Chapter 12

Functions of Several Variables; Partial Differentiation; Total Differentiation

-------------------▷ 2

67

Let d = diameter of the cylinder
l = length of the cylinder

Then the volume V is $V = \frac{\pi}{4}d^2 l$, i.e. $V = f(d,\ l)$

V is a function of the two independent variables d and l.

The tolerance of the volume is given by

$$\delta V = \sum \frac{\partial V}{\partial x_j} \mathrm{d}x_j$$
$$\delta V = V_d \delta d + V_l \delta l$$

We evaluate the partial derivatives and insert the given values.

$$V_d = \tfrac{\pi}{2} d\,l,\ V_d(80,\ 120) = 15\,079.65$$
$$V_l = \tfrac{\pi}{4} d^2,\ V_l(80,\ 120) = 5026.55$$
$$\delta V = (15079.65 + 5026.55)0.15 = 3015.93\,\mathrm{mm}^3$$
$$V(80,\ \ 120) = \tfrac{\pi}{4} \times 80^2 \times 120 = 603\,185.79\,\mathrm{mm}^3$$

Percentage error in $V = \dfrac{\delta V}{V} \times 100 = \dfrac{301\,593}{603\,185.79} = 0.5\%$

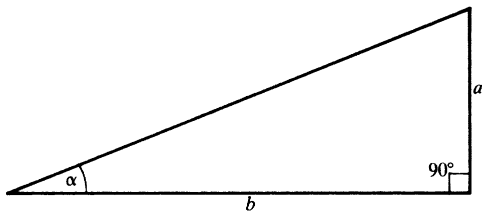
-------------------▷ 68

Before starting a new chapter quickly recapitulate the previous one, writing down all important facts from memory. Then check them with the text and any notes you made when you were studying that particular chapter. 2

This shouldn't take you more than 5 minutes, but don't skip it.

------------------▷ 3

Now try the following problem. 68



A field in the shape of a right-angled triangle is measured: b is found to be 75.5 m and $\alpha = 25.5°$ with maximum errors of 0.5 m and 0.75° respectively. Calculate the percentage error in calculating the length a and the area A of the field.

Hint: Calculate in radians and convert $\delta\alpha$ into radians first.

Solution ------------------▷ 72

Further hints wanted ------------------▷ 69

12.1 The Concept of Functions of Several Variables

3

Objective: Concept of a function of several variables, determination of surfaces of functions of two independent variables.

A very effective way of studying mathematical and physical concepts is to work with similar problems to those in the text, in parallel with it.

As you study the text carry out all the necessary operations using the function

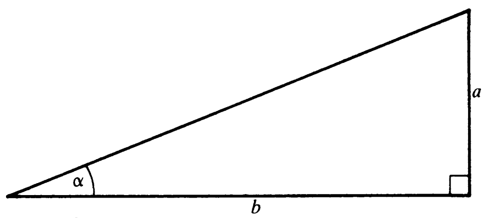
$$z = f(x, \ y) = e^{-(x^2+y^2)}$$

READ: **12.1 Introduction**
12.2 Functions of several variables
Textbook pages 339–340

-------------------▷ 4

69

If you are in difficulty remember that from our knowledge of trigonometry we have:
Length: $a = b \tan \alpha$
Area: $A = \frac{ab}{2} = \frac{b^2 \tan \alpha}{2}$



Now try to evaluate $\frac{\delta a}{a} = ...$ $\frac{\delta A}{A} = ...$

The relative error is given by $\frac{\delta a}{a}$ and $\frac{\delta A}{A}$.

The percentage error is given by $\frac{\delta a}{a} \times 100$ and $\frac{\delta A}{A} \times 100$.

Given $b = 75.5\,\text{m} \quad \alpha = 25.5°$

$$\delta b = 0.5\,\text{m} \quad \delta\alpha = 0.75°$$

Calculate the relative errors and hence the percentage errors.

Solution -------------------▷ 72

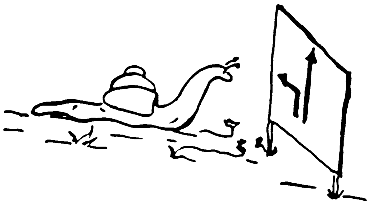
Detailed solution -------------------▷ 70

Have you been working with the function

4

$$z = f(x,\ y) = e^{-(x^2+y^2)}$$

in parallel with the text?



☐ Yes ▷ 6

☐ No ▷ 5

Consider the error in the length a of the triangle. By differentiating partially, first with respect to b and then with respect to α, we find

70

$\frac{\partial a}{\partial b} = \tan\alpha = \tan 25.5° = 0.48$ to two decimal places

$$\frac{\partial a}{\partial \alpha} = \frac{b}{\cos^2\alpha} = \frac{75.5\,\mathrm{m}}{\cos^2 25.5°} = 92.68$$

Hence the error δa in a due to errors $\delta b = 0.5$, $\delta\alpha = 0.75°$ is

$$\delta a = \frac{\partial a}{\partial b}\delta b + \frac{\partial a}{\partial \alpha}\delta\alpha = 0.48 \times 0.5 + \frac{92.68}{57.3} \times 0.75 = 1.45\,\mathrm{m}$$

(Remember the factor 57.3 to convert $0.75°$ to radians!)

Percentage error in the length a

$$\frac{\delta a}{a} \times 100 = \frac{1.45 \times 100}{75.5 \tan 25.5°} = 4.03\%$$

▷ 71

That's a pity. 5

To solve a problem in parallel with the one in the text is cumbersome. But it does lead to a better understanding of the subject matter; in the long run it saves time.

So try to sketch the surface given by the equation

$$z = \mathrm{e}^{-(x^2+y^2)}$$

following steps analogous to those in section 12.2.

▷ 6

Now for the error in the area. 71

$$\frac{\partial A}{\partial b} = b \tan\alpha = 75.5 \tan 25.5 = 36.01$$

$$\frac{\partial A}{\partial \alpha} = \frac{b^2}{2\cos^2\alpha} = \frac{75.5^2}{2\cos^2 25.5^\circ} = 3498.54$$

Hence $\delta A = \frac{\partial A}{\partial b}\mathrm{d}b + \frac{\partial A}{\partial \alpha}\mathrm{d}\alpha = 36.01 \times 0.5 + \frac{3498.54 \times 0.75}{57.3} = 63.80\,\mathrm{m}^2$

The percentage error in the measurement of the area is

$$\frac{\delta A}{A} \times 100 = \frac{6380}{\frac{1}{2} \times 75.5^2 \tan 25.5^\circ} = 4.69\%$$

▷ 72

Well done!

6

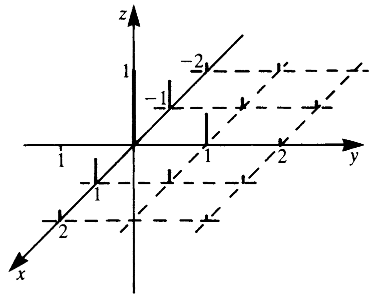
To carry out a calculation in parallel with the text instead of reading quickly may seem tiresome but if you do it you will reap the benefit.

Here are some hints for the solution of the problem to sketch

$$z = e^{-(x^2+y^2)}$$

The values have been rounded, e.g. $e^{-1} \approx 0.4$; $e^{-4} \approx 0.02$.

y \ x	0	1	2
0	1	0.4	0.02
1	0.4	0.1	0.007
2	0.02	0.007	0.0003



-------------------▷ 7

Percentage error in the length a: 4.03%

72

Percentage error in the area A: 4.69%

In case of difficulties work through the detailed solution given in frames 69, 70 and 71 again.

Write down the two properties of the gradient:

Given the function $z = f(x, y)$

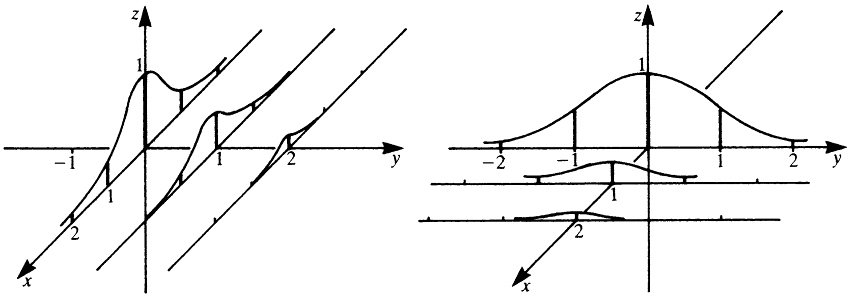
— The gradient is a .... vector to the contour lines. Thus it points in the direction of the ....

— The absolute value of the gradient is proportional to ....

-------------------▷ 73

Your sketches might be similar to these

7



The surface is similar to the one in the text. However, it approaches zero very quickly for large values of x and y.

The function is a Gauss surface. This type of surface occurs in probability theory.

------------------▷ 8

— The gradient is a *normal* vector to the contour line. Thus it points in the direction of the greatest change in z.

73

— The absolute value of the gradient is proportional to the change in z per unit length in its direction.

Given: $z = f(x, y)$

Obtain

grad $f(x, y) =$ ....

