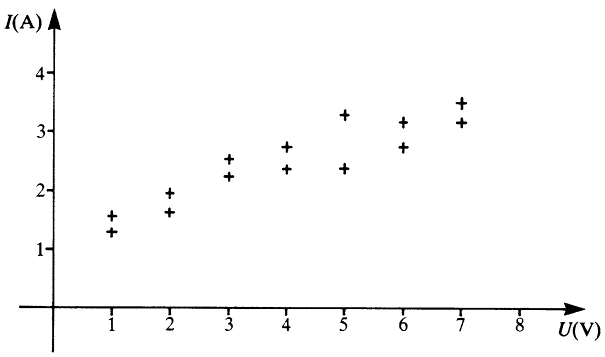
The standard deviation of the mean value of a series of measurements is the smaller the larger the number of measurements n. The following relation is valid:

$$\sigma_M = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is called ....

--------------------▷ 26

66



The figure shows a number of data points representing the current I flowing in a light bulb as a function of the voltage. By eye, sketch what you consider to be the best curve through these data points. Draw a curve free hand.

--------------------▷ 67

$$\sigma_M = \frac{\sigma}{\sqrt{n}}$$ 26

the sampling error, or mean error of the mean value.

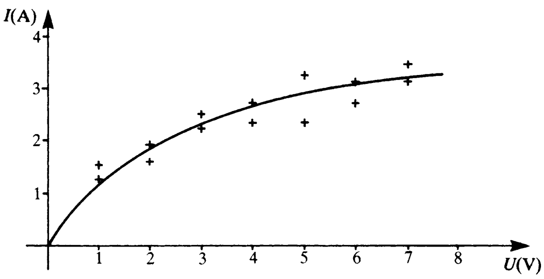
Without the assistance of the example in the textbook, calculate the error of the mean value given the following data:

A series of 11 measurements (wire diameter) has a mean value $\bar{d} = 0.142\,\text{mm}$ and a variance $\sigma^2 = 0.046 \times 10^{-4}\,\text{mm}^2$.

$\sigma_M = $ ....

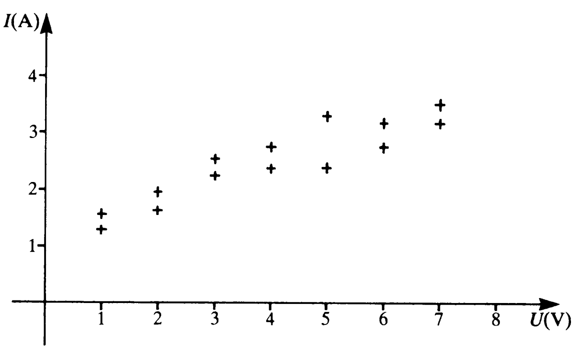
-------------------▷ 27

67



The figure shows a free-hand curve through the data points. We consider this curve to represent the mean values through these values.

Now replace this curve by the best straight line through the points.



-------------------▷ 68

$$\sigma_M = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{4.6 \times 10^{-6}}{11}} \approx 0.00065\,\text{mm}$$ 27

You may find such calculations a little boring, but without doing them you will not get the feel of the problem and you will probably deceive yourself as to your understanding of the subject. It is only with practice that you will acquire this understanding.

In practice it is usual to quote the mean value and the sampling error of a series of readings in the following way:
wire diameter $d = \mu \pm \sigma_M$
in our case, $d =$ ....

-------------------▷ 28

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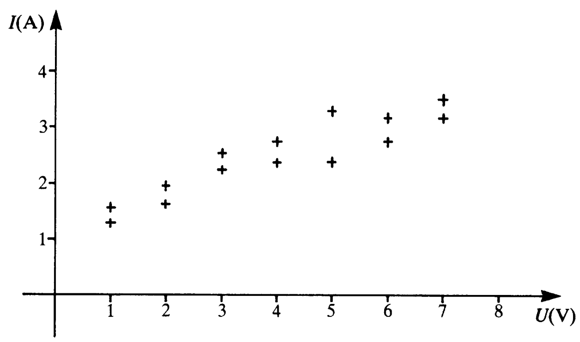


Depending on the measurements and on theoretical consideration the curve which may best fit the data could be, for example, a parabola, an exponential or a logarithmic function or even a sine function. However, we often do not have a definite idea about the shape of the curve. In such situations the simplest curve is a straight line.

