Chapter 6
Integral Calculus

--------------------▷ 2

93

Hint:
The aim of the substitution is to reduce a complicated integral to a standard one. Look again carefully at the example:

$$\int (2-3x)^7\,\mathrm{d}x$$

It looks very much like $\int u^7\,\mathrm{d}u$ which is a standard integral.

Now what is your choice for u?

$$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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Preliminary note: The integral calculus can be approached in two different ways: 2

(i) Analytically: in this approach integration is formally defined as the inverse of differentiation.
(ii) The calculation of an area under a given curve leads to the integral calculus.

These approaches are shown to be equivalent and are discussed in sections 6.1 and 6.2 of the textbook.

--------------------▷ 3

$$u = 2 - 3x$$

94

Solve the integral $\int (2-3x)^7 \, dx =$ ....

--------------------▷ 95

6.1 The Primitive Function Fundamental Problem of the Integral Calculus

3

Objective: Concepts of primitive function, boundary conditions, integration.

READ: **6.1 The primitive function**
6.1.1 Fundamental problem of the integral calculus
Textbook pages 147–149

--------------------▷ 4

$$\int (2-3x)^7 \, dx = -\tfrac{1}{24}(2-3x)^8 + C$$

95

Correct --------------------▷ 97

Detailed solution required --------------------▷ 96

Let the function be denoted by $f(x)$.
We require a primitive function $F(x)$.
What relationship exists between these two functions?

.... = ...

4

-------------------▷ 5

To evaluate $\int (2-3x)^7 \, \mathrm{d}x$

96

let $u = 2 - 3x$.

$\frac{\mathrm{d}u}{\mathrm{d}x} = -3$, therefore $\mathrm{d}x = -\frac{1}{3}\,\mathrm{d}u$

The relation between $\mathrm{d}u$ and $\mathrm{d}x$ follows naturally from the equation relating x and u. With the substitution the integral becomes

$$\int (2-3x)^7 \, \mathrm{d}x = -\frac{1}{3}\int u^7 \, \mathrm{d}u = -\frac{1}{24}u^8 + C$$

Finally we express the solution in terms of the original variable x.

The solution is

$$\int (2-3x)^7 \, \mathrm{d}x = -\frac{1}{24}(2-3x)^8 + C$$

-------------------▷ 97

$$F'(x) = f(x)$$

5

Integration is the inverse of differentiation.

This means that if we differentiate a given function and integrate the new function we get back the original function except for an additive constant.

Differentiate and integrate successively the function

$$y = x^3$$

Differentiating: $y' =$ ....

Integrating: $y =$ ....

--------------------▷ 6

Evaluate $\int e^{2ax}\,dx =$ ....

97

--------------------▷ 98

$$y' = 3x^2$$
$$y = x^3 + C$$

6

Let us have a look at another example.

Primitive functions, or simply primitives, are usually written with capital letters. Remember the constant added to the primitive!

$$F(x) = y = x^2 + 4$$

Differentiating: $F'(x) = f(x) = y' =$ ....

Integrating: $F(x) = y =$ ....

--------------------▷ 7

$$\int e^{2ax}\,dx = \frac{1}{2a}e^{2ax} + C$$

98

Evaluate $\int \cos^2 x\,dx$

This is an important integral we frequently encounter.

It can be solved in many ways: Transform $\cos^2 x$ using the addition formula, or use the method shown in the textbook for $\int \sin^2 x\,dx$, or use the relation $\cos^2 x = 1 - \sin^2 x$ and use the known result of

$$\int \sin^2 x\,dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) + \mathrm{C}$$
$$\int \cos^2 x\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 99

$$F'(x) = 2x$$
$$F(x) = x^2 + C$$

7

If we differentiate and integrate successively a given function then we get back the original function except for an additive constant.

Let's follow these operations with the help of the next example.
We start with the function

$$y = \sin(2\pi x)$$

Differentiating, we obtain

$$F'(x) = f(x) = 2\pi \cos(2\pi x)$$

Integrating, we find

$$F(x) = \sin(2\pi x) + C$$

Can you name the primitive of $f(x) = \cos x$?

$$F(x) = \text{..............................}$$

--------------------▷ 8

$$\int \cos^2 x \, \mathrm{d}x = \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x) + C$$

99

Correct --------------------▷ 101

Detailed solution using the substitution method --------------------▷ 100

If

$$f(x) = \cos x$$
$$F(x) = \sin x + C$$

8

If your result does not agree, satisfy yourself of the correctness of the solution by differentiating $F(x)$. You should obtain $f(x)$.

Difficulties may occur because of the notation. We have to memorise: the primitive is denoted by $F(x)$ and its derivative by $f(x)$. We always use this notation in the textbook; it is very common. Integrating means trying to find the .... of a function.

The given function is the derivative of the ....

--------------------▷ 9

We know that

100

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Therefore

$$\int \cos^2 x \, dx = \frac{1}{2}\int (1 + \cos 2x) \, dx = \frac{1}{2}\int dx + \frac{1}{2}\int \cos 2x \, dx$$

For the second integral let $u = 2x$, $dx = \frac{1}{2} du$. The integral now becomes

$$\int \cos^2 x \, dx = \frac{1}{2}\int dx + \frac{1}{4}\int \cos u \, du$$
$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

--------------------▷ 101

primitive
primitive

9

In the following you will find additional explanations of the graphical representation and the link between the integral and differential calculus.

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Now let us look at integrals of the type

101

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$$

The numerator is the differential coefficient of the denominator.

Substitute $u = \ldots\ldots\ldots\ldots\ldots\ldots$

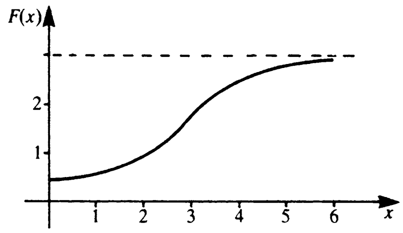
$\mathrm{d}u = \ldots\ldots\ldots\ldots\ldots\ldots$

and evaluate the integral

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 102

Given the function $F(x)$:



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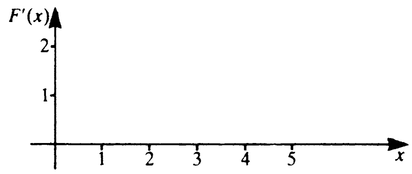
Sketch the graph of the function $F'(x)$

i.e. of the derivative in the interval $0 \leq x \leq 6$

In other words, differentiate the given curve.



------------------▷ 11

$$u = f(x), \qquad du = f'(x)\,dx, \qquad \frac{du}{dx} = f'(x)$$

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$$

102

Evaluate

(a) $\int \cot x\,dx \quad =$ ....

(b) $\int \frac{2+x}{x^2+4x}\,dx =$ ....

I have evaluated the integrals ------------------▷ 103

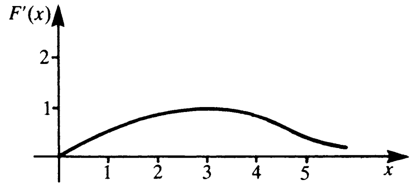
I need help ------------------▷ 104

11

To sketch the curve we have three reference points: At $x = 0$, $F(x)$ has a horizontal tangent, which means

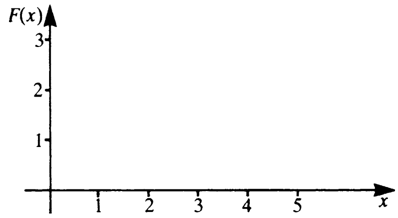
$$F'(x) = 0$$

At $x = 3$, the slope of $F(x)$ has its greatest value, about 1.



For $x \gg 3$ the curve approaches a horizontal asymptote its slope tends to zero.

Sketch two functions $F(x)$ using the information given above for the derivative $F'(x)$. This operation corresponds to integration. One curve should pass through the origin and a second curve through the point (0, 1).



--------------------▷ 12

103

(a) $\int \cot x \, dx = \ln|\sin x| + C$

(b) $\int \frac{2+x}{x^2+4x} dx = \frac{1}{2} \ln\left|x^2+4x\right| + C$

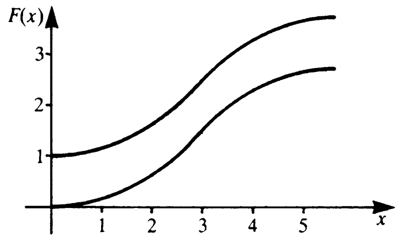
Correct --------------------▷ 106

Detailed solution required --------------------▷ 104

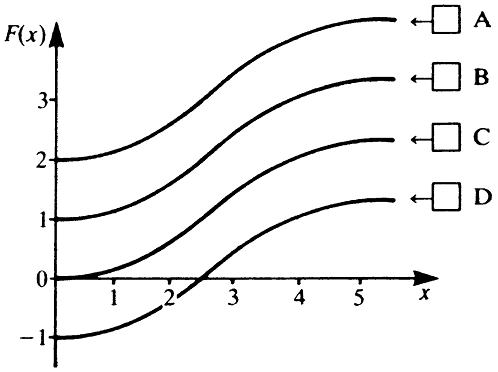
12

The actual details of the graphs are not important but the trend should be correct.

Check the horizontal tangent at $x = 0$ and for $x \gg 3$.



Which of the curves $F(x)$ shown are also solutions of the curve $F'(x)$ described in the previous frame?



13

104

(a) $\int \cot x \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x$

Let $u = \sin x$, $\quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$

Therefore $\mathrm{d}x = \frac{\mathrm{d}u}{\cos x}$

Substituting in the original integral, we have

$$\int \frac{\cos x}{\sin x} \, \mathrm{d}x = \int \frac{\mathrm{d}u}{u} = \ln|u| + C = \ln|\sin x| + C$$

Alternatively, we could have used the formula

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln|f(x)| + C$$

Try it as an exercise!

105

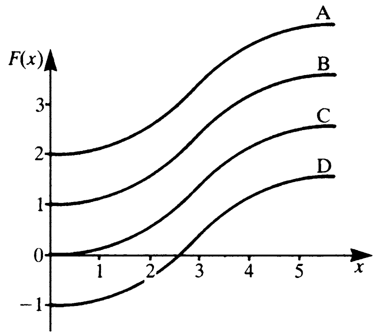
All curves are integral curves of the same derivative. They only differ by an additive constant. Name boundary conditions for the four curves A, B, C and D at $x = 0$:

A, $F(0) =$ ....

B, $F(0) =$ ....

C, $F(0) =$ ....

D, $F(0) =$ ....



13

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(b) $\int \frac{2+x}{x^2+4x} \, dx$

105

Let $u = x^2 + 4x$, $\quad \frac{du}{dx} = 2x + 4 = 2(x+2)$

Therefore $dx = \frac{du}{2(x+2)}$

Substituting in the integral we find

$$\int \frac{2+x}{x^2+4x} \, dx = \int \frac{2+x}{u} \frac{du}{2(x+2)} = \frac{1}{2} \int \frac{du}{u}$$
$$= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln\left|x^2+4x\right| + C$$

-------------------▷ 106

Curve A, $F(0) = 2$
Curve B, $F(0) = 1$
Curve C, $F(0) = 0$
Curve D, $F(0) = -1$

14

15

Whenever you have to evaluate an integral whose integrand is a fraction you should always check whether the numerator is the differential coefficient of the denominator or can be modified to become the differential coefficient of the denominator. 106

Now to integrals of the type $\int f(g(x))g'(x)\,\mathrm{d}x$.

