# Chapter 8
# Taylor Series and Power Series

→ 2

48

The series for $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

If we wish to compute the value of the cosine function at $x = 1$ (i.e. 1 radian = 57.3 degrees) then

$$\begin{aligned}\cos 1 &= 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots \\ &= 1 - 0.5 + 0.0417 - 0.0014 + \ldots\end{aligned}$$

If we take $n = 2$ as a first approximation then

$$\cos 1 = 1 - 0.5 + R_2(1) = 0.5 + R_2(1)$$

Now we evaluate the error.

Since the general form of the remainder is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

it follows that with $n$ = 2 we have

$R_2(1) =$ ................

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## 8.1 Expansion of a Function in a Power Series

2

**Objective:** Concepts of power series, factorials, Maclaurin's series, Taylor's series, interval of convergence.

**READ:** **8.1 Introduction**
**8.2 Expansion of a function in a power series**
**8.3 Interval of convergence of power series**
**Textbook pages 229–235**

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$$R_2(1) = \frac{f^{(3)}(\xi)}{3!} = \frac{\sin\xi}{6}, \quad 0 < \xi < 1$$

49

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We do not know the exact value of $\xi$, but we can be certain that the error will not exceed the value of $R_2(1)$ with $\xi = 1$, since the sine function is monotonically increasing in the interval (0, 1).

Hence the error will not be greater than

$$|R_2(1)| = \left|\frac{\sin\xi}{6}\right| = \left|\frac{\sin(1)}{6}\right| = \frac{0.842}{6} \approx 0.14$$

Our approximation gave cos $1 = 0.5$
The exact value is 0.5403
What is the actual error $E$?

$$E = \ldots\ldots\ldots\ldots\ldots$$

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Name at least three concepts which were newly introduced in this section: 3

(1) .................
(2) .................
(3) .................

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$$E = 0.5403 - 0.5 = 0.0403$$ 50

Note: This is less than 0.14, which we predicted as the largest possible value.

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The approximation for cos 1 can be improved if we take $n = 4$.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$
$$\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} = 1 - 0.5 + 0.0417 = 0.5417$$

What is the actual error $E$ now?

$E =$ .............

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(1) Power series
(2) Maclaurin's series
(3) Interval of convergence

4

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Give three reasons for expanding a function in a power series:

(1) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
(2) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .
(3) . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

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$$E = 0.5403 - 0.5417 = -0.0014$$

51

Note: The error is estimated to be less than

$$|R_4(1)| = \left|\frac{f^{(5)}(\xi)}{5!}\right| = \left|-\frac{\sin(\xi)}{120}\right| \leq \frac{0.842}{120} \approx 0.0070$$

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(1) The first terms of a power series are often suitable for obtaining an approximate value of the function. 5
(2) Power series can be differentiated and integrated term by term.
(3) We can use power series to calculate the values of many functions.

The expression $n!$ is read as . . . . . . . . . . . . . . . . . .
The expression $n!$ means . . . . . . . . . . . . . . . . . .

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Some remarks on human memory will follow. Do you want to skip them?. 52

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During an oral exam the examiner asks a student:

> Explain to me the relationship between differentiation and integration and write down the symbol for the indefinite integral.

The student hesitates, and hesitates . . .

Finally the examiner states, 'Integration is the inverse operation to differentiation. The general solution of the integration is the indefinite integral. Here are two alternatives. How should it be written?'

$$\text{A} \int f(x)\mathrm{d}x = F(x)$$

$$\text{B} \int f(x)\mathrm{d}x = F(x) + C$$

The student replies: 'Yes, solution B is the correct one. I understood that well at the time.' To this the examiner says: 'But you didn't know when I asked you just now.'
Who is right?

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factorial $n$ 6

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1)n$$

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Work out the following, simplifying where possible:

$$\begin{aligned} 5! &= \ldots\ldots\ldots\ldots\ldots \\ \frac{7!}{5!} &= \ldots\ldots\ldots\ldots\ldots \\ \frac{(n+1)!}{n!} &= \ldots\ldots\ldots\ldots\ldots \\ \frac{9!}{11!} &= \ldots\ldots\ldots\ldots\ldots \end{aligned}$$

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Both are right; things are a bit more complicated. 53

The student had understood the matter at the time of studying it. He quickly recognised the correct solution again.

The examiner emphasised, and rightly so, that the question wasn't answered without considerable help. The student was neither in a position to describe the relationship nor to write down the symbol *actively*.

Conclusion:

> Recognition is easier than reproduction. But reproduction and application are the objectives of our learning.

Anyone who believes that, a year from now, he will be able to reproduce everything he now understands, is greatly mistaken.

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$$
\begin{aligned}
5! &= 120\\
\frac{7!}{5!} &= \frac{1\times2\times3\times4\times5\times6\times7}{1\times2\times3\times4\times5} = 6\times7 = 42\\
\frac{(n+1)!}{n!} &= \frac{1\times2\times3\times\ldots\times n\times(n+1)}{1\times2\times3\times\ldots\times n} = n+1\\
\frac{9!}{11!} &= \frac{1\times2\times3\times..\times9}{1\times2\times3\times\ldots\times10\times11} = \frac{1}{110}
\end{aligned}
$$

7

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You can sometimes simplify an expression involving factorials by cancelling factors common to both numerator and denominator.

Did you make some mistakes?

Yes --------------------▷ 8

No --------------------▷ 11

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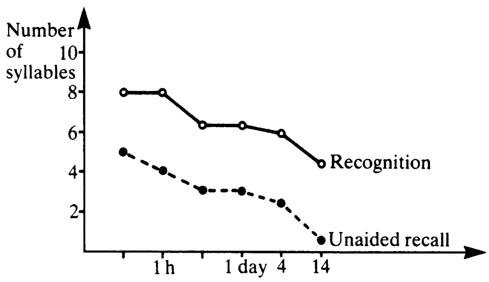
54

In an experiment A. Miles (1960) gave 60 people 10 syllables each to learn. This learning material was studied three times in succession. At two different intervals each person was examined according to two different methods.

(1) Unaided recall: number of syllables reproduced without aid.
(2) Recognition: number of learned syllables which could be recognised from an extensive list.