The technical name for it is: ....

-------------------▷ 69

$$d = 0.142 \pm 0.00065\,\text{mm}$$

28

The standard deviation of the mean value can often be rounded.

Reason: The standard deviation of the mean value is a measure of the accuracy of the mean value, and this measure is itself the result of an estimation. It is therefore senseless to give too many digits.

Describe in your own words the significance of the standard deviation of the mean value in our example.

....

....

--------------------▷ 29

---

a regression line

69

Have you done and understood the example in the textbook?

☐ Yes --------------------▷ 76

☐ No --------------------▷ 70

You could have written: 29

The true value is, with a 68% probability, in the interval $(0.142 - 0.00065)$ mm to $(0.142 + 0.00065)$ mm. This interval is called the confidence interval.

Let us compute the standard deviation of the mean value in the example concerning the necklace (cf. frame 13).

After 10 measurements the volume V of the necklace was found to be

$$\bar{V} = 2.60 \times 10^3\,\text{mm}^3$$

Standard deviation of the individual measurements

$$\sigma = 0.13 \times 10^3\,\text{mm}^3$$

Standard deviation of the mean value: $\sigma_M =$ ....

Hence the volume is: $V =$ ....

------------------▷ 30

It may be useful to carry out a simple example step by step. 70

The following values of current and voltage have been measured:

U_i (V)	2	3	4	5	6
I_i (A)	1.3	1.7	2.1	2.1	2.9

We want to find the regression line.

First, which products do we have to compute and sum? Set up a table!

In the textbook we obtained the regression line for an $x - y$ system of coordinates. Now the variables are U and I.

------------------▷ 71

$$\sigma_M = \frac{\sigma}{\sqrt{10}}$$
$$\sigma_M = 0.04 \times 10^3\,\text{mm}^3$$
$$V = (2.60 \pm 0.04) \times 10^3\,\text{mm}^3$$

30

We can therefore expect the true value to lie in the range from $2.56 \times 10^3\,\text{mm}^3$ to $2.64 \times 10^3\,\text{mm}^3$ with a probability of 68%.

This implies that the true value lies outside this interval with a probability of 32%. Such uncertainty is often too large in practice.

Instead, one frequently gives the interval which contains the true value with 95% certainty.
Upper limit: ....
Lower limit: ....

--------------------▷ 31

You should have drawn up a table like this: 71

	$U_i\,(\text{V})$	$U_i^2\,(\text{V}^2)$	$I_i\,(\text{A})$	$U_i I_i\,(\text{VA})$
	2		1.3	
	3		1.7	
	4		2.1	
	5		2.4	
	6		2.9	
Σ				

Complete the table.

--------------------▷ 72

Upper limit: $2.68 \times 10^3\,\text{mm}^3$
Lower limit: $2.52 \times 10^3\,\text{mm}^3$

31

The accuracy of the true value can be enhanced by increasing the number of individual measurements. Given 10 individual measurements in a particular case, how many measurements do we need to bring down the standard deviation of the mean value to

(a) one-half? $n_a = \ldots\ldots\ldots\ldots\ldots\ldots$

(b) one-third? $n_b = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 32

72

	U_i (V)	U_i^2 (V^2)	I_i (A)	$U_i I_i$ (VA)
	2	4	1.3	2.6
	3	9	1.7	5.1
	4	16	2.1	8.4
	5	25	2.4	12.0
	6	36	2.9	17.4
Σ	20	90	10.4	45.5

The mean values of the voltage and the current can be computed:

$$\bar{U} = \ldots\ldots\ldots\ldots\ldots$$
$$\bar{I} = \ldots\ldots\ldots\ldots\ldots$$

------------------▷ 73

(a) $n_a = 40$
(b) $n_b = 90$

32

Correct → 36

Wrong, or explanation required → 33

$$\bar{U} = 4\,\text{V}(\text{corresponds to } \bar{x})$$
$$\bar{I} = 2.08\,\text{A}(\text{corresponds to } \bar{y})$$

73

Here is the table once more:

	$U_i\,(\text{V})$	$U_i^2\,(\text{V}^2)$	$I_i\,(\text{A})$	$U_i\,I_i\,(\text{VA})$
	2	4	1.3	2.6
	3	9	1.7	5.1
	4	16	2.1	8.4
	5	25	2.4	12.0
	6	36	2.9	17.4
Σ	20	90	10.4	45.5

Inserting the sum in the equation $a = \dfrac{\sum x_i\,y_i - n\bar{x}\,\bar{y}}{\sum x_i^2 - n\bar{x}^2}$

$a =$ ....
$b =$ ....

