Chapter 14
Transformation of Coordinates; Matrices

→ 2

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(3) Learning plateau — something can be done about it!

Within this book it is possible to mention just a few points about the complex structure of learning difficulties.

- The division of the field of study into different sections very often leads to overall coherence being lost. Try to structure the study material when you are revising! Terms and essential statements should be seen in relation to each other. Revision from the notes you have written yourself is especially effective.
- Lack of work planning is frequently conductive to uneconomical learning. A change in certain working and living habits may lead to greater subjective satisfaction in one's personal capacity. In some passages we have tried to give some advice on this topic.

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14.1 Transformation of Coordinates; Matrices

2

Objective: Concepts of transformation, rotation.

In the introduction the importance of the choice of a suitable coordinate system is pointed out. It can save a considerable amount of work or even make possible the solution of a problem.

READ: 14.1 Introduction
Textbook pages 403–406

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14.6 Matrix Algebra

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Objective: Concepts of matrices, columns and rows of a matrix, square matrices, addition, subtraction and multiplication of matrices, multiplication of matrices by vectors and scalars.

READ: 14.4 Matrix algebra
Textbook pages 415–420

Don't forget to work in parallel with the text, taking notes and doing the examples yourself. This section is quite long and contains new concepts and rules. You may therefore wish to split it into a number of subsections with short breaks in between.

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What kinds of transformations are mentioned? 3
Can you write them down from memory?

(1) ....
(2) ....

→ 4

Write down the columns and rows of the matrix 40

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 6 & 2 \end{pmatrix}$$

Columns: ....
Rows: ....

→ 41

(1) Shift or translation
(2) Rotation

4

In textbooks usually a suitable coordinate system has already been chosen by the authors when solving a problem. Part of the work has already been done.

However, if you have to solve a problem yourself from scratch, you must decide upon a suitable coordinate system. You may also have to transform one coordinate system into another, hence the importance of coordinate transformations which we will consider in what follows.

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Columns: $\begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

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Rows: $(1 \quad 2 \quad 4)$, $(-3 \quad 6 \quad 2)$

When we consider a single row or column of a matrix we speak of a *row vector* or a *column vector*.

Specify the types and the orders of the following two matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 8 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 5 & 2 & 1 \\ 7 & 3 & 9 \end{pmatrix}$$

A is a .... matrix of order ....
B is a .... matrix of order ....

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14.2 Parallel Shift of Coordinates: Translation

5

Objective: Concepts of shift of coordinates and of transformed vectors.

READ: **14.2 Parallel shift of coordinates: translation**
Textbook pages 406–409

You should aim to follow the reasoning in your own words by writing the arguments down in your note book. It will help you to sort out your difficulties.

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A is a square matrix of order 2×2, $(n \times n)$
B is a rectangular matrix of order 2×3, $(m \times n)$

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Given: the four matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \\ 2 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \\ 2 & 0 \end{pmatrix}, \ \mathbf{D} = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & 0 \end{pmatrix}$$

Which matrices are equal?

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A sphere of radius $R = 2$ units has its center at the point O′ = (3, 2, 4). Its equation is

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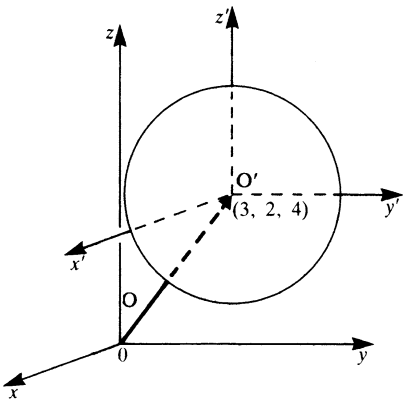

$$4 = (x-3)^2 + (y-2)^2 + (z-4)^2$$

The following transformation represents a shift of the origin O to the point O′:

$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

The equation becomes

$$4 = x'^2 + y'^2 + z'^2$$

The coordinates of the original position vector to the center of the sphere were (3, 2, 4).

After the transformation the coordinates of the new position vector to the center of the sphere are:........................

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Matrices **A** and **C** are equal since

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(i) they are of the same order, 3×2 in this case,

(ii) the elements of **A** are equal to the corresponding elements of **C**,

Generally speaking

$$a_{ik} = c_{ik} \qquad i = 1, 2, 3, \ldots, m$$
$$k = 1, 2, 3, \ldots, n$$

Add the two matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -3 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

What are the necessary conditions for adding (or subtracting) matrices?

....

C = A + B =

....

