Contents

Contents

Contents

Chapter 1
Vector Algebra I: Scalars and Vectors

Dear reader. This study guide will help you to study efficiently with the textbook. Your study periods will alternate between the textbook and this study guide. All you need will be explained in due time. Now start and turn overleaf to page

--------------------▷ 2

1.1 Scalars and Vectors

2

Your first task is to study in the textbook Sect. 1.1, Scalars and vectors. When you have completed this simple and limited task carry on with this programmed study guide.

The introduction to each section in this study guide will always have the same structure. The objectives are named in catchwords, so you know in advance what you will learn.

Thus, for this section:

Objective: Concepts of scalars, vectors, directed segments, line of action, free vector.

READ: 1.1 Scalars and vectors
Textbook pages 1–4

Afterwards, move on to the next frame in this study guide, i.e. go to frame 3

--------------------▷ 3

3

After you have studied a particular section in the textbook you are asked to answer questions in the programmed study guide. Thus you test your progress. Even if you have understood the text you may not always retain all of it.

You should know and remember the following concepts:

Definition of a scalar quantity:
Scalars are defined by their ....

Definition of a vector quantity:
Vectors are defined by their .... and ....

Write down your answers in a separate sheet. The dotted lines always indicate that answers are required. You will find the answers on the top of the next frame.

Go to frame 4.

--------------------▷ 4

At this place you will always find our answers for you to check your own.

4

Scalars are defined by their *magnitude*.
Vectors are defined by their *magnitude* and *direction*.

A vector can be represented geometrically by a ....
A vector from the origin of a coordinate system to a point P is called ....
A vector of magnitude 1 is called ....

When learning, be strong-minded; look up our answers only after you have completed yours!
You certainly know by now that the arrow points to the number of the frame to come.

--------------------▷ 5

directed line
position vector
unit vector

5

Of the following, which are scalars and which are vectors?

mass	....
temperature	....
field strength	....
force	....
gravitational field strength	....
density	....
pressure	....
time	....
displacement	....
velocity	....
acceleration	....

6

6

mass	scalar
temperature	scalar
field strength	vector
force	vector
gravitational field strength	vector
density	scalar
pressure	scalar
time	scalar

Displacement, velocity and acceleration are all vectors.

Now decide how you will proceed:

If everything you have done so far was correct then go to 8

If you made a mistake and thought that pressure was a vector then go to 7

Pressure is a scalar, not a vector, because it has no preferred direction; it acts in all directions. 7

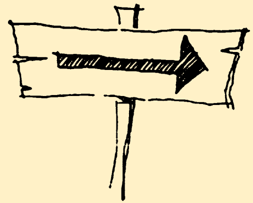
The scalar quantity *pressure* and the vector quantity *force* are connected in this case by a physical relationship.

Consider a gas inside a cylinder; each point in the cylinder is under the same pressure. It has no direction. But the pressure exerts a force on the wall of the cylinder. The direction of this force is not determined by the pressure but by the direction of the wall, and it always acts in a direction perpendicular to the wall.

8

Vector quantities are conveniently represented by means of arrows. With the help of an arrow we can represent .... and .... of the physical quantity.

8



9

magnitude and *direction*

9

As an exercise in notation, which symbols are used to represent vectors?

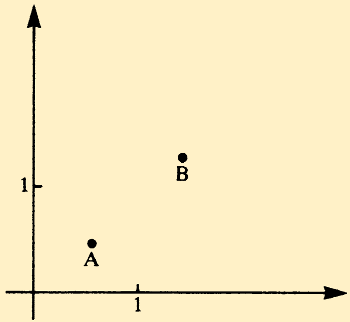
☐	$\mathbf{b}$	☐	$\overrightarrow{PQ}$
☐	$\left\lvert\overrightarrow{PQ}\right\rvert$	☐	PQ

Tick the appropriate boxes!

10

$\mathbf{b}, \overrightarrow{PQ}$

10



A car travels from A to B. Can this change in position be represented by a vector?

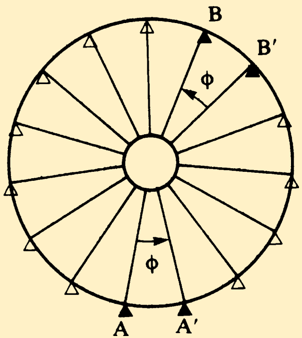
☐ Yes

☐ No

11

Yes

11



The diagram shows a big wheel often found in fairgrounds. Neil is sitting in gondola A and Mary in gondola B. The wheel rotates through an angle ϕ. Insert the vectors $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$.

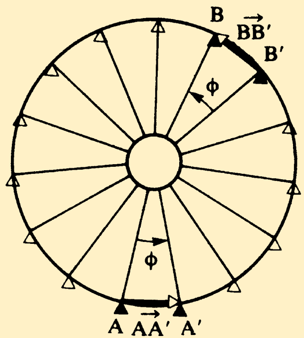
Have both vectors the same direction?

☐ Yes

☐ No

12

No



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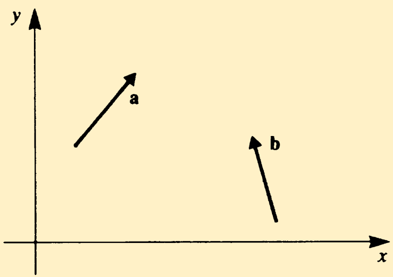
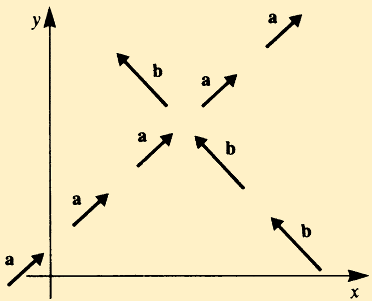


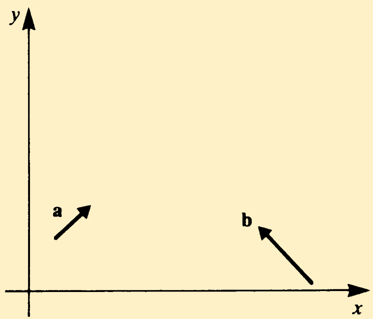
Given two vectors **a** and **b**. Shift **a** and **b** along their respective directions and plot equivalent vectors **a** and **b**.

13



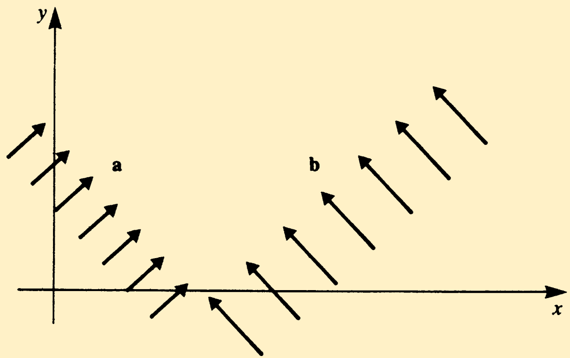
The line which defines the direction of the vectors is called the *line of action*.

13



Plot equivalent vectors which are shifted parallel to **a** and **b**.

14



14

Free vectors are considered equal if they have the same magnitude and direction.

Vectors can be shifted:

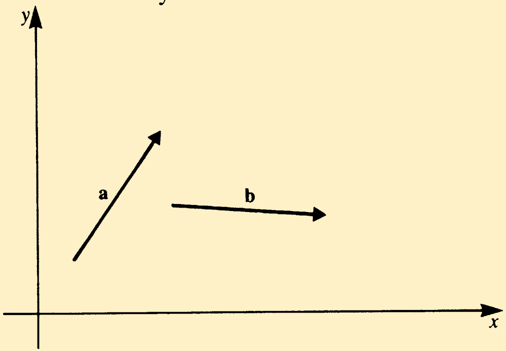
(a) along their ....
(b) .... to themselves.

15

(a) direction or line of action
(b) parallel

15

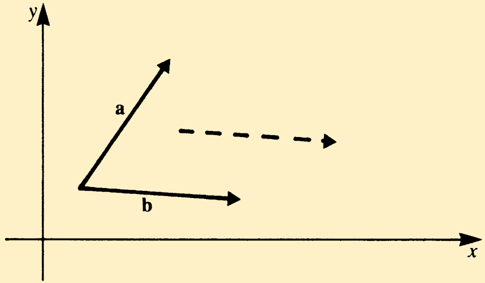
The reason why we wish to shift vectors is that addition and subtraction then become more convenient.

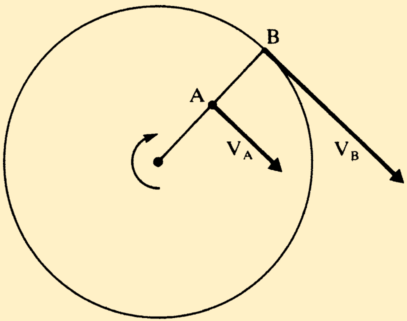


In the diagram, shift vector **b** so that its starting point coincides with that of **a**.

16

16





Vectors can be used to indicate the instantaneous velocity of points on a rotating disk.

In this diagram the scale is

$$5\,\text{mm} \mathrel{\widehat{=}} 1\,\text{m/s}.$$

Evaluate from the drawing the magnitude of the velocity of points A and B.

Velocity of A = ...
Velocity of B = ...

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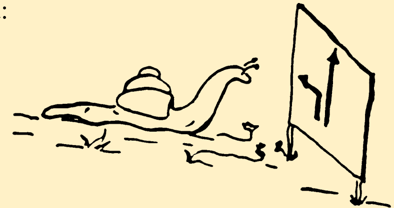
$$V_A = 2.5\,\mathrm{m/s}, \quad V_B = 5\,\mathrm{m/s}$$

17

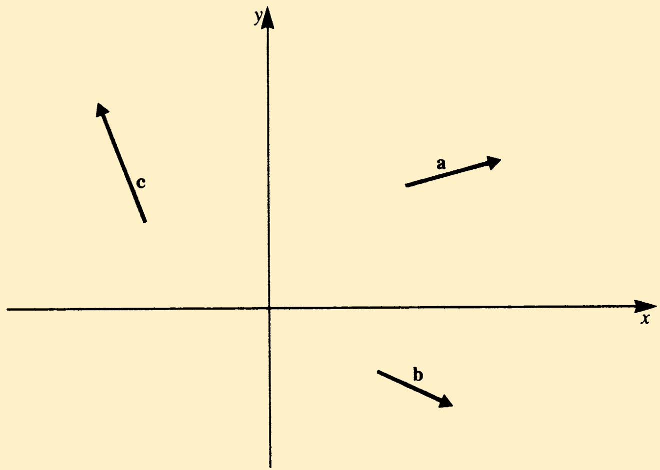
Now decide for yourself how to proceed:



No difficulties; go to 19

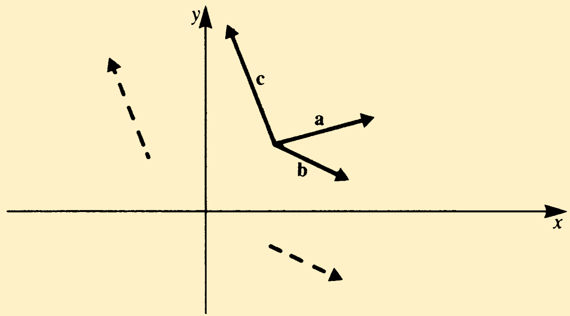
Further example required 18

18

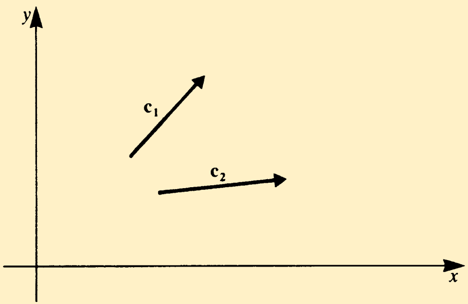


Shift **c** and **b** so that all three vectors begin at the starting point of **a**.

19



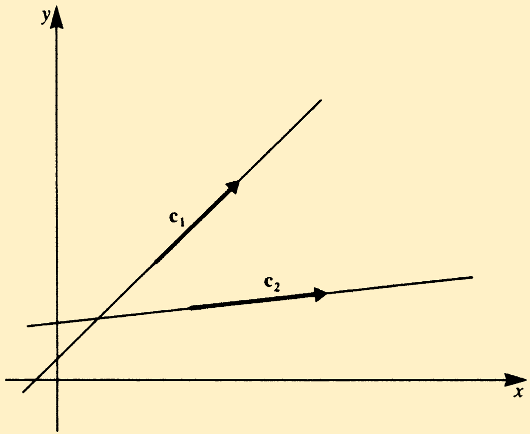
19



Draw in the lines of action of the vectors $\mathbf{c}_1$ and $\mathbf{c}_2$.

20

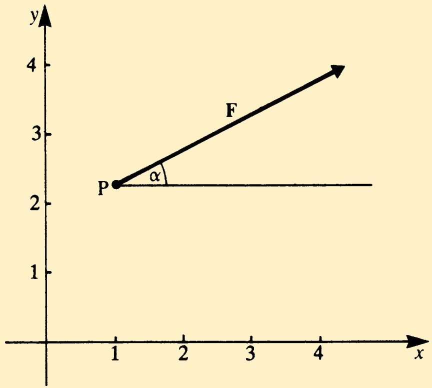
20



21

A force is applied at **P**; if in the drawing 1 unit = 1 newton what is the magnitude of the force **F**?

21



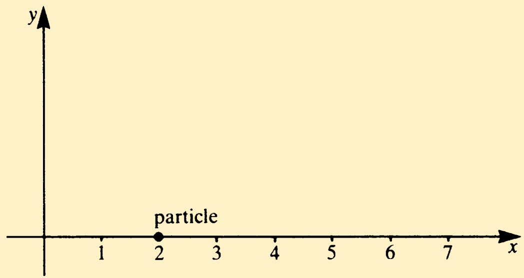
F = ...

22

$$\mathbf{F} = 3.8\,\mathrm{N}$$

22

A particle is moving along the positive x-axis with a velocity of $4\,\mathrm{m/s}$ and it is located at that instant at $x = 2$. If on the drawing 1 unit $= 1\,\mathrm{m/s}$, draw in the velocity vector.



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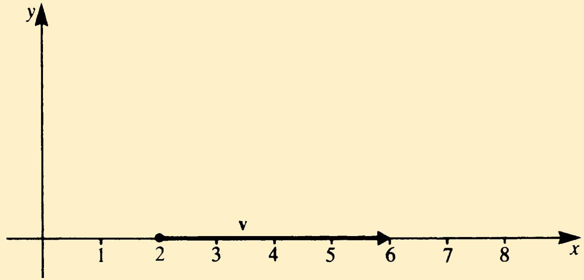


23

24

1.2 Addition and Subtraction of Vectors

24

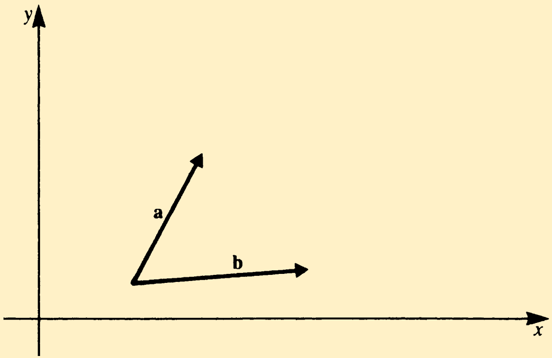
Objective: Concepts of geometrical addition, vector sum, resultant vector, negative vector, vector difference.

The principle of geometrical addition and subtraction of vectors is extremely useful.

READ: 1.2 Addition of vectors
1.3 Subtraction of vectors
Textbook pages 4–7

Then move on to frame 25 in this study guide.

25



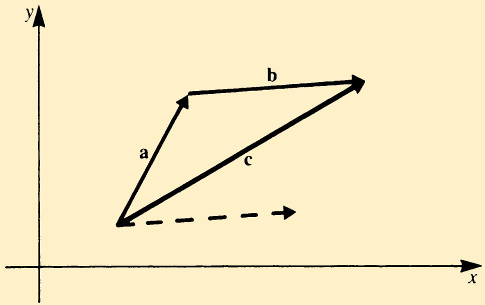
Add the vectors **a** and **b**:

25

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

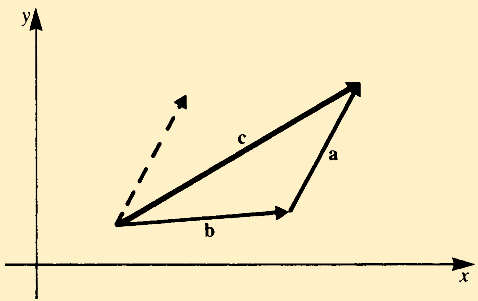
The vector **c** is called the
............................,
....
or ....

26



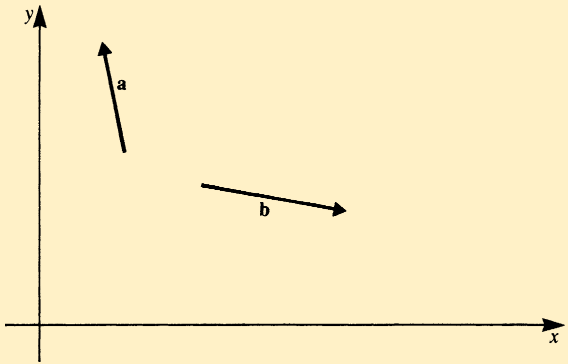
Resultant vector, vector sum or resultant.

26



Interchanging the order of addition does not affect the result.

27



Draw the vector sum
$\mathbf{c} = \mathbf{a} + \mathbf{b}$

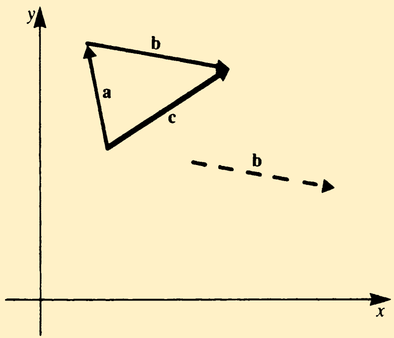
27

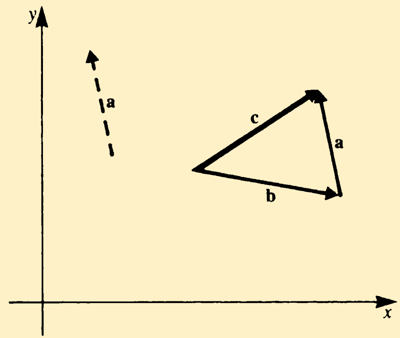
Other names for the vector sum are:
....
....

28

resultant vector
resultant

The two solutions shown are equivalent:

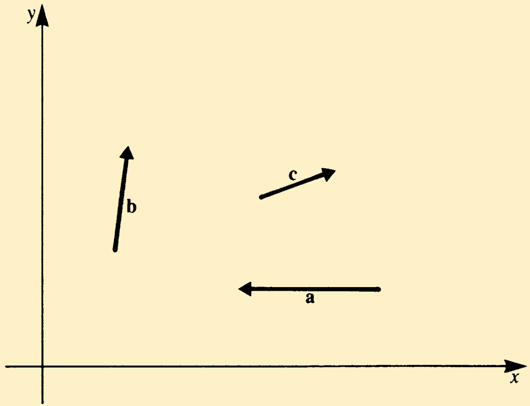




29

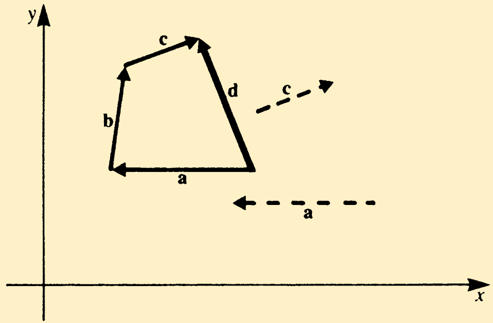
Add geometrically the vectors **a**, **b** and **c**: $\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$

29



30

30



The addition of vectors consists of forming a closed chain, resulting in a polygon. To do this vectors must be shifted. The order is of no importance.

Now decide:

Addition of vectors grasped → 33

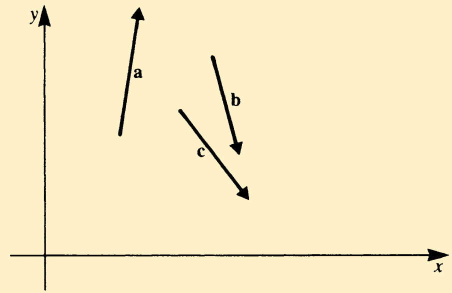
Further explanation required → 31

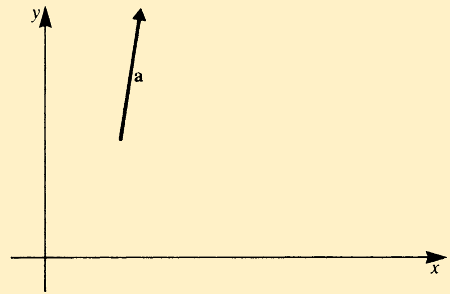
31

The addition of vectors involves two steps:
Step 1: Formation of a continuous chain:

(a) shift vector **b** so that it starts at the tip of vector **a**.
(b) now shift vector **c** so that it starts at the tip of vector **b**.

Form the chain.





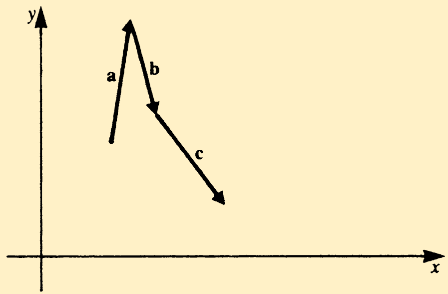
32

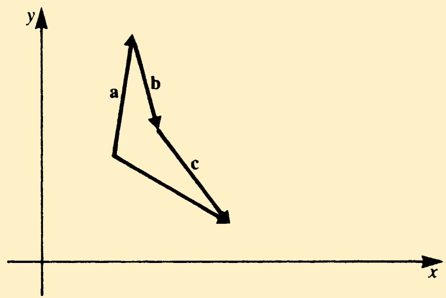
32

Step 2: Link up the starting point and end point of the chain.

The vector sum is then the closing vector, i.e. to the tip of vector **c**.

quad The procedure for the graphical addition of vectors is quite simple. We only need to form a chain of the arrows in the sense of each arrow. The last arrow placed on the chain always starts at the tip of the preceding one. The order is immaterial.





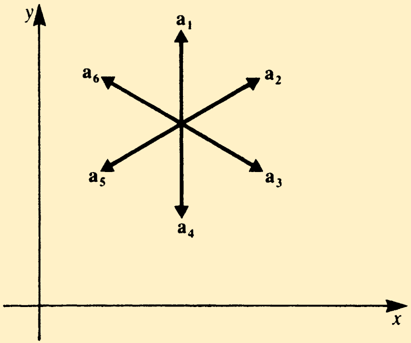
33

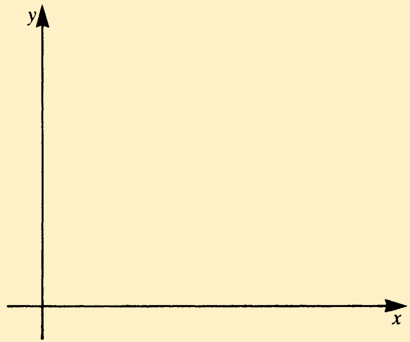
The vector sum is independent of the order in which we add the vectors.

Draw the vector polygon in the right-hand diagram. Add all vectors.

What is the resultant vector?

33





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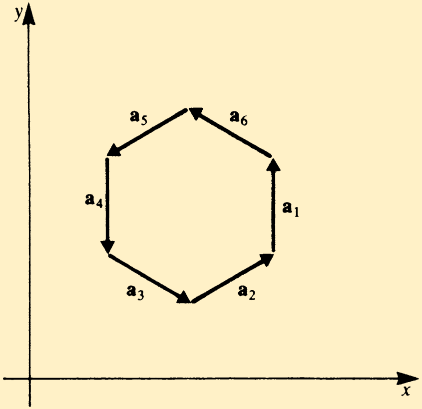
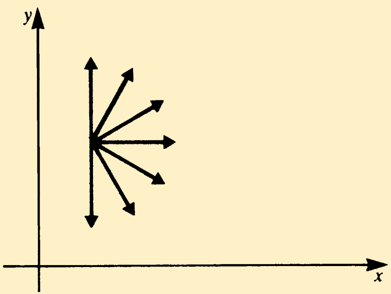


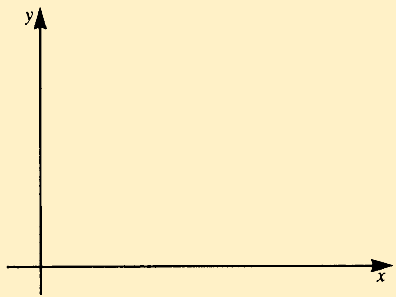
In this case the vector sum, or resultant vector, is zero. Any different order in which we add the vectors yields the same result.
For example, by adding

$$(\mathbf{a}_1 + \mathbf{a}_4) + (\mathbf{a}_2 + \mathbf{a}_5) + (\mathbf{a}_3 + \mathbf{a}_6)$$

we can see immediately that the resultant vanishes.

Now do the same for the following case:





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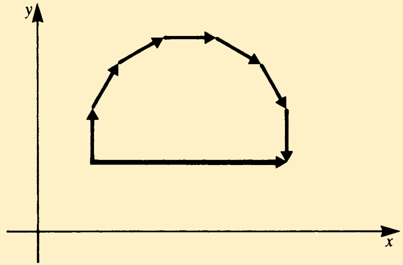

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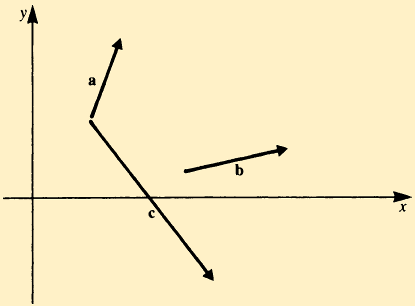
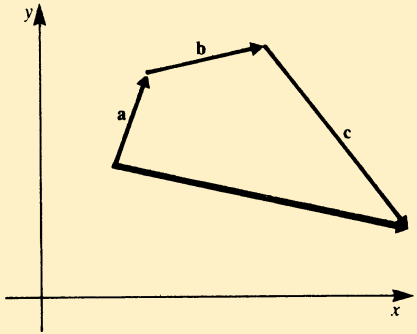


Add geometrically the three vectors **a**, **b** and **c**.

36

36

We form a chain with the three vectors as shown.
The vector sum is thus uniquely defined.



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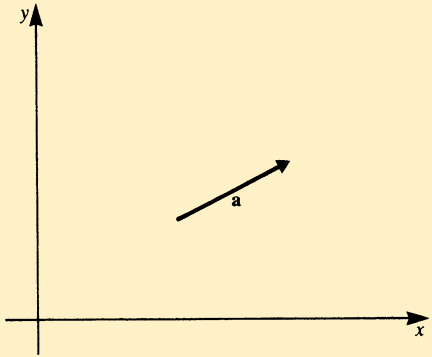
Given the vector **a**, draw the vector −**a**.

37

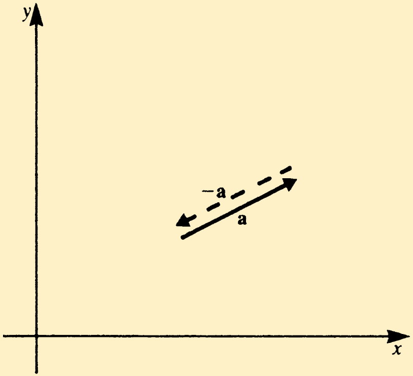


−**a** is called ....

38

38

$-\mathbf{a}$ is called the negative of the vector $\mathbf{a}$.



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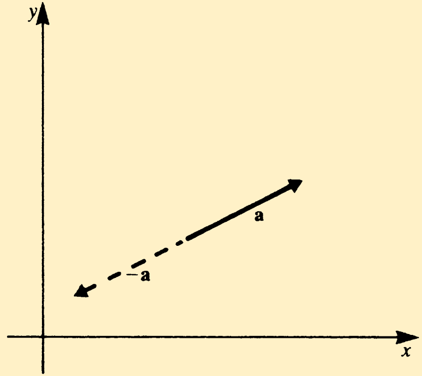
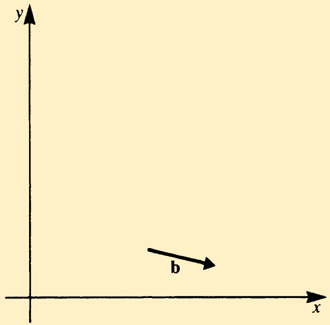
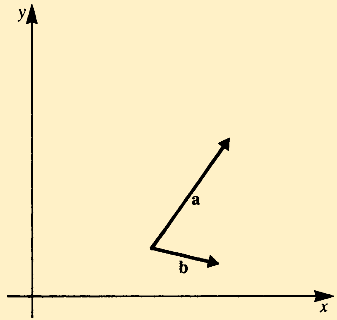


39

The concept of a negative vector is very useful because it helps us to perform vector subtraction; subtraction is thereby reduced to the addition of vectors. 39

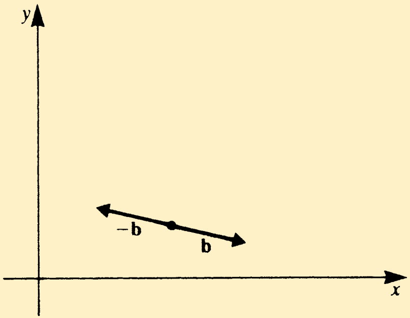


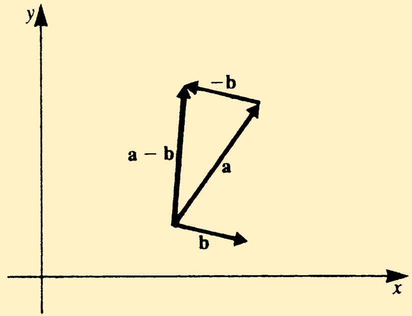
Draw the vector $-\mathbf{b}$



Draw the vector $\mathbf{a} - \mathbf{b}$

40





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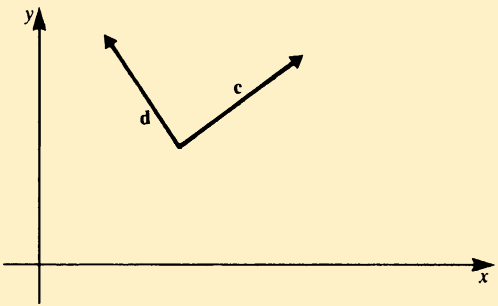
Draw $\mathbf{c} - \mathbf{d} = \mathbf{f}$ in the diagram.



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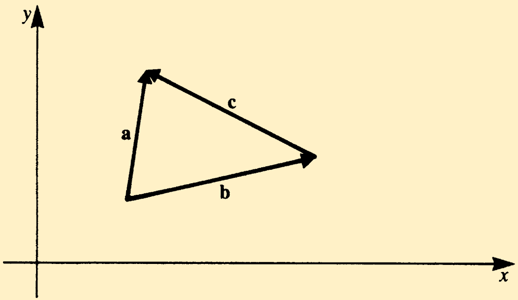
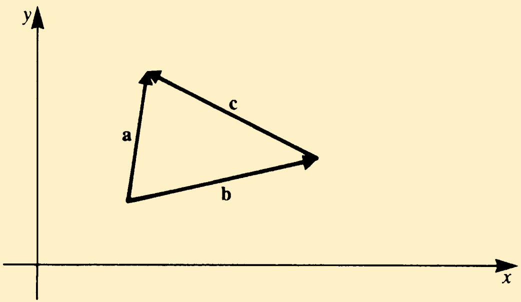


$$\mathbf{f} = \mathbf{c} - \mathbf{d}$$
$$= \mathbf{c} + (-\mathbf{d})$$

41



c is a difference vector. Write down the vector equation:

$\mathbf{c} =$ ....

-------------------▷ 42

$$\mathbf{c} = \mathbf{a} - \mathbf{b}$$

You certainly know now that in many cases the decision on how to proceed with this study guide is up to you. From now on you will just be given the options:

No mistakes 48

Some mistakes, or further explanation required 43

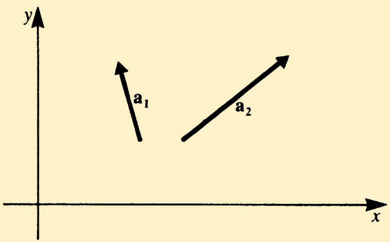
The formation of the difference of two vectors, $\mathbf{a} - \mathbf{b}$, can be reduced to addition with the help of a negative vector, 43

i.e. $$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

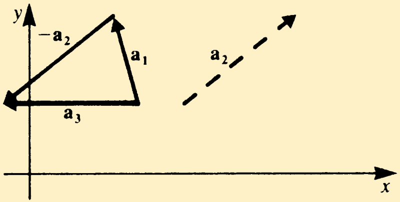
This leads to the following procedure to obtain the difference vector $\mathbf{a} - \mathbf{b}$:

Step 1: Draw the negative vector $-\mathbf{b}$
Step 2: Add the negative vector to $\mathbf{a}$
Step 3: Vector $\mathbf{a}$ plus the negative of vector $\mathbf{b}$ is the difference vector $\mathbf{a} - \mathbf{b}$.

Draw the difference vector $\mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2$



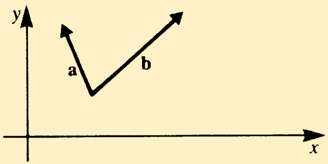
44

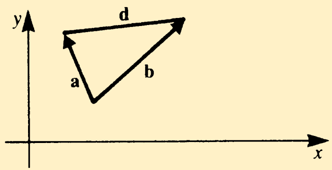


$\mathbf{a}_3 = \mathbf{a}_1 - \mathbf{a}_2$

44

An equivalent method for evaluating the difference vector $\mathbf{a} - \mathbf{b}$ requires the following procedure:





Step 1: Draw **a** and **b**.

Step 2: Connect the arrowheads of both vectors. The connection is the difference vector.

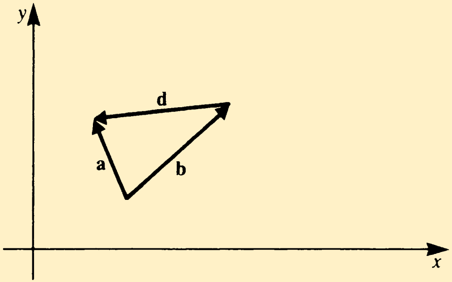
Step 3: Obtain the sense of the difference vector. To do this rearrange the equation:

$\mathbf{d} = \mathbf{a} - \mathbf{b}$ to $\mathbf{b} + \mathbf{d} = \mathbf{a}$

The sense of **d** has to satisfy this equation.

45

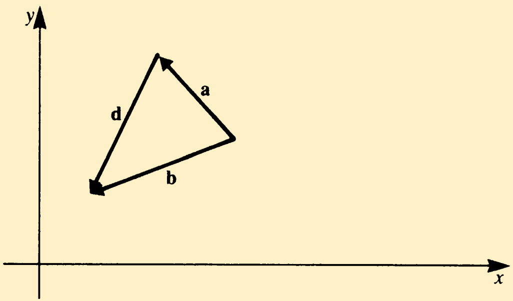
Draw in the sense.



45

$\mathbf{a} = \mathbf{b} + \mathbf{d}$

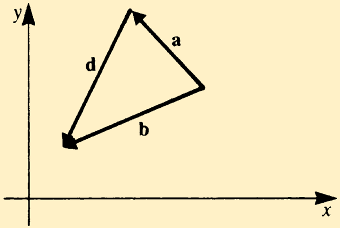
One more exercise. Write down the correct equation for the vectors shown in the figure.



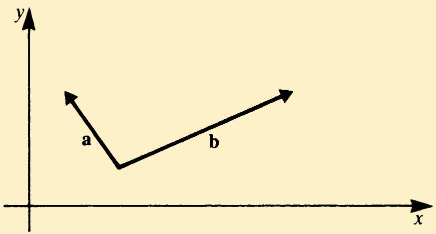
46

46

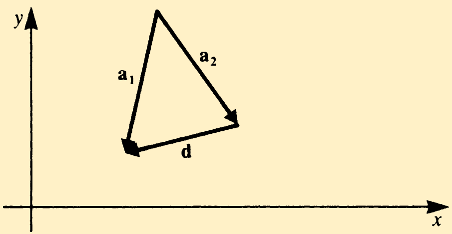
The correct equation is



$\mathbf{d} = \mathbf{b} - \mathbf{a}$
or $\mathbf{a} + \mathbf{d} = \mathbf{b}$
or $\mathbf{a} + \mathbf{d} - \mathbf{b} = 0$



$\mathbf{d} = \mathbf{a} - \mathbf{b}$



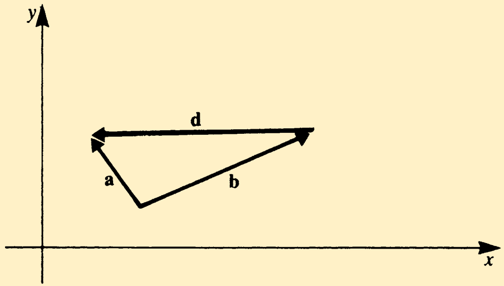
Draw in the vector difference,

Write down the equation

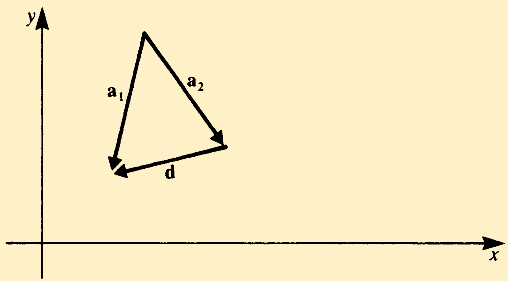
$\mathbf{d} =$ ....

-------------------▷ 47

47



$$\mathbf{d} = \mathbf{a} - \mathbf{b}$$



$$\mathbf{d} = \mathbf{a}_1 - \mathbf{a}_2$$

48

Further exercises will be found in the textbook. 48

If you have experienced no difficulty, go through those exercises — not now but after two or three days. Then the exercises will be more effective.

49

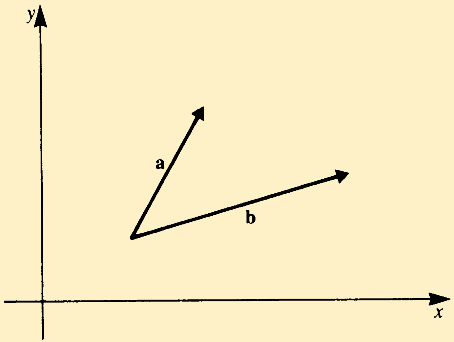
1.3 Components and Projection of a Vector

49

Objective: Concepts of projection, components, projecting one vector on to another.

READ: 1.4 Components and projection of a vector
Textbook pages 7–9

50

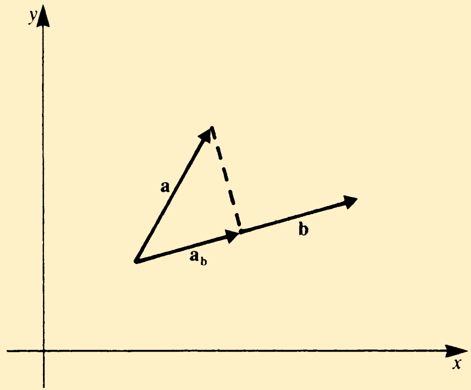


50

Draw the projection of **a** on to **b**.

51

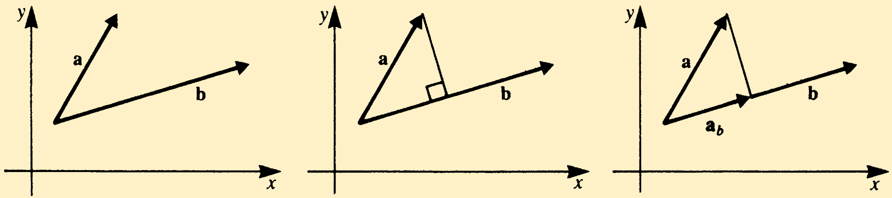
51



Answer correct 54

Mistakes, or further explanation required 52

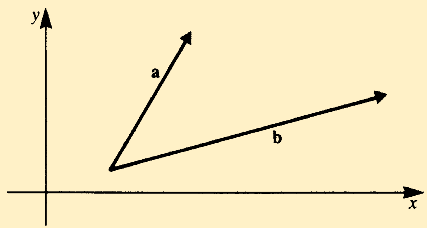
52



The projection of vector **a** on to vector **b** is obtained in two steps.

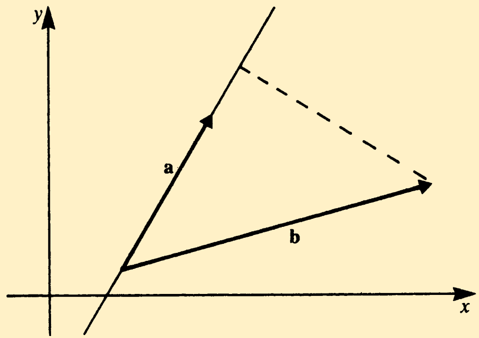
Step 1: Drop a perpendicular from the tip of **a** on to **b**.
Step 2: The projection $\mathbf{a_b}$ is the line segment from the starting point to the point of intersection with the perpendicular.

Now draw the projection of **b** on to **a**.



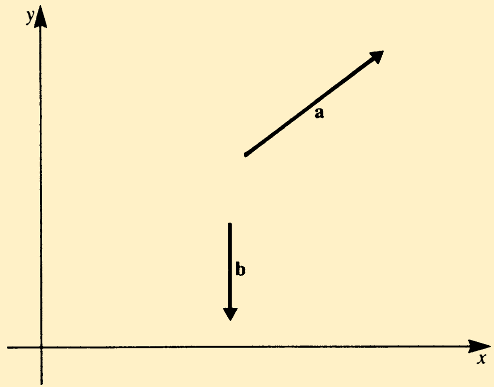
53

53



In this case we had to extend the line of action of **a** and then drop the perpendicular from the tip of **b**.

54

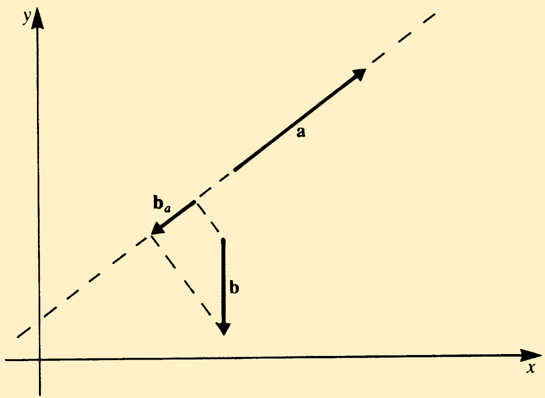


54

Draw the projection of **b** on to **a**.

55

55



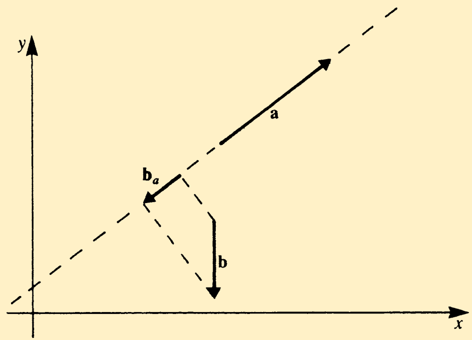
All correct 57

Further explanation required 56

In this example the vectors **a** and **b** do not have the same starting point. 56

The projection of **b** on to **a** requires three steps:

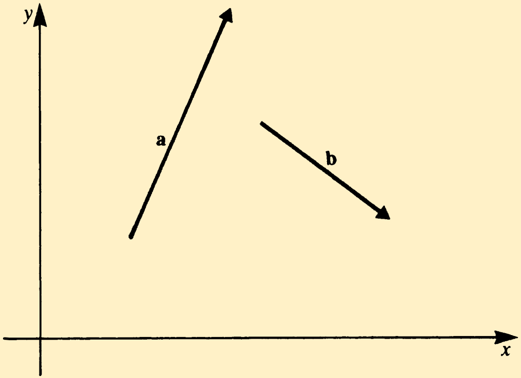
Step 1: Extend the line of action of **a**.
Step 2: Drop perpendiculars from the start and the tip of vector **b** on to the line of action of **a**.
Step 3: Draw in the projection as shown.



57

Draw the projection of **a** on to **b**.

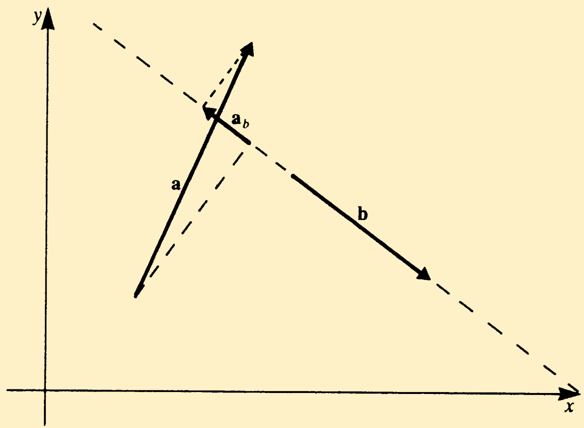
57



If you are still experiencing difficulty read the relevant parts of the textbook once more, and then solve this exercise with the help of the construction given in the textbook.

58

58

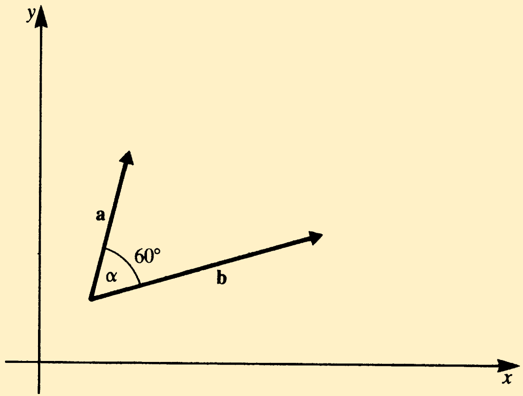


The difficulty in this case is that the line of action of **b** crosses that of **a**. But the principle is still the same, namely to drop perpendiculars from the start and tip of **a** on to the line of action of **b**, as shown in the drawing.

59

Numerical calculation of the projection of a vector:
If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and $\alpha = 60°$, calculate the magnitude of the projection of **a** on to **b**.

59

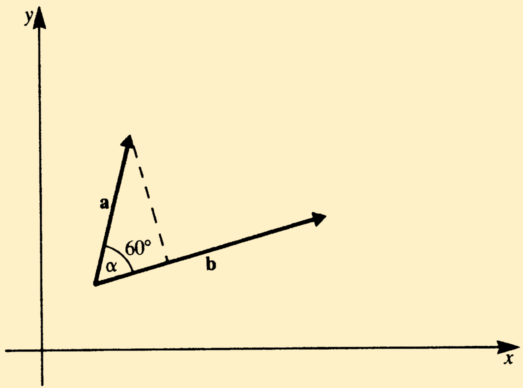


$$a_b = \ldots\ldots\ldots\ldots\ldots\ldots$$

Hint : $\sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos 60° = 0.5$

60



60

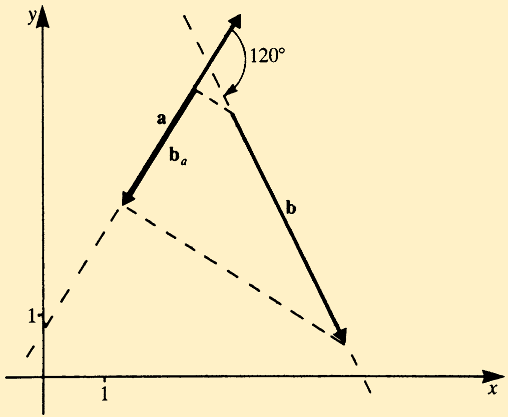
$$\begin{aligned} a_b &= a\cos\alpha \\ &= 3\cos 60^\circ \\ &= 3\times 0.5 = 1.5 \end{aligned}$$

Note that the magnitude of **b** is of no importance.

61

Obtain graphically the numerical value (magnitude) of the projection of **b** on to **a** and check it mathematically: $a = 3,\ b = 4,\ \alpha = ?$

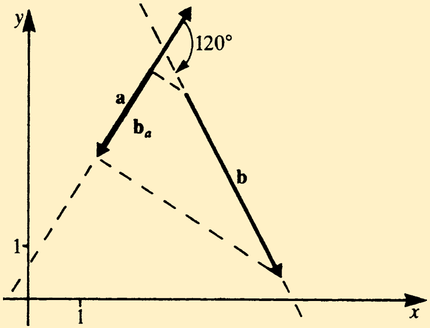
61



$b_a =$ ....

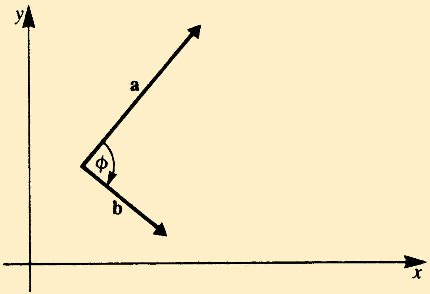
In case of difficulty refer to the textbook.

62



62

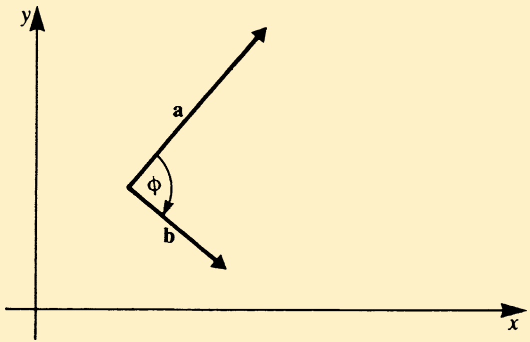
$$\begin{aligned} b_a &= b\cos 60^\circ \\ &= 4\times 0.5 \\ &= 2 \end{aligned}$$



Obtain graphically and mathematically the numerical value of the projection of **a** on to **b**:

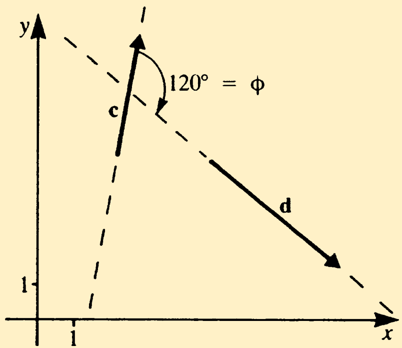
$$a = 4,\ b = 2,\ \phi = \frac{\pi}{2}$$
$$a_b = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

63



$a_b = 0$, because $\cos\frac{\pi}{2} = 0$

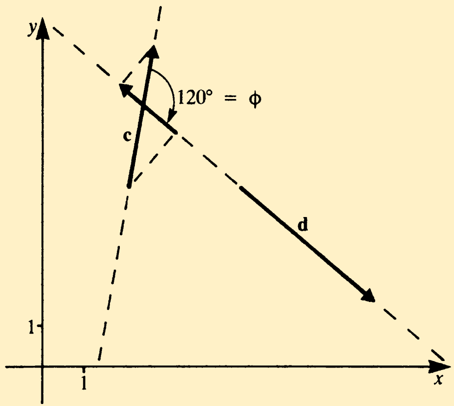
63



Obtain graphically and mathematically the projection of **c** on to **d**:

$c = 4,\ d = 5,\ \phi = 120°$

64



64

$$
\begin{aligned}
c_d &= c\cos 120^\circ \\
&= 4\times(-0.5) \\
&= -2
\end{aligned}
$$

65

1.4 Planning working periods and breaks.

65

All creatures get tired, human beings included.

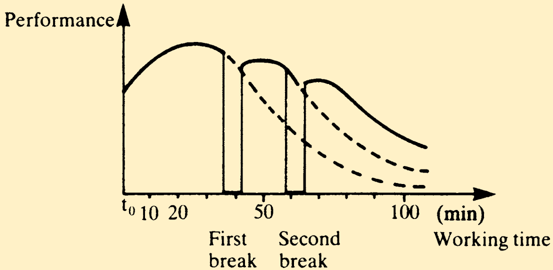
Occasionally we have to have a break, particularly if we are tired. Should we have a break when we are so tired that our eyelids are drooping? Yes, of course.

But it is important to have a break before that stage is reached. Having breaks in good time can delay a decrease in concentration and therefore in performance and achievement.

Typical findings of psychological experiments are presented on the next page.

--------------------▷ 66

The performance of students at study has been determined experimentally and is represented as a function of working time on the graph. 66



The broken lines indicate the trend without a break.

Breaks delay a loss in performance.

That means you should divide your work into well-defined working periods. In this programmed study guide breaks are suggested. The extent of a working period depends on the difficulty of the content and is not easy to forecast. Hence the working periods in the study guide are chosen to be short rather than long.

The most profitable working periods for total concentration lie between 20 and 40 minutes.

--------------------▷ 67

After studying a section you should check what you have retained. It is no use reading something and immediately forgetting the most important aspects. 67

Before a break write down the keywords from the section just studied; this is most useful. If you find that you have already forgotten what you have just read then read the material in the textbook once more.

Therefore, before you start a break: *Check that you have achieved the objectives of the section by expressing them in your own words*.

The next frame shows the findings of a psychological experiment, but you may leave it out.

Continue with the programmed study guide --------------------▷ 70

Investigation of active versus passive learning modes --------------------▷ 68

Experimental design:

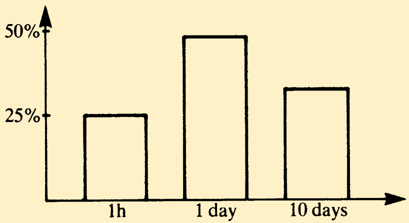
68

Group A: A section in a textbook is read four times.
Group B: The same section in the textbook is read twice and reproduced actively after each reading.

The ability to recall the material is checked after certain periods of time.
Result: The diagram shows the *difference* in the ability to recall subject matter between the groups.

Group B (active reproduction) performs better at any time.

Conclusion: Actively acquired knowledge is easier to recall than passively acquired knowledge.



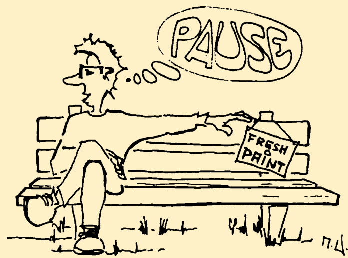
Recall performance of group B compared with that of group A.

69

To avoid failure it is important not to attempt to take 'too big a bite'. It is better to study a limited number of pages in the textbook at a time, and to check one's progress at the end of each study period. 69

If you can, it is useful to work with friends because you can help and check each other. Not only is it easier to test someone else, but working with friends also helps you to express yourself orally on the subject matter.

You should also remember that breaks form an integral part of your studies. They should be properly organised and they should not be so long that your work is completely interrupted.



----> 70

1.5 Component Representation in Coordinate Systems

70

Objective: Concepts of unit vectors, component of a vector, notation of vectors, position vectors.

READ: **1.4 Component representation in coordinate systems**
1.5.1 Position vector
1.5.2 Unit vectors
1.5.3 Component representation of a vector
Textbook pages 9–12

71

Vectors whose magnitude is 1 are called 71

The representation of a vector in the form $\mathbf{a} = (3, \quad 1, \quad 2)$ is called ....