They were tested immediately after this learning period and then 1 hour, 6 hours, 1 day, 4 days and 14 days later.

The diagram shows the results

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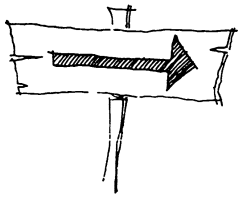
The symbol $n!$ (read as factorial $n$) is an abbreviation for the product of the first $n$ natural numbers 8

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

What is $(n-2)!$?

$$(n-2)! = \ \ldots\ldots\ldots\ldots\ldots$$



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Similar dependencies are also found in the case of meaningful subject matter. Free reproduction is more difficult than recognition. 55

In exam situations facts concerning specific questions must be actively reproduced. Incidentally, this also goes for a large number of situations in which learned material has to be applied. As in the experiment with meaningless syllables, with meaningful subject matter too the difference between the skill shown in unaided recall and that shown in recognition is still great. Here there exists the possibility that we subjectively deceive ourselves: we often mistake those facts that we once studied, but thereafter only recognise, for facts which have been well memorised.

This is often self-deception.

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$$(n-2)! = 1 \times 2 \times 3 \times \ldots \times (n-3)(n-2)$$

9

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Here are some more examples for you! Remember to simplify if you can.

(1) $\dfrac{n!}{(n-2)!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(2) $\dfrac{3!5!}{6!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\dfrac{100!}{101!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

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56

Let us assume that, through intensive reading, you have understood a fact. That is to say that the terms can be actively reproduced and the operations which were learned can be carried out. A well-known safeguard against forgetting things is revision — a process you are now familiar with. At the end of every work section it is recommended that you go over the contents again and try to write down all the keywords from memory before you stop for a break.

The second phase of revision is to check, after an interval of perhaps a week, whether you can actively reproduce the most important contents of the previous lesson.

If you have difficulty here it is important that you repeat the lesson once more.

In order not to forget this, put a slip of paper into the textbook saying 'Lesson must be repeated'.

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10

(1) $\dfrac{n!}{(n-2)!} = \dfrac{1\times 2\times 3\times \ldots \times (n-2)(n-1)n}{1\times 2\times 3\times \ldots \times (n-2)} = (n-1)n$

(2) $\dfrac{3!5!}{6!} = \dfrac{1\times 2\times 3\times 1\times 2\times 3\times 4\times 5}{1\times 2\times 3\times 4\times 5\times 6} = \dfrac{1\times 2\times 3}{6} = 1$

(3) $\dfrac{100!}{101!} = \dfrac{1\times 2\times 3\times \ldots \times 100}{1\times 2\times 3\times \ldots \times 100\times 101} = \dfrac{1}{101}$

In case of difficulties consult the textbook.

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## 8.3 Expansion of a Function $f(x)$ at an Arbitrary Position. Applications of Series. Approximations

57

**Objective:** Evaluation of the first terms of a Taylor's series at $x_0 \neq 0$, application of series.

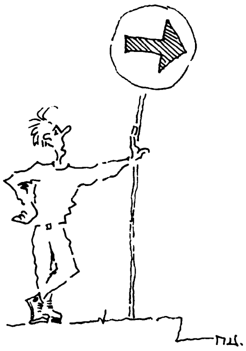
**READ:** **8.5 Expansion of a function $f(x)$ at an arbitrary position**
**8.6 Applications of series**
**8.6.1 Polynomials as approximations**
**Textbook pages 237–241**

------------------▷ 58

Write down the general form of Maclaurin's series for a function $f(x)$. You may have to look at the textbook again. If you do, don't just look at it, write it down! This will help you to fix it in your mind.

11

$f(x) =$ ..........................................................................................



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Having read the relevant section in the textbook write down the formula for Taylor's series:

58

$f(x) =$ .............

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$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots$$ 12

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Use the series to expand $\cos x$ up to the term $n = 4$ in accordance with the steps below.

**Step 1:** Obtain derivatives $f', f'', f''', f^{(4)}$.
**Step 2:** Calculate the values of the function and its derivatives at $x = 0$.
**Step 3:** Substitute the values of $f(0),\ f'(0),\ \dots, f^{(4)}(0)$ in Maclaurin's series.

$$\cos x \approx \dots\dots\dots\dots$$

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$$f(x) = f(x_0) + f'(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \dots$$ 59

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Suppose you needed to calculate sin 47° and you did not have tables or a scientific calculator at hand. You must expand the sine function at an appropriate position.

Which $\alpha$ (or $x_0$) will be suitable?

$$\alpha_0 = \dots\dots\dots\dots$$

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$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$ 13

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Correct ⟶ 15

Wrong, further explanation required ⟶ 14

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$\alpha_0 = 45°$, i.e. $x_0 = \frac{\pi}{4}$, is a good choice because $\sin 45°$ is known from a simple triangle, i.e. $\sin 45° = \frac{1}{\sqrt{2}}$, and the values of the derivatives are known too. The differences $(\alpha - \alpha_0)$, or $(x - x_0)$, will be small. 60

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What steps are needed to obtain the first four terms of the expansion?

**Step 1:** ..........................................................................

**Step 2:** ..........................................................................

**Step 3:** ..........................................................................

⟶ 61

To express cos $x$ in a power series according to Maclaurin's expansion we have to proceed as follows: 14

**Step 1:** Obtain the derivatives

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

**Step 2:** Obtain the values of the derivatives at $x = 0$

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \end{aligned}$$

**Step 3:** Substitute the values of $f'(0), \ldots, f^{(4)}(0)$ in Maclaurin's series

$$f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0) + \frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots$$

$$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

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**Step 1:** Obtain the derivatives $f'(x), f''(x), f'''(x)$, etc... 61
**Step 2:** Calculate the values of the function and its derivatives at $x = x_0$.
**Step 3:** Substitute the values $f'(x_0), f''(x_0)$, etc. in the Taylor's series.

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Now proceed to compute the first four terms of the expansion for sin 47°. Remember that you must express the angles in radians: $x = \alpha \dfrac{\pi}{180}$

$$\sin 47° = \ldots\ldots\ldots$$

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Use Maclaurin's series to expand $f(x) = \frac{1}{(1+x)^2}$ up to the third term, i.e. $n = 3$. 15

What steps should you follow?