------------------▷ 74

We shall now have a look at the sketching of functions of two variables. We saw in Chapter 5 how to do this in the case of a function of one variable, where we looked for key features. 8

Before proceeding, go back to the section on curve sketching in that chapter for a quick revision if you are not sure of how to proceed.

-------------------▷ 9

74

$$\text{grad } f(x, \ y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

or

$$\text{grad } f(x, \ y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

If $f(x, \ y) = x^2 + y^2$

grad $f(x, \ y) =$ ....

If $u = x + y$

grad $u =$ ....

-------------------▷ 75

A function of two variables represents a surface in space and to sketch it we proceed in a manner similar to that for a single variable. The process is, however, more lengthy since a surface in space is a more complex geometrical figure than a plane curve. 9

To sketch the surface we can use either of two methods:

Method (a)

Draw up a table of values, a matrix, thus:

x \ y	0	1	2	...
0				
1	$z = f(x = 1, y = 0)$		$z = f(x = 1, y = 2)$	
2				
⋮				

For each pair of values $(x,\ y)$ there corresponds a value for z in accordance with the equation $z = f(x, y)$. The points x, y, z are then placed in the coordinate system and connected to each other in the x-direction and in the y-direction. In this way we build up the surface.

------------------▷ 10

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grad $f(x,\ y) = (2x,\ 2y)$ or grad $f(x,\ y) = 2x\mathbf{i}, 2y\mathbf{j}$ 75

grad $u = (1,\ 1)$ *Note:* grad u is constant.

Obtain the gradient of the function

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

grad $z =$ ....

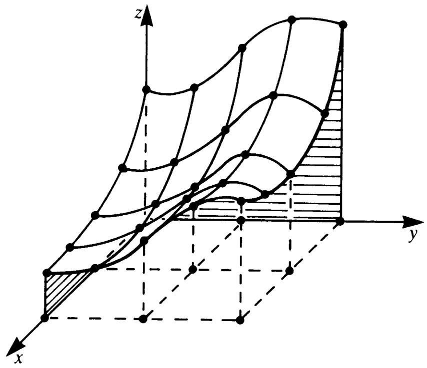
Find the equation of the contour lines of the given function for $z = C$.

$$y = \text{......................}$$

------------------▷ 76

The figure shows such a sketch

10



→ 11

76

$$\text{grad } z = \left(-\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}, -\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} \right)$$

Contour line: $y = 3\sqrt{(1-c^2) - \frac{x^2}{4}}$

Obtain the direction of a tangent to the contour line with $z = C$.

$$\mathbf{t} = \left(1, \ \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$\mathbf{t} = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$

Direction of $\mathbf{t}$ and grad z: they are .... to each other.

→ 77

Method (b) 11

We look for particular features such as:

intersection with the x–z plane, by setting $y = 0$
intersection with the y–z plane, by setting $x = 0$
intersection with the x–y plane, by setting $z = 0$
and the intersections with planes parallel to:

the x–y plane, by setting $z = z_0$, a particular value
the x–z plane, by setting $y = y_0$
the y–z plane, by setting $x = x_0$

the behaviour of the function as $x \to \pm\infty$ and $y \to \pm\infty$.

These intersection curves serve to sketch the surface. Sometimes it is even possible to guess where the surface has a maximum and/or minimum. The calculation of minima and maxima is shown later in section 12.6.

------------------▷ 12

77

$$\mathbf{t} = \left(1, -\frac{3x}{4\sqrt{(1-c^2) - \frac{x^2}{4}}}\right).$$

$\mathbf{t}$ and grad z are perpendicular to each other.

Now obtain

$\mathbf{t}\cdot$grad z = ...

Remember: grad $z = \left(-\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}, -\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\right)$

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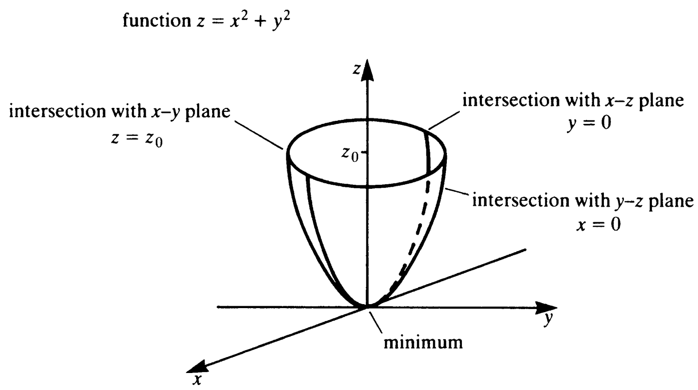

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--------------------▷ 13

$$\mathbf{t} \cdot \mathrm{grad} z = 0$$

78

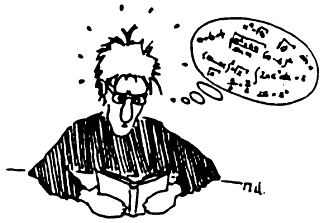
The following type of function describes the potential energy in gravitational and electrical fields. Obtain the gradient.

$$P = \frac{P_0}{\sqrt{x^2 + y^2 + z^2}}$$

grad $P =$ ....

--------------------▷ 79

With a pocket calculator it is quite easy to establish large tables of functional values. Nevertheless method (b) is most important. Often we only need a rough picture of the surface. Furthermore, the salient points are often of theoretical and practical importance. 13



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$$\text{grad}\,P = \left(-\frac{P_0 x}{(x^2+y^2+z^2)^{3/2}}, -\frac{P_0 y}{(x^2+y^2+z^2)^{3/2}}, -\frac{P_0 z}{(x^2+y^2+z^2)^{3/2}}\right)$$ 79

Obtain the magnitude of grad P.

$$|\text{grad}\,P| = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 80

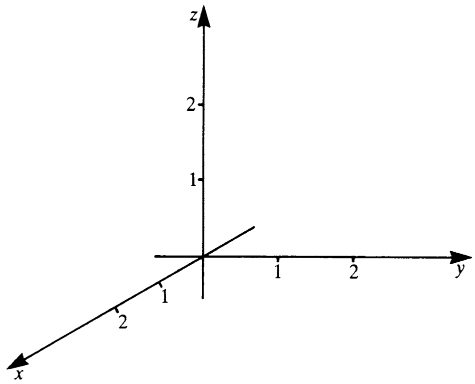
Given the function

$$z = x^2 + y^2$$

14

sketch the intersections with

(a) the x–z plane $y = 0$
(b) the y–z plane $x = 0$

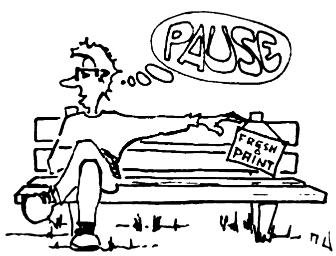


--------------------▷ 15

$$|\text{grad}\, P| = P_0 \frac{1}{x^2 + y^2 + z^2}$$

80

Note: The gradient of a potential is equivalent to the acting force **F**. This relation is often used in mechanics and electricity.



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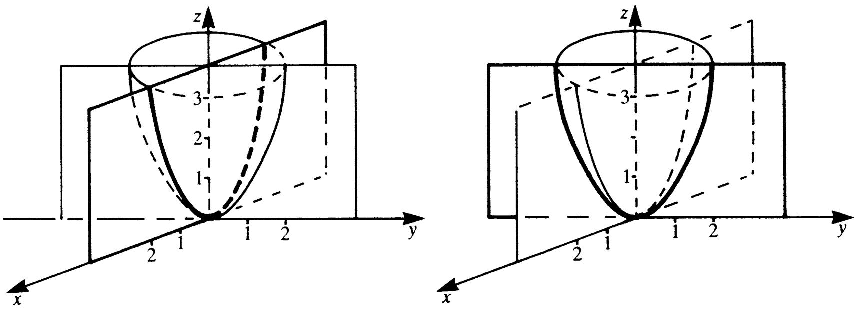
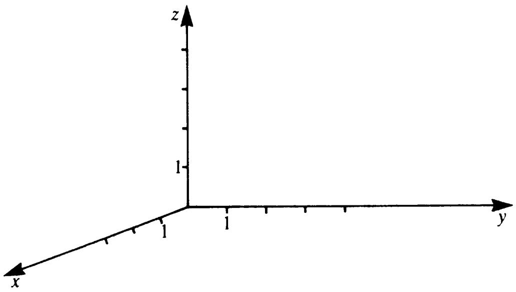
15



Explanation: The intersection with the x–z plane is obtained by setting $y = 0$, giving $z = x^2$. This is the equation of a parabola. The intersection with the y–z plane is obtained by setting $x = 0$, giving $z = y^2$.

Now sketch a few intersections with planes parallel to the x–y plane at $z = 1, 2, 3$ and 4: $z = x^2 + y^2$.



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12.4 Total Derivative

81

Objective: Concept of total derivative.