--------------------▷ 72

unit vectors
component representation

72

Write down three ways of writing the unit vectors along the x-, y- and z-axes of a Cartesian coordinate system:

....	....	....
....	....	....
....	....	....

73

$$\mathbf{i},\ \mathbf{j},\ \mathbf{k}$$
$$\mathbf{e}_x,\ \mathbf{e}_y,\ \mathbf{e}_z$$
$$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3$$

73

Let $p_x,\ p_y,\ p_z$ be the components of a position vector **p**.

Write down the vector **p** in terms of its components and the unit vectors.

$$\mathbf{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The abbreviated form is

$$\mathbf{p} = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots) \text{ or } \mathbf{p} = \begin{pmatrix} \ldots\ldots\ldots \\ \ldots\ldots\ldots \\ \ldots\ldots\ldots \end{pmatrix}$$

74

$$\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$$

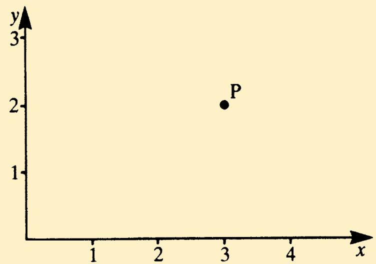
$$\mathbf{p} = (p_x,\ p_y,\ p_z) \text{ or } \mathbf{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

74

Given the point P = (3, 2)

The position vector **p** has the two .... $p_x\mathbf{i}$ and $p_y\mathbf{j}$.

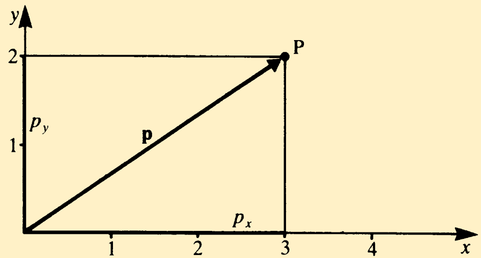
Draw in the components p_x and p_y as well as **p**.



75

components

75



What is the component representation of **p**?

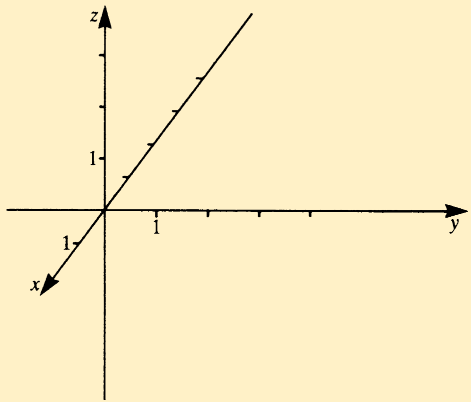
$$\mathbf{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

76

76

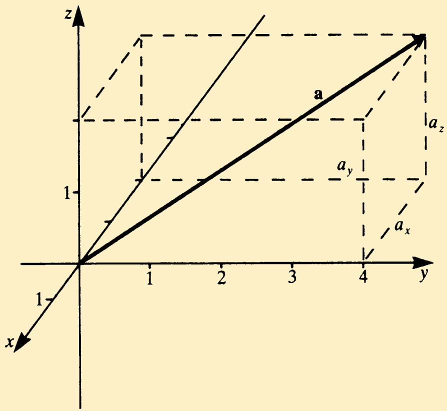
$$\mathbf{p} = (3, \quad 2) \text{ or } \mathbf{p} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

In the three-dimensional coordinate system shown draw the vector **a** by adding the components:
$\mathbf{a} = (-2, \quad 4, \quad 2)$



77

77



Check with the help of the drawing that the order does not matter.

Having difficulties? --------------------▷ 78

All correct? --------------------▷ 79

Go back to the textbook and try to transfer all statements to the two-dimensional case; you should find it easier. 78

Sketch figures analogous to those in the textbook for the two-dimensional case.

--------------------▷ 79

1.6 Representation of the Sum of Two Vectors in Terms of their Components

79

1.7 Subtraction of Vectors in Terms of their Components

Objective: Addition and subtraction of vectors in terms of their components.

Many physical quantities can be represented by vectors. So far we have added and subtracted these vectors graphically. The addition and subtraction can also be carried out analytically.

This is achieved by expressing each vector as components along the axes of a coordinate system. Then we can add and subtract components along each axis.

READ: **1.5.4 Representation of the sum of two vectors in terms of their components**
1.5.5 Subtraction of vectors in terms of their components
Textbook pages 12–14

80

Given two vectors in terms of their components: 80

$$\mathbf{a} = (3, \quad 4), \ \mathbf{b} = (2, \quad -2)$$

Find

$$\mathbf{c} = \mathbf{a} + \mathbf{b}$$

$$\mathbf{c} = (\ldots\ldots, \quad \ldots\ldots)$$

Do this analytically, even though it could be solved graphically.

81

$$\mathbf{c} = (5, \quad 2)$$

81

Here are further examples of the same type:

A: $\mathbf{a} = (-2, \quad 1), \quad \mathbf{b} = (1, \quad 3)$
$\mathbf{c} = \mathbf{a} + \mathbf{b} = (\ldots\ldots.., \quad \ldots\ldots..)$

B: $\mathbf{v}_1 = (15\,\mathrm{m/s}, \quad 10\,\mathrm{m/s})$
$\mathbf{v}_2 = (2\,\mathrm{m/s}, \quad -5\,\mathrm{m/s})$

$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2 = (\ldots\ldots\ldots., \; \ldots\ldots\ldots.)$

82

A: $\mathbf{c} = (-1, \quad 4)$
B: $\mathbf{v} = (17\,\mathrm{m/s}, \quad 5\,\mathrm{m/s})$

82

Vectors are added analytically by adding their components. We need to assign units in the case of vectors used to represent physical quantities.

The last example was a case in point because it concerned velocities. Therefore the components represented velocities as well.

Let
$$\mathbf{a} = (4, \quad 2)$$
$$\mathbf{b} = (2, \quad 2)$$

Express the difference vector $\mathbf{d} = \mathbf{a} - \mathbf{b}$ in component form.

$$\mathbf{d} = (\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots)$$

83

$$\mathbf{d} = (4-2, \quad 2-2) = (2, \quad 0)$$

83

Analytically, the subtraction of vectors is carried out by subtracting their components.

Do the following exercises:

$$\mathbf{v}_1 = (5\,\mathrm{m/s}, \quad 5\,\mathrm{m/s}); \quad \mathbf{v}_2 = (10\,\mathrm{m/s}, \quad 2\,\mathrm{m/s});$$

$$\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2 = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$\mathbf{F}_1 = (2.5\,\mathrm{N}, \quad 0\,\mathrm{N}); \quad \mathbf{F}_2 = (1\,\mathrm{N}, \quad 2\,\mathrm{N});$$

$$\mathbf{F}_1 + \mathbf{F}_2 = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

$$\mathbf{F}_1 - \mathbf{F}_2 = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

84

84

$$\mathbf{v}_3 = (-5\,\text{m/s}, \quad 3\,\text{m/s})$$
$$\mathbf{F}_1 + \mathbf{F}_2 = (3.5\,\text{N}, \quad 2\,\text{N})$$
$$\mathbf{F}_1 - \mathbf{F}_2 = (1.5\,\text{N}, \quad -2\,\text{N})$$

In the last example we were dealing with velocities and forces. Velocities are measured in metres/second and forces in newtons.

We shall now consider vectors in space; this means that each vector has three components:

$$\mathbf{a} = (1, \quad 2, \quad 1)$$
$$\mathbf{b} = (2, \quad 1, \quad 0)$$
$$\mathbf{c} = \mathbf{a} + \mathbf{b} = (\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots)$$
$$\mathbf{d} = \mathbf{b} - \mathbf{a} = (\ldots\ldots, \quad \ldots\ldots, \quad \ldots\ldots)$$

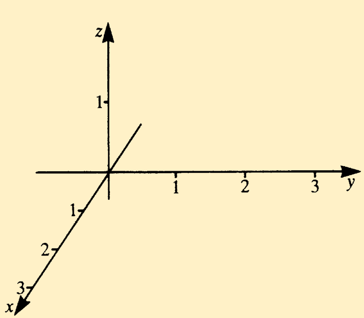
85

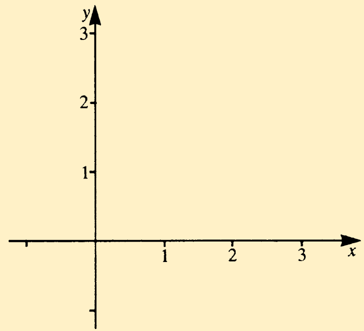
$$\mathbf{c} = \mathbf{a} + \mathbf{b} = (3, \quad 3, \quad 1)$$
$$\mathbf{d} = \mathbf{b} - \mathbf{a} = (1, \quad -1, \quad -1)$$

85

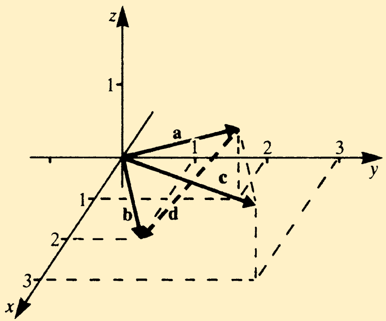
You can check the analytical result with the help of a drawing.

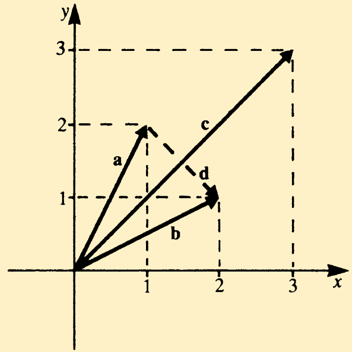
If you find the three-dimensional case too difficult then carry out the graphical method two-dimensionally with $\mathbf{a} = (1, \quad 2)$ and $\mathbf{b} = (2, \quad 1)$.





86





86

The component method enables us to carry out addition and subtraction analytically and this is obviously very useful.

87

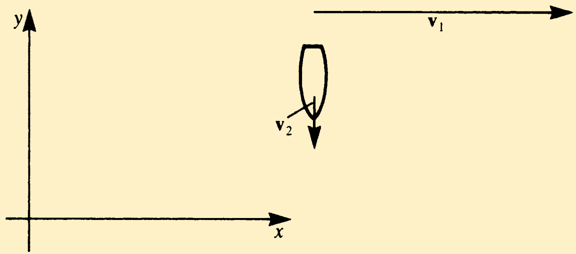
A boat is crossing a river. The river has a velocity $\mathbf{v}_1 = (10\,\mathrm{m/s}, \quad 0)$.

87

The velocity of the boat relative to the water is $\mathbf{v}_2 = (0, \quad -2\,\mathrm{m/s})$.

Then the absolute velocity of the boat, i.e. the velocity of the boat relative to the river bank, is composed of the velocity of the water plus the velocity of the boat relative to the water.

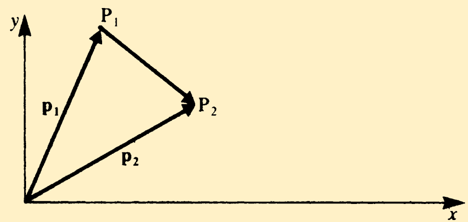
$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2; \text{ components of } \mathbf{v} = (\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots)$$



88

$$\mathbf{v} = (10\,\mathrm{m/s}, \quad -2\,\mathrm{m/s})$$

88



Given two points P_1 and P_2 with position vectors $\mathbf{p_1}$ and $\mathbf{p_2}$; in component representations: $\mathbf{p_1} = (p_{1x}, \quad p_{1y})$, $\mathbf{p_2} = (p_{2x}, \quad p_{2y})$.

We require the vector which connects P_1 and P_2.

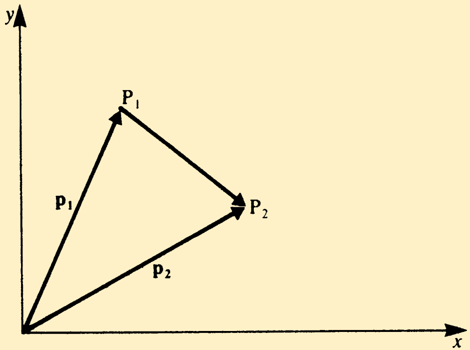
The connecting vector is the difference vector of the position vectors $\mathbf{p_1}$ and $\mathbf{p_2}$, i.e. $\overrightarrow{P_1P_2}$. It is the difference vector of the two position vectors $\mathbf{p_1}$ and $\mathbf{p_2}$:

$$\overrightarrow{P_1P_2} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Component representation:

$$\overrightarrow{P_1P_2} = (\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots)$$

--------------------▷ 89



$$\overrightarrow{P_1P_2} = \mathbf{p_2} - \mathbf{p_1}$$
$$\overrightarrow{P_1P_2} = (p_{2x} - p_{1x}, \quad p_{2y} - p_{1y})$$

89

The equation can easily be checked with the help of the above drawing and with a slight rearrangement:

$$\mathbf{p_1} + \overrightarrow{P_1P_2} = \mathbf{p_2}$$

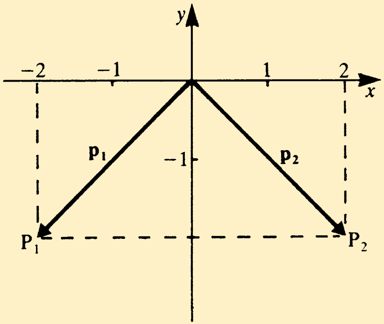
Given $\mathbf{p_1} = (1, \quad 4)$ and $\mathbf{p_2} = (3, \quad 3)$

$$\overrightarrow{P_1P_2} = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

-------------------▷ 90

$$\overrightarrow{P_1P_2} = \mathbf{p_2} - \mathbf{p_1} = (2, \quad -1)$$

90



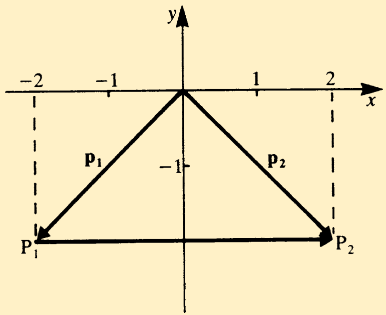
$$\mathbf{p_1} = (-2, \quad -2)$$
$$\mathbf{p_2} = (2, \quad -2)$$

(a) Draw the vector $\overrightarrow{P_1P_2}$ which links P_1 and P_2.

(b) Component representation:

$$\overrightarrow{P_1P_2} = (\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots)$$

91

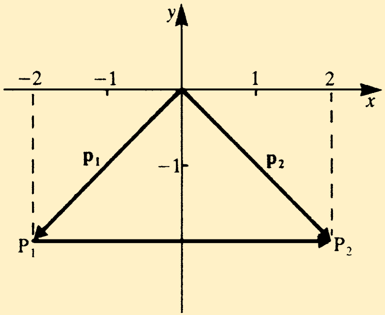


91

$\overrightarrow{P_1P_2} = (4, \quad 0)$

All correct --------------------▷ 95

Errors, detailed explanation required --------------------▷ 92



$\overrightarrow{P_1P_2}$ is a vector which starts at P_1 and ends at P_2 with the arrow head at P_2.

Difficulties may be due to the signs.

92

From the diagram we gather

$$\mathbf{p}_2 = \mathbf{p}_1 + \overrightarrow{P_1P_2}$$

Transforming yields

$$\overrightarrow{P_1P_2} = \mathbf{p}_2 - \mathbf{p}_1$$

93

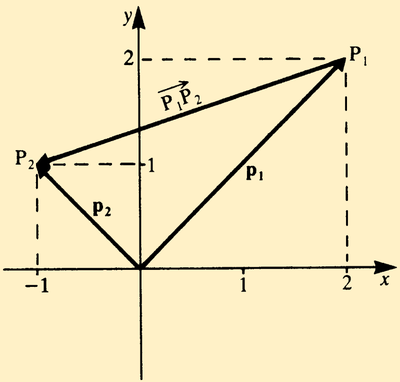
93

With

$$\mathbf{p_1} = (-2, \quad -2)$$
$$\mathbf{p_2} = (2, \quad -2)$$

then

$$\overrightarrow{P_1P_2} = (2-(-2), \quad -2-(-2)) = (4, \quad 0)$$



Now try again.

$$\mathbf{p_1} = (2, \quad 2)$$
$$\mathbf{p_2} = (-1, \quad 1)$$
$$\overrightarrow{P_1P_2} = (\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots)$$

94

$$\overrightarrow{P_1P_2} = (-3, \quad -1)$$

94

Remember:

The components of the vector from a point P_1 to a point P_2 are given by the coordinates of the tip of the arrow minus the coordinates of the start of the arrow.

Further exercises are in the textbook.

95

Some remarks about working in a group and working alone. 95

Working alone is advisable if facts are to be learnt thoroughly, calculations copied, proofs studied and texts read intensively.

Working in a group is advisable for:

— the solution of exercises and problems with the help of methods just acquired,
— the discussion of results,
— the preparation and identification of problems.

Working in a group should alternate with working alone.

Working in a group cannot replace working alone; similarly, working alone cannot replace those benefits gained by working in a group.

Many students believe that the necessity for expressing facts during group discussions is a good preparation for examinations. They are right, provided of course that in a group 'nonsense' is labelled as 'nonsense'. This means that if someone explains something wrongly he/she must be corrected to ensure that wrong concepts are not passed on.

Time for a break!

--------------------▷ 96

1.8 Multiplication of a Vector by a Scalar. Magnitude of a Vector

96

Objective: Multiplication of a vector by a scalar, magnitude of a vector in terms of its components, distance between two points.

READ: **1.6 Multiplication of a vector by a scalar**
1.7 Magnitude of a vector
Textbook pages 14–17

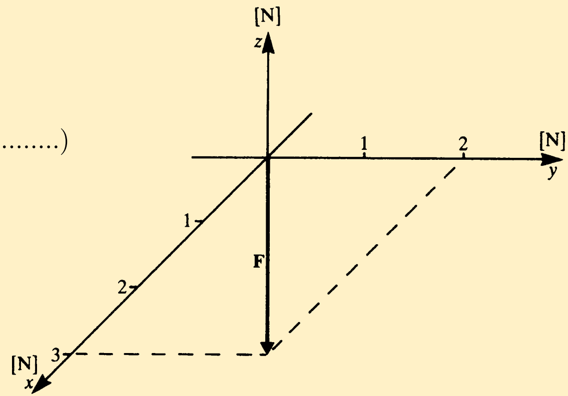
97

97

Let the force

$\mathbf{F} = (3\,\text{N}, \quad 2\,\text{N}, \quad 0\,\text{N})$

be increased by a factor of 2.5;
then $2.5\mathbf{F} = (\ldots\ldots, \quad \ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots)$



98

$$2.5\mathbf{F} = (7.5\,\mathrm{N}, \quad 5\,\mathrm{N}, \quad 0\,\mathrm{N})$$

98

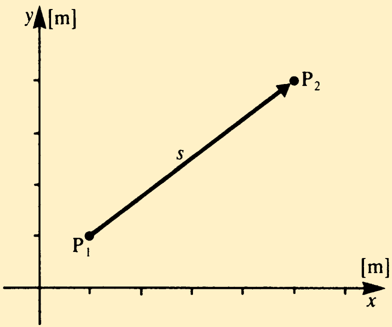
The vector $0 = (0, \quad 0, \quad 0)$ is called ....

99

a null vector

99

A mechanical digger is crawling along with uniform velocity from a position $\mathbf{P}_1 = (1\,\text{m}, \quad 1\,\text{m})$ to a position $\mathbf{P}_2 = (5\,\text{m}, \quad 4\,\text{m})$ in a time of 50 seconds.



The change in position

$$\overrightarrow{P_1P_2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The magnitude of the distance covered

$$s = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The magnitude of the velocity

$$v = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The velocity $\mathbf{v} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 100

$$\overrightarrow{P_1P_2} = (4\,\mathrm{m}, \quad 3\,\mathrm{m})$$
$$s = \sqrt{16\,\mathrm{m}^2 + 9\,\mathrm{m}^2} = 5\,\mathrm{m}$$
$$v = 5\,\mathrm{m}/50\,\mathrm{s} = 0.1\,\mathrm{m/s}$$
$$\mathbf{v} = \left(\frac{4}{50}\,\mathrm{m/s}, \quad \frac{3}{50}\,\mathrm{m/s}\right)$$

100

If you made mistakes try once more to solve the problem with the help of the textbook.

--------------------▷ 101

Given the vector $\mathbf{b} = (b_x, \quad b_y, \quad b_z)$

its magnitude is $b = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

101

Numerical example

$$\mathbf{b} = (1, \quad 2, \quad 1)$$

$$b = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

102

$$b = \sqrt{\left(b_x^2 + b_y^2 + b_z^2\right)}$$
$$b = \sqrt{6} = 2.45 \quad \text{(to two decimal places)}$$

102

Calculation of unit vectors:
A unit vector has a magnitude equal to ....

Given the vector $\mathbf{a} = (4, \quad 2, \quad 4)$
then its magnitude

$$a = \text{............................}$$

-------------------▷ 103

1

$$a = \sqrt{16+4+16} = \sqrt{36} = 6$$

103

The unit vector in the direction of **a** is denoted by $\mathbf{e}_a$.

The unit vector has a magnitude of 1. The direction of the unit vector in the direction **a** is obtained by multiplying **a** by the scalar $\frac{1}{a}$.

Obtain the unit vector $\mathbf{e}_a$ for $\mathbf{a} = (4, \quad 2, \quad 4)$.

$$\mathbf{e}_a = \dots\dots\dots\dots\dots\dots$$

104

$$\mathbf{e}_a = \left(\tfrac{4}{6}, \quad \tfrac{2}{6}, \quad \tfrac{4}{6}\right) = \left(\tfrac{2}{3}, \quad \tfrac{1}{3}, \quad \tfrac{2}{3}\right)$$

104

As a check calculate the magnitude of $\mathbf{e}_a$:

$$\mathbf{e}_a = \text{............................}$$

105

$$\mathbf{e}_a = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = 1$$

105

Calculate the unit vector for $\mathbf{a} = (3, \quad 4)$

$$\mathbf{e}_a = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

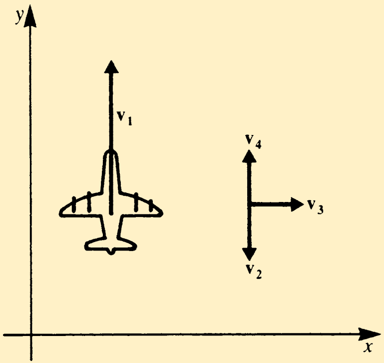
106

$$\mathbf{e}_a = (0.6, \quad 0.8)$$

106

In case of difficulty try to solve the problem with the help of the textbook.

107



An aircraft is flying on a northerly course. Its velocity relative to the air is

107

$$\mathbf{v_1} = (0, \quad 200\,\mathrm{km/h}).$$

Calculate the velocity of the aircraft relative to the ground for the following air velocities:

$\mathbf{v_2} = (0, \quad -50\,\mathrm{km/h})$, head wind
$\mathbf{v_3} = (50\,\mathrm{km/h}, \quad 0)$, cross wind
$\mathbf{v_4} = (0, \quad 50\,\mathrm{km/h})$, tail wind

$\mathbf{v_1} + \mathbf{v_2} =$ ....
$\mathbf{v_1} + \mathbf{v_3} =$ ....
$\mathbf{v_1} + \mathbf{v_4} =$ ....

------------------▷ 108

With a head wind, $\mathbf{v_1} + \mathbf{v_2} = (0, \quad 150\,\mathrm{km/h})$
With a cross wind, $\mathbf{v_1} + \mathbf{v_3} = (50\,\mathrm{km/h}, \quad 200\,\mathrm{km/h})$
With a tail wind, $\mathbf{v_1} + \mathbf{v_4} = (0, \quad 250\,\mathrm{km/h})$

108

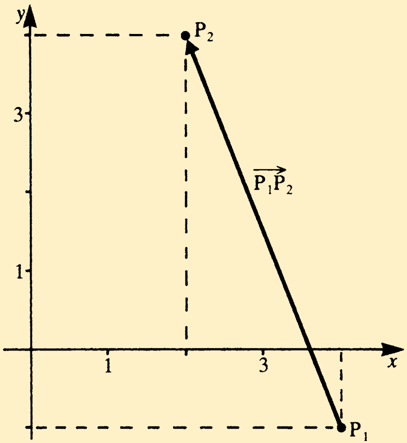
Calculate the magnitude of the velocity relative to the ground for each of the three cases above:

$|\mathbf{v_1} + \mathbf{v_2}| = \ldots\ldots\ldots\ldots\ldots$ head wind
$|\mathbf{v_1} + \mathbf{v_3}| = \ldots\ldots\ldots\ldots\ldots$ cross wind
$|\mathbf{v_1} + \mathbf{v_4}| = \ldots\ldots\ldots\ldots\ldots\ldots$ tail wind

109

$$|\mathbf{v_1} + \mathbf{v_2}| = \sqrt{(150\,\text{km/h})^2} = 150\,\text{km/h}$$
$$|\mathbf{v_1} + \mathbf{v_3}| = \sqrt{50^2 + 200^2} = 206.16\,\text{km/h}$$
$$|\mathbf{v_1} + \mathbf{v_4}| = \sqrt{250^2} = 250\,\text{km/h}$$

109



Given the points $P_1 = (4, \quad -1)$ and $P_2 = (2, \quad 4)$:

$\overrightarrow{P_1P_2} = (\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots)$

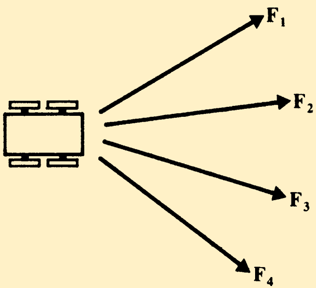
Distance between the points $= \ldots\ldots\ldots$

----------------▷ 110

$$\begin{aligned} \overrightarrow{P_1P_2} &= (-2, \quad 5) \\ \left|\overrightarrow{P_1P_2}\right| &= \sqrt{29} \approx 5.39 \end{aligned}$$

110

A carriage is pulled by four men.



The components of the four forces $\mathbf{F_1}$, $\mathbf{F_2}$, $\mathbf{F_3}$ and $\mathbf{F_4}$ are

$$\begin{aligned} \mathbf{F_1} &= (200\,\mathrm{N}, \quad 150\,\mathrm{N}) \\ \mathbf{F_2} &= (180\,\mathrm{N}, \quad 0) \\ \mathbf{F_3} &= (250\,\mathrm{N}, \quad -50\,\mathrm{N}) \\ \mathbf{F_4} &= (270\,\mathrm{N}, \quad -200\,\mathrm{N}) \end{aligned}$$

Resultant force $\mathbf{F}$ = ...

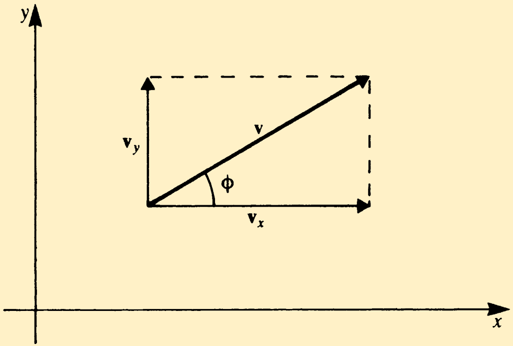
and $|\mathbf{F}|$ = ...

111

$$\mathbf{F} = (900\,\text{N}, \quad -100\,\text{N})$$
$$|\mathbf{F}| = \sqrt{900^2 + 100^2} = 905.5\,\text{N}$$

111

Calculation of the components of a vector, given its magnitude and its angle.



What is **v** in component form?

$\mathbf{v} = (\ldots\ldots\ldots\ldots\ldots\ldots, \quad \ldots\ldots\ldots\ldots\ldots\ldots)$

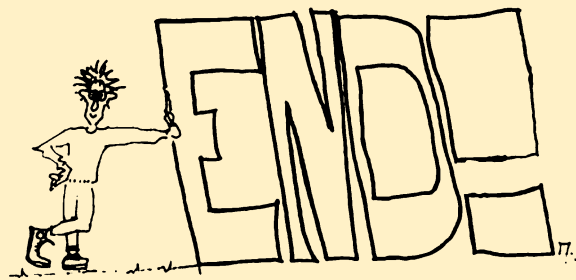
112

$$\mathbf{v} = (v \cos\phi, \quad v \sin\phi)$$

If you have mastered the exercises then it is pointless to do more of the same type; you will not learn more.

But the same exercises may be more difficult if they appear in a different context.

The mathematical methods you have just learnt are frequently used in physics and engineering, although you may find other notations for the same subject matter. Therefore, in the study guide, we sometimes change notation and give exercises related to previous sections of the textbook.



Chapter 2
Vector Algebra II: Scalar and Vector Products

A well-known maxim when delivering a lecture is:

2

Say what you are going to say.
Say it.
Say what you have said.

In other words:

— before starting a lecture give a brief explanation of what it is about;
— go through the subject matter in detail;
— at the end summarise the lecture.

By following this maxim it is easier for the audience to learn and retain the content.

The maxim ensures that important facts are repeated .... times.

3

three 3

The method is useful for another reason: if the essential facts are given at the beginning and repeated at the end we have a better chance of realising which aspects are fundamental, and this means that we have established priorities.

The maxim is thus also useful for the learner if he applies it to his own way of learning. It is just as useful to recall the essential facts of the last lecture before starting a new one.

Write down, briefly, the essential points of the chapter on Vector Algebra I, including basic formulae and symbols.

Stop working after 5 minutes.

-------------------▷ 4

Your keywords may differ but the following illustrates what you could have written: 4

1 Vectors have magnitude and direction; geometrical representation: directed line segments, e.g. arrows.
2 The addition of vectors is in accordance with the parallelogram or triangle law. For more than two vectors a chain is formed.
3 Subtracting a vector is equivalent to adding the negative of it.
4 Projection of a vector **a** on to a vector **b**:
Drop perpendicular lines from the start and end of vector **a** on to the line of action of **b**.
5 A vector may be expressed in terms of its components. The addition of two vectors **a** and **b** would be

$$\mathbf{a}+\mathbf{b}=(a_x+b_x, \quad a_y+b_y, \quad a_z+b_z)$$

6 The magnitude of a vector in terms of its components is

$$a=\sqrt{a_x^2+a_y^2+a_z^2}$$

7 The unit vector $\mathbf{e}_a$ of a vector **a** has magnitude unity and the direction of **a**.

Thus $$\mathbf{e}_a=\frac{\mathbf{a}}{a}$$

5

It does not matter if your brief résumé does not coincide with the one just given. 5

Your own may be more concise or more lengthy and it may contain other concepts, such as free vectors, bound vectors, multiplication of a vector by a scalar, etc. What matters most is that you are able to summarise the chapter recently studied. This is a useful general rule: before you start something new, recall what you have learnt in the last section.

6

2.1 The Scalar Product

6

Objective: Concepts of scalar product, inner product, calculation of the scalar product.

In the textbook the scalar product is derived from a mechanics example and then the rule is generalised.

READ: 2.1 Scalar product
2.1.1 Application: Equation of a line and a plane
2.1.2 Special cases
2.1.3 Commutative and distributive laws
Textbook pages 23–27

Then return to the study guide.

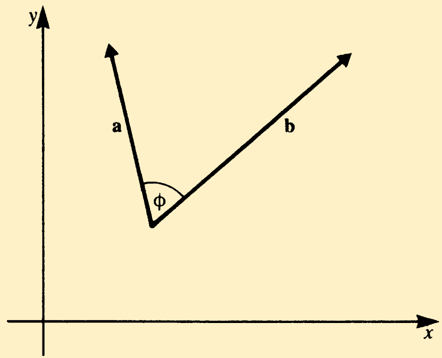
7

The inner or .... product of two vectors can be obtained if the .... and the .... are given. 7

8

scalar
magnitude, included angle

8



State the formula for the inner product of the vectors **a** and **b**.

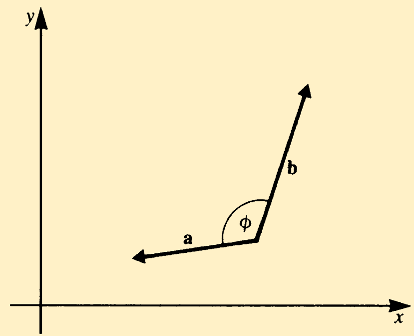
$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

9

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \phi$$

9

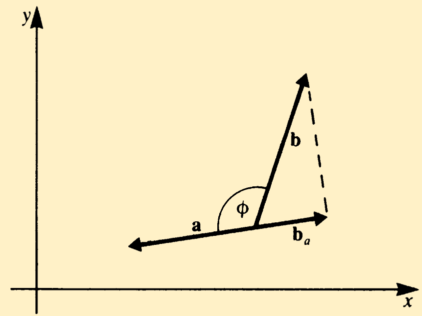
The scalar product of two vectors **a** and **b** is equal to the product of the magnitude of vector **a** and the magnitude of the projection of .... on ....



Complete the sketch in such a way that it fits the above statement.

10

The scalar product of two vectors **a** and **b** is equal to the product of the magnitude of vector **a** and the magnitude of the projection of **b** on **a**.



10

In the above example the sign of the scalar product is:

☐ positive

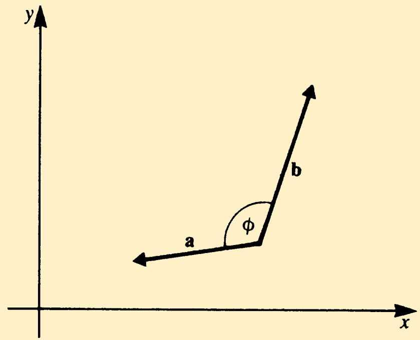
☐ negative

11

negative: $\cos\phi$ is negative. The projection of **b** on **a** is opposite to **a** in direction.

11

The scalar product of the vectors **a** and **b** is also equal to the product of the magnitude of **b** and the magnitude of the projection of .... on ....



The vectors **a** and **b** are the same as those in the last example. Complete the sketch for this case. The scalar product of the vectors is:

☐ positive

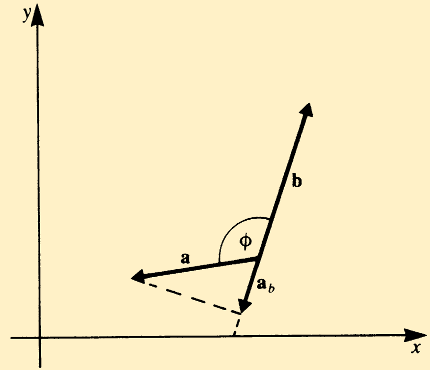
☐ negative

12

12

The scalar product of the vectors **a** and **b** is equal to the product of the magnitude of **b** and the magnitude of the projection of **a** on **b**.

The scalar product of the vectors is negative.



All correct 16

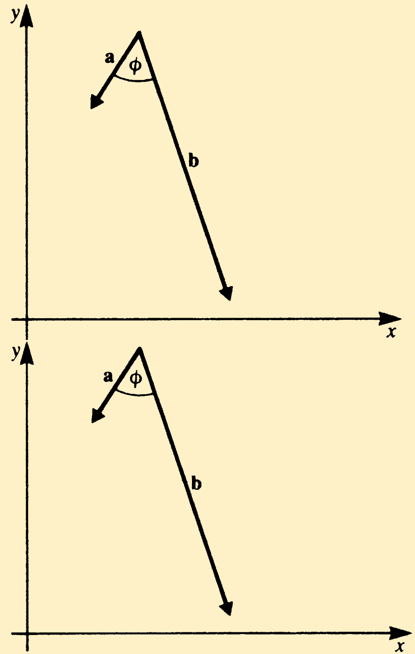
Further explanation required, or you have made errors 13

13

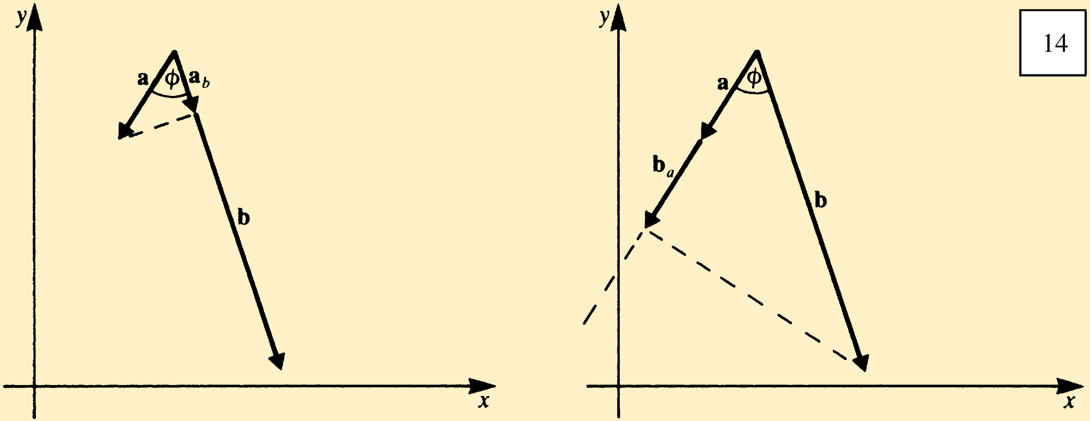
In order to evaluate the scalar product we need to know the projection of **a** on **b** or of **b** on **a**.

(i) Given **a** and **b** draw the projection of **a** on **b**.
$\mathbf{a}_b$ has a
magnitude = ...

(ii) Draw the projection of **b** on **a**.
$\mathbf{b}_a$ has a
magnitude = ...



14



14

$\mathbf{a}_b = a \cos \phi$ $\mathbf{b}_a = b \cos \phi$

The length of the projection of a vector depends on the included angle.

If the included angle is greater than 90° or $\pi/2$ the projected vector is opposite to the vector on which it is projected since the cosine is negative.

--------------------▷ 15

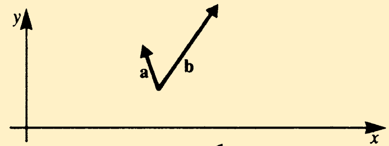
15

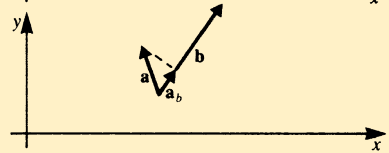
Evaluation of the scalar product.

Step 1: Choose **b** as reference vector.

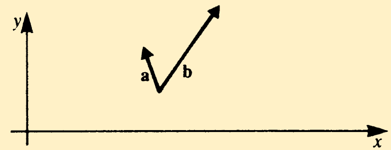
Step 2: Project **a** on the reference vector.

Step 3: The scalar product is obtained by multiplying the reference vector, **b**, by the component of the other vector, $\mathbf{a}_b$, on the reference vector.



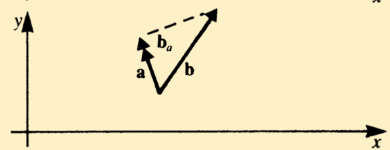


The reference vector can also be **a**. Complete the diagram with **a** as reference.



This is the solution.

We have the line of action of **a** as reference.



16

Different symbols are used in the literature for the scalar product. Pick out three correct notations from the following: 16

☐ $\mathbf{a} \cdot \mathbf{b}$

☐ $\mathbf{a}, \quad \mathbf{b}$

☐ $<\mathbf{a}, \quad \mathbf{b}>$

☐ $\mathbf{a} \times \mathbf{b}$

☐ $(\mathbf{a}, \quad \mathbf{b})$

17

$$\mathbf{a} \cdot \mathbf{b}$$

$$(\mathbf{a}, \mathbf{b})$$

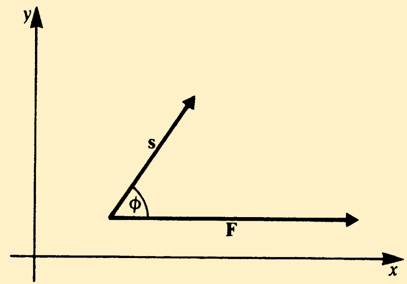
$$<\mathbf{a}, \quad \mathbf{b}>$$

17

A body is moved a distance **s** by a force **F** applied to it.

Calculate the work done by the force, given:

$$F = 6\,\text{N}$$
$$s = 2\,\text{m}$$
$$\phi = \pi/3 \text{ radians}$$



Distance is measured in metres (m), force in newtons (N).

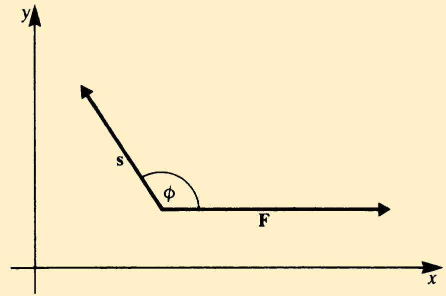
The work done is $\mathbf{F} \cdot \mathbf{s} =$ ....

18

$$\mathbf{F} \cdot \mathbf{s} = 6 \times 2 \cos \frac{\pi}{3} = 6\,\mathrm{N\,m}$$

18

If $\mathbf{F} = 6\,\mathrm{N}$
$\mathbf{s} = 2\,\mathrm{m}$
$\phi = 120^\circ$
then $\mathbf{F} \cdot \mathbf{s} =$ ....



19

$$\mathbf{F} \cdot \mathbf{s} = -6\,\mathrm{N\,m}$$

19

All correct → 23

Units not clear, explanation required → 20

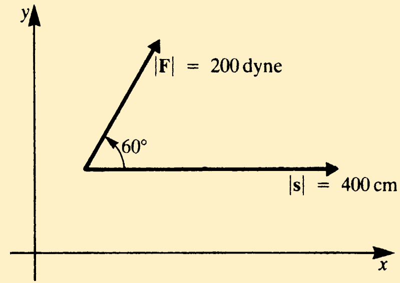
Concept of mechanical work not clear, explanation required → 22

20

Most quantities encountered in physics and engineering need to be given magnitudes and units. For vectors we need direction as well. Vectors are represented by arrows and operations can be performed with these arrows. Each arrow carries with it a unit, and when carrying out calculations we have to consider the units.

Examples:

	(SI unit)
force	newton (N)
velocity	m/s
displacement	m
electrical field intensity	V/m



A force applied to a body has a magnitude of 200 dyn and a displacement of 400 cm at an angle of 60° to the line of action of the force. Since $\cos\ 60° = 0.5$ the work done (W) is:

$W =$ ....

-------------------▷ 21

$$W = 40\,000\,\text{dyn cm} = 0.004\,\text{N m}$$ 21

In this case it was important not to forget the units!

Concept of mechanical work is not clear, explanation required

22

No difficulties

23

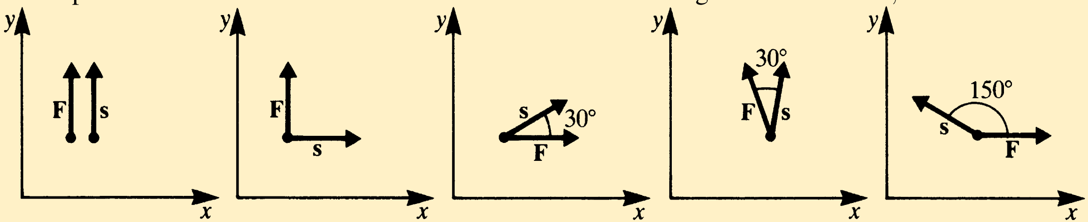
22

The mechanical work done by a force is

— the product of that force and the component of the displacement; or
— the product of the component of the force in the direction of the displacement and the displacement.

Work is considered to be positive if the displacement is in the same direction as the force and negative if these directions are opposite.

Thus mechanical work is obtained by forming the scalar product of the force and the displacement of the point of action. Calculate the work done **F** for the following cases if $F = 1\,\mathrm{N}$, $s = 1\,\mathrm{m}$.



$W_1 =$ .... $W_2 =$ .... $W_3 =$ .... $W_4 =$ .... $W_5 =$ ....

23

$W_1 = 1\,\mathrm{N\,m}, \quad W_2 = 0, \quad W_3 = 0.87\,\mathrm{N\,m}, \quad W_4 = 0.87\,\mathrm{N\,m}, \quad W_5 = -0.87\,\mathrm{N\,m}$

23

The inner or scalar product is an arithmetical operation which we have introduced by means of an example from mechanics.

But the concept of a scalar product is not restricted to applications in mechanics.

Given $a = 2$
$b = 1$
$\phi = 135°$
Calculate $\mathbf{a} \cdot \mathbf{b} =$ ....

ϕ		$\cos\phi$	$\sin\phi$
$0 = 0.00$	$0°$	1	0
$\frac{\pi}{6} = 0.52$	$30°$	0.866	0.500
$\frac{\pi}{4} = 0.79$	$45°$	0.707	0.707
$\frac{\pi}{3} = 1.05$	$60°$	0.500	0.866
$\frac{\pi}{2} = 1.57$	$90°$	0	1

24

$$\mathbf{a} \cdot \mathbf{b} = 2 \times 1 \ \cos\ 135^\circ = 2 \times 1(-0.707) = -1.414$$

24

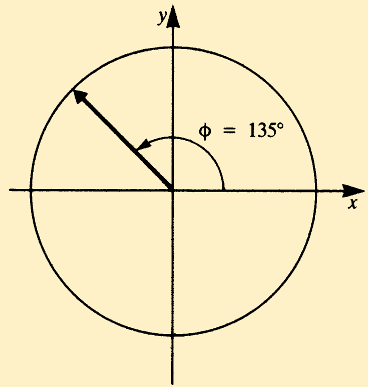
Correct result ▷ 27

Wrong result, explanation wanted ▷ 25

The scalar product is sometimes negative since the cosine of a particular angle ϕ can be negative. By using the unit circle you can easily determine the right sign. Let us consider an example: The unit circle is drawn and the angle $\phi = 135°$ is marked. 25

☐ $\cos 135° = 0.707$

☐ $\cos 135° = -0.707$



26

$$\cos 135^\circ = -0.707$$

26

Observing the sign of the cosine, try again to solve:

$$a = 2$$
$$b = 1$$
$$\phi = 135^\circ \quad \cos\phi = -0.707$$

$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

27

$$\mathbf{a} \cdot \mathbf{b} = 2 \times 1 \times (-0.707) = -1.414$$

27

Given: $a = 2, \quad b = 4$.

Calculate the scalar products for the following included angles.

$\alpha = 45^\circ$

$\mathbf{a} \cdot \mathbf{b} =$ ....

$\alpha = \frac{\pi}{2}$

$\mathbf{a} \cdot \mathbf{b} =$ ....

$\alpha = 120^\circ$

$\mathbf{a} \cdot \mathbf{b} =$ ....

ϕ		$\cos\phi$	$\sin\phi$
$0 = 0.00$	0°	1	0
$\frac{\pi}{6} = 0.52$	30°	0.87	0.5
$\frac{\pi}{4} = 0.79$	45°	0.71	0.71
$\frac{\pi}{3} = 1.05$	60°	0.50	0.87
$\frac{\pi}{2} = 1.57$	90°	0	1

28

$$\alpha = 45^\circ \mathbf{a}\cdot\mathbf{b} = 4\times 2\times 0.71 = 5.68$$
$$\alpha = \tfrac{\pi}{2} \mathbf{a}\cdot\mathbf{b} = 4\times 2\times 0 = 0$$
$$\alpha = 120^\circ \mathbf{a}\cdot\mathbf{b} = 4\times 2\times(-0.5) = -4$$

28

The following special cases are worth memorising:

— The scalar product of parallel vectors is equal to the product of their magnitudes.
— The scalar product of perpendicular vectors is zero.

It follows that:

If the scalar product of two vectors is zero, then the vectors are .... to each other, unless one (or both) of the vectors are ....

29

perpendicular
zero

29

The scalar product of a vector with itself implies parallelism:

$$\mathbf{a} \cdot \mathbf{a} = a^2$$

Given: $c = 3, \quad a = 3.$

If $\mathbf{c} \cdot \mathbf{a} = 9$, the included angle $\phi =$ ....

If $\mathbf{c} \cdot \mathbf{a} = 0$, the included angle $\phi =$ ....

Obtain $\mathbf{c} \cdot \mathbf{c} =$ ....

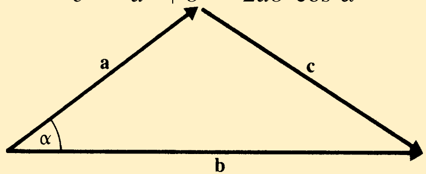
30

$$\phi = 0$$
$$\phi = 90^\circ$$
$$\mathbf{c} \cdot \mathbf{c} = 9$$

30

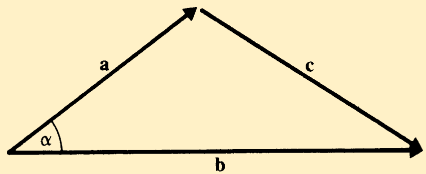
Try to prove the cosine rule by yourself:

$$c^2 = a^2 + b^2 - 2ab\ \cos\ \alpha$$



Successful ----------> 32

Hint required ----------> 31



We express **c** in terms of **a** and **b**

31

$$\mathbf{b} = \mathbf{a} + \mathbf{c}$$

hence $\mathbf{c} = \mathbf{b} - \mathbf{a}$.

We now form

$$\mathbf{c} \cdot \mathbf{c} = (\mathbf{b} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{a}),$$

hence $c^2 =$ ....

Check your result with the help of the textbook.

32

2.2 The Scalar Product in Terms of the Components of the Vectors

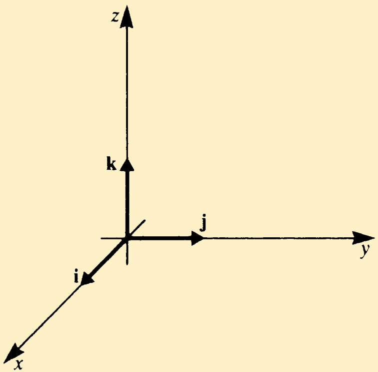
32

Objective: To calculate the scalar product of two vectors when their components are known.

In the textbook it is shown that the calculations of the scalar product can be simplified if the components of the vectors are known.

READ: 2.1.4 Scalar product in terms of the components of the vectors
Textbook pages 27–29

----------------▷ 33



The unit vectors along the Cartesian axes are shown. Obtain:

33

$\mathbf{i} \cdot \mathbf{i} = \ldots\ldots$

$\mathbf{i} \cdot \mathbf{j} = \ldots\ldots$

$\mathbf{i} \cdot \mathbf{k} = \ldots\ldots$

$\mathbf{j} \cdot \mathbf{i} = \ldots\ldots$

$\mathbf{j} \cdot \mathbf{j} = \ldots\ldots$

$\mathbf{j} \cdot \mathbf{k} = \ldots\ldots$

$\mathbf{k} \cdot \mathbf{i} = \ldots\ldots$

$\mathbf{k} \cdot \mathbf{j} = \ldots\ldots$

$\mathbf{k} \cdot \mathbf{k} = \ldots\ldots$

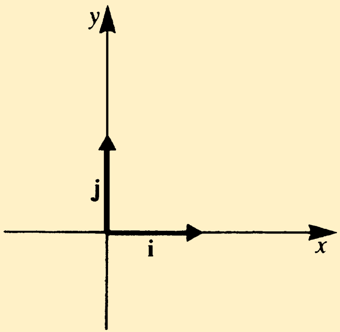
--------------------▷ 34

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}=1 & \mathbf{j}\cdot\mathbf{i}=0 & \mathbf{k}\cdot\mathbf{i}=0 \\ \mathbf{i}\cdot\mathbf{j}=0 & \mathbf{j}\cdot\mathbf{j}=1 & \mathbf{k}\cdot\mathbf{j}=0 \\ \mathbf{i}\cdot\mathbf{k}=0 & \mathbf{j}\cdot\mathbf{k}=0 & \mathbf{k}\cdot\mathbf{k}=1 \end{array}$$

34

All correct --------------------▷ 39

Errors or difficulties, or more examples wanted --------------------▷ 35



Let us look at the problem in the $x-y$ plane. The unit vectors have magnitude 1 and are directed along the axes as shown.

35

$\mathbf{i} \cdot \mathbf{i}$ is the scalar product of the unit vector with itself; both have the same direction.

Result: $\mathbf{i} \cdot \mathbf{i} = 1 \times 1 \times \cos 0^\circ = 1$

$\mathbf{i} \cdot \mathbf{j}$: The vectors are perpendicular to each other; therefore their scalar product is zero.

$$\mathbf{i} \cdot \mathbf{j} = 1 \times 1 \times \cos 90^\circ = 0$$

Now try the following for yourself:

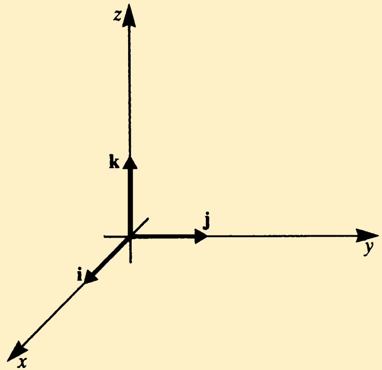
$\mathbf{j} \cdot \mathbf{j} =$ ....

$\mathbf{j} \cdot \mathbf{i} =$ ....

--------------------▷ 36

$$\mathbf{j} \cdot \mathbf{j} = 1$$
$$\mathbf{j} \cdot \mathbf{i} = 0$$

36



Now consider again the three-dimensional case and evaluate the scalar products of the following unit vectors:

$$\mathbf{k} \cdot \mathbf{i} = \ldots\ldots\ldots\ldots$$
$$\mathbf{k} \cdot \mathbf{j} = \ldots\ldots\ldots\ldots$$
$$\mathbf{k} \cdot \mathbf{k} = \ldots\ldots\ldots\ldots$$

37

$\mathbf{k}\cdot\mathbf{i} = 0$, the vectors are perpendicular to each other
$\mathbf{k}\cdot\mathbf{j} = 0$, also perpendicular to each other
$\mathbf{k}\cdot\mathbf{k} = 1$, the vectors have the same direction

37

If

$$\mathbf{a} = (1, \quad 4)$$
$$\mathbf{b} = (3, \quad 1)$$
$$\mathbf{a}\cdot\mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots$$

38

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 3 + 4 \cdot 1 = 7$$

38

We end the explanations, but if you still have difficulties read section 2.1.4 once more and/or ask someone for help.

39

If

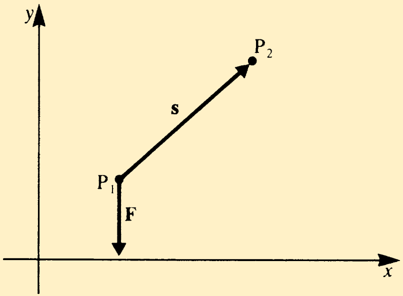
$$\mathbf{a} = (a_x, \quad a_y, \quad a_z)$$
$$\mathbf{c} = (c_x, \quad c_y, \quad c_z)$$
$$\mathbf{a} \cdot \mathbf{c} = \text{..............................}$$

39

40

$$\mathbf{a} \cdot \mathbf{c} = a_x c_x + a_y c_y + a_z c_z$$

40



$\mathbf{F} = (0, \quad -5\,\mathrm{N})$

$\mathbf{s} = (3\,\mathrm{m}, \quad 3\,\mathrm{m}) = \overrightarrow{P_1 P_2}$

The displacement is given by the vector **s**.