**Step 1:** ..........................................................................................

**Step 2:** ..........................................................................................

**Step 3:** ..........................................................................................

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$$\sin 47^\circ \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(2\frac{\pi}{180}\right) - \frac{1}{2\sqrt{2}}\left(2\frac{\pi}{180}\right)^2 - \frac{1}{6\sqrt{2}}\left(2\frac{\pi}{180}\right)^3 \quad \text{up to } n = 3$$ 62

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Correct -------------------▷ 64

Error; detailed solution required -------------------▷ 63

16

**Step 1:** Obtain the derivatives $f'(x), f''(x), f'''((x))$.
**Step 2:** Obtain the values of the function and its derivatives at $x = 0$.
**Step 3:** Substitute these values in Maclaurin's series.

$$f(x) \approx \sum_{n=0}^{n=3} \frac{f^{(n)}(0)}{n!} x^n$$

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Now execute the steps

$$f(x) = \frac{1}{(1+x)^2}$$

**Step 1:**

$$f'(x) = \ldots\ldots\ldots\ldots$$
$$f''(x) = \ldots\ldots\ldots\ldots$$
$$f'''(x) = \ldots\ldots\ldots\ldots$$

------------------▷ 17

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63

Here is the solution in detail.

**Step 1:** The derivatives are

$$f(x) = \sin x \qquad f''(x) = -\sin x$$
$$f'(x) = \cos x \qquad f'''(x) = -\cos x$$

**Step 2:** The values of the function and its derivatives at $x_0 = 45\frac{\pi}{180}$

$$f(x_0) = \frac{1}{\sqrt{2}} \qquad f''(x_0) = \frac{-1}{\sqrt{2}}$$
$$f'(x_0) = \frac{1}{\sqrt{2}} \qquad f'''(x_0) = \frac{-1}{\sqrt{2}}$$

**Step 3:** Substitute these values in Taylor's series:

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots$$
$$\sin 47° \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(2\frac{\pi}{180}\right) - \frac{1}{2\sqrt{2}}\left(2\frac{\pi}{180}\right)^2 - \frac{1}{6\sqrt{2}}\left(2\frac{\pi}{180}\right)^3 \quad \text{up to } n=3$$

------------------▷ 64

$$f'(x) = \frac{-2}{(1+x)^3}$$
$$f''(x) = \frac{6}{(1+x)^4}$$
$$f'''(x) = \frac{-24}{(1+x)^5}$$

17

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Correct ------------------> 23

Wrong ------------------> 18

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We assume you have a simple (non-scientific) calculator. Calculate, as an approximation, the value of sin 47° to five decimal places taking (a) the first two terms of the expansion and (b) the first three terms of the expansion.

64

What are the errors in each case, knowing that the exact value (to five decimal places) of sin 47° = 0.73135?

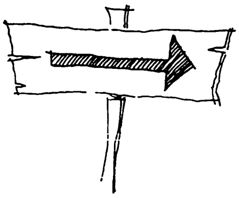
(a) sin 47° = ............. error = .............
(b) sin 47° = ............. error = .............

------------------> 65

Where did you go wrong? We must try to analyse your error, but we can only do this if we know the reason for your difficulties. 18

Never let errors rest; they have to be eliminated, because they cannot go away by themselves!

Even errors due to carelessness should not be allowed to increase.



Error found --------------------▷ 23

Explanation of calculation --------------------▷ 19

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(a) 0.73179 error 0.00044 65

(b) 0.73136 error 0.00001

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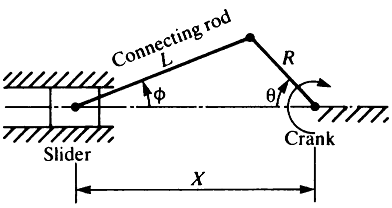
Now we give an example which is of special interest to mechanical engineers; other readers may skip it and go to --------------------▷ 76




The figure shows the slider crank mechanism as used in the petrol and diesel engine as well as in reciprocating pumps and compressors. There are hundreds of millions of such mechanisms throughout the world. As it is a very important device we propose to examine its kinematics.

Using simple geometry express the displacement $x$ of the slider (or piston) as a function of $R$, $L$, the crank angle $\theta$ and the connecting rod angle $\phi$, as shown.

$$x = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 66

Your error may have occurred during differentiation of $f(x) = \dfrac{1}{(1+x)^2}$ 19

To differentiate this function we can use the quotient rule (look it up again if you have forgotten it), or we can write the function as $f(x) = (1+x)^{-2}$ and use the function of a function rule. The latter form is much easier to apply in this instance.

$$f(x) = (1+x)^{-2} = u^{-2};\ u = 1+x$$
$$f'(x) = \frac{df}{du} \times \frac{du}{dx} = -2u^{-3} \times 1 = -2u^{-3}$$
$$= -2(1+x)^{-3} = \frac{-2}{(1+x)^3}$$

Similarly $f''(x) = +6(1+x)^{-4} = \dfrac{6}{(1+x)^4}$

and $f'''(x) = -24(1+x)^{-5} = -\dfrac{24}{(1+x)^5}$

--------------------▷ 20

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$$x = R\ \cos\ \theta + L\ \cos\ \varphi$$ 66

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It is more convenient in practice to express $x$ as a function of $R$, $L$ and $\theta$ only. Eliminate $\varphi$ and obtain

$$x = \ \ldots\ldots\ldots\ldots$$

Solution --------------------▷ 68

Hints and explanation --------------------▷ 67

If you used the quotient rule you should have obtained the same result. **20**

$$f'(x) = \frac{0 \times (1+x)^2 - 2 \times (1+x)}{(1+x)^4} = \frac{-2}{(1+x)^3}$$
$$f''(x) = \frac{0 \times (1+x)^3 - (-2) \times 3 \times (1+x)^2}{(1+x)^6} = \frac{6}{(1+x)^4}$$
$$f'''(x) = \frac{0 \times (1+x)^4 - 6 \times 4 \times (1+x)^3}{(1+x)^8} = \frac{-24}{(1+x)^5}$$