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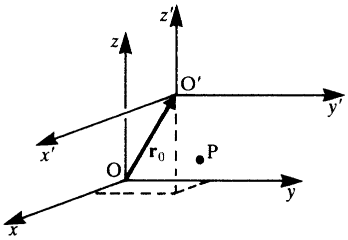
(0, 0, 0) 7



What are the new coordinates of the position vector to the point P = (5, 7, 2) after the coordinate transformation:

$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

$\mathbf{r}' = (2, \ 5, \ -2)$

--------------------▷ 8

$\mathbf{r}' = (8, \ 9, \ 6)$

--------------------▷ 9

The conditions for adding (or subtracting) matrices are that they must be of the same order, $m \times n$. 44

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} 4-3 & 0+1 & 2-1 \\ 2+0 & 0+1 & 4-2 \\ 0+1 & 1+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Now for another exercise:

Given the matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 2 & 0 & -4 & 2 \\ 0 & 11 & 0 & 10 \end{pmatrix}$

Multiply **A** by the number 3, a scalar:

$3\mathbf{A} =$

....

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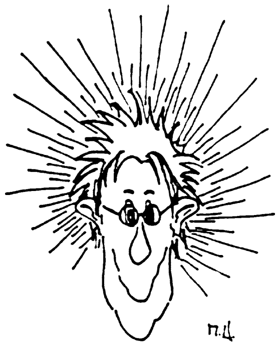
Correct!
If we insert the transformation



$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

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in the equation for the coordinates x, y and z of the point P(5, 7, 2), we obtain $\mathbf{r}' = (2,\ 5,\ -2)$

I would like another exercise --------------------▷ 11

I want to carry on --------------------▷ 13

$$3\mathbf{A} = \begin{pmatrix} 12 & 0 & 6 & 3 \\ 6 & 0 & -12 & 6 \\ 0 & 33 & 0 & 30 \end{pmatrix}$$

The result is obtained by multiplying every element of the matrix by the number 3.

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Now suppose we wish to multiply a matrix $\mathbf{A}$ by a vector $\mathbf{r}$ given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$ is a matrix of order 2×2, and $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is a 2×1 column vector.

Then we obtain:

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -3 & 6 \end{pmatrix}$$ --------------------▷ 46

$$\begin{pmatrix} 4+8 \\ -3+6 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$$ --------------------▷ 48

$$\begin{pmatrix} 4 & 2 \\ -12 & 6 \end{pmatrix}$$ --------------------▷ 46

$$\begin{pmatrix} 4+2 \\ -12+6 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$ --------------------▷ 49

Wrong, unfortunately! 9

In section 14.1 we derived transformation formulae for the shift of a position vector (x, y, z) to the position vector (x', y', z') in the new coordinate system.

For a shift by a vector $\mathbf{r}_0 = (x_0, y_0, z_0)$ the transformation equations are:

$$\begin{aligned} x' &= x - x_0 \\ y' &= y - y_0 \quad \text{or} \quad \mathbf{r}' = \mathbf{r} - \mathbf{r}_0 \\ z' &= z - z_0 \end{aligned}$$

In our problem we had $x_0 = 3, \ y_0 = 2, \ z_0 = 4$, or a $\mathbf{r}_0 = (3, 2, 4)$.

The point P in the x–y–z system was given by $x = 5, \ y = 7$, and $z = 2$, or $\mathbf{r} = (5, 7, 2)$.

Now obtain the position vector $\mathbf{r}'$ of the point P in the $x' - y' - z'$ coordinate system.

$$\mathbf{r}' = (x', y', z') = \text{........................}$$

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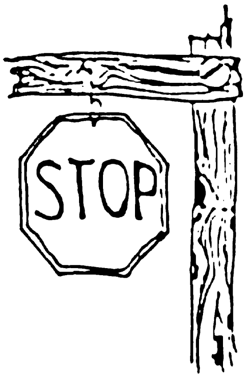
Wrong! 46



The expression **Ar** is also a vector!

Read once more the definition of the product **Ar** in section 14.4, and try again with the following example:

$$\mathbf{Ar} = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \text{............................}$$

Hint: It is useful to follow the scheme given in the textbook for multiplication of matrices, even if one matrix is a vector.