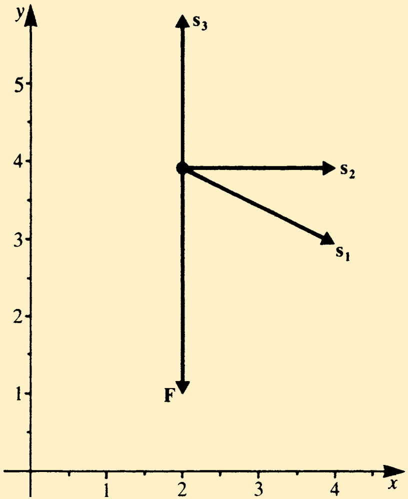
Calculate the work done in a displacement from P_1 to P_2.

$W = \mathbf{F} \cdot \mathbf{s} =$ ....

41

$$W = 0 \times 3 + (-5) \times 3 = -15\,\mathrm{N\,m}$$

41



Given: $\mathbf{F} = (0, \quad -5\,\mathrm{N})$.
Calculate the work done if:

$\mathbf{s_1} = (2\,\mathrm{m}, \quad -1\,\mathrm{m})$
$\mathbf{s_2} = (2\,\mathrm{m}, \quad 0\,\mathrm{m})$
$\mathbf{s_3} = (0\,\mathrm{m}, \quad 2\,\mathrm{m})$

$W_1 =$ ....
$W_2 =$ ....
$W_3 =$ ....

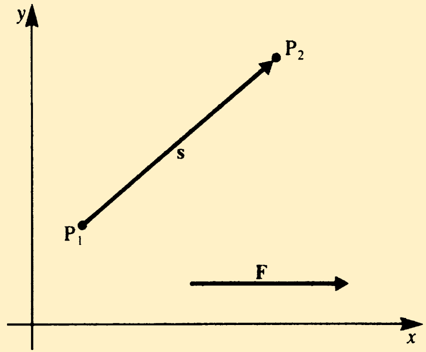
42

$$W_1 = 5\,\text{N m}, \quad W_2 = 0, \quad W_3 = -10\,\text{N m}$$

42

All correct 47

Errors, or further examples wanted 43



Consider a force **F** such that

43

$$\mathbf{F} = (F_x, \quad F_y) = (200\,\mathrm{N}, \quad 0)$$

The force is applied to a body and moves it from P_1 to P_2.

The displacement **s** has components s_x and s_y.

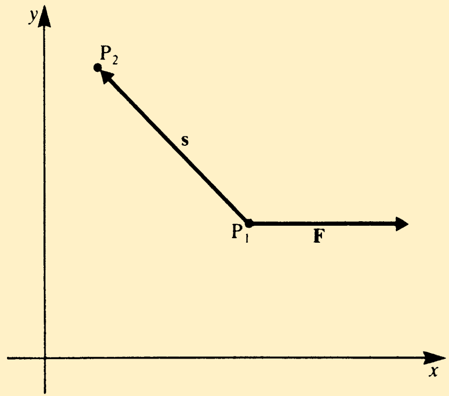
$$\mathbf{s} = (s_x, \quad s_y)$$
$$= (2\,\mathrm{km}, \quad 2\,\mathrm{km})$$

$W = \mathbf{F} \cdot \mathbf{s} =$ ....

44

$$\begin{aligned}\mathbf{F}\cdot\mathbf{s} &= (F_x s_x + F_y s_y)\\ &= (200\,\mathrm{N}\times 2\,\mathrm{km} + 0\,\mathrm{N}\times 2\,\mathrm{km})\\ &= 400\,\mathrm{N\ km} = 400\,000\,\mathrm{N\ m} = 4\times 10^5\,\mathrm{N\ m}\end{aligned}$$

44



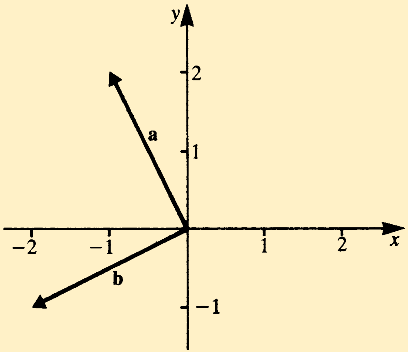
We now consider a different displacement **s** for the same force.

$$\mathbf{F} = (200\,\mathrm{N}, \quad 0)$$
$$\mathbf{s} = (-2\,\mathrm{km}, \quad 2\,\mathrm{km})$$
$$\mathbf{F}\cdot\mathbf{s} = \ldots\ldots\ldots\ldots\ldots\ldots$$

45

$$\mathbf{F} \cdot \mathbf{s} = -400\,\text{N Km} = -400\,000\,\text{N m} = -4 \times 10^5\,\text{N m}$$

45



Given: $\mathbf{a} = (-1, \quad 2)$

$\mathbf{b} = (-2, \quad -1)$

Obtain: $\mathbf{a} \cdot \mathbf{b} =$ ....

The vectors are .... to each other.

46

$$\mathbf{a} \cdot \mathbf{b} = 2 - 2 = 0$$

46

The vectors are perpendicular to each other.

Write down the scalar product of the following vectors:

$$\mathbf{F} = (F_x, \quad F_y, \quad F_z)$$
$$\mathbf{s} = (s_x, \quad s_y, \quad s_z)$$
$$\mathbf{F} \cdot \mathbf{s} = \text{................................}$$

Check your answer yourself using the textbook

47

Obtain the magnitude of the vector 47

$$\mathbf{c} = (3, \quad 2, \quad -2)$$

$$c = \text{..............................}$$

48

$$c = \sqrt{(3^2 + 2^2 + (-2)^2} = \sqrt{17} = 4.12 \text{ to 2 d.p.}$$

48

Calculate the scalar product of the two vectors

$\mathbf{a} = (4, \quad 1)$

$\mathbf{b} = (-1, \quad 4)$

$\mathbf{a} \cdot \mathbf{b} =$ ....

What is the value of the angle between $\mathbf{a}$ and $\mathbf{b}$?

$\phi =$ ....

49

$\mathbf{a} \cdot \mathbf{b} = 0, \quad \phi = 90^\circ$

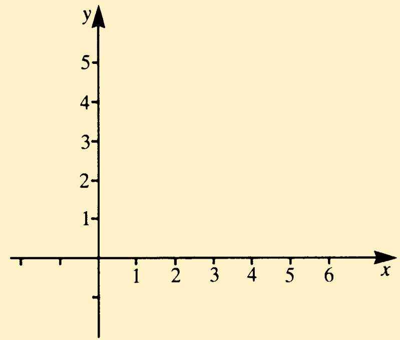
The vectors are perpendicular to each other.

49

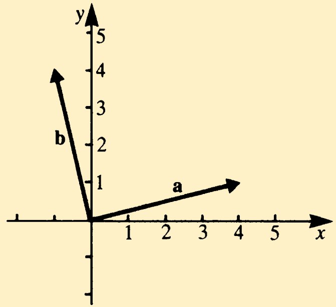
Check the result geometrically by completing the diagram

$\mathbf{a} = (4, \quad 1)$

$\mathbf{b} = (-1, \quad 4)$



50



50

The scalar product vanishes for perpendicular vectors. This relation is frequently used to check whether two vectors are perpendicular to each other. If the components of the vectors are known, we form the scalar product to see if it vanishes.

Given:

$$\mathbf{a} = (a_x, \quad a_y)$$
$$\mathbf{a}_1 = (-a_x, \quad -a_y)$$
$$\mathbf{a}_2 = (-a_x, \quad a_y)$$
$$\mathbf{a}_3 = (a_y, \quad -a_x)$$
$$\mathbf{a}_4 = (-a_y, \quad a_x)$$

Which of the vectors are perpendicular to **a**? ....

--------------------▷ 51

a and $\mathbf{a_3}$ as well as **a** and $\mathbf{a_4}$ are perpendicular. 51

Let $\mathbf{F} = (1\,\text{N}, \quad -1\,\text{N}, \quad 2\,\text{N})$.

Which position vectors are perpendicular to **F**?

$$\begin{aligned}
\mathbf{s_1} &= (2\,\text{m}, \quad 1\,\text{m}, \quad 1\,\text{m}) \\
\mathbf{s_2} &= (-1\,\text{m}, \quad 1\,\text{m}, \quad 1\,\text{m}) \\
\mathbf{s_3} &= (1\,\text{m}, \quad 1\,\text{m}, \quad -2\,\text{m}) \\
\mathbf{s_4} &= (3\,\text{m}, \quad 1\,\text{m}, \quad -1\,\text{m})
\end{aligned}$$

Perpendicular vectors ....

52

$\mathbf{s}_2$ and $\mathbf{s}_4$ are perpendicular to $\mathbf{F}$. 52

All correct 54

Additional explanation

The scalar product of two vectors which are perpendicular to each other is zero. We use this fact to check whether two vectors $\mathbf{F}$ and $\mathbf{s}$ are perpendicular to each other.

Let $\mathbf{F} = (1\,\mathrm{N}, \quad -1\,\mathrm{N}, \quad 2\,\mathrm{N})$.

It is required to check if $\mathbf{s} = (2\,\mathrm{m}, \quad 1\,\mathrm{m}, \quad 1\,\mathrm{m})$ is perpendicular to $\mathbf{F}$.

To check, we know that the scalar product must be zero.

$$\begin{aligned} \text{Now } \mathbf{F}\cdot\mathbf{s} &= 1\,\mathrm{N}\times 2\,\mathrm{m} - 1\,\mathrm{N}\times 1\,\mathrm{m} + 2\,\mathrm{N}\times 1\,\mathrm{m} \\ &= 3\,\mathrm{Nm} \end{aligned}$$

Are $\mathbf{F}$ and $\mathbf{s}$ perpendicular to each other?

☐ Yes
☐ No

53

No. 53

Since $\mathbf{F} \cdot \mathbf{s}$ is not zero the vectors are not perpendicular to each other. If we have to check whether two vectors are perpendicular to each other, we evaluate the scalar product. If $\mathbf{F} \cdot \mathbf{s} = 0$ then the vectors $\mathbf{F}$ and $\mathbf{s}$ are perpendicular to each other, provided $\mathbf{F} \neq 0$ and $\mathbf{s} \neq 0$.

Further exercises can be found at the end of each chapter in the textbook.
If you have difficulties working through the exercises always consult the corresponding section of the textbook.

--------------------▷ 54

54

Here are a few comments on working independently and in a group.

Group work and working alone are not mutually exclusive methods of study. They complement each other.

Independent study is appropriate when facts must be memorised, when calculations have to be checked, when proofs must be studied and when coherent material must be worked out.

Group work is suitable:

— in the preparation phase, for identifying and analysing problems;
— for the discussion of results and for solving new problems with new methods;
— as a means of checking up on others.

Group work can prepare for and guide private study. Working in a group is particularly fruitful when the work is prepared by the individuals so that all members of the group can take part in the discussion — all being equally competent to do so. Group work cannot replace private study and, similarly, private study cannot replace certain functions of work done in a group.

--------------------▷ 55

2.3 The Vector Product; Torque

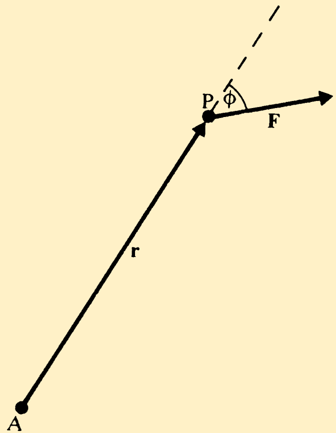
55

Objective: Concepts of vector product, outer product, torque.

It is shown in the textbook how to calculate the torque for the general case of arbitrary forces and any points of application.

READ: **2.2.1 Torque**
2.2.2 Torque as a vector
Textbook pages 30–32

--------------------▷ 56



A force **F** is applied to a body at point P causing it to rotate about an axis through A.

56

In order to calculate the torque we resolve the force into its component perpendicular to the radius vector **r** and its component in the direction of **r**.

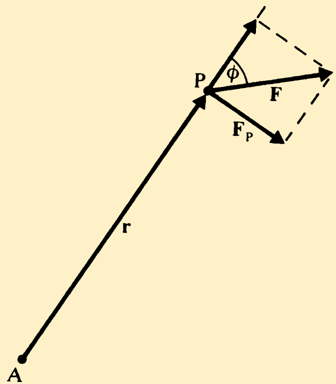
Complete the diagram by drawing the components of **F**.
The magnitude of the component perpendicular to **r** is

$$F_{\mathrm{P}} = \text{..............................}$$

57

$$F_P = F \sin \phi$$

57



The component of **F** in the direction of **r** does not contribute to the torque; it has no turning effect on the body. Only the component perpendicular to **r** is to be considered.

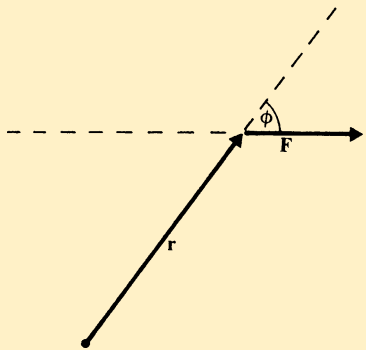
Hence the magnitude of the torque C is:

$C =$ ....

58

$$C = rF \sin \phi$$

58



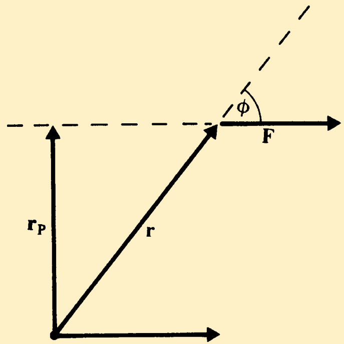
Draw the components of **r** in the sketch, one perpendicular to **F** and one parallel to **F**. The magnitude of the former is:

$r_P =$ ....

59

$$r_P = r \sin \phi$$

59



By this construction the problem is again reduced to the special case of force and radius being perpendicular to each other.

Hence the magnitude of the torque is:

$$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 60

$$C = rF \sin \phi$$

60

When calculating torque the vectors cannot be considered as free vectors. We can only shift them along their line of action. A parallel shift of vectors is not permissible in this case.

Torque is

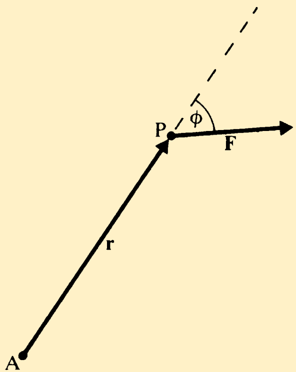
☐ a scalar

☐ a vector

--------------------▷ 61

vector

61



Determine the magnitude and direction of the torque defined in the figure.

Magnitude, $C =$ ....

Direction of **C**:

(1) **C** is .... to **r** and **F**.

(2) When **r** is rotated towards **F** in accordance with the right-hand rule, a right-handed screw moves forward in the direction ....

62

$C = rF \sin\phi$ (1) Perpendicular
(2) perpendicularly into the plane of the page.

62

The right-hand screw rule is perhaps a little difficult to formulate clearly; it is easier to demonstrate. We proceed as follows:

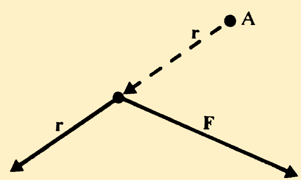
(1) We shift **r** and **F** along their line of action to a common point.
(2) **r** is turned in the shortest way so that it coincides with **F**.
(3) The rotation takes place in the way a right-handed screw would move.

Anyone who has had occasion to use a screwdriver will have developed a feel for right-handed screw threads; we know from experience that such a screw would move forward and we use this fact to define the direction of **C**.

The torque in the new figure on the right

$\mathbf{C} = \mathbf{r} \wedge \mathbf{F}$

is perpendicular to the page of this text and points

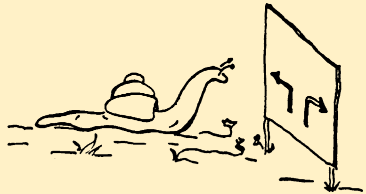


☐ upwards
☐ downwards

63

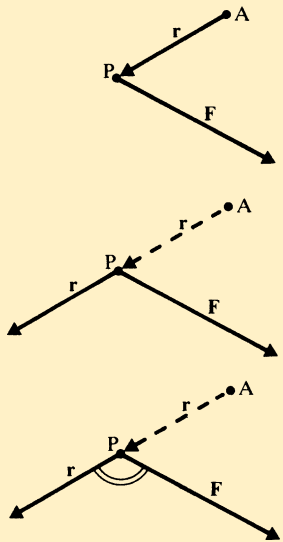
The vector **C** points upwards.

63



Further explanation required 64

I want to go on 65



64

We proceed one step at a time.

Given: the shaft axis A of a body, the force **F** and the point of application **P**.

Step 1: Shift **r** along its line of action so that **r** and **F** meet at a common point, **P**.

Step 2: A right-handed screw at **P** operating from below the page would rotate **r** towards **F** and the screw would move out of the paper towards the reader; this is the direction of the torque vector **C**.

65

2.4 Definition of the Vector Product

65

Objective: Concepts of outer product, vector product, null vector, calculation of the vector product (magnitude and direction).

READ: **2.2.3 Definition of the vector product**
2.2.4 Special cases
2.2.5 Anti-commutative law for vector products
Textbook pages 32–34

--------------------▷ 66

The need to calculate a torque leads to the need to define a new product of vectors. 66

We call this product .... product

or .... product

To distinguish this product from the inner or scalar product we need new symbols.

Two such symbols are:

....

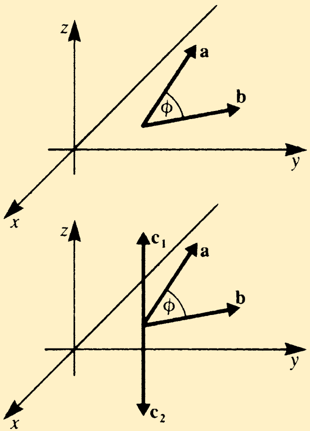
or ....

67

Vector product or outer product

If $\mathbf{A}$ and $\mathbf{B}$ are two vectors,
the vector product is written $\mathbf{A} \times \mathbf{B}$ or $\mathbf{A} \wedge \mathbf{B}$.

67



Given: two vectors $\mathbf{a}$ and $\mathbf{b}$ in the $x-y$ plane, and ϕ the angle between them. If $\mathbf{c}$ is their vector product then

$\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$ has the following properties:

Magnitude $c =$ ....

It is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

Its direction follows the right-hand rule and is:

☐ $\mathbf{c}_1$

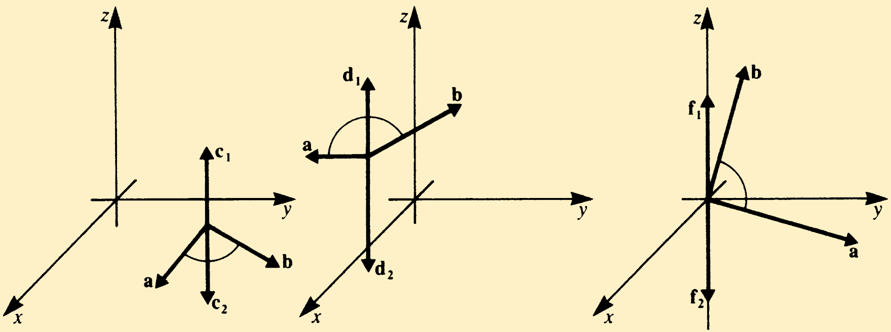
☐ $\mathbf{c}_2$

--------------------▷ 68

$$|\mathbf{c}| = |\mathbf{a} \wedge \mathbf{b}| = ab \sin\phi, \quad \mathbf{c}_2$$

68

Indicate the direction of the vector product $\mathbf{a} \wedge \mathbf{b}$ for the cases below. $\mathbf{a}$ and $\mathbf{b}$ lie in the $x-y$ plane.



☐ $\mathbf{c}_1$
☐ $\mathbf{c}_2$

☐ $\mathbf{d}_1$
☐ $\mathbf{d}_2$

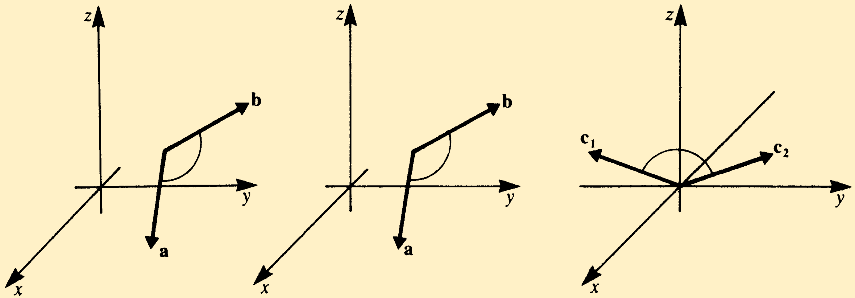
☐ $\mathbf{f}_1$
☐ $\mathbf{f}_2$

-------------------▷ 69

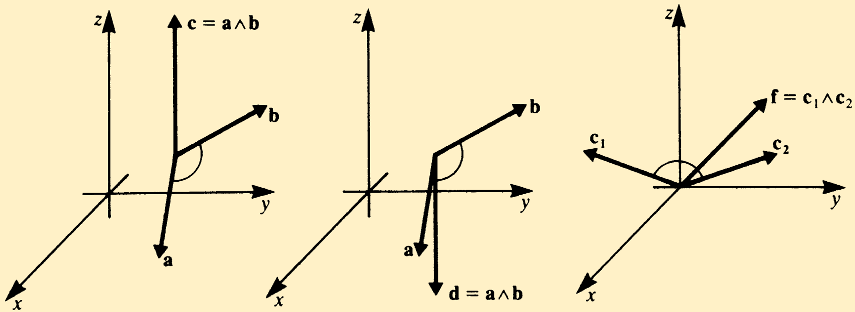
$\mathbf{c}_1$, $\mathbf{d}_2$, $\mathbf{f}_1$

69

Insert in the sketches below the directions of the vector products. **a** and **b** lie in the $x-y$ plane and $\mathbf{c}_1$ and $\mathbf{c}_2$ in the $y-z$ plane.



70

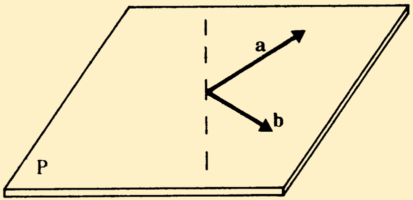


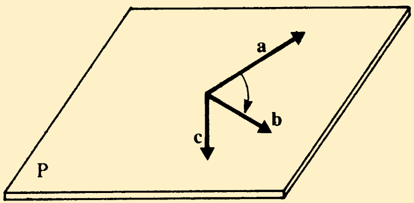
70

All correct: right-hand screw rule fully understood ---▷ 75

Errors, further explanation required ---▷ 71

The result of the vector product $\mathbf{a} \wedge \mathbf{b}$ is a new vector $\mathbf{c}$.$\mathbf{a}$ and $\mathbf{b}$ define a plane P as shown in the figure. $\mathbf{c}$ is perpendicular to that plane and its direction is defined by the right-handed screw rule. 71



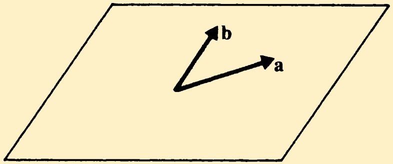


Steps required:

1: Turn the first vector, $\mathbf{a}$ in this case, towards the second vector $\mathbf{b}$ until they coincide. Take the shortest route. The direction of the vector product $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$ is the same as the displacement of a right-handed screw.

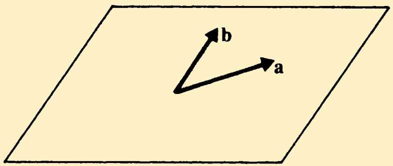
2: To determine the sense of the vector $\mathbf{c}$ we imagine the way a right-handed corkscrew would advance as we turn it.

-------------------▷ 72



Insert the direction of the vector product $\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$. The order of the vectors is important. To obtain the direction of $\mathbf{c}$ we always start with the first vector, i.e. $\mathbf{a}$ towards $\mathbf{b}$. Thus the direction of $\mathbf{c}$ depends on the order.

72



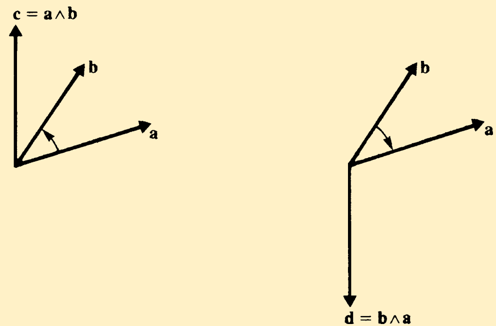
Insert now the direction of

$$\mathbf{d} = \mathbf{b} \wedge \mathbf{a}$$

Note the change in the order.

--------------------▷ 73

73



The direction of the vector product depends on the order in which we take the vectors. This is quite different from the scalar product where

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

For the vector product, however,

$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

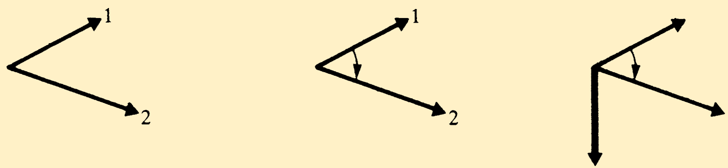
74

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

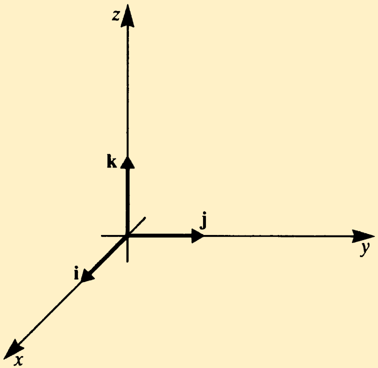
74

To obtain the direction of the vector product proceed as follows:

(1) Consider the first vector.
(2) Rotate the first vector towards the second in the shortest way.
(3) Imagine this rotation to take place as if you were turning a right-handed screw. The advance of this screw defines the direction of the vector product.



75



The diagram shows a three-dimensional coordinate system with the unit vectors **i**, **j** and **k** on the axes.

75

Determine the magnitude of the vector product

$|\mathbf{i} \wedge \mathbf{j}| =$ ....

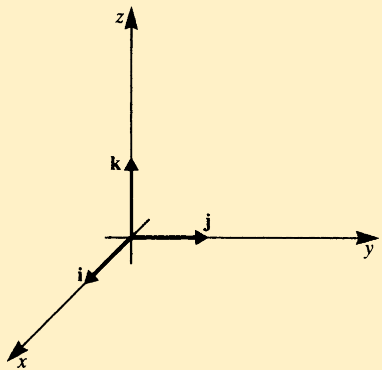
The vector product

$$\mathbf{i} \wedge \mathbf{j} = \text{...............................}$$

76

$$|\mathbf{i} \wedge \mathbf{j}| = 1$$
$$\mathbf{i} \wedge \mathbf{j} = \mathbf{k}$$

76



Write down the vector products of the following unit vectors:

$\mathbf{j} \wedge \mathbf{i} =$ ....

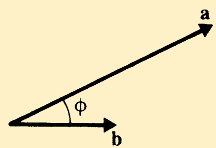
$\mathbf{i} \wedge \mathbf{k} =$ ....

$\mathbf{j} \wedge \mathbf{k} =$ ....

77

$$\begin{aligned} \mathbf{j}\wedge\mathbf{i} &= -\mathbf{k} \quad (\text{opposite of } \mathbf{i}\wedge\mathbf{j}) \\ \mathbf{i}\wedge\mathbf{k} &= -\mathbf{j} \\ \mathbf{j}\wedge\mathbf{k} &= \mathbf{i} \end{aligned}$$

77



Given: $a = 4$

$b = 2$

$\phi = \frac{\pi}{6}$

$|\mathbf{a}\wedge\mathbf{b}| =$ ....

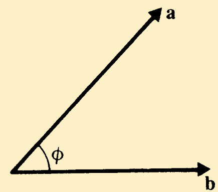
78

$$|\mathbf{a} \wedge \mathbf{b}| = ab \sin \phi = 4 \times 2 \times \sin 30^\circ = 4 \times 2 \times 0.5 = 4$$

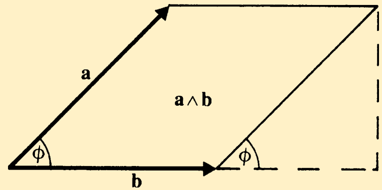
78

The magnitude of the vector product has a geometrical meaning; it is the area of a certain plane surface.

Sketch the area determined by $\mathbf{a} \wedge \mathbf{b}$.



79



79

We had to know that $|\mathbf{a} \wedge \mathbf{b}| = ab\ \sin\ \phi$.

We can learn it in two ways:

(1) By learning the geometrical meaning, i.e. that $\mathbf{a} \wedge \mathbf{b}$ represents the surface area of a parallelogram. Hence with the help of a diagram we can reconstruct the formula.

(2) By memorising the formula.

The first method forces us to reason, and hence understand the problem, and is far superior to the second one.

80

The vector product (or outer product) has magnitude

80

$$|\mathbf{a} \wedge \mathbf{b}| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The scalar product (or inner product) has magnitude

$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Try to derive both formulae geometrically.

--------------------▷ 81

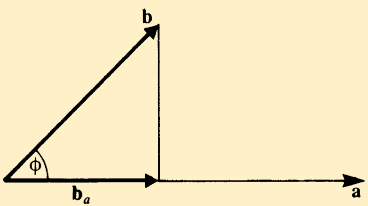
$$|\mathbf{a} \wedge \mathbf{b}| = ab \ \sin \phi$$
$$\mathbf{a} \cdot \mathbf{b} = ab \ \cos \phi$$

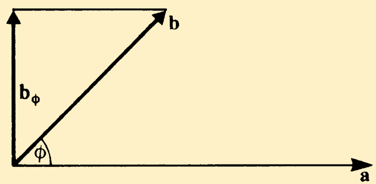
81

The two diagrams below also help us to calculate the scalar and vector products. For **a** and **b**, the magnitude of the:

scalar product = **a** times the projection of **b** on to **a**;
vector product = **a** times the projection of **b** on to the perpendicular to **a**.

$\mathbf{a} \cdot \mathbf{b} =$ .... $|\mathbf{a} \wedge \mathbf{b}| =$ ....





82

$$\mathbf{a} \cdot \mathbf{b} = ab \ \cos \ \phi$$
$$|\mathbf{a} \wedge \mathbf{b}| = ab \ \sin \ \phi$$

82

Using the definitions you should be able to answer the following questions:

$$\mathbf{a} \wedge \mathbf{a} = \text{..............................}?$$

$$\mathbf{a} \cdot \mathbf{a} = \text{..............................}?$$

83

$$\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$$
$$\mathbf{a} \cdot \mathbf{a} = a^2$$

83

It is time to have a break!

Before the break try to recapitulate briefly what you have learned. Write down the important concepts.

Now fix the duration of the break.

During the break it is important that you do something totally different.

Go and make yourself a cup of coffee or tea, do some exercises (physical ones!) or go for a short walk. If you prefer, go and play your piano or your guitar or put a record on your stereo.

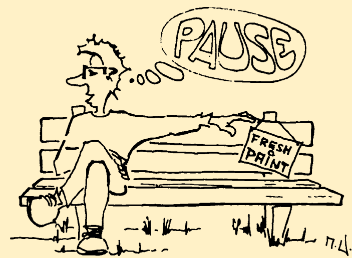
Do something different during your break! Read on for an explanation.

84

Psychologists have demonstrated that learning is impaired if similar subjects are studied at the same time. For example, a foreign correspondent is studying Spanish and Italian at the same time. She believes that the similarity between the two languages will promote the learning process. Unfortunately she is mistaken; she notices that when studying Spanish, Italian words keep coming into her mind and vice versa. This makes her feel unsure because she does not know if the words that come to her mind belong to Spanish or Italian. This phenomenon is called *interference*. Such interference obstructs the learning process, and increases the learning time and affects one's confidence. 84

Interference is prevented by doing something totally different during your break, something which has nothing to do with mathematics.

So have a break now and fix its duration.



End of your break: ....

--------------------▷ 85

It is much easier to fix the duration of the break than to keep to it. 85

Look at your watch and compare the time that you fixed with the time now.

Are they in agreement?

If yes: splendid!

If not: it is not too serious.

However, the difference between intentions and actions should not be allowed to accumulate.

--------------------▷ 86

2.5 Resultant of Several Torques Applied to a Body Components of the Vector Product

86

The first section is intended to show you an application of the vector product.

The second shows the calculation of the vector product if the vectors are represented by their components.

READ: 2.2.6 Components of the vector product
Textbook pages 34–35

Write things down for yourself as you read. If you are not familiar with determinants follow the example in the textbook by applying equation (2.11b).

--------------------▷ 87

With the help of the textbook or from your own notes evaluate the vector product $\mathbf{a} \wedge \mathbf{b}$ for

87

$$\mathbf{a} = (2, \quad 1, \quad 1) \quad \text{and} \quad \mathbf{b} = (-1, \quad 2, \quad 1):$$

$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

88

$$\mathbf{a} \wedge \mathbf{b} = (1 \times 1 - 1 \times 2)\mathbf{i} + (-1 \ - 2)\mathbf{j} + (4 + 1)\mathbf{k} = -\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

88

Note: We have used equation (2.11a) of the textbook.

Write the answer in the shorthand way of expressing a vector, i.e.

$$\mathbf{a} \wedge \mathbf{b} = (........,\ ....,\ ....)$$

89

$$\mathbf{a} \wedge \mathbf{b} = (-1, \quad -3, \quad 5)$$

89

A body rotates about the x-axis with an angular velocity $\omega = (\omega, \quad 0, \quad 0)$.

Calculate the velocity of a point $\mathrm{P} = (1, \quad 1, \quad 0)$.

Hint: the velocity $\mathbf{v} = \omega \wedge \mathbf{r}$

$\mathbf{r}$ is the position vector of P.

$\mathbf{v} = (\ldots\ldots.,\quad \ldots\ldots.,\quad \ldots\ldots.)$

90

$$\mathbf{v} = (0, \quad 0, \quad \omega)$$

90

The following are exercises on the whole chapter.

Write down the formulae for the vector product

$$\mathbf{a} \wedge \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and for the scalar product

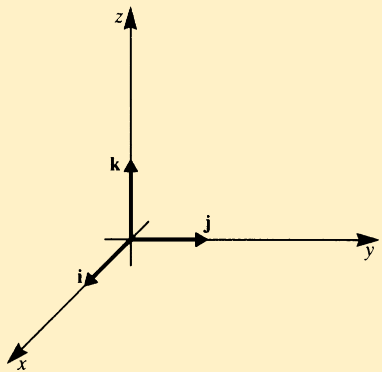
$$\mathbf{a} \cdot \mathbf{b} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

given the magnitude of the vectors and the included angle ϕ.

--------------------▷ 91

$$\mathbf{a} \wedge \mathbf{b} = ab \sin \phi$$
$$\mathbf{a} \cdot \mathbf{b} = ab \cos \phi$$

91



Write down the vector products of the unit vectors:

$$\mathbf{i} \wedge \mathbf{j} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{i} \wedge \mathbf{k} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{i} \wedge \mathbf{i} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and the scalar products

$$\mathbf{i} \cdot \mathbf{j} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{i} \cdot \mathbf{k} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{i} \cdot \mathbf{i} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 92

$$\mathbf{k}, \quad -\mathbf{j}, \quad \mathbf{0}$$
$$0, \quad 0, \quad 1$$

92

Which maxim for the delivery of a lecture was cited at the beginning of this chapter?

....

....

....

93

Say what you are going to say, say it, say what you have said. 93

The vector $\mathbf{a} = (2, \quad 3, \quad 1)$ has magnitude $a =$ ....

94

$a = \sqrt{14} = 3.74$ (to two decimal places)

94

A body moves from P_0 to P_1 under the action of a force $\mathbf{F} = (5\,\text{N}, \quad 0)$. The change in its position is $\mathbf{s} = (3\,\text{m}, \quad 0)$.

The work $W =$ ....

--------------------▷ 95

$$W = 15\,\mathrm{N\,m}$$

To solve such problems it is important to consider the direction in which the displacement takes place.

Extreme cases occur whenever:

(1) displacement and force have the same or opposite directions, or
(2) displacement and force are perpendicular to each other.

It is advisable to draw sketches.

Some final remarks:

Planning your work is important; it helps you to learn efficiently and to save time and energy.

You can fix times and dates and monitor them.

A notebook and pen help!

Chapter 3
Functions

3.1 The mathematical concept of functions and its meaning in physics and engineering

For many readers, most parts of chapter 3 will be a mere recapitulation of well-known facts.
If, however, you are not sufficiently familiar with its content, we suggest that you carefully take notes of all new concepts and notations.

Now start to study the textbook

READ 3.1 The mathematical concept of functions and its meaning in physics and engineering
Textbook page 39-42

When you have finished return to this study guide frame ------------------▷ 2

2

After having read the textbook, answer the following questions in order to ensure that you have grasped the main ideas.
Very often we understand facts but we fail to memorize them.

The expression $y = f(x)$ is called …………..
The particular parts are named as follows

x: ……………

y: ……………

$f(x)$ …………

Check your answers by going to frame 3 ------------------▷ 3

functional expression
x is the independent variable
y is the dependent variable
$f(x)$ is the defining expression

3

The set of all x values for which the function is defined is called ....

The set of all corresponding y-values is called …………..

-------------------▷ 4

Domain of definition, or simply domain

range or codomain

4

If you are still unsure how to use these concepts and notations consult your notes or the textbook.
Click your answer

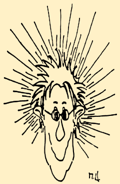
With a function we assign to a given x-value:

☐ one and only one y------------------value ------------------▷ 5

☐ one or more y------------------values ------------------▷ 6

You are right. Functions assign to a given x-value one and only one y-value

5



Which are functions?

$y = x^2 + 2$ ☐

$y = \pm\sqrt{x^2 + 2}$ ☐

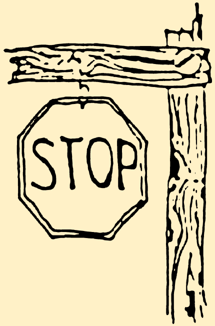
$y = \frac{1}{x}$ ☐

$y = \frac{1}{x} \pm \sqrt{x}$ ☐

$y = \frac{1}{x^2 + 1}$ ☐

Now go to frame 6

6



Sorry your answer is not right. Functions assign to a given x-value one and only one y-value. Functions are unambiguous. But unfortunately during the last decades the term function changed a bit. Thus, in engineering literature you may find the terms "two-valued" or "many valued function." In modern terminology these are "relationships."

An ambiguous relation is $y = \sqrt{x+3}$

For $x = 1$ this results in $y = \pm 2$

Is the following term unambiguous?

$$y = \left(4 \pm \sqrt{\frac{1}{x}}\right)^2$$

☐ yes

☐ no

------------------▷ 7

No: $y=\left(4\pm\sqrt{\frac{1}{x}}\right)^2$ is ambiguous

7

Select and mark the functions.

- ☐ $y=x^2+2$
- ☐ $y=\pm\sqrt{x^2+2}$
- ☐ $y=\dfrac{1}{x}$
- ☐ $y=\dfrac{1}{x}\pm\sqrt{x}$
- ☐ $y=\dfrac{1}{x^2+1}$

8

Only the following are functions

8

$y = x^2 + 2$ $\qquad y = \frac{1}{x}$ $\qquad y = \frac{1}{x^2 + 1}$

The symbol $\pm$ means that there are two values. But often this symbol is omitted because everyone is supposed to know that a square root has two solutions. You can transform the unambiguous term $y = \sqrt{x^2 + 2}$ to a function if you limit yourself to using only the positive or the negative result of the root. In this case you must denote this.

Transform the expression $y = \frac{1}{\sqrt{x}}$ into functions.

$y_1 =$ ……… $\qquad y_2 =$ ………

------------------▷ 9

$$y_1 = +\frac{1}{\sqrt{x}} \qquad y_2 = -\frac{1}{\sqrt{x}}$$

9

The equation $y = x^2$ is a function.

Its argument is ……………..
Its dependent variable is ..……………..
Its domain of definition is ……………
Its codomain is ……………

--------------------▷ 10

10

x
y
$-\infty \leq x \leq +\infty$
$o \leq y \leq +\infty$

Control of your knowledge using simple questions is important to eliminate errors or misunderstanding from the beginning. Lectures cannot help you with this task.
If your answer was wrong this is not important but you must do something to eliminate the reason for your error.

------------------ ▷ 11

3.2 Graphical representation of functions

11

READ **3.2. Graphical representation of functions**
3.2.1 Coordinate system, position vector
3.2.2 The linear function; The straight line
Textbook page 42–44

After your study return to this study guide

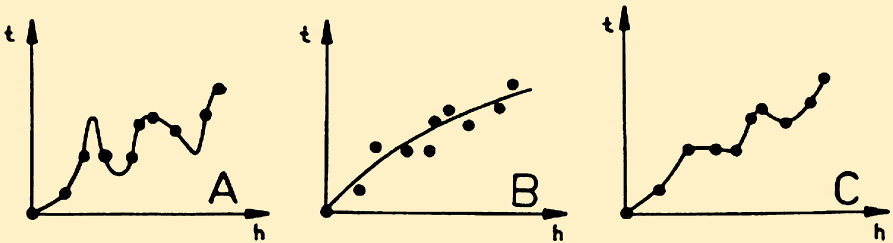
------------------▷ 12

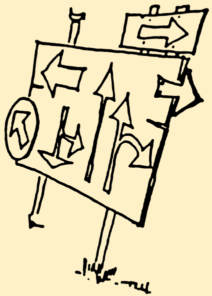
12

A short introductory and entertaining problem will be presented first.

A body falls from a certain height h. The body hits the ground after a time t. Both values are measured.

The three graphs show the measured values and a curve to represent the result.
Which is the best curve?





☐ A --------------------▷ 13

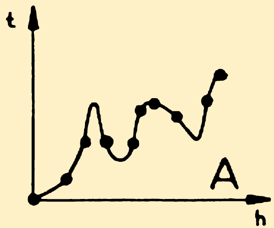
☐ B --------------------▷ 14

☐ C --------------------▷ 15

No, no, no

13

Perhaps you are curious to know what will be said here.
You chose this curve



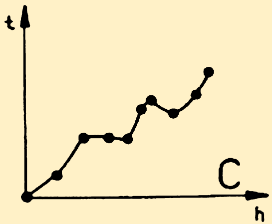
It is highly improbable that this curve with its irregular pattern represents the relationship between height of fall and time of fall.

Go back and try again

-------------------▷ 14

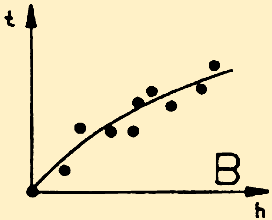
You chose this curve

14



This curve represents all measured values, but it would not be chosen by physicists or engineers. We know that the accuracy of measurements depends on the used instruments and we always have errors in measurements. In our case we assume that the time of fall increases with the height of fall. Thus, we prefer curve B as the best fitting curve and regard the deviations from the fitting curve as errors of measurement. In chapter 21 "Theory of errors" we will discuss this topic again.

------------------▷ 15



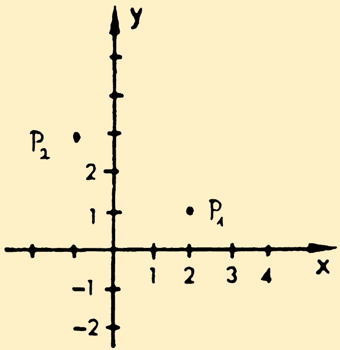
15

Very good. This was the right choice.
We know that all measurements are subject to measurement errors.
To construct fitting curves involves knowledge of the used instruments and insight into the physical relationships.
In chapter 21 “Theory of errors” we will treat methods to deal with errors of measurement.

------------------▷ 16

Give the coordinates of point P_1 and P_2

16



P_1 = ………………..

P_2 = ……………….

The *x*-coordinate is named ……………

The *y*-coordinate is named ……………

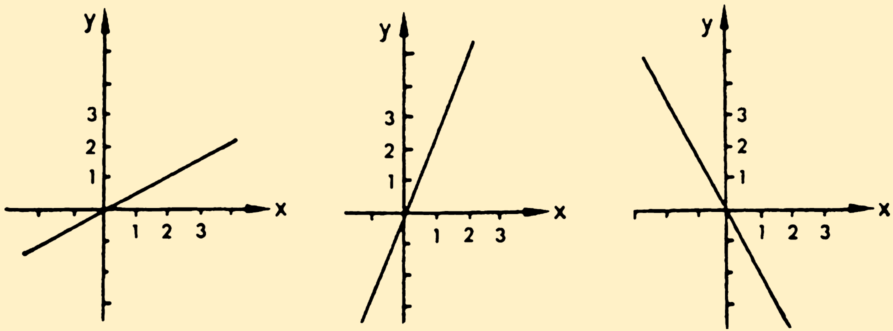
------------------▷ 17

$P_1 = (2, 1)$ $P_2 = (-1, 3)$ 17

17

abscissae
ordinate

Give the equations of the three straight lines



$y_1 = \ldots.$ $y_2 = \ldots.$ $y_3 = \ldots$

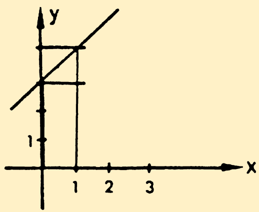
If this question is too simple for you skip it and go to 23

Answers and more exercises 18

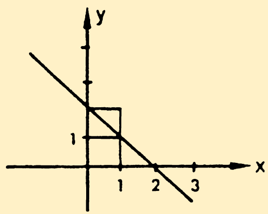
$y_1 = 0.5x$ $y_2 = 2.5x$ $y_3 = -2x$

18

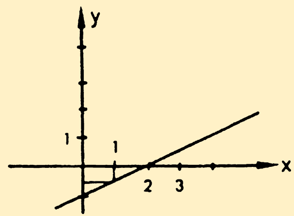
It is more difficult to give the equation if the straight line does not cross the origin of the coordinate system.
In this case we determine first the constant b and afterwards the slope *a*.
Give the equations



$y_1 =$ ....



$y_2 =$ ....



$y_3 =$ ....

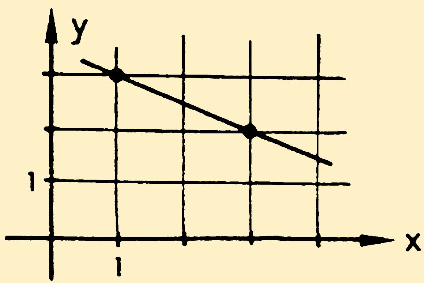
------------------▷ 19

$y_1 = x + 3$ $y_2 = x + 2$ $y_3 = \frac{1}{2}x - 1$

19

In case of difficulties go back to the textbook and solve this task according to the text. Having done this solve the following exercises in which you can calculate the slope by dividing the increase or decrease of y for a given section of the abscissae. To do this you have to choose appropriate sections of the abscissae.

Determine the slope



$y = ax + b$

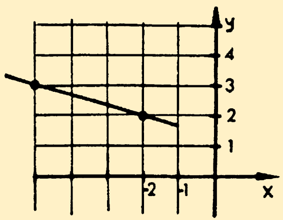
$a = \ldots\ldots\ldots\ldots$

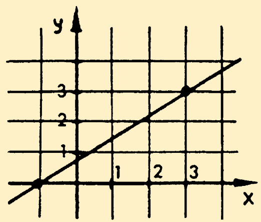
-------------------▷ 20

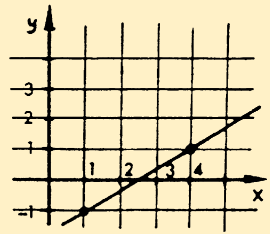
$a = -\frac{1}{2}$

20

Determine the slope a for the given straight lines.







$a_1 = \ldots.$ $a_2 = \ldots.$ $a_3 = \ldots.$

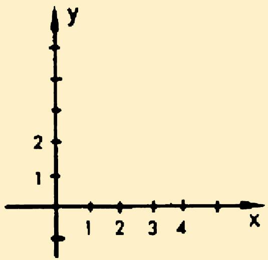
--------------------▷ 21

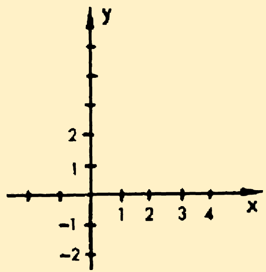
$a_1 = -\dfrac{1}{3}$ $a_2 = \dfrac{3}{4}$ $a_3 = \dfrac{2}{3}$

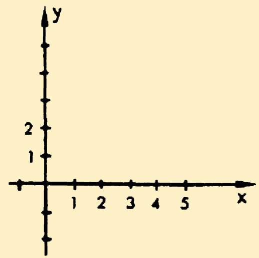
21

Give the graphical representation

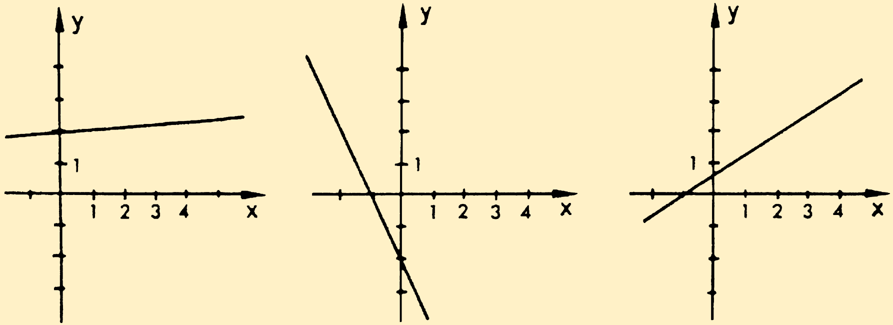
$y_1 = 0{,}1x + 2$ $y_2 = -2x - 2$ $y_3 = \dfrac{x+1}{2}$







------------------ ▷ 22



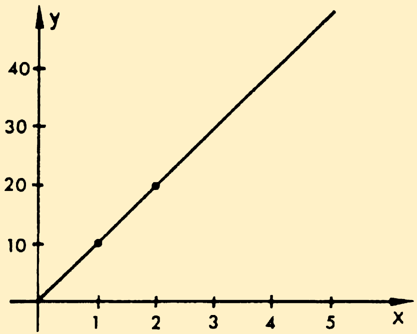
22

In case of difficulties study again section 3.3.1 in the textbook.

23

In practice the calibration of the coordinate axis has to be chosen according to the problem. Give the function that represents the graph.

23

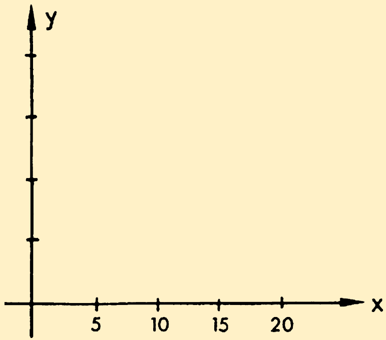


$y = \ldots\ldots\ldots\ldots$

--------------------▷ 24

$y = 10x$

24

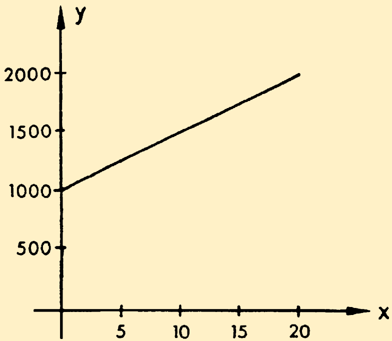


Given $y = 50x + 1000$

Domain of x: $0 \le x \le 20$

Give an appropriate calibration for the ordinate and sketch the straight line

-------------------▷ 25



25

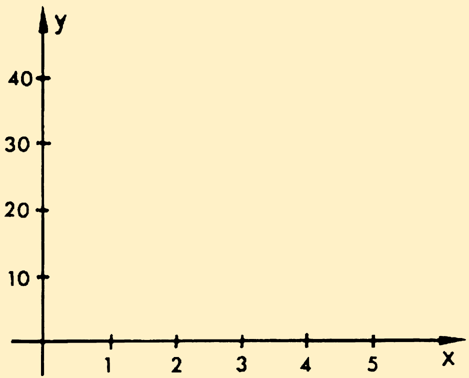
Everything correct --------------------▷ 30

Explanation wanted --------------------▷ 26

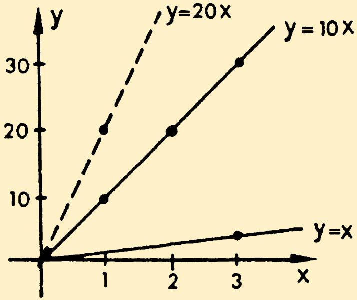
The calibration of the coordinate axis can be chosen deliberately. As a rule the axis is calibrated to aid observation of the relevant features of a curve.
Sketch the straight lines for

26

$y_1 = x$ $\qquad$ $y_2 = 10x$ $\qquad$ $y_3 = 20x$

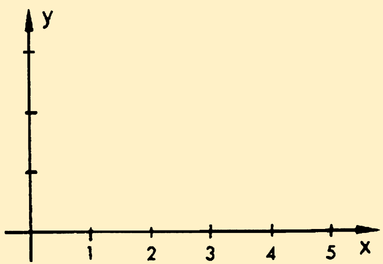


27



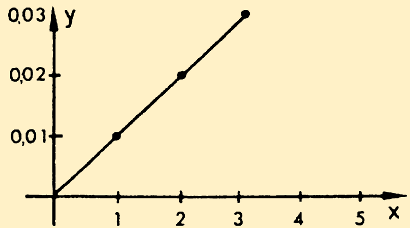
27

Good observables are the functions $y_2 = 10x$ and $y_3 = 20x$



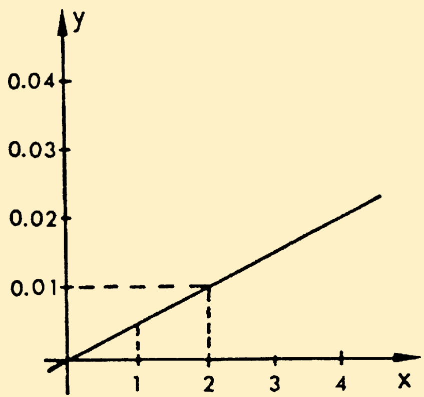
Calibrate the ordinate to obtain a good representation of the function $y = 0.01x$
Domain: $0 \leq x \leq 5$

--------------------▷ 28



The rule is simple. The domain must have space on the x-axis and the codomain must have enough space on the y-axis

28



If we have to determine the function for a given straight line we first determine the slope using the expression

$$a = \frac{y_s - y_1}{x_2 - x_1}$$

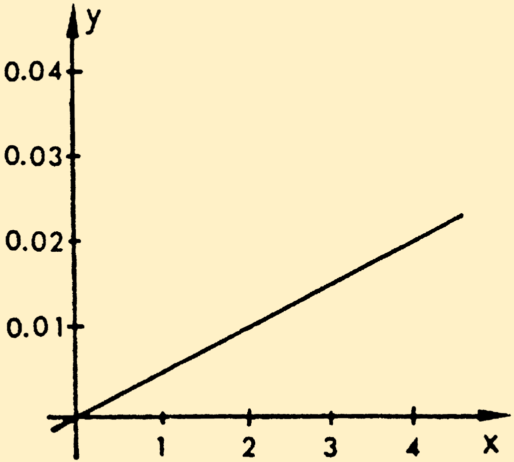
It makes sense to use the origin and an appropriate point of the line.