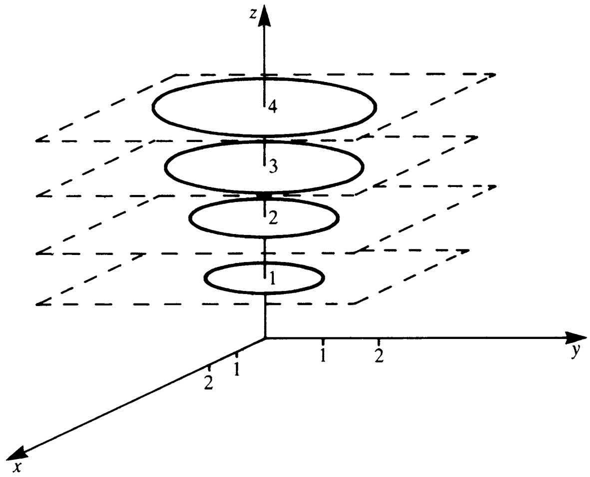
This section extends the concept of total differential of a function of two variables to two special cases.

In the first case it is assumed that the two variables both depend on a third. In the second case implicit functions are considered.

READ: 12.5 Total derivative
12.5.1 Explicit functions
12.5.2 Implicit functions
Textbook pages 360–363

------------------▷ 82

16



The intersecting curves with z = constant are circles.

17

82

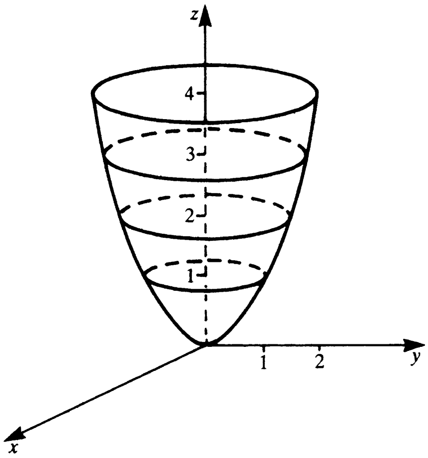
If $z = f(x, y)$ and $x = x(t)$
$y = y(t)$

Obtain $\dfrac{dz}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

This expression is called ....

83

17



By examining the intersecting curves we conclude that the equation $z = x^2 + y^2$ represents a paraboloid.

------------------▷ 18

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ 83

total derivative or total differential coefficient

Let $z = x + y$ and $x = \sin t$
$y = \sin^2 t$

Obtain:
The total differential
....
The total derivative
....

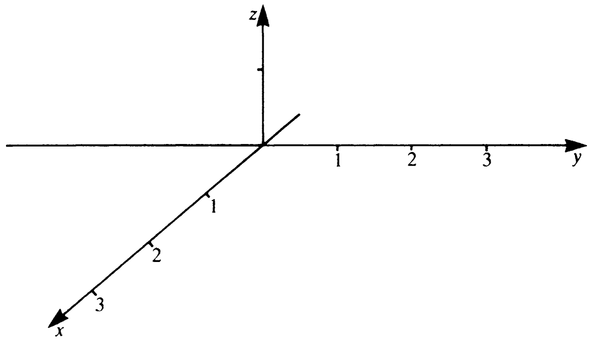
------------------▷ 84

Sketch the surface whose equation is 18

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

First plot the intersecting curve in the y–z plane.

$$z(0, y) = \text{..............................}$$



-------------------▷ 19

Total differential 84

$$\mathrm{d}z = \mathrm{d}x + \mathrm{d}y$$

Total derivative

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \cos t + 2\sin t \cos t$$

The function $z = x + y$ represents geometrically a ....
The function $z = x + y$, with further stipulations $x = \sin t$

$$y = \sin^2 t$$

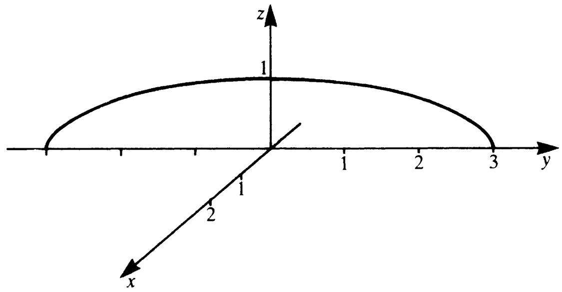
represents a curve on the surface. It is a function of one variable only. The total derivative is the differential coefficient with respect to ....

This function can be represented by a graph in the z–t coordinate system.

-------------------▷ 85

$$z(0, y) = \sqrt{1 - \frac{y^2}{9}}$$ 19

This is an ellipse.



Now plot the intersecting curve in the x–z plane in the drawing above.

$$z(x,\ 0) = \text{...........................}$$

------------------▷ 20

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surface in space; in this case it is a plane.

The total derivative is the differential coefficient with respect to t. 85

Consider an oil tank, the form of which is a cylinder of radius r and height h. Its volume is, of course,

$$V = \pi r^2 h$$

For arbitrary values of r and h this is a function of two variables. The size of the metal construction depends on the temperature t.

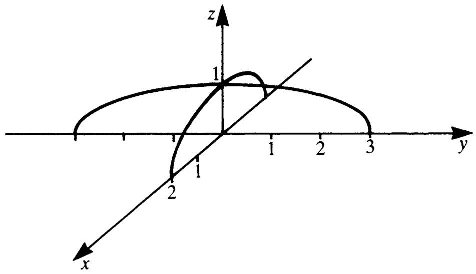
Let $h = h_0(1 + 16 \times 10^{-6} t)$
$r = r_0(1 + 16 \times 10^{-6} t)$

Obtain the total derivative of V.

$$\frac{dV}{dt} = \text{..}$$

------------------▷ 86

$z(x,0) = \sqrt{1 - \frac{x^2}{4}}$. This is an ellipse too. 20



Now add the intersecting curve in the $x - y$ plane i.e.

$$0 = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

Plot it in the figure.
Write the equation $y = y(x) =$ ....

--------------------▷ 21

$$\frac{dV}{dt} = \pi 2rhr_0 \times 16 \times 10^{-6} + \pi r^2 h_0 \times 16 \times 10^{-6}$$ 86

Since $r \approx r_0$ and $h \approx h_0$ we can write

$$\frac{dV}{dt} \approx \pi r^2 h(2 \times 16 \times 10^{-6} + 16 \times 10^{-6}) \approx 3\pi r^2 h \times 16 \times 10^{-6}$$
$$\approx 3V \times 16 \times 10^{-6}$$

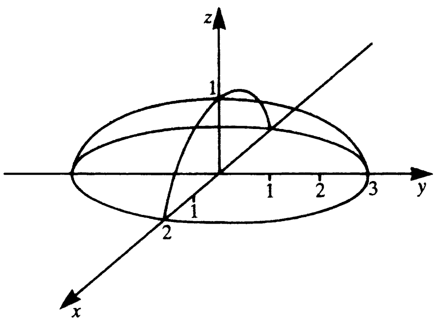
Note: The same result could have been obtained by inserting the equation for $h(t)$ and $r(t)$ into $V = \pi r^2 h$ and differentiating with respect to t.

Given the result above

$$\frac{dV}{dt} \approx 3V \times 16 \times 10^{-6}$$

what is the percentage increase in volume if the temperature rises from 0° to 30°C?
Percentage increase: $\frac{\Delta V}{V} 100 \approx$ ....

--------------------▷ 87



$y = 3\sqrt{1 - \frac{x^2}{4}}$ an ellipse in the x–y plane. 21

The function

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

represents the top half of an ellipsoid, above the x–y plane.

▷ 22

Percentage increase: $\frac{\Delta V}{V} 100 \approx \frac{\mathrm{d}V}{\mathrm{d}t} \Delta t \frac{1}{V} \times 100$ 87

$$\frac{\Delta V}{V} \times 100 \approx 3 \times 16 \times 10^{-6} \times 30 \times 100 = 0.144\%$$

The percentage increase is small. It can be neglected in most practical cases.

Let $z = x^2 + y^2$
$x = e^t$
$y = e^{-t}$

Obtain the total differential

$$\mathrm{d}z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Obtain the total derivative

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

▷ 88

In case of errors try to identify their cause by retracing your steps. Remember that if $x = 0$ we have a plane coincident with the y–z plane but if $x = x_0$, some constant value, we have a plane parallel with the y–z plane but at a distance x_0 along the x-axis from the origin of the coordinate axes. Similarly if $y = 0$, $y = y_0$, and if $z = 0$, $z = z_0$. These planes will cut the surface along the curves defined by the equation for z. If $x = 0$ then $z = f(y)$ only and we have a plane curve in the y–z plane and so on with the other variables. 22

As a further exercise find the intersection curves for the given example

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \text{ for } x = 1, \quad y = 1$$

--------------------▷ 23

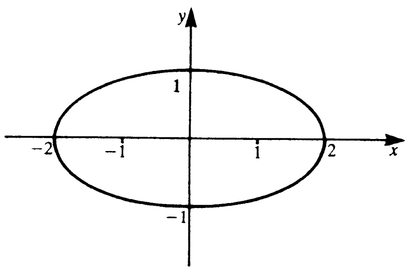
$$dz = 2x\,dx + 2y\,dy$$

$$\frac{dz}{dt} = 2e^{2t} - 2e^{-2t}$$

88

Let us now turn our attention to implicit functions.

Given: $x^2 + 4y^2 = 4$. It is the equation of an ellipse.