The integrand is a product, one factor being the differential coefficient of the inner function.

To obtain the general solution

Let $u = \ldots\ldots\ldots\ldots$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \ldots\ldots\ldots\ldots$$

Therefore $\mathrm{d}x = \ldots\ldots\ldots\ldots$

Hence $\int f(g(x))g'(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots$

107

15

Obtain the derivative of

$$y = x^3 + 5, \quad y' = \text{............................}$$

Obtain the primitive of

$$F'(x) = f(x) = 3x^2, \quad F(x) = \text{............................}$$

Obtain the derivative of

$$y = 3x + 2, \quad y' = \text{............................}$$

Obtain the primitive of

$$F'(x) = f(x) = 3, \quad F(x) = \text{............................}$$

------------------▷ 16

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$$u = g(x)$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = g'(x)$$
$$\mathrm{d}x = \frac{\mathrm{d}u}{g'(x)}$$
$$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u$$

If your result is not correct, consult the textbook and go through the derivation.

------------------▷ 108

$$y' = 3x^2 \qquad y' = 3$$
$$F(x) = x^3 + C \qquad F(x) = 3x + C$$

16

After finding the primitive function a constant C must always be added.
In the examples above the first part contained the solution.

Now obtain the primitive of the following functions:

$f_1(x) = 2x \qquad F_1(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

$f_2(x) = x^2 \qquad F_2(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

Now change the notation; instead of f use g and instead of x use t.

$g(t) = t + 1 \qquad G(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 17

Evaluate

108

$$\int \frac{\cosh^5 x - 3\cosh^2 x - 7}{\cosh^4 x} \sinh x \, dx$$

First look for a suitable substitution

$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and then evaluate the integral ....

-------------------▷ 109

$F_1(x) = x^2 + C$
$F_2(x) = \frac{1}{3}x^3 + C$
$G(t) = \frac{1}{2}t^2 + t + C$

Don't Forget the Constant!

17

Calculation of the constant from a given boundary condition:
Let $f(x) = x + 1$
The primitive function is

$$F(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

Boundary condition: the primitive should pass through the point (0, 0).

Of the infinite number of solutions, called integral curves, only one curve will pass through that point; its equation is:

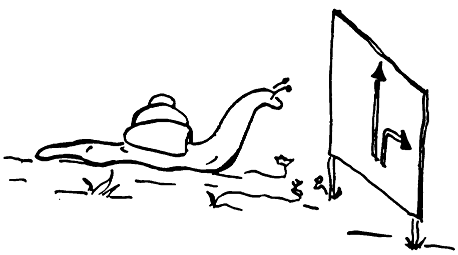
$$F(x) = \ldots\ldots\ldots\ldots\ldots\ldots \qquad C = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 18

Let $u = \cosh x$
The solution of the integral is

109

$$\frac{1}{2}\cosh^2 x + \frac{3}{\cosh x} + \frac{7}{3\cosh^3 x} + C$$



Correct

------------------▷ 111

Detailed solution required

------------------▷ 110

$$F(x) = \frac{x^2}{2} + x + C$$
$$F(x) = \frac{x^2}{2} + x, \qquad C = 0$$

18

Both correct → 20

Further explanation required → 19

Let $u = \cosh x$. 110

$$\frac{du}{dx} = \sinh x, \qquad dx = \frac{du}{\sinh x}$$

Substituting in the integral we have

$$\begin{aligned}
\int \frac{\cosh^5 x - 3\cosh^2 x - 7}{\cosh^4 x} \sinh x \, dx &= \int \frac{u^5 - 3u^2 - 7}{u^4} du \\
&= \int \left(u - \frac{3}{u^2} - \frac{7}{u^4} \right) du \\
&= \frac{u^2}{2} + \frac{3}{u} + \frac{7}{3u^3} + C \\
&= \frac{1}{2}\cosh^2 x + \frac{3}{\cosh x} + \frac{7}{3\cosh^3 x} + C
\end{aligned}$$

→ 111

19

The value of the constant to satisfy a given boundary condition:

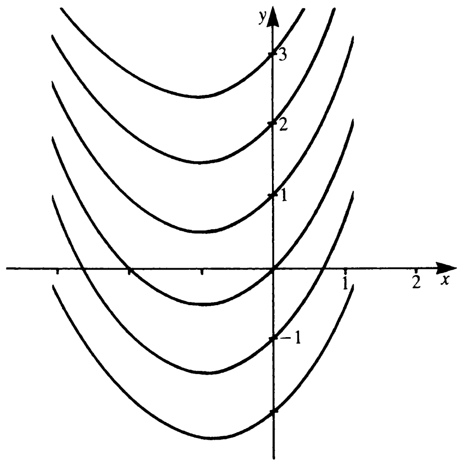
Consider the primitive function

$$F(x) = \frac{x^2}{2} + x + C$$

This represents a family of parabolas.

If the boundary condition calls for $F(x)$ to pass through the point (0, 0), i.e. the origin, we can obtain the constant as follows:

When $x = 0$, $y = F(0) = 0$



Step 1: Insert $x = 0$ and $y = 0$ in the primitive and obtain $0 = 0 + 0 + C$.

Step 2: Solve for the constant. In this case $C = 0$.

→ 20

111

We now consider integrals of the type:

$$\int R(\sin x, \quad \cos x, \quad \tan x, \quad \cot x)\,\mathrm{d}x$$

where the integrand is a rational function of the trigonometric functions. What substitution would you make in this case? Consult the textbook.

$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

→ 112

Let the derivative be $y' = x$. 20

The primitive function $y = \dots\dots\dots\dots$

Boundary condition: the primitive should pass through the point $P = (1, \quad 2)$.

Calculate the constant of integration:

$$C = \dots\dots\dots\dots$$
$$y(x) = \dots\dots\dots\dots$$

--------------------▷ 21

$$u = \tan\frac{x}{2}$$ 112

With this substitution the integral becomes

$$\int R_1(u)\,\mathrm{d}u$$

where R_1 is a rational function of u.

In the textbook we expressed $\sin x$, $\cos x$, $\tan x$, $\cot x$ in terms of this new function.

Evaluate

$$\int \frac{\mathrm{d}x}{1+\cos x} = \dots\dots\dots\dots$$

using the substitution $u = \tan\frac{x}{2}$.

--------------------▷ 113

$$y(x) = \frac{x^2}{2} + C$$
$$C = \frac{3}{2}$$
$$y(x) = \frac{x^2}{2} + \frac{3}{2}$$

21

Correct solution

--------------------▷ 25

Still having difficulties, want further explanation

--------------------▷ 22

$$\int \frac{dx}{1+\cos x} = \tan\frac{x}{2} + C$$

113

If you have had some difficulties consult the textbook. There a similar integral is evaluated; go through its solution carefully.

Exercises are a kind of self-assessment, showing if we are able to apply the methods we have just learnt.

Unfortunately we tend to forget; repeating the methods helps to overcome forgetfulness. Hence exercises are particularly valuable when they are done the following day, a week later and a month later, say.

You will find a number of exercises in the textbook.

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22

Given $y' = x$. We require that the primitive should pass though the point $P = (1, \quad 2)$.

Step 1: Obtain the primitive of $y' = x$.

The primitive is $y = \frac{x^2}{2} + C$

C is not known numerically at this stage.

Step 2: Obtain the value of the constant.

The primitive $y = \frac{x^2}{2} + C$ is required to pass through the point whose coordinates are $x = 1$, $y = 2$. We need to substitute these values in the equation, thus:

$$2 = \frac{1}{2} \times 1^2 + C$$

and solving for C, $C = \frac{3}{2}$.

The solution is: $y(x) = \frac{1}{2}x^2 + \frac{3}{2}$

--------------------▷ 23

114

Now you may have a break. But before starting the break you ought to do two things:

(1) ....

(2) ....

--------------------▷ 115

Given $$y'(x) = -\frac{3}{4}x^2$$ 23

Obtain the primitive function $y(x)$ which goes through the point P(1, -3).

Step 1: Obtain the primitive

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Step 2: Evaluate the constant, i.e. substitute $x = 1$ and $y = -3$ and solve for C.

$$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, $y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

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(1) Recall what you have just learnt, making sure you have understood the subject matter. 115
(2) Fix the duration of the break or the time when you intend to resume work.

These two things must become a habit, not only now but whenever you are studying a subject using texts.

(1) Give in your own words a résumé of what you have just learnt.
(2) Write down the time when you will continue working.

End of the break ...

--------------------▷ 116

$$y(x) = -\frac{1}{4}x^3 + C, \quad C = -\frac{11}{4}$$

24

You will find further problems in the textbook.

Before the next short break read some hints on the study technique of *selective reading*.

First case for the application of selective reading:

Let's assume you are familiar with most of the content of a certain chapter. In such a case intensive reading is *not* necessary. You know the facts already. Here, something else is important. You must read the text through with a view to finding the new material which is introduced, defined or deduced there. It is a question of quickly picking out any new points from a large number of familiar or known ones.

--------------------▷ 25

Let us now carry on. Check the time against the one you fixed at the start of the break.

116

--------------------▷ 117

Second case for the application of selective reading: 25

Suppose you are looking for a specific piece of information in a long text. There is a danger that you will be distracted from your aim, by finding some piece of interesting but irrelevant information which you consequently begin to read. Who can honestly say he hasn't been distracted in this way? Perhaps, for example, you once looked up the headword SYNERGETICS and read the entries for SOLIPSISM and SYNAGOGUE as well. Deviation from target-directed searching is known as the 'encyclopaedia effect'.

Selective reading as a time-saving study technique demands the separation of information which is currently relevant from that which is currently irrelevant. One should not even be aware of the existence of the irrelevant material.

Now practise selective reading:
Look for the numerical value of Euler's number e.
If necessary use the index.
The numerical value of e is ....

-------------------▷ 26

6.6 Integration by Partial Fractions

117

Objective: Understanding the principle of partial fractions. Concept of partial fractions, proper and improper fractions.

The general procedures may seem cumbersome, but you will understand the procedures with the examples.

READ: 6.5.7 Integration by partial fractions
Real and unequal roots
Textbook pages 172–177

-------------------▷ 118

$$e = 2.71828\ldots\ldots\ldots$$

26

The technique used in selective reading is exactly the opposite of that used in intensive reading. Different aims are pursued.

In the case of selective reading, the text is glanced over carefully, your attention being directed exclusively towards the particular information being sought after.

If you come across a part of the text which you already know, try to read it selectively. However, if it's new to you then you must change your reading mode and read intensively.