$a = \ldots\ldots.$ $\qquad$ $y = \ldots\ldots..$

------------------▷ 29

$$a = \frac{0.01}{2} = 0.005 \qquad y = 0.005x$$

29



Verify for yourself that the slope does not depend on the chosen interval. Repeat the calculation of a by choosing $x_1 = 0$ and $x_2 = 1$ and then $x_1 = 2$, $x_2 = 3$, and then $x_1 = 0$, $x_2 = 3$

$$a = \frac{y_2 - y_1}{x_2 - x_1}$$

$a_1 = \ldots\ldots$

$a_2 = \ldots\ldots$

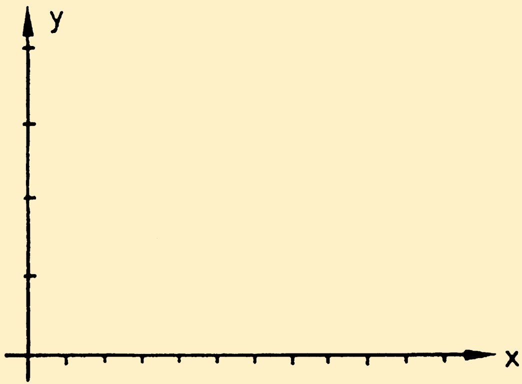
$a_3 = \ldots\ldots$

------------------▷ 30

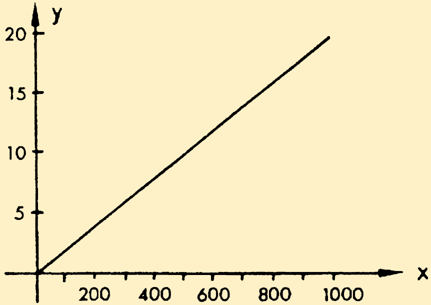
All slopes a are the same: $a = 0.005$

30

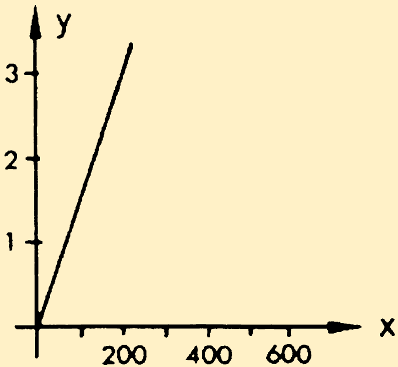
Calibrate the axis to represent $y = 0.02x$ for the domain $0 \le x \le 1000$



31



31



Give the equation of the given graph

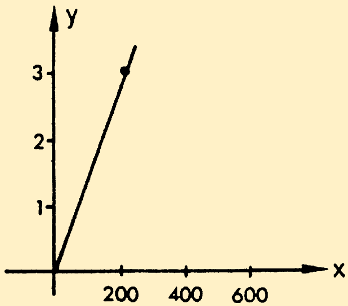
$y = \ldots\ldots$

-------------------▷ 32

$$y = \frac{3}{200}x = 0.01x$$ 32

All correct ------------------▷ 35

Further explanation wanted ------------------▷ 33



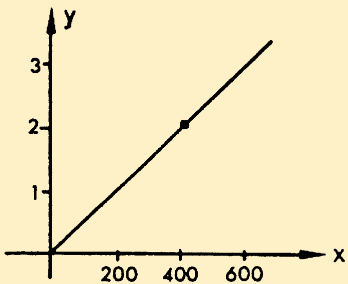
33

We choose in the graph two points

$x_1 = 0$ $y_1 = 0$

$x_2 = 200$ $y_2 = 3$

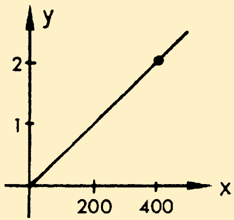
According to these points we calculate the slope of $y = ax$



$$a = \frac{y_2 - y_1}{x_2 - x_1} \qquad a = \frac{3}{200}$$

Give the function of the graph to the left

--------------------▷ 34



$$y = \frac{2}{400} = 0.005x$$

34

To determine the function of a given graph of a straight line you must select two points.
Then you must calculate the quotient of

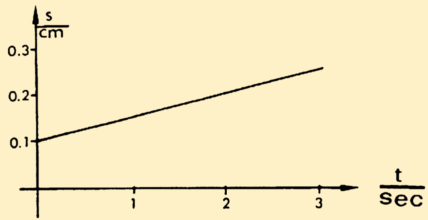
- the difference of the y-values
- the difference of the x-values

Of course you have to observe the calibration.

------------------▷ 35

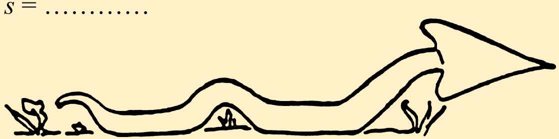
In practice the notations x and y are substituted by variables with different dimensions. Determine the function of the given graph

35



The graph represents the movement of a snail

$s =$ …………



36

$$s = 0.05\frac{cm}{s} \cdot ts + 0.1cm$$

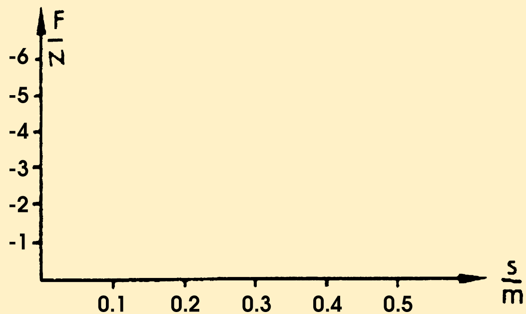
36

A spring is fixed at one end and stretched at the other end. This results in a force which opposes the displacement.
The paired values of force and displacement are tabled beneath.

displacement force

m	N
0	0
0.1	–1.2
0.2	–2.4
0.3	–3.6
0.4	–4.8
0.5	–6.0

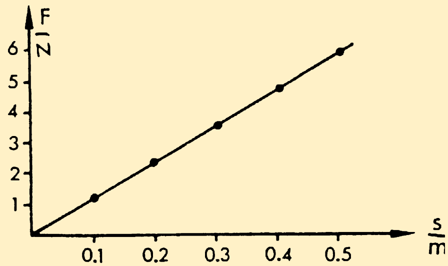
Sketch the graph and give the function



$F =$ ....

$a =$ ....

--------------------▷ 37



$$F = a \cdot s = -12\frac{sN}{m}$$

$$a = -12\frac{N}{m}$$

37

With practice you will gain experience in choosing appropriate calibrations.

--------------------▷ 38

3.2.1 Graph plotting

38

Again you may find contents which you are familiar with from school. Whether you study this section quickly or carefully depends on your level of knowledge.

READ **3.2.3 Graph plotting**
Pages 44–46

Afterwards return to this study guide

--------------------▷ 39

Find the zeros of the function
$y = x^2 - 4$
Zeros …………

39

40

$x_1 = +2$
$x_2 = -2$

40

Find the pole for $y = \dfrac{1}{x+1} - 1$

Pole: ……..

----------------------▷ 41

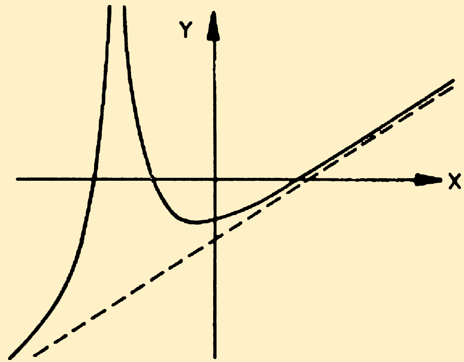
$x = -1$

41

How many zeros has the sketched function?

....

The dashed line is named ....



------------------▷ 42

3 zeros
asymptote

42

Given the function

$$y = \frac{1}{x^2 - 4}$$

This function has

………. zero(s)

………. pole(s)

………. asymptote(s)

--------------------▷ 43

No zeros
2 poles
1 asymptote

43

Write down in your own words:

To calculate poles ....

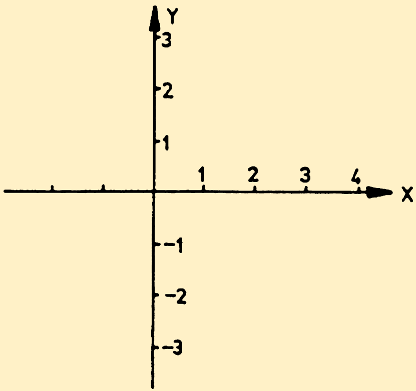
....

-------------------▷ 44

Poles are found by calculating the zeros in the denominator of a fraction. The numerator of which is not zero.

44

Sketch the function $y = \frac{2}{1}$



The function has

…. zero(s)

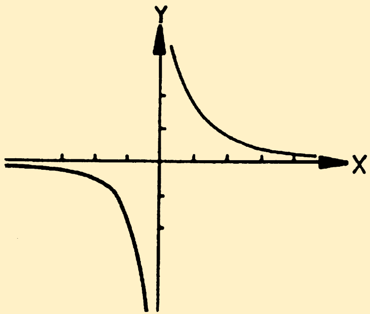
…. pole(s)

…. asymptote(s)

45

No zeros a pole no asymptotes

45



$y = \dfrac{2}{x}$ is a hyperbola

It has two separate parts

Is $y = \dfrac{a}{x} + b$ a hyperbola as well?

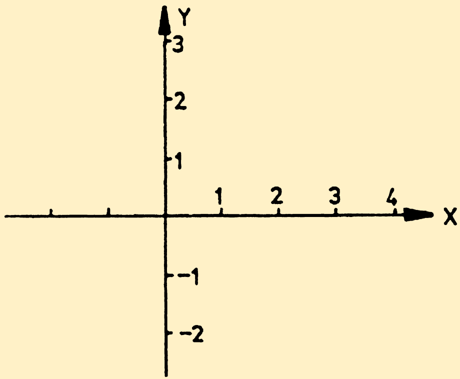
☐ yes

☐ no

--------------------▷ 46

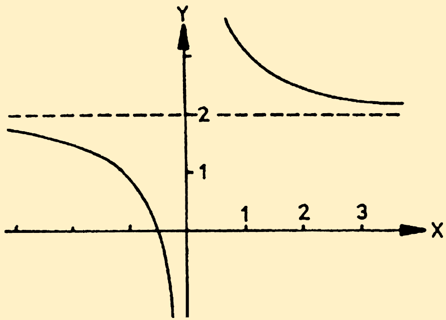
Yes, it is shifted in the y-direction

46



Sketch a representation of $y = \frac{1}{x} + 2$

▷ 47



Hopefully you sketched both parts of the hyperbola, which has

47

.............zero(s)

.............asymptote(s)

.............pole(s)

48

48

1 zero
1 asymptote
1 pole

You may sketch more functions and determine zeros, poles, and asymptotes. This is up to you. If you still feel unsure we suggest you continue with the exercises.

It is quite an annoying fact that if we have enough competence we like to do the exercises. We even enjoy it. But in this case we do not need more training. In the following you will find more exercises.

------------------▷ 49

Given some functions: $y = x^2 + x + 1$ 49

$$y = \frac{1}{x^2 + x + 1}$$

$$y = \frac{1}{x^2}$$

Sketch the graphs!

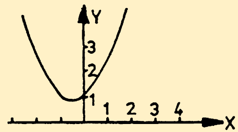
Solutions and further exercises ------------------▷ 50

If you solved the exercises and characterized them as quite easy ------------------▷ 56

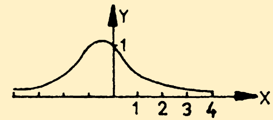
Given three functions

50

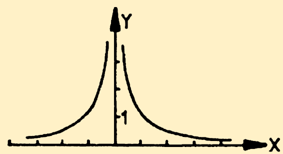
$$y_1 = x^2 + x + 1$$



$$y_2 = \frac{1}{x^2 + x + 1}$$



$$y_3 = \frac{1}{x^2}$$



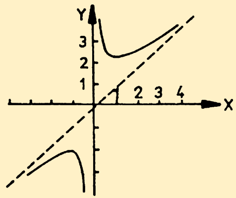
Sketch the graphs

$$y_1 = \frac{1}{x} + x \qquad y_2 = -\frac{1}{x} \qquad y_3 = \frac{3}{x} - 2$$

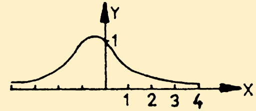
Solutions and further exercises --------------------▷ 51

If you solved the exercises and characterized them as quite easy --------------------▷ 56

$$y_1 = \frac{1}{x} + x$$

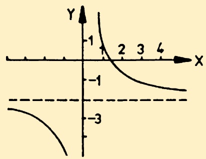


$$y_2 = -\frac{1}{x}$$



51

$$y_3 = \frac{3}{x} - 2$$



More exercises ▷ 52

If these exercises are simple go to ▷ 56

Do you remember the correct names?

52

The x-axis is named ....

The y-axis is named ....

Calculate the zeros

a) $y = x - 2$
zero(s): ....

b) $y = x^2 - 4$
zero(s):

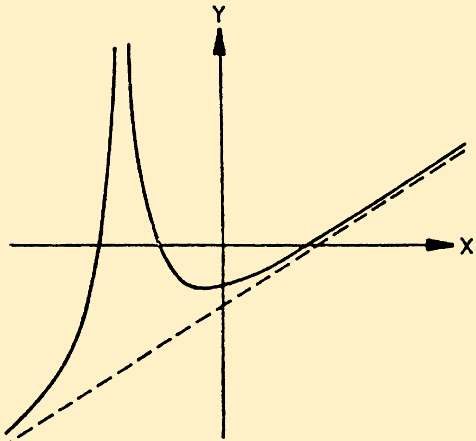
--------------------▷ 53

....

abscissae ordinate

53

Zeros: a) $x = 2$ b) $x_1 = +2$
$x_2 = -2$



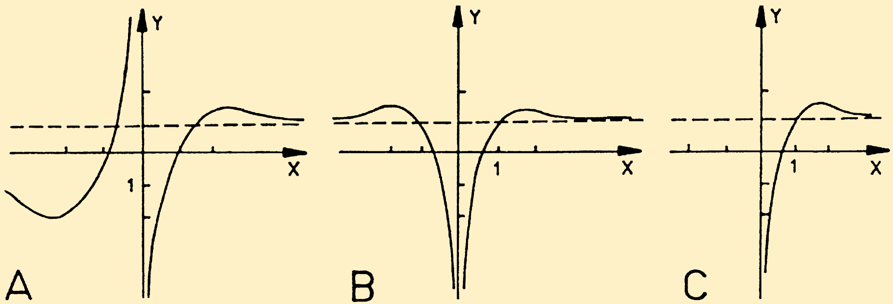
The plotted function has
.........zeros and .... poles
The dotted line is named
....

54

3 zeros 1 pole asymptote

54

Which is the plot of $y = \dfrac{x^2 - 1}{x^4} + 1$



The function has …. zeros …. poles ….. asymptote

55

Plot B

55

2 zeros 1 pole 1 asymptote

If you solved all problems correctly congratulations.

-------------------▷ 56

3.3 Quadratic Equations

56

Quadratic equations should and may be known from school. But since this topic is often used a rehearsal may be worthwhile.

Read **3.3 Quadratic Equations**
Textbook page 47–48

-------------------▷ 57

57

Given
$ax^2 + bx + c = o$
Calculate the roots.

$x_1 =$ ………..

$x_2 =$ ………..

In case of difficulties consult the textbook.

--------------------▷ 58

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

58

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Give the roots for the equation

$$x^2 + px + q = o$$

$x_1 =$ ....

$x_2 =$ ....

-------------------▷ 59

$$x_1 = \frac{-p + \sqrt{p^2 - 4q}}{2}$$

$$x_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}$$

59

Given $2x_2 + 2x - 4 = o$

$x_1 =$ …………..

$x_2 =$ …………….

Solution --------------------▷ 63

Help and hints wanted --------------------▷ 60

Given $2x^2 + 2x - 4 = 0$

60

You may use the formula

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2q}$$

Write down

$a = \ldots\ldots$ $b = \ldots.$ $c = \ldots..$

Inserting gives

$x_1 = \ldots\ldots\ldots$

$x_2 = \ldots\ldots\ldots$

-------------------▷ 61

$a = 2$ $b = 2$ $c = -4$ 61

$x_1 = 1$
$x_2 = -2$

You may try the other formula as well.
Given $2x^2 + 2x - 4 = 0$
Dividing by 2 gives

……………. = 0

Thus p = ………… q = ……………

------------------▷ 62

$x^2 + x - 2 = 0$

62

$p = 1$ $q = -2$

Inserting in the given formula you obtain

$x_1 = \ldots\ldots\ldots\ldots..$

$x_2 = \ldots\ldots\ldots\ldots\ldots$

-------------------▷ 63

$x_1 = 1$ $x_2 = -2$ 63

Given

$$3x^2 - 9x - 30 = 0$$

Try to use both formulae to obtain x_1 and x_2

Hints welcome ------------------▷ 64

Solution ------------------▷ 65

Given $3x \ \ -9x-30=0$

To apply the first formula you write down

$a = \ldots.$ $b = \ldots.$ $c = \ldots.$

Then insert into the given formula to obtain

$x_1 = \ldots\ldots.$

$x_2 = \ldots\ldots.$

To try the other formula as well divide the equation above by 3 to get q and p and to obtain.

$x_1 = \ldots\ldots.$

$x_2 = \ldots\ldots.$

----------------------▷ 65

$x_1 = 5$
$x_2 = -2$

3.4 Parametric changes of functions and its graphs

66

READ **3.4 Parametric changes of functions and its graphs**
Textbook page 49–50

Having studied

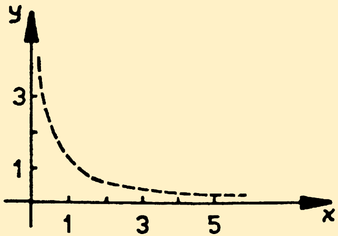
--------------------▷ 67

67

In the following we regard the function

$y_1 = f(x) = \frac{1}{x}$

We will limit our considerations to one part of the hyperbola

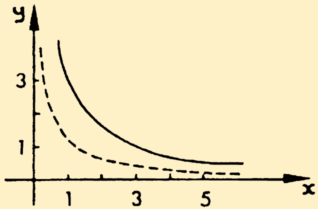


Multiplication of the function with a constant, e.g. 3.
Sketch the graph
$y_2 = 3 \cdot f(x)$

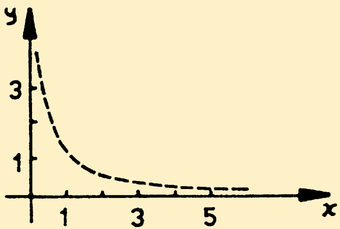
------------------▷ 68

$$y_2 = 3f(x) = \frac{3}{x}$$

68



Addition of a constant to a function:

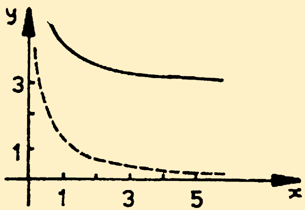


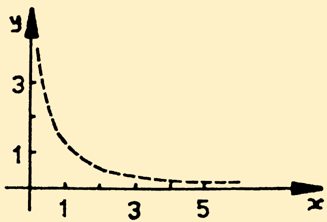
Given $y_1 = f(x) = \dfrac{1}{x}$

Sketch
$y_2 = f(x) + 3 = \ldots.$

------------------ ▷ 69

$y_2 = f(x) + 3 = \frac{1}{x} + 3$





Multiplication of the argument by a constant.

Given $y_1 = f(x) = \frac{1}{x}$

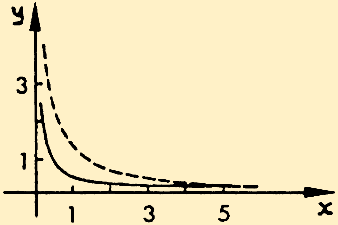
Multiplying the argument by 3 gives $y_2 = \ldots\ldots\ldots$

Sketch the modified function.

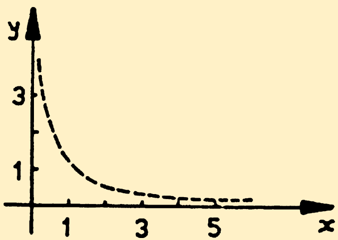
------------------▷ 70

$y_2 = \frac{1}{3x}$

70



Addition of a constant to the argument



Given $y_1 = f(x) = \frac{1}{x}$

Adding the constant 3 to the argument results in

$y_2 = f(x+3) = \ldots$

Hint: We have to replace the argument by $(x + 3)$

Sketch the function y_2

Modification of the graph:

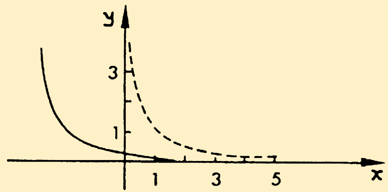
□ shift to the right

□ shift to the left

------------------▷ 71

$$y_2 = \frac{1}{x+3}$$

Shift to the left



71

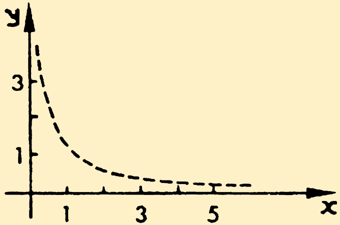
In the following exercises we will modify the function by $c = -3$
Decide for yourself

I do not need the exercise ----------> 73

I want the exercise ----------> 72

Multiplication of the argument by a constant $c = -3$

72



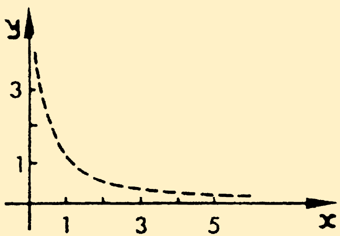
Given $y_1 = f(x) = \frac{1}{x}$

Multiply the argument by $c = -3$

$y_2 = f(x \cdot (-3)) = ...$

Sketch the function

Addition of a constant to the function: c – 3



Given $y_1 = \frac{1}{x}$

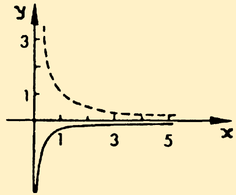
$y_2 = f(x) - 3 = ...$

Sketch y_2

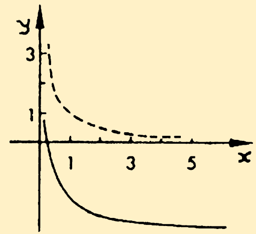
73

73

$y_2 = \dfrac{1}{-3x}$

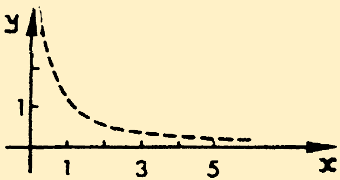


$y_2 = \dfrac{1}{x} - 3$



Addition of a constant c to the argument $c = -3$

Given $y_1 = f(x) = \dfrac{1}{x}$



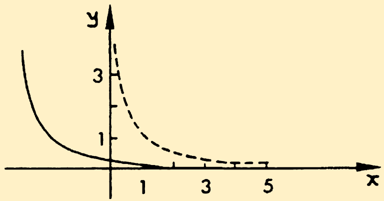
$y_2 = f(x-3) = \ldots\ldots\ldots\ldots$

74

Contents | First Page | Previous View | Next View | Print | Exit

$$y_2 = \frac{1}{x-3}$$

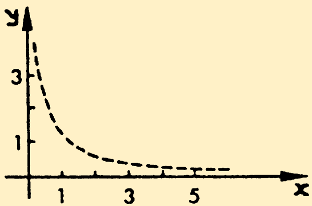
74



Multiplication of the function by a constant.

$e \cdot g: \quad c = -3$

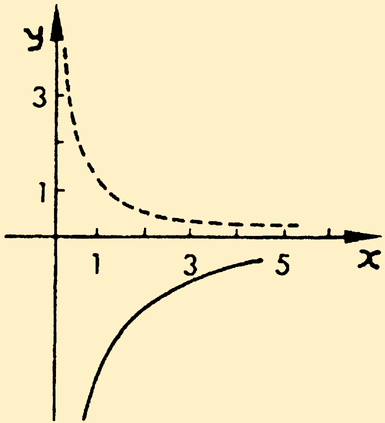
$y_1 = \frac{1}{x}$ $\qquad y_2 = c \cdot y_1 = \ldots\ldots\ldots$



Sketch the new function y_2

75

$y_2 = -\frac{3}{x}$



75

76

3.5 Inverse Functions

76

The concept of the inverse function is often used and it helps in obtaining new functions

READ **3.5. Inverse functions**
Textbook page 50–51

77

To obtain an inverse function we proceed in two steps:

1. ……………………...

2. …………………..

77

78

1. Interchange x and y to obtain a new function
2. Solve the new function for y

78

Obtain the inverse function y^{-1} for

$$y = 1 - \frac{1}{x}$$

Choose the correct solution

□ $x^{-1}(y) = \dfrac{1}{1-y}$ --------------------▷ 79

□ $y^{-1}(x) = \dfrac{1}{1-x}$ --------------------▷ 81

Sorry, this is the wrong choice. 79

We suggest you study again section 3.4 “inverse functions” in the textbook. Consulting the textbook obtain the inverse functions of

$y_1 = \frac{1}{x+1}$ $y^{-1}(x) = ...$

$y_2 = e^{2x}$ $y^{-1}(x)f = ...$

--------------------▷ 80

$$y^{-1}1(x) = \frac{1}{x} - 1$$ 80

$$y_2^{-1}(x) = \frac{1}{2}\ln x$$

Try again to get the inverse function of

$$y = 1 - \frac{1}{x}$$

$y^{-1} =$ ....

--------------------▷ 81

$y^{-1}(x) = \dfrac{1}{1-x}$ is the correct solution 81

Obtain the inverse function of

$$y = 27x^3$$

$y^{-1}(x) = ...\ldots\ldots\ldots\ldots..$

------------------▷ 82

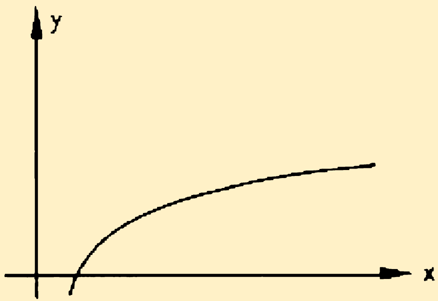
$$y^{-1}(x) = \frac{3\sqrt{x}}{3}$$

82

The inverse function is a new function. You obtain the inverse function in two steps

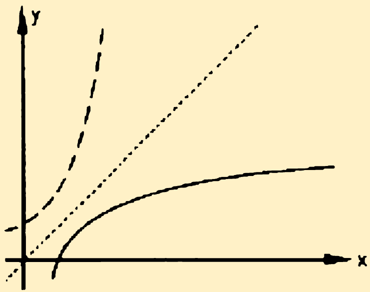
Step 1: ……………………

Step 2: …………………



Given the graph of a function. Sketch the graph of the inverse function.
If you feel unsure, please consult the textbook again

-------------------▷ 83



83

Give the geometrical procedure to obtain the inverse function:

....

....

-------------------▷ 84

The graph of the inverse function is obtained by reflecting the line $f(x)$ in the line $y = x$ which bisects the first and third quadrant.

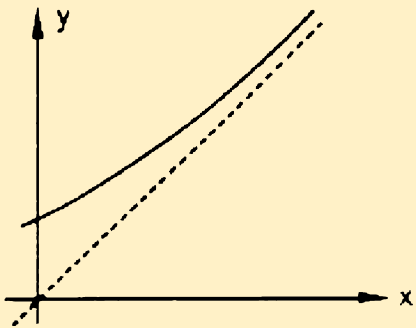
84

Or any different wording with the same meaning

Obtain the inverse functions

$y_1 = \dfrac{1}{x+1}$ $y_1^{-1}(x) = \ldots\ldots\ldots$

$y_2 = 5x+1$ $y_{-12}(x) = \ldots\ldots\ldots\ldots$

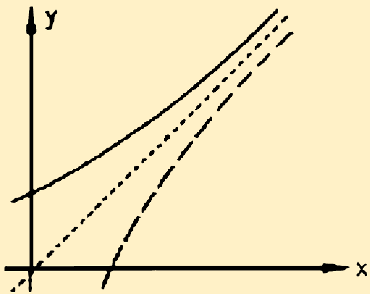


Sketch the graph of the inverse function for the plotted function.

--------------------▷ 85

$$y_1^{-1}(x) = \frac{1}{x} - 1 \qquad y_2^{-1}(x) = \frac{x-1}{5}$$

85



You certainly know by now that in case of difficulties you should always consult the textbook first.

Successful students eliminate with this technique misunderstandings.

------------------▷ 86

3.6 Trigonometric or Circular Functions

86

An important prerequisite to understanding trigonometric functions is that you know to express angles in radians.

READ **3.6.1 Unit circle**
Textbook page 52

Then go to

------------------ ▷ 87

Complete the table

87

Degrees		radians
180^0	=	…….rad
…..	=	2π rad
57^o	=	….rad
……	=	2 rad

--------------------▷ 88

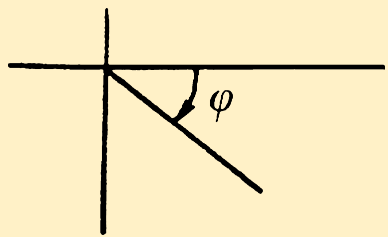
$180^0 = \pi$ rad
$360^0 = 2\pi$ rad
$57^0 = 1$ rad
$115^0 = 2$ rad

88

Angles in clockwise direction are

☐ positive

☐ negative



Convert in radians

$1^0 \approx ..$ rad
$45^0 \approx ..$ rad

--------------------▷ 89

Negative

$1^0 \approx 0.017$ rad
$45^0 \approx 0.78$ rad

Hint: the notation of angles may vary slightly in different books.

I am not familiar with these conversions and want more exercises --------------------▷ 90

I am familiar with the conversion of angles --------------------▷ 95

In practice you should regard carefully how angles are denoted. It is very useful indeed to be familiar with the conversion of degrees into radians and vice versa.
Convert:

90

$$1^0 \approx \ldots \text{ rad}$$
$$90^0 = \ldots \text{ rad}$$
$$180^0 = \ldots. \text{ rad}$$
$$360^0 = \ldots. \text{ rad}$$

------------------▷ 91

91

$$1^0 \approx 0.017 \text{ rad}$$
$$90^0 = \frac{\pi}{2} \text{ rad} \approx 1.57 \text{ rad}$$
$$180^0 = \pi \text{ rad} \approx 3.14 \text{ rad}$$
$$360^0 = 2\pi \text{ rad} \approx 6.28 \text{ rad}$$

It is useful to always remember the following relationship:

The angle around four quadrants of a circle is 2π radians or 360^0.

In other words:
An angle of 360^0 has in radians the value of the circumference of the circle namely 2π.

A positive angle is

- ☐ clockwise
- ☐ anticlockwise

In case of remaining difficulties and errors go back to the textbook and solve the exercises once again

------------------▷ 92

Anticlockwise.
Hint: The definition of the sign of the angle is a convention.

92

Convert the angles. Calculate the values approximately. It matters only that you gain a certain familiarity with these conversions. Later on you will use your calculator

radians		degrees
3.14	=	……….
1	≈	……….
0.1	≈	……….
1.79	≈	……….

------------------▷ 93

93

radians		degrees
3.14	$\approx$	180^0
1	$\approx$	57^0
0.1	$\approx$	$5{,}7^0$
1.79	$\approx$	102^0

If we denote the angles in degrees α and radians in φ then you can write down the conversion formulae:

$\alpha = ...\varphi$
$\varphi = ...\alpha$

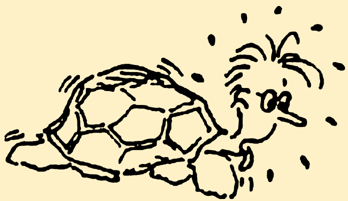
Hint: always remember $2\pi rad \approx 360^0$

------------------▷ 94

$$\alpha = \frac{360^0}{2\pi}\varphi \qquad \varphi = \frac{2\pi}{360}\alpha$$

94

We repeated many times: You should be familiar with these relationships because conversions are often used and it helps you if you can do it from the beginning.



------------------▷ 95

3.6.1 Sine function

95

The following section is more extensive. Probably you will know some content from school. Please take notes of all new definitions and rules. If you are tired during your study you may have short breaks.

READ **3.6.2 Sine function**
Pages 53–58

--------------------▷ 96

96

We assume you remember the geometrical definition of the sine.
But the definition of the sine – function regarding the unit circle may be new to you.
The sine function is called a …… function. It is defined for angles or in other words its domain is

- ☐ $0 \leq \varphi \leq 2\pi$
- ☐ $0 \leq \varphi \leq \infty$
- ☐ $-\infty \leq \varphi \leq +\infty$

------------------ ▷ 97

Trigonometric function
Domain of the sine function
$-\infty \leq \varphi \leq +\infty$

97

The value of $y = \sin x$ never surpasses the value $y_{\max} = \ldots.$

The value of $y = \sin x$ never is less the $y_{\min} = \ldots.$

The codomain of $\sin x$ is given by $\ldots\ldots\ldots\ldots \leq \sin x \leq \ldots\ldots.$

--------------------▷ 98

$y_{\max} = +1$
$y_{\min} = -1$
$-1 \leq \sin x \leq +1$

98

We can denote this result

$|\sin x| \leq 1$

Give some zeros of $y = \sin x$

$x_{\text{zero}} = \ldots = \ldots = \ldots. = \ldots$

-------------------▷ 99

$x_{zeros} = 0, \pm\pi, \pm 2\pi, \pm 3\pi$ and so on.

99

Give two different notations for the argument

$y = \sin \ldots.$

$y = \sin \ldots.$

The sine function has the period ……

-------------------▷ 100

$y = \sin\varphi$, $y = \sin x$ or similar notations

The period is 2π

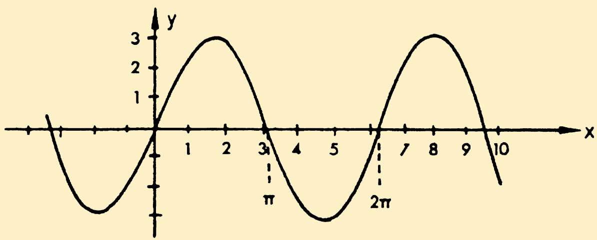
100

Given the function

$y = A\sin x$

A is the …………

The plot represents $y =$ ……………..



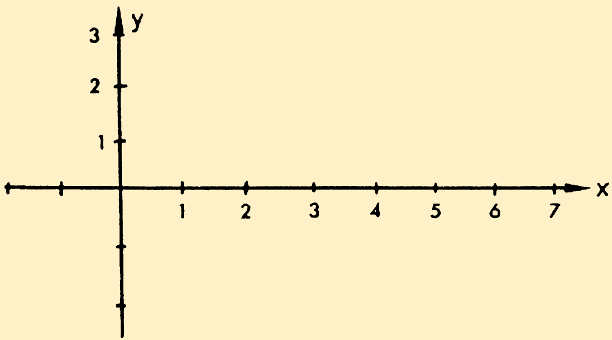
-------------------▷ 101

Amplitude
$y = 3\sin x$

Sketch by free hand the two functions

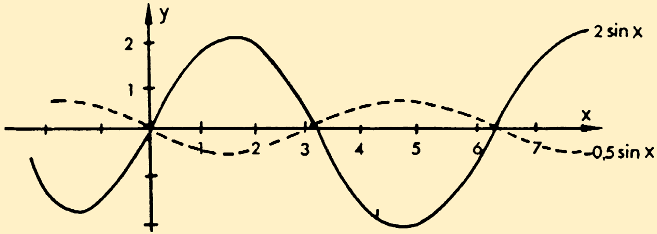
$$y_1 = 2\sin x$$
$$y_2 = 0.5\sin x$$

Accuracy does not matter. The plots must be fundamentally correct.



102

102



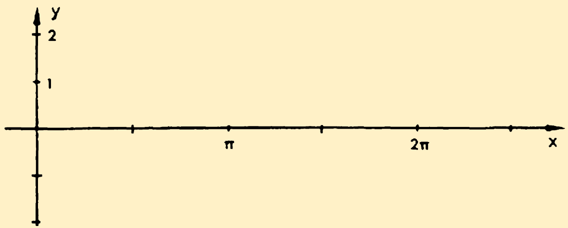
All correct ----------------------▷ 105

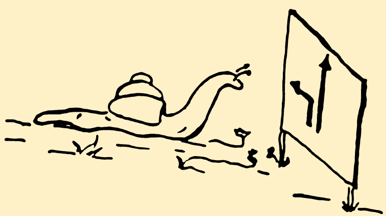
In case of errors ----------------------▷ 103

Read again in the textbook the section "amplitude." The amplitude can be negative.
Sketch by free hand

$y = -2\sin x$

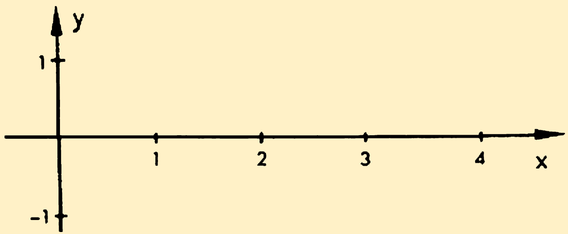
103





104

104



In case of remaining difficulties you may do the following exercise.
Sketch the function $y = \sin x$ on a separate sheet of paper.
Then construct the plot of
$y = (-2) \cdot \sin x$
All values must be multiplied by the factor –2.

Thus, you will obtain the solution shown above.

------------------▷ 105

The sine function is a periodic function with a period of 2π .
If we add to the argument the value 2π we obtain the same value.
Thus, $\sin x = \sin(x + 2\pi)$
Given $y = \sin(bx)$. Which value must you add to the argument x to obtain the same value of the function again?

105

$$\sin(b[\ x + x_{period}\ ]) = \sin(bx)$$

x_{period} = ...……………….

--------------------▷ 106

$$\frac{2\pi}{b}$$

106

The function $y = \sin bx$ has the period $\frac{2\pi}{b}$

Correct answer ------------------▷ 109

You need help or further explanation ------------------▷ 107

107

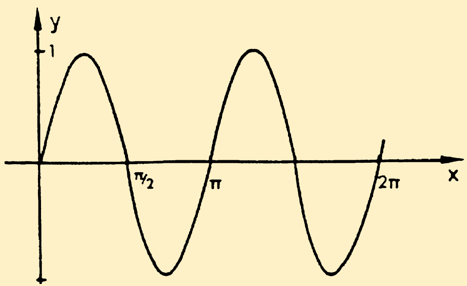
$y = \sin x$ has the following zeros: $x = 0. \pm \pi . \pm 2\pi ....$

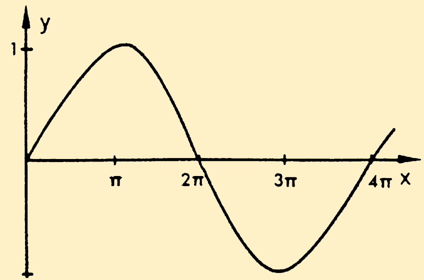
The period equals twice the distance between zeros.

The function $y = \sin bx$ has zeros at $bx = 0. \pm \pi . \pm 2\pi ....$

The distance between zeros is $\frac{\pi}{b}$

Thus, the period is $\frac{2\pi}{b}$





Analyze the graphs and give the function and the period:

$y_1 = ...$ $y_2 = ...$

$period : y_{period} = ...$ period: $y_{2\,period} = ...$

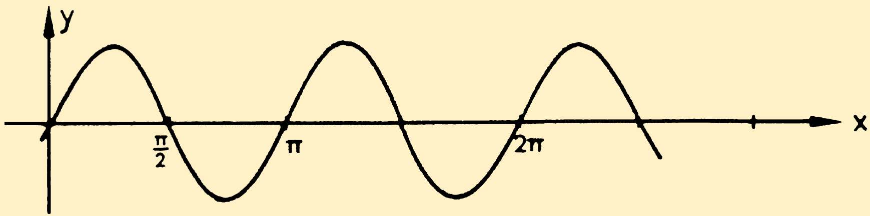
-------------------▷ 108

108

$y_1 = \sin(2x)$ $y_2 = \sin\frac{x}{2}$

$period_1 : \pi$ $period_2 : 4\pi$

Here is the detailed solution for the first exercise.



The general formula of the sine functions is $y = (b \cdot x)$

The period of the plotted function is $x_{period} = \pi$

Thus $b \cdot x_{period} = 2\pi$

We insert x_{period} and obtain: $b \cdot \pi = 2\pi$

Thus: $\mathrm{b} = 2$

Finally, the solution is: $y_1 = \sin 2x$

--------------------▷ 109

Calculate the periods of the following sine functions 109

$y_1 = 5\sin(2x)$ $\quad x_{period} = ...$

$y_2 = 0.5\sin(2x)$ $\quad x_{period} = ...$

$y_3 = 0.5\sin(2\pi x)$ $\quad x_{period} = ...$

----------------- ▷ 110

$x_{1\,period} = \pi$

$x_{2\,period} = \pi$

$x_{3\,period} = 1$

Try to reconstruct the period of the function $y = A \cdot \sin(bx)$

$x_{period} = ...$

Hint: A period is completed if the argument of the sine function increases by 2π . In case of doubt go back to the textbook.

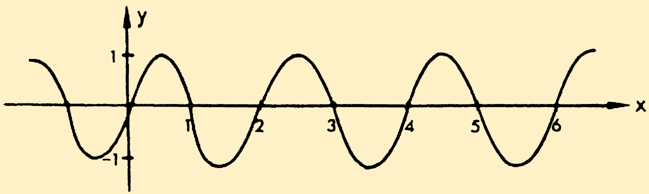
--------------------▷ 111

111

$$x_{period} = \frac{2\pi}{b}$$

This results from $b \cdot x_{period} = 2\pi$

Given the plotted function



Period = ……

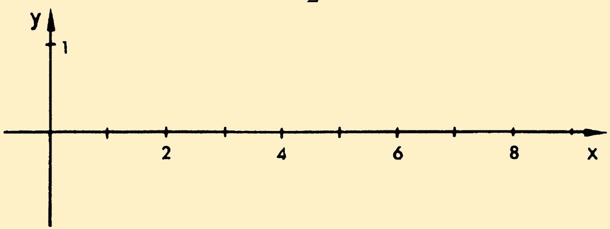
Function: $y =$ ……

--------------------▷ 112

Period = 2
Function: $y = \sin(\pi x)$

112

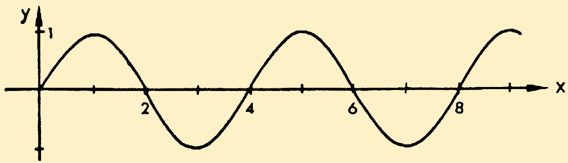
Sketch free-handed $y = \sin(\frac{1}{2}x)$



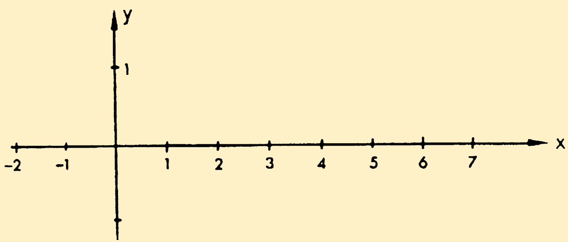
-------------------▷ 113

113

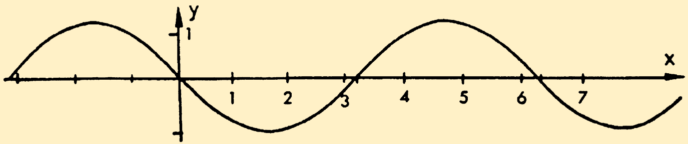
Sketch of $y = \sin\left(\frac{1}{2}\pi x\right)$



Plot the function $y = \sin(x + \pi)$



------------------▷ 114



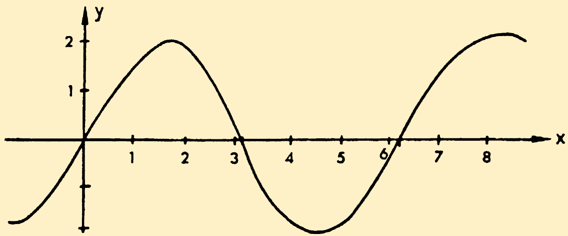
114

Hint: With $y = \sin(x + \pi)$ the argument of the sine has its first zero if $x = -\pi$.
At this point starts the normal sine function.
The curve is shifted by the constant π to the left.

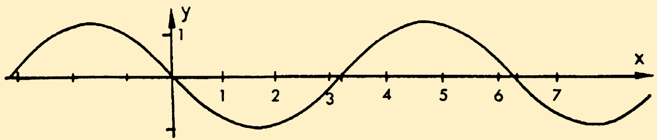
--------------------▷ 115

Sketch the function $y = \sin(\pi + x)$

115



116



116

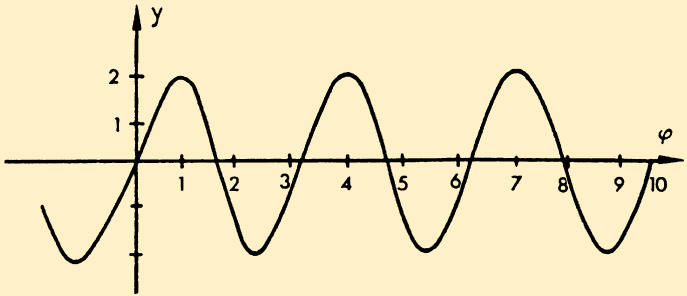
Hint: given: $y = \sin(\pi + x)$. The term in brackets has its zero for $x = -\pi$. At this point starts loosely speaking the curve if we start with sin(o). The sine-curve is shifted by the distance π to the left.

--------------------▷ 117

You should measure your competence and your vocabulary by answering some test questions.

Give the function which is plotted beneath

117



y = ...

------------------▷ 118

$y = 2\sin x$

118

Given $y = A\sin\varphi$. Write down the names

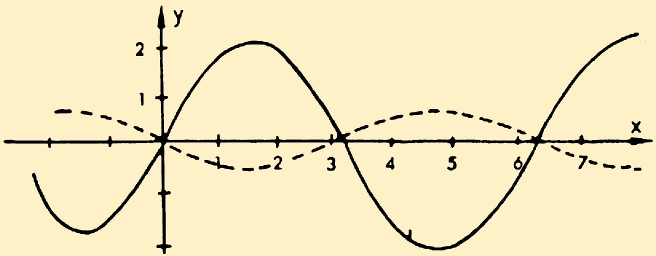
φ is the ....
A is the ....

-------------------▷ 119

φ is the argument or independent variable
A is the amplitude

119

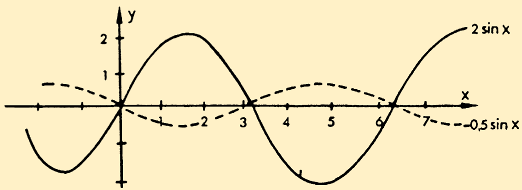
The functions of the curve beneath are $y_1 =$ .... $y_2 =$ ....



--------------------▷ 120

$y_1 = 2\sin x$ $\qquad$ $y_2 = -\frac{1}{2}\sin x$

120



The function $y_1 = 2\sin\rho$ has the period $\rho_{period} =$ ....

The function $y_2 = A\sin bx$ has the period $x_{period} =$ ....

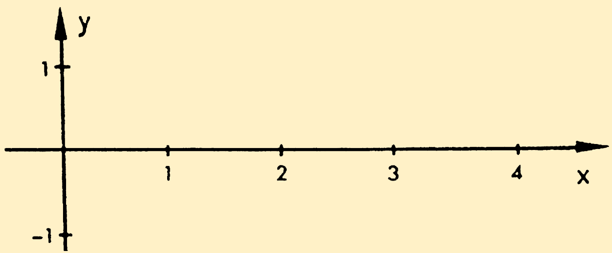
----------------------▷ 121

121

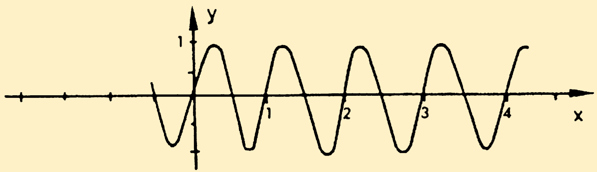
$$\varphi_{period} = \pi$$

$$y_2 = x_{period} = \frac{2\pi}{b}$$

Sketch free-handed the function $y = \sin(2\pi x)$. Your plot must not be perfect. It only matters that it is basically correct.

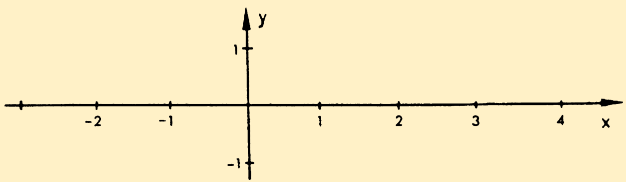


--------------------▷ 122



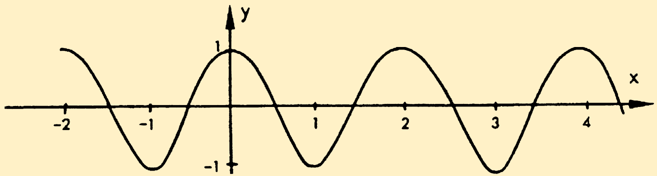
122

Sketch free-handed the function $y = \sin\left[\pi x + \frac{\pi}{2}\right]$



--------------------▷ 123

123



It is important to be familiar with the effect of b and c in the function
$y = A\sin(bx + c)$
b determines the period. Positive c shifts the graph to the left.
In case of remaining difficulties we suggest you repeat the section “sine function” in the textbook and solve the exercises in this study guide once more.

------------------▷ 124

3.6.2 Cosine Function

124

Relationships between the sine and the cosine function.
Divide your study and have breaks.

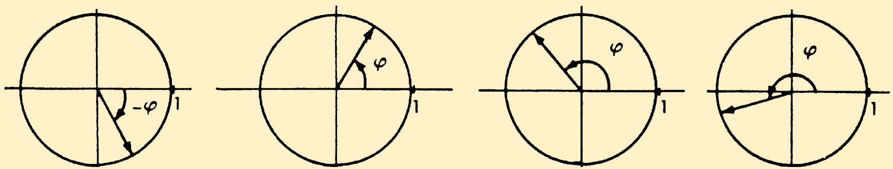
READ **3.6.3 Cosine Function**
3.6.4 Relationships between the sine and cosine function
Textbook pages 58–60

Having done

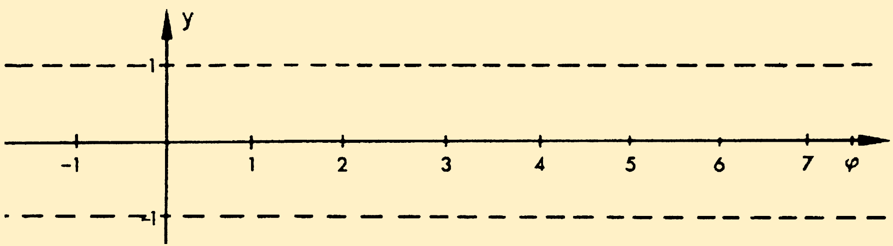
------------------▷ 125

125

Sketch the cosine of the angle φ into the plots:

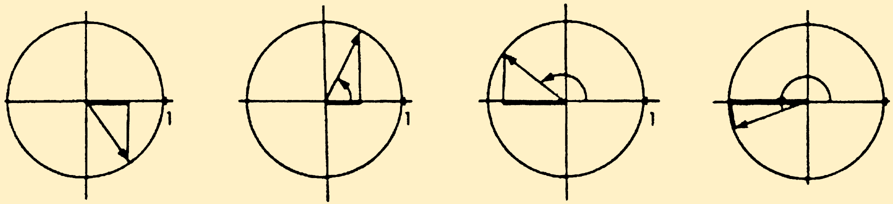


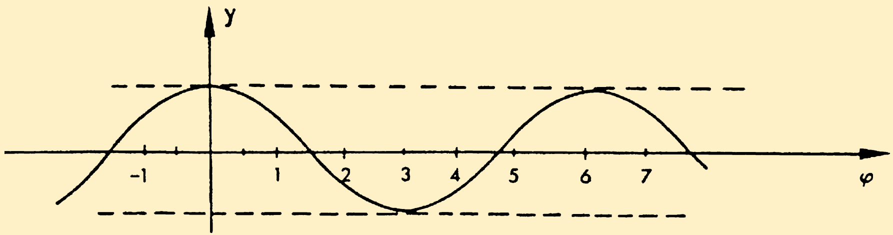
Sketch the function $y = \cos\varphi$



126

126

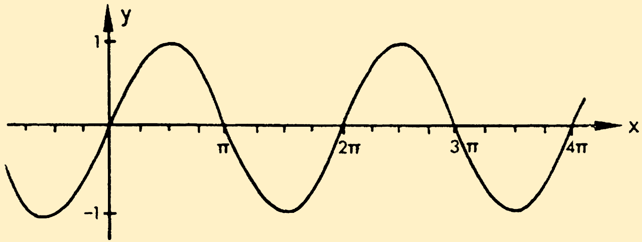




Go ahead to 127

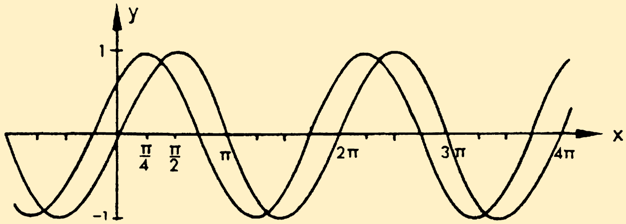
The sketch beneath shows $y = \sin x$

127



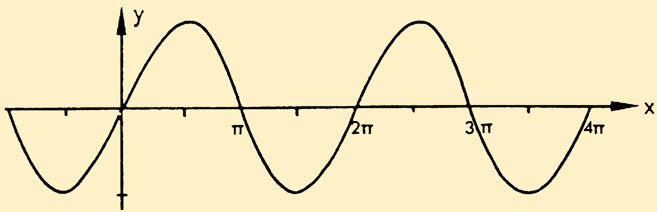
Plot in the given sketch the function $y = \sin(x + \frac{\pi}{4})$

-------------------▷ 128



128

The sketch below shows $y = \sin x$.



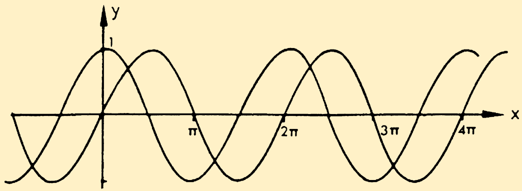
Plot in the given sketch the function $y = \sin(x + \frac{\pi}{2})$

You know the function you plotted this time. It is the ……… function

------------------▷ 129

cosine function

129



We consider the transition from the sine function to the cosine function.
To obtain the function $y = \sin x$ from $y = \cos x$ you shift the cosine function to the right.
Written down this looks like this

$\cos(x\ldots.) = \sin x$

To obtain $y = \cos x$ from $y = \sin x$ you shift the sine function to the left.

$\sin(x\ldots.) = \cos x$

------------------▷ 130

$$\cos(x-\frac{\pi}{2})=\sin x$$ 130

$$\left(x+\frac{\pi}{2}=\cos x\right)$$

The sine function and the cosine function are to some extent similar functions.
If they are shifted by the phase $\frac{\pi}{2}$ they happen to coincide. Thus, both functions may be used to represent mechanical or electrical oscillations.