------------------▷ 47

$$\mathbf{r}' = (x', \ y', \ z') = (2, \ 5, \ -2)$$

10

Error, or help required ▷ 11

All correct ▷ 13

Following the scheme we obtain

47

$$\begin{array}{cc} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \mathbf{Ar} = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 0+2 \\ 3+8 \end{pmatrix}; \quad \mathbf{Ar} = \begin{pmatrix} 2 \\ 11 \end{pmatrix} \end{array}$$

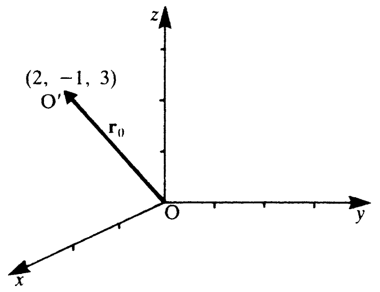
Now compute

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

▷ 49

Read section 14.1 again, and then do the following exercise: **11**

The x–y–z– coordinate system is shifted by the vector



$$\mathbf{r}_0 = (2, \; -1, \; 3)$$

(a) Fill in on the sketch the new coordinate system.

(b) Determine the new coordinates of the position vector $\mathbf{r}_1 = (2, \; 1, \; 2)$:

$$\mathbf{r}_1' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(c) A point P with position vector

$$\mathbf{r}_2 = (2, \; -2, \; 4)$$

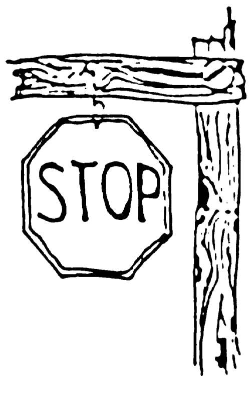
will have as its new coordinates:

$$x_2' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_2' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$z_2' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 12

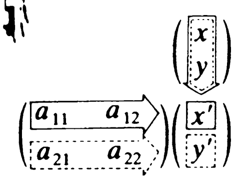
Wrong, unfortunately! **48**

The definition of the product **Ar** is:



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Remember the scheme:



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

Using this expression compute again

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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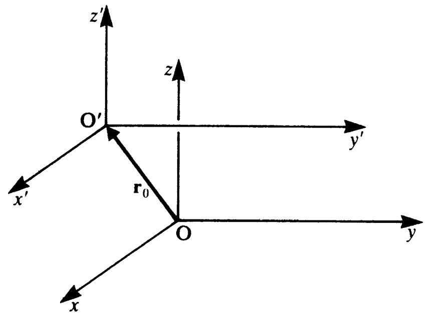

(b) **r** becomes

$$\mathbf{r}'_1 = (0,\ \ 2,\ \ -1)$$

(c)

$$x'_2 = 0$$
$$y'_2 = -1$$
$$z'_2 = 1$$

12

→ 13

You found the correct solution:

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

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Evaluate the product

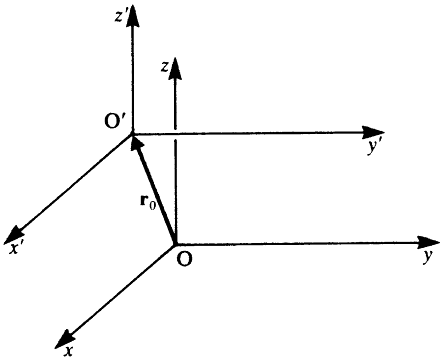
$$\begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & -3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} =$$

....

→ 50

Another example! 13

The transformation from an x–y–z coordinate system to an $x' - y' - z'$ system is accomplished by a shift, a translation, of the origin O to the new origin O′ by the vector $\mathbf{r}_0 = (0,\ 1,\ 3)$.



A position vector

$$\mathbf{r} = (1,\ 13,\ -4)$$

becomes, as a result of this transformation, the vector $\mathbf{r}'$ given by:

$$\mathbf{r}' = (-1,\ -12,\ 7)$$

------------------▷ 14

$$\mathbf{r}' = (1,\ 12,\ -7)$$

------------------▷ 15

$$\mathbf{r}' = (1,\ 14,\ -1)$$

------------------▷ 16

$$\begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & -3 & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 3\times 4+0\times 6+2\times 5 \\ -1\times 4+1\times 6-2\times 5 \\ 2\times 4-3\times 6+0\times 5 \end{pmatrix} = \begin{pmatrix} 22 \\ -8 \\ -10 \end{pmatrix}$$ 50

Multiplication of two matrices.
Which products of the following matrices are possible?

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 1 \end{pmatrix},\ \mathbf{C} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 8 \end{pmatrix}$$

☐ **AB**

☐ **BC**

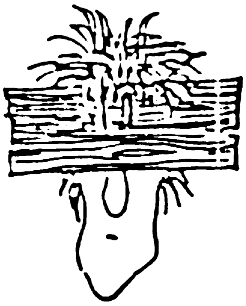
☐ **AC**

------------------▷ 51

No! 14

You have found $\mathbf{r}' = \mathbf{r}_0 - \mathbf{r}$.