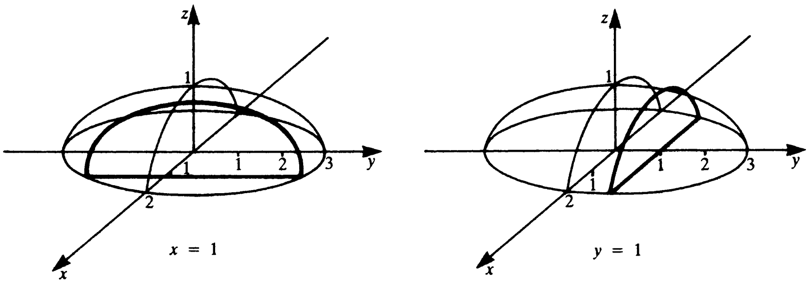
Obtain $\frac{dy}{dx}$.

$\frac{dy}{dx} =$ ....

--------------------▷ 89

$$z(1, \ y) = \sqrt{\frac{3}{4} - \frac{y^2}{9}}; \qquad z(x, \ 1) = \sqrt{\frac{8}{9} - \frac{x^2}{4}}$$ 23

Both equations represent an ellipse



You should now have a reasonable grasp of how to sketch a function of two variables. It is a surface in three-dimensional space.

We cannot sketch a function of three variables because we would need a four-dimensional space.

------------------▷ 24

$$\frac{dy}{dx} = -\frac{2x}{8y}$$ 89

Hint: We have used the formula $\dfrac{dy}{dx} = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}$

for $f(x, \ y) = x^2 + 4y^2 - 4$, $\dfrac{\partial f}{\partial x} = 2x$, $\dfrac{\partial f}{\partial y} = 8y$

To obtain $\dfrac{dy}{dx}$ as a function of x only we must use the given equation $x^2 + 4y^2 = 4$, solve for y and insert into the expression $\dfrac{dy}{dx}$.

Obtain $\dfrac{dy}{dx} = -\dfrac{2x}{8y}$ as a function of x only:

$$\frac{dy}{dx} = \text{..}$$

------------------▷ 90

12.2 Partial Differentiation
Higher Partial Derivatives

24

Objective: Concepts of partial differentiation and higher partial derivatives.

READ: 12.3 Partial differentiation
12.3.1 Higher partial derivatives
Textbook pages 346–351

While reading the textbook do all calculations with the functions

$$z = \frac{1}{1+x^2+y^2} \text{ and } u = \frac{x}{y} + 2z$$

parallel with the text

-------------------▷ 25

$$\frac{dy}{dx} = -\frac{2x}{8\sqrt{1-\frac{x^2}{4}}}$$

90

This result could also have been obtained by using the procedure outlined in Chapter 5, section 5.9.1 (implicit functions and their derivatives).

Now we give some hints on revision techniques and on preparing for examinations. Decide for yourself:

I'd like to find out a bit about exams and how to revise for them -------------------▷ 91

I don't have an exam for a few terms; I'd prefer to continue -------------------▷ 96

The symbols for the partial derivative of a function $f(x, y)$ with respect to x are .... and ....
The symbols for the partial derivative with respect to y are .... and 25

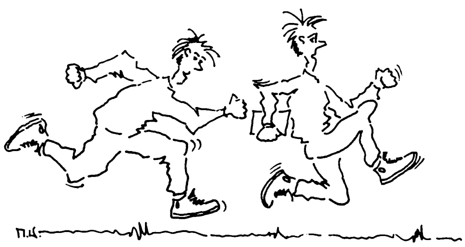
------------------▷ 26

Examinations and their preparation. 91

Points of view about exams vary from suggestions for their total abolition to requests for stricter control and higher performance standards. We will not discuss the pros and cons of examinations but we must certainly tackle the problems that examinations pose.

Preparation for exams almost always takes place under pressure. This circumstance is conditioned partly by the individual, partly by the institution.

Here we would like to give a few, perhaps trivial, pieces of advice which may help to reduce the stress experienced under exam conditions.



------------------▷ 92

$$\frac{\partial f}{\partial x}, f_x \qquad \frac{\partial f}{\partial y}, f_y$$ 26

Remember that the partial derivative of a function of several variables such as

$$z = f(x, \ y)$$

means that you differentiate the function with respect to a particular variable, x for instance, treating the others as if they were constants.

Now for some practice: $z = f(x, \ y) = x^2 + y^2$

Differentiating with respect to x yields

$\frac{\partial f}{\partial x} = 2x + y^2$ --------------------▷ 27

$\frac{\partial f}{\partial x} = 2x + 2y$ --------------------▷ 28

$\frac{\partial f}{\partial x} = 2x$ --------------------▷ 30

---

(1) Analyse the demands and your competence. 92

Success in examinations depends largely on careful *analysis* and *planning*. In order to achieve this, you should first consider the following:

(a) What demands are made of you in the exam?
(b) Which of those demands can you already meet?
(c) Which knowledge do you still lack?

Then you should try to estimate how much time you need to acquire the desired knowledge. We recommend that you double the estimated time allowance since one normally underestimates the work load considerably and, besides that, you need to set aside a little extra time as a safety measure. You never know what may crop up!

--------------------▷ 93

27

You have made a mistake!

Partial differentiation with respect to x, i.e. $\frac{\partial f}{\partial x}$ or f_x, means that we regard *all* variables as constants *except* x.

In this case y^2 is treated as a constant and since the derivative of a constant is zero it follows that

$$\frac{\partial}{\partial x}(x^2 + \text{constant}) = 2x$$

Hence

$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

Try these!

$$f(x,\ y) = x + y$$
$$f_x = \text{..........................}$$
$$g(x,\ y) = x + xy$$
$$g_x = \text{..........................}$$

--------------------▷ 29

93

(2) Plan your revision.

From the estimated time allowance, a *rough written study plan* for your exam preparation can be drawn up.

In the case of mathematics, for example, it is sufficient to link chapters with weeks. This helps you to distribute the material to be studied evenly over the available time. Of course it is much more difficult to keep to the study plan than it is to draw it up. This is because the further away an event (e.g. an examination) lies, the less seriously one takes it. With the help of a plan you can check to what extent the 'actual state' and 'desired state' truly correspond.

--------------------▷ 94

You have made a mistake! 28

You differentiated x^2 correctly but you appear to have differentiated y^2 with respect to y as well.

Partial differentiation with respect to x means that we regard all other variables as constants. Thus regard y as a constant and differentiate with respect to x only.

$$\frac{\partial}{\partial x}(x^2 + \text{constant}) = 2x$$

Hence

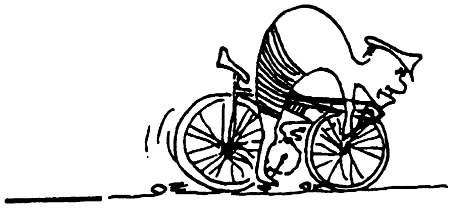
$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

Now try

$f(x,\ y) = x + y \qquad f_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and

$g(x,\ y) = x + xy \qquad g_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



29

(3) Have a schedule for your revision. 94

Let us assume that we have notes from a textbook or course. They can be revised chapter by chapter.

First step: working through the chapter;
Second step: active recall and check with the aid of notes;
Third step: solution of exercises and problems;
Fourth step: deepening of your knowledge in certain specific fields.

A rather good work technique is to do the general preparation in a small group. Verbal communication of the meaning of terms and interrelations consolidates active knowledge. The next step is the solution of exercises and questions.

Group work enables you, among other things, to estimate your personal standard of knowledge by comparing it with that of your fellow students in the group.

95

$$f_x = 1$$
$$g_x = 1 + y$$

29

Hints (in case you need them):
y is regarded as constant. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x,\ y) = \frac{\partial}{\partial x}(x+y) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y = 1 + 0$$
$$\text{and } \frac{\partial g}{\partial x} = gx = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}(xy) = 1 + y\frac{\partial}{\partial x}x = 1 + y$$

Now calculate again:

$$f(x,\ y) = x^2 + y^2 \qquad f_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 30

Minimum demand for an exam: knowledge of the given terms and interrelations. This is a prerequisite for any deepening of understanding in the given field of study.

95

Skill in presenting and solving problems, and in the drawing of parallels between the different special fields of study within the subject, give above-average results — for you yourself and in the assessment of your performance.

------------------▷ 96

$2x$ is the correct solution 30

If $z = f(x, \ y) = x^2 + y^2 + 5$

$$\frac{\partial z}{\partial y} = f_y = \text{..}$$

-------------------▷ 31

12.5 Maxima and Minima of Functions of Two or More Variables 96

Objective: Concepts of maxima and minima of a function of two or more variables, calculation of the maxima and minima of given functions.

READ: 12.6 Maxima and minima of functions of two or more variables
Textbook pages 363–396

-------------------▷ 97

$$\frac{\partial f}{\partial y} = fy = 2y$$ 31

Correct

--------------------▷ 32

Wrong: here is the solution for you to follow:
In this case we regard x as a constant and differentiate with respect to y, so that

$$\frac{\partial}{\partial y}(x^2 + y^2 + 5) \equiv \frac{\partial}{\partial y}(y^2 + \text{two constants}) = 2y$$

--------------------▷ 32

97

The function $z = f(x, y)$ depends on the independent variables x and y; it represents a surface in space whose shape may be quite complex. There may be 'hills' and 'valleys', i.e. points where the surface has a maximum or a minimum value.