------------------▷ 131

131

There is no error or contradiction if in one textbook you read:
The oscillation of a pendulum is represented by $s = s_0 \cos(\omega t)$
And in another textbook you read the oscillation of a pendulum is represented by
$A = A_0 \sin(\omega t)$
The representations only differ in two points:

- the amplitude is denoted differently. This is of no importance.
- The position of the pendulum for $t = o$. For $t = o$ the function s has its maximum while A has its zero.

It is obvious that this does not change the basic characteristics of the oscillation.

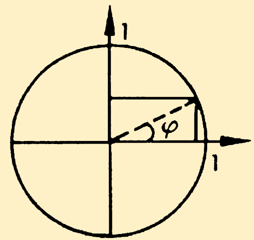
------------------▷ 132

The relationship between sine function and cosine function results from an analysis using the unit circle and the theorem of Pythagoras 132

$\sin^2\varphi + \cos^2\varphi = \ldots\ldots\ldots$

$\sin^2\varphi = \ldots\ldots\ldots$

$\cos^2\varphi = \ldots\ldots\ldots$



----------------- ▷ 133

133

$\sin^2\varphi + \cos^2\varphi = 1$
$\sin^2\varphi = \cos^2\varphi$
$\cos^2\varphi = \sin^2\varphi$

It is easy to memorize: $\sin^2\varphi + \cos^2\varphi = 1$
Often you will find the following notation

$\sin\varphi = \sqrt{1-\cos^2\varphi}$
$\cos\varphi = \sqrt{1-\sin^2\varphi}$

In this case the sign of the root must not be forgotten
$\cos\ \varphi$ is positive in the ….. and ….. quadrant.
Thus, in this case you must write $\cos^2\varphi = ...\sqrt{1-\sin^2\varphi}$

$\cos^2\varphi$ is negative in the …. and …. quadrant.
Thus, in this case you must write $\sin\varphi = ...\sqrt{1-\cos^2\varphi}$

------------------▷ 134

$\cos\varphi$ is positive in the first and fourth quadrant. In this case you must write. 134

$\cos\varphi = +\sqrt{1-\sin^2\varphi}$

$\cos\varphi$ is negative in the second and third quadrant. In this case you must write

$\cos\varphi = -\sqrt{1-\sin^2\varphi}$

-------------------▷ 135

3.6.3 Tangent and Cotangent

135

3.6.4 Addition formulae

With this section two new functions are presented. They are combinations of the trigonometric functions which you are now familiar with.

READ **3.6.5 Tangent and Cotangent**
3.6.6 Addition formulae
Textbook pages 61–64

Go to

136

The quotient $\frac{\sin\varphi}{\cos\varphi}$ is named ....

Give the definition:

$\cot\varphi = \ldots\ldots\ldots\ldots$

136

---------------------▷ 137

$\tan\varphi$

$\cot\varphi = \dfrac{\cos\varphi}{\sin\varphi}$

137

You can obtain the important features of $\tan\varphi = \dfrac{\sin\varphi}{\cos\varphi}$ by analyzing nominator and denominator of the fraction.

Regarding the domain of $\varphi \quad 0 \le \varphi \le 2\pi$

The tangent function has

zeros for $\varphi =$..... ....

poles for $\varphi =$..... ....

If $\tan\varphi = 1 \quad \varphi =$....

Regarding the domain $0 \le \varphi \le \infty$

The tangent function has

zeros for $\varphi =$..... ....

poles for $\varphi =$..... ....

If $\tan\varphi = 1 \qquad \varphi =$.....

------------------▷ 138

138

Domain of φ:

$0 \le \varphi \le 2\pi$
zeros $\varphi = o, \pi, 2\pi$
poles $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}$
$\tan\varphi = 1$ for $\varphi = \frac{\pi}{4}, \frac{5}{4}\pi$

Domain of $\varphi : 0 \le \varphi \le \infty$
zeros $\varphi = 0, \pi, 2\pi, 3\pi, 4\pi.....$
poles $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, ...$
$\tan 1$ for $\varphi = \frac{\pi}{4}, \frac{5}{4}\pi, \frac{9}{4}\pi, \frac{13\pi}{4}....$

In case of difficulties consult the textbook and analyze the drawings. Let φ start with $\varphi = 0$ and let it increase up to 2π

------------------▷ 139

Express the sine by the cosine and vice versa. There are some solutions. Find at least two. 139

$\sin\varphi = \ldots\ldots\ldots$

$\sin\varphi = \ldots\ldots\ldots$

$\cos\varphi = \ldots\ldots\ldots$

$\cos\varphi = \ldots\ldots\ldots$

--------------------▷ 140

140

$$\sin\varphi = \cos(\varphi - \frac{\pi}{2}) = -\cos(\varphi + \frac{\pi}{2})$$

$$\sin\varphi = \sqrt{1-\cos^2\varphi}$$

$$\cos\varphi = \sin(\varphi + \frac{\pi}{2}) = -\sin(\varphi - \frac{\pi}{2})$$

$$\cos\varphi = \sqrt{1-\sin\ \ \varphi}$$

Simplify using the table in the textbook

a) $\dfrac{\sin(\omega_1+\omega_2)+\sin(\omega_1-\omega_2)}{\cos(\omega_1+\omega_2)+\cos(\omega_1-\omega_2)}$

b) $\cos(45^0+\delta)+\cos(45^0-\delta)$

c) $\dfrac{\cos^2\varphi}{\sin 2\varphi}$

----------------------▷ 141

a) $\dfrac{2\sin\omega_1 \cdot \cos\omega_2}{2\cos\omega_1 \cos\omega_2} = \tan\omega_1$

b) $2\cos 45^0 \cos\delta = \sqrt{2} \cdot \cos\delta$

c) $\dfrac{\cos^2\varphi}{\sin 2\varphi \cos\varphi} = \dfrac{1}{2}\cot\varphi$

141

Hint: Having solved the exercises you may solve them as well with a computer program if you are familiar with it.

--------------------▷ 142

142

Substitution of variables often helps to understand equations.
In physics we use different variables

t = time	v = velocity
ρ = density	g = gravitational acceleration
h = height	p = pressure
E = electrical field	a = acceleration

The equation for pressure as a function of depth reads $p = \rho \cdot g \cdot h$

If we substitute pressure p by y, depth h by x and the product of constants $\rho \cdot g$ by a we obtain a well-known equation

$y = \ldots.$

------------------▷ 143

$y = a \cdot x$

143

This equation is easily understood because we are familiar with it.
To understand relationships in physics or engineering we must understand the mathematical relationship.
Our understanding is often more easily reached if we substitute unfamiliar notations with familiar ones like x, y, z, or substitute complex terms with simpler ones.
This procedure is performed in three steps.
Step 1: Substitution of unfamiliar symbols with familiar ones.
Step 2: Work, calculate and discuss the relationship
Step 3: Resubstitute the variables

------------------ ▷ 144

3.7 Inverse trigonometric functions

144

Here we use the concept of inverse functions to obtain a new type of function

READ **3.7 Inverse trigonometric functions**
Textbook pages 64–66

145

The expression $arc\sin 1 = y$ means
y is the …. whose …. has the value … 145

Try to solve for y: …….
$y = \text{arc}\cos 1$
$y =$ …..

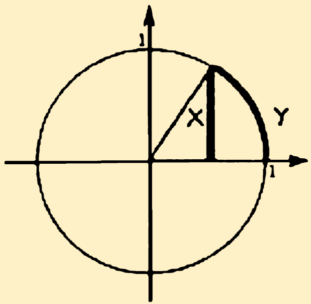
-------------------▷ 146

y is the angle whose value is 1 146

$y = \text{o}$ or $y = 0^0$

No difficulties go to --------------------▷ 148

For more explanation --------------------▷ 147



In the unit circle is marked the angle y in radians and its sine.

147

In this case we may say y is the angle in radians whose sine has the value x.

This is the meaning of the expression

$y = \text{arc}\sin x$

Difficulties may arise if you do not discern between $\sin(x)$ and $\sin x$ in the equation above. $\sin(x)$ is the sine of the angle x.

In the equation above $\sin x$ means x is the value of the sine whose angle is required.

148

Let us reiterate. 148

In the equation
y = arc sin x the term "sin x" means.
The sine has the value x

In the equation
y = arc cos x the term "cos x" means
The …. has the value ….

------------------▷ 149

The cosine has the value x

149

This notation is new, unfamiliar, and more difficult than the underlying mathematics. Try to formulate the question in your mind before solving the exercises

$\varphi = arc \cos 0.5 = \ldots$
$y = arc \quad \sin 1 = \ldots.$
$\alpha = arc \sin 0.5 = \ldots.$

φ	α	$\cos \alpha$ $\cos \varphi$	$\sin \alpha$ $\sin \varphi$
$0 = 0{,}00$	$0°$	1	0
$\frac{\pi}{6} = 0{,}52$	$30°$	0,87	0,5
$\frac{\pi}{4} = 0{,}78$	$45°$	0,71	0,71
$\frac{\pi}{3} = 1{,}05$	$60°$	0,50	0,87
$\frac{\pi}{2} = 1{,}56$	$90°$	0	1

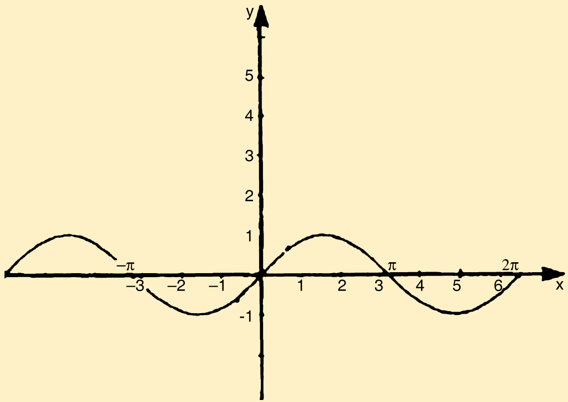
------------------▷ 150

150

$\varphi = \frac{2}{3}\pi rad$ or $\varphi = 60^0$

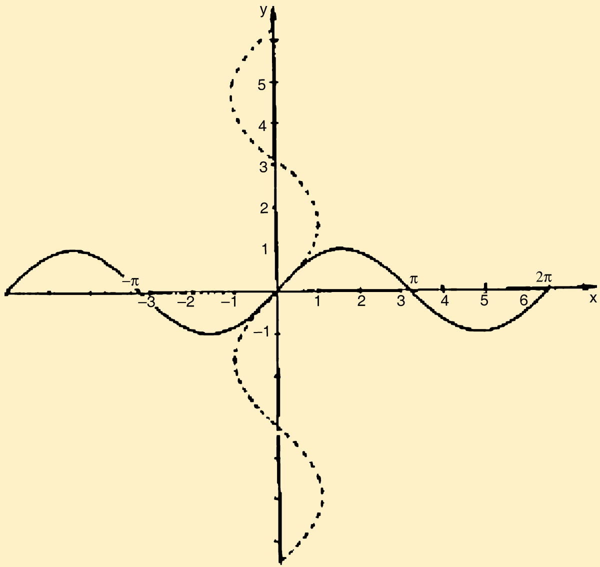
$y = \frac{\pi}{2} rad$ or $y = 90^0$

$\alpha = \frac{\pi}{6} rad$ or $y = 30^0$



Plotted are two periods of the sine function sketch the inverse.

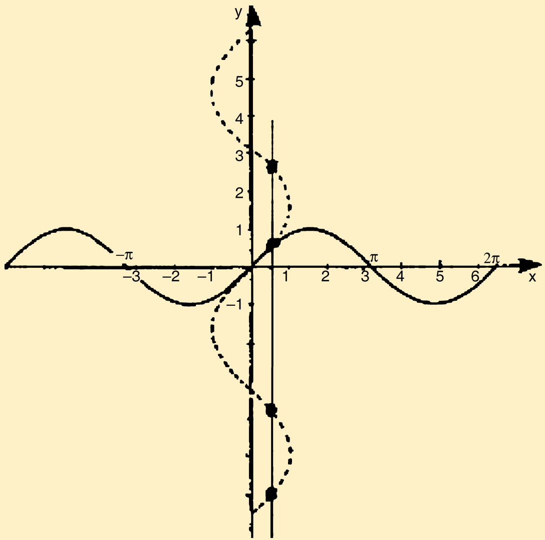
151



The sketched inverse is a
☐ function
☐ relation
Mark the values of the inverse for
$x = 0.5$

151

152

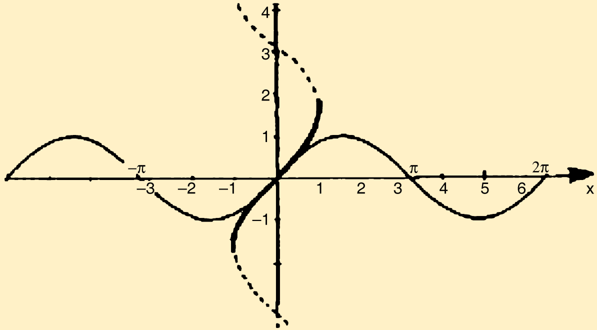


Relation
Hint: For $x = 0.5$
we have four values sketched but
more exist.
Now mark the main values for
which the inverse may be regarded
as a function.

152

------------------▷ 153

153



a) $y_1 = arc\cos 0{,}71 \qquad y_1 = ...$

b) $y_2 arc\sin 0.87 \qquad y\ = ...$

φ	α	cos α cos φ	sin α sin φ
0 = 0,00	0°	1	0
$\frac{\pi}{6}$ = 0,52	30°	0,87	0,5
$\frac{\pi}{4}$ = 0,78	45°	0,71	0,71
$\frac{\pi}{3}$ = 1,05	60°	0,50	0,87
$\frac{\pi}{2}$ = 1,56	90°	0	1

------------------▷ 154

154

$$y_1 = \frac{\pi}{4} rad \quad \text{or} \quad y_1 = 45^0$$

$$y_2 = -\frac{\pi}{3} rad \text{ or } \quad y_2 = -60^0$$

If we denote angles by φ or α , you may solve the same questions

$y_1 = arc\sin 0.5 \qquad \varphi_1 = ...$

$\varphi_2 = arc\sin - 0.5 \qquad \varphi_2 = ...$

$\alpha = arc\cos - 0.71 \qquad \alpha = ...$

------------------▷ 155

$$\varphi_1 = \frac{\pi}{6} rad \qquad \text{or} \quad \varphi_1 = 30^0$$

155

$$\varphi = -\frac{\pi}{6} rad \qquad \text{or} \quad \varphi = -30^0$$

$$\alpha = \frac{3\pi}{4} \qquad \text{or} \quad \alpha = 135^0$$

At last a question regarding the arc tan relation

$y = arc \tan 1 \qquad y = \ldots.$

$\varphi = arc \tan o \qquad \varphi = \ldots.$

$\alpha = arc \tan -1 \qquad \alpha = \ldots..$

------------------▷ 156

$$\varphi_1 = \frac{\pi}{4} rad \text{ or } \quad y = 45^0$$

$$\varphi = o\, rad \text{ or } \quad \varphi = 0^0$$

$$\alpha = \frac{\pi}{4} rad \text{ or } \quad \alpha = -45^0$$

Hint: $\tan\frac{\pi}{4} = \tan 45^0 = 1$

We use arc relations if we know certain values of sin, cos, or tan and want to know the related angles.

------------------▷ 157

3.8 Functions of a Function (Composition)

157

Often the concept of composition explained in the textbook helps to simplify equations

READ **3.8. Function of a Function (Composition)**
Textbook pages 66–67

Having completed your study go to

--------------------▹ 158

A function of a function or in other words a composition is a special form of substitution. 158
Given two equations:

$y = f(u)$

$u = g(x)$

In this case u substitutes $g(x)$

Then $y = f(u) = f[g(x)]$

$f(u)$ is called ………

$g(x)$ is called ……..

-------------------▷ 159

$f(u)$ is called outer function
$g(x)$ is called inner function

159

Let us solve a function of a function

Given
$y = u^2 - 1$
$u = x^2 + 1$

$y = \ldots\ldots$

---------------------▷ 160

$y = x^4 + 2x^2$

160

Given:

$$y = \sin(u + \pi)$$
$$u = \frac{\pi}{2}x$$
$$x = 1$$

Calculate $y = f[u(x)]$ for x = 1

$y(x = 1) =$.....

--------------------▷ 161

$y = -1$ 161

Solution correct --------------------▷ 165

Further explanations wanted --------------------▷ 162

162

Given:

$y = \sin(u + \pi)$

$u = \frac{\pi}{2} x$

$x = 1$

For $x = 1$ we obtain $u = \frac{\pi}{2}$

Inserting u into the given function results in

$y = \sin(\frac{\pi}{2} + \pi) = \sin(\frac{3}{2}\pi) = ...$

--------------------▷ 163

$y = -1$

163

Now try to solve $y = u + \sqrt{u}$

$$u = \frac{x^4}{6}$$

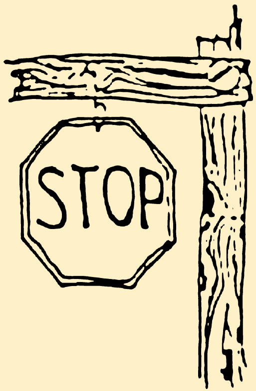
$y = \ldots\ldots$

--------------------▷ 164

164

$y = 2$

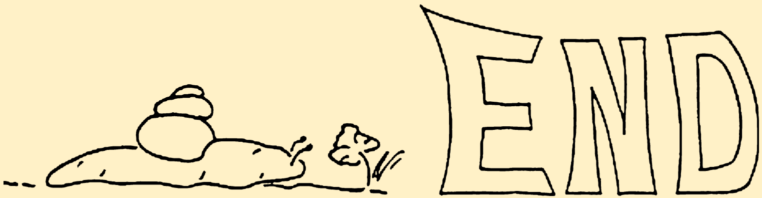
If you are unsure go back to frame 165 and try again



165

You have reached the end of this chapter

If all subject matter has been new to you, it will have taken you quite some time.
But you have finished it and made an important step forward.



of Chapter 3

Chapter 4
Exponential, logarithmic and hyperbolic functions

0

--------------------▷ 1

4.1 Powers, Exponential Function

1

First you will study a section in the textbook. For many users this section will be known from school. But if the material covered is not a mere repetition of well-known facts, carefully take notes and copy the important rules into your notebook.

Study in the Textbook **4.1.1 Powers; Exponential Function**
4.1.2 Laws of Indices or Exponents
Textbook pages 69–71

When done, proceed to ------------------▷ 2

Expand the following expressions

2

$a^4 = \ldots.\ldots\ldots$

$b^{-2} = \ldots.\ldots\ldots$

3

$$a^4 = a \cdot a \cdot a \cdot a$$

$$b^{-2} = \frac{1}{b \cdot b}$$

3

Now, let us repeat the vocabulary

The term b^m is called ……………….

b is called the ……………………….

m is called …… or ……………..

-------------------▷ 4

b^m is called power
b is named base
m is named exponent or index
Hint: These names should be known by heart

4

The definition of powers to a negative exponent is derived by looking at the results of consecutively dividing a given power by its base.
Express the following power as a fraction.

$x^{-3} = ...$

------------------▷ 5

$$x^{-3} = \frac{1}{x^3}$$

5

Let us reiterate. The expression

10^x is called …………..

10 is called ……………

x is called …. or ………

-------------------▷ 6

Contents | First Page | Previous View | Next View | Print | Exit

Power
Base
Exponent or index

6

Write down the power for:

a) base: x
Exponent: 3 …………
Power: ………………

b) base A
Exponent x …………
Power: ……………..

7

a) x^3
b) a^x

7

It will be very useful if you understand the origin of the rules. In the following we will deliberately change the notations. The relationships remain unchanged. Transform the following terms:

Product:	$a^x \cdot a^y = \ldots\ldots.$	$A^{t_1} \cdot A^{t_2} = \ldots..$
Quotient:	$\frac{b^m}{b^n} = \ldots..$	$\frac{S^n}{S^m} = \ldots..$
Power:	$(x^n)^m = \ldots.$	$(a^x)^y = \ldots.$
Root:	$\sqrt[a]{x^b} = \ldots..$	$\sqrt[x]{a^y} = \ldots\ldots$

------------------▷ 8

Product:	a^{x+y}	$A^{(t_1+t_2)}$	8
Quotient:	b^{m-n}	s^{n-m}	
Power:	$x^{n\cdot m}$	$a^{x\cdot y}$	
Root:	$x^{\frac{b}{a}}$	$a^{\frac{y}{x}}$	

In case of difficulties consult the textbook and solve the exercise again. Do not be disturbed by the different notations.
In practice notations change in accordance with the given problems.
Now solve

a) $27^o = \ldots\ldots\ldots\ldots\ldots\ldots$ b) $(3^3)^0 = \ldots\ldots\ldots\ldots$

c) $(2^2)^3 = \ldots\ldots\ldots\ldots\ldots$ d) $1^5 = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 9

a) 1 b) 1

c) 64 d) 1

9

Next, compute the values or simplify the expressions respectively.

a) $3^4 \cdot 3^{-3} = \ldots\ldots\ldots\ldots\ldots$

b) $10^6 \cdot 10^8 \cdot 10^{-1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

c) $b^{-m} = \ldots\ldots\ldots\ldots\ldots\ldots$

d) $e^{-1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

e) $4^{\frac{1}{2}} = \ldots\ldots\ldots\ldots\ldots$

------------------▷ 10

a) 3

b) 10

c) $c\dfrac{1}{b^m}$

d) 2

e) $\dfrac{1}{e}$

10

If you had difficulties with the previous exercises, solve them consulting the textbook. You should gain a certain familiarity with these transformations.

If the tasks so far were easy go straight to ------------------▷ 17

If, however, you want to improve your efficiency, you are invited to go to ------------------▷ 11

Solve

11

a) $\sqrt[x]{A} =$ .... b) $27^{\frac{1}{3}} =$ ....

c) $(y^2)^3 =$ .... d) $(0.1)^0 =$ ....

e) $10^3 \cdot 10^{-3} \cdot 10^2 =$ .... f) $\alpha^{-3} =$ ....

--------------------▷ 12

12

a) $A^{\frac{1}{x}}$ b) 3

c) y^6 d) 1

e) 10^6 f) $\frac{1}{\alpha^3}$

Further exercises wanted ------------------▷ 13

No difficulties so far ------------------▷ 16

Solve

13

a) $2^{-3} =$ ....   b) $27^{0} =$ ....

c) $e^{0} =$ ....   d) $3^{-1} =$ ....

e) $\alpha^{-3} =$ ....   f) $y^{\frac{2}{3}} =$ ....

--------------------▷ 14

14

a) $\frac{1}{2^3} = \frac{1}{8}$ b) 1

c) 1 d) $\frac{1}{3}$

e) $\frac{1}{\alpha^3}$ f) $(\sqrt[3]{y})^2$

Transform following this example $x^n \cdot x^m = x^{m+n}$

a) $b^n b^m =$ .... b) $(y^n)^m =$ ....

c) $\frac{A^n}{A^m} =$ .... d) $\sqrt[n]{C^m} =$ ....

----------------- ▷ 15

a) b^{n+m} b) A^{n-m} 15

c) $y^{n \cdot m}$ d) $C^{\frac{m}{n}}$

Last exercise:

a) $4^{\frac{1}{2}}$ b) $(3^{0})^{2}$

c) $3^{4} \cdot 3^{-3}$ d) $10^{-6} \cdot 10^{-8} \cdot 10^{1}$

e) e^{-1}

We do not give the answers this time. In case of doubt ask a fellow student or consult the textbook again.

------------------▷ 16

4.1.1 Exponential function

16

The exponential function is a basic requisite for further studies.

Study in the textbook
4.1.3 Binomial theorem
4.1.4 Exponential function
Pages 71–73

Having done this

-------------------▷ 17

The function $y = 10^x$ is called ....

17

Which of the following functions increases most rapidly for x $\rightarrow \infty$?

Insert values for

$x = 10$, $x = 100$, $x = 1000$

☐ $y_1 = x^{100}$

☐ $y_2 = 10^x$

------------------▷ 18

Exponential function

18

$y_2 = 10^x$

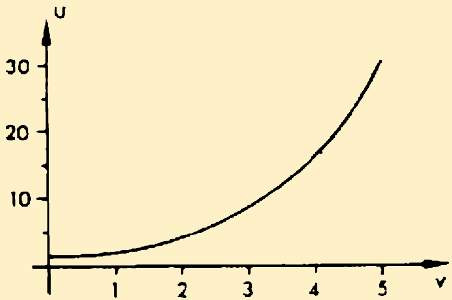
Hint: For $x = 1000$ we obtain for $y_1 : (1000)^{100} = 10^{300}$ and for $y_2 : 10^{1000}$

y_2 exceeds y_1 significantly

If we substitute the familiar notations x and y by other notations the mathematical relationships do not change.
Complex equations often seem difficult if unfamiliar notations are used.
In these cases it often helps to substitute the given notations by x and y and to solve the seemingly easier equations and then resubstitute.

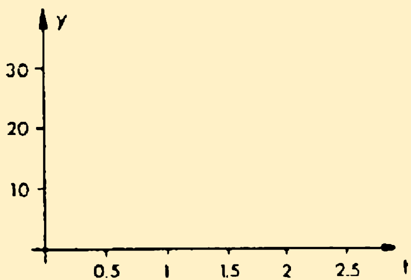
Sketch the function $u = 2^r$

-------------------▷ 19



19

Sketch the exponential function
$y = 2^{at}$ with $a = 2$



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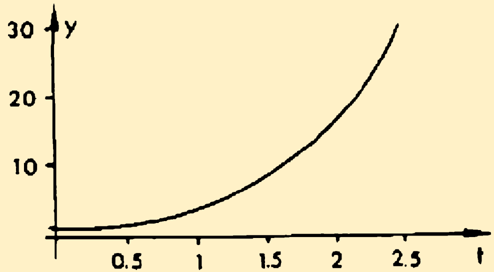

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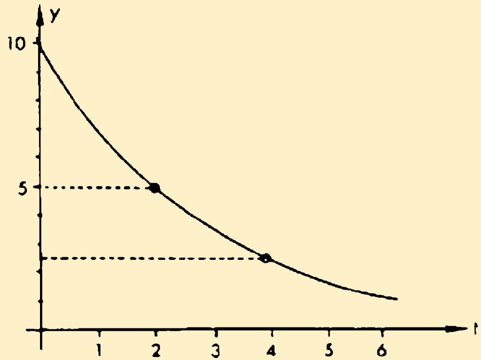


Solve the following task.
In case of difficulties consult the textbook.
The plot represents the general exponential function

$$y = A \cdot e^{-\frac{t}{t_n}}$$

Determine A and t_h (t_h = half life value)

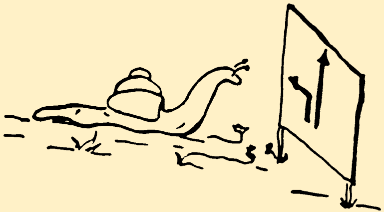
Use the given points.

$y =$ ……………….

--------------------▷ 21

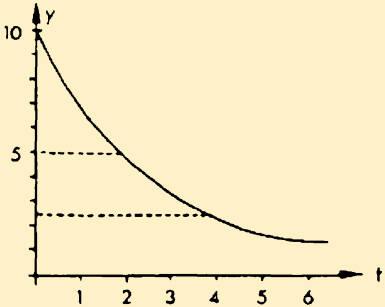
$y = 10 \cdot 2^{-\frac{t}{2}}$

21



All correct ------------------▷ 24

Explanation wanted ------------------▷ 22



In the textbook this exponential function is explained. t is the time. The initial value for t = o is A = ……………..

From the plot you may read that the curve decreases to half of its initial value at t_h = ...

22

Hint:

At $t = 0$ the term $A \cdot e^{\frac{t}{th}} = A \cdot e^{0} = ...$

Thus, the function is $y =$ ………………

------------------▷ 23

23

$A = 10$

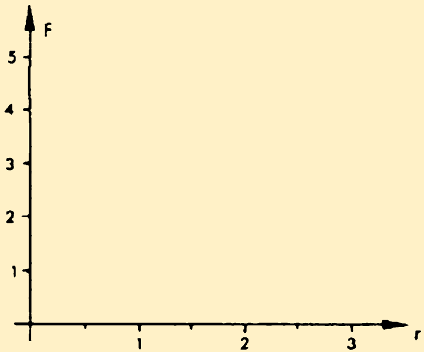
$t_h = 2$

$y = 10 \cdot 2^{-\frac{t}{2}}$

Hint: The values have been taken from the graph.

Since $A = 10$ and $t_h = 2$ we insert into $y = A \cdot e^{-\frac{t}{t_h}}$ and obtain $y = 10 \cdot 2^{-\frac{t}{2}}$

--------------------▷ 24



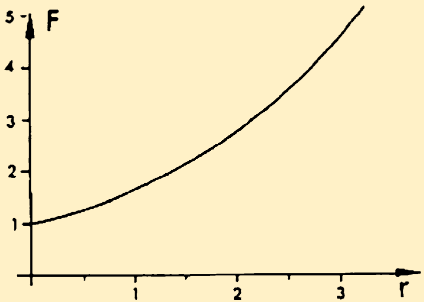
Sketch the exponential function given below

24

$F = e^{0.5t}$ $e = 2.72$

In case you are not familiar with this notation you may substitute F by y and t by x
Perhaps the equation will be more familiar to you

------------------ ▷ 25



25

We suggest a break. You may work with fairly good concentration for 20 minutes up to 60 minutes. There are great differences in optimal individual working periods.
If you are interested in your work these periods may be longer.
Find out how long you may work with concentration. It is important to divide your tasks and to have a short break - and to end the short break in due time..

------------------▷ 26

Short breaks have a duration of 5–15 minutes. With longer breaks the difficulties of warm up rise again. 26
It is of some importance what you do during the break. Mark appropriate activities

☐ Water your flowers, have a cup of tea, prepare a coffee, do some physical exercise

☐ solve mathematical problems, read a different chapter in your textbook.

------------------▷ 27

Appropriate activities are to water flowers, prepare tea or coffee, do some physical exercise. 27

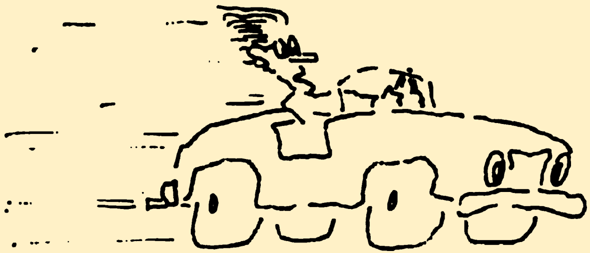
Learning and memorizing a certain matter will be prevented if you concentrate during your break on a similar matter. Solving other mathematical problems is very similar to your learning. It has a negative effect on your learning. This phenomenon is well known in psychology and is named interference.
Another suggestion. Write down on a separate sheet of paper the planned end of your break.

Now enjoy your break.

------------------▷ 28

Short break

28



Having ended your break go to

--------------------▷ 29

Before continuing compare the actual time with the time you planned. There may be a difference. Breaks tend to increase. This does not matter. But in the long run these differences between planned lengths of breaks and the real lengths should not increase too much. 29

------------------▷ 30

4.2 Logarithm, logarithmic function

30

Logarithms are quite difficult when learned for the first time; however if you are familiar with this matter you will proceed quickly.

READ **4.2.1 Logarithm**
Textbook pages 76–77

Then go to

--------------------▷ 31

Taking logarithms is a new operation. To take logarithms is to solve the equation $y = a^x$ for x. This means given: ………….. and ……………
wanted ………..… 31

32

32

$y = a^x$
given: a and y

wanted: x

In another notation the task of taking logarithms can be expressed as well:
The equation $a^y = x$ is to be solved for y.
Up to now we are unable to find a solution.
Thus, we must create a new operation. In mathematics this new operation is called “Taking logarithms.”

--------------------▷ 33

Let us take the equation $a^y = x$ 33

In the textbook you found the following definition:

The logarithm of a given number x to a base a is the exponent of the power to which this base must be raised to equal this number x.

For this exponent we use the symbol

$\log_a x$

In other words: The term "$\log_a x$" is a power or an exponent.

$a^{(\log_a x)} = \ldots\ldots\ldots\ldots\ldots$

------------------▷ 34

$a^{(\log_a x)} = x$

34

You must memorize:
For a given base
the logarithm of a certain number is the exponent to equal this number.

The definition and the meaning of logarithms are important and you must be familiar with them. The logarithms for base 10 are called ………… and abbreviated ………………….

------------------▷ 35

common logarithm abbreviation: lg

35

Let us regard common logarithms.

Calculate:

$10^{\lg 5} = \ldots\ldots\ldots\ldots\ldots$

$10^{\lg 20} = \ldots\ldots\ldots\ldots\ldots$

$10^{\lg 3.14} = \ldots\ldots\ldots\ldots\ldots$

-------------------▷ 36

$10^{\lg 5} = 5$
$10^{\lg 20} = 20$
$10^{\log 3.14} = 3.14$

36

Logarithms with base 2 are called dyadic logarithms or logarithms to the base of 2 and abbreviated ....

Calculate

$2^{ld\,4} = \ldots\ldots\ldots$

$2^{ld\,100} = \ldots\ldots\ldots$

$2^{ld\,b} = \ldots\ldots\ldots$

------------------▷ 37

Logarithms to the base of 2 are abbreviated ld

$2^{ld\,4} = 4$

$2^{ld\,100} = 100$

$2^{ldb} = b$

37

Logarithms with base e are called …………. and abbreviated ……………….

Calculate:

$e^{\ln 6} = ...$

$e^{\ln a} = ...$

$e^{\ln 10} = ...$

--------------------▷ 38

Natural Logarithms ln

38

$e^{\ln 6} = 6$

$e^{\ln a} = a$

$e^{\ln 10} = 10$

With logarithms the base has always to be defined. Give the names of the logarithms with the following base.

base 2: ………………..

base e: ………………..

base 10: ………………

------------------▷ 39

39

base 2: dyadic logarithms or logarithms to the base of 2
base e: natural logarithms
base 10: common logarithms

If you want to take logarithms you have three options

1. You use your pocket calculator or your computer. This is convenient and precise.

2. You use a plot of the logarithmic function. This is convenient but not precise.

3. You use a table. This is precise but inconvenient.

For some values you can calculate the logarithms without help

$ld\,2 = \ldots\ldots\ldots\ldots\ldots\ldots$

$\ln e^x = \ldots\ldots\ldots\ldots\ldots\ldots$

$\lg 100 = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 40

40

$ld\,2 = 1$ since $2^1 = 2$

$\ln e^x = x$ since $e^x = e^x$

$\lg 100 = 3$ since $10^3 = 100$

An operation used later on is to take logarithms of equations. In this case the operation is to be applied to both sides of the equation.
By this operation equations sometimes simplify.
Example: given $e^y = e^{ax}$
The base is (and must be) the same for both sides. In this case the exponents on both sides must be equal:
$y = ax$
Thus, we have just taken the logarithm of the equation since $\ln e^y = y = \ln e^{ax} = ax$
Take the logarithms
$e^a = e^{b+c}$

…………..=……………

----------------▷ 41

a = b + c

41

Take the logarithm of the equation

$10^{y} = 10^{bx}$

Using common logarithms we get

$\lg 10^{y} = \lg 10^{bx}$

Thus $y = bx$

Take logarithms of the following equations

$2^{y} = 2^{cx}$ $\qquad y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$e^{a} = e^{\omega(t+t_0)}$ $\qquad a = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 42

$y = cx$ $a = \omega(t + t_0)$

42

To take logarithms of an equation means to regard exponents if the bases are equal.

Example: $2^7 = 2^{x+1}$

We take the logarithms:

$$ld\,2^7 = ld\,2^{x+1}$$
$$7 = x + 1$$
$$x = 6$$

Calculate

$$10^{(2y+1)} = 10^{x-3}$$
$$y = \ldots\ldots\ldots\ldots$$

------------------ ▷ 43

$y = \frac{1}{2}(x-4)$ 43

In the foregoing examples we had on both sides of the equation exponents of the same base. This is not always the case.

Example $y = e^{-ax}$. Can you take the logarithm? You may think this is impossible. But in this case a trick helps.

Write down y as a power of e.

$y = e^{\ln y}$

Thus, we have on both sides the same base

$e^{\ln y} = e^{-ax}$

Now you can take the logarithm of the equation:

…………..=…………….

------------------▷ 44

$\ln y = -ax$ 44

If you have difficulties taking logarithms of an equation you often may use this trick. Try in an intermediary step to write both sides as powers of the same base.

Given $y = e^a$

Intermediate step

$e^{\ln y} = e^a$

Result: $\ln y = a$

Given: $y = e^{a+x}$

…………..=…………..

------------------▷ 45

$\ln y = a + x$

45

Take logarithms of the following equations

$y = e^{\frac{1}{x}}$ …………=………………

$y = 2^{a \cdot x}$ …………=………………

$y = 10^{(-x+5)}$ …………=………………

Choose appropriate bases

------------------▷ 46

46

$\ln y = \frac{1}{x}$

$ldy = a \cdot x$

$\lg y = -x + 5$

We give some more exercises.
Hint: if exercises seem easy you do not need more of them. If exercises seem difficult you need more of them

$y = e^{(\alpha x+\beta)}$ …………=………………

$b \cdot y = e^{a \cdot x} \cdot e^{c \cdot x}$ …………=………………

$a \cdot y = 10^{0.1x}$ …………=………………

$y = e^{(\ln x - \ln a)}$ …………=………………

-------------------▷ 47

47

$\ln y = \alpha x + \beta$

$\ln(by) = (a + c)x$ or $\ln y = (a + c)x - \ln b$

$\lg(ay) = 0.1x$ or $\lg y = 0.1x - \lg^{a}$

$\ln y = \ln x - \ln a$

It may be time to have a short break. The reader sketched will have a short break. What is he doing? Have a guess............

------------------▷ 48

Perhaps he recapitulates the new concepts of the preceding section. Perhaps he writes down the time for the end of his break.

48

-------------------▷ 49

4.2.1 Operations with logarithms

49

The basic reasoning of operations with logarithms is quite simple. All operations have to be done with logarithms instead of with the original values.
Thus a product of two values will be the sum of its logarithms.

READ **4.2.2 Operations with logarithms**
Textbook pages 78–79

-------------------▷ 50

Can you write down

50

a) $\ln(a \cdot b) = \ldots\ldots\ldots\ldots\ldots\ldots$

b) $\ln\frac{a}{b} = \ldots\ldots\ldots\ldots\ldots\ldots$

c) $\lg(A \cdot B) = \ldots\ldots\ldots\ldots\ldots\ldots$

d) $\lg\frac{y}{x} = \ldots\ldots\ldots\ldots\ldots\ldots$

51

a) $\ln(a \cdot b) = \ln a + \ln b$ 51

b) $\ln \frac{a}{b} = \ln a - \ln b$

c) $\lg(A \cdot B) = \log A + \log B$

d) $\lg \frac{y}{x} = \lg x - \lg y$

In case of difficulties calculate the examples consulting the textbook.

------------------ ▷ 52

In the next examples we deliberately use different notations. The objective is to obtain a certain familiarity with using different notations.

52

$$\lg(x \cdot y) = \ldots\ldots\ldots\ldots\ldots$$

$$ld(N_1 \cdot N_2) = \ldots\ldots\ldots\ldots$$

$$\lg\frac{A \cdot B}{C} = \ldots\ldots\ldots\ldots\ldots$$

$$\ln\frac{a \cdot \beta \cdot \gamma}{\delta} = \ldots\ldots\ldots\ldots$$

------------------▷ 53

53

$$\lg xy = \lg x + \lg y$$

$$ldN_1 \cdot N_2 = ldN_1 + +ldN_2$$

$$\lg \frac{AD}{c} = \lg A + \lg B - \lg C$$

$$\ln \frac{\alpha \cdot \beta \cdot \gamma}{\delta} = \ln \alpha + \ln \beta + \ln \gamma - \ln \delta$$

Calculate

$\ln 5^x =$ ....

$\lg x^2 =$ ....

$\lg a^{\frac{1}{2}} =$ ....

------------------ ▷ 54

$$\ln 5^x = x \ln 5$$

54

$$\lg x^2 = 2 \lg x$$

$$\lg a^{\frac{1}{2}} = \frac{1}{2} \lg a$$

Transform:

a) $\ln 2^x = \ldots\ldots\ldots\ldots\ldots\ldots$

b) $\lg \sqrt{x} = \ldots\ldots\ldots\ldots\ldots\ldots$

c) $\lg \sqrt[3]{x} = \ldots\ldots\ldots\ldots\ldots$

d) $ld(4 \cdot 16) = \ldots\ldots\ldots\ldots\ldots$

------------------- ▷ 55

a) $\ln 2^x = x \ln 2$ b) $\lg \sqrt{x} = \frac{1}{2} \lg x$ 55

c) $\lg \sqrt[3]{x} = \frac{1}{3} \lg x$ d) $ld(4 \cdot 16) = ld4 + ld16 = 6$

Can you give the rules for the general care base a

Multiplication: …………

Division: ……………….

Power: …………………

Root: ………………….

------------------▷ 56

56

Multiplication: $\log_a AB = \log_a A + \log_a B$

Division: $\log_a \dfrac{A}{B} = \log_a A - \log_a B$

Power: $\log_a A^m = m \log_a A$

Root: $\log_a \sqrt[n]{A} = \dfrac{1}{n} \log_a A$

Want to proceed ------------------ ▷ 58

Want more exercises ------------------ ▷ 57

Transform

57

a) $\ln(C \cdot D) = \ldots\ldots\ldots\ldots\ldots\ldots$

b) $\lg y^2 = \ldots\ldots\ldots\ldots\ldots$

c) $ld\,2 \cdot 32 = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 58

a) $\ln C + LnD$

b) $2\lg y$

c) 6

58

Transform

a) $ld\sqrt{x} = \ldots\ldots\ldots\ldots\ldots$

b) $\ln\left(e^{2x} \cdot e^{5x}\right) = \ldots\ldots\ldots\ldots\ldots$

c) $\lg \frac{1}{10^x} = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 59

59

a) $\frac{1}{2} ldx$

b) $7x$

c) $-x$

Calculate by taking logarithms

$C = 10^{3x+1}$ $x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$A = e^{(r \cdot t)}$ $t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$16 = 2^{x+2}$ $x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 60

$$x = \frac{1}{3}(\lg C - 1)$$

60

$$t = \frac{\ln A}{r}$$

$$x = 2$$

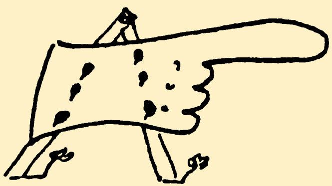
If you encounter difficulties calculating exercises there is a golden rule. Write down the given exercises on a separate sheet and go back to the textbook.

Try to solve the problem with reference to the textbook and exercise at the same time.
You must not divide your attention by turning over pages many times.
You memorize the operations with logarithms much easier if you understand the relations with the laws of exponents treated earlier

--------------------▷ 61

Can you express the basic reasoning of the operations with logarithms in your own words? Perhaps you may explain this for a fellow student. 61

This is the advantage of working in a team of fellow students. You often need to explain something to others. It is not enough to understand subject matter. You must be able to express it in your own words. At the end you will need this competence during your examinations.



------------------▷ 62

4.2.2 Logarithmic function

62

READ **4.2.3 Logarithmic function**
Textbook pages 79–80

Having done

--------------------▷ 63

At what point intersect all logarithmic functions?

63

$x =$ ....

$y =$ ....

Has the logarithmic function a pole?

☐ yes

☐ no

Has the logarithmic function an asymptote

☐ yes

☐ no

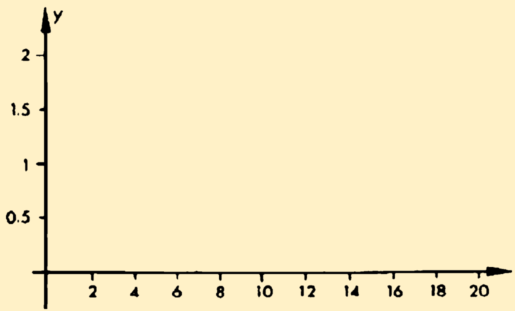
------------------ ▷ 64

Intersection of all logarithmic functions at $x = 1$ $y = 0$

64

Pole at $x = 0$

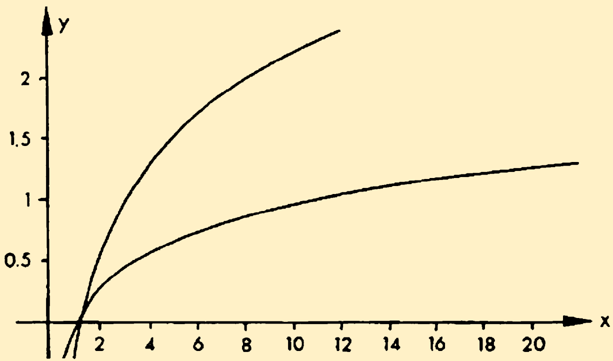
No asymptote



Sketch the graph of

$y = \lg x$
$y = \ln x$

------------------▷ 65



65

66

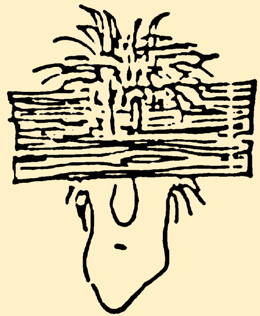
The logarithmic function is the inverse of the exponential function.
Is the exponential function the inverse of the logarithmic function? 66

☐ yes 68

☐ no 67

Sorry you are wrong. We obtain the inverse of a function by reflecting it in the line $y = x$ which bisects the first quadrant and vive versa. Go back to section 3.4 “inverse functions” and read it again. The relation is symmetrical.

67



68

Yes, you are right. The exponential function is the inverse function of the logarithmic function.
We remember that we get the inverse function by reflecting the original function in the line $y = x$ which bisects the first quadrant.

68

Not every function has an inverse function.
If possible give the inverse function for

$y_1 = 3^{2x}$ $\qquad$ $y_1^{-1} = \ldots\ldots\ldots\ldots.$

$y_2 = 4x^2$ $\qquad$ $y_2^{-1} = \ldots\ldots\ldots\ldots$

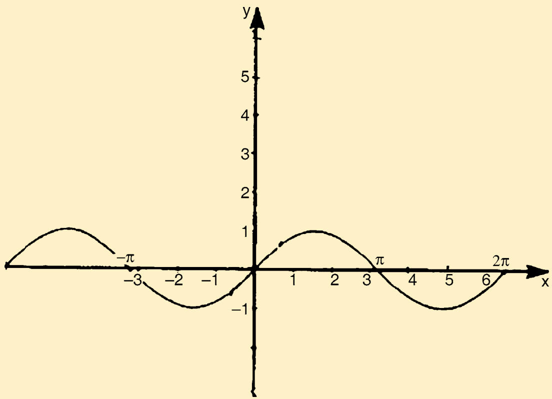
Hint:
Remember the definition of a function.

------------------▷ 69

69

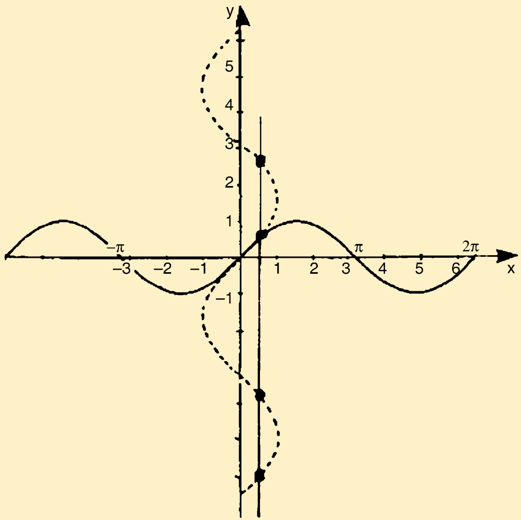
$y_1^{-1} = \frac{1}{2}\log_3{}^x$

y_2 has no inverse function since the expression $y_2^{-1} = \pm\frac{1}{2}\sqrt{x}$ does not represent a function because it is ambiguous.



On the other side the sine curve is plotted. You may reflect it in the line $y = x$ which bisects the first quadrant.
Sketch the reflected curve. Is the reflected curve a function?

70



70

The reflected curve is ambiguous. It is a relation not a function. We can get a function if we restrict the domain and the codomain.

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

You know this function already:

$y = arc\sin x$ and you know its meaning.

Complete:

y is the angle ....

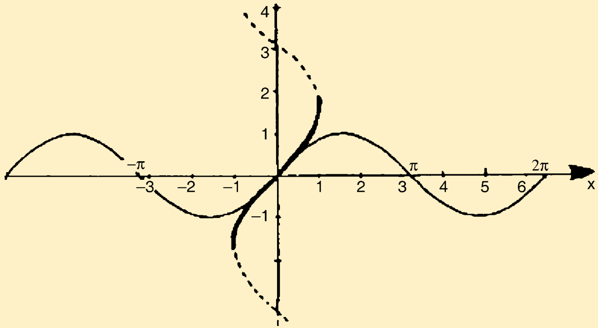
....

Sketch in the plot above with a heavy line the function for the restricted domain and codomain.

------------------▷ 71

y is the angle whose sine is x.

71



72

4.3 Hyperbolic functions and inverse Hyperbolic Functions

72

Since these functions are used in advanced mathematics chapters you may skip this section for the time being. Especially if you found it difficult to master the preceding two introductory chapters.
In this case you may return to this section later.
But if all of the preceding material was known you should study this section now.

I want to skip section 4.3 ------------------▷ 88

I want to study the section on hyperbolic functions

READ **4.3. Hyperbolic functions and inverse hyperbolic functions**
Textbook pages 80–84

Having done ------------------▷ 73

Give the definition of

73

sinh x = ...

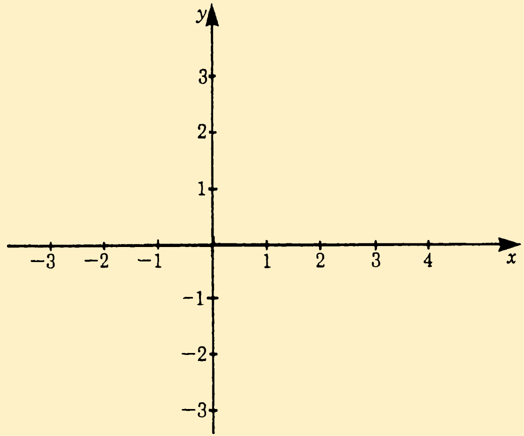
-------------------▷ 74

$$\sinh x = \frac{e^{x} - e^{-x}}{2}$$

74

Sketch $y_1 \sinh 2x$ with a dashed line

and $y_2 \sinh \dfrac{x}{2}$ with a dotted line



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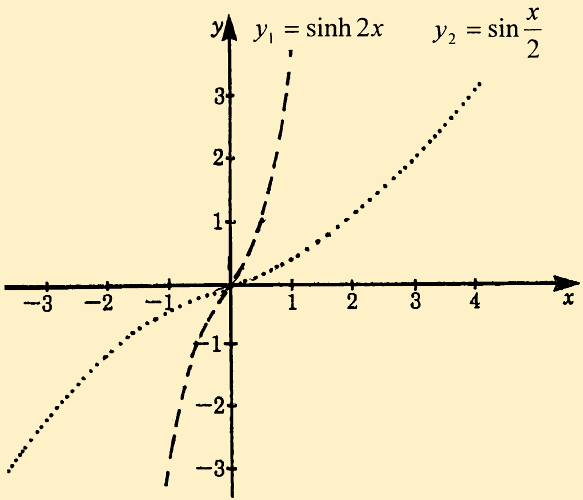
75



Give the definition of $\cosh x =$ ....

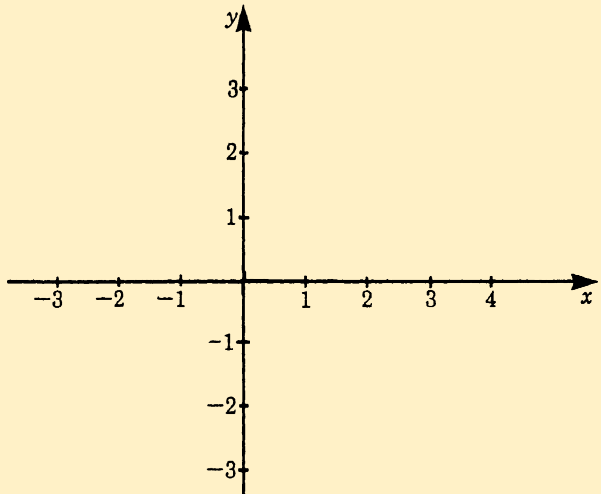
76

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

76

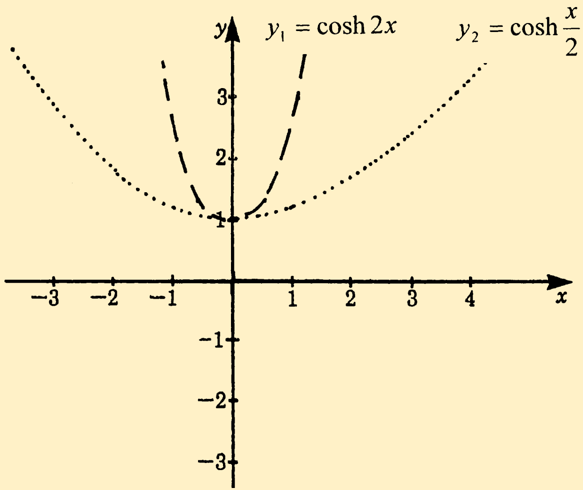
Sketch $y_1 = \cosh 2x$ with a dashed line

and $y_2 = \cosh \frac{x}{2}$ with a dotted line



77

77



Calculate $(\cosh 2x)^2 - (\sinh 2x)^2 = $ ....

78

$(\cosh 2x)^2 - (\sinh 2x)^2 = 1$

78

Give the definition of $\tanh x$. Try first to answer without consulting the textbook

$\tanh x =$ .... = ... = ...

------------------▷ 79

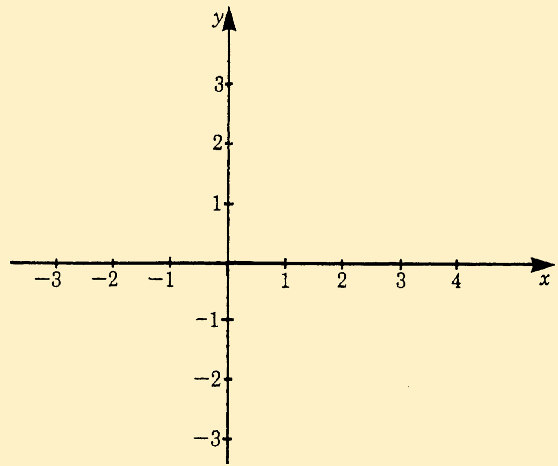
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

79

Try to sketch

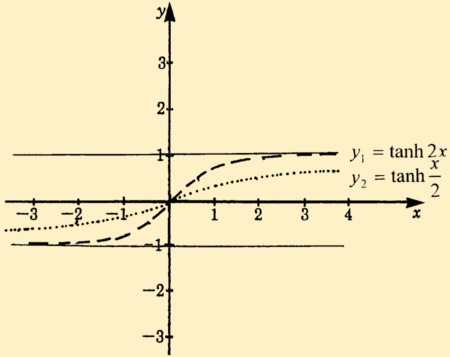
$y_1 = \tanh 2x$ with a dashed line and

$y_2 = \tanh \dfrac{x}{2}$ with a dotted line



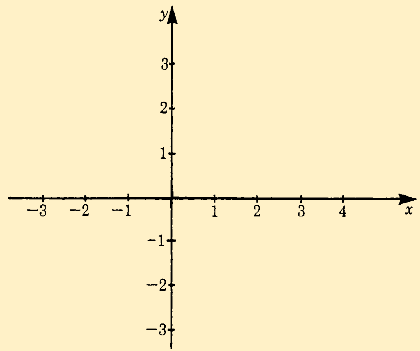
80

80

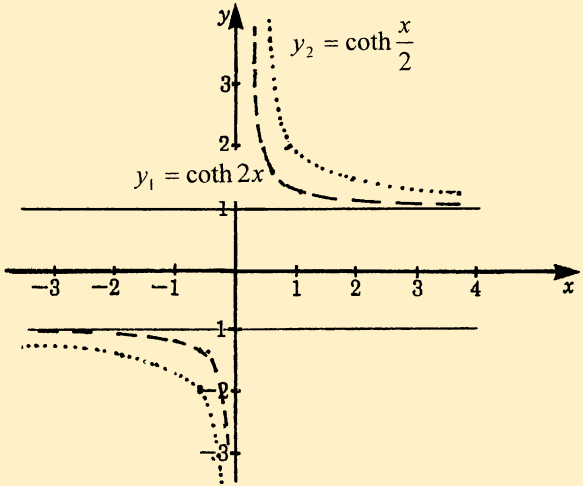


Try to sketch $y_1 = \coth x$ with a dashed line

and $y_2 = \coth \dfrac{x}{2}$ with a dotted line



------------------▷ 81



81

In the textbook we give the definition of the inverse hyperbolic functions.