-------------------▷ 27

It is important that you should understand the principle of decomposition into partial fractions. 118

Express the following in partial fractions:

$$\frac{1}{1-x^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 121

If you require a detailed explanation

-------------------▷ 119

6.2 The Area Problem: The Definite Integral Fundamental Theorem of the Differential and Integral Calculus

27

Objective: Concepts of definite integrals, integrand, limits of integration.

Operations: evaluation of simple definite integrals, inserting limits.

The next section is longer than usual. Divide your work into two parts. After the first part — say after having completed section 6.3 — have a break and check that you have understood the new concepts by reproducing them in your own words. If you do not succeed, do not go any further but go back to that particular part.

READ: **6.2 The area problem: the definite integral**
6.3 Fundamental theorem of the differential and integral calculus
6.4 The definite integral
Textbook pages 149–161

------------------▷ 28

119

Step 1: Find the roots of the denominator and then express it as the product of factors of the lowest possible degree:

$$1-x^2=0$$

The roots are $x_1=1$, $x_2=-1$.
The denominator can be expressed thus:

$$(1-x^2)=(1+x)(1-x)$$

Step 2: Express as partial fractions:

$$\frac{1}{1-x^2}=\frac{A}{1+x}+\frac{B}{1-x}$$

Step 3: Clear the fractions:

$$\frac{1}{1-x^2}=\frac{A(1-x)}{(1+x^2)}+\frac{B(1+x)}{(1-x^2)}$$
$$1=A(1-x)+B(1+x)$$

To calculate the values of A and B substitute the values of the roots x_1, x_2 successively; this yields

$A=$ ...., $B=$ ....

------------------▷ 120

The expression $\int_a^b f(x)\,\mathrm{d}x$ is called 28

a is called the ....

b is called the ....

$f(x)$ is called the ....

$\mathrm{d}x$ (known from Chapter 5) is called .... --------------------▷ 29

120

$$A = \frac{1}{2}, \quad B = \frac{1}{2}$$

The fraction becomes: $\dfrac{1}{1-x^2} =$ .... --------------------▷ 121

29

$\int_a^b f(x)\,\mathrm{d}x$ = definite integral
a = lower limit of integration
b = upper limit of integration
$f(x)$ = integrand
$\mathrm{d}x$ = differential

You have to know these concepts. Make sure that you can assign the right meaning to each one!

--------------------▷ 30

121

$$\frac{1}{1-x^2} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

Here is another problem (in case of difficulty try to solve the problem by using the textbook and following the steps described).

Evaluate $\int \frac{\mathrm{d}x}{x(2x+3)} =$ ....

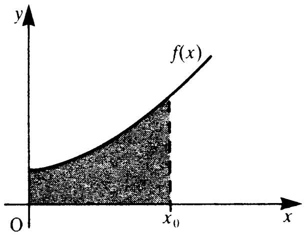
--------------------▷ 123

Explanation required

--------------------▷ 122

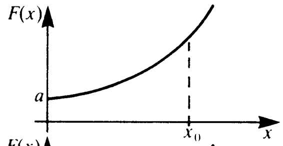
30

The shaded area is bounded by $f(x)$. The area function $F(x)$ gives the area under the curve $f(x)$ between 0 and x_0.

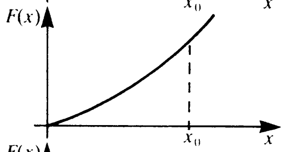


Which of the graphs represents the area function:

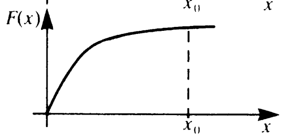
$$F(x) = \int_0^{x_0} f(x)\,\mathrm{d}x$$



→ 31



→ 32



→ 33

122

Step 1: Express the integrand as partial fractions.
The integrand is already expressed in factors. Thus the roots of the denominator are $x_1 = 0$, $x_2 = -\frac{3}{2}$.

Step 2: Rewrite the integrand as the sum of partial fractions

$$\frac{1}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3} = \frac{A(2x+3)}{x(2x+3)} + \frac{Bx}{x(2x+3)}$$

Step 3: Calculate the values of the constants by inserting successively the roots

$A =$ ...., $B =$ ....

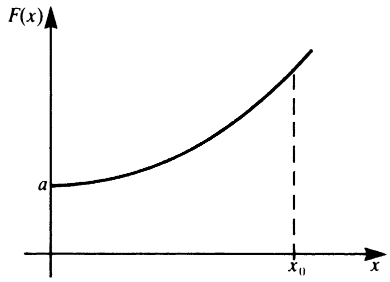
Now evaluate the integral

$$\int \frac{\mathrm{d}x}{x(2x+3)} = \text{..}$$

→ 123

31

Wrong, unfortunately.

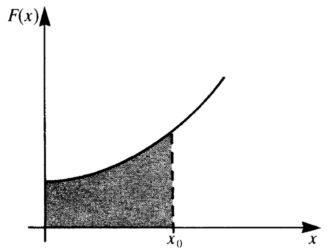


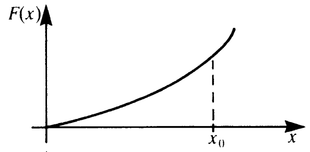
The area function $F(x)$ must pass through the origin.

If the left- and right-hand side limits of integration coincide, then the area under the curve shrinks to a line and consequently the magnitude of the area is zero, i.e. $F(0)=0$

But here $F(0)=a$.

Now decide once more what the area function should look like:





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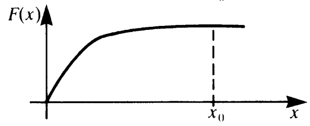


→ 33

123

$$A=\frac{1}{3},\quad B=-\frac{1}{3}$$

$$\int\frac{\mathrm{d}x}{x\,(2x+3)}=\frac{1}{3}\int\frac{\mathrm{d}x}{x}-\frac{1}{3}\int\frac{\mathrm{d}x}{(2x+3)}$$
$$=\tfrac{1}{3}\ln|x|-\tfrac{1}{6}\ln|2x+3|+C$$

In case of difficulties try to solve the example using the textbook.

→ 124

Correct!
It is true: $F(0) = 0$, and in addition the functional values of $F(x)$ increase as x increases. 32

Skip a frame; go to --> 34

We now discuss an example where the integrand is an improper fractional rational function. You will recall that we have to transform the integrand into the sum of a polynomial and a proper fractional rational function. 124

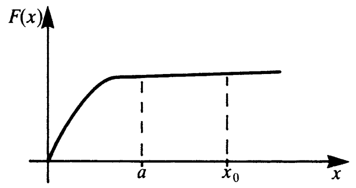
I would like to do the example --> 125

I know the work and would like to skip the example --> 127

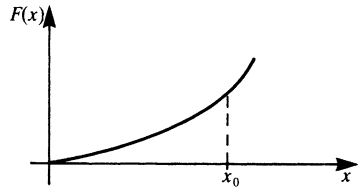
33

Unfortunately not entirely correct!

It is true that $F(x)$ goes though zero. Furthermore the function $F(x)$ has to be monotonically increasing, which means that it grows as x increases, since the area under the curve $f(x)$ increases if the limit is moved to the right. However, the function $F(x)$ chosen by you is constant from position a onwards.



The correct graph is



------------------▷ 34

125

The example is:

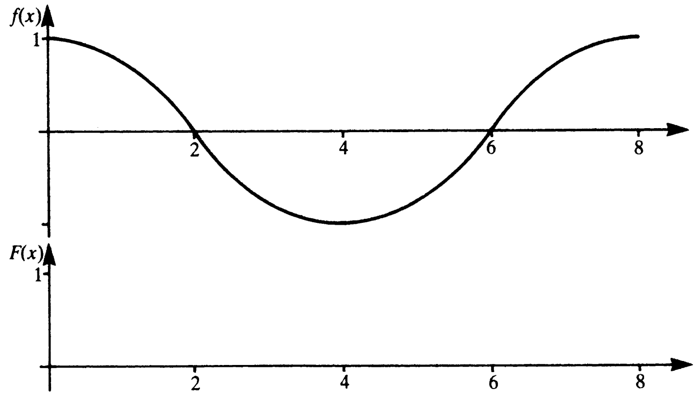
$$\int \frac{x^3-1}{x(x+1)}\,dx$$

The integrand is an improper fraction. We must divide the numerator by the denominator to ensure that the numerator is of lower degree than the denominator before we can express the integrand as partial fractions.

$$\frac{x^3-1}{x^2+x} = \underbrace{x-1}_{\substack{\text{integral}\\\text{rational}\\\text{function}}} + \underbrace{\frac{x-1}{x(x+1)}}_{\substack{\text{proper fractional}\\\text{rational function}}}$$

------------------▷ 126

Sketch the area function for $f(x)$: 34



If you have found the solution ⟶ 37

If you require help ⟶ 35

Now we can solve the integral 126

$$\int \frac{x^3-1}{x^2+x} = \int x\,\mathrm{d}x - \int \mathrm{d}x + \int \frac{x-1}{x(x+1)}\,\mathrm{d}x$$

The last integral can be solved by establishing partial fractions

$$\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

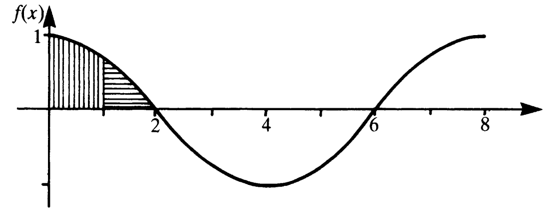
Evaluate A and B, solve the integral and check your result:

$$\int \frac{x^3-1}{x(x+1)} = \frac{x^2}{2} - x - \ln|x| + 2\ln|x+1| + C$$

Decomposition into partial fractions is carried out step by step as shown in the textbook.

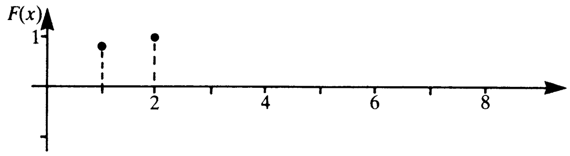
⟶ 127

Here is a hint. Given the function $f(x)$, what is the area function $F(x)$? (We have to pay particular attention to the fact that the area below the x-axis is, by definition, negative.) To obtain the area function we divide the graph into a number of intervals of arbitrary length. 35



The area function is zero at $x = 0$. The area up to the first interval is about 0.7. The area of the second interval is about 0.3, and therefore the area of the first and the second intervals is about 1. The area of the third interval is negative and has to be deducted.

Complete the area curve for yourself by taking a number of intervals.



------------------▷ 36

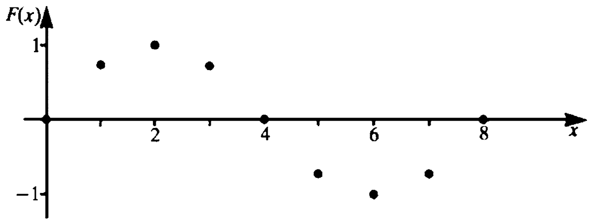
6.7 Partial Fractions: Real and Repeated Roots, Complex Roots 127

The next two sections in the textbook show cases of partial fractions. The basic procedure is the same. Complex numbers are dealt with in Chapter 9. If you are not familiar with complex numbers skip that part of section 6.5.6 and return to it later.