The correct relation is $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ whose coordinates are



$$x' = x - x_0$$
$$y' = y - y_0$$
$$z' = z - z_0$$

--------------------▷ 13

The products **AC** and **BC** are possible. 51

Multiplication of two matrices is only possible if the number of columns of the first matrix (**A** and **B** in this case) is equal to the number of rows of the second matrix (**C** in this case).

It follows that **AB** is not possible.

Multiplying two matrices: First obtain an expression for the element c_{11} in

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

$$c_{11} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 52

Correct!
Since $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$

15

--------------------▷ 17

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

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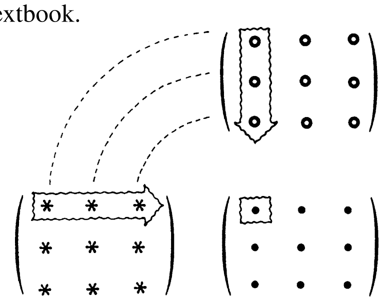
The multiplication of two matrices can be remembered by using the following scheme, given in the textbook.



c_{11} can be considered as the product of the row vector **a** and the column vector **b**, and similarly for finding each element c_{ik}. Using this scheme obtain c_{33} and indicate it on a diagram similar to the one in this frame.

$c_{33} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 53

16

Wrong!

Instead of $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ you have found the vector $\mathbf{r}' = \mathbf{r} + \mathbf{r}_0$.

Was it an oversight?

--------------------▷ 13

Do you have difficulties in understanding?

--------------------▷ 11

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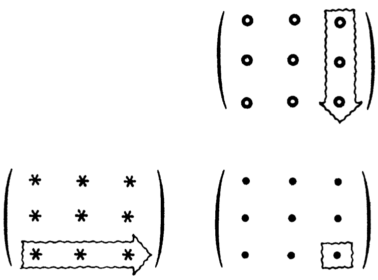
$$c_{33} = a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$$



Because multiplication of matrices may be a little difficult at first it is helpful to adopt a scheme such as the one above, in which the correspondence between column vectors and row vectors is obvious.

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The next exercise is optional. Decide for yourself if you need it. One *needs* exercises even if they appear to be difficult! 17

I'll skip the next exercise

--------------------▷ 23

Next exercise

--------------------▷ 18

Evaluate **AB** given 54

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix}$$

AB = ...

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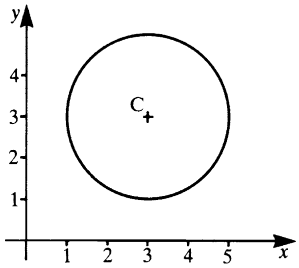

The circle shown has a radius $R = 2$ units and its center C has coordinates (3, 3).

18

(a) What is the equation of the circle? ....

(b) By what vector $\mathbf{r}_0$ must the coordinate system be shifted in order to obtain the following equation for the circle:

$$x'^2 + y'^2 = 4?$$

$$\mathbf{r}_0 = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 19

55

$$\begin{array}{cc} & \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix} & \begin{pmatrix} 0-2 & 0+8 \\ 18-1 & 0+4 \end{pmatrix} = \begin{pmatrix} -2 & 8 \\ 17 & 4 \end{pmatrix} \end{array}$$

Evaluate

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix} =$$

....

Hint: Use the scheme!

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(a) $(x-3)^2+(y-3)^2=4$

or

$x^2-6x+y^2-6y=-14$

(b) $\mathbf{r}_0=(3,\ 3)$

19

-------------------▷ 20

$$\begin{pmatrix} 2 & 3 & -2 \\ 2 & 8 & 2 \\ 4 & 7 & 0 \end{pmatrix}$$

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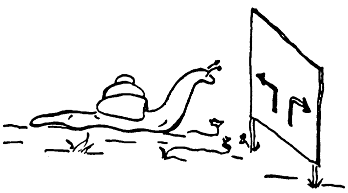
Here is the detailed solution:

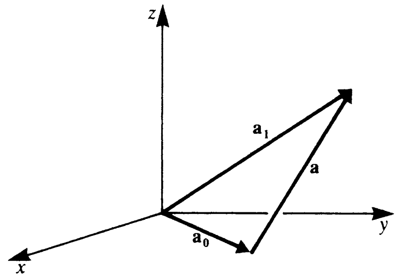
$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0\times2+1\times0+1\times1+1\times1 & 0\times1-1\times1+1\times1+1\times3 & 0\times0+1\times0-1\times2+1\times0 \\ 0\times2+0\times0-1\times1+3\times1 & 0\times1-0\times1-1\times1+3\times3 & 0\times0+0\times0+1\times2+3\times0 \\ 1\times2+0\times0+0\times1+2\times1 & 1\times1-0\times1+0\times1+2\times3 & 1\times0+0\times0-0\times2+2\times0 \end{pmatrix}$$