If you wish to derive one of them --------------------▷ 82

If you want to skip derivation of $\sinh x^{-1}$ --------------------▷ 87

Given the hyperbolic sine: $f(x) = \sinh x = \dfrac{1}{2}\left(e^{x} - e^{-x}\right)$ 82

We may write it $y = \dfrac{1}{2}\left(e^{x} - e^{-x}\right)$

To obtain the inverse function we change y and x

$x =$ ……………………

------------------▷ 83

83

$$x = \frac{1}{2}\left(e^{y} - e^{-y}\right)$$

$$2x = \left(e^{y} - e^{-y}\right)$$

Multiplying by e^{y} gives

$$2x \cdot e^{y} = \ldots\ldots\ldots\ldots\ldots\ldots$$

84

$2xe^{y} = \left(e^{2y} - 1\right)$

84

Substituting $e^{y} = a$

gives

$2xa = \ldots\ldots\ldots\ldots\ldots$

85

$2xa = a^2 - 1$

85

Thus, we get a quadratic function for a.
We already know to solve it:

$a = x + \sqrt{x^2 + 1}$

Since $a = e^y$ we obtain
$e^y = x + \sqrt{x^2 + 1}$

To obtain the inverse function y we take the logarithm and obtain $y =$……………..

------------------▷ 86

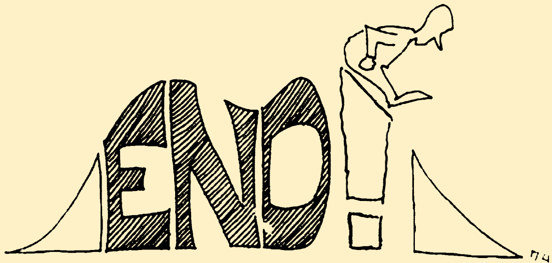
86

$y = \ln\left(x + \sqrt{x^2 + 1}\right)$

Well done.

If you worked through this rather rough matter you will use hyperbolic functions later on in advanced mathematics.

87



You have reached the end of chapter 4

Chapter 5
Differential Calculus

5.1 Sequences and Limits

2

Objective: Concepts of sequences of numbers, limit of a sequence of numbers, convergent sequence of numbers, divergent sequence of numbers.

Depending on your previous knowledge, now choose:
Concepts are known, go to

--------------------▷ 18

Concepts are not known, or revision is needed:

READ: **5.1.1 The concept of sequence**
5.1.2 Limit of a sequence
Textbook pages 87–90

--------------------▷ 3

The expression
$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$
is called ....
a_n is the ....

$\boxed{3}$

$\boxed{4}$

a sequence of numbers
general term

4

State the first five terms of the sequence of numbers, given that

$$a_n = \frac{(-1)^n}{(1+n)^2}$$

....

5

$$-\frac{1}{2}, \frac{1}{5}, -\frac{1}{10}, \frac{1}{17}, -\frac{1}{26}, \ldots$$

5

The above sequence of numbers is a

☐ convergent sequence

☐ divergent sequence

Has the sequence a limiting value?

☐ Yes

☐ No

If yes, which one? ....

6

convergent sequence

6

Yes: $\lim\limits_{n\to\infty} \frac{(-1)^n}{1+n^2} = 0$

If the general term of a sequence of numbers tends to a fixed value as n grows beyond all bounds ($n \to \infty$) this value is called ....

Calculate

$$\lim_{n\to\infty} \frac{1}{n+10} = \ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 7

the limiting value
0

7

If you are experiencing difficulties with the concepts, then you are advised to study the textbook again.

When doing this, write down the concepts on a separate sheet of paper with short explanations. With the help of keywords try to reproduce the definitions aloud.

8

Calculate

8

$$\lim_{n\to\infty} \frac{1}{n} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty} \frac{2}{n} + 3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

9

$$\lim_{n\to\infty}\frac{1}{n}=0$$
$$\lim_{n\to\infty}\frac{2}{n}+3=3$$

9

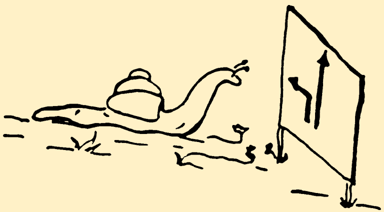
Calculate the following three limits:

$$(1)\ \lim_{n\to\infty}\frac{1}{\sqrt{n}}=\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(2)\ \lim_{n\to\infty}\left(3+\frac{1}{n^2}\right)=\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(3)\ \lim_{n\to\infty}\frac{(-1)^n}{2^n}=\ldots\ldots\ldots\ldots\ldots\ldots$$

10

0
3
0

10



No errors ▷ 14

Errors, or further explanation on calculating limits required ▷ 11

One method that is successful in many cases for the determination of limits is as follows: 11

We try to transform the numerator and denominator of the expression so that we obtain integral powers of $\frac{1}{n}$.

The reason is clear; we know that for $\frac{1}{n}$ the limit vanishes because the denominator grows larger and larger. All the terms disappear in the limit. The limiting value of the expression follows from the remainder.

Example:

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}$$

By a simple transformation we have

$$\lim_{n\to\infty}\sqrt{\frac{1}{n}}=\sqrt{0}=0$$

Try the following for yourself:

$$\lim_{n\to\infty}\frac{n}{3+n}=\ldots\ldots\ldots\ldots\ldots\ldots$$

$\dashrightarrow$ 12

1

12

Let us calculate step by step:

$$\lim_{n\to\infty} \frac{n}{3+n} = ?$$

We transform the expression in order to obtain terms involving $\frac{1}{n}$; thus:

$$\lim_{n\to\infty} \frac{n}{3+n} = \lim_{n\to\infty} \left(\frac{n}{n} \times \frac{1}{\frac{3}{n}+1} \right) = \lim_{n\to\infty} \frac{1}{\frac{3}{n}+1} = \frac{1}{0+1} = 1$$

What is $\lim\limits_{n\to\infty} \frac{n+2}{n+4} = ?$...........................

13

1

13

Another sequence of numbers whose limiting value is 0 is:

$$a_n = \frac{1}{2^n}$$

Here the denominator grows beyond all bounds as $n \to \infty$, and the term vanishes.

Generally:

$$\lim_{n\to\infty} \frac{1}{c^n} = 0, \quad \text{if} \quad c > 1$$

Calculate the limiting value of

$$\lim_{n\to\infty} 2\frac{3^n}{3+3^n} = \text{.....................}$$

14

2

14

Determine

$$\lim_{n\to\infty} \frac{3n^2 - 2}{2n + n^2}$$

The result is ....

17

I need further help

15

The exercise was $\lim_{n\to\infty} \frac{3n^2 - 2}{2n + n^2}$ 15

We try to transform the numerator and denominator so that we obtain integral powers of $\frac{1}{n}$. We factorise out the highest power of n; here it is n^2, so that we have

$$\lim_{n\to\infty} \frac{n^2\left(\ldots\ldots\right)}{n^2\left(\ldots\ldots\right)}$$

Fill in the brackets!

16

$$\lim_{n\to\infty} \frac{n^2\left(3-\dfrac{2}{n^2}\right)}{n^2\left(\dfrac{2}{n}+1\right)}$$

16

If you have had difficulties, convince yourself of the correctness of the expression by carrying out the multiplications.

The expression now becomes:

$$\lim_{n\to\infty} \frac{3-\dfrac{2}{n^2}}{\dfrac{2}{n}+1} = \dots\dots\dots\dots\dots\dots$$

Further hint: Determine the limiting values for the numerator and denominator separately. You should be able to do this!

-------------------▷ 17

3

17

The principle behind the determination of limiting values is often as has just been described. The expression is transformed so that we obtain terms which we know to disappear in the passage to the limit.

Terms such as

$$\frac{1}{n},\ \frac{1}{\sqrt{n}},\ c^{-n},\ \frac{1}{2^n},\ \text{and more}$$

can be treated in this way.

You will find more exercises in the textbook. You should do the exercises until you have no difficulty with their solution.

18

Test yourself!

The expression $a_1, a_2, \ldots, a_n, a_{n+1}$ is called ....

a_n is the ....

18

$$\lim_{n\to\infty} \frac{n^2+2n+1}{5n^2+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{n\to\infty} \frac{1}{2+2^{-n}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Which sequence is divergent?

☐ $a_n = \dfrac{(-1)^n}{n^2}$

☐ $b_n = n(n-1)^n$

19

a sequence of numbers

19

a_n = the general term

$\frac{1}{5}$

$\frac{1}{2}$

a_n : convergent sequence of numbers

b_n : divergent sequence of numbers

If you still have doubts, try the appropriate exercises at the end of Chapter 5 of the textbook.

20

5.2 Limit of a Function, Continuity

20

Objective: Concepts of limit of a function, continuity.

Concepts are known

34

Concepts are new:

READ: **5.1.3 Limit of a function**
5.1.4 Examples for the practical determination of limits
5.2 Continuity
Textbook pages 91–93

21

The concept of the limit of a sequence of numbers can be extended to functions. 21

We consider limits of functions for $x \longrightarrow \infty$.

We also consider limits for $x \longrightarrow x_0$.

This means that the limits are calculated for a fixed value of x, e.g. $x_0 = 0$.

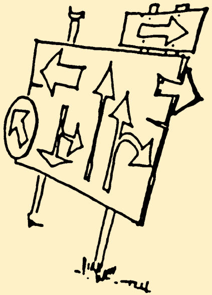
We indicate this under the lim sign as follows: $\lim\limits_{x \to x_0}$

22

Given that $f(x) = \frac{1}{x^2}$,
it is required to calculate the limiting value

22

$$\lim_{x \to 2} = \frac{1}{x^2}$$



☐ 0 ---------------------▷ 23

☐ $\frac{1}{2}$ ---------------------▷ 24

☐ $\frac{1}{4}$ ---------------------▷ 25

☐ ∞ ---------------------▷ 26

Wrong, unfortunately!

23

The limiting value of the function as $x \to 2$ was required; this means the value of the function at $x_0 = 2$.

Your error was probably that you calculated the limiting value for $x \to \infty$.

From now on we have to be very careful and be certain of the value of x_0 the limit is required for. This is clearly stated under the lim sign.

Calculate again:

$$\lim_{x \to 2} \frac{1}{x^2} =$$

☐ $\frac{1}{2}$ ⊳ 24

☐ $\frac{1}{4}$ ⊳ 25

☐ ∞ ⊳ 26

There is an error.

24

It is possible that you have calculated the value of $\lim\limits_{x \to 2} \frac{1}{x}$.

But the given function is $f(x) = \frac{1}{x^2}$.

Calculate again:

$$\lim_{x \to 2} \frac{1}{x^2} =$$

- ☐ 0 --→ 23
- ☐ $\frac{1}{4}$ --→ 25
- ☐ ∞ --→ 26

Correct!

25

We must always pay attention to the value of x_0 for which the limit of $f(x)$ is required.

27

Wrong, unfortunately! 26

This result can only be obtained when calculating the limit for $x \to 0$. But the required limit was for $x \to 2$; this means that we wish to calculate the value of the function at $x_0 = 2$.

Try again

$\lim\limits_{x \to 2} \frac{1}{x^2} =$

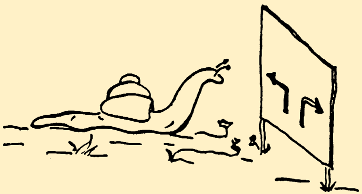
☐ 0 ▷ 23

☐ $\frac{1}{2}$ ▷ 24

☐ $\frac{1}{4}$ ▷ 25

Calculate the following limit: 27

$$\lim_{x\to 0} \frac{x^2+6x}{2x} = \dots\dots\dots\dots\dots\dots$$



Solution found ⟶ 29

Help is required ⟶ 28

Wanted: $\lim\limits_{x \to o} \frac{x^2+6x}{2x}$ 28

In this expression both the numerator and the denominator approach the value zero as $x \to 0$.

This gives the indeterminate expression $\frac{0}{0}$.

We have to try to transform the expression in order to obtain another expression which has a determinate form. One way, in this case, is to factorise out x from the numerator and denominator. The remainder will then be determinate.

$$\lim_{x \to 0} \frac{x^2+6x}{2x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

----------------------▷ 29

$$\lim_{x \to 0} \frac{x^2 + 6x}{2x} = 3$$

29

All correct

31

Detailed explanation required

30

It was required to obtain

30

$$\lim_{x \to 0} \frac{x^2 + 6x}{2x}$$

In this expression both the numerator and denominator approach zero as $x \to 0$, resulting in an indeterminate form. We therefore factorise out x so that the expression becomes

$$\frac{x^2 + 6x}{2x} = \frac{x(x+6)}{2x} = \frac{x+6}{2} = \frac{x}{2} + 3$$

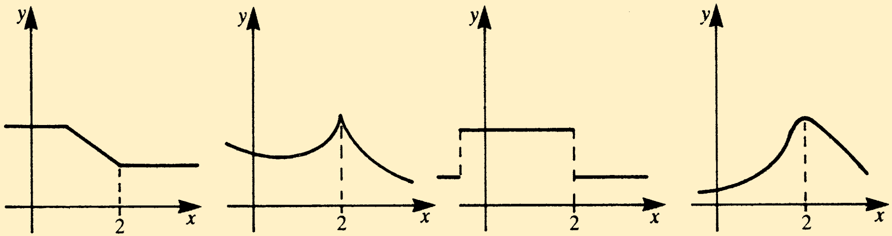
We know that $\lim_{x \to 0} \frac{x}{2} = 0$,

consequently $\lim_{x \to 0} \frac{x^2 + 6x}{2x} = \lim_{x \to 0} \left(\frac{x}{2} + 3 \right) = 3$

31

Which of the following functions are not continuous at $x = 2$?

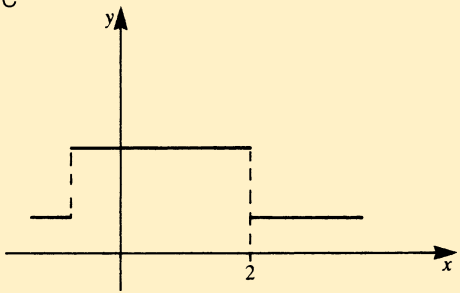
31



☐ A ☐ B ☐ C ☐ D

32

C



This function is discontinuous where $x = 2$.

32

At a point of discontinuity the function 'jumps'. The limit of the function when approached from the left is, in the given case, different from the limit when approached from the right.

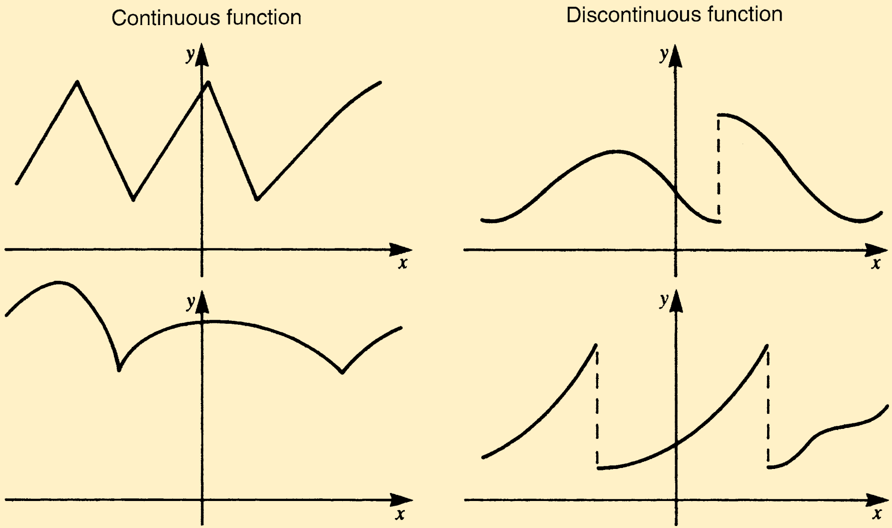
Is it allowable for a continuous function to possess a sharp point or cusp?

☐ Yes
☐ No

33

Yes

33



34

We now have a short test to see if you have understood the subject of limits.

34

$$\lim_{x\to 0} \frac{x^2+1}{x-1} = \ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to 2} \left(\frac{1}{x}\right) = \ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to 0} \frac{x^2+10x}{2x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{x\to\infty} \mathrm{e}^{-x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

35

-1

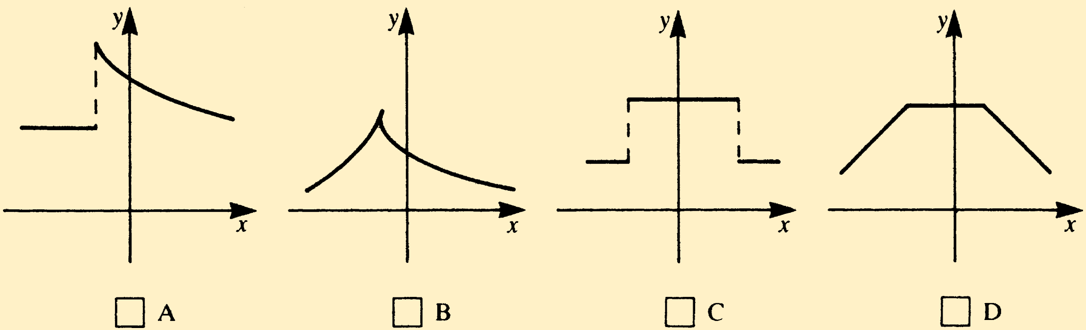
$\frac{1}{2}$

5

0

35

Which functions are continuous?

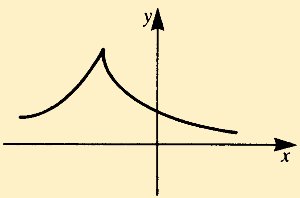


36

Continuous functions:

36

B



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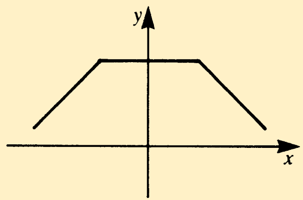


Have you made errors or experienced difficulties?

No difficulties, answers correct --------------------▷ 37

Yes, I experienced difficulties. In this case it is advisable to consult the textbook.

READ: 5.1.3 Limit of a function
5.1.4 Examples for the practical determition of limits
5.2 Continuity
Textbook pages 91–93

Then try the exercises in this study guide again. --------------------▷ 21

As you progress through this programmed text there are two important aspects concerning working periods and breaks: 37

(i) fixing the end of the break, and
(ii) . . . .

38

observing the end of the break

In general, the start of a break accords with one's inclination.

The end of a break does not always accord with one's inclination.

38

39

Success in learning depends very much on attentiveness and concentration on the subject matter. 39

Concentration depends on many factors:

Tiredness in which case a break or sleep is needed.

Interest in the subject.

Attitude towards study.

Timing of working periods.

Disturbances (noise, interruptions).

Physical and *psychological* state.

The influence of these factors on concentration is obvious and can be demonstrated experimentally, but we can modify their effects within certain limits.

Therefore, let us examine these effects.

40

Tiredness can be counteracted by a limited break after a defined working period. 40

Interest in the subject normally increases with progress but decreases with failure. Thus giving correct answers tends to increase your interest.

Timing: For one working period, limit your target; e.g. one section in the textbook. Do not try to take too big a bite at a time.

Do not always regard a *disturbance* as a welcome distraction. The learner who is concentrating on a piece of work has the right to scare off intruders in a friendly but determined way.

--------------------▷ 41

From the foregoing it follows that: 41

Absorbing, digesting and memorising subject matter are very much dependent on attention and concentration.

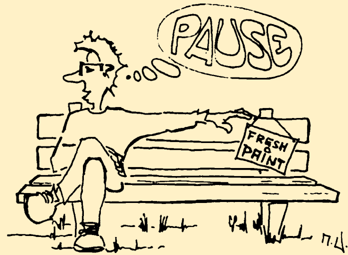
If you have problems with your concentration, then you must first trace the disturbances which affect your learning process and result in weakness in concentration, then try to discover the causes of them and take action accordingly.

Often a change in the working plan is beneficial.

--------------------▷ 42

Now a break is advisable!

42



43

5.3 Series

43

Objective: Concepts of series, geometric series, finite series, summation sign.

READ: **5.3 Series**
5.3.1 Geometric series
Textbook pages 94–96

44

The infinite series: $1+4+9+16+$ .... is abbreviated to 44

Example of a geometric series: ....
....

45

$\sum_{n=1}^{\infty} n^2$ or, using a different variable, by $\sum_{j=1}^{\infty} j^2$

45

$$a + aq + aq^2 + \ldots + aq^{r-1} \text{ or}$$

$$1 + x + x^2 + x^3 + \ldots + x^n$$

Given the *sequence* of odd numbers:

$$1,\ 3,\ 5,\ 7, \ldots,\ 19$$

write down the corresponding *series*.

. . . .

--------------------▷ 46

$$1+3+5+7+\ldots+19$$

46

Let the sum of this series be s_r, then

$$s_r = 1+3+5+7+\ldots+19$$

Express this series using the summation sign!

$$s_r = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Instead of n we could take v, for example, as the variable.

We have to learn to manage expressions with different symbols.

--------------------▷ 47

$$s_r = \sum_{v=0}^{9} (2v+1) = \sum_{v=1}^{10} (2v-1)$$ 47

Completely correct ▷ 51

Error in specifying the bounds ▷ 49

Error in obtaining an expression for the general term ▷ 48

First let us consider the general term of a very simple sequence, the sequence of positive even numbers: 48

$$2,\ 4,\ 6,\ 8,\ \dots,\ 20$$

The factor of the general term of this sequence is

$$a = 2$$

If we use the same variable n as in the textbook we obtain

$$a_n = 2n$$

49

The given series was $1+3+5+7+\ldots+19$ 49

The series should be expressed using the summation sign. Let us take v as the variable.

(i) **Solution:** The general term is $a_v = 2v - 1$.

$$\begin{aligned} \text{Proof: for } v = 1, &\quad a_1 = 1 \\ \text{for } v = 2, &\quad a_2 = 3 \\ &\vdots \\ \text{for } v = 10, &\quad a_{10} = 19 \end{aligned}$$

In this case the bounds are $v = 1$ to $v = 10$, hence $s_r = \sum_{v=1}^{10} (2v-1)$

(ii) **Alternative solution:** The general term is $a_v = 2v + 1$.

$$\begin{aligned} \text{Proof: for } v = 1, &\quad a_1 = 1 \\ \text{for } v = 2, &\quad a_2 = 3 \\ &\vdots \\ \text{for } v = 10, &\quad a_{10} = 19 \end{aligned}$$

In this case the bounds are $v = 0$ to $v = 9$, hence $s_r = \sum_{v=0}^{9} (2v+1)$

50

Write with the summation sign: 50

A) $3+7+11+\ldots+31 =$ ....
B) $5+5^2+5^3+\ldots+5^{11} =$ ....

The solution is given below this time:

$$\text{A)}\ \sum_{v=1}^{8}(4v-1) \qquad \text{or} \qquad \sum_{v=0}^{7}(4v+3)$$

$$\text{B)}\ \sum_{v=1}^{11}5^v$$

51

Given the following series: 51

$$\begin{aligned} s_r &= 5 \times \frac{1}{2} + 5 \times \frac{1}{4} + 5 \times \frac{1}{8} + \ldots \\ &= 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + 5\left(\frac{1}{2}\right)^3 + \ldots \end{aligned}$$

Such a series is called: ....

52

a geometric series

52

With the aid of the textbook calculate the sum of the series

$$S = 5 + 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \ldots$$

$$S = 5\sum_{v=0}^{\infty}\left(\frac{1}{2}\right)^v$$

$$S = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

53

$$S = 5 \times \frac{1}{\left(1 - \frac{1}{2}\right)} = 5 \times 2 = 10$$

53

Did you obtain this result?

☐ Yes 55

☐ No 54

You need some assistance. Here it is. 54

The infinite series

$$a + aq + aq^2 + \ldots = \sum_{v=0}^{\infty} aq^v$$

converges for $|q| < 1$. It then has the value

$$S = a\left(\frac{1}{1-q}\right)$$

In our example the series is

$$5 + 5\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right)^2 + \ldots$$

It follows that $a = 5$ and $q = \frac{1}{2}$ by comparison. The series converges to the limit

$$S = 5 \times \frac{1}{1-\frac{1}{2}} = 5 \times 2 = 10$$

-------------------▷ 55

In the programmed study guide, we often mention 'concepts' and 'operations' which we have previously studied in the relevant sections in the textbook. 55

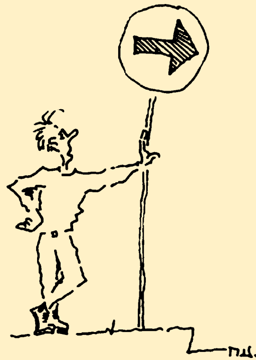
Do you find that after reading a particular section with the impression of understanding the subject you are sometimes unable to completely recall the concepts afterwards?

☐ Yes
57

☐ No
56

You are remarkably talented!

56



Nevertheless, read on!

57

57

Some remarks on reading skills:

Intensive reading

No one is able to retain all that he reads. For an average person the rate at which information is perceived is 10 to 20 times greater than the rate at which it is stored in the memory. In other words, we are able to perceive, to read, to hear and to grasp and to understand much more than we memorise. For instance, try to reproduce a lecture which you have understood on a subject of great interest to you!

Everyone is always surprised how little has been retained.

It is the aim of many reading techniques to enable you to .... more of the subject.

--------------------▷ 58

It is the aim of many reading techniques to enable you to *memorise* more of the subject.

58

You may have discovered that there is a particular pattern in this programmed study guide. Questions about new concepts are asked and are followed by exercises. The pattern is:

— A new concept is presented.
— This new concept is then written down by you from memory.

The reason for this: What we write down is memorised better.

59

What we write down is memorised better than what we merely read. 59

Understanding is all-important in mathematics. We understand things better if we know the concepts used in a test or in a lecture. Mathematics, physics and engineering are coherent subjects which require a special technique for their study. What 'coherent' means is best illustrated by an example.

In section 5.1.4 we calculated the following limit:

$$\lim_{x\to 0} \frac{\sin x}{x}$$

You can follow the chain of reasoning only if you know the concepts of limit and of sine and cosine. The concept of trigonometric functions can only be understood if you know what a function is. And a function can only be understood if you know, at least, the fundamental operations of arithmetic.

Such a sequence can be extended and the meaning should be immediately clear. We can only understand facts if certain prerequisites are known. Subjects in which there exist many relations with very long chains of prerequisites are called .... subjects.

--------------------▷ 60

coherent

60

A person who wants to know where Tokyo is located does not need to know where Paris is, or a person who wants to learn where Tunis is does not need to know how long the River Nile is. These geographical data are not coherent.

The degree of coherence of a subject has an influence on the most appropriate method of study. With a coherent subject we need to study intensively. (At school the teacher took care of these things, but when you study by yourself you have to take over this role to some extent.) Mathematics and physics are coherent subjects.

To learn intensively means, follow the subject matter actively:

(i) you should not accept anything that you have not understood;
(ii) you recognise, sum up, extract and repeat fundamental concepts and rules.

This last point is explained in the next frame.

--------------------▷ 61

In the textbook new concepts are often written in italics. Definitions and rules are set out. 61

What then is the best way to learn new concepts and definitions?

☐ By careful reading

--------------------▷ 62

☐ By reading again and again until you know them by heart

--------------------▷ 63

☐ By extracting (taking notes) and repeating

--------------------▷ 64

Reading with care is good but it may lead to self-deception. In order to ensure that a new concept has been mastered we need checks. **62**

In this study guide such checks are given and you are required to carry them out. Further on we shall talk about a method of study which will help you to carry out these checks yourself.

Here is a hint: mutual questioning and solving problems with fellow students is helpful in carrying out these checks.

Reading with care is not sufficient.

How do we learn new concepts and definitions?

☐ By reading again and again until we know the subject by heart

------------------▷ 63

☐ By extracting (taking notes) and repeating

------------------▷ 64

Reading a definition or an explanation of a new concept again and again until you know it word for word is a bit dull. The danger is that you learn the words but not the meaning. 63

An effective procedure is to extract new concepts and definitions. Extracting (taking notes) means writing down key words. The most important part of the text is taken out. By doing so we have to think and digest the content. The notes only need to be such that we can later reconstruct the meaning.

64

Yes, good. In fact, extracting is the most effective method of memorizing something new. Extracting means to take out of the text the most salient parts. These are mostly new concepts, rules and definitions as well as short explanations. These extracts only need to be so detailed that we can later reconstruct the meaning. 64

65

Active learning is more effective than passive learning.

On the next page we describe a psychological experiment to illustrate the effect of active learning.

65

--------------------▷ 66

If you are not interested in the experiment and its results, proceed to

--------------------▷ 67

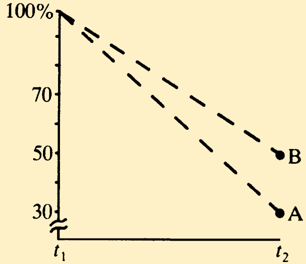
Two groups of students A and B write at time t_1 a dictation of foreign words.

66

Group A is given back the dictation with the errors underlined in red. They then have to look up the words they got wrong in order to carry out the corrections. If they have made mistakes in the punctuation they have to look up the rules.

Group B is given back the corrected dictation where the correct way of writing and punctuation has been inserted.

After four weeks (time t_2) they are both given a second dictation. The results are shown in the figure below.



Ordinate: relative number of total errors.

At time t_1 it is the same for both groups.

The decrease in the relative number of errors in group A is attributed to the more active way of learning.

--------------------▷ 67

We not only have to know new concepts and rules, we also need to know how to apply them. 67

By taking notes we learn

☐ to apply the subject matter

68

☐ to memorize the subject matter

69

This is not entirely correct.

By means of extracts it is easier to memorise new concepts. With notes we have the possibility of reproducing the subject matter later on. In doing so we do not necessarily learn how to apply these new concepts. 68

We can only apply those concepts and rules that we know. For the time being then we only deal with the first step, i.e. getting to know.

69

Extracting makes it easier to memorise and retain. 69

If you want to apply new concepts, definitions and rules in a *different* context then you must have them firmly fixed in your memory.

For this purpose extracts are helpful. The method of making extracts is not difficult and like all important methods it is very simple.

Extracting information is an active form of learning.

Now, as an exercise, extract the most salient aspects of section 5.3 in the textbook.

--------------------▷ 70

Have you written down the salient aspects?

70

☐ No, I did not have a piece of paper handy

71

☐ No, I already know the salient aspects

72

☐ Yes

73

No comments

71

73

Well, if you already know the salient aspects of the work the exercise is pointless. You should start to make extracts as soon as new subject matter is presented to you. Please do it! 72

73

You could have written down: 73

Relation between sequence and series;
leading term, last term;
infinite sequence/finite sequence;
geometric series.

We shall use this technique frequently.
From now on you should make a short extract from each text studied intensively.
These extracts can be collected in some kind of order and kept in a binder, folder or filing cabinet.

--------------------▷ 74

5.4 Differentiation of a Function

Objective: Concepts of slope of a curve, difference quotient, differential quotient, derivative, differential.

When learning from the textbook use a scribbling pad to perform calculations. Check the transformations.

A chain of reasoning is better understood and retained if an active part is taken, e.g. if you take notes. It may be tiresome but it is well worth while.

READ: 5.4 Differentiation of a function
Textbook pages 96–102

75

Have you made an extract of section 5.4 and taken notes?

75

☐ Yes

--------------------▷ 76

☐ No

--------------------▷ 78

Splendid. You have applied one of the important techniques of study.

What are the following symbols called?

76

$$\frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$f'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

77

77

$\frac{\Delta y}{\Delta x}$ = difference quotient

$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = differential coefficient

$\frac{dy}{dx}$ = differential coefficient

dx = differential of the independent variable

dy = differential of the dependent variable

$f'(x)$ = derivative of the function $f(x)$

With the aid of your notes it should have been possible to write down the concepts.

80

It is a pity that you did not make extracts. But if you already knew the content of the section very well then you were right not to do so. 78

We make extracts if we want to learn new facts. This method of working is highly recommended.

Now try, from memory, to name the following symbols:

$$\frac{\Delta y}{\Delta x} = \text{..........................}$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \text{..........................}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{..........................}$$

$$\mathrm{d}x = \text{..........................}$$

$$\mathrm{d}y = \text{..........................}$$

$$f'(x) = \text{..........................}$$

------------------▷ 79

79

$\frac{\Delta y}{\Delta x}$ = difference quotient

$\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ = differential coefficient

$\frac{\mathrm{d}y}{\mathrm{d}x}$ = differential coefficient

$\mathrm{d}x$ = differential of the independent variable

$\mathrm{d}y$ = differential of the dependent variable

$f'(x)$ = derivative of the function $f(x)$

All correct ⟶ 81

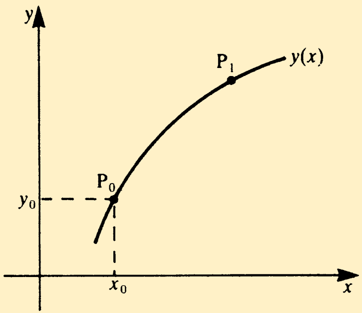
Errors ⟶ 80

While taking notes you memorise new concepts more easily. At the same time you are forced to distinguish between the essential and the non-essential. Finally, you will have in your notes all the important keywords. 80

If you take notes and collect them together they will help you to recall the subject matter more quickly before an examination.

READ again and take notes **5.4 Differentiation of a function**
Textbook pages 96–102

81



The derivative of a function $y(x)$ at the point x_0 has a geometrical significance. Pick out the *wrong* sentence!

81

(a) The derivative $y'(x_0)$ indicates the slope of the secant through $P_0(x_0, y_0)$ and an arbitrary point P_1 on the curve $y(x)$

--------------------▷ 82

(b) The derivative $y'(x_0)$ indicates the slope of the tangent to the curve $y(x)$ at the point $P_0(x_0, y_0)$

--------------------▷ 83

(c) The derivative indicates the slope of the curve $y(x)$ at the point $P_0(x_0, y_0)$

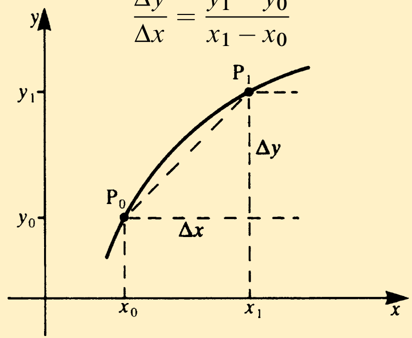
--------------------▷ 85

You are right

82

The slope of a secant through the point P_0 and a point P_1 is not given by the derivative but by the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$



→ 86

Wrong, unfortunately

83

You have not yet entirely understood the geometrical meaning of the derivative of a function.
Have a look again at sections 5.4.2 and 5.4.3 in the textbook and clarify the following statements:

1. The derivative $f'(x)$ of a function indicates the slope of:
   (a) the chord
   (b) the tangent
2. The expression $\frac{\Delta y}{\Delta x}$ is represented by the slope of:
   (a) the chord
   (b) the tangent

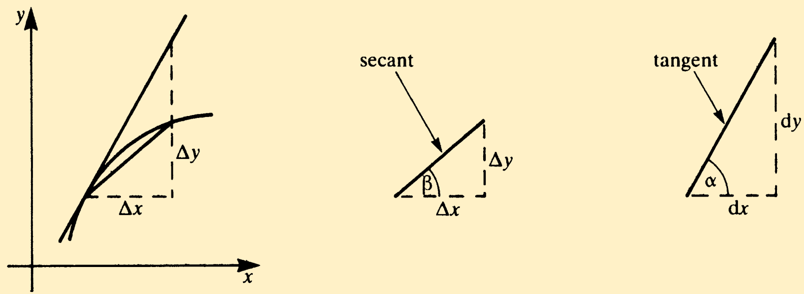
--------------------▷ 84

$f'(x)$ = slope of the tangent

84

$\dfrac{\Delta y}{\Delta x}$ = slope of the chord

If you have still made mistakes, then here is some advice.
Have a look at the following sketches.
On the right are drawn the secant and the tangent.
The angles α and β are the measures of the corresponding slopes.



86

Wrong, unfortunately

85

The question asked for the *wrong* statement about the geometrical significance of the derivative $f'(x)$ of a function at a point x_0. You picked out the *right* answer: the derivative does indeed indicate the slope of the curve at the point $P_0(x_0, \; y_0)$.

Try again; go back to

81

In the textbook we describe the concept of instantaneous velocity. 86

We have to distinguish very clearly between instantaneous velocity and average velocity. In daily life we often distinguish between them very poorly.

The reading on the speedometer of a car shows the .... velocity.

When we speak of average travelling speed we mean the .... velocity, as a rule.

--------------------▷ 87

Speedometer reading: instantaneous velocity
Average travelling speed: average velocity

87

As an aside, let us mention that Newton discovered the differential calculus when investigating velocities and motions (1665–1676, theory of fluxions). At about the same time Leibniz developed the same calculus when investigating mathematical problems (1673–1676).

For derivatives with respect to time Newton used the dot above the variable:

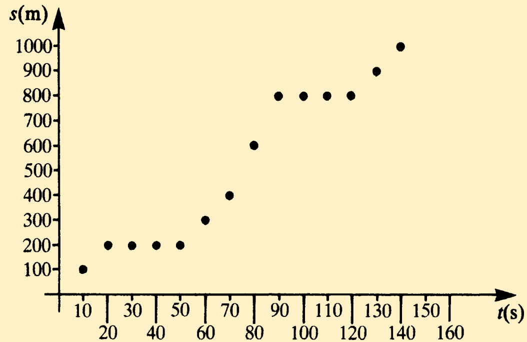
$$\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{s}$$

The passage to the limit $\mathrm{d}t \to 0$ is one of the fundamental mathematical abstractions in science. We are not able to measure arbitrarily small times. This abstraction is, however, confirmed by the conclusions.

88

A car is being driven along a straight main road with many traffic lights. The position of the car is measured at intervals of 10 s and plotted on a graph. The abscissa represents the time and the ordinate the distance covered.

88



The car has stopped ☐ once

☐ twice

☐ three times

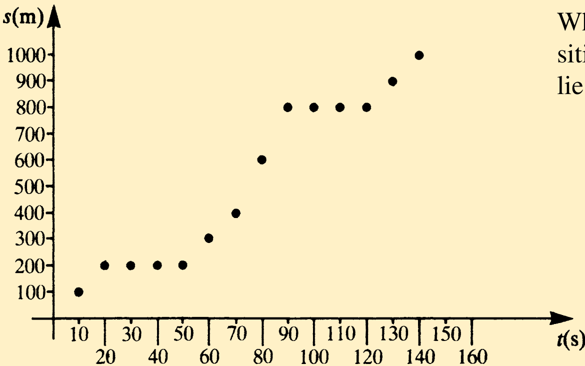
☐ four times

The duration of each stop at the traffic lights is .... seconds.

89

twice
about 30 seconds

89



When the car is at a stop time elapses but its position remains constant. The corresponding points lie on a horizontal line.

Draw a curve through the points on the distance–time graph.

The average speed of the car in the time interval from 60 to 70s after the start of the journey is ....

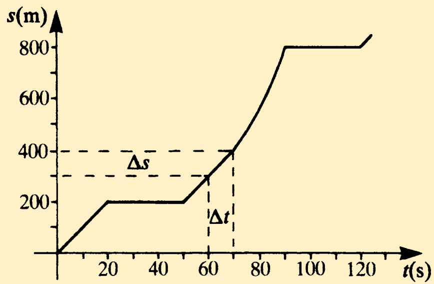
--------------------▷ 90

Average speed $= \dfrac{\Delta s}{\Delta t} = 10\,\mathrm{m/s}$

90

Newton found that the concept of the difference quotient was not sufficient in order to describe the instantaneous velocity, i.e. the velocity at any given instant t. In order to overcome this problem he was led to the concept of the differential coefficient.

The average speed is obtained by dividing the distance covered by the time taken. It is geometrically identical to the determination of the slope of a secant of a curve. The determination of the instantaneous velocity is geometrically identical to the determination of the slope of the tangent to a curve.



The difference quotient indicates the slope of the ....

The differential coefficient indicates the slope of the ....

91

secant
tangent

91

Above all it is important for you to have understood the fundamental idea which led to the solution of the tangent problem discussed in section 5.4.

I have understood the fundamental idea

--------------------▷ 96

I have not understood everything, I need additional explanation

--------------------▷ 92

92

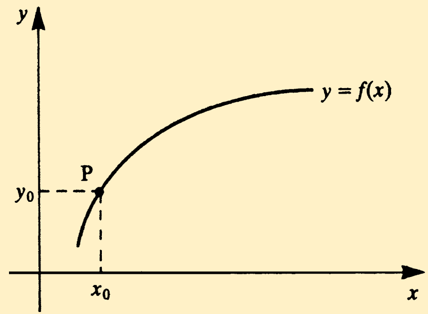
The solution of the tangent problem is now explained in a different way.

Problem: The slope of a curve at a point is to be determined.

Coordinates of the point P = $(x_0,\ y_0)$

Alternatively: $P = (x_0,\ f(x_0))$

The y value is calculated in accordance with the functional relation.



93

93

We draw a secant by joining P to some other point Q.

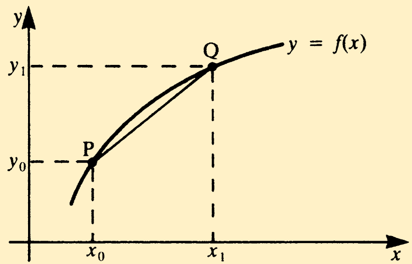
Q has coordinates $Q = (x_1, y_1)$

Alternatively: $Q = (x_1, f(x_1))$

We call Δx the difference in the x-values, i.e. $\Delta x = x_1 - x_0$.

Are you now able to express the coordinates of the point Q in terms of x_0 and Δx?

$$Q = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad f(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots))$$



94

$$Q = (x_0 + \Delta x, f(x_0 + \Delta x))$$

94

We are now in a position to state the slope of the secant.

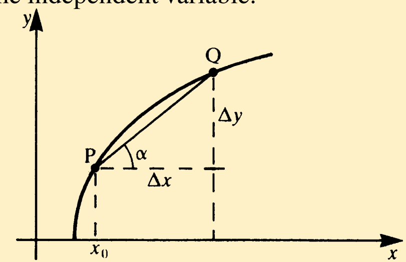
Δy is yet to be defined:

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

Then the slope of the secant is $\tan\alpha = \dfrac{\Delta y}{\Delta x} = \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$

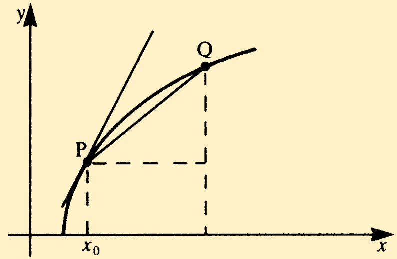
This is a fundamental equation. The numerator expresses the difference between the functional values. The denominator expresses the difference in the independent variable.

As $\Delta x \to 0$ Q is moved closer and closer to P, and we go from the slope of the secant to that of the tangent.



On the graph draw the tangent to the curve at P.

95



95

Now we want to move from the slope of the secant to that of the tangent.

We have to carry out the move to the limit. This is the reason why we discussed the problem of finding limits at the beginning of this chapter. The problem in essence consists of transforming the expression for the slope of the secant in such a way that the limit can be seen to exist. This is done in the textbook for a simple case, namely $f(x) = x^2$.

For many functions the fundamental idea of the proof is always the same; we evaluate the expression $f(x+\Delta x)$ and subtract from it the expression $f(x)$ and obtain Δy. According to the problem, we have to undertake transformations that will enable us to move to the limit. We shall explain this process in greater detail further on. The important thing now is to understand that the tangent problem requires the determination of the limit as $\Delta x \to 0$.

--------------------▷ 96

The differential $\mathrm{d}y$ of the function $y = f(x)$ is defined as

$$\mathrm{d}y = \text{.....................................}$$

96

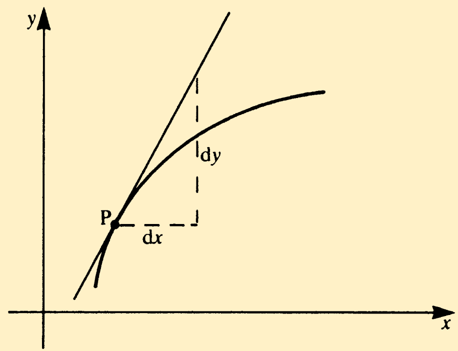
97

$$dy = f'(x)dx$$

97

The slope of the tangent is considered as the ratio of small differentials. If the slope of the tangent is determined it is possible to choose a dx to obtain a corresponding dy for the tangent by the equation

$$dy = f'(x)dx$$

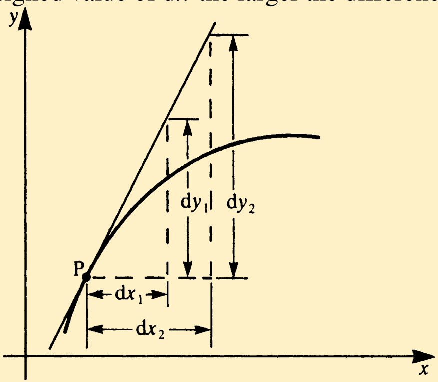


98

The differential coefficient $f'(x) = \dfrac{dy}{dx}$ measures the slope of the tangent to the curve at P. 98

The quotient $\dfrac{dy}{dx}$ is independent of the value assigned to dx.

If we assign a value to dx and the slope of the tangent f' is known then we can calculate the value of the differential dy. These differentials are always measured to the tangent. The larger the assigned value of dx the larger the difference between the tangent and the curve.



$$f'(x) = \frac{dy_1}{dx_1} = \frac{dy_2}{dx_2}$$

99

Let $y = x^2$; then $y' = 2x$

Express dy in terms of y' and dx: 99

$$\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

100

$$y(x) = x^2, \quad y' = \frac{dy}{dx} = 2x$$

100

$$dy = y' dx$$
$$dy = 2x dx$$

Does your result agree with the one above?

Yes 103

No, or further explanation required 101

Let us work out another example: 101

$$y = 3x^4$$

We form the difference quotient $\Delta y = f(x + \Delta x) - f(x)$

$$\Delta y = 3(x + \Delta x)^4 - 3x^4$$

or $\Delta y = 3(4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4)$
Dividing by Δx gives

$$\frac{\Delta y}{\Delta x} = 3(4x^3 + 6x^2\Delta x + 4x(\Delta x)^2 + (\Delta x)^3)$$

As $\Delta x \to 0$ all the terms vanish except for the first one.
Hence

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 \text{ and } \mathrm{d}y = 12x^3\mathrm{d}x$$

102

Further example: $y = 2x^2 + 2$

102

We form the difference quotient

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{2(x+\Delta x)^2 + 2 - (2x^2+2)}{\Delta x} \\ &= \frac{4x\Delta x + 2(\Delta x)^2}{\Delta x} = 4x + 2\Delta x\end{aligned}$$

As $\Delta x \to 0$ it follows that

$$\lim_{\Delta x \to 0} (4x + 2\Delta x) = 4x$$

In order to obtain the difference Δy in the value of the function we have to calculate f for the position x and the position $x + \Delta x$ and form the difference. To obtain the difference quotient we have to divide Δy by Δx. Subsequently we carry out the limiting process.

As Δx approaches zero some terms are usually negligible compared with other terms.

--------------------▷ 103

Try to obtain the differential coefficient of the function $y = x^3$. Follow the procedure shown in the previous examples. 103

Note that if $\Delta x \rightarrow 0$, then $(\Delta x)^2$ and $(\Delta x)^3 \rightarrow 0$ also.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

104

If $y = x^3$ then $\frac{dy}{dx} = 3x^2$ 104

Now it is time to have a BREAK

... and we know how to do it properly.

-------------------▷ 105

5.5 Calculating Differential Coefficients

105

Objective: Application of the rules of differentiation, forming the derivatives of simple functions.

Mastery of the rules of differentiation is an important tool for the scientist and the engineer. Read on for a further hint.

106

Learning technique: intensive reading, extracting (taking notes). 106

If we study a text intensively we must do so actively, e.g. by writing down proofs.

New concepts, definitions and symbols should be extracted.

It is recommended that you collect these extracts and keep them in a ring binder or suitable folder. Later we can use them as a basis for recalling the relevant subject matter.

A first recall should take place before each ....

--------------------▷ 107

break

107

Abstracts and summaries are not exercises in calligraphy but they have to be readable.

A further hint: in order to write extracts clearly (take notes) it is recommended that you write concepts on the left and underline them. Place explanations on the right, using keywords.

Thus you can use your notes more efficiently as a learning aid, to memorise concepts and symbols and to test your knowledge.

First run through: Cover concepts and symbols.
From your explanation you should be able to recall the concepts.
Second run through: Cover explanations.
You should be able to recall the explanations by heart according to the concepts.

It is a sensible alternative idea to write down the concepts on the front side of a card and the explanations in keywords on the back. That way we build up a useful system for learning and revising.

------------------▷ 108

READ (and take notes): **5.5 Calculating differential coefficients**
5.5.1 Derivatives of power functions; constant factors
5.5.2 Rules for differentiation
Textbook pages 102–108

108

109

109

In sections 5.5.1 and 5.5.2 the following rules are dealt with:

Differentiation of

1. ....
2. ....
3. ....
4. ....
5. ....
6. ....
7. ....

----------------------▷ 110

Differentiation of

110

1. power function
2. constant factor
3. sum (sum rule)
4. product (product rule)
5. quotient (quotient rule)
6. function of a function (chain rule)
7. inverse function

Did you recall all the rules?
You should have been able to do so with the help of your notes.

111

The technique of differentiation requires practice.
For this we need:

111

(a) knowledge of the general rules for differentiation;
(b) knowledge of the derivatives of simple functions.

For the time being we shall consider power functions.
Differentiate the following power functions:

1. $y = 5$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
2. $y = x^n$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
3. $y = \frac{1}{x^n}$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
4. $y = \sqrt{x}$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
5. $y = x^{-5/3}$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

- - - - - - - - ▷ 112

112

1. $y' = 0$
2. $y' = nx^{n-1}$
3. $y' = (-n)x^{-(n+1)} = -\dfrac{n}{x^{n+1}}$
4. $y' = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$
5. $y' = \left(-\frac{5}{3}\right)x^{-8/3} = \dfrac{-5}{3\sqrt[3]{x^8}}$

☐ All correct 116

☐ Error in exercise 1 113

☐ Errors in exercises 2–5 114

113

The derivative of a constant is zero.

We have to distinguish between an additive constant and a constant in a product.

Examples:

$$y = a \qquad y' = 0$$

$$y = ax \qquad y' = a$$

$$y = cx^2 \qquad y' = \ldots\ldots\ldots$$

$$y = c + x^2 \qquad y' = \ldots\ldots\ldots$$

Check your answers with the help of the textbook.

Errors in exercises 2–5 $\dashrightarrow$ 114

Exercises 2–5 carried out correctly $\dashrightarrow$ 116

All the exercises were concerned with differentiating a power function 114

$$y = x^n$$

n can be an arbitrary number, it does not have to be integral. The derivative is

$$y' = nx^{n-1}$$

You should know this rule by heart.

Calculate the derivatives of:

$y = x^2$ $\quad$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = \sqrt[3]{x}$ $\quad$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = x^{-2}$ $\quad$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 115

115

$$y' = 2x$$

$$y' = \tfrac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$y' = -2x^{-3} = -\frac{2}{x^3}$$

0

Write down the general expression for the derivative of

$$y = x^n \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Check your answer with the help of the textbook.

116

We now go a step further and differentiate functions composed of power functions. 116

Using your extracts:
If $y = 3x^2 + 2x + 4$
find $y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

117

$$y' = 6x + 2$$

117

We used: sum rule

Differentiate

$$y = 2(x^2 + 1)\sqrt{x}$$

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

----------------▷ 119

Further help required

----------------▷ 118

118

To differentiate

$$y = 2(x^2 + 1)\sqrt{x}$$

We note that it is the product of two functions, namely $2(x^2 + 1)$ and $\sqrt{x}$. We apply the product rule.

You may substitute

$$\begin{aligned} 2(x^2 + 1) &= u(x) \\ \sqrt{x} &= v(x) \end{aligned}$$

Then obtain $u'(x)$ and $v'(x)$ and apply the product rule.

$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

119

$$y' = 4x\sqrt{x} + 2(x^2+1)\frac{1}{2}\frac{1}{\sqrt{x}} = \frac{5x^2+1}{\sqrt{x}}$$

119

We used: product rule, sum rule

Differentiate

$$y = \frac{x}{x^2+1}$$

$$y' = \dots\dots\dots\dots\dots\dots$$

120

$$y' = \frac{1-x^2}{(x^2+1)^2}$$

120

We used: quotient rule

When differentiating, be patient, concentrate and take it steadily; this way you will make fewer mistakes in lengthy exercises.

$$y = \frac{(x+1)\sqrt{x}}{(x-1)}$$

$$y' = \text{............................}$$

121

$$y' = \frac{(x-1)\left[\sqrt{x} + (x+1)\frac{1}{2\sqrt{x}}\right] - (x+1)\sqrt{x}}{(x-1)^2}$$ 121

Rearranging yields:

$$y' = \frac{x^2 - 4x - 1}{2\sqrt{x}(x-1)^2}$$

We now apply the chain rule.

Let us differentiate the function

$$y = (x^2 + 2)^5$$

We could multiply out the expression and apply the rule for power functions $y = x^n$. The chain rule makes the work easier. Substitute $x^2 + 2 = g(x)$ so that

$$y = (x^2 + 2)^5 = (g(x))^5$$

then $y' = \ldots\ldots\ldots\ldots\ldots$

--------------------▷ 123

If you require assistance

--------------------▷ 122

This example is just like the one in the textbook in the section concerning the chain rule. 122

Look at the textbook, study the example and substitute so that:

$$y = (x^2+2)^5 \text{ and with } g = (x^2+2)$$

$$y = g^5$$

then $\dfrac{\mathrm{d}y}{\mathrm{d}g} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and $\dfrac{\mathrm{d}g}{\mathrm{d}x} = g' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

By the chain rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

123

$$y' = 5(x^2+2)^4 2x = 10x(x^2+2)^4$$

123

Derivative of the inverse function.