Our task is to determine where these maxima and minima occur.

Suppose that x is kept constant at some value x_0; then $z = f(x_0, \ y)$ becomes a function of the single variable y, say $z = F(y)$. What is the first condition that z should have a maximum or a minimum and what does it mean geometrically?

The condition is that $\frac{dF}{dy} =$ ....

Geometrically it means that ....
....

--------------------▷ 98

Differentiate partially the function $f(x, \ y) = 2x + 4y^3$ 32

$$\frac{\partial f}{\partial x} \text{ or } f_x = \text{............................}$$

$$\frac{\partial f}{\partial y} \text{ or } f_y = \text{............................}$$

-------------------▷ 33

$$\frac{\mathrm{d}F}{\mathrm{d}y} = 0$$ 98

Geometrically, it means that the slope of the tangent to the curve is zero; the tangent is horizontal, i.e. parallel to the y-axis.

Similarly, if y is kept constant at $y = y_0$, then $z = (x, \ y_0)$ becomes a function of the single variable x, say $z = G(x)$. The first condition to be satisfied for a maximum or a minimum is:

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \text{..............................}$$

Geometrically, this means ....
....

-------------------▷ 99

$$f_x = 2, \quad f_y = 12y^2$$

33

Correct

-------------------▷ 34

Wrong: In this case go back to the textbook, section 12.3. Solve the example consulting the textbook. Find the cause of your error.

$f(x, y) = 2x + 4y^3$
$f_x =$ ....

-------------------▷ 34

$$\frac{dG}{dx} = 0$$

99

The slope of the tangent to the curve is zero; the tangent is horizontal, i.e. parallel to the x-axis.

Now consider the function $z = f(x, y)$. For this function to have a maximum or a minimum we must have:

....

....

-------------------▷ 100

Obtain the partial derivatives of the function

34

$$f(x, \ y) = x^3 + 5xy - \frac{1}{2}y^2 + 3$$

$f_x =$ ....
$f_y =$ ....

--------------------▷ 35

$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

100

Further explanation wanted on why both derivatives have to be zero?

☐ No --------------------▷ 102

☐ Yes --------------------▷ 101

$$f_x = 3x^2 + 5y, \qquad f_y = 5x - y$$

35

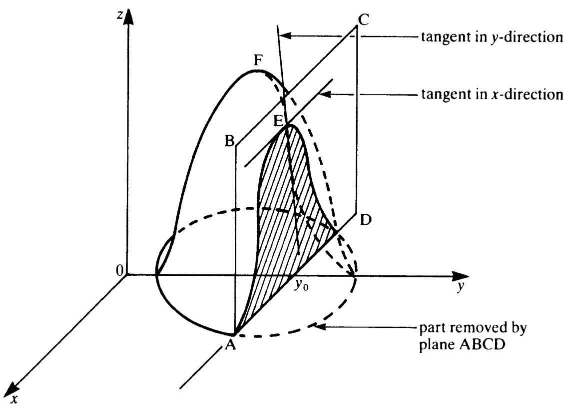
If $f(x, y) = 2x^3 \sin 2y$
$f_x =$ ....
$f_y =$ ....

36

With $z = f(x, y)$ we have a surface in space perhaps like the one shown below.

101

Suppose we cut through the surface with a plane $y = y_0$, ABCD. The intersection curve is a function of x. The slope of the tangent to it at E is zero. But the surface does not possess a maximum at E because the tangent in the y-direction is not zero. At F the tangents in the x and y directions are both zero. At F the surface has a maximum. For a maximum and a minimum we need:



$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

102

$$f_x = 6x^2 \sin 2y$$
$$f_y = 4x^3 \cos 2y$$

36

One final example!

If $u = x^2 - \sin y \cos z$

$$\frac{\partial u}{\partial x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\partial u}{\partial y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\partial u}{\partial z} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 37

The surface $z = f(x, y)$ possesses maxima and minima if $f_x = 0$ *and* $f_y = 0$ simultaneously.

For what values of x and y will the function $z = x^2 + xy + y^2 - 6x + 2$ have a maximum or a minimum?

102

$x = \ldots\ldots\ldots\ldots$, $y = \ldots\ldots\ldots\ldots$

Solution ------------------▷ 104

Explanation and detailed solution wanted ------------------▷ 103

37

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = -\cos y \cos z$$
$$\frac{\partial u}{\partial z} = \sin y \sin z$$

If you had any difficulties you might revise the rules of differentiation of functions of one variable in Chapter 5.

------------------▷ 38

103

The conditions for a maximum or a minimum are

$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

Since $z = f(x,\ y) = x^2 + xy + y^2 - 6x + 2$ we have

$$f_x = 2x + y - 6 = 0 \qquad [2]$$

and $$f_y = x + 2y = 0 \qquad [3]$$

To calculate the values of x and y we must solve equations 2 and 3 simultaneously.
What are these values?

$x =$ ...., $y =$ ....

------------------▷ 104

A few remarks concerning the necessity of doing exercises. 38

In the study guide you are asked to solve a number of exercises and this helps to sort out your difficulties. Your aim should be to become independent of the study guide as you progress with the subject matter.

You should gradually work independently by selecting the exercises according to your needs.

Exercises act like red lights to warn you when you are experiencing difficulties. You must not ignore them; they are telling you that you missed something when you were studying the text and the worked exercises, perhaps owing to lack of concentration or because you did not work in parallel with the text as we explained earlier.

-------------------▷ 39

104

$$x = 4, \quad y = -2$$

In case of difficulty work through the detailed solution given in frame 103.

We must now determine the conditions for a maximum, and those for a minimum.

Do you remember these conditions?

Yes -------------------▷ 107

I want a detailed explanation -------------------▷ 105

What is the geometrical meaning of 39

(1) the partial derivative $\frac{\partial f}{\partial x}$?

(2) the partial derivative $\frac{\partial f}{\partial y}$?

(3) What are the symbols for the partial derivatives of a function $f(x, y)$ with respect to x and y?

(4) What are the rules for partial differentiation with respect to x and with respect to y?

------------------▷ 40

Suppose that at $x = x_1$ and $y = y_1$ the slopes of the tangent along the x and y axes are zero, i.e. the tangent plane formed by these two tangents is horizontal. To discover whether the function $z = f(x,\ y)$ at x_1 and y_1 has a maximum or a minimum we could proceed as follows. Let x_1 be increased by a small amount h, positive or negative, and y_1 be increased by an equally small positive or negative amount k. 105

We now examine the value

$$\Delta f = f(x_1 + h,\ y_1 + k) - f(x_1,\ y_1)$$

To do this we expand the function into a power series. The expansion is given without proof:

$$\begin{aligned} f(x_1 + h,\ y_1 + k) = f(x_1,\ y_1) + f_x(x_1,\ y_1)h + f_y(x_1,\ y_1)k \\ + \frac{1}{2!} f_{xx}(x_1 y_1)h^2 + \frac{2}{2!} f_{xy}(x_1 y_1)hk + \frac{1}{2!} f_{yy}(x_1,\ y_2)k^2 + \ldots \end{aligned}$$

If we insert into

$$\Delta f = f(x_1 + h, y_1 + k) - f(x_1, y_1)$$

we obtain

$$\Delta f = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 106

40

(1) $\frac{\partial f}{\partial x}$ gives the slope of the tangent to the surface $z = f(x, y)$ in the x-direction.

(2) $\frac{\partial f}{\partial y}$ gives the slope of the tangent to the surface $z = f(x, y)$ in the y-direction.

(3) $\frac{\partial f}{\partial x}$ or f_x $\frac{\partial f}{\partial y}$ or f_y.

(4) If $z = f(x, y)$ is differentiated partially with respect to x it means that we regard y as a constant and carry out the process of differentiation in the usual way.
Similarly when differentiating partially with respect to y, x is considered as a constant. We carry out the process of differentiation regarding y as variable.

Example: If $z = f(x, \ y) = \sin x + \cos y$

then $\frac{\partial z}{\partial x} = ...,\quad \frac{\partial z}{\partial y} = ...$

--------------------▷ 41

106

$$\Delta f = f_x(x_1 y_1) + f_y(x_1 y_1)k + \frac{1}{2!} f_{xx}(x_1, \ y_1)h^2 + \frac{2}{2!} f_{xy}(x_1, \ y_1)hk$$
$$+ \frac{1}{2!} f_{xx}(x_1, \ y_1)k^2 + \ldots$$

Compare this equation with the equivalent one in the textbook. They are the same. Only the notation is different. Now work through section 12.6 of the textbook again.

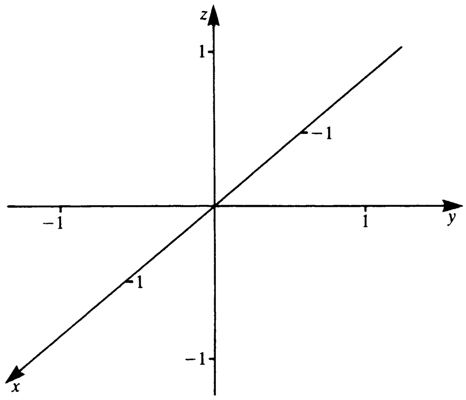
--------------------▷ 107

$$\frac{\partial z}{\partial x} = \cos x, \quad \frac{\partial z}{\partial y} = -\sin y$$ 41

In what follows we shall look once more at the geometrical meaning of the partial derivatives f_x and f_y, using the sphere of unit radius as an example.