Correct -------------------▷ 59

Wrong -------------------▷ 57



The vector **a** starts at

$$\mathbf{a}_0 = (1,\ 1,\ 0)$$

and ends at

$$\mathbf{a}_1 = (1,\ 3,\ 2)$$

20

Give

(a) the components of $\mathbf{a} = \dots\dots\dots\dots\dots\dots$

(b) the magnitude of **a**, $a = \dots\dots\dots\dots\dots\dots$

The coordinate system is now shifted by the vector $\mathbf{u} = (1,\ 1,\ 1)$.

(1) The starting point of the vector **a** is now $\mathbf{a}'_0 = \dots\dots\dots\dots\dots\dots\dots\dots$

(2) The end point of **a** is now $\mathbf{a}'_1 = \dots\dots\dots\dots\dots\dots\dots\dots\dots$

(3) Components of $\mathbf{a} = \dots\dots\dots\dots\dots\dots$

------------------▷ 21

You have either made a computational error or you have still not grasped the rule for multiplying matrices. If the latter is the case you should study once more section 14.4 in the textbook and then do the following exercises. Remember the scheme given in the textbook.

57

(a) $\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \dots\dots\dots\dots\dots\dots\dots$

(b) $\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \dots\dots\dots\dots\dots\dots\dots$

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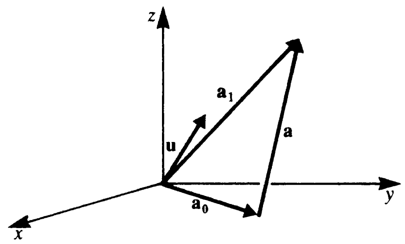

(a) $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_0 = (0,\ 2,\ 2)$

(b) $a = \sqrt{0 + 2^2 + 2^2} = 2\sqrt{2}$

(1) $\mathbf{a}'_0 = (0,\ 0,\ -1)$

(2) $\mathbf{a}'_1 = (0,\ 2,\ 1)$

(3) $\mathbf{a} = (0,\ 2,\ 2)$

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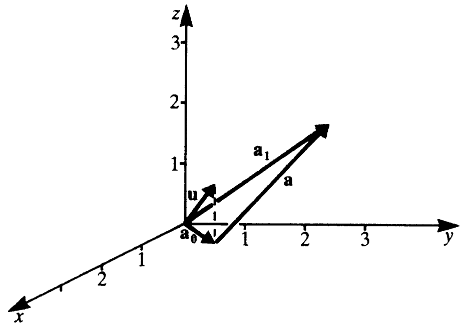

Fill in the sketch of the new coordinate system as a result of a shift by the vector $\mathbf{u} = (1,\ 1,\ 1)$ and check the numerical results.

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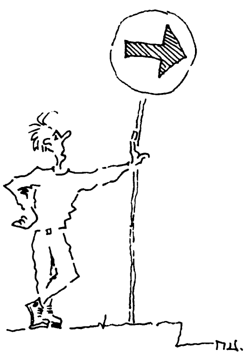
(a) $\begin{pmatrix} 2 & 6 & 2 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

58



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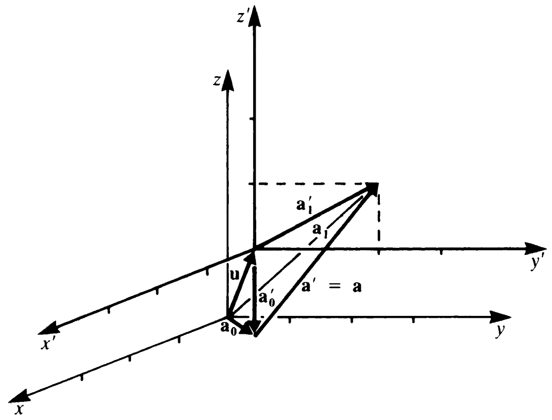
22



23

14.7 Rotations Expressed in Matrix Form

59

Objective: Concepts of matrix representation of the transformation equations, rotation of coordinates.