Given: a function $f(x)$

$$y = 4x + 3$$

First obtain the inverse function $f^{-1}(x)$:

Write it down in two ways.

$$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In case of difficulty consult your textbook, Chapter 1, section 1.4.

The concept of inverse functions is fundamental for the next exercises.

124

$$x = 4y + 3$$
$$y = \frac{x-3}{4} = f^{-1}(x)$$

124

Given: $f(x) = 4x + 3$

Obtain the derivative of the inverse function $f^{-1}(x)$.

Use the rule given in the textbook and check the result by direct computation; differentiating a linear function is easy.

The inverse function reads

$$x = 4y + 3$$
$$y = \frac{x-3}{4}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

125

Applying the rule: $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dfrac{1}{4}$ 125

Direct differentiation of the inverse function gives $\dfrac{dy}{dx} = \dfrac{1}{4}$ too.

Given: $f(x) = y = x^3$

The inverse function $f^{-1}(x)$ reads $x = \dots\dots\dots\dots\dots\dots$

$y = \dots\dots\dots\dots\dots\dots$

Find the derivative:

Step 1: Obtain $\dfrac{dx}{dy} = \dots\dots\dots\dots$

Step 2: Express $\dfrac{dx}{dy}$ as a function of x; $\dfrac{dx}{dy} = \dots\dots\dots\dots$

Step 3: Insert into the formula $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}} = \dots\dots\dots\dots$

126

$$x = y^3; \quad y = \sqrt[3]{x}; \quad \frac{\mathrm{d}x}{\mathrm{d}y} = 3y^2;$$

126

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 3(\sqrt[3]{x})^2 = 3\sqrt[3]{x^2}; \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3\sqrt[3]{x^2}}$$

You can obtain the derivative of inverse functions as follows. Follow these steps with $f(x) = y = x^2 + 2 \qquad (x \geq 0)$

Step 1: Write down the inverse function in both forms

$x = \ldots\ldots\ldots\ldots\ldots \quad y = \ldots\ldots\ldots\ldots\ldots$ Obtain the derivative $\frac{\mathrm{d}x}{\mathrm{d}y} = \ldots\ldots\ldots\ldots\ldots$

Step 2: Express the derivative $\frac{\mathrm{d}x}{\mathrm{d}y}$ as a function of x

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Step 3: Insert into the formula $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 127

127

Step 1: $x = y^2 + 2 \qquad y = \sqrt{x-2}$

Step 2: $\dfrac{dx}{dy} = 2y = 2\sqrt{x-2}$

Step 3: $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x-2}}$

The derivative of the inverse function is a subtle concept, but following the given steps is quite simple.

The necessity of this concept will become clearer later on when the derivatives of, for example, trigonometric and exponential functions are known. We suggest you reread the arguments given in the textbook carefully and try to understand fully the geometrical reasoning. It may prove to be wise to return to this section in the textbook where the derivatives of specific inverse functions are discussed.

--------------------▷ 128

5.6 Differentiation of Fundamental Functions (Part 1)

128

Objective: Differentiation of trigonometric functions and inverse trigonometric functions.

READ (and take notes): **5.5.3 Differentiation of fundamental functions:**

1. Trigonometric functions

2. Inverse trigonometric functions

Textbook pages 108–114

129

First let us practise differentiating trigonometric functions.
Differentiate: 129

$$y = 3\sin x$$
$$y' = \dots\dots\dots\dots\dots\dots$$
$$y = 2\cos x$$
$$y' = \dots\dots\dots\dots\dots\dots$$

130

$$y' = 3\cos x$$
$$y' = -2\sin x$$

130

Obtain the derivative of the tangent function using the quotient rule.
Remember that $\sin^2 x + \cos^2 x = 1$.

$$y = \tan x = \frac{\sin x}{\cos x}$$
$$y' = \text{...........................}$$

--------------------▷ 131

$$y' = \frac{1}{\cos^2 x}$$ 131

As a help, if needed, here is the detailed working:

$$y = \frac{\sin x}{\cos x}$$

$$y' = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

Another way to represent y' is:

$$y' = 1 + \tan^2 x$$

Did you obtain the correct result?

☐ Yes - - - - ▷ 134

☐ No - - - - ▷ 132

Given:

$$y = \sin x \qquad y' = \cos x$$
$$y = \cos x \qquad y' = -\sin x$$

132

Obtain the derivative of the cotangent function by using the quotient rule.

$$y = \cot x = \frac{\cos x}{\sin x} \qquad (\textit{Hint:}\ \sin^2 x + \cos^2 x = 1)$$
$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 133

$$y' = -\frac{1}{\sin^2 x} = -1 - \cot^2 x$$

133

The detailed working was:

$$y = \frac{\cos x}{\sin x}$$

$$y' = \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = -\frac{\sin^2 x + \cos^2 x}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

Another way of representing y' is:

$$y' = -\cot^2 x - 1$$

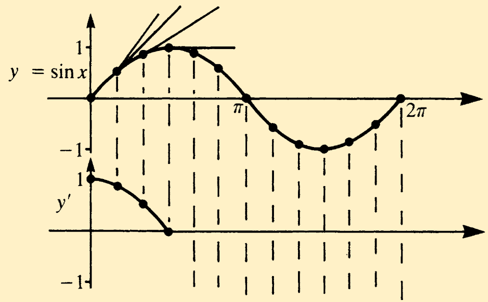
134

Differentiation of trigonometric functions requires knowing that:

134

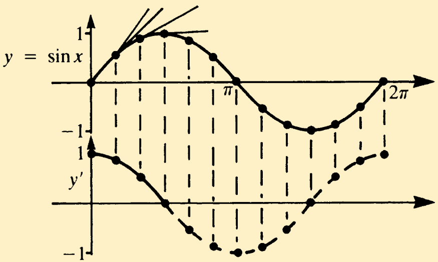
$$y = \sin(ax) \qquad y' = a\cos(ax)$$
$$y = \cos(ax) \qquad y' = -a\sin(ax)$$

We should also understand the geometrical meaning. Hence, referring to the figure which shows the function sin x, we draw the slope at four points.



Now plot on the second diagram the values of the slopes of the tangent for each point.

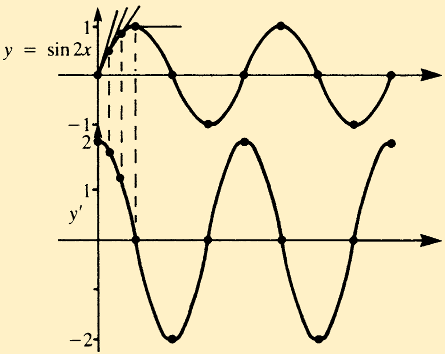
135



Notice that we obtain the cosine function

135

$$y' = \cos x$$



In a similar way we consider the slope of the function $y = \sin(2x)$. The tangents are steeper. In fact all slopes are doubled. This geometrical approach shows clearly that

$$y' = 2\cos(2x)$$

Differentiate:

$y = 7\sin(cx)$ $\qquad$ $y = \frac{1}{5}\cos(6x)$

$y' = \ldots\ldots\ldots\ldots\ldots\ldots$ $\qquad$ $y' = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 136

$$y' = 7c\cos(cx)$$
$$y' = -\frac{6}{5}\sin(6x)$$

136

All correct

-------------------▷ 139

If you made some mistakes differentiate the following:

$$y_1 = 3\sin\left(\tfrac{1}{3}x\right)$$
$$y_2 = 4\cos(2x)$$

with the aid of the textbook.

Then go to

-------------------▷ 137

$$y_1' = \cos\left(\tfrac{1}{3}x\right)$$
$$y_2' = -8\sin(2x)$$

137

You might still have some difficulties.

If we continue now without overcoming them we will not save time, but create more difficulties for you in the future. We must, therefore, isolate them before going any further.

Obtain the derivatives of the following functions:

(1)	power rule	$y = 4x^3$	$y' = \ldots\ldots\ldots\ldots$
(2)	power or quotient rule	$y = \dfrac{1}{2x}$	$y' = \ldots\ldots\ldots\ldots$
(3)	power rule	$y = 3x^{-1/2} + x^{1/2}$	$y' = \ldots\ldots\ldots\ldots$
(4)	chain rule	$y = 7\sin(ax)$	$y' = \ldots\ldots\ldots\ldots$

------------------▷ 138

(1) $y' = 3(4x^2)$

(2) $y' = -\dfrac{1}{2x^2}$

(3) $y' = -\frac{3}{2}x^{-3/2} + \frac{1}{2}x^{-1/2}$

(4) $y' = 7a\cos(ax)$

138

You will find further exercises in the textbook, at the end of each chapter.
You know by now that exercises are useful if they seem difficult to you!

139

139

An alternating current is given by the expression

$$I = I_0 \sin(\omega t + \phi)$$

I_0, ω and ϕ are constants.

Obtain the derivative with respect to the time t

$$\frac{dI}{dt} = \text{..........................}$$

140

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I_0 \omega \cos(\omega t + \phi)$$ 140

All correct 142

I need further explanation 141

You were asked to differentiate with respect to time t: 141

$$I = I_0 \sin(\omega t + \phi)$$

Let us first rename the variables and constants so that:

$$t = x \quad \omega = a$$
$$I = y \quad \phi = c$$
$$I_0 = y_0$$

Hence $y = y_0 \sin(ax + c)$

Now we apply the chain rule $g = (ax + c)$:

$$y = y_0 \sin g$$

$$\frac{dy}{dg} = y_0 \cos g \quad \frac{dg}{dx} = a$$

$$\frac{dy}{dx} = \frac{dy}{dg}\frac{dg}{dx} = y_0 \cos(ax + c)a$$

It follows that with the original notation:

$$\frac{dI}{dt} = \omega I_0 \cos(\omega t + \phi)$$

142

Differentiation of the inverse trigonometric functions.

142

$$y = A\sin^{-1}(x) \qquad y' = \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y = \cos^{-1}(ax) \qquad y' = \dots\dots\dots\dots\dots\dots\dots\dots$$

$$y = \tan^{-1}(ax) \qquad y' = \dots\dots\dots\dots\dots\dots\dots\dots$$

143

$$y' = \frac{A}{\sqrt{1-x^2}}$$ 143

$$y' = \frac{-a}{\sqrt{1-a^2x^2}}$$

$$y' = \frac{a}{1+a^2x^2}$$

The proof is analogous to that in the textbook.

Use it to obtain the derivative of the inverse tangent function.

$$y = \tan^{-1}(x) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

You found the solution ------------------▷ 146

You need help ------------------▷ 144

To differentiate

$$y = \tan^{-1}(x)$$

144

Step 1: $x = \tan y$
hence

$$\frac{dx}{dy} = \frac{1}{\cos^2 y} = 1 + \tan^2 y$$

Step 2: Express $\frac{dx}{dy}$ as a function of x

$$\frac{dx}{dy} = \text{..............................}$$

Step 3: Apply the formula $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\frac{dy}{dx} = \text{......................................}$$

----------------------▷ 146

If you are in doubt try to express $\cos^2 y$ as a function of x using

$$x = \frac{\sin y}{\cos y} = \tan y$$

----------------------▷ 145

We know that $\frac{dx}{dy} = \frac{1}{\cos^2 y} = 1 + \tan^2 y$ 145

Since $x = \tan y$

$$x^2 + 1 = \tan^2 y + 1$$

Inserting into the formula $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$ yields

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

146

$$\frac{\mathrm{d}x}{\mathrm{d}y} = 1 + x^2, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} = \frac{1}{1+x^2}$$

146

Now for a general revision.

In the following examples the more usual notations have been changed; all of them are simple exercises. If you have any doubts replace the unfamiliar symbols by the known ones, i.e. x and y.

$$U = V^2 \qquad U' = \ldots\ldots\ldots$$

$$E_{\text{kin}} = \frac{m}{2}v^2 \qquad \frac{\mathrm{d}}{\mathrm{d}v}(E_{\text{kin}}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$s = \frac{g}{2}t^2 \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$p = \rho h \qquad \frac{\mathrm{d}p}{\mathrm{d}h} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A = A_0 \sin(\omega t) \qquad \frac{\mathrm{d}A}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$S = A_0 \cos(\omega t) \qquad \frac{\mathrm{d}s}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

----------------------▷ 147

$$U' = 2V$$

$$\frac{\mathrm{d}}{\mathrm{d}v}(E_{\mathrm{kin}}) = mv$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = gt$$

$$\frac{\mathrm{d}p}{\mathrm{d}h} = \rho$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A_0 \omega \cos(\omega t)$$

$$\frac{\mathrm{ds}}{\mathrm{d}t} = -A_0 \omega \sin(\omega t)$$

147

148

5.7 Differentiation of Fundamental Functions (Part 2)

148

Objective: Differentiation of the exponential function, the logarithmic function and the hyperbolic functions.

READ: 5.5.3 Differentiation of fundamental functions
Exponential and logarithmic functions
Hyperbolic functions
Textbook pages 108–114

--------------------▷ 149

Obtain the derivatives:

149

$y = 3e^x$ $y' = \dots\dots\dots\dots\dots\dots$

$y = e^{2x}$ $y' = \dots\dots\dots\dots\dots\dots$

$y = 2\ln x$ $y' = \dots\dots\dots\dots\dots\dots$

$y = \ln(3x)$ $y' = \dots\dots\dots\dots\dots\dots$

150

$$y' = 3e^{x} \qquad y' = 2e^{2x}$$

$$y' = \frac{2}{x} \qquad y' = \frac{3}{3x} = \frac{1}{x}$$

150

The charge of a capacitor is given by

$$Q = Q_0 e^{-\frac{t}{RC}}$$

where R = resistance
C = capacitance
Q = charge

The current is given by $\frac{dQ}{dt} = \dot{Q}$

$$\frac{dQ}{dt} = \ldots\ldots\ldots\ldots\ldots$$

151

$$\frac{dQ}{dt} = -\frac{Q_0}{RC}\,e^{-\frac{t}{RC}}$$

151

Obtain the derivatives of the hyperbolic functions

$$y = A\cosh(ax) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = 3\tanh(2x) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

152

$$y' = Aa\sinh(ax)$$
$$y' = 6(1 - \tanh^2 2x)$$

152

Obtain the derivative of the hyperbolic cosine function:

$$y = \cosh x = \frac{1}{2}(e^x + e^{-x})$$
$$y' = \dots\dots\dots\dots\dots\dots$$

153

$$y' = \frac{1}{2}(e^x - e^{-x}) = \sinh x$$ 153

The derivatives of the inverse hyperbolic functions have a special significance in the integral calculus. We shall not, however, derive them here. But for the reader who has a particular interest we shall prove how to obtain the derivative of the inverse hyperbolic tangent.

Do you wish to omit the proof? --------------------▷ 157

Would you like to go through the proof? --------------------▷ 154

The proof follows exactly that of the example in the textbook. 154

To differentiate:

$$y = \tanh^{-1} x$$

Step 1: Obtain

$$x = \text{..............................} \qquad \frac{\mathrm{d}x}{\mathrm{d}y} = \text{.....................................}$$

155

$$x = \tanh y \quad \frac{\mathrm{d}x}{\mathrm{d}y} = 1 - \tanh^2 y$$ 155

Step 2: Substitute for $\tanh^2 y$. $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Can you carry on?
Here is a little more help:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}y}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

156

156

Since
$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

and
$$\frac{dx}{dy} = 1 - x^2$$

the result is:
$$\frac{dy}{dx} = \frac{1}{1 - x^2}$$

157

Now you should be able to solve the exercises in the textbook without too much difficulty. 157

It is important to be able to apply the product rule, the quotient rule and the chain rule with confidence.

Decide for yourself how many exercises you want to do. You may wish to solve the odd-numbered ones first; if you do not experience any difficulty you may assume that you have understood the subject matter.

--------------------▷ 158

5.8 Higher Derivatives

158

Objective: Generalisation of the concept of differentiation, second derivative, formation of higher derivatives.

READ: 5.6 Higher derivatives
Textbook pages 114–115

159

In the following examples we have changed the notation. In one of them the second derivative is required; in the last the fourth derivative is required. 159

$g(\phi) = a\sin\phi + \tan\phi \qquad g'(\phi) = \ldots\ldots\ldots\ldots\ldots\ldots$

$v(u) = u^2 e^u \qquad v'(u) = \ldots\ldots\ldots\ldots\ldots\ldots$

$f(x) = \ln x \qquad f''(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

$h(x) = x^5 + 2x^2 \qquad h^{(4)}(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 160

$$g'(\phi) = a\cos\phi + \frac{1}{\cos^2\phi}$$
$$v'(u) = e^u(2u + u^2)$$
$$f''(x) = -\frac{1}{x^2}$$
$$h^{(4)}(x) = 120x$$

160

All correct ⟶ 163

Errors or difficulties ⟶ 161

Higher derivatives are calculated as follows: 161

Consider the function $y = \ln x$

The first derivative $y' = \dfrac{1}{x}$

The second derivative is obtained by differentiating once more the first derivative with respect to x. Hence

$$y''(x) = \frac{\mathrm{d}}{\mathrm{d}x} y'(x) = \left(\frac{1}{x}\right)' = \frac{-1}{x^2}$$

Similarly for higher derivatives. They are obtained by differentiating successively. Thus the fourth derivative of

$$\begin{aligned} h(x) &= x^5 + 2x^2 : \\ h'(x) &= 5x^4 + 4x \\ h''(x) &= 20x^3 + 4 \\ h'''(x) &= 60x^2 \\ h^{(4)}(x) &= 120x \end{aligned}$$

162

Proficiency in differentiating requires practice. If you are still experiencing difficulties in solving the exercises it is a sign that you have not yet mastered the subject matter. It is therefore necessary for you to practise with more exercises. 162

You will find them at the end of the chapter in the textbook.

163

There now follows a short recapitulation. 163

Differentiate the function of a function:

$$y(x) = f(g(x)) = \sqrt{2x^3 + 5}$$

To do this we apply the *chain rule*.

Recall it, please!

$$y = f(g(x)) \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

164

$$y' = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x} \qquad \text{or} \qquad \frac{\mathrm{d}f}{\mathrm{d}g}g'(x)$$

164

To differentiate $y(x) = \sqrt{2x^3 + 5}$
let $g(x) = 2x^3 + 5$
so that $f(g) = \sqrt{g}$
Obtain the derivative

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

165

$$y'(x) = \frac{3x^2}{\sqrt{2x^3+5}}$$

165

All correct

167

Different result, or difficulties

166

The function $y = f(g(x)) = \sqrt{2x^3+5}$ is made up of 166

$$g(x) = 2x^3 + 5$$
$$f(g) = \sqrt{g}$$

To use the chain rule we have to multiply $f'(g) = \frac{\mathrm{d}f}{\mathrm{d}g}$ ('outer derivative') with $g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x}$ ('inner derivative').

Consequently we obtain the following derivatives:

$$g'(x) = \frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2x^3+5) = 6x^2$$
$$f'(g) = \frac{\mathrm{d}f}{\mathrm{d}g} = \frac{\mathrm{d}}{\mathrm{d}g}(\sqrt{g}) = \frac{1}{2\sqrt{g}}$$

The chain rule gives: $y' = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$

Substituting leads to $y'(x) = \frac{1}{2\sqrt{g}}6x^2 = \frac{3x^2}{\sqrt{2x^3+5}}$

----------------------▷ 167

Obtain the following derivatives:

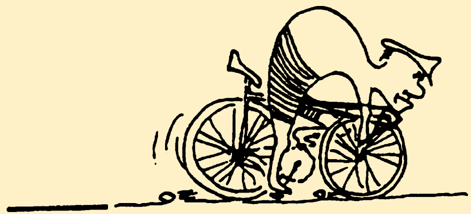
167

$$y = (3x^2 + 2)^2 \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = a \sin(bx + c) \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = e^{2x^3 - 4} \qquad y' = \ldots\ldots\ldots\ldots\ldots\ldots$$

168

$$y' = 12x(3x^2+2)$$
$$y' = ab\cos(bx+c)$$
$$y' = 6x^2 e^{2x^3-4}$$

168



169

You will frequently have to use the

169

— product rule
— quotient rule
— chain rule

Now do the exercises in the textbook corresponding to section 5.6.
For the solutions of the exercises use the table of derivatives at the end of chapter 5 in the textbook.

170

There are two ways of plotting a curve defined by $f(x)$: 170

(1) We tabulate many values of the function. This was explained in Chapter 1. The procedure is laborious and time consuming.

(2) We look for *particular features* of the curve in order to obtain a picture of the *quantitive nature* of the graph.

The second method is most important; it enables us to get a quick picture of the general character of the function. It is usually referred to as curve sketching.

Section 5.7 of the text book deals with the determination of these particular features.

--------------------▷ 171

5.9 Extreme Values and Points of Inflexion; Curve Sketching

171

READ: **5.7.1 Maximum and minimum values of a function**
5.7.2 Further remarks on points of inflexion (contraflexure)
5.7.3 Curve sketching
Textbook pages 115–123

172

(1) What do we call the points of intersection of a function $y(x)$ with the x-axis? 172

(2) What do we call the position x_0 for which in the neighborhood of x_0 the following inequality is valid?

$$f(x) > f(x_0)$$

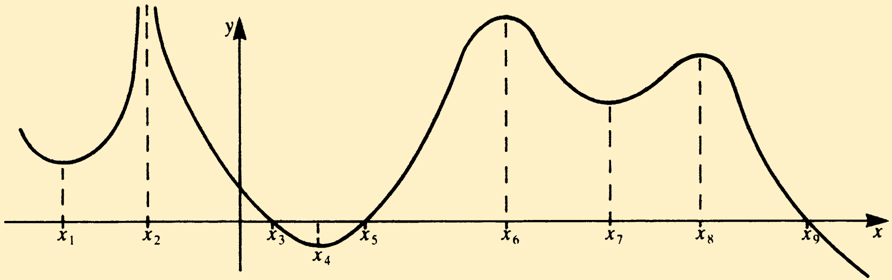
....

----------------------▷ 173

(1) Zeros
(2) Local minimum

173

The graph in the figure shows a fairly complicated curve.



Name the x-values of the following salient points:

Zeros ....
Local maxima ....
Local minima ....
Pole ....

------------------▷ 174

Zeros: $x_3, \quad x_5, \quad x_9$ Local minima: $x_1, \quad x_4, \quad x_7$

Local maxima: $x_6, \quad x_8$ Pole: x_2

174

All correct ----------> 180

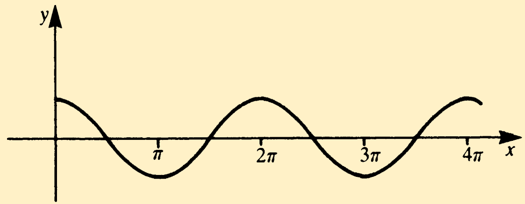
Zeros wrong ----------> 175

Extreme values wrong ----------> 177

Pole wrong ----------> 179

Read once more the definition concerning zero positions in the textbook. Then state the position of all the zero values of the cosine function in the interval 0 to 4π. 175

$y = \cos x$



The positions of the zero values are: ....

176

$$\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5ß}{2}, \quad \frac{7\pi}{2}$$

176

Finding the positions where the function is zero is easy: they occur at the points of intersection of the curve with the x-axis.

All correct --------------------▷ 180

Extreme values wrong --------------------▷ 177

Pole wrong --------------------▷ 179

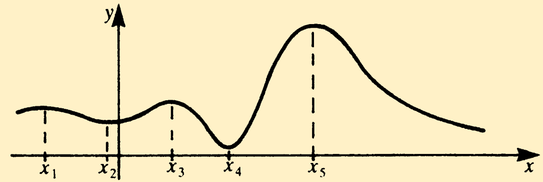
177

Identify the *local maxima* of the graph shown below.

A local *maximum* occurs at a point where the curve has a *peak*.

Similarly a local *minimum* occurs where the curve has a *trough* like the bottom of a valley.

We call certain maxima and minima local because it is possible that at other points there are maxima (or minima) having greater (or smaller) values. A maximum is not absolute unless it is the highest compared with all other maxima. Similarly for an absolute minimum.



Identify the:

Local maxima ....

Local minima ....

Absolute maximum ....

Absolute minimum ....

178

Local maxima: $x_1, \quad x_3, \quad x_5$
Local minima: $x_2, \quad x_4$
Absolute maximum: x_5
Absolute minimum: x_4

178

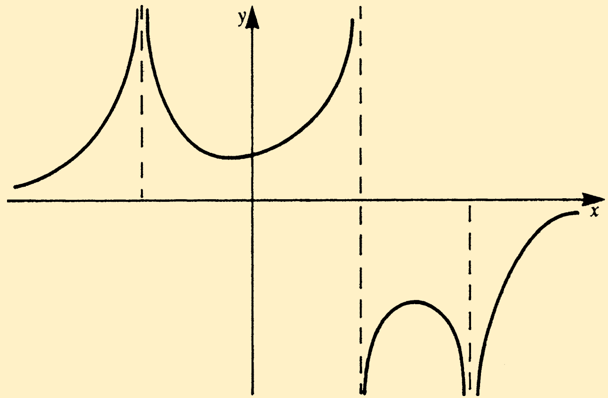
All correct 180

Pole wrong 179

A pole occurs at a position where the function tends to plus or minus infinity. The function under consideration had one pole position. The function shown here has three pole positions.

179

At one position the function tends to plus infinity from both sides, at a second position it tends to plus infinity on one side and minus infinity on the other side, while at the third position it tends to minus infinity from both sides.



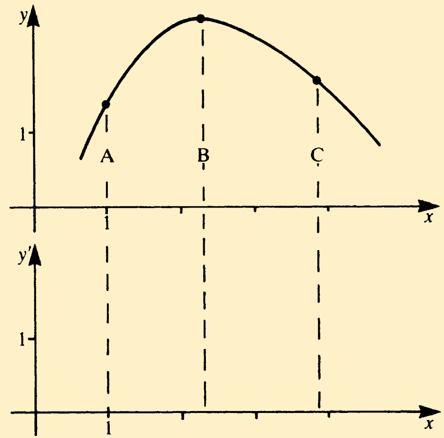
180

One of the advantages of the differential calculus is that it aids us in finding maxima and minima. 180

The function shown below has one maximum. Draw the tangents to the curve at A, B and C.

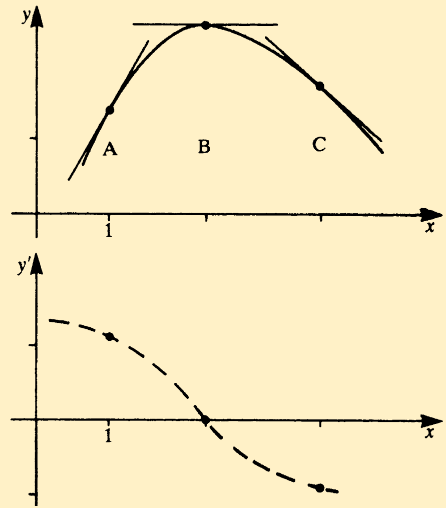
Then plot the value of the slope of the tangent at each point.

This is the function $y'(x)$.



181

Leaving aside the question of the scale of the diagram, we observe that the values of the slope on the left of B are positive while those on the right are negative.

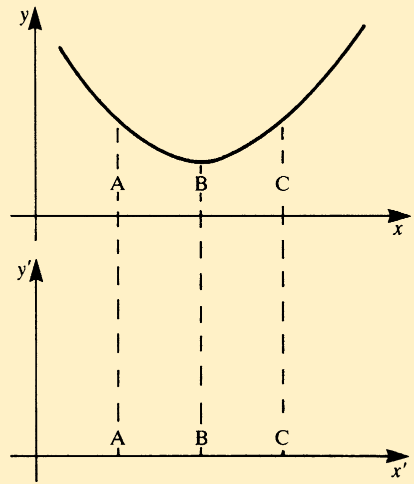


181

Plotted in the lower diagram is the curve representing y'.

182

Let us now carry out the same procedure in the case of a minimum.

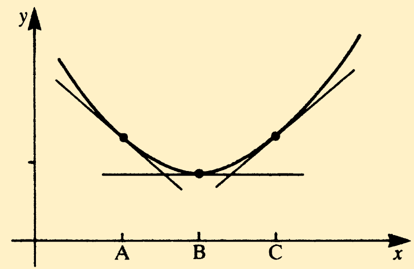


182

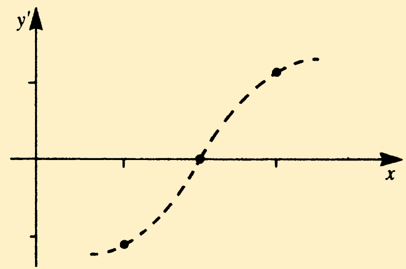
Draw on the curve the tangents at A, B and C and hence sketch the curve of y'.

183

Drawing tangents is not difficult.
Estimating the values of the slopes can only be approximate. What is important is that they are negative to the left of B and positive to the right of B.



183



We also notice that the slope of the curve for y' is different in this case.
The values of y' increase from left to right.
Recall that for a maximum, y' decreases in value from left to right.

184

We now know the conditions required to determine the location x_0 of a maximum or a minimum. 184

In both cases, *the tangent is horizontal*.

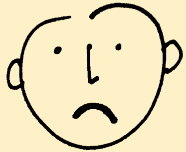
Mathematically this means: $y'(x_0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

185

$$y' x_0 = 0$$

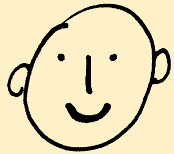
185

For a *maximum*: the value of the derivative decreases from left to right.
Mathematically this means: $y''(x_0) < 0$



max = neg

For a *minimum*: the value of the derivative increases form left to right.
Mathematically this means: $y''(x_0) > 0$



min = pos

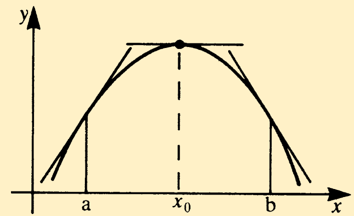
If you have not yet understood everything

----------------▷ 186

Otherwise

----------------▷ 190

We repeat once more the chain of reasoning:



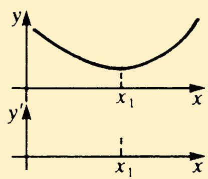
186

(1) At the point x_0 the tangent to the curve $y(x)$ is horizontal, i.e. the slope is zero. This is expressed mathematically by $y' = 0$

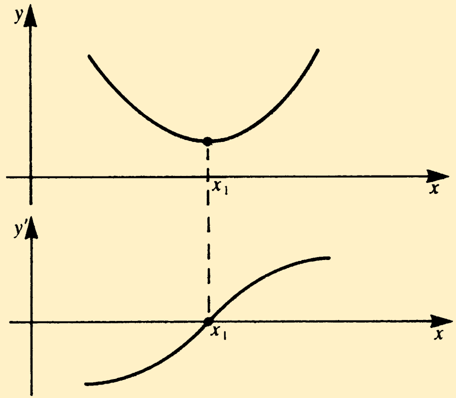
(2) In the interval (a, x_0) the slope of the curve $y(x)$ is positive. Thus in this interval $y' > 0$, but the slope of the tangent decreases from left to right.

(3) In the interval (x_0, b) the slope of the curve $y(x)$ is negative. Thus in this interval $y' < 0$, and the slope of the tangent decreases from left to right.

Sketch the curve for $y'(x)$ in the neighbourhood of a minimum x_1.



187



187

Look at the neighbourhood of x_1 in the diagram above.

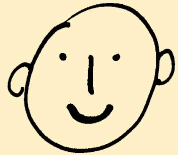
Which of the following is true?

☐ $y''(x) < 0$

☐ $y''(x) > 0$

188

$y'' > 0$



188

To determine whether a curve has a maximum or a minimum we need to remember two things:

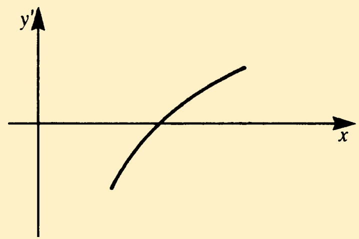
(1) The slope at an extreme value is zero: $y' = 0$.
(2) Whether there is a maximum or a minimum can be determined by investigating the nature of the slope.

You should remember that for a maximum the slope is positive before and negative after the point where the maximum occurs.

This implies that the slope is getting smaller, i.e. $y'' < 0$.

For a minimum the reverse is true.

The figure shows the slope y' of a curve.



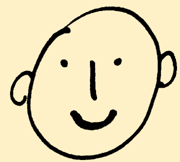
The curve y has a

☐ minimum

☐ maximum

189

minimum



189

Remember: the mouth represents the curve, the smiling expression represents the positive value of y''.

190

To determine characteristic points of a curve the following conditions apply:

190

(1) Zero values: $y = 0$
(2) Local maxima: $y' = 0, \quad y'' < 0$
(3) Local minima: $y' = 0, \quad y'' > 0$

For what value x_E has the function $y = x^2$ an extreme value?

The curve is a parabola.
To obtain the required value:

(1) Find the derivative

$$y' = \text{..............................}$$

(2) Solve the equation
$y' = 0 =$ .... $x_E =$ ....

191

$y' = 2x$
$0 = 2x$
hence $x_E = 0$

191

Is there a maximum or a minimum at that point?

$y'' = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

☐ $y'' > 0$
☐ $y'' < 0$

☐ Minimum
☐ Maximum

$\dashrightarrow$ 192

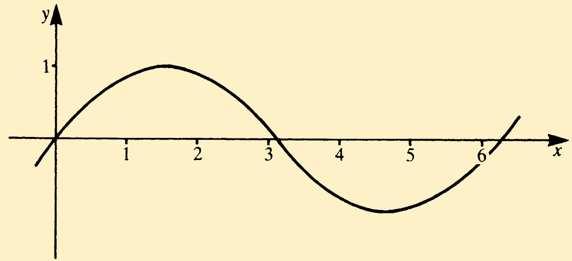
$y'' = 2 > 0$
Minimum

192

For what values of x in the range $0 \leq x \leq 2\pi$ is the sine function $y = \sin x$ zero?

....

Obtain the derivative.



For what values of x in the interval $0 \leq x \leq 2\pi$ has the function
$y = \sin x$
maxima ....
minima ....

193

193

In the interval $0 \leq x \leq 2\pi$:

$y = 0$ at $x = 0$

at $x = \pi$

and at $x = 2\pi$

Maximum at $x = \frac{\pi}{2}$

Minimum at $x = \frac{3\pi}{2}$

The steps required to obtain the position of a maximum or minimum are:

Step 1: $y' = \cos x$

Step 2: $y' = 0$ for $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

Step 3: $y'' = -1$ for $x = \frac{\pi}{2}$, hence a maximum

$y'' = +1$ for $x = \frac{3\pi}{2}$, hence a minimum

194

It is often important to determine the extreme values of a function. 194

Whether there is a minimum or a maximum often follows from the nature of the problem, and for this we do not need to apply the second condition.

The steps are:

— obtain the derivative
— set the derivative = 0, and solve the equation $y' = 0$.

Here is an example for you.

$$y = x^3 + x^2$$

$$y' = \dots\dots\dots\dots$$

Extreme values occur at

$$x_{E1} = \dots\dots\dots\dots$$
$$x_{E2} = \dots\dots\dots\dots$$

--------------------▷ 196

If you require help --------------------▷ 195

Given: $y = x^3 + x^2$

Required: extreme values

For the extreme values: $y'(x_{\mathrm{E}}) = 0$

195

Step 1: Derive y'

$$y' = 3x^2 + 2x$$

Step 2: $y' = 0$

$$0 = 3x^2 + 2x$$

Solve this equation for the values of x.

We can factorise out x to obtain

$0 = x(3x+2)$

You should be able to solve this equation for x.

$x_{\mathrm{E1}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$x_{\mathrm{E2}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 196

$$x_{E1} = 0$$
$$x_{E2} = -\frac{2}{3}$$

196

Obtain the value of x for which the function is zero and where there is an extreme value:

$$y = -x^2 + 2x$$

Zero ....

Extreme values ....

--------------------▷ 200

If you need help with the zero

--------------------▷ 197

If you need help with the extreme values

--------------------▷ 198

You have found the solution

--------------------▷ 200

197

To obtain the values of x for which the function is zero we set $y = 0$.
Hence if $y = -x^2 + 2x$
then $y = 0$ leads to

$$0 = -x^2 + 2x \quad or \quad x^2 - 2x = 0$$

$$x(x-2) = 0$$

hence $x_1 = 0$
$x_2 = 2$

Now calculate the extreme values of the function

$$y = -x^2 + 2x$$

Extreme values ....

--------------------▷ 200

If you need help

--------------------▷ 198

Given: $y = -x^2 + 2x$
Required: Extreme values x_E

198

Step 1: Obtain the derivative. That is the slope of a tangent, y' : ....
Step 2: Set $y' =$ .... $= 0$

and solve for x.

$= -2x + 2$

giving $x_E =$ ....

--------------------▷ 199

$$x_E = 1$$

199

Now obtain the extreme values of the function

$$y = 2x^2 + 4x$$

Obtain the derivative y'.

Set $y' = 0$.

Show that the function has an extreme value at

$$x_E = -1$$

Check your solution by yourself.

200

The function $y = -x^2 + 2x$
is zero at $x_1 = 0$ and $x_2 = 2$;
it has an extreme value at $x_E = 1$

200

As the last example of this sequence, sketch the function

$$y(x) = \frac{2}{e^{-(x+1)} + 1}$$

First establish a few salient points!

Solution found --------------------▷ 203

Hints on zeros, poles and asymptotes --------------------▷ 201

Hints on extreme values and points of inflexion --------------------▷ 202

Zeros: $y(x_z) = 0$ 201

The numerator does not vanish (and the denominator is always positive). Thus there are no zeros.

Poles: The denominator does not vanish. Thus there are no poles.

Asymptotes: We examine the behaviour of the function as

$x \to +\infty$ and $x \to -\infty$

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \frac{2}{e^{-(x+1)} + 1} = 2$$

Thus the asymptote for $x \to \infty$ is $y = 2$.

The other asymptote is found in the same way: for $x \to -\infty$ the asymptote is $y = 0$.

Full solution established

--------------------▷ 203

Hints on extreme values and points of inflexion

--------------------▷ 202

Extreme values: $y'(x_E) = 0$

202

$$y'(x) = \frac{2e^{-(x+1)}}{(e^{-(x+1)}+1)^2}$$

The numerator never vanishes. Thus the function has no extreme values.
Points of inflexion: $y''(x_i) = 0$

$$y''(x) = \frac{-2e^{-(x+1)}(e^{-(x+1)}+1) + 4e^{-(x+1)}e^{-(x+1)}}{(e^{-(x+1)}+1)^3}$$

After rearranging we get

$$y''(x) = \frac{2e^{-(x+1)}[2e^{-(x+1)} - e^{-(x+1)} - 1]}{(e^{-(x+1)}+1)^3}$$

The bracket in the numerator must be zero for

$$y''(x_i) = 0$$

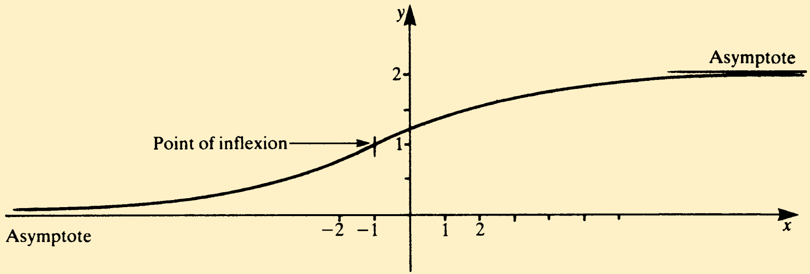
$0 = e^{-(x_i+1)} - 1$
Since $e^{-(x_i+1)} = 1 = e^0$
Thus the function has a point of inflexion at $x_i = -1$

203

There are no zeros and no poles. 203

Asymptotes $x \to \infty \quad y = 2$

Point of inflexion $x_i = -1$ $\quad x \to -\infty \quad y = 0$



If you have mastered the subject you do not need more exercises at the moment. If you had difficulties some more exercises should be useful to get more practice. In any case, have a look at some exercises in one or two weeks' time. As you already know, further exercises are given in the textbook at the end of Chapter 5.

-------------------▷ 204

5.10 Applications of the Differential Calculus

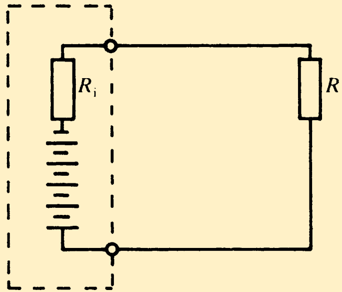
204

Section 5.8 in the textbook is intended to give you an impression of possible applications. As there are many applications for the physicist and engineer, it is impossible to present an exhaustive summary. Some of the examples might be studied intensively in later periods when you are facing advanced problems. In any case this section may be useful when you want to review the concept of curvature and the l'Hôpital's rule.

READ: 5.8 Applications of differential calculus
5.8.1 Extreme values
5.8.2 Increments
5.8.3 Curvature
5.8.4 Determination of limits by differentiation: l'Hôpital's rule
Textbook pages 123–129

205

We start with a maximum problem in electricity. Consider a battery or an amplifier. Its voltage is V_0, its internal resistance is R_i. 205



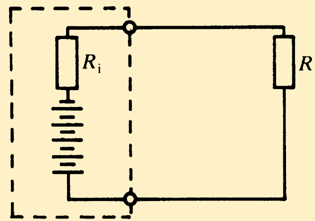
The problem is to find the value for R at which the electrical power consumed by R is a maximum.

Want to solve the problem without help --------------------▷ 210

Want hints --------------------▷ 206

The circuit is shown again:

206



The box on the left represents the battery or amplifier. If there is a current I the voltage applied to R is V_0 diminished by the voltage applied to R_i. Thus the voltage applied to R is

$$V_R = V_0 - IR_i$$

The electrical power consumed by R is then

$$L = V_R I$$

Now we want the maximum of L depending on R.

Want to solve the problem on my own --------------------▷ 210

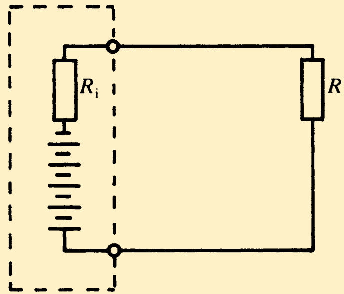
More help is needed --------------------▷ 207

207

The voltage applied to R is $V_R = V_0 - R_i I$
The power consumed by R is $L = V_R I$

The current I is given by Ohm's law: $I = \dfrac{V_0}{R_i + R}$

We express L in terms of R:



$$L = (V_0 - R_i I) I$$

$$L = \left(V_0 - R_i \frac{V_0}{R_i + R}\right) \frac{V_0}{R_i + R}$$

$$L = \frac{R V_0^2}{(R_i + R)^2}$$

We look for the maximum of L with respect to the variable R.

Want to solve the problem on my own

--------------------▷ 210

Want further hints

--------------------▷ 208

Wanted: the maximum of L 208

$$L = \frac{RV_0^2}{(R_\mathrm{i}+R)^2}$$

Let us change the variables:

$$L = y$$
$$R = x$$
$$y = \frac{V_0^2 x}{(R_\mathrm{i}+x)^2}$$

The necessary condition for y being maximal is $y' = 0$.

I can solve the problem now 210

Further hints wanted 209

209

The given equation reads

$$y = \frac{V_0^2 x}{(R_\mathrm{i} + x)^2}$$

We differentiate:

$$y' = \frac{V_0^2 (R_\mathrm{i} + x)^2 - V_0 x 2(R_\mathrm{i} + x)}{(R_\mathrm{i} + x)^4} = V_0^2 \frac{R_\mathrm{i} - x}{(R_\mathrm{i} + x)^3}$$

An extreme value is given if $y' = 0$.

$$0 = \frac{R_\mathrm{i} - x}{(R_\mathrm{i} + x)^3} \quad \text{i.e. } x = R_\mathrm{i}$$

Since we substituted $R = x$ we now have the solution:

$$R = R_\mathrm{i}$$

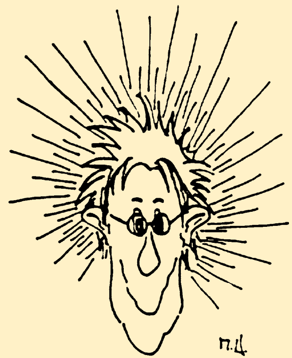
210

$$R = R_{\mathrm{i}}$$

210

From the context it is obvious that this is a maximum and not a minimum. But you may prove it for yourself by determining the second derivative.

The maximum electrical power that can be supplied by a battery is therefore reached if the resistance in the external circuit is the same as the internal resistance. In this case half of the electric power is consumed within the battery or amplifier.



Note also that the result is independent of V_0.

211

211

Now let us tackle a problem on errors.

To measure the depth of a well we drop a stone. Since we can hear it hitting the water we measure the time of the fall.

The time is $2 \pm 0.2\,\mathrm{s}$.

What is the corresponding error in the depth?

What is the relative error?

(The time of travel of the acoustic signal may be neglected; the approximate value of $g = 9.81$ m/s.)

I want to solve the problem on my own

--------------------▷ 213

Further hints

--------------------▷ 212

The time of fall is $2 \pm 0.2\,\text{s}$. Thus the relative error in measurement is 10%. 212

The depth is

$$s = \frac{g}{2}t^2 = \frac{9.81}{2}\,\text{m/s}^2(2\,\text{s})^2$$
$$s = 19.62\text{m}$$

The error

$$\Delta s = \frac{\mathrm{d}s}{\mathrm{d}t}\Delta t = gt\Delta t$$
$$\Delta s = \text{..............................}$$

and the relative error

$$\frac{\Delta s}{s} = \text{..........................}$$

213

The error $\Delta s = \pm g t \Delta t \approx \pm 4\text{m}$ 213

The relative error $\dfrac{\Delta s}{s} = 2\dfrac{\Delta t}{t} \approx 20\%$

Now we turn to an example on curvature.

Let us calculate the radius of curvature of a circle. The result is already known. The radius of curvature of a circle is the radius of the circle. Nevertheless this problem is an interesting one because it provides a check for the formula derived in the textbook.

Equation of a circle: $x^2 + y^2 = R_0^2$

$$y = \sqrt{R_0^2 - x^2}$$

First write down the formula for the radius of curvature. In case of doubt use the textbook.

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 214

$$R = \frac{[1 + (y')^2]^{3/2}}{y''}$$

214

Calculate R with

$$y = \sqrt{R_0^2 - x^2}$$

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

217

Want further hints

215

Given:

$$R = \frac{[1+(y')^2]^{3/2}}{y''}$$

$$y = \sqrt{R_0^2 - x^2}$$

215

First find the derivatives

$$y' = \text{..............................}$$

$$y'' = \text{..............................}$$

Derivatives found

217

Further hints wanted

216

$$y' = -\frac{x}{\sqrt{R_0^2 - x^2}}$$

$$y'' = -\frac{-R_0^2}{(R^2 - x^2)^{3/2}}$$

216

These derivatives are to be inserted into the formula

$$R = \frac{[1 = (y')^2]^{3/2}}{y''}$$

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

217

$$R = -R_0$$

217

The sign indicates that the curve is concave downwards.
The radius of a circle is the radius of curvature.
R is a constant in this case.

What is the radius of curvature of the parabola $y = x^2$?

$$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

218

$$R = \frac{(1+4x^2)^{3/2}}{2}$$ 218

In this case R is not a constant. The radius R of curvature grows with increasing magnitude of x. For $x = 0$ we have $R = \frac{1}{2}$. The curvature $\frac{1}{R}$ decreases with increasing magnitude of x.

--------------------▷ 219

5.11 Further Methods for Calculating Differential Coefficients

219

This section covers slightly more advanced methods. They are useful if you have to deal with complicated functions. For the first pass through the textbook you may skip this section but you should return to it before having completed differential equations (Chapter 10).

READ: 5.9 Further methods for calculating differential coefficients
Textbook pages 129–130

220

Given: $2x^2 + 3y^3 = 27$.

This is an implicit function.

Calculate y' using one of the methods shown in the textbook. 220

$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

→ 223

Hints wanted

→ 221

We want the derivative y' of the *implicit* function

221

$$2x^2 + 3y^3 = 27$$

We may use the method for differentiating implicit functions or we may use logarithmic differentiation. We start with the first method:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$$
$$\frac{\mathrm{d}}{\mathrm{d}y}(y^3) = 3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{.............................}$$

----------------------▷ 223

Further hints wanted

----------------------▷ 222

Given: $2x^2+3y^3=27$

222

We differentiate all terms with respect to x and solve for $\dfrac{dy}{dx}$:

$$\frac{d}{dx}(2x^2 1)+\frac{d}{dx}(3y^3)=\frac{d}{dx}(27)$$
$$4x+9y^2\frac{dy}{dx}=0$$

Now solve for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx}=\dots\dots\dots\dots\dots\dots$$

223

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4x}{9y^2}$$

223

Now try logarithmic differentiation for the same function. It is more complicated in this case.

$$2x^2 + 3y^3 = 27$$

$$2x^2 = 27 - 3y^3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

225

Hints needed 224

224

Given: $2x^2 = 27 - 3y^3$

We take logarithms:

$$\ln 2 + 2\ln x = \ln(27 - 3y^3)$$

We differentiate with respect to x

$$\frac{2}{x} = \frac{-3 \times 3y^2}{27 - 3y^3}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Inserting $2x^2$ for $(27 - 3y^3)$ from the first equation yields

$$\frac{2}{x} = \frac{-9y^2}{2x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 225

$$\frac{dy}{dx} = -\frac{4x}{9y^2}$$

225

Given the function:

$$y = (x+1)^3(x+2)$$

There are several different methods for solving this problem, but it is easiest to use logarithmic differentiation.

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 228

Explanation wanted

--------------------▷ 226

$$y = (x+1)^3(x+2)$$

We take logs on both sides:

$$\ln y = 3\ln(x+1) + \ln(x+2)$$

Now we differentiate with respect to x and solve for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Then we substitute back $(x+1)^3(x+2) = y$

$$y' = \text{...............................}$$

228

Detailed calculation

227

227

$$y=(x+1)^3(x+2)$$

Taking logarithms

$$\ln y=3\ln(x+1)+\ln(x+2)$$

Differentiating with respect to x yields:

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{x+1}+\frac{1}{x+2}$$

Solving for $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=y\frac{3(x+2)+(x+1)}{(x+1)(x+2)}$$

Substituting back y yields

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\dots\dots\dots\dots\dots\dots$$

228

$$y' = (4x+7)(x+1)^2$$

228

This should have been easy. You can check the answer by applying the rule for differentiating a product.

Here is a very different function:

$$y = \frac{xe^{2x}}{(x+3)^2}$$

$$y' = \text{..........................}$$

229

$$y' = \frac{e^{2x}(2x^2+5x+3)}{(x+3)^3}$$ 229

Correct 231

Wrong, detailed solution 230

230

The function to differentiate is $y = \dfrac{x\mathrm{e}^{2x}}{(x+3)^2}$

Step 1: Take logs on both sides: $\ln y = \ln x + 2x - 2\ln(x+3)$

Step 2: Differentiate with respect to x, remembering that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln y = \frac{\mathrm{d}}{\mathrm{d}y}(\ln y)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln y = \frac{1}{y}y' = \frac{1}{x} + 2 - \frac{2}{x+3}$$

Step 3: Multiply through by y so that $y' = \dfrac{x\mathrm{e}^{2x}}{(x+3)^2}\left(\dfrac{1}{x} + 2 - \dfrac{2}{x+3}\right)$

Step 4: Simplify by placing the terms in the brackets under a common denominator $x(x+3)$ and obtain

$$y' = \frac{\mathrm{e}^{2x}}{(x+3)^3}(2x^2 + 5x + 3)$$

231

5.12 Parametric Functions and their Derivatives

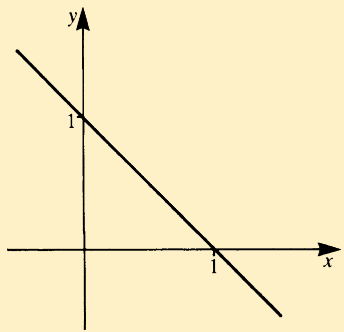
231

In the first part of this section the concept of a parametric function will be explained using well-known examples from physics. This concept is very powerful.

In the second part of the section we shall deal with the derivatives of parametric functions.

READ: 5.10 Parametric functions and their derivatives
5.10.1 Parametric form of an equation
Textbook pages 131–136

--------------------▷ 232



Give the parametric form of the straight line in the figure. Denote the parameter by λ:

$x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

232

233

$$x = \lambda$$
$$y = 1 - \lambda$$

233

Eliminate the parameter and get back to the usual form of the equation of this straight line.

$y =$ ....

234

$$y = 1 - x$$

234

Correct

236

Explanation wanted

235

We start with the parametric form

235

$$x = \lambda$$
$$y = 1 - \lambda$$

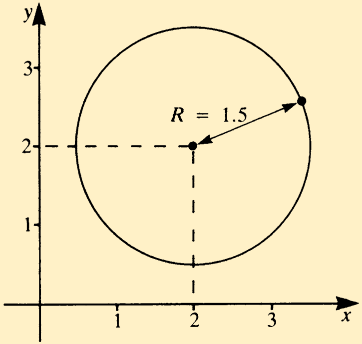
We want to eliminate the parameter in order to obtain one single equation. In this case this is quite easy.
First express λ in terms of x:

$$\lambda = x$$

Second, insert λ in the second equation:

$$y = 1 - \lambda = 1 - x$$

236



A point moves on the circle.
Give the parametric equation of the curve.
Denote the parameter by ϕ.

236

$x =$ ....

$y =$ ....

237

$$x = 2 + 1.5\cos\phi$$
$$y = 2 + 1.5\sin\phi$$

237

Correct

239

Explanation wanted

238

The example is similar to one given in the textbook. 238

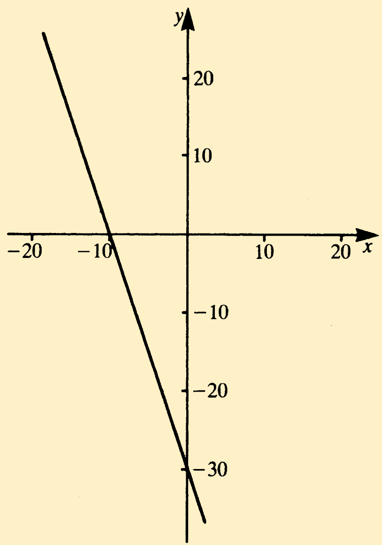
The difference is that the center of the circle has the coordinates (2, 2).

The equations are obtained by adding the coordinates of the center to those of the circle centered at the origin.

$$x = 2 + 1.5\cos\phi$$

$$y = 2 + 1.5\sin\phi$$

239



Give parametric equations of the straight line in the diagram.

Denote the parameter by t.

$x =$ ....

$y =$ ....

239

240

$$x = -10 + 10t$$
$$y = -30t$$

240

Other equations are possible. The following equations are also valid:

$$x = 5t$$
$$y = -30 - 15t$$

In case of doubt or difficulty consult the textbook.

The vector **a** can be chosen arbitrarily; thus different equations may result. But all give the same equation in the usual form if one eliminates the parameter. Eliminate the parameter in both pairs of equations above and show that in both cases the result is

$$y = -3x - 30$$

------------------▷ 241

5.13 Derivatives of Parametric Functions

241

Try to calculate the examples in the second part of this section without using the textbook.