READ: 6.5.6 Real and repeated roots
Complex roots
Textbook page 168–172

------------------▷ 128

The area function is obtained by drawing a curve through the points: 36



37

Evaluate $\int \frac{\mathrm{d}x}{x^3(x+1)}$ 128

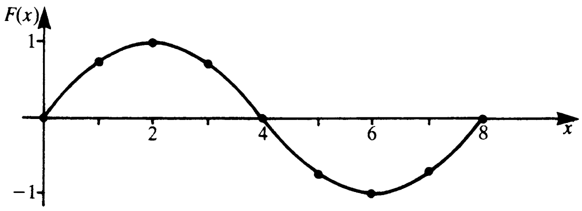
To express the integrand in partial fractions we need the roots of the denominator. They are

$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots$, $x_2 = \ldots\ldots\ldots\ldots\ldots\ldots$,
$x_3 = \ldots\ldots\ldots\ldots\ldots\ldots$, $x_4 = \ldots\ldots\ldots\ldots\ldots\ldots$

The roots of the denominator are .... but three are ...., therefore we have

$$\frac{1}{x^3(x+1)} = \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots$$

129



37

We started with the cosine function, the equation being $f(x) = \cos(......)$.

The area function is a ....

Its equation is $F(x) =$ ....

(Note the period: the value of the argument after one complete period must be equal to 2π.)

▷ 38

$$x_1 = x_2 = x_3 = 0, \quad x_4 = -1$$

129

The roots are *real* but three are *equal* (or *repeated*). Hence we must set up

$$\frac{1}{x^3(x+1)} = \frac{N(x)}{D(x)} = \frac{A_1}{x^3} + \frac{A_2}{x^2} + \frac{A_3}{x} + \frac{B}{x+1}$$

Explanation:

The roots $x_1 = x_2 = x_3 = 0$ in the denominator are repeated. If we set up

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x} + \frac{E}{x+1}$$

we see that after clearing the fractions on the right the common denominator does not agree with $N(x)$. Agreement will occur only if we set up the fractions shown above.

I found the solution ▷ 134

I require further explanation ▷ 130

$$f(x) = \cos\left(\frac{\pi x}{4}\right)$$ 38

sine function

$$F(x) = \frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right)$$

I understand the area function problem

--------------------▷ 43

I require a graphical explanation of the relationship between the area function and the integral function

--------------------▷ 39

Clearing the fractions 130

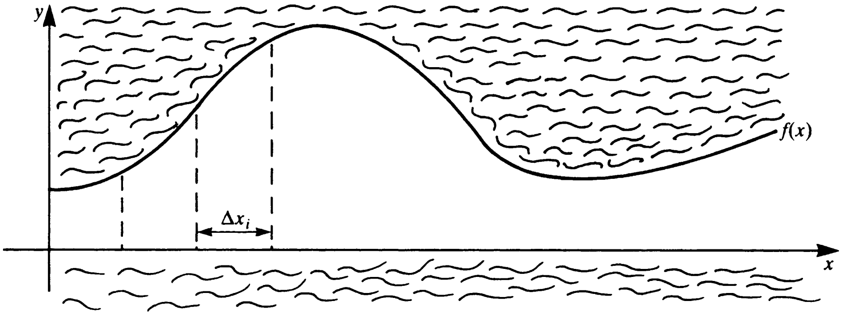
$$\frac{1}{x^3(x+1)} = \frac{A_1}{x^3} + \frac{A_2}{x^2} + \frac{A_3}{x} + \frac{B}{x+1}$$

Or multiplying by $D(x) = x^3(x+1)$ we have:

$1 =$ ....

--------------------▷ 131

A peninsula is bounded on one side by a straight coastline, the x-axis, and on the other side by a curved coastline. We call it $f(x)$. 39



The peninsula has to be cleared of weeds and shrubs. The workers divide the work by lines perpendicular to the x-axis and separated by equal amounts Δx_i and clear the area bounded by two perpendicular lines each day. The area cleared during a day can be approximately calculated by multiplying Δx_i by $f(x)$, the width of the peninsula at the position the work was stopped in the evening. This area is plotted on a chart by a very officious administrative officer every day. This chart is called the curve of *reclaimed land*.

------------------▷ 40

$$1 = A_1(x+1) + A_2x(x+1) + A_3x^2(x+1) + Bx^3$$ 131

Since this identity is true for all values of x we can substitute successively

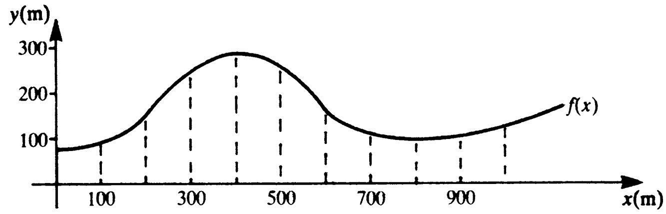
$x = x_1 = 0$	hence	$1 =$ ....
$x = x_4 = -1$		$1 =$ ....
$x = 1$		$1 =$ ....
$x = 2$		$1 =$ ....

We can choose the values of x conveniently.

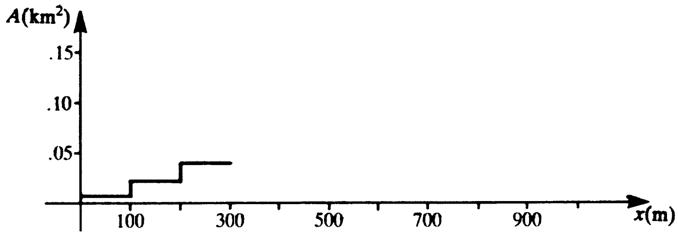
------------------▷ 132

What is the shape of this curve? 40

The coastline is reproduced below:



Complete the diagram showing the area of land reclaimed:



------------------▷ 41

For $x = x_1 = 0$ we have $1 = A_1$ 132

$x = x_4 = -1$ $\quad 1 = -B$

$x = 1$ $\quad 1 = 2A_1 + 2A_2 + 2A_3 + B$

$x = 2$ $\quad 1 = 3A_1 + 6A_2 + 12A_3 + 8B$

From the first two equations we find

$A_1 =$ ....

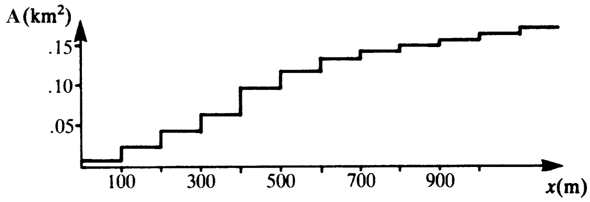
$B =$ ....

Inserting these values in the last two equations yields

$A_2 =$ ....

$A_3 =$ ....

------------------▷ 133



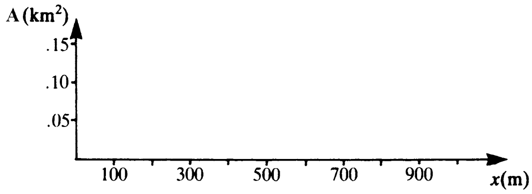
41

The area cleared every day $f(x_i)\Delta x_i$ is added on the chart. Thus the amount of land reclaimed grows each day. After the nth day the total area of land reclaimed is given by

$$L = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The smaller the intervals the more accurate the computation. This means that if the workers report the area cleared two or three times a day the actual area cleared will be computed more accurately. If the process were carried further the chart representing the area of land reclaimed would become a smooth curve, i.e. the integral curve.

Now sketch the integral curve:



------------------▷ 42

$$A_1 = 1, \quad A_2 = -1$$
$$A_3 = 1, \quad B = -1$$

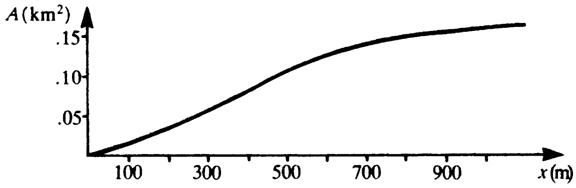
133

With these values we can evaluate the integral, hence

$$\int \frac{dx}{x^3(x+1)} = \int \frac{dx}{x^3} - \int \frac{dx}{x^2} + \int \frac{dx}{x} - \int \frac{dx}{x+1}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 134



42

The passage from a discontinuous curve to a continuous integral curve is obtained by means of a limiting process.

Mathematically this is

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_0^x f(x)\,\mathrm{d}x$$

--------------------▷ 43

$$\int \frac{\mathrm{d}x}{x^2(x+1)} = -\frac{1}{2x^2} + \frac{1}{x} + \ln|x| - \ln|x+1| + C$$

134

The following example uses complex numbers, which are dealt with in Chapter 9. If you are not familiar with these skip the example and go to

--------------------▷ 138

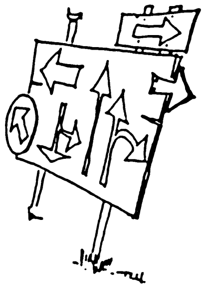
Otherwise

--------------------▷ 135

You should be able to solve the following problem by inspection if you have understood the relation between the differential and integral calculus. 43

If
$$F(x) = \int_0^x (3x^2 + 2)\,\mathrm{d}x$$

the derivative $F'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



I have found the solution --------------------▷ 45

I am not absolutely sure --------------------▷ 44

Evaluate $\int \frac{7x^2 - 10x + 37}{(x+1)(x^2 - 4x + 13)}\,\mathrm{d}x$ using partial fractions. 135

Proceed as before; first obtain the roots of the denominator.
The roots are
$x_1 = \ldots\ldots\ldots\ldots$, $x_2 = \ldots\ldots\ldots\ldots$, $x_3 = \ldots\ldots\ldots\ldots$

The partial fractions are

$$\frac{7x^2 - 10x + 37}{(x+1)(x^2 - 4x + 13)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 136

44

Here is a hint: According to the fundamental theorem of the differential and integral calculus, if the area function is

$$F(x) = \int_0^x f(x)\mathrm{d}x$$

then
$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(x)\,\mathrm{d}x = f(x)$$

We carry out successively two operations which cancel out.

Differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}(\;)$

Integration: $\int_0^x (\;)\,\mathrm{d}x$

Our problem was

Given: $F(x) = \int_0^x (3x^2+2)\,\mathrm{d}x$

Required: $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (3x^2+2)\,\mathrm{d}x$

$F'(x) =$ ....

--------------------▷ 45

136

$$x_1 = -1, \quad x_2 = 2+3\mathrm{j}, \quad x_3 = 2-3\mathrm{j}$$

We express the integrand by

$$\frac{7x^2-10x+37}{(x+1)(x^2-4x+13)} = \frac{A}{x+1} + \frac{Px+Q}{x^2-4x+13}$$

Now calculate the constants A, P and Q in the same way as before and then evaluate the integral.