READ: 14.5 Rotations expressed in matrix form
Textbook pages 421–423

60

14.3 Rotation in a Plane

23

Objective: Concepts of rotation and successive rotations; rotations about one axis of the coordinate system for the three-dimensional case.

READ: 14.3.1 Rotation in a plane
Textbook pages 409–412

Follow the text carefully, reading without pen and paper is day-dreaming! Write down the transformation equations for the rotation of a two-dimensional coordinate system through a given angle.

------------------▷ 24

Set up the transformation matrix for a rotation of the two-dimensional coordinate system through an angle of 180°. You may use the formulae you wrote down as you studied the text. 60

....

------------------▷ 61

Let the rectangular $x - y$ coordinate system be rotated through an angle $\phi = \dfrac{\pi}{2}$. **24**

What are the components of the vector $\mathbf{r} = (1,\ 2)$ in the new coordinate system?

Do not hesitate to use the equations you wrote down in your notebook. You will use them again later on.

$$\mathbf{r}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 25

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ **61**

Here is the solution:

The transformation matrix is

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Inserting the values for $\phi = 180^\circ = \pi$ radians yields

$$\begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The result could also have been obtained pictorially.

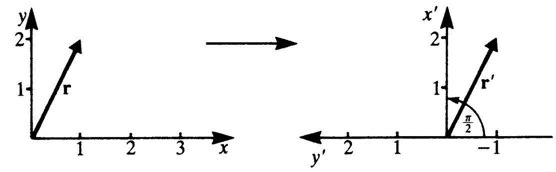
--------------------▷ 62

$$\mathbf{r}' = (2,\ -1)$$

25

This exercise can be solved in two ways:

(a) Sketch the system of coordinates before and after the rotation thus:



Notice that $\mathbf{r}' = (2,\ -1)$

(b) Use the transformation equations

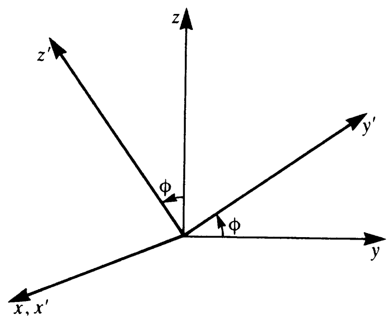
$x' = x\cos\phi + y\sin\phi$

$y' = -x\sin\phi + y\cos\phi$

and insert the values $x = 1,\ \ y = 2,\ \ \phi = \dfrac{\pi}{2}$ to obtain $x' = 2$ and $y' = -1$

I'd like to carry on -------------------▷ 28

I'd like another exercise -------------------▷ 26



Set up the transformation matrix for a rotation in three-dimensional space about the x-axis, the angle of rotation being ϕ.

62

Transformation matrix

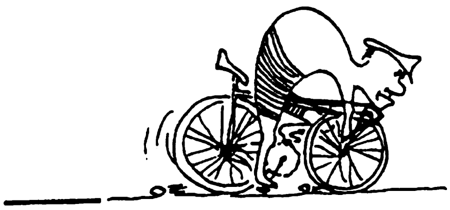
A =

....

-------------------▷ 63

The coordinate system is rotated through an angle $\phi = \frac{\pi}{3}$. Obtain the vector $\mathbf{r}'$ given that $\mathbf{r}$ = $(-2, 1)$. **26**

$$\mathbf{r}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



-------------------▷ 27

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$ **63**

It is obtained as follows:

The transformation equations for rotation about the x-axis are

$$\begin{aligned} x' &= x \\ y' &= y\cos\phi + z\sin\phi \\ z' &= -y\sin\phi + z\cos\phi \end{aligned}$$

In matrix notation these are

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}}_{\text{Transformation matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

-------------------▷ 64

$$\mathbf{r}' = (-1 + \tfrac{1}{2}\sqrt{3},\ \ \sqrt{3} + \tfrac{1}{2})$$ 27

Solution:

It is achieved more quickly by using the transformation equations:

We were given $\phi = \dfrac{\pi}{3}$, $\mathbf{r} = (x,\ \ y) = (-2,\ \ 1)$. Substituting in the equations yields:

$$x' = -2\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$
$$y' = 2\sin\frac{\pi}{3} + \cos\frac{\pi}{3}$$

Since $\cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and $\sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$ we get

$$\mathbf{r}' = \left(-1 + \tfrac{1}{2}\sqrt{3},\ \ \sqrt{3} + \tfrac{1}{2}\right)$$

Difficulties? If so, read section 14.3.1 once more and do the examples without the help of the textbook.