The differentiation is clear --------------------▷ 24

Still having difficulties in differentiating --------------------▷ 21

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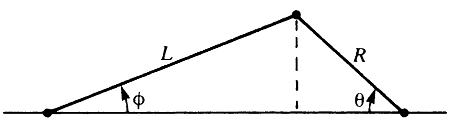
Given: $x = R\ \cos\ \theta + L\ \cos\ \varphi$ **67**

To eliminate $\varphi$ use the fact that $L\ \sin\ \varphi = R\ \sin\ \theta$.
Remembering that $\cos^2\ \varphi + \sin^2\ \varphi = 1$, thus we have




$$\cos\phi = \sqrt{1 - \sin^2\phi}$$
$$= \sqrt{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta}$$

Hence

$$x = R\ \cos\ \theta + L\ \cos\ \varphi = \dots\dots\dots\dots\dots\dots$$

--------------------▷ 68

The $n$th term in Maclaurin's series is 21

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$f^{(n)}(0)$ is the value of the $n$th derivative of the function at $x = 0$.

Hence to obtain the series for $f(x)$ we need the higher derivatives

$$f'(x), f''(x), \ldots, f^{(n)}(x).$$

Since you are experiencing difficulties with the process of differentiating you should interrupt this section of the work for the time being.

Read section 5.6 in the textbook to revise the concept of higher derivatives, and read again the section regarding the quotient rule and the derivative of a function of a function (chain rule).

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$$x = R \cos\theta + L\sqrt{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta}$$ 68

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If your result does not agree with the one above go back to --------------------▷ 67

The ratio $\frac{R}{L}$ is small, usually about $\frac{1}{3}$ or $\frac{1}{4}$.

Use the binomial expansion to obtain an approximate value for the square root term. (The first three terms will be sufficient.) Then complete the expression for the displacement $x$ of the slider.

$$x = R \cos\theta + L \;\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 70

Hints and further explanations --------------------▷ 69

The function in our case is a quotient; to differentiate it we can, on the one hand, apply the quotient rule. On the other hand, since the function can be written 22

$$f(x) = (1+x)^{-2} = u^{-2}$$

we can also use the function of a function rule.

You need some practice in differentiating! Try to apply both rules to obtain the first four derivatives of the function

$$\begin{aligned} f(x) &= \frac{1}{(1+x)^2} \\ f'(x) &= \dots\dots\dots\dots\dots\dots \\ f''(x) &= \dots\dots\dots\dots\dots\dots \\ f'''(x) &= \dots\dots\dots\dots\dots\dots \\ f^{(4)}(x) &= \dots\dots\dots\dots\dots\dots \end{aligned}$$

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We are asked to evaluate the first three terms of the binomial expansion of $\sqrt{1-\left(\frac{R}{L}\right)^2 \sin^2\theta}$ 69

**Step 1:** The first three terms of the binomial expansion are

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$$

**Step 2:** $\sqrt{1-\left(\frac{R}{L}\right)^2 \sin^2\theta} = \left(1-\left(\frac{R}{L}\right)^2 \sin^2\theta\right)^{1/2}$

$$\text{hence} \quad n = \frac{1}{2}, \quad x = -\left(\frac{R}{L}\right)^2 \sin^2\theta$$

**Step 3:** Substitute in the binomial expansion and obtain

$$\left(1-\left(\frac{R}{L}\right)^2 \sin^2\theta\right)^{1/2} = 1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{R}{L}\right)^4 \sin^4\theta$$

**Step 4:** The complete expression for the displacement $x$ of the slider is

$$x = R\cos\theta + L(\dots\dots\dots\dots\dots\dots)$$

------------------▷ 70

$$f'(x) = \frac{-2}{(1+x)^3}$$
$$f''(x) = \frac{6}{(1+x)^4}$$
$$f'''(x) = \frac{-24}{(1+x)^5}$$
$$f^{(4)}(x) = \frac{120}{(1+x)^6}$$

23

In case of further difficulties you should revise the calculation of derivatives using the study guide for chapter 5.

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$$x = R\cos\theta + L\left(1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta - \frac{1}{8}\left(\frac{R}{L}\right)^4 \sin^4\theta ...\right)$$

70

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If your result is wrong

------------------▷ 69

Note: in practice only the first two terms of the expansion are significant, except for very high performance engines.

Use the first two terms of the expansion to obtain an expression for the acceleration $\ddot{x}$ of the slider.

Notation: $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$ where $t$ is the time. Remember that $\theta$ is also a function of time but $\dot{\theta}$ is supposed to be constant.

$\ddot{x} = $ ................................

Solution found

------------------▷ 73

Hints and detailed solution

------------------▷ 71

The first three derivatives of the function $f(x) = \frac{1}{(1+x)^2}$ are 24

$$f'(x) = \frac{-2}{(1+x)^3}, \quad f''(x) = \frac{6}{(1+x)^4}, \quad f'''(x) = \frac{-24}{(1+x)^5}$$

Thus step 1 is completed.

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**Step 2:** Substituting $x = 0$ in each yields

$$f(0) = 1; \quad f'(0) = -2; \quad f''(0) = 6; \quad f'''(0) = -24$$

**Step 3:** Inserting these values in Maclaurin's expansion we find

$$\frac{1}{(1+x)^2} \approx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 25

---

With the first two terms of the expansion we have 71

$$x = R\cos\theta + L - \frac{L}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta$$

We differentiate $x$ twice to obtain the acceleration.

Obtain $\dot{x} = \frac{dx}{dt}$ first, remembering that

$\frac{d}{dt}\sin^2\theta = \left(\frac{d}{d\theta}\sin^2\theta\right)\frac{d\theta}{dt}$ (function of a function rule):

$$\dot{x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 72

$$\frac{1}{(1+x)^2} \approx 1 - 2x + \frac{6}{2!}x^2 - \frac{24}{3!}x^3 \text{ up to } n = 3$$
$$\approx 1 - 2x + 3x^2 - 4x^3$$

25

---

Note: In many cases it is sufficient to determine the first three or four terms of Maclaurin's series in order to infer the form of the complete series. In our case we will have

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \dots$$

------------------▷ 26

---

$$\dot{x} = -R\dot{\theta}\left(\sin\theta \frac{R}{2L}\sin 2\theta\right)$$

72

A detailed explanation of this differentiation is given below. Skip it if you obtained this result.