--------------------▷ 137

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x (3x^2+2)\,\mathrm{d}x = 3x^2+2$$ 45

Differentiation and integration are called inverse operations, i.e. they neutralise each other when carried out in succession.

As another example of inverse mathematical operations consider the problem of squaring a number and then taking the square root:

$$\sqrt{a^2} = a$$

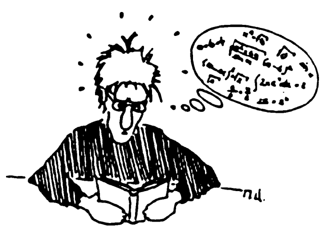
Similarly, $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(x)\mathrm{d}x = f(x)$

------------------▷ 46

$$A = 3, \quad P = 4, \quad Q = -2$$ 137

and using the table of integrals

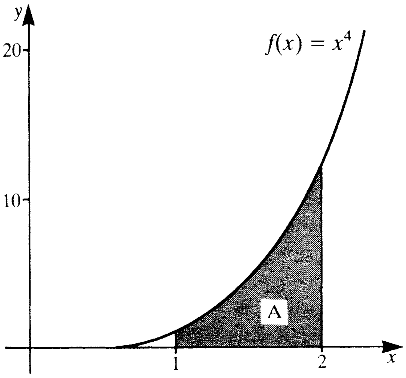
$$\int \frac{7x^2-10x+37}{(x+1)(x^2-4x+13)}\,\mathrm{d}x = 3\int\frac{\mathrm{d}x}{1+x} + \int\frac{4x-2}{x^2-4x+13}\,\mathrm{d}x$$
$$= 3\ln|1+x| + 2\ln\left|x^2-4x+13\right| + 2\tan^{-1}\left(\frac{x-2}{3}\right) + C$$



If your solution does not agree, follow the worked example in section 6.5.6 in the textbook. It is similar to the one above and you should then be able to obtain the correct solution.

------------------▷ 138

Given the function $f(x) = x^4$. Calculate the area A, shown shaded.
How would you proceed? 46



Split the interval into small equidistant points, i.e.

$$x_1 = 1,\ x_2,\ x_3,\ \ldots,\ x_n = 2$$

and obtaining the area A as a limiting value

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

--------------------▷ 47

Look for a primitive function $F(x)$ of $f(x)$; the area A is then given by

$$A = F(2) - F(1)$$

--------------------▷ 48

Integration by partial fractions follows a definite pattern. First make sure that the numerator is of lower degree than the denominator. 138

Step 1: Find the roots of the denominator of the integrand.
Step 2: Rewrite the original integrand as a sum of the partial fractions.
Step 3: Finally, calculate the values of the constants which appear with each partial fraction. Now you can evaluate the corresponding integrals, and hence the original integral.

Now it is time for a break.

--------------------▷ 139

You are quite right. It *is* possible to calculate the area that way, but this method is cumbersome. We use this method if the function cannot be integrated in a simple manner. It is much easier to look for the primitive of $f(x) = x^4$ and then calculate the area A. 47

------------------▷ 48

6.8 Rules for Solving Definite Integrals Substitution 139

Objective: Practice of different methods of evaluating definite integrals.

READ: **6.6 Rules for solving definite integrals**
6.7 Mean value theorem
Textbook pages 177–180

------------------▷ 140

Correct. 48

We can calculate the area $A = \int_1^2 f(x)\,\mathrm{d}x$ by finding the primitive function $F(x)$ of $f(x)$.

Then

$$A = \int_1^2 f(x)\,\mathrm{d}x = F(2) - F(1) = \Big[F(x)\Big]_1^2$$

Now complete the problem: calculate the area A:

Primitive function $F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Area $A = F(2) - F(1)$

$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

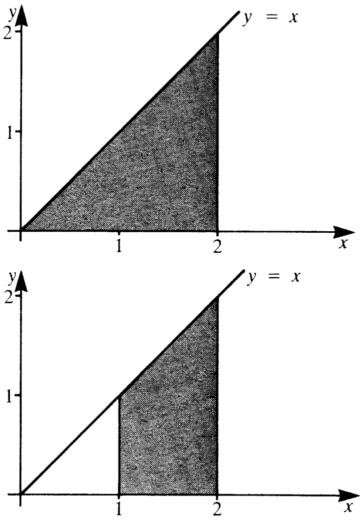
--------------------▷ 49

Evaluate $\int_0^{1.76} x^3\,\mathrm{d}x + \int_{1.76}^{2} x^3\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ 140

--------------------▷ 141

$$F(x) = \frac{1}{5}x^5 + C$$
$$A = F(2) - F(1) = \left(\frac{1}{5}2^5 + C\right) - \left(\frac{1}{5} + C\right) = 6.2$$

49



Calculate the area under the function $y = x$ in the two intervals (0, 2) and (1, 2).

$$\int_0^2 x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_1^2 x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 50

$$\int_0^{1.76} x^3\,\mathrm{d}x + \int_{1.76}^2 x^3\,\mathrm{d}x = \int_0^2 x^3\,\mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^2 = 4$$

141

The following generally holds true

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

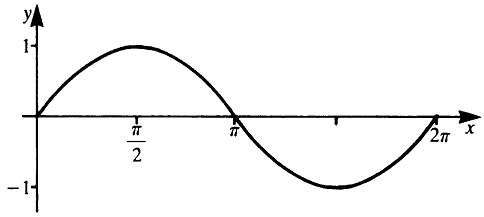
--------------------▷ 142

$$\int_0^2 x\,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_0^2 = 2 - 0 = 2$$

$$\int_1^2 x\,\mathrm{d}x = \left[\frac{1}{2}x^2\right]_1^2 = 2 - \frac{1}{2} = 1.5$$

50

Calculate the absolute value of the area bounded by the sine function for the intervals shown:



(1) $0 \leq x \leq \pi/2$ Area = ...

(2) $0 \leq x \leq \pi$ Area = ...

(3) $0 \leq x \leq 2\pi$ Area = ...

------------------▷ 51

$$\int_a^b f(x)\,\mathrm{d}x$$

142

$$\int_0^x \left(\frac{ax^3}{4} + bx^2 + c\right)\mathrm{d}x = \text{.....................................}$$

------------------▷ 143

(1) Area $= \int_0^{\pi/2} \sin x \, \mathrm{d}x = \left[-\cos x\right]_0^{\pi/2} = 0 - (-1) = 1$ 51

(2) Area $= \int_0^{\pi/2} \sin x \, \mathrm{d}x = \left[-\cos x\right]_0^{\pi} = 1 - (-1) = 2$

For the third problem we have to subdivide the interval into two sections, as the absolute value of the area is required; hence

(3) Area $= \left|\int_0^{\pi} \sin x \, \mathrm{d}x\right| + \left|\int_{\pi}^{2\pi} \sin x \, \mathrm{d}x\right| = |2| + |-2| = 4$

-------------------▷ 52

$$\frac{ax^4}{16} + \frac{bx^3}{3} + cx$$ 143

If $F(x) = \mathrm{e}^x$
then $F'(x) = f(x) = \mathrm{e}^x$
Evaluate

$$\int_0^1 \mathrm{e}^x \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

$\mathrm{e}^x + C$

-------------------▷ 144

$\mathrm{e} - 1$

-------------------▷ 145

A vehicle has a uniform acceleration $a = 2\,\mathrm{m/s^2}$ from rest. What is the magnitude of the velocity v (its speed) after 5 seconds? 52

$$v = \int_0^5 a\,\mathrm{d}t$$
$$v(5) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

What is the distance s covered in that time?

$$s = \int_0^5 v\,\mathrm{d}t$$
$$s(5) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 53

Wrong; you are not paying attention. 144

A definite integral $\int_a^b f(x)\,\mathrm{d}x$ with fixed limits a and b is equal to the difference $F(b) - F(a)$ if $F(x)$ is a primitive function of $f(x)$; i.e.

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

The given function was $f(x) = \mathrm{e}^x$; its primitive function is $F(x) = \mathrm{e}^x$. In order to calculate the value of the integral you have to insert the limits. Hence

$$\int_0^1 \mathrm{e}^x\,\mathrm{d}x = F(1) - F(0)$$

but $F(1) = \mathrm{e}^1 = \mathrm{e}$
and $F(0) = \mathrm{e}^0 = 1$
Therefore the value of the integral is ....

-------------------▷ 145

$$v(5) = \Big[at\Big]_0^5; \quad v(5) = 10\,\mathrm{m/s} = 36\,\mathrm{km/h}$$ 53

$$s(5) = \int_0^5 v\,\mathrm{d}t = \int_0^5 at \quad \mathrm{d}t = \left[\frac{at^2}{2}\right]_0^5; \quad s(5) = 25\,\mathrm{m}$$

Calculate absolute areas corresponding to the following integrals:

(1) $f(x) = 3\cos x$

(a) $\int_0^{\pi/2} f(x)\mathrm{d}x$ (b) $\int_{-\pi/2}^{\pi/2} f(x)\mathrm{d}x$ (c) $\int_0^{\pi} f(x)\mathrm{d}x$

(2) $f(x) = x - 2$

(a) $\int_{-2}^{0} f(x)\mathrm{d}x$ (b) $\int_0^{2} f(x)\mathrm{d}x$ (c) $\int_0^{4} f(x)\mathrm{d}x$

(d) $\int_2^{4} f(x)\mathrm{d}x$

Watch the limits!

--------------------▷ 54

---

e − 1 is correct 145

If $f(x) = x^2$, then the indefinite integral (general solution) is

$$\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + 2$$

This result is ☐ correct --------------------▷ 147

☐ wrong --------------------▷ 146

54

(1) (a) 3 (b) 6 (c) $6 = |3| + |-3|$
(2) (a) $|-6|$ (b) $|-2|$ (c) $4 = |-2| + |2|$ (d) 2

The areas below the x-axis would be assigned a negative value. In such cases we take the absolute value.

You will find more problems in the textbook.

--------------------▷ 55

146

You are right!

The indefinite integral defines a *family* of primitive functions which differ from each other by arbitrary constants. Hence we should write:

$$\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 + C$$

or generally

$$\int f(x) \, \mathrm{d}x = F(x) + C$$

Skip the next frame

--------------------▷ 148

Now it is time for a break. At the end of a working period do we simply close the book and start the break? 55

☐ Yes

☐ No

--------------------▷ 56

The arbitrary constant has been given the value 2. It is not the general solution. 147

$\frac{1}{3}x^3 + 2$ is only one possible primitive function of $f(x) = x^2$.

The indefinite integral defines a family of primitive functions; hence we must write

$$\int x^2 \, \mathrm{d}x = \frac{1}{3}x^3 + C$$

where C depends on a boundary condition.

Generally

$$\int f(x) \, \mathrm{d}x = F(x) + C$$

--------------------▷ 148

No! Of course not! 56

Before starting a break you should check that you know all the concepts and rules of that part of the work you have just completed. Use your own notes or extracts.