READ: 5.10.2 Derivatives of parametric functions
Textbook pages 136–142

242

A point moves on a straight line. Its movement is given by the equations

242

$$x = -10\,\mathrm{cm} + 10\,(\mathrm{cm/s})t$$
$$y = -30\,(\mathrm{cm/s})t$$

Give the magnitude of the velocity:

$$v = \ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: Obtain the components of the velocity first.

243

$$v = 31.6\,\text{cm/s} = \sqrt{100+900}\,\text{cm/s}$$

243

A point moves on a helix. The position vector is given by

$$\mathbf{r}(t) = \left(R\cos\omega t\,,\quad R\sin\omega t\,,\quad \frac{t}{2\pi}\right)$$

Evaluate the

velocity $\mathbf{v}(t) =$ ....

acceleration $\mathbf{a}(t) =$ ....

244

$$\mathbf{v}(t) = (-R\omega \sin \omega t, \quad R\omega \cos \omega t, \frac{1}{2\pi})$$
$$\mathbf{a}(t) = (-R\omega^2 \cos \omega t, \quad -R\omega^2 \sin \omega t, \quad 0)$$

244

Now give the magnitude of velocity and acceleration.

$$|\mathbf{v}(t)| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$|\mathbf{a}(t)| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

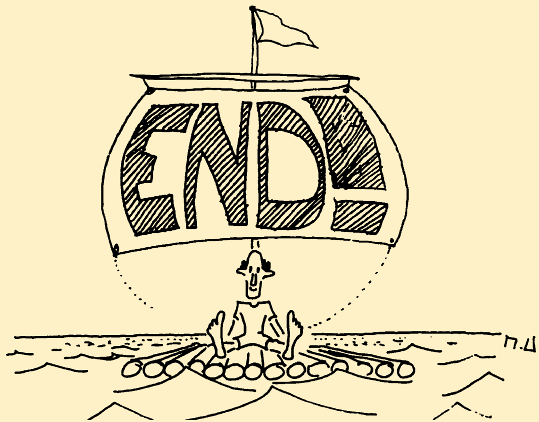
----------------▷ 245

$$v = \sqrt{R^2\omega^2 + \left(\frac{1}{2\pi}\right)^2}$$

$$a = R\omega^2$$

This chapter has turned out to be rather lengthy. The reason is that the differential calculus is of fundamental importance for many areas of applied mathematics.

But before you have your break, which you do deserve now without doubt, do recapitulate the most salient points of Chapter 5.



Chapter 6
Integral Calculus

2

Preliminary note: The integral calculus can be approached in two different ways:

(i) Analytically: in this approach integration is formally defined as the inverse of differentiation.

(ii) The calculation of an area under a given curve leads to the integral calculus.

These approaches are shown to be equivalent and are discussed in sections 6.1 and 6.2 of the textbook.

--------------------▷ 3

6.1 The Primitive Function Fundamental Problem of the Integral Calculus

3

Objective: Concepts of primitive function, boundary conditions, integration.

READ: **6.1 The primitive function**
6.1.1 Fundamental problem of the integral calculus
Textbook pages 147–149

4

Let the function be denoted by $f(x)$.
We require a primitive function $F(x)$.
What relationship exists between these two functions?

.... = ...

4

5

$$F'(x) = f(x)$$

5

Integration is the inverse of differentiation.

This means that if we differentiate a given function and integrate the new function we get back the original function except for an additive constant.

Differentiate and integrate successively the function

$$y = x^3$$

Differentiating: $y' =$ ....

Integrating: $y =$ ....

6

$$y' = 3x^2$$
$$y = x^3 + C$$

6

Let us have a look at another example.

Primitive functions, or simply primitives, are usually written with capital letters. Remember the constant added to the primitive!

$$F(x) = y = x^2 + 4$$

Differentiating: $F'(x) = f(x) = y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Integrating: $F(x) = y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 7

$$F'(x) = 2x$$
$$F(x) = x^2 + C$$

7

If we differentiate and integrate successively a given function then we get back the original function except for an additive constant.

Let's follow these operations with the help of the next example.

We start with the function

$$y = \sin(2\pi x)$$

Differentiating, we obtain

$$F'(x) = f(x) = 2\pi\cos(2\pi x)$$

Integrating, we find

$$F(x) = \sin(2\pi x) + C$$

Can you name the primitive of $f(x) = \cos x$?

$$F(x) = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

8

If

$$f(x) = \cos x$$
$$F(x) = \sin x + C$$

8

If your result does not agree, satisfy yourself of the correctness of the solution by differentiating $F(x)$. You should obtain $f(x)$.

Difficulties may occur because of the notation. We have to memorise: the primitive is denoted by $F(x)$ and its derivative by $f(x)$. We always use this notation in the textbook; it is very common. Integrating means trying to find the .... of a function.

The given function is the derivative of the ....

9

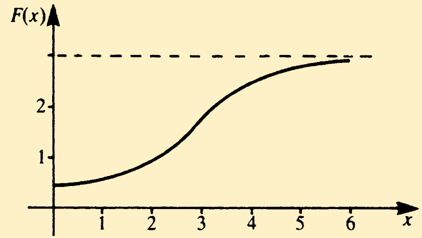
primitive
primitive

9

In the following you will find additional explanations of the graphical representation and the link between the integral and differential calculus.

10

Given the function $F(x)$:

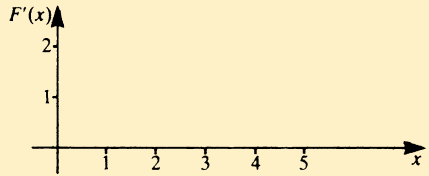


10

Sketch the graph of the function $F'(x)$

i.e. of the derivative in the interval $0 \le x \le 6$

In other words, differentiate the given curve.

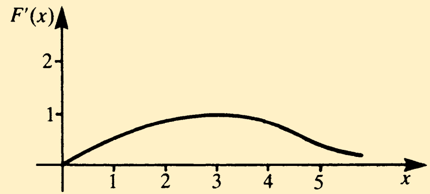


11

To sketch the curve we have three reference points: At $x = 0$, $F(x)$ has a horizontal tangent, which means

$$F'(x) = 0$$

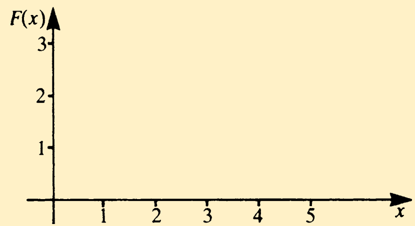
At $x = 3$, the slope of $F(x)$ has its greatest value, about 1.



11

For $x \gg 3$ the curve approaches a horizontal asymptote its slope tends to zero.

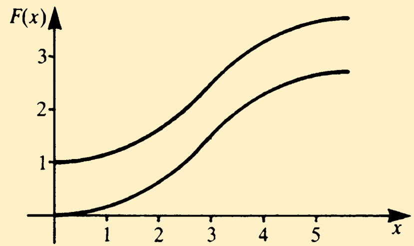
Sketch two functions $F(x)$ using the information given above for the derivative $F'(x)$. This operation corresponds to integration. One curve should pass through the origin and a second curve through the point (0, 1).



12

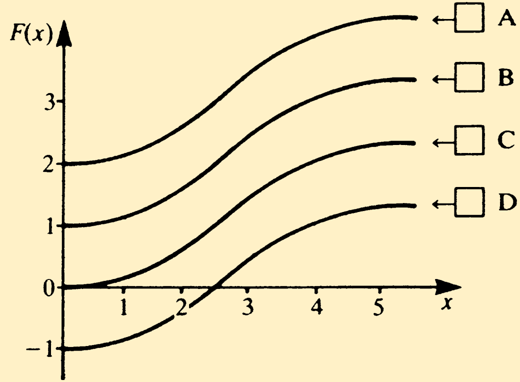
The actual details of the graphs are not important but the trend should be correct.

Check the horizontal tangent at $x = 0$ and for $x \gg 3$.



12

Which of the curves $F(x)$ shown are also solutions of the curve $F'(x)$ described in the previous frame?



13

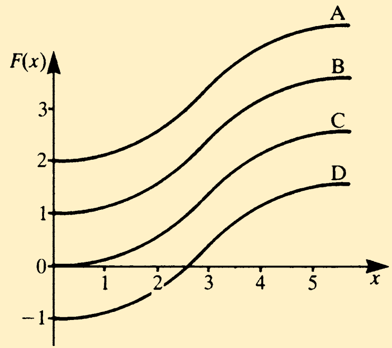
All curves are integral curves of the same derivative. They only differ by an additive constant. Name boundary conditions for the four curves A, B, C and D at $x = 0$:

A, $F(0) =$ ....

B, $F(0) =$ ....

C, $F(0) =$ ....

D, $F(0) =$ ....



13

14

Curve A, $F(0) = 2$
Curve B, $F(0) = 1$
Curve C, $F(0) = 0$
Curve D, $F(0) = -1$

14

15

Obtain the derivative of

15

$$y = x^3 + 5, \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Obtain the primitive of

$$F'(x) = f(x) = 3x^2, \quad F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Obtain the derivative of

$$y = 3x + 2, \quad y' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Obtain the primitive of

$$F'(x) = f(x) = 3, \quad F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 16

$$y' = 3x^2 \qquad y' = 3$$
$$F(x) = x^3 + C \qquad F(x) = 3x + C$$

16

After finding the primitive function a constant C must always be added.
In the examples above the first part contained the solution.

Now obtain the primitive of the following functions:

$$f_1(x) = 2x \qquad F_1(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_2(x) = x^2 \qquad F_2(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now change the notation; instead of f use g and instead of x use t.

$$g(t) = t + 1 \qquad G(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 17

$$F_1(x) = x^2 + C$$
$$F_2(x) = \frac{1}{3}x^3 + C$$
$$G(t) = \frac{1}{2}t^2 + t + C$$

Don't Forget the Constant!

17

Calculation of the constant from a given boundary condition:

Let $f(x) = x + 1$

The primitive function is

$$F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Boundary condition: the primitive should pass through the point (0, 0).

Of the infinite number of solutions, called integral curves, only one curve will pass through that point; its equation is:

$$F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 18

$$F(x) = \frac{x^2}{2} + x + C$$ 18

$$F(x) = \frac{x^2}{2} + x, \qquad C = 0$$

Both correct

- - - - - - - - - - ▷ 20

Further explanation required

- - - - - - - - - - ▷ 19

19

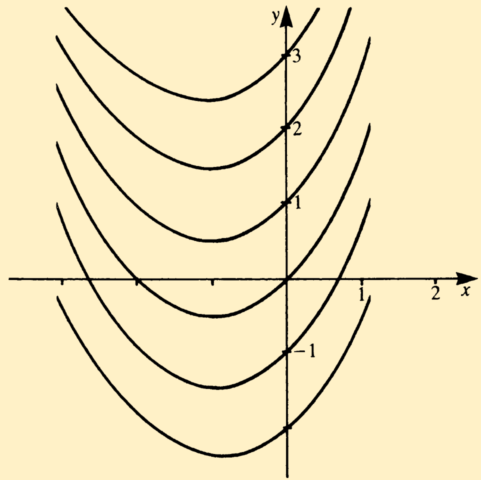
The value of the constant to satisfy a given boundary condition:

Consider the primitive function

$$F(x) = \frac{x^2}{2} + x + C$$

This represents a family of parabolas.

If the boundary condition calls for $F(x)$ to pass through the point (0, 0), i.e. the origin, we can obtain the constant as follows:



When $x = 0$, $y = F(0) = 0$

Step 1: Insert $x = 0$ and $y = 0$ in the primitive and obtain $0 = 0 + 0 + C$.

Step 2: Solve for the constant. In this case $C = 0$.

20

Let the derivative be $y' = x$.

The primitive function $y =$ ....

Boundary condition: the primitive should pass through the point $P = (1, \quad 2)$.

Calculate the constant of integration:

$$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

20

21

$$y(x) = \frac{x^2}{2} + C$$

$$C = \frac{3}{2}$$

$$y(x) = \frac{x^2}{2} + \frac{3}{2}$$

21

Correct solution 25

Still having difficulties, want further explanation 22

Given $y' = x$. We require that the primitive should pass though the point $P = (1, \quad 2)$.

22

Step 1: Obtain the primitive of $y' = x$.

The primitive is $y = \dfrac{x^2}{2} + C$

C is not known numerically at this stage.

Step 2: Obtain the value of the constant.

The primitive $y = \dfrac{x^2}{2} + C$ is required to pass through the point whose coordinates are $x = 1$, $y = 2$. We need to substitute these values in the equation, thus:

$$2 = \frac{1}{2} \times 1^2 + C$$

and solving for C, $C = \frac{3}{2}$.

The solution is: $y(x) = \frac{1}{2}x^2 + \frac{3}{2}$

23

Given $$y'(x) = -\frac{3}{4}x^2$$

23

Obtain the primitive function $y(x)$ which goes through the point P(1, -3).

Step 1: Obtain the primitive

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Step 2: Evaluate the constant, i.e. substitute $x = 1$ and $y = -3$ and solve for C.

$$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, $y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

24

$$y(x) = -\frac{1}{4}x^3 + C\,, \quad C = -\frac{11}{4}$$ 24

You will find further problems in the textbook.

Before the next short break read some hints on the study technique of *selective reading*.

First case for the application of selective reading:

Let's assume you are familiar with most of the content of a certain chapter. In such a case intensive reading is *not* necessary. You know the facts already. Here, something else is important. You must read the text through with a view to finding the new material which is introduced, defined or deduced there. It is a question of quickly picking out any new points from a large number of familiar or known ones.

--------------------▷ 25

Second case for the application of selective reading: 25

Suppose you are looking for a specific piece of information in a long text. There is a danger that you will be distracted from your aim, by finding some piece of interesting but irrelevant information which you consequently begin to read. Who can honestly say he hasn't been distracted in this way? Perhaps, for example, you once looked up the headword SYNERGETICS and read the entries for SOLIPSISM and SYNAGOGUE as well. Deviation from target-directed searching is known as the 'encyclopaedia effect'.

Selective reading as a time-saving study technique demands the separation of information which is currently relevant from that which is currently irrelevant. One should not even be aware of the existence of the irrelevant material.

Now practise selective reading:
Look for the numerical value of Euler's number e.
If necessary use the index.
The numerical value of e is ....

--------------------▷ 26

$$e = 2.71828..........$$

26

The technique used in selective reading is exactly the opposite of that used in intensive reading. Different aims are pursued.

In the case of selective reading, the text is glanced over carefully, your attention being directed exclusively towards the particular information being sought after.

If you come across a part of the text which you already know, try to read it selectively. However, if it's new to you then you must change your reading mode and read intensively.

27

6.2 The Area Problem: The Definite Integral Fundamental Theorem of the Differential and Integral Calculus

27

Objective: Concepts of definite integrals, integrand, limits of integration.

Operations: evaluation of simple definite integrals, inserting limits.

The next section is longer than usual. Divide your work into two parts. After the first part — say after having completed section 6.3 — have a break and check that you have understood the new concepts by reproducing them in your own words. If you do not succeed, do not go any further but go back to that particular part.

READ: **6.2 The area problem: the definite integral**
6.3 Fundamental theorem of the differential and integral calculus
6.4 The definite integral
Textbook pages 149–161

--------------------▷ 28

The expression $\int_a^b f(x)\,dx$ is called 28

a is called the ....

b is called the ....

$f(x)$ is called the ....

dx (known from Chapter 5) is called ....

29

Contents | First Page | Previous View | Next View | Print | Exit

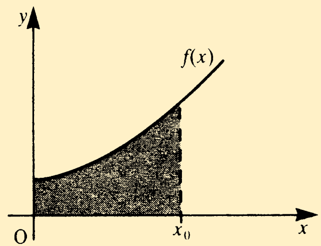
29

$\int_a^b f(x)\,\mathrm{d}x$	= definite integral
a	= lower limit of integration
b	= upper limit of integration
$f(x)$	= integrand
$\mathrm{d}x$	= differential

You have to know these concepts. Make sure that you can assign the right meaning to each one!

$\dashrightarrow$ 30

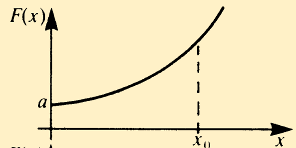
The shaded area is bounded by $f(x)$. The area function $F(x)$ gives the area under the curve $f(x)$ between 0 and x_0.



30

Which of the graphs represents the area function:

$$F(x) = \int_0^{x_0} f(x)\,\mathrm{d}x$$



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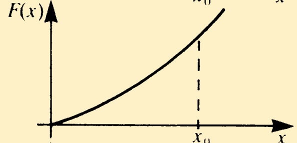

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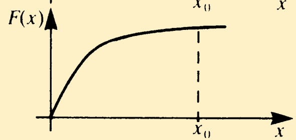
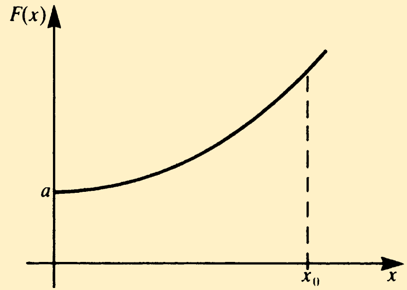


33

31

Wrong, unfortunately.

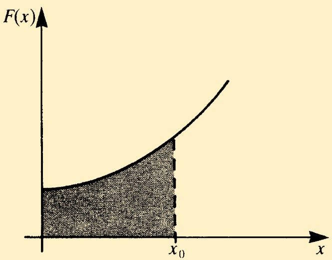
The area function $F(x)$ must pass through the origin.

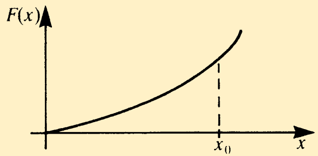


If the left- and right-hand side limits of integration coincide, then the area under the curve shrinks to a line and consequently the magnitude of the area is zero, i.e. $F(0) = 0$

But here $F(0) = a$.

Now decide once more what the area function should look like:





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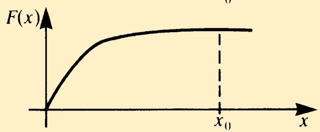


33

Correct!
It is true: $F(0) = 0$, and in addition the functional values of $F(x)$ increase as x increases.

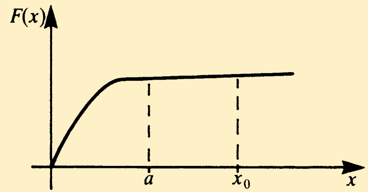
32

Skip a frame; go to 34

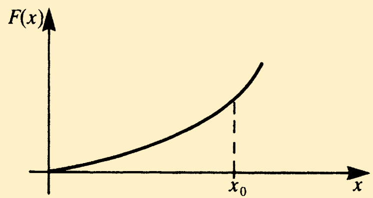
Unfortunately not entirely correct!

33

It is true that $F(x)$ goes though zero. Furthermore the function $F(x)$ has to be monotonically increasing, which means that it grows as x increases, since the area under the curve $f(x)$ increases if the limit is moved to the right. However, the function $F(x)$ chosen by you is constant from position a onwards.



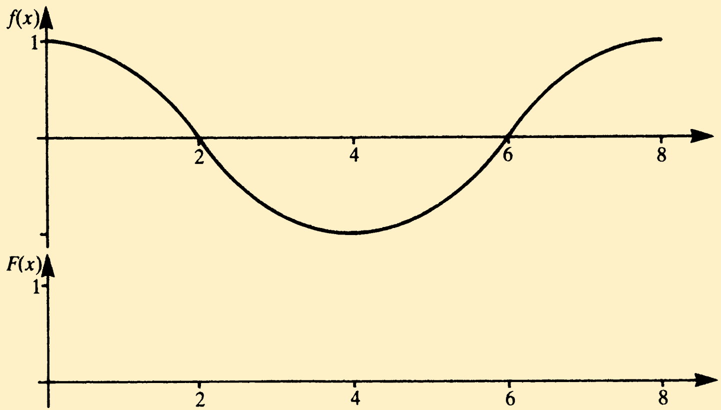
The correct graph is



34

Sketch the area function for $f(x)$:

34

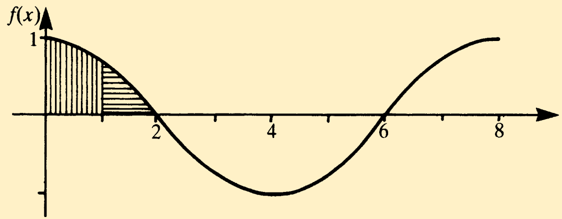


If you have found the solution ----------------▷ 37

If you require help ----------------▷ 35

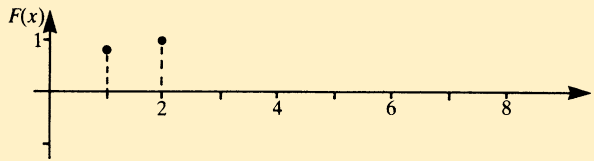
Here is a hint. Given the function $f(x)$, what is the area function $F(x)$? (We have to pay particular attention to the fact that the area below the x-axis is, by definition, negative.) To obtain the area function we divide the graph into a number of intervals of arbitrary length.

35



The area function is zero at $x = 0$. The area up to the first interval is about 0.7. The area of the second interval is about 0.3, and therefore the area of the first and the second intervals is about 1. The area of the third interval is negative and has to be deducted.

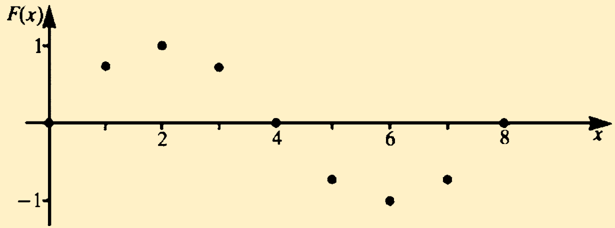
Complete the area curve for yourself by taking a number of intervals.



36

The area function is obtained by drawing a curve through the points:

36



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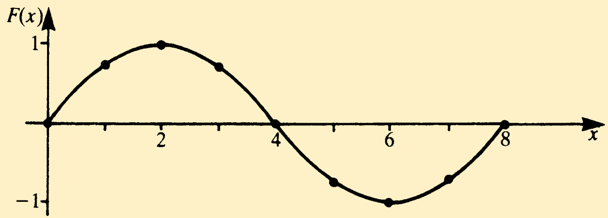


37

We started with the cosine function, the equation being $f(x) = \cos(......)$.

The area function is a ....

Its equation is $F(x) =$ ....

(Note the period: the value of the argument after one complete period must be equal to 2π.)

38

$$f(x) = \cos\left(\frac{\pi x}{4}\right)$$

38

sine function

$$F(x) = \frac{4}{\pi}\sin\left(\frac{\pi x}{4}\right)$$

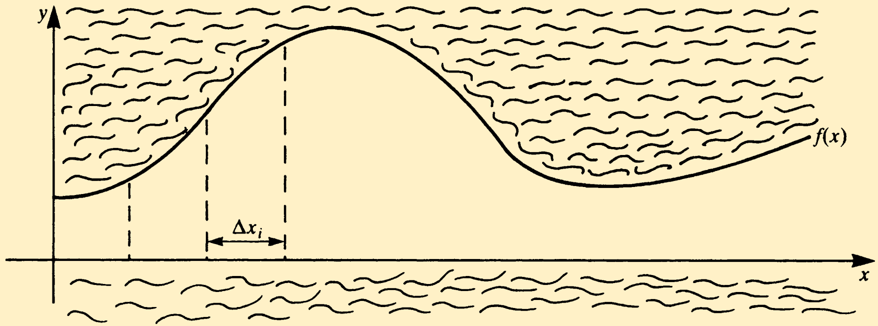
I understand the area function problem

--------------------▷ 43

I require a graphical explanation of the relationship between the area function and the integral function

--------------------▷ 39

A peninsula is bounded on one side by a straight coastline, the x-axis, and on the other side by a curved coastline. We call it $f(x)$. 39



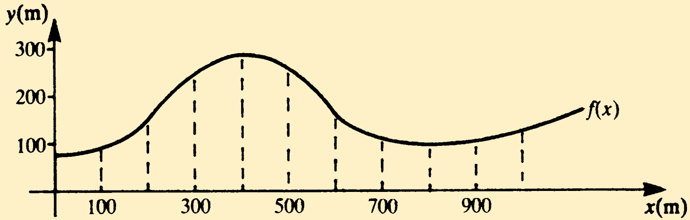
The peninsula has to be cleared of weeds and shrubs. The workers divide the work by lines perpendicular to the x-axis and separated by equal amounts Δx_i and clear the area bounded by two perpendicular lines each day. The area cleared during a day can be approximately calculated by multiplying Δx_i by $f(x)$, the width of the peninsula at the position the work was stopped in the evening. This area is plotted on a chart by a very officious administrative officer every day. This chart is called the curve of *reclaimed land*.

40

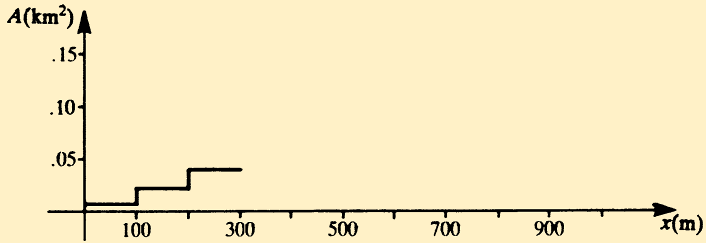
What is the shape of this curve?

The coastline is reproduced below:

40



Complete the diagram showing the area of land reclaimed:



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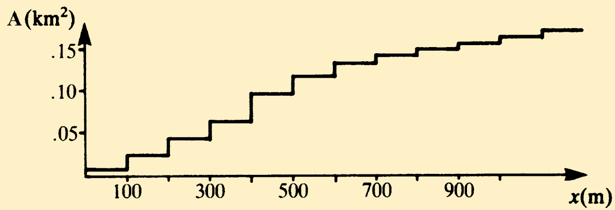


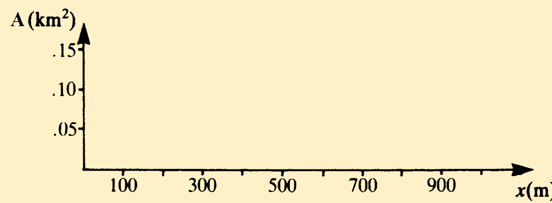
41

The area cleared every day $f(x_i)\Delta x_i$ is added on the chart. Thus the amount of land reclaimed grows each day. After the nth day the total area of land reclaimed is given by

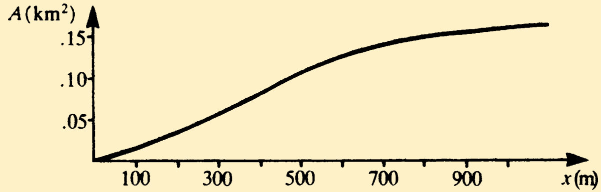
$$L = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

The smaller the intervals the more accurate the computation. This means that if the workers report the area cleared two or three times a day the actual area cleared will be computed more accurately. If the process were carried further the chart representing the area of land reclaimed would become a smooth curve, i.e. the integral curve.

Now sketch the integral curve:



42



42

The passage from a discontinuous curve to a continuous integral curve is obtained by means of a limiting process.

Mathematically this is

$$\lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \int_0^x f(x)\,dx$$

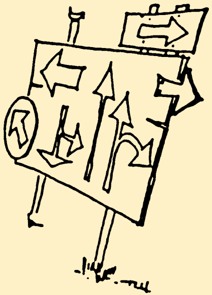
43

You should be able to solve the following problem by inspection if you have understood the relation between the differential and integral calculus.

43

If
$$F(x) = \int_0^x (3x^2+2)\,\mathrm{d}x$$

the derivative $F'(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



I have found the solution ------------------▷ 45

I am not absolutely sure ------------------▷ 44

Here is a hint: According to the fundamental theorem of the differential and integral calculus, if the area function is 44

$$F(x) = \int_0^x f(x) \mathrm{d}x$$

then
$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(x) \,\mathrm{d}x = f(x)$$

We carry out successively two operations which cancel out.

Differentiation: $\frac{\mathrm{d}}{\mathrm{d}x}(\ )$

Integration: $\int_0^x (\ )\,\mathrm{d}x$

Our problem was

Given: $F(x) = \int_0^x (3x^2+2)\,\mathrm{d}x$

Required: $\frac{\mathrm{d}}{\mathrm{d}x} F(x) = F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (3x^2+2)\,\mathrm{d}x$

$F'(x) =$ ....

45

$$F'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \int_0^x (3x^2 + 2)\,\mathrm{d}x = 3x^2 + 2$$

45

Differentiation and integration are called inverse operations, i.e. they neutralise each other when carried out in succession.

As another example of inverse mathematical operations consider the problem of squaring a number and then taking the square root:

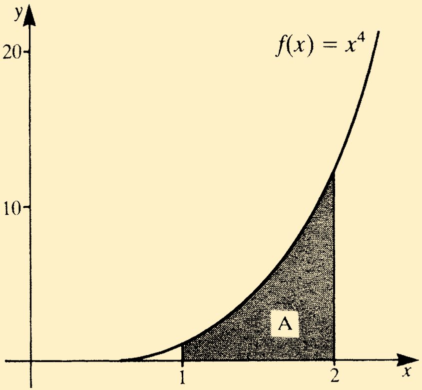
$$\sqrt{a^2} = a$$

Similarly, $\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x f(x)\mathrm{d}x = f(x)$

46

Given the function $f(x) = x^4$. Calculate the area A, shown shaded. How would you proceed?

46



Split the interval into small equidistant points, i.e.

$$x_1 = 1,\ x_2,\ x_3,\ \ldots,\ x_n = 2$$

and obtaining the area A as a limiting value

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

--------------------▷ 47

Look for a primitive function $F(x)$ of $f(x)$; the area A is then given by

$$A = F(2) - F(1)$$

--------------------▷ 48

You are quite right. It *is* possible to calculate the area that way, but this method is cumbersome. We use this method if the function cannot be integrated in a simple manner. It is much easier to look for the primitive of $f(x) = x^4$ and then calculate the area A. 47

48

Correct. 48

We can calculate the area $A = \int_1^2 f(x)\,dx$ by finding the primitive function $F(x)$ of $f(x)$.

Then

$$A = \int_1^2 f(x)\,dx = F(2) - F(1) = \Big[F(x)\Big]_1^2$$

Now complete the problem: calculate the area A:

Primitive function $F(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Area $A = F(2) - F(1)$

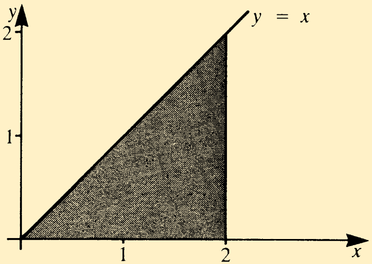
$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

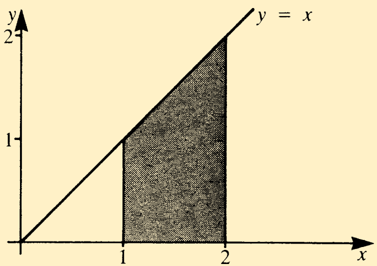
----------------------▷ 49

$$F(x) = \frac{1}{5}x^5 + C$$

$$A = F(2) - F(1) = \left(\frac{1}{5}2^5 + C\right) - \left(\frac{1}{5} + C\right) = 6.2$$

49





Calculate the area under the function $y = x$ in the two intervals (0, 2) and (1, 2).

$$\int_0^2 x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int_1^2 x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

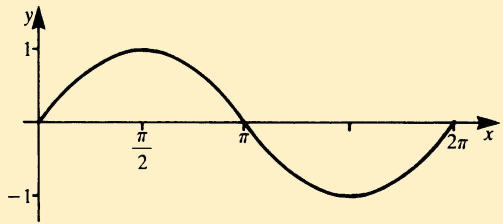
----------------▷ 50

$$\int_0^2 x\,dx = \left[\frac{1}{2}x^2\right]_0^2 = 2 - 0 = 2$$

$$\int_1^2 x\,dx = \left[\frac{1}{2}x^2\right]_1^2 = 2 - \frac{1}{2} = 1.5$$

50

Calculate the absolute value of the area bounded by the sine function for the intervals shown:



(1) $0 \leq x \leq \pi/2$ Area = ...

(2) $0 \leq x \leq \pi$ Area = ...

(3) $0 \leq x \leq 2\pi$ Area = ...

--------------------▷ 51

51

(1) Area $= \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi/2} = 0 - (-1) = 1$

(2) Area $= \int_0^{\pi/2} \sin x \, dx = \Big[-\cos x\Big]_0^{\pi} = 1 - (-1) = 2$

For the third problem we have to subdivide the interval into two sections, as the absolute value of the area is required; hence

(3) Area $= \left|\int_0^{\pi} \sin x \, dx\right| + \left|\int_{\pi}^{2\pi} \sin x \, dx\right| = |2| + |-2| = 4$

52

A vehicle has a uniform acceleration $a = 2\,\mathrm{m/s}^2$ from rest. What is the magnitude of the velocity v (its speed) after 5 seconds? 52

$$v = \int_0^5 a\,\mathrm{d}t$$
$$v\,(5) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

What is the distance s covered in that time?

$$s = \int_0^5 v\,\mathrm{d}t$$
$$s\,(5) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

53

$$v(5) = \Big[at\Big]_0^5; \quad v(5) = 10\,\mathrm{m/s} = 36\,\mathrm{km/h}$$

$$s(5) = \int_0^5 v\,\mathrm{d}t = \int_0^5 at \quad \mathrm{d}t = \left[\frac{at^2}{2}\right]_0^5; \quad s(5) = 25\,\mathrm{m}$$

53

Calculate absolute areas corresponding to the following integrals:

(1) $f(x) = 3\cos x$

(a) $\int_0^{\pi/2} f(x)\mathrm{d}x$ (b) $\int_{-\pi/2}^{\pi/2} f(x)\mathrm{d}x$ (c) $\int_0^{\pi} f(x)\mathrm{d}x$

(2) $f(x) = x - 2$

(a) $\int_{-2}^{0} f(x)\mathrm{d}x$ (b) $\int_0^{2} f(x)\mathrm{d}x$ (c) $\int_0^{4} f(x)\mathrm{d}x$

(d) $\int_2^{4} f(x)\mathrm{d}x$

Watch the limits!

54

(1) (a) 3 (b) 6 (c) $6 = |3| + |-3|$
(2) (a) $|-6|$ (b) $|-2|$ (c) $4 = |-2| + |2|$ (d) 2

54

The areas below the x-axis would be assigned a negative value. In such cases we take the absolute value.

You will find more problems in the textbook.

55

Now it is time for a break. At the end of a working period do we simply close the book and start the break? 55

☐ Yes

☐ No

56

No! Of course not!

Before starting a break you should check that you know all the concepts and rules of that part of the work you have just completed. Use your own notes or extracts.

56

57

6.3 Methods of Integration

57

Objective: Concepts of indefinite integrals, principle of verification, standard integrals, primitive functions.
Operations: integrating.

READ: **6.5 Methods of integration**
6.5.1 Principle of verification
6.5.2 Standard integrals
6.5.3 Constant factor and the sum of functions
Textbook pages 161–163

58

The primitive of the function $f(x)$ is called ....
The symbol for it is ....

58

59

an indefinite integral

$$\int f(x)\,\mathrm{d}x$$

59

There are a number of standard integrals that you should know by heart.

Can you complete the following list?

Function	*Standard integral*
x^n	....
$\sin x$	....
e^x	....
$\dfrac{1}{x}$	....

60

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C\,, \quad n \neq -1$$

60

$$\int \sin x \, dx = -\cos x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \frac{1}{x} \, dx = \ln |x| + C\,, \quad x \neq 0$$

Evaluate the following integrals; the notation has been changed:

$$\int t^2 \, dt = \dots\dots\dots\dots\dots\dots\dots$$

$$\int \cos \phi \, d\phi = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\int e^u \, du = \dots\dots\dots\dots\dots\dots\dots\dots$$

61

$$\int t^2\,\mathrm{d}t = \tfrac{1}{3}t^3 + C$$

$$\int \cos\phi\,\mathrm{d}\phi = \sin\phi + C$$

$$\int \mathrm{e}^u\,\mathrm{d}u = \mathrm{e}^u + C$$

61

Many integrals can be solved with little effort using tables of standard integrals. You will find such a table at the end of Chapter 6 of the textbook, and this should be sufficient in many cases. It is important that you learn how to use such tables.

Using the table, evaluate:

$$\int \frac{1}{(x-a)^2}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

62

$$\int \frac{1}{(x-a)^2}\,\mathrm{d}x = -\frac{1}{x-a} + C$$ 62

You may have solved the last integral by yourself without using the table.

But here is a hard one, for which you will probably need the table:

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x = \dots\dots\dots\dots\dots\dots$$

63

$$\int \frac{1}{1+\sin x}\mathrm{d}x = \tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+C$$

63

In case you have had difficulties here are some hints:

(1) In the table of standard integrals the constant has been omitted.
(2) The integrand is on the left-hand side and the integral on the right-hand side.
(3) When using the table look for the function first so that you get used to the list.
On the top left in the table at the end of chapter 6 you find:

$f(x)$	$\int f(x)\mathrm{d}x = cx$
c	cx

It means:

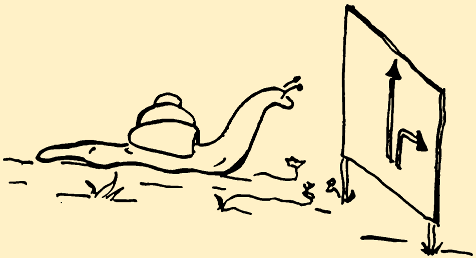
$$\int c\,\mathrm{d}x = cx$$

64

When using the table you should be selective. 64

Selective reading is a special form of 'quick reading'.

Quick reading is useful when looking for a particular piece of information or to obtain a general overview of the text.



Hints on 'quick reading' wanted

-------------------▷ 65

I wish to skip the hints

-------------------▷ 66

The normal reading speed is about 130–150 words per minute. The eyes pass over the lines in jerks and during a normal line of 80–90 letters stop 4–6 times. During a stop a small number of words is perceived simultaneously. With practice it is possible to reduce the number of stops and to increase the number of words perceived simultaneously. Thus closely spaced lines are easier to read. 65

With training it is possible to increase the reading speed to 250–400 words per minute. 'Quick reading' and 'intensive reading' are opposite techniques; mathematical knowledge is acquired by the latter. Knowing different reading techniques enables you to increase your own speed by choosing the correct reading technique to meet your objective.

--------------------▷ 66

Evaluate the following integrals using the table of integrals.

66

$$\int \frac{1}{x-a}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \frac{1}{\cos^2 - x}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \frac{a}{x^2+a^2}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \sin^2\phi\,\mathrm{d}\phi = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int a^t\,\mathrm{d}t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 67

67

$\ln|x-a|+C$

$\tan x + C$

$\tan^{-1}\left(\frac{x}{a}\right)+C$

$\frac{1}{2}(\phi - \sin\phi\cos\phi)+C$

$\frac{a^t}{\ln a}+C$

Further problems for you to solve

(1) $\int t^3\,\mathrm{d}t \quad = \ldots\ldots\ldots\ldots\ldots$

(2) $\int \frac{\mathrm{d}z}{\cos^2 z} = \ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\int u\,\mathrm{d}u \quad = \ldots\ldots\ldots\ldots\ldots\ldots$

I have found the solution ----------▷ 69

I am still having difficulty with the different notations ----------▷ 68

In physics and engineering one uses notations which are best suited to the particular problem. Therefore, there are many different notations: t, z, u, $\ldots$ You can overcome this difficulty if you substitute the following scheme for the notation you are familiar with: 68

(i) substitution: replace t, z, u, $\ldots$ by x.
(ii) execute the mathematical operation.
(iii) replace x by t, z, u .....

Now evaluate

(1) $\int t^3\,\mathrm{d}t = \ldots\ldots\ldots\ldots$

(2) $\int \frac{\mathrm{d}z}{\cos^2 z} = \ldots\ldots\ldots\ldots\ldots$

(3) $\int u\,\mathrm{d}u = \ldots\ldots\ldots\ldots\ldots$

--------------------▷ 69

(1) $\dfrac{t^4}{4} + C$

69

(2) $\tan z + C$

(3) $\dfrac{u^2}{2} + C$

In the table we find

$$\int \tan x \, \mathrm{d}x = -\ln|\cos x|$$

Verify the relationship!

Complete:

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln\cos x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 70

$$\frac{\mathrm{d}}{\mathrm{d}x}(-\ln\cos x) = \frac{-1}{\cos x}(-\sin x) = \frac{\sin x}{\cos x} = \tan x$$ 70

In this way we can verify all integrals in the table. 71

6.4 Integration by Parts

71

Some of the standard integrals shown in the table at the end of Chapter 6 are obtained by using the method of integration by parts.

There is one efficient way of achieving understanding that is quite simple. Work through the examples in the textbook. And after that try to do the *same* examples without the help of the textbook. If you did not know this technique try it with some examples.

READ: 6.5.4 Integration by parts: product of two functions
Textbook page 163–166

--------------------▷ 72

Write down the formula for integration by parts

72

$$\int uv' \, \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

73

$$\int uv' \, \mathrm{d}x = uv - \int vu' \, \mathrm{d}x$$

73

This is the basic formula. To execute it we have to interpret the original integral skillfully. In the textbook the first example we looked at was:

$$\int x\mathrm{e}^x \, \mathrm{d}x$$

In this case we set $x = u$ and $\mathrm{e}^x = v'$. The reason should be clear; we produce an integral we can solve on the right-hand side.

Evaluate the integral for yourself without looking at the textbook.

$$\int x\mathrm{e}^x \, \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 74

$$\int x\mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + C = \mathrm{e}^x(x-1) + C$$ 74

I want to carry on 77

I am not absolutely sure of the underlying rule 75

Additional explanation of the method of integration by parts:

Integration by parts is a skillful exploitation of the rule for differentiating a product.

75

Example:

Let

$$f(x) = x \sin x$$
$$u = x$$
$$v = \sin x$$

The function can be written as

$$f(x) = uv$$

The product rule for differentiation is

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \frac{\mathrm{d}}{\mathrm{d}x}(uv) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

---------------------▷ 76

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u'v + uv'$$

76

Now we reverse the process and integrate both sides giving

$$\int \frac{\mathrm{d}}{\mathrm{d}x}(uv)\,\mathrm{d}x = \int u'v\,\mathrm{d}x + \int uv'\,\mathrm{d}x$$

Differentiation followed by integration neutralise each other. Thus we get:

$$uv = \int u'v\,\mathrm{d}x + \int uv'\,\mathrm{d}x$$

So far we have achieved nothing, but the real trick is to transform the equation so that we have an integral on the left-hand side, i.e. the integral we wish to evaluate, and an integral which we can evaluate on the right-hand side.

The transformation yields

$$\int uv'\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 77

Formula for integration by parts: 77

$$\int uv' \,\mathrm{d}x = uv - \int vu' \,\mathrm{d}x$$

It is useful to know this formula by heart.

Evaluate the following using the technique of integration by parts:

$$\int \ln x \,\mathrm{d}x = \dots\dots\dots\dots\dots\dots$$

Hints wanted --------------------▷ 78

Solution found --------------------▷ 79

The formula is $\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x$

78

Hint: Set

$$u = \ln x$$
$$v' = 1 \text{ and hence } v = x$$

Now evaluate the integral

$$\int \ln x\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

79

$$\int \ln x \, dx = x \ln x - x + C$$

79

If your answer is correct

81

Further explanation required

80

$$\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x$$

80

with
$$u = \ln x,\ u' = \frac{1}{x}$$
$$v' = 1, \quad v = x$$

Hence we have

$$\int \ln x\,1\,\mathrm{d}x = (\ln x)x - x\left(\frac{1}{x}\right)\mathrm{d}x = x\ln x - \int \mathrm{d}x$$
$$= x\ln x - x + C$$

81

81

Using integration by parts, evaluate

$$\int \sin^4 x \, \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In case of difficulties consult the worked example in the textbook.

82

$$\int \sin^4 x \, dx = -\frac{1}{4} \cos x \sin^3 x - \frac{3}{8} \sin x \cos x + \frac{3}{8} x + C$$

82

Correct solution $\dashrightarrow$ 85

Further help required $\dashrightarrow$ 83

$\int \sin^4 x =$ .... Set $u = \sin^3 x$ $\quad u' = 3\sin^2 x \cos x$

$v' = \sin x \quad v = -\cos x$

83

Substituting, we find $\int \sin^4 x\,dx = -\cos x \sin^3 x + 3\int \sin^2 x \cos^2 x\,dx$

Since $\sin^2 x + \cos^2 x = 1$, then $\cos^2 x = 1 - \sin^2 x$.

Substituting in the right-hand integral gives

$$\int \sin^4 x\,dx = -\cos x \sin^3 x + 3\int \sin^2 x\,dx - 3\int \sin^4 x\,dx$$

Rearranging yields $\displaystyle\int \sin^4 x\,dx = -\frac{1}{4}\cos x\,\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx$

We have already evaluated $\displaystyle\int \sin^2 x\,dx$ in the textbook: $\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{1}{2}x + C$

Hence $\displaystyle\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin^3 x - \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C$

84

Some comments on the solution:

84

The integral to evaluate is reduced from $\int \sin^4 x\,\mathrm{d}x$ to $\int \sin^2 x\,\mathrm{d}x$. The same technique would enable us to evaluate $\int \sin^8 x\,\mathrm{d}x$, for instance, reducing it to $\int \sin^6 x\,\mathrm{d}x$, and by repeated application down to $\int \sin^2 x\,\mathrm{d}x$. In the textbook the corresponding reduction formula is stated.

Let us recapitulate:

Integration by parts is based on the formula

$$\int uv'\,\mathrm{d}x = uv - \int vu'\,\mathrm{d}x$$

To apply it successfully we have to factorise the integral into u and $v\prime$ in such a way that the right-hand integral can be solved (either directly or by further reduction).

--------------------▷ 85

85

Now you should take a break.

A period of 20 to 60 minutes of concentrated study is about right if you are to make good progress. The length of the appropriate study period varies with different people. But remember that short breaks are beneficial.

The optimum time in your case is for you to decide; you are the best judge in the end. It is important that you should organise your work, and the number and duration of breaks.

What do you think the duration of this break should be?

☐ 5 minutes

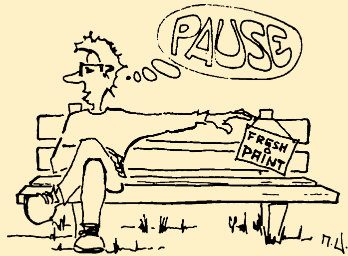
☐ 15 minutes

☐ 30 minutes

------------------▷ 86

Before carrying on with your studies, check the time. How does it compare with the time you fixed before starting the break? There may be a difference between the two times; it is not serious if it is small. 86

As a rule you should keep to the time you decided on, unless there is a good reason why the break was longer. Under normal circumstances you must not allow the duration of the breaks to increase.



87

6.5 Integration by Substitution

87

Objective: Evaluation of integrals by means of a substitution.

The basic idea is developed in the beginning of the section. Substitution is quite a simple technique. Generally speaking, integration is harder in many cases than differentiation.

READ: **6.5.5 Integration by substitution**
6.5.6 Substitution in particular cases
Textbook pages 166–172

88

With the help of a substitution evaluate

88

$$\int \sin(ax+b)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

I have done it!

90

I could do with some help!

89

Go back to the textbook, and go through the examples once more. If you still have difficulties with the concept of composition of functions read section 2.3 of Chapter 2 of the textbook again. 89

Having done this evaluate

$$\int \sin(ax+b)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

90

$$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + C$$

90

Integration by substitution requires the following steps:

Example: To solve $\int \sin(4\pi x)\,\mathrm{d}x$

Step 1: Choose a substitution $u = 4\pi x$

Step 2: Substitute

(a) in the function $\sin(4\pi x) = \sin \mathrm{d}u$

(b) for the differential $\frac{\mathrm{d}u}{\mathrm{d}x} = 4\pi \;\; \mathrm{d}x = \frac{1}{4\pi}\mathrm{d}u$

Step 3: Integrate with respect to the new variable $\frac{1}{4\pi}\int \sin u\,\mathrm{d}u = -\frac{1}{4\pi}\cos u + C$

Step 4: Substitute the original variable in the solution $\int \sin(4\pi x)\,\mathrm{d}x = -\frac{1}{4\pi}\cos(4\pi x) + C$

91

The aim of the method of integration by substitution is to find a suitable substitution to reduce a complicated integral to a standard one. It is only with practice that you will be able to find suitable substitutions. 91

In the textbook we have grouped together certain types of integrals which can be evaluated by substitution and considered as standard types. What are they?

(a) ....
(b) ....
(c) ....
(d) ....

--------------------▷ 92

(a) $\int f(ax+b)\,\mathrm{d}x$ (b) $\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$

(c) $\int f(g(x))g'(x)\,\mathrm{d}x$ (d) $\int R(\sin x, \cos x, \tan x)\,\mathrm{d}x$

92

Consider integrals of the type: $\int f(ax+b)\,\mathrm{d}x$.

Solve: $\int (2-3x)^7 \mathrm{d}x =$ ....

What substitution would you choose to evaluate the integral?

Substitute $u =$ ....

I have found u 94

I am not sure 93

Hint:
The aim of the substitution is to reduce a complicated integral to a standard one. Look again carefully at the example: 93

$$\int (2-3x)^7 \, dx$$

It looks very much like $\int u^7 \, du$ which is a standard integral.

Now what is your choice for u?

$u =$ ....

--------------------▷ 94

$$u = 2 - 3x$$ 94

Solve the integral $\int (2-3x)^7 \, dx =$ ....

95

$$\int (2-3x)^7 \, \mathrm{d}x = -\frac{1}{24}(2-3x)^8 + C$$ 95

Correct ▷ 97

Detailed solution required ▷ 96

96

To evaluate $\int (2-3x)^7 \, dx$

let $u = 2 - 3x$.

$\frac{du}{dx} = -3$, therefore $dx = -\frac{1}{3} du$

The relation between du and dx follows naturally from the equation relating x and u. With the substitution the integral becomes

$$\int (2-3x)^7 \, dx = -\frac{1}{3} \int u^7 \, du = -\frac{1}{24} u^8 + C$$

Finally we express the solution in terms of the original variable x.

The solution is

$$\int (2-3x)^7 \, dx = -\frac{1}{24}(2-3x)^8 + C$$

97

Evaluate $\int \mathrm{e}^{2ax}\,\mathrm{d}x =$ ....

97

98

$$\int e^{2ax}\,dx = \frac{1}{2a}e^{2ax} + C$$ 98

Evaluate $\int \cos^2 x\,dx$

This is an important integral we frequently encounter.

It can be solved in many ways: Transform $\cos^2 x$ using the addition formula, or use the method shown in the textbook for $\int \sin^2 x\,dx$, or use the relation $\cos^2 x = 1 - \sin^2 x$ and use the known result of

$$\int \sin^2 x\,dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) + C$$

$$\int \cos^2 x\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

- - - - - - - - - - ▷ 99

$$\int \cos^2 x \, dx = \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$$ 99

Correct --------------------▷ 101

Detailed solution using the substitution method --------------------▷ 100

We know that

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

100

Therefore

$$\int \cos^2 x \, dx = \frac{1}{2} \int (1 + \cos 2x) \, dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx$$

For the second integral let $u = 2x$, $dx = \frac{1}{2} du$. The integral now becomes

$$\begin{aligned} \int \cos^2 x \, dx &= \frac{1}{2} \int dx + \frac{1}{4} \int \cos u \, du \\ &= \frac{1}{2} x + \frac{1}{4} \sin 2x + C \end{aligned}$$

101

Now let us look at integrals of the type 101

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$$

The numerator is the differential coefficient of the denominator.

Substitute

$$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathrm{d}u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and evaluate the integral

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

102

$$u = f(x), \qquad du = f'(x)\,dx, \qquad \frac{du}{dx} = f'(x)$$

102

$$\int \frac{f'(x)}{f(x)}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|f(x)| + C$$

Evaluate

(a) $\int \cot x\,dx \quad = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(b) $\int \frac{2+x}{x^2+4x}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

I have evaluated the integrals ⟶ 103

I need help ⟶ 104

(a) $\int \cot x \, \mathrm{d}x = \ln|\sin x| + C$

(b) $\int \frac{2+x}{x^2+4x} \, \mathrm{d}x = \frac{1}{2} \ln\left|x^2+4x\right| + C$

103

Correct --------------------▷ 106

Detailed solution required --------------------▷ 104

(a) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$ 104

Let $u = \sin x$, $\frac{du}{dx} = \cos x$

Therefore $dx = \frac{du}{\cos x}$

Substituting in the original integral, we have

$$\int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C$$

Alternatively, we could have used the formula

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

Try it as an exercise!

105

(b) $\int \frac{2+x}{x^2+4x}\,\mathrm{d}x$ 105

Let $u = x^2 + 4x$, $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x + 4 = 2(x+2)$

Therefore $\mathrm{d}x = \frac{\mathrm{d}u}{2(x+2)}$

Substituting in the integral we find

$$\int \frac{2+x}{x^2+4x}\,\mathrm{d}x = \int \frac{2+x}{u}\,\frac{\mathrm{d}u}{2\,(x+2)} = \frac{1}{2}\int \frac{\mathrm{d}u}{u}$$

$$= \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln\left|x^2+4x\right| + C$$

106

Whenever you have to evaluate an integral whose integrand is a fraction you should always check whether the numerator is the differential coefficient of the denominator or can be modified to become the differential coefficient of the denominator. 106

Now to integrals of the type $\int f(g(x))g'(x)\,\mathrm{d}x$.

The integrand is a product, one factor being the differential coefficient of the inner function.

To obtain the general solution

Let

$$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Therefore $\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hence $\int f(g(x))g'(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 107

107

$$u = g(x)$$
$$\frac{du}{dx} = g'(x)$$
$$dx = \frac{du}{g'(x)}$$
$$\int f(g(x))g'(x)dx = \int f(u)du$$

If your result is not correct, consult the textbook and go through the derivation.

108

Evaluate

$$\int \frac{\cosh^5 x - 3\cosh^2 x - 7}{\cosh^4 x} \sinh x \, \mathrm{d}x$$

108

First look for a suitable substitution

$$u = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and then evaluate the integral ....

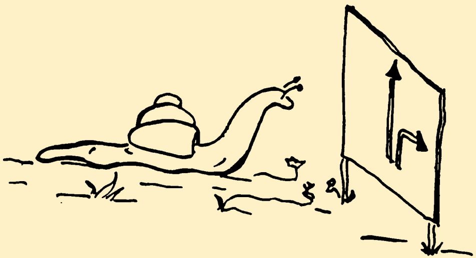
109

Let $u = \cosh x$

The solution of the integral is

$$\frac{1}{2}\cosh^2 x + \frac{3}{\cosh x} + \frac{7}{3\cosh^3 x} + C$$

109



Correct

111

Detailed solution required

110

110

Let
$$u = \cosh x.$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sinh x, \qquad \mathrm{d}x = \frac{\mathrm{d}u}{\sinh x}$$

Substituting in the integral we have

$$\begin{aligned}
\int \frac{\cosh^5 x - 3\cosh^2 x - 7}{\cosh^4 x} \sinh x \, \mathrm{d}x &= \int \frac{u^5 - 3u^2 - 7}{u^4} \, \mathrm{d}u \\
&= \int \left(