→ 74

In this case the starting point was 10 readings from which the standard deviation of the mean value was obtained. 33

We now ask ourselves: how many measurements should we carry out in order to halve this standard deviation?

The formula $\sigma_M = \frac{\sigma}{\sqrt{n}}$ contains the answer!

If we increase n the deviation σ_M decreases. If we want to double the denominator we must multiply n by 4, and if we want to treble the denominator we must multiply n by ....

------------------▷ 34

$a = 0.39\,\text{amps/volt}$
$b = 0.52\,\text{amps}$ 74

Correct ------------------▷ 76

I need assistance ------------------▷ 75

9 (a ninefold increase in n) 34

An increase in the number of measurements reduces the standard deviation of the mean value; this means that the difference between the true value and our mean value will probably decrease.

How many measurements are necessary to bring down the sampling error from $0.04 \times 10^3\,\text{mm}^3$ to $0.01 \times 10^3\,\text{mm}^3$?

We originally had $n = 10$ measurements.

Now we need $n = \ldots\ldots\ldots\ldots\ldots\ldots$ measurements.

------------------▷ 35

Perhaps your difficulty lies in understanding the equation for calculating a: a slight confusion with symbols possibly? In this case it is advisable to replace U by x and I by y in the equation. 75

At the end of the computation you can substitute back for U and I.

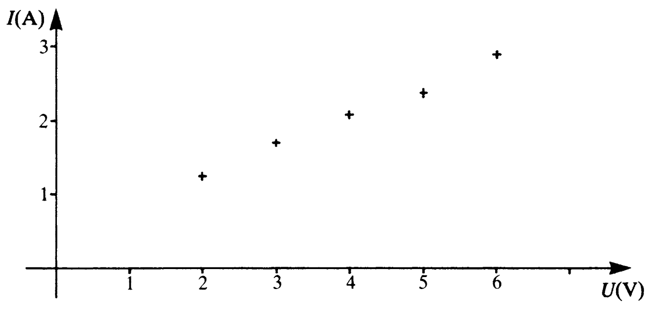
------------------▷ 76

We need approximately $n = 160 = 10 \times 4^2$ measurements. 35

Now compute the standard deviation of the mean value for exercises 22.2(a) and (b) in the textbook.

--------------------▷ 36

Here are the measured values in a coordinate system: 76



Try to fit a straight line through the points by eye. Then plot the straight line obtained by the method of least squares, namely

$$I = 0.39U + 0.52$$

--------------------▷ 77

22.3 Normal Distribution, Distribution of Random Errors

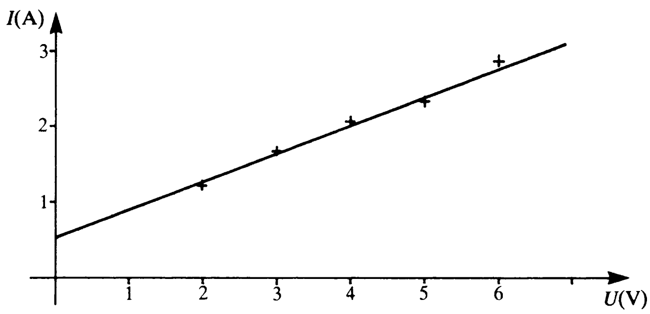
36

Objective: Concept of elementary error.

This section on the normal distribution borders on aspects of Chapter 20, where we defined the normal distribution as the limiting case of the binomial distribution.

READ: 22.5 Normal distribution Distribution of random errors
Textbook pages 557–558

--------------------▷ 37



77

In the textbook there follows a section on correlation and the correlation coefficient. These concepts are developed to measure the quality of the approximation of data sets by a regression line. But since this problem is of special interest only the study guide ends here.

This chapter on the theory of errors is quite important. We therefore repeat the content very briefly, but you should have made your own brief notes as you read the textbook!