We require $\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$

Now $$\frac{dx}{d\theta} = -R\sin\theta + 0 - \frac{L}{2}\left(\frac{R}{L}\right)^2 \frac{d}{d\theta}(\sin^2\theta)$$

but $$\frac{d}{d\theta}(\sin^2\theta) = 2\sin\theta\cos\theta = \sin 2\theta$$

therefore $$\dot{x} = -R\dot{\theta}\left(\sin\theta + \frac{R}{2L}\sin 2\theta\right)$$

---

Now differentiate once more, remembering that

$$\frac{d\dot{\theta}}{dt} = \ddot{\theta} = 0$$
$$\ddot{x} = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

------------------▷ 73

Consider the series $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$ 26

If we replace $x$ by $-x$ we find

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots$$

Thus we find the series for

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

Now consider the power series for $e^x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Obtain the series for $e^{-x}$

$e^{-x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 27

---

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{d\theta}\frac{d\theta}{dt} = -R\dot{\theta}\left(\cos\theta + \frac{R}{L}\cos 2\theta\right)\dot{\theta}$$ 73

$$= -R\dot{\theta}^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right)$$

---

In case of difficulties go through the explanation again, beginning with frame 68.

--------------------▷ 74

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

27

---

Express the function ln $(1 + x)$ as a Maclaurin's series up to $n = 3$.
What steps should you follow?

**Step 1:** ..........................................................................................................

**Step 2:** ..........................................................................................................

**Step 3:** ..........................................................................................................

-------------------▷ 28

---

74

With the help of Maclaurin's and Taylor's series, as well as the binomial expansion, we can derive approximate expressions for some functions. This is particularly helpful when dealing with complicated expressions but we have to ensure that the error involved is kept within limits which are acceptable in practice, e.g. 1%, 5% or 10%. It depends very much on the nature of the problem and therefore no general rule can be given.

The table of approximations for typical functions at the end of section 8.6.3 in the textbook contains the errors for the first and second approximations.

Give the first and second approximations for cos $x$ using this table.

First approximation: $\cos x \approx$........................

Second approximation: $\cos x \approx$........................

-------------------▷ 75

**Step 1:** Obtain the derivatives 28
$f'(x)$, $f''(x)$, $f'''(x)$
**Step 2:** Obtain the values of the function and its derivatives at $x = 0$, i.e.
$f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$
**Step 3:** Substitute in Maclaurin's series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

---

Now execute the steps:

**Step 1:**
$f(x) = \ln(1+x)$
$f'(x) = \dots\dots\dots$
$f''(x) = \dots\dots\dots$
$f'''(x) = \dots\dots\dots$

**Step 2:**
$f(0) = \dots\dots\dots$
$f'(0) = \dots\dots\dots$
$f''(0) = \dots\dots\dots$
$f'''(0) = \dots\dots\dots$

**Step 3:**
$\ln(1+x) \approx \dots\dots\dots$

------------------▷ 29

---

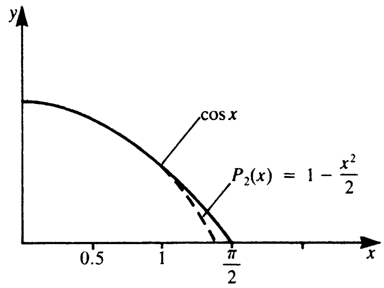
First approximation: $\cos x \approx 1 - \frac{x^2}{2!}$ 75

Second approximation: $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

---

For small angles, i.e. small $x$, the first approximation for the cosine is

$$\cos x \approx 1 - \frac{x^2}{2!}$$




Let us consider the error made by using this approximation for $x = 0.5$ (radians).

$$\cos 0.5 = 0.8776 \text{(exact value)}$$

$$P_2(0.5) = 1 - \frac{0.5^2}{2} = 0.8750$$

The difference is $\cos(0.5) - P_2(0.5) = 0.0026$ thus the error is smaller than 3%.

Now compute the error, using the same approximation, when $x = 0.75$ radians.
$\cos 0.75 = 0.7317 \quad E = cos 0.75 - P_2(0.75) = \dots\dots\dots$

------------------▷ 76

29

$$\begin{aligned} f(x) &= ln(1+x) & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\ f''(x) &= \frac{-1}{(1+x)^2} & f''(0) &= -1 \\ f'''(x) &= \frac{2}{(1+x)^3} & f'''(0) &= 2 \end{aligned}$$

$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$

---

Correct 34

Errors; explanation required 30

---

76

$$E = 0.7317 - 0.7188 = 0.0129$$

---

The function $f(x) = \sqrt{1+x}$ is to be replaced by an approximate expression in the range $0 \le x \le 0.5$ with an error not greater than 1%. Which is the simplest approximation that can be used?

Use the table of approximations for typical functions given in the textbook.

☐ First approximation 77

☐ Second approximation 78

**Step 1:** The first derivative of the function $\ln(1+x)$ can be obtained by the chain rule. 30

$$f(x) = \ln(1+x) = \ln g \quad \text{where } g = 1+x$$
$$\text{then } f(x) = \frac{1}{g} g' = \frac{1}{1+x} \quad \text{since } g' = 1$$

The next derivatives are obtained in the same way or by using the quotient rule;
hence $f''(x) = \dfrac{1}{(1+x)^2}$.

$f'''(x) =$ ..............................

-------------------▷ 31

---

Wrong 77

---

The first approximation for $(1+x)^{1/2}$ is $1 + \frac{1}{2}x$; it has an error not exceeding 1% in the range

$$0 \leq x \leq 0.32$$

The second approximation $1 + \frac{1}{2}x - \frac{1}{8}x^2$ has an error not exceeding 1% in the range

$$0 \leq x \leq 0.66$$

which more than meets the requirement. Hence only the second approximation is acceptable.
Look back at the table of approximations for typical functions in the textbook to check this statement.