--------------------▷ 57

Evaluate the integral $\int_1^2 (3x-4)^2\,dx$ by means of the substitution $t = 3x - 4$. 148

In the textbook we used the letter u for the substitution but we can obviously use whatever letter we care to choose. A change in notation should not cause you any problems.

In terms of the new variable t the integral and its *limits* are given by:

$$\int_1^2 (3x-4)^2\,dx = \int_1^2 t^2 dt$$

--------------------▷ 149

$$= \int_1^2 \frac{1}{3} t^2\,dt$$

--------------------▷ 150

$$= \int_{-1}^2 \frac{1}{3} t^2\,dt$$

--------------------▷ 153

I want a detailed explanation of the solution

--------------------▷ 151

6.3 Methods of Integration

57

Objective: Concepts of indefinite integrals, principle of verification, standard integrals, primitive functions.
Operations: integrating.

READ: **6.5 Methods of integration**
6.5.1 Principle of verification
6.5.2 Standard integrals
6.5.3 Constant factor and the sum of functions
Textbook pages 161–163

--------------------▷ 58

149

You made a mistake, you forgot to substitute for $\mathrm{d}x$.
Since $t = 3x - 4$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 3, \quad \text{therefore} \quad \mathrm{d}x = \frac{1}{3}\,\mathrm{d}t$$

You also have to change the limits of integration. The new limits are calculated by substituting in the equation for t as a function of x.

$$\int_1^2 (3x-4)^2\,\mathrm{d}x = \int_1^2 \frac{1}{3}t^2\,\mathrm{d}t$$

--------------------▷ 150

$$= \int_{-1}^2 \frac{1}{3}t^2\,\mathrm{d}t$$

--------------------▷ 153

Detailed explanation

--------------------▷ 151

The primitive of the function $f(x)$ is called ....
The symbol for it is ....

58

--------------------▷ 59

The limits of integration are wrong.
The new limits are obtained as follows:
since $t = 3x - 4$, then
when $x_1 = 1$ we have $t_1 = 3 \times 1 - 4 = -1$
when $x_2 = 2$ we have $t_2 = 3 \times 2 - 4 = 2$
The integral becomes

$$\int_{-1}^{2} \frac{1}{3} t^2 \, \mathrm{d}t$$

150

I want to go on --------------------▷ 154

I want a detailed explanation of the method of substitution --------------------▷ 151

an indefinite integral 59

$$\int f(x)\,\mathrm{d}x$$

There are a number of standard integrals that you should know by heart.
Can you complete the following list?

Function	*Standard integral*
x^n	....
$\sin x$	....
e^x	....
$\frac{1}{x}$	....

------------------▷ 60

To help you understand the method we will consider a different example and go through the solution step by step. As you follow the steps complete the right-hand side, which was the original problem. 151

Step	*Example*	*Exercise*
	$\int_0^2 (2x+3)\,\mathrm{d}x$	$\int_1^2 (3x-4)^2\,\mathrm{d}x$
(1) Select a substitution	$t = 2x+3$	$t = 3x-4$
(2) Differentiation with respect to the original variable	$\frac{\mathrm{d}t}{\mathrm{d}x} = 2$	$\frac{\mathrm{d}t}{\mathrm{d}x} =$ ....
(3) Find the value of dx	$\mathrm{d}x = \frac{1}{2}\,\mathrm{d}t$	$\mathrm{d}x =$ ....
(4) Calculate thenew limits	$t_1 = 2\times 0+3 = 3$ $t_2 = 2\times 2+3 = 7$	$t_1 =$ .... $t_2 =$ ....
(5) The new integral is:	$\int_3^7 \frac{1}{2}t\,\mathrm{d}t$	$\int$ ....dt = ...

------------------▷ 152

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int e^x\,dx = e^x + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C, \quad x \neq 0$$

60

Evaluate the following integrals; the notation has been changed:

$$\int t^2\,dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\int \cos\phi\,d\phi = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\int e^u\,du = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 61

We were required to evaluate $\int_1^2 (3x-4)^2 dx$

152

(1) Substitution: $t = 3x - 4$

(2) Differentiation: $\frac{dt}{dx} = 3$

(3) Differential: $dx = \frac{1}{3}\,dt$

(4) New limits: $t_1 = 3 \times 1 - 4 = -1$
$t_2 = 3 \times 2 - 4 = 2$

(5) New integral: $\frac{1}{3}\int_{-1}^{2} t^2\,dt = 1$

--------------------▷ 154

$$\int t^2\,\mathrm{d}t = \tfrac{1}{3}t^3 + C$$
$$\int \cos\phi\,\mathrm{d}\phi = \sin\phi + C$$
$$\int \mathrm{e}^u\,\mathrm{d}u = \mathrm{e}^u + C$$

61

Many integrals can be solved with little effort using tables of standard integrals. You will find such a table at the end of Chapter 6 of the textbook, and this should be sufficient in many cases. It is important that you learn how to use such tables.

Using the table, evaluate:

$$\int \frac{1}{(x-a)^2}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 62

Correct!

153

The value of the integral is:

$$\int_1^2 (3x-4)^2\mathrm{d}x = \frac{1}{3}\int_{-1}^{2} t^2\,\mathrm{d}t = \left[\frac{1}{9}t^3\right]_{-1}^{2} = \frac{8}{9} + \frac{1}{9} = 1$$

Note that we could have evaluated the same integral without a substitution:

$$\begin{aligned}\int_1^2 (3x-4)^2\mathrm{d}x &= \int_1^2 (9x^2-24x+16)\mathrm{d}x \\ &= \left[3x^3-12x^2+16x\right]_1^2 \\ &= (24-48+32)-(3-12+16) = 8-7 = 1\end{aligned}$$

--------------------▷ 154

$$\int \frac{1}{(x-a)^2}\,\mathrm{d}x = -\frac{1}{x-a} + C$$ 62

You may have solved the last integral by yourself without using the table.
But here is a hard one, for which you will probably need the table:

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 63

154

Evaluate the following definite integrals:

(1) $\int_0^{1/2} \sin \pi x \,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots$

(2) $\int_1^2 av^2\,\mathrm{d}v = \ldots\ldots\ldots\ldots\ldots$

(3) $\int_0^3 \mathrm{e}^{-\gamma t}\,\mathrm{d}t = \ldots\ldots\ldots\ldots\ldots$

--------------------▷ 155

$$\int \frac{1}{1+\sin x} dx = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right) + C$$

63

In case you have had difficulties here are some hints:

(1) In the table of standard integrals the constant has been omitted.
(2) The integrand is on the left-hand side and the integral on the right-hand side.
(3) When using the table look for the function first so that you get used to the list.
On the top left in the table at the end of chapter 6 you find:

$f(x)$	$\int f(x) dx = cx$
c	cx

It means:
$\int c\, dx = cx$

--------------------▷ 64

$$(1)\ \frac{1}{\pi}, \quad (2)\ \frac{7a}{3}, \quad (3)\ \frac{1}{\gamma}(1 - e^{-3\gamma})$$

155

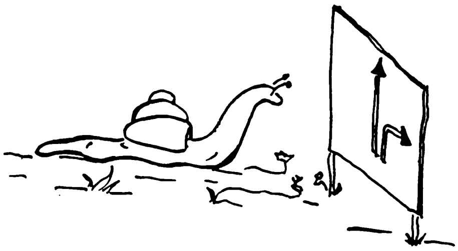
I want to carry on --------------------▷ 158

The notations are troubling me --------------------▷ 156

When using the table you should be selective. 64

Selective reading is a special form of 'quick reading'.

Quick reading is useful when looking for a particular piece of information or to obtain a general overview of the text.



Hints on 'quick reading' wanted

------------------▷ 65

I wish to skip the hints

------------------▷ 66

As we have said before, physical quantities are denoted by certain convenient letters, e.g. time t, mass m, force F, velocity v, frequency ω, modulus of elasticity E, pressure p, density ρ, volume V, etc … The list is as long as you like. 156

If you are not familiar with the notation of a variable it is advisable to replace it by a familiar one.

Replace the variable of integration by x and evaluate the integrals:

$$\int \frac{1}{2}\cos\phi \, d\phi = \ldots\ldots\ldots\ldots$$

and

$$\int (3\sin\alpha + \cos\alpha)\, d\alpha = \ldots\ldots\ldots\ldots$$

------------------▷ 157

The normal reading speed is about 130–150 words per minute. The eyes pass over the lines in jerks and during a normal line of 80–90 letters stop 4–6 times. During a stop a small number of words is perceived simultaneously. With practice it is possible to reduce the number of stops and to increase the number of words perceived simultaneously. Thus closely spaced lines are easier to read. 65

With training it is possible to increase the reading speed to 250–400 words per minute. ‘Quick reading’ and ‘intensive reading’ are opposite techniques; mathematical knowledge is acquired by the latter. Knowing different reading techniques enables you to increase your own speed by choosing the correct reading technique to meet your objective.

------------------▷ 66

157

$$\int \frac{1}{2}\cos\phi\,\mathrm{d}\phi \to \frac{1}{2}\int \cos x\,\mathrm{d}x = \frac{1}{2}\sin x + C \to \frac{1}{2}\sin\phi + C$$

$$\int (3\sin\alpha + \cos\alpha)\,\mathrm{d}\alpha \to \int (3\sin x + \cos x)\,\mathrm{d}x = -3\cos x + \sin x + C$$
$$\longrightarrow -3\cos\alpha + \sin\alpha + C$$

The procedure should be clear. It often helps to facilitate the integration.

Step 1: Replace the unfamiliar variable by a familiar one, e.g. x.
Step 2: Integrate, use the table of integrals.
Step 3: Replace the familiar variable by the original one.

------------------▷ 158

Evaluate the following integrals using the table of integrals. 66

$$\int \frac{1}{x-a}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \frac{1}{\cos^2 - x}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \frac{a}{x^2+a^2}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \sin^2\phi\,\mathrm{d}\phi = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int a^t\,\mathrm{d}t = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 67

You will find more exercises in the textbook. Remember that doing a lot of examples immediately following the learning of a particular aspect of the work is not too helpful. It is more beneficial if you do some the following day, a week later or a month later, or even a year later! 158

-------------------▷ 159

$\ln|x-a|+C$ 67

$\tan x+C$

$\tan^{-1}\left(\frac{x}{a}\right)+C$

$\frac{1}{2}(\phi-\sin\phi\cos\phi)+C$

$\frac{a^t}{\ln a}+C$

Further problems for you to solve

(1) $\int t^3\,\mathrm{d}t \quad = \ldots\ldots\ldots\ldots\ldots$

(2) $\int \frac{\mathrm{d}z}{\cos^2 z} = \ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\int u\,\mathrm{d}u \quad = \ldots\ldots\ldots\ldots\ldots\ldots$

I have found the solution

------------------▷ 69

I am still having difficulty with the different notations

------------------▷ 68

6.9 Improper Integrals

159

Objective: Concept of improper integrals.