--------------------▷ 78

The judgement as to the accuracy of the mean value of a series of readings is based on the assumption that the dispersion of the test data about this value is a normal distribution. 37

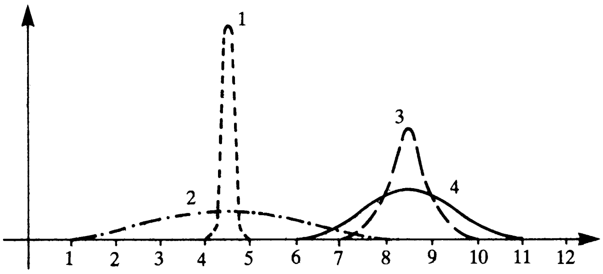
This assumption might appear at first glance to be quite arbitrary; it has however been confirmed by observation. If we carry out a large number of measurements under identical conditions and draw a graph of the results we find that, in very many cases, the distributions follow the Gaussian bell-shaped curve.

-------------------▷ 38

---

(1) With the help of the theory of errors the magnitude of .... can be estimated. 78
(2) The quantity which is a measure of the dispersion of the individual values about the mean is called ....
(3) The dispersion of the test data can be estimated on the basis of random samples.
(4) The square root of the variance is also a measure of the dispersion; it is called ....

-------------------▷ 79



38

The figure shows four different Gaussian distributions which differ in the position of the mean value and in the value of the standard deviation. Arrange the distributions 1, 2, 3 and 4 in the order of increasing σ from the smallest to the largest value.

--------------------▷ 39

79

(1) random errors
It is important to realise that the theory of errors does not make any allowance for systematic (constant) errors.

(2) variance
A series of readings is a random sample out of a parent population of all possible test data.

(4) standard deviation
68% of all test data deviate by less than one standard deviation about the mean.

--------------------▷ 80

1, 3, 4, 2

39

We can improve our retention of things through visualisation. This is a technique which depends very much on the individual but is particularly valuable for people who have a good imagination. Such a person often uses it without thinking that it is something special.

I would like to skip the following instructions ------------------▷ 44

I would like to learn more about imagination ------------------▷ 40

80

The most important conclusions are:

(5) The dispersion of the mean value is less than that of the individual values; hence the mean value is .... than the individual values. For this reason we carry out a number of series of measurements in science and engineering.

(6) The variance and the dispersion of the mean value can be determined by using the individual test data, and the following relation then holds true $\sigma_M =$ ....

(7) A quantity is computed from a series of measurements of several other quantities. These quantities are subject to errors; hence the error in the required quantity is obtained in accordance with the ....

(8) The method of least squares leads us to a deeper understanding of curve fitting. We have only dealt with the simplest case by fitting a straight line through the data points.
Such a straight line is often referred to in the literature as a ....

------------------▷ 81

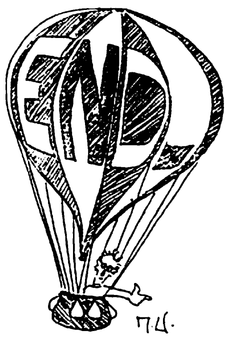
40

The rule is a simple one: we create a mental picture of the facts presented to us. Example: we can imagine the Gaussian distribution as a pointed bell-shaped curve which changes from being pointed to being flatter as the value of the standard deviation increases. (At the same time the maximum value decreases.) The reverse process can also be imagined. You saw this in frame 39; close your eyes for a minute and imagine these curves and you will virtually 'see' them being recreated for you by your memory. You can achieve this by concentrating hard!

------------------▷ 41

81

(5) more reliable
(6) $\sigma_M = \dfrac{\sigma}{\sqrt{n}}$
(7) error propagation law
(8) regression line

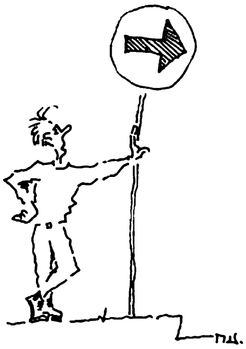


Many people — not all — succeed in their first attempt to create such a mental picture of the facts presented to them. If you did not achieve this in your first attempt, try again; don't give up! 41

The person who succeeds will probably not forget what he has learned.

Bower, an American psychologist, found on the basis of empirical investigation that only 30% to 50% of facts are retained without a mental picture, whereas people using this technique retained 50% to 80% of the facts. Hence the benefits of this technique justify the effort required.

As an exercise, try to imagine how the Gaussian distribution shifts to the right or left as the mean value μ changes.



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Please continue on page 1
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