Many people find the geometrical illustration of mathematics very useful in gaining an understanding of a particular topic.

First sketch the upper half of the sphere of unit radius and then place tangents at the north pole P(0, 0, 1) in the x and y directions.



--------------------▷ 42

The mathematical condition to determine whether a function $z = f(x, y)$ of two independent variables possesses a maximum or a minimum are: 107

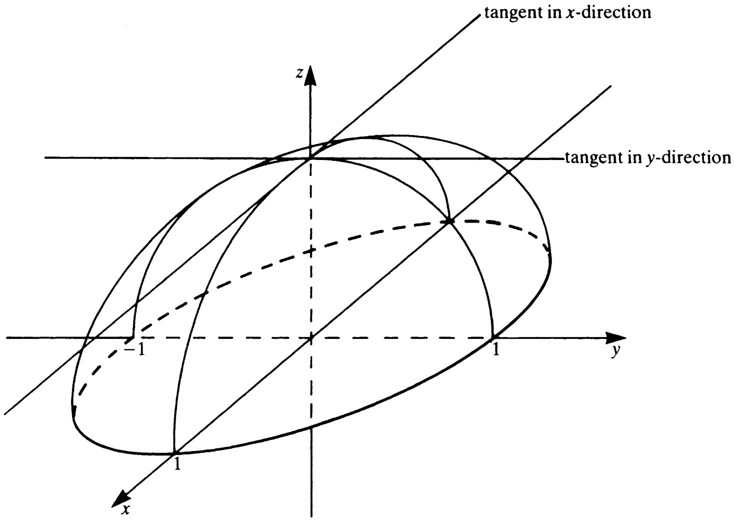
for a maximum ....

for a minimum ....

and in both cases ....

--------------------▷ 108

Correct your own sketch if necessary, or copy this diagram. 42



43

$f_{xx} < 0$ and $f_{yy} < 0$ for a maximum 108
$f_{xx} > 0$ and $f_{yy} > 0$ for a minimum
and $f_{xy}^2 - f_{xx} f_{yy} < 0$ in both cases
If you did not agree, or couldn't recall it, go back to section 12.6 or your notes.

We established earlier that the function

$$z = x^2 + xy + y^2 - 6x + 2$$

has a maximum or a minimum at $x = 4$, $y = -2$.

We want to decide whether it is a maximum or a minimum.

First we establish the derivatives at the point $(4, -2)$:

$$f_{xx}(4, -2) = \ldots\ldots\ldots\ldots$$
$$f_{yy}(4, -2) = \ldots\ldots\ldots\ldots$$
$$f_{xy}(4, -2) = \ldots\ldots\ldots\ldots$$

109

Now calculate the slope of the tangent at the point P(0, 0, 1) in the x-direction. For this we 43

(a) determine the partial derivative with respect to x, f_x, and
(b) insert the values for x and y at P(0, 0, 1) in f_x.

Note: The equation of the upper half of the unit sphere is

$$z = \sqrt{1 - x^2 - y^2}$$
$$f_x(0, 0) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 45

Difficulties or explanations wanted --------------------▷ 44

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$ 109

Solution correct --------------------▷ 110

If you had difficulties with the calculation of second derivatives the detailed solution is given here.

$$z = f(x, y) = x^2 + xy + y^2 - 6x + 2$$

We obtain the second derivatives.

$$f_x = 2x + y - 6, \qquad f_y = x + 2y$$
$$f_{xx} = 2, \qquad f_{yy} = 2$$

The second derivatives are constant. Thus they do not depend on the point (4, −2).

--------------------▷ 110

Given: $z = \sqrt{1 - x^2 - y^2}$ 44

Required: $\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{1 - x^2 - y^2}}$; y is regarded as a constant.

Now we insert the values: $x = 0,\ \ y = 0$; i.e.

$$\frac{\partial z}{\partial x}(0,\ \ 0) = \frac{0}{1 - 0 - 0} = 0$$

This means that the tangent is horizontal at that point.

--------------------▷ 45

Knowing the second derivatives $f_{xx} = 2$; $f_{xy} = 1$; $f_{yy} = 2$, decide whether the function has a maximum or a minimum at $x = 4$, $y = -2$. 110

☐ Maximum --------------------▷ 111

☐ Minimum --------------------▷ 112

$$f_x(0, \ 0) = 0$$ 45

The tangent runs horizontally since its slope is zero.

It was $z = \sqrt{1 - x^2 - y^2}$

Now calculate $f_y(0, \ 0) =$ ....

Solution --------------------▷ 47

Further explanation wanted --------------------▷ 46

Wrong! 111

For a maximum $f_{xx} < 0$ and $f_{yy} < 0$. In our case $f_{xx} > 0$ and $f_{yy} > 0$, so the function has a minimum at $x = 4, \quad y = -2$.

--------------------▷ 112

In this case x is treated as a constant. We must differentiate with respect to y. 46

$$z = \sqrt{1 - x^2 - y^2}$$
$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

Now we insert $x = 0$, $y = 0$. Thus the slope $\frac{\partial z}{\partial y}$ is zero, which means that the tangent is horizontal.

Given: $z = (1 - x)^2 + (1 - y)^2$

Obtain $\frac{\partial z}{\partial y}$ for $x = 1,\ y = 1$.

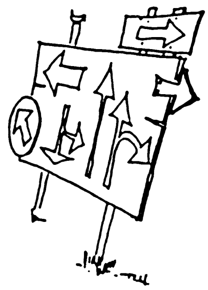
$$\frac{\partial z}{\partial y}(1,\ 1) = \text{........................}$$

--------------------▷ 47

Minimum is correct since $f_{xx} > 0$ and $f_{yy} > 0$. 112

But now the crucial question.

Are *all* the conditions satisfied? Have we established a minimum?



☐ No --------------------▷ 114

☐ Yes --------------------▷ 113

-2

47

Now let us discuss higher partial derivatives.

In the textbook, section 12.3, we obtained the second partial derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

of the function

$$f\,(x,\ y,\ z) = \frac{x}{y} + 2z$$

We obtained $f_{yx} = -\frac{1}{y^2}$

Now obtain the derivative

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$ of that same function and compare it with f_{yx}

$f_{xy} =$ ....

--------------------▷ 48

Wrong, unfortunately.

113

We have not yet examined the sign of

$$f_{xy}^2 - f_{xx} f_{yy} \quad \text{at} \quad x = 4,\ \ y = -2$$

--------------------▷ 114

$$f_{xy} = -\frac{1}{y^2}, \; f_{xy} = f_{yx}$$ 48

Explanation: Since $f(x, y, z) = \frac{x}{y} + 2z$

$\frac{\partial f}{\partial x} = \frac{1}{y}$; y and z are considered as constants.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = -\frac{1}{y^2}$$

We observe that $f_{xy} = f_{yx}$.

This statement holds true for most functions encountered in physics and engineering provided that their partial derivatives are continuous.

Does $f_{zx} = f_{xz}$ hold true for that function?

$f_{zx} =$ ....

$f_{xz} =$ ....

--------------------▷ 49

To establish a minimum, finally, we check the last condition that must be satisfied. 114

Remember: $f_{xx}(4, -2) = 2$

$f_{yy}(4, -2) = 2$

$f_{xy}(4, -2) = 1$

$$f_{xy}^2 - f_{xx} f_{yy} = \text{........................}$$

--------------------▷ 115

Yes: $f_{zx} = 0, \ \ f_{xz} = 0$ 49

The second derivative is independent of the order of differentiation provided that the first derivative is continuous and that the second derivative exists.

If you want another exercise ------------------▷ 50

If you wish to carry on ------------------▷ 52

$$f_{xy}^2 - f_{xx} f_{yy} = -3 < 0$$ 115

The condition is satisfied; a minimum is established.

Now for another example. Given $z = x^3 y^2 (6 - x - y)$, determine whether this function has a maximum or a minimum.