Task: Evaluation of convergent improper integrals

The concept of the definite integral is extended to include cases in which the limits of integration are infinite. One such integral with an infinite limit frequently encountered is:

$$\int_{x_0}^{\infty}\frac{\mathrm{d}x}{x^2}$$

Example: The work done while projecting a body outside the gravitational field of the Earth.

READ: 6.8 Improper integrals
Textbook pages 181–183

------------------▷ 160

In physics and engineering one uses notations which are best suited to the particular problem. Therefore, there are many different notations: t, z, u, ... You can overcome this difficulty if you substitute the following scheme for the notation you are familiar with: 68

(i) substitution: replace t, z, u, ... by x.
(ii) execute the mathematical operation.
(iii) replace x by t, z, u .....

Now evaluate

(1) $\int t^3 \, dt = \ldots\ldots\ldots\ldots$

(2) $\int \frac{dz}{\cos^2 z} = \ldots\ldots\ldots\ldots$

(3) $\int u \, du = \ldots\ldots\ldots\ldots$

-------------------▷ 69

An integral in which at least one limit of integration tends to infinity is called an improper integral. 160

Can such an integral have a finite value?

☐ Yes -------------------▷ 162

☐ No -------------------▷ 161

69

(1) $\frac{t^4}{4} + C$

(2) $\tan z + C$

(3) $\frac{u^2}{2} + C$

In the table we find

$$\int \tan x \, \mathrm{d}x = -\ln|\cos x|$$

Verify the relationship!

Complete:

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln\cos x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 70

You are wrong. An improper integral may have a finite value.

161

Perhaps you are getting tired. There was a lot of material to cover; a break might be a good thing.

The last section dealt with improper integrals and it was shown that such integrals can have finite values even if one of the limits tends to infinity. This does not hold true for all such integrals. An improper integral with a finite value is said to be convergent.

-------------------▷ 162

$$\frac{d}{dx}(-\ln\cos x) = \frac{-1}{\cos x}(-\sin x) = \frac{\sin x}{\cos x} = \tan x$$ 70

In this way we can verify all integrals in the table.

--------------------▷ 71

You are right. Improper integrals may have finite values! 162

The integral $\int_a^\infty \frac{dx}{x^2}$ is of special interest to physicists.

What is its value?

$$\int_a^\infty \frac{dx}{x^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 163

6.4 Integration by Parts

71

Some of the standard integrals shown in the table at the end of Chapter 6 are obtained by using the method of integration by parts.

There is one efficient way of achieving understanding that is quite simple. Work through the examples in the textbook. And after that try to do the *same* examples without the help of the textbook. If you did not know this technique try it with some examples.

READ: 6.5.4 Integration by parts: product of two functions
Textbook page 163–166

→ 72

$$\int_a^\infty \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_a^\infty = -\frac{1}{\infty} + \frac{1}{a} = \frac{1}{a}$$

163

The improper integral $\int_a^\infty \frac{dx}{x^2}$ occurs frequently in physics, for instance in the calculation of the energy required to:

(i) take a body ‘outside’ the gravitational field of the Earth (i.e. to infinity),
(ii) remove an electron from an atom.

The name improper integral should not disturb you! It is only a special case of the definite integral and simply means that one of the limits tends to infinity.

→ 164

Write down the formula for integration by parts 72

$$\int uv' \, dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 73

What is the value of the improper integral 164

$$\int_a^\infty \frac{1}{x} \, dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (a > 0)?$$

I have found the solution --------------------▷ 167

I want further explanation --------------------▷ 165

$$\int uv' \, dx = uv - \int vu' \, dx$$ 73

This is the basic formula. To execute it we have to interpret the original integral skillfully. In the textbook the first example we looked at was:

$$\int x e^x \, dx$$

In this case we set $x = u$ and $e^x = v'$. The reason should be clear; we produce an integral we can solve on the right-hand side.

Evaluate the integral for yourself without looking at the textbook.

$$\int x e^x \, dx = \text{...}$$

-------------------▷ 74

Further explanation: 165

Step 1: We start with a definite integral and evaluate it:

$$\int_a^b \frac{1}{x} \, dx = \Big[\ln x\Big]_a^b, \quad a \text{ and } b \text{ positive}$$

Step 2: We insert the limits and carry out the limiting process $b \longrightarrow \infty$:

$$\int_a^\infty \frac{1}{x} \, dx = \lim_{b \to \infty} \int_a^b \frac{1}{x} \, dx = \lim_{b \to \infty} (\ln b - \ln a)$$

Does the term including b converge towards a finite value?

☐ Yes -------------------▷ 166

☐ No -------------------▷ 167

$$\int x e^x \, dx = x e^x - e^x + C = e^x (x - 1) + C$$ 74

I want to carry on

--------------------▷ 77

I am not absolutely sure of the underlying rule

--------------------▷ 75

No! It does not, for the simple reason that as $b \longrightarrow \infty$ the value of In b grows beyond all bounds, i.e. In $b \longrightarrow \infty$. This improper integral is *not* convergent. 166

Hence $\int_a^\infty \frac{dx}{x} = \Big[\ln x\Big]_a^\infty =$ ....

--------------------▷ 167

Additional explanation of the method of integration by parts: 75

Integration by parts is a skillful exploitation of the rule for differentiating a product.

Example:

$$f(x) = x \sin x$$

Let

$$u = x$$
$$v = \sin x$$

The function can be written as

$$f(x) = uv$$

The product rule for differentiation is

$$\frac{d}{dx} f(x) = \frac{d}{dx}(uv) = \text{.....................................}$$

--------------------▷ 76

$$\int_a^\infty \frac{dx}{x} = \infty$$ 167

All clear --------------------▷ 168

Detailed solution --------------------▷ 165

$$\frac{d}{dx}(uv) = u'v + uv'$$

76

Now we reverse the process and integrate both sides giving

$$\int \frac{d}{dx}(uv)\,dx = \int u'v\,dx + \int uv'\,dx$$

Differentiation followed by integration neutralise each other. Thus we get:

$$uv = \int u'v\,dx + \int uv'\,dx$$

So far we have achieved nothing, but the real trick is to transform the equation so that we have an integral on the left-hand side, i.e. the integral we wish to evaluate, and an integral which we can evaluate on the right-hand side.

The transformation yields

$$\int uv'\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 77

168

If you need a little more practice try the following problems. They are similar to those we have discussed already.

(1) $\displaystyle\int_4^\infty \frac{ds}{s^2} = \ldots\ldots\ldots\ldots$

(2) $\displaystyle\int_{10}^\infty \frac{dx}{x} = \ldots\ldots\ldots\ldots$

(3) $\displaystyle\int_1^\infty \frac{dx}{(2+x)^2} = \ldots\ldots\ldots\ldots$

(4) $\displaystyle\int_1^\infty \frac{d\gamma}{\gamma^4} = \ldots\ldots\ldots\ldots$

--------------------▷ 169

Formula for integration by parts: 77

$$\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x$$

It is useful to know this formula by heart.

Evaluate the following using the technique of integration by parts:

$$\int \ln x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hints wanted --------------------▷ 78

Solution found --------------------▷ 79

---

(1) $\frac{1}{4}$ (2) ∞ (3) $\frac{1}{3}$ (4) $\frac{1}{3}$ 169

There are more problems in the textbook!

You should now know whether you can solve the problems easily, in which case you need not do any more, or whether you do not find the solutions easily, in which case more exercises are needed.

--------------------▷ 170

The formula is $\int uv'\,dx = uv - \int vu'\,dx$ 78

Hint: Set $u = \ln x$

$v' = 1$ and hence $v = x$

Now evaluate the integral

$$\int \ln x\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 79

In previous chapters we have discussed three study techniques; what are they? 170

Write them down, using keywords:

(1) ....

(2) ....

(3) ....

--------------------▷ 171

$$\int \ln x \, dx = x \ln x - x + C$$ 79

If your answer is correct --------------------▷ 81

Further explanation required --------------------▷ 80

---

(1) Balance between periods of study and breaks, keeping to an established timetable. 171
(2) Intensive reading:
Taking notes of new concepts, definitions and rules; inthe case of mathematical derivations, doing them yourself.
(3) Selective reading:
Searching quickly for new information; glancing over large sections of text looking for particular material.

Presumably you wrote these down in your own words. You should know these techniques by now. Remember that it is not sufficient to know them; the problem is to apply them!

--------------------▷ 172

$$\int uv' \, dx = uv - \int vu' \, dx$$ 80

with
$$u = \ln x, \; u' = \frac{1}{x}$$
$$v' = 1, \quad v = x$$

Hence we have

$$\int \ln x \, 1 \, dx = (\ln x)x - x\left(\frac{1}{x}\right) dx = x \ln x - \int dx$$
$$= x \ln x - x + C$$

--------------------▷ 81

6.10 Line Integrals

172

This section may be of more interest to the physicist than to the engineer.

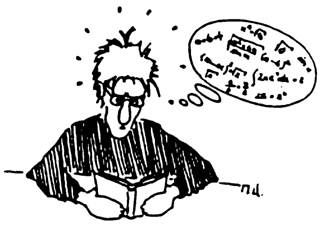
The basic concept of the line integral is quite easy to understand, even if the notation of some expressions may seem cumbersome.

READ: 6.9 Line integrals
Textbook pages 183–186

--------------------▷ 173

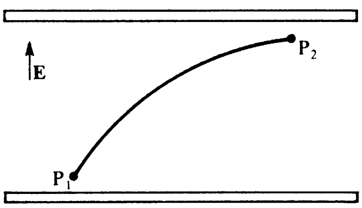
Using integration by parts, evaluate 81

$$\int \sin^4 x \, dx = \text{..........................}$$



In case of difficulties consult the worked example in the textbook.

------------------▷ 82



The figure shows a capacitor. An electrical charge q is moved from point 173

$\mathbf{P}_1 = (x_1, \; y_1)$ to point

$\mathbf{P}_2 = (x_2, \; y_2)$. The force acting on the charge is

$$\mathbf{F} = \mathbf{E}q$$

We want to evaluate the work done.

Write the general expression for the line integral

$$W = \text{...................}$$

------------------▷ 174

$$\int \sin^4 x \, dx = -\tfrac{1}{4} \cos x \sin^3 x - \tfrac{3}{8} \sin x \cos x + \tfrac{3}{8} x + C$$

82

Correct solution

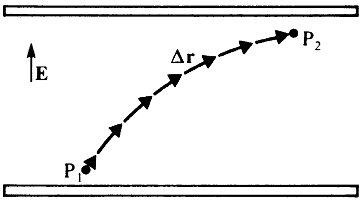
--------------------▷ 85

Further help required

--------------------▷ 83

$$W = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_1} \mathbf{E}q \cdot d\mathbf{r}$$

174



We can approximate the line integral by a sum. The work done could be calculated for each path element. Analytically the line integral can be solved if we succeed in transforming it into known and solvable integrals.