-------------------▷ 78

$$f'''(x) = \frac{2}{(1+x)^3}$$ 31

---

**Step 2:** Obtain the values of the function and its derivatives at $x = 0$.

$$f'(0) = \ln(1+0) = 0$$
$$f'(0) = \frac{1}{1+0} = 1$$
$$f''(0) = -\frac{1}{(1+0)^2} = -1$$
$$f'''(0) = \frac{2}{(1+0)^3} = 2$$

**Step 3:** Substitute these values in Maclaurin's series. (In case of difficulties return to the textbook to check the formula.)

$$f(x) = \ln(1+x) = 0 + 1 \times x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

-------------------▷ 32

---

Correct 78

---

Approximations are frequently used to compute particular values of the exponential, logarithmic and trigonometrical functions when tables or scientific calculators are not available. To illustrate this point suppose that in a particular problem the value of $e^{0.2}$ is required, i.e. the value of $e^{x_0}$ where $x_0 = 0.2$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

As a first approximation we have

$$e^{x_0} \approx 1 + x_0 = 1 + 0.2 = 1.2$$

As a second approximation we have

$$e^{x_0} \approx 1 + x_0 + \frac{x_0{}^2}{2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 79

A quite powerful expansion is the binomial expansion. 32
You know from the textbook:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \dots$$

The exponent $n$ may be a fraction.

Obtain the following expansions by applying this formula:

(1) $\sqrt{1+x} = \dots\dots\dots\dots$

(2) $\sqrt{1+x^2} = \dots\dots\dots\dots$

(3) $\frac{1}{\sqrt{1-x^3}} = \dots\dots\dots\dots$

Solutions found -------------------▷ 34

Hints and detailed explanation required -------------------▷ 33

---

1.22 79

---

Fractions whose denominator does not differ greatly from unity can easily be calculated by means of an approximation.

*Example:*

$$\frac{1}{0.94} = \frac{1}{1-0.06}$$

Using the expansion for $\frac{1}{1-x}$ we have

$$\frac{1}{1-x} \approx \dots\dots\dots\dots, \qquad \frac{1}{1-0.06} \approx \dots\dots\dots\dots$$

What is the percentage error $E$? Use the table!

☐ $E < 1\%$

☐ $E < 10\%$

-------------------▷ 80

Each one of the given problems can be solved by applying the general form of the binomial series. 33

We compare the given problem with the general formula for $(a+b)^n$ in order to find the actual substitutions for $a$, $b$ and $n$.

Problem (1): $\sqrt{1+x} = (1+x)^{1/2}$ yields $a=1,\ b=x,\ n=\frac{1}{2}$

Problem (2): $\sqrt{1+x^2} = (1+x^2)^{1/2}$ *yields* $a=1,\ b=x^2,\ n=\frac{1}{2}$

Problem (3): $\frac{1}{\sqrt{1+x}^3} = (1+x^3)^{-1/2}$ yields $a=1,\ b=x^3,\ n=-\frac{1}{2}$

---

Now try again:

$$\sqrt{1+x} = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sqrt{1+x^2} = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{1}{\sqrt{1+x^3}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 34

---

$$\frac{1}{1-x} \approx 1+x; \frac{1}{1-0.06} \approx 1.06$$ 80

$E < 1\%$ for all $x$ in the range $0 \leq x \leq 0.1$

---

Approximate: $\frac{1}{\sqrt{0.6}} = \frac{1}{\sqrt{1-0.4}}$

In the table you will find the approximation for the function

$$\frac{1}{\sqrt{1+x}}$$

You can use this expansion if you substitute $x$ by $-x$. Thus

$$\frac{1}{\sqrt{1+(-x)}} \approx \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 81

34

$$(1)\quad \sqrt{1+x} = 1+\frac{1}{2}x+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}x^2+\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{1}{3!}x^3+\ldots$$
$$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\ldots$$
$$(2)\quad \sqrt{1+x^2} = 1+\frac{x^2}{2}-\frac{x^4}{8}+\frac{x^6}{16}-\ldots$$
$$(3)\quad \frac{1}{\sqrt{1-x^3}} = 1+\left(-\frac{1}{2}\right)(-x^3)+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}(-x^3)^2$$
$$+\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}(-x^3)^3+\ldots$$
$$= 1-\frac{x^3}{2}+\frac{3}{8}x^6-\frac{5}{16}x^9+\ldots$$

Note: By this method many functions can be reduced to a binomial series.

---

If $f(u) = \sin u$ then the general form of this function as a series is

$$f(u) = \sum_{n=0}^{\infty} \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sin u = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 35

---

$$\frac{1}{\sqrt{1+(-x)}} \approx 1+\frac{x}{2}+\frac{3}{8}x^2$$

81

---

Now calculate

$$\frac{1}{\sqrt{1-0.4}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

☐ The error is less than 1%

☐ The error is less than 10%

☐ The error exceeds 10%

Use the table in the textbook!

--------------------▷ 82

$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)u^n}{n!}$$ 35
$$\sin = u - \frac{u^3}{3!} + \frac{u^5}{5!}$$

---

A power series (such as Maclaurin's) does not always converge for all values of the variable. The interval of convergence can be determined in a way similar to that shown by the examples in the textbook.

Important series such as the exponential functions $e^x$, $e^{-x}$ and the trigonometric functions $\sin x$, $\cos x$ are convergent for all values of $x$.

We will now consider the solution of an example on convergence in detail. You may skip it if you wish.

Example on convergence --------------------▷ 36

I will skip it --------------------▷ 38

---

$$\frac{1}{\sqrt{1-0.4}} = \approx 1.26;\ \text{truevalue} = 1.291$$ 82

The error is less than 10%, but more than 1%.

---

Calculate $\sqrt{1.4}$ using an approximation with an error less than 1%.