Solution ------------------▷ 119

Further explanation and detailed solution wanted ------------------▷ 116

If $f(x,\ y,\ z) = x^2y + y^2 + z^2x$ 50

$f_{zx} = \dots\dots\dots\dots$ $f_{xx} = \dots\dots\dots\dots$

$f_{xz} = \dots\dots\dots\dots$ $f_{yy} = \dots\dots\dots\dots$

$f_{yx} = \dots\dots\dots\dots$ $f_{zz} = \dots\dots\dots\dots$

$f_{xy} = \dots\dots\dots\dots$

------------------▷ 51

Let us go through the solution step by step. 116

Step 1: The function is $z = f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$. The first condition for a maximum or a minium is that the partial derivatives should be equal to zero, hence we have

$$f_x = \frac{\partial f}{\partial x} = 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(18 - 4x - 3y) = 0$$
$$f_y = \frac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3x^3y^2 = x^3y(12 - 2x - 3y) = 0$$

Step 2: Solve the two simultaneous equations to get the points of extreme value.
One solution is $x = y = 0$.
The other solution is found by solving

$$18 - 4x - 3y = 0$$
$$12 - 2x - 3y = 0$$

Subtracting yields $6 - 2x = 0$; hence $x = 3$, and substituting in one of the equations gives $y = 2$. Thus the second solution is $x = 3,\ \ y = 2$

------------------▷ 117

$$f_{zx} = f_{xz} = 2z \qquad f_{xx} = 2y$$
$$f_{yx} = f_{xy} = 2x \qquad f_{yy} = 2$$
$$f_{zz} = 2x$$

51

In case you do not agree with these answers the solution is given below for f_{zx} and f_{xx}.
Given: $f(x, y, z) = x^2y + y^2 + z^2x$
Partial derivative f_{zx}:

Step 1: $f_z = 2zx$ (x and y are regarded as constants)
Step 2: $f_{zx} = 2z$ (z is regarded as a constant)

Partial derivative f_{xx}:

Step 1: $f_x = 2xy + z^2$ (y and z are regarded as constants)
Step 2: $f_{xx} = 2y$ (y and z are regarded as constants)

------------------▷ 52

Step 3: To examine the condition we need f_{xx}, f_{yy}, f_{xy} at the points (0, 0) and (3, 2).

117

$f_{xx}(0,0) = \ldots\ldots\ldots\ldots$ $f_{xx} = (3,2) = \ldots\ldots\ldots\ldots$
$f_{yy}(0,0) = \ldots\ldots\ldots\ldots$ $f_{yy} = (3,2) = \ldots\ldots\ldots\ldots$
$f_{xy}(0,0) = \ldots\ldots\ldots\ldots$ $f_{xy} = (3,2) = \ldots\ldots\ldots\ldots$

------------------▷ 118

52

Straight on

--------------------▷ 54

About the necessity of exercises:

In this study guide examples are worked out. Some of the foreseeable difficulties you may have are dealt with in detail in the text.

However, with regard to study techniques, your aim should be to try to work without the help of the given examples, using the techniques involved independently.

This means you choose and attempt the exercises according to necessity. Doing exercises is particularly important when they continue to prove difficult for you.

Exercises are indicators which objectively reveal existing problems of understanding that are not recognised subjectively. It is also important that you carry out corrections promptly.

--------------------▷ 53

118

$$f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3 \qquad f_{xx}(0,\ 0) = 0 \qquad f_{xx} = (3,\ 2) = -144$$
$$f_{yy} = 12x^3 - 2x^4 - 6x^3y \qquad f_{yy}(0,\ 0) = 0 \qquad f_{yy} = (3,\ 2) = -162$$
$$f_{xy} = 36x^2y - 8x^3y - 9x^2y^2 \qquad f_{xy}(0,\ 0) = 0 \qquad f_{xy} = (3,\ 2) = -108$$

Step 4: For a maximum, $f_{xx} < 0$ and $f_{yy} < 0$,
for a minimum, $f_{xx} > 0$ and $f_{yy} > 0$,
and $f_{xy}^2 - f_{xx}f_{yy} < 0$ in both cases.
Let us check first for the point (3, 2):

$$f_{xx} = -144, \quad f_{yy} = -162, \quad f_{xy} = -108$$

and $f_{xy}^2 - f_{xx}f_{yy} = -1164$

We therefore conclude that at the point $x = 3$, $y = 2$ the function possesses a ....

--------------------▷ 119

If you do have difficulty with some exercises this may be due to one of two things: 53

(1) You have not mastered the contents of the textbook.
(2) The exercise requires the use of additional operations — knowledge of which we assume you already have but which you do not in fact possess.

Try to analyse your difficulties and to eliminate them with the aid of the relevant sections of the textbook.

At the end of the chapters in the textbook further exercises are given. Work out a sufficiently large number of these exercises.

------------------▷ 54

---

maximum 119

Now we check for the point $x = 0$, $y = 0$. We found $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 0$.

Thus the condition $f_{xy}^2 - f_{xx} f_{yy} < 0$ is not fulfilled.

The test gives no information.

------------------▷ 120

12.3 Total Differential

54

Objective: Concepts of total differential of a function of two or three variables, contour lines, gradient.

READ: **12.4.1 Total differential of functions**
12.4.2 Application: Small tolerances
12.4.3 Gradient
Textbook pages 352–360

Remember to be active as you read the textbook. Work in parallel with the text by doing the examples on a piece of paper.

------------------▷ 55

Maximum at (3, 2).

Maximum, minimum or saddle point at (0, 0)?

No decision yet.

120

Let us recapitulate.

The condition for extreme values $f_x = 0$ and $f_y = 0$ gave two solutions. We established a maximum at (3, 2). But at the point $x = 0$, $y = 0$ the necessary condition for a maximum or a minimum is not fulfilled. We obtained

$$f_{xy}^2 - f_{xx} f_{yy} = 0$$

This does not agree with the condition

$$f_{xy}^2 - f_{xx} f_{yy} < 0$$

So far we cannot decide the question. The exercises might have been a bit exhaustive, and in the end disappointing. But after all you succeeded with this quite demanding section.

------------------▷ 121

The total differential of a function $f(x, \ y, \ z)$ is defined as: 55

$$\mathrm{d}f = \ldots\ldots\ldots\ldots\ldots\ldots$$

For small values of $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ the following approximation holds good in practice

$$\Delta f \approx \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 56

12.6 Wave Functions

121

The one-dimensional wave function is a function of two variables, a length and a time. It is a special but quite important case of a function of two variables.

READ: 12.7 Applications: Wave function and wave equation
Textbook pages 369–375

------------------▷ 122

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$
$$\Delta f \approx \mathrm{d}f$$

56

(Δf being the actual change in the function.)

Obtain an expression for the total differential of the function

$$f(x,\ y,\ z) = \frac{x}{y} + z$$
$$\mathrm{d}f = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 58

Explanation and help wanted --------------------▷ 57

Write down the wave function for a one-dimensional wave travelling on a string in the x direction with amplitude A, velocity v and wavelength λ:

122

$$f(x,\ t) = \ldots\ldots\ldots\ldots\ldots\ldots$$

What is the circular frequency in terms of v or v and λ?

$$\omega = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 123

The total differential of a function is defined to be 57

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

If $f = \frac{x}{y} + z$, the derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial f}{\partial z} = 1$$

Substituting in the above equation yields

$$\mathrm{d}f = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 58

$$f(x,\ t) = A\sin(2\pi\frac{x}{\lambda} - \omega t)$$ 123

or

$$f(x,\ t) = A\sin(\frac{2\pi}{\lambda}(x - vt))$$
$$\omega = 2\pi v = 2\pi\frac{v}{\lambda}$$

Given a wave function

$$f(x,\ t) = A\sin(\frac{2\pi x}{\lambda} - \omega t)$$

Can it also be written using the cosine function?
If yes, write it down

$$f(x,\ t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 124

$$\mathrm{d}f = \frac{\mathrm{d}x}{y} - \frac{x}{y^2}\mathrm{d}y + \mathrm{d}z$$

58

In case of errors go back to the explanation given in frame 57.

The projection of lines of constant z-value on to the x–y plane is called ....

Given a surface, which we know already to be a paraboloid:

$$z = x^2 + y^2$$

Calculate the projection of the line $z = 9$ on to the x–y plane.

$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 59

Yes

$$f(x,\ t) = A\cos\left(\frac{2\pi x}{\lambda} - \omega t + \frac{\pi}{2}\right)$$

124

Given a wave function

$f(x,\ t) = 0.1\cos(4\pi x - 8\pi t)$, ($x$ in metres, t in seconds)

What is the wavelength? $\lambda =$ ....

What is the frequency? $\nu =$ ....

What is the phase velocity? $v =$ ....

What is the time of one oscillation of a given point x_0? $T =$ ....

-------------------▷ 125

a contour line

$$y = \pm\sqrt{9 - x^2}$$

59

Hint: We had to solve $z = 9 = x^2 + y^2$ for y.

The function $f(x,\ y) = z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$ represents the top half of an ellipsoid. You know it already.

Obtain the contour lines for

$z = 0$ $y =$ ....
$z = \frac{1}{4}$ $y =$ ....
$z = \frac{1}{2}$ $y =$ ....
$z = \frac{3}{4}$ $y =$ ....
$z = 1$ $y =$ ....

--------------------▷ 60

$$\lambda = \frac{1}{2}\,\mathrm{m},\ \ \nu = 4\,\mathrm{s}^{-1},\ \ v = 2\,\mathrm{m/s},\ \ T = \frac{1}{4}\,\mathrm{s}$$

125

Given the same wave function.
$f(x,t) = 0.1\cos(4\pi x - 8\pi t)$

It is to describe the displacement of a fixed point on a string or cable. Obtain the displacement and velocity v of the point $x = 2$:

$f(2,\ t)$ = ...
$v(2,\ t)$ = ...
Maximum velocity = ...