-------------------▷ 28

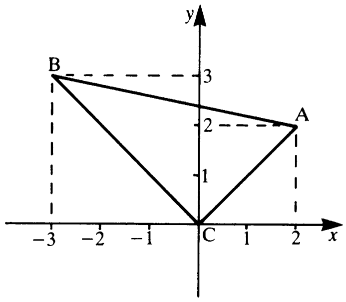
14.8 Special Matrices

64

Objective: Concepts of the transpose of a matrix, null matrix, diagonal matrix, unit matrix, symmetric and skew-symmetric matrices, inverse matrix.

READ: **14.6 Special matrices**
14.7 Inverse matrix
Textbook pages 423–427

-------------------▷ 65



Given the right-angled triangle whose corners are: 28

$$A = (2,\ 2)$$
$$B = (-3,\ 3)$$
$$C = (0,\ 0)$$

Rotate the system of coordinates in such a way that the points A and B coincide with the axes.

(a) Determine the angle of rotation: $\tan\phi =$ ....
(b) Carry out the transformation for A and B.
$A' =$ ....
$B' =$ ....
(c) Fill in the new position on the drawing.

-------------------▷ 29

Obtain the transpose of the matrix 65

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix}$$

$\mathbf{A}^{\mathrm{T}} =$ ....

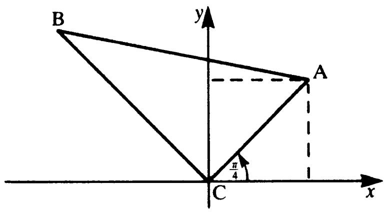
and

$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} =$ ....

-------------------▷ 66

29

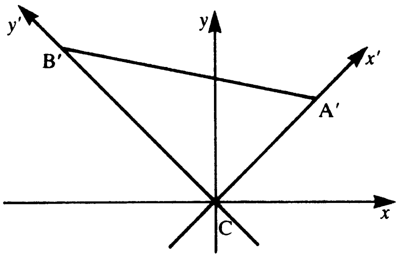
(a) $\tan \phi = 1$



(b) $A' = \left(2\sqrt{2},\ 0\right)$

$B' = \left(0,\ 3\sqrt{2}\right)$

(c)



30

66

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad (\mathbf{A}^{\mathrm{T}})^{\mathbf{T}} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix} = \mathbf{A}$$

Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 3 & -1 \end{pmatrix}$$

Obtain

$\mathbf{AB} =$ .... and $(\mathbf{AB})^{\mathrm{T}} =$ ....

67

14.4 Successive Rotations

30

We have just been considering the essential steps concerning a rotation in two dimensions.

The result of two rotations carried out successively through angles ϕ and ψ is equivalent to a single rotation through an angle $(\phi + \psi)$.

This statement is proved in section 14.3.2. You now have a choice:

Skip the proof and carry on immediately

-------------------▷ 31

Read the proof

READ: 14.3.2 Successive rotations
Textbook pages 412–413

-------------------▷ 31

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 19 & 2 \end{pmatrix} \quad (\mathbf{AB})^{\mathrm{T}} = \begin{pmatrix} 3 & 19 \\ 4 & 2 \end{pmatrix}$$

67

Given: the following square matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Give the name of each matrix.

A : ....
B : ....
C : ....

-------------------▷ 68

14.5 Rotation in Three-Dimensional Space

31

Objective: Concepts of spatial rotations, transformation of a vector when the coordinate system is rotated about the z- or x-axis.

READ: **14.3.3 Rotations in three-dimensional space**
Textbook pages 418–420

--------------------▷ 32

68

A: null matrix
B: diagonal matrix
C: unit matrix

Correct --------------------▷ 69

Wrong: Read the definitions in the textbook again. Take notes! --------------------▷ 69

The three-dimensional system is rotated about the z-axis. 32

Let the angle of rotation be $\frac{\pi}{2}$. Find the components of the position vector $\mathbf{r} = (2,\ 3,\ 1)$ in the transformed system.

$$\mathbf{r}' = \text{........................}$$

--------------------▷ 33

If 69

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Obtain

$$\mathbf{AI} =$$

....

and

$$\mathbf{IA} =$$

....

--------------------▷ 70

$$\mathbf{r}' = (3, \ -2, \ 1)$$

33

Explanation: There are two ways of proceeding:

(a) For a rotation about the z-axis through an angle $\phi = \frac{\pi}{2}$ the z-axis does not change, the x-axis is rotated into the y-axis and the y-axis into the negative x-axis. It therefore follows that

$$\begin{aligned} x' &= y = 3 \\ y' &= -x = -2 \\ z' &= z = 1 \end{aligned}$$

hence $\mathbf{r}' = (3, \ -2, \ 1)$

(b) We can use the transformation equations

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \\ z' &= z \end{aligned}$$

and substitute the numerical values for x, y and ϕ, and obtain

$$\mathbf{r}' = (3, \ -2, \ 1)$$

------------------▷ 34

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$
$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

70

i.e. the matrix **A** is unchanged.