Solution --------------------▷ 84

Hints and detailed solution --------------------▷ 83

Determine the interval of convergence for the following series 36

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \pm \frac{x^n}{n} \ldots$$

This series converges for all values of $x$ if and only if

$$x < R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Obtain

$$\left| \frac{a_n}{a_{n+1}} \right| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 37

We wish to compute $\sqrt{1.4}$. 83

We transform the number under the square root sign in order to obtain an expression which is included in the table of approximations.

$$\sqrt{1.4} = \sqrt{1+0.4}$$

In the table we find

$$\sqrt{1+x} \approx 1 + \frac{x}{2}; \text{ for } x = 0.4 \text{ the error exceeds } 1\%$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}; \text{ for } x = 0.4 \text{ the error exceeds } 1\%$$

Now calculate

$$\sqrt{1+0.4} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 84

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n} = 1 + \frac{1}{n}$$ 37

---

Now obtain the radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|1+\frac{1}{n}\right| = \ldots\ldots\ldots\ldots\ldots$$

The answer is: $R = 1$

Hence the series for $\ln(1+x)$ will converge for $-1 < x < 1$. The end points are not included, of course. (ln(0) is not even defined.)

------------------▷ 38

---

$$\sqrt{1+0.4} \approx 1 + \frac{0.4}{2} - \frac{0.4^2}{8} = 1.18$$ 84

---

Have a break!
You should by now be able to decide for yourself when to have a break and how long it should last. Furthermore, you should stick to the duration you have fixed for it.

Remember to do something quite different during a break; you must give your brain a rest!
After your break

------------------▷ 85

## 8.2 APPROXIMATE VALUES OF FUNCTIONS

38

**Objectives:** Concepts of approximate polynomials, remainder.

**READ: 8.4 Approximate values of functions**
**Textbook pages 235–237**

-------------------▷ 39

**INTEGRATION BY MEANS OF SERIES**

85

**READ: 8.6.2 Integration of functions when expressed as power series**
**8.6.3 Expansion in a series by integrating**
**Textbook pages 242–244**

-------------------▷ 86

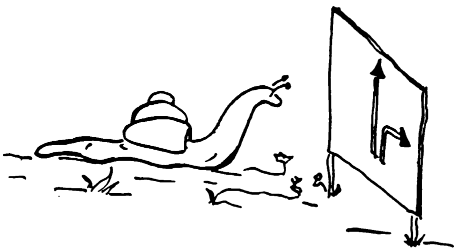
Let the expansion of a function into a power series be broken off after $n$ terms. What are the names of the two parts of this power series? 39

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}_{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} + \underbrace{a_{n+1} x^{n+1} + \ldots}_{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

--------------------▷ 40

If an integral cannot be solved by any of the well-known methods, and provided that the series is convergent, it is useful to expand a function in a series and to integrate the series. 86

Calculate the value of $\int_0^{0.53} \sqrt{1+x^3}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



Solution --------------------▷ 89

Hints and further explanation wanted --------------------▷ 87

approximate polynomial $P_n(x)$ of the $n$th degree
remainder: $R_n(x)$ 40

---

We shall now deal with the approximate polynomial.

Given the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

Write down the first four approximate polynomials:

$P_1(x) = \dots\dots\dots\dots\dots$

$P_2(x) = \dots\dots\dots\dots\dots$

$P_3(x) = \dots\dots\dots\dots\dots$

$P_4(x) = \dots\dots\dots\dots\dots$

--------> 41

---

To solve $\int_0^{0.53} \sqrt{1+x^3}\,dx$ proceed as follows: 87

We expand the integrand in a series. To do this we can use the binomial expansion which was derived in section 8.2 of the textbook.
Hence the series for $(1+x^3)^{1/2}$ is

$$(1+x^3)^{1/2} = \dots\dots\dots\dots\dots \quad |x| < 1$$

Now we integrate term by term

$$\int_0^x (1+x^3)^{1/2}\,dx = \dots\dots\dots\dots\dots$$

--------> 88

$$P_1(x) = 1 + x$$
$$P_2(x) = 1 + x + \frac{x^2}{2!}$$
$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$
$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

41

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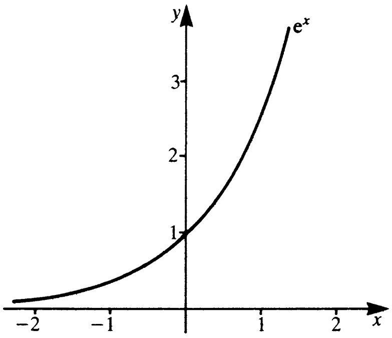



The figure shows the graph of the function

$$y = e^x$$

Draw on the diagram the first approximation

$$P_1(x) = 1 + x$$

42

---

$$(1+x^3)^{1/2} = 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 - \frac{5}{128}x^{12} + \ldots \qquad \text{provided } |x| < 1$$
$$\int_0^x (1+x^3)^{1/2}\mathrm{d}x = x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5x^{13}}{1664} + \ldots \qquad \text{provided } |x| < 1$$

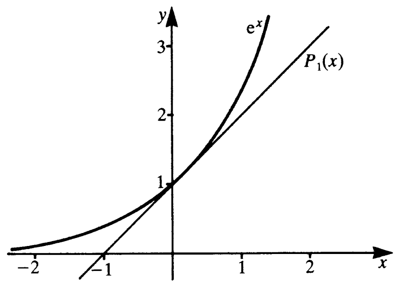
88

---

At this stage we introduce the limits for $x$, i.e. $x = 0$ and $x = 0.53$ so that

$$\int_0^{0.53} (1+x^3)^{1/2}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

89




42

The approximation $P_1(x) = 1 + x$ is a straight line which is tangential to the curve of $y = e^x$ at $x = 0$. At $x = 0$ the slopes of the function and of the polynomial are the same.

The coefficient $a_1$ of the approximate polynomial $P_1(x) = a_0 + a_1 x = 1 + x$ was chosen to satisfy the first derivative of the function $e^x$. A better approximation to the function in the neighbourhood of $x = 0$ is

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

The graph of this function is a ................................

--------------------▷ 43

---

$$\int_0^{0.53} (1 + x^3)^{1/2} dx = 0.53 + \frac{0.53^4}{8} - \frac{0.53^7}{56} + \ldots \approx 0.5398$$

89

Calculate the value of the integral

$$y = \int_0^{0.4} \sin x \sqrt{1 + x^3} dx$$

to four decimal places.

It cannot be solved by a well-known method. Thus it is useful to express the integrand as a power series and to integrate term by term:

$$y = \int_0^{0.4} \sin x \sqrt{1 + x^3} dx = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found

--------------------▷ 91

Further explantion wanted

--------------------▷ 90

parabola **43**

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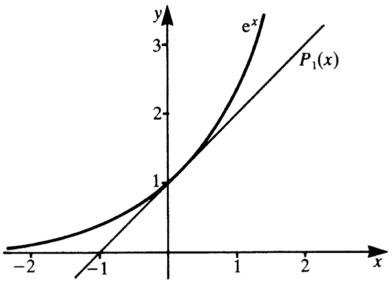


Draw on the diagram the approximation

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

------------------▷ **44**

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The integrand is a product. Both factors are series which you have encountered already: **90**

$$\sin x(1+x^3)^{1/2} = \underbrace{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)}_{\substack{\text{Series for } \sin x \text{ which} \\ \text{you can look up}}}\underbrace{\left(1 + \tfrac{1}{2}x^3 - \tfrac{1}{8}x^6 + \dots\right)}_{\substack{\text{Binomial series for} \\ \sqrt{1+x^3} = (1+x^3)^{1/2}}}$$

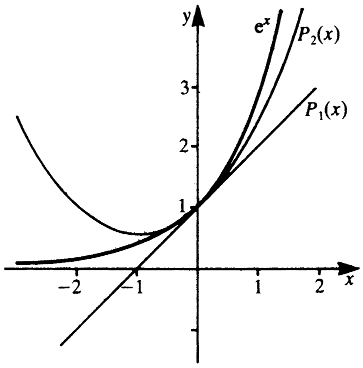
These series are both convergent for $|x| < 1$. We now multiply the two series and obtain a new series:

$$\sin x(1+x^3)^{1/2} = x - \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^5}{120} - ...$$

We integrate term by term, giving

$$y = \int_0^{0.4}\left(x - \frac{x^3}{6} + \frac{x^4}{2} + \frac{x^5}{120} - ...\right)\mathrm{d}x = \left[\frac{x^2}{2} - \frac{1}{24}x^4 + \frac{1}{10}x^5 + \frac{1}{720}x^6 - ...\right]_0^{0.4}$$
$$= \frac{1}{2}(0.4)^2 - \frac{1}{24}(0.4)^4 + \frac{1}{10}(0.4)^5 + \frac{1}{720}(0.4)^6 - ...$$
$$= ......................................$$

------------------▷ **91**




44

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The parabola is a better approximation to $e^x$. It has the same slope at $x = 0$ as well as the same curvature, i.e. the second derivatives of the function and the polynomial are the same at $x = 0$.
$f''(0) =$ ......................... $P_2''(0) =$ .........................

--------------------▷ 45

---

$$y = 0.080$$

91

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We conclude this section by using an integral to obtain the expansion in a series for a given function.
We use an example which has been solved previously in a different way:
Obtain a power series for $\ln(1+x)$, knowing that

$$\ln(1+x) = \int \frac{1}{1+x} dx$$
$$\ln(1+x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 93

Hints and detailed solution --------------------▷ 92

$$f''(0) = e^0 = 1,\ P_2''(x) = 1$$

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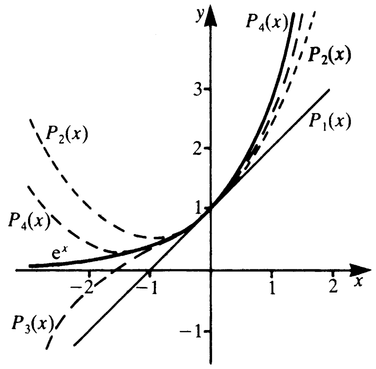
The third approximation to $e^x$ is given by $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

It is a polynomial of the third degree, which is even better than the two previous ones.




The diagram shows the graph of the function

$$f(x) = e^x$$

with the four approximations

$$P_1(x),\ P_2(x),\ P_3(x)\ and\ P_4(x).$$

It demonstrates quite clearly that the higher the degree of the approximate polynomial the more closely the approximations fit the graph.

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The required series is obtained by expanding the integrand in a series using the binomial expansion. We have

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$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 - x^4 - \ldots$$

Hence

$$\ln(1+x) = \int_0^x (1+x)^{-1}dx = \int_0^x (1 - x + x^2 - x^3 + x^4 - \ldots)dx$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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We cut off the series for $e^x$ after $n = 4$ so that 46

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

There is an error whose value can be estimated by the expression

$$R_n = \frac{f^{(n+1)}(,)}{(n+1)!} x^{n+1}$$

It is called ..................

In our case $f(x) = e^x$

$$R_4 = \text{..................}$$

--------------------▷ 47

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{... provided} -1 < x \leq 1$$ 93

Finally, a few comments on how to plan your work. In an experiment school children were presented with a subject to study on their own. The children were divided into two groups.

Experimental group: A work plan was given to the pupils according to which the written subject matter had to be worked out.

Control group: These pupils were given the material with only general instructions.

Immediately after the lesson, as well as 10 days later, the children were tested to determine to what extent the subject matter could be reproduced:

| Reproduction | With plan | Without plan |
|---|---|---|
| after the lesson | 65% | 61% |
| 10 days later | 46% | 26% |

--------------------▷ 94

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The remainder (or Lagrange's form of the remainder)

$$R_4 = \frac{e^{\xi}x^5}{5!}, \ 0 < \xi < x$$

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You should remember that when you cut off a series after $n$ terms you automatically create an error. This error can be estimated. It will be as small as you like since you can yourself fix the order of the approximation in a practical situation.

If you would like to do an example on evaluating the error, one is coming up!

I would rather carry on ▷ 52

I would like to do the example ▷ 48

Please continue on page 1
(bottom half)

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The children who had learned in accordance with the plan had learned more efficiently. The results lend themselves to generalisation. Study planning is

(a) situation analysis(how much time is available, what is your own personal capacity?);
(b) analysis of aim (what must be learned, how well must it be learned, which are the priorities and where do they lie?).

By means of a study plan, which is basically a time schedule and an analysis of objectives, a complex task can be divided up, and priorities can be established.

Such a form of work planning can already be made with the help of a simple pocket calendar.

This study guide aids you to plan and divide up your work. That is why it is so effective. In the long term, however, you should attempt to take on this job yourself. There are not always study guides around!

of Chapter 8