--------------------▷ 175

$\int \sin^4 x =$ .... Set $u = \sin^3 x \qquad u' = 3\sin^2 x \cos x$

$v' = \sin x \qquad v = -\cos x$

83

Substituting, we find $\int \sin^4 x\,\mathrm{d}x = -\cos x \sin^3 x + 3\int \sin^2 x \cos^2 x\,\mathrm{d}x$

Since $\sin^2 x + \cos^2 x = 1$, then $\cos^2 x = 1 - \sin^2 x$.

Substituting in the right-hand integral gives

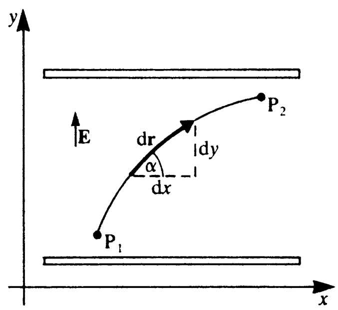
$$\int \sin^4 x\,\mathrm{d}x = -\cos x \sin^3 x + 3\int \sin^2 x\,\mathrm{d}x - 3\int \sin^4 x\,\mathrm{d}x$$

Rearranging yields $\int \sin^4 x\,\mathrm{d}x = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\,\mathrm{d}x$

We have already evaluated $\int \sin^2 x\,\mathrm{d}x$ in the textbook: $\int \sin^2 x\,\mathrm{d}x = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$

Hence $\int \sin^4 x\,\mathrm{d}x = -\frac{1}{4}\cos x \sin^3 x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$

--------------------▷ 84



To calculate:

175

$$W = q\int_{P_1}^{P_2} \mathbf{E}\cdot \mathrm{d}\mathbf{r}$$

We assume a homogeneous electrical field in the y-direction:

$$\mathbf{E} = (0, \quad E).$$

The path element is $\mathrm{d}\mathbf{r} = (\mathrm{d}x, \quad \mathrm{d}y)$

Thus

$$W = q\int \mathbf{E}\cdot \mathrm{d}\mathbf{r} = q\int_{P_1}^{P_2} \ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 176

84

Some comments on the solution:

The integral to evaluate is reduced from $\int \sin^4 x \, dx$ to $\int \sin^2 x \, dx$. The same technique would enable us to evaluate $\int \sin^8 x \, dx$, for instance, reducing it to $\int \sin^6 x \, dx$, and by repeated application down to $\int \sin^2 x \, dx$. In the textbook the corresponding reduction formula is stated.

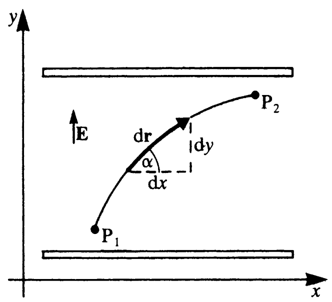
Let us recapitulate:

Integration by parts is based on the formula

$$\int uv' \, dx = uv - \int vu' \, dx$$

To apply it successfully we have to factorise the integral into u and v' in such a way that the right-hand integral can be solved (either directly or by further reduction).

------------------▷ 85



176

$$W = q \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{r} = q \int_{y_1}^{y_2} E \, dy$$

Only the movement in the y-direction contributes to the work W.

Inserting the limits yields

$$\int_{y_1}^{y_2} E dy = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 177

85

Now you should take a break.

A period of 20 to 60 minutes of concentrated study is about right if you are to make good progress. The length of the appropriate study period varies with different people. But remember that short breaks are beneficial.

The optimum time in your case is for you to decide; you are the best judge in the end. It is important that you should organise your work, and the number and duration of breaks.

What do you think the duration of this break should be?

☐ 5 minutes

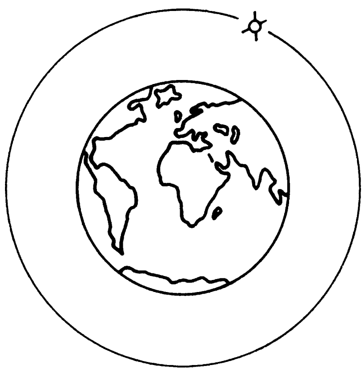
☐ 15 minutes

☐ 30 minutes

--------------------▷ 86

177

$$\int_{y_1}^{y_2} E\,\mathrm{d}y = E(y_2 - y_1)$$



A satellite moves in a circular orbit. The work done during one rotation around the Earth is given by the line integral

$$W = \int_{\text{circle}} \mathbf{F} \cdot \mathbf{r}\,\mathrm{d}\mathbf{r}$$

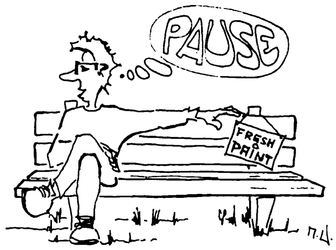
For the direction of $\mathbf{F}$ and d$\mathbf{r}$ you can give the value of this line integral:

$$W = \int_{\text{circle}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 178

Before carrying on with your studies, check the time. How does it compare with the time you fixed before starting the break? There may be a difference between the two times; it is not serious if it is small. 86

As a rule you should keep to the time you decided on, unless there is a good reason why the break was longer. Under normal circumstances you must not allow the duration of the breaks to increase.



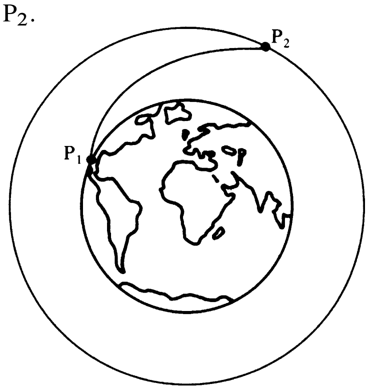
--------------------▷ 87

178

$$W = \int \mathbf{F} \cdot d\mathbf{r} = 0$$

Explanation: since **F** and d**r** are perpendicular the scalar product vanishes.

The figure shows the Earth and the path of another satellite. The satellite starts at P_1. It reaches its orbit at P_2.



Can you give the work to be done against the gravitational forces?

r_1 = radius of the Earth

r_2 = radius of the orbit

$$\mathbf{F} = \gamma \frac{mM}{r^2} \frac{\mathbf{r}}{r}.$$

Remember that $\frac{\mathbf{r}}{r}$ is a unit vector with a radial direction.

$W = \ldots\ldots\ldots\ldots\ldots\ldots$ --------------------▷ 183

Explanation --------------------▷ 179

6.5 Integration by Substitution

87

Objective: Evaluation of integrals by means of a substitution.

The basic idea is developed in the beginning of the section. Substitution is quite a simple technique. Generally speaking, integration is harder in many cases than differentiation.

READ: 6.5.5 Integration by substitution
6.5.6 Substitution in particular cases
Textbook pages 166–172

-------------------▷ 88

We ask for the work to be done against the gravitational forces. We do not consider the work done to accelerate the satellite. 179
The work is given by the line integral

$$W = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$$

Now remember that the gravitational field is a conservative field. Thus for given points P_1 and P_2 the work

☐ depends on the geometric form of the path

☐ does not depend on the geometric form of the path

-------------------▷ 180

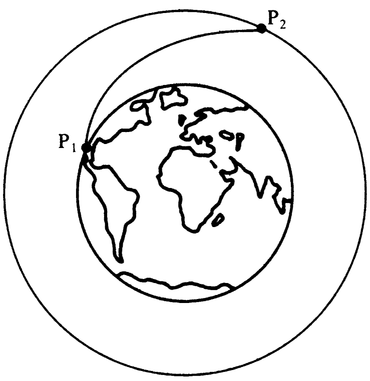
With the help of a substitution evaluate

88

$$\int \sin(ax+b)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$$

I have done it! --------------------▷ 90

I could do with some help! --------------------▷ 89



The work does not depend on the geometric form of the path. It depends only on the coordinates of P_1 and P_2.

180

In this case you may find a path for which the line integral is easy to evaluate.

Sketch it on the drawing.

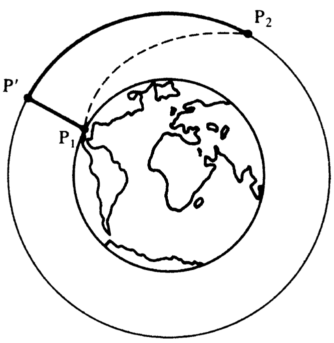
--------------------▷ 180

Go back to the textbook, and go through the examples once more. If you still have difficulties with the concept of composition of functions read section 2.3 of Chapter 2 of the textbook again. 89

Having done this evaluate

$$\int \sin(ax+b)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 90



The path is composed of two elements. First, a path in radial direction from P_1 to P'; second, an arc of a circle from P' to P_2. 181

The work done can be calculated for both parts. Remember that

$$\mathbf{F} = \gamma \frac{mM}{r^2} \frac{\mathbf{r}}{r}$$

$$W_1 = \int_{P_1}^{P'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$W_2 = \int_{P'}^{P_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 182

$$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+C$$ 90

Integration by substitution requires the following steps:

Example: To solve $\int \sin(4\pi x)\,\mathrm{d}x$

Step 1: Choose a substitution $u = 4\pi x$

Step 2: Substitute

(a) in the function $\sin(4\pi x) = \sin \mathrm{d}u$

(b) for the differential $\frac{\mathrm{d}u}{\mathrm{d}x} = 4\pi \ \ \mathrm{d}x = \frac{1}{4\pi}\mathrm{d}u$

Step 3: Integrate with respect to the new variable $\frac{1}{4\pi}\int \sin u\,\mathrm{d}u = -\frac{1}{4\pi}\cos u + C$

Step 4: Substitute the original variable in the solution $\int \sin(4\pi x)\,\mathrm{d}x = -\frac{1}{4\pi}\cos(4\pi x)+C$

--------------------▷ 91

$$W_1 = \gamma m M \left(\frac{1}{r_1} - \frac{1}{r'}\right)$$ 182

$W_2 = 0$ since d**r** is perpendicular to **F**.

--------------------▷ 183

The aim of the method of integration by substitution is to find a suitable substitution to reduce a complicated integral to a standard one. It is only with practice that you will be able to find suitable substitutions. 91

In the textbook we have grouped together certain types of integrals which can be evaluated by substitution and considered as standard types. What are they?

(a) ....
(b) ....
(c) ....
(d) ....

--------------------▷ 92

$$U = \gamma m M \left(\frac{1}{r_1} - \frac{1}{r'} \right)$$ 183

If you want a detailed explanation --------------------▷ 179

Now you deserve a break! But remember:
It is important to recapitulate before you have a break.

(a) $\int f(ax+b)\,\mathrm{d}x$ (b) $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$

(c) $\int f(g(x))g'(x)\,\mathrm{d}x$ (d) $\int R(\sin x, \cos x, \tan x)\,\mathrm{d}x$

92

Consider integrals of the type: $\int f(ax+b)\,\mathrm{d}x$.

Solve: $\int (2-3x)^7\mathrm{d}x =$ ....

What substitution would you choose to evaluate the integral?

Substitute $u =$ ....

I have found u --------------------▷ 94

I am not sure --------------------▷ 93

Please continue on page 1
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