Given: the two matrices

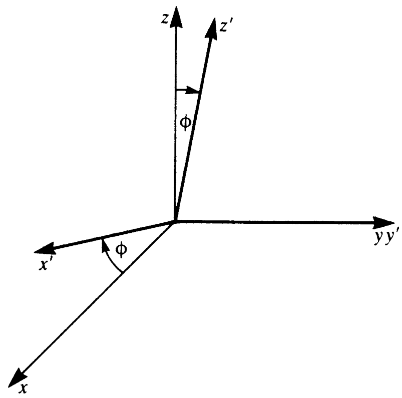
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{pmatrix} \ \mathbf{B} = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

Which matrix is

(a) symmetric?
(b) antisymmetric, i.e. skew-symmetric?

A : ....
B : ....

------------------▷ 71



Derive the transformation equations for a rotation about the y-axis through an angle ϕ: **34**

$x' = \ldots\ldots\ldots\ldots\ldots\ldots$
$y' = \ldots\ldots\ldots\ldots\ldots\ldots$
$z' = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 35

A: symmetric, since $a_{ij} = a_{ji}$ **71**
B: skew-symmetric, since $b_{ij} = -b_{ji}$

Given the matrix **A**, then
$\mathbf{A}^{-1}$ is called .... matrix of **A**.

$$\text{If} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$$

obtain

$$\mathbf{A}\mathbf{A}^{-1} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{A}^{-1}\mathbf{A} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 72

$$x' = x\cos\phi + z\sin\phi$$
$$y' = y$$
$$z' = -x\sin\phi + z\cos\phi$$

35

Explanation: During a rotation about the y-axis the y component of a vector $\mathbf{r} = (x,\ y,\ z)$ is unchanged. The projection $\mathbf{r}_{xz} = (x,\ y)$ of the vector $\mathbf{r}$ in the $x-z$ plane is transformed in accordance with the equations given in section 14.3.1 by replacing y by z.

While working through the mathematical programme, phases may arise where, in spite of your efforts to pick up the subject matter, you find you are making no progress.

Comments on learning curves and learning plateaus

-------------------▷ 36

Otherwise

-------------------▷ 39

$\mathbf{A}^{-1}$ is called the *inverse* matrix of $\mathbf{A}$.

72

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The calculation of the inverse matrix is not shown in this chapter. It will be shown in Chapter 15.

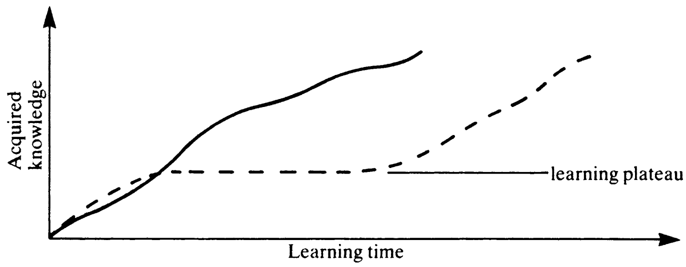
-------------------▷ 73

36

(1) Learning curves
Learning processes can be plotted on graphs, the so-called learning curves. In principle they show the quantitative increase in knowledge, dependent on learning time.

The form these learning curves take can be very varied. It is, for example, determined by individual peculiarities, previous knowledge and by the nature of the subject matter.

The diagram shows two typical learning curves. One of them shows a learning plateau:



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73

Before closing your book, notes, etc., you should recapitulate the concepts and rules you have just been learning. You should then have a break and do something totally different; we have said this to you on a number of occasions. It does make good sense!

Don't forget to do the exercises given in the textbook; remember that 'practice makes perfect'!

END OF CHAPTER 14

37

(2) Learning plateaus

'Learning plateau' is the description given to a time phase in which no progress in learning can be subjectively determined. It frequently corresponds to a state of discontent. A learning plateau of this kind does not mean that the utmost limit of your learning capacity has already been reached — to determine such a limit is hardly possible, anyway.

This phase of what appears to be ineffectiveness sometimes indicates a transition between two levels of competence. This means that the knowledge and all its possible applications, acquired during the preceding period of study, must first be integrated before any further progress begins.

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Please continue on page 1
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