Solution --------------------▷ 127

Further hints --------------------▷ 126

60

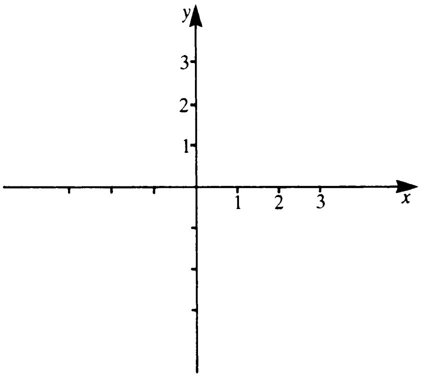
$z=0,\ y=3\sqrt{1-\frac{x^2}{4}};\qquad z=\frac{1}{4},\ y=3\sqrt{\frac{15}{16}-\frac{x^2}{4}}$

$z=\frac{1}{2},\ y=3\sqrt{\frac{3}{4}-\frac{x^2}{4}};\qquad z=\frac{3}{4},\ y=3\sqrt{\frac{7}{16}-\frac{x^2}{4}}$

$z=1,\ y=0$ since $0=-\frac{x^2}{4}-\frac{y^2}{9}$ is valid for $x=0$ and $y=0$ only

---

Sketch these contour lines.



------------------▷ 61

126

Given $f(x,\ t)=0.1\cos(4\pi x-8\pi t)$

The given function $f(x,\ t)$ represents the displacement of a point of the string depending on position x and time t. The function describes a wave with the phase velocity $v=2\,\mathrm{m/s}$.

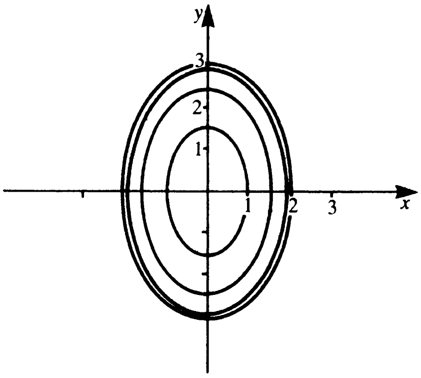
Yet what is wanted is not the phase velocity but the velocity $\mathbf{v}$ of a point of the string at $x=2$. The function $f(2,\ t)$ describes the oscillation of that given point.

$$f=(2,\ t)=0.1\cos(8\pi-8\pi t)=0.1\cos(-8\pi t)$$

$$\mathrm{v}=\frac{\mathrm{d}}{\mathrm{d}t}f(2,\ t)=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{v}_{\max}=\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 127



61

If $z = 2x^2 + 5xy + 3y^2$

calculate the percentage error in taking the differential dz as an approximation to Δz when $x = 2.5$, $y = 5.75$, d$x = 0.25$, d$y = 0.15$.

$$\text{Percentage error} = \frac{\text{True value} - \text{Approximate value}}{\text{True value}} = \frac{\Delta z - \mathrm{d}z}{\Delta z}$$

Solution -------------------▷ 64

Further explanation wanted -------------------▷ 62

$$\mathbf{v} = -0.1(8\pi)\sin(-8\pi t)$$
$$\mathbf{v}_{\max} = 0.1 \times 8\pi = 2.512\,\mathrm{m/s}$$

127

Given the one-dimensional wave equation

$$c^2 \frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial^2 f(x,t)}{\partial t^2}$$

show by verification that the following wave function is a solution:

$$f(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Solution -------------------▷ 129

Detailed explanation wanted -------------------▷ 128

$$z = 2x^2 + 5xy + 3y^2$$

62

To calculate the percentage error, if we take the total differential $\mathrm{d}z$ as an approximation for Δz when $x = 2.5$; $y = 5.75$, $\mathrm{d}x = 0.25$, $\mathrm{d}y = 0.15$:

$$\text{Percentage error} = \frac{\text{True value} - \text{Approximate value}}{\text{True value}} = \frac{\Delta z - \mathrm{d}z}{\Delta z}$$

Step 1: Calculation of the approximate value. This is the approximate change in z if we use the total differential and insert the given values.

$$\mathrm{d}z = (4x + 5y)\mathrm{d}x + (5x + 6y)\mathrm{d}y$$
$$\mathrm{d}z = (4 \times 2.5 + 5 \times 5.75)0.25 + (5 \times 2.5 + 6 \times 5.75)\,0.15$$
$$\mathrm{d}z = 16.7375$$

------------------▷ 63

Given: $f(x,\ t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$

128

We want to show that it satisfies the relation

$$c^2 \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$$

We differentiate partially

$$\frac{\partial^2 f}{\partial x^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$
$$\frac{\partial^2 f}{\partial t^2} = -A\left(\frac{2\pi}{\lambda}v\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Inserting yields

$$-c^2 A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] = -A\left(\frac{2\pi}{\lambda}v\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$c^2 = v^2$ and thus
$v = c$

------------------▷ 129

63

Step 2: Calculation of the true change in z (true value).

$$\Delta z = f(x+\Delta x,\ y+\Delta y) - f(x,\ y)$$

$$f(2.5, 5.75) = 2(2.5)^2 + 5\times 2.5\times 5.75 + 3(5.75)^2 = 183.5625$$

$$f(2.5+0.25, 5.75+0.15) = 2(2.75)^2 + 5\times 2.75\times 5.9 + 3(5.9)^2 = 200.68$$

Thus

$$\begin{aligned}\Delta z &= 200.68 - 183.5625\\ &= 17.1175\end{aligned}$$

Step 3: Calculation of the error by using dz as an approximation for the true change of the function Δz.

$$\Delta z - \mathrm{d}z = 17.1175 - 16.7375$$

Now calculate the percentage error:

$$\begin{aligned}\text{Percentage error} &= \frac{\text{True value} - \text{Approximate value}}{\text{True value}} \times 100\\ &= \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

--------------------▷ 64

The wave function is a solution of the wave equation with $v = c$. 129

Obtain the superposition of two waves with opposite velocities and equal frequencies:

$$f_1(x,\ t) = \sin\left(\frac{2\pi}{\lambda}x - \omega t\right)$$

$$f_2(x,\ t) = \sin\left(\frac{2\pi}{\lambda}x + \omega t\right)$$

$$f(x,\ t) = f_1(x,\ t) + f_2(x,\ t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

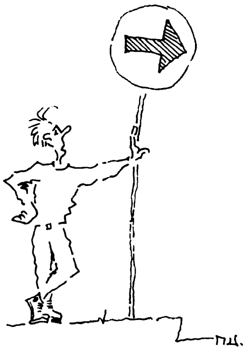
Solution --------------------▷ 131

Further explanation wanted --------------------▷ 130

$$\text{Percentage error} = \frac{\Delta z - \mathrm{d}z}{\Delta z} \times 100 = \frac{0.38 \times 100}{17.1175} = 2.22\%$$

64

If $u = f(x_1, \; x_2, \; x_3, \ldots, x_n)$ is a function of n independent variables x_i then provided that the change in their values, i.e. Δx_i, is small, the error in taking the total differential $\mathrm{d}u$ as an approximation for the actual change Δu in the function is acceptable in many practical situations.



------------------▷ 65

130

Superposition means addition. We simply add both waves.

$$f = \sin\left(\frac{2\pi}{\lambda}x - \omega t\right) + \sin\left(\frac{2\pi}{\lambda}x + \omega t\right)$$

Using the addition formulae we obtain

$$\begin{aligned} f &= \sin\frac{2\pi}{\lambda}x\cos\omega t - \cos\frac{2\pi}{\lambda}x\sin\omega t + \sin\frac{2\pi}{\lambda}x\cos\omega t + \cos\frac{2\pi}{\lambda}x\sin\omega t \\ &= 2\sin\frac{2\pi}{\lambda}x\cos\omega t \end{aligned}$$

------------------▷ 131

The length and diameter of a small cylinder are found to be 120 mm and 80 mm respectively when measured. If there is a probable tolerance of 0.15 mm in each measurement what is the approximate percentage tolerance in the value of the volume? 65

$$\frac{\delta V}{V} \times 100 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

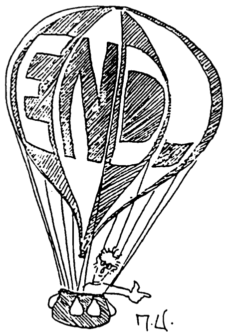
------------------▷ 66

$$f(x,t) = 2\sin\frac{2\pi}{\lambda}x\cos\omega t$$ 131

The resulting wave is called a standing wave.

The wave oscillates with the frequency ω at any position $x = x_0$. The amplitude of the wave is a function of x. The wave is at rest if $\sin\frac{2\pi}{\lambda}x$ becomes zero. These positions are called nodes.

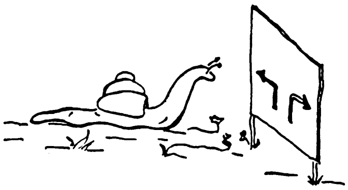
For a node $\frac{2\pi}{\lambda}x = n2\pi$. Thus nodes occur at $x = n\lambda$ (n = 1, 2, 3, ...).



Standing waves can be interpreted as the superposition of two waves travelling in opposite directions.
In this book only rudimentary but interesting and basic aspects of waves could be included. More advanced textbooks must be consulted when this subject is needed, e.g. in your studies of mechanics, acoustics or electricity.

END OF CHAPTER 12

$$\frac{\delta V}{V} \times 100 = 0.5\%$$ 66



Correct ------------------> 68

Wrong, or explanation wanted ------------------> 67

Please continue on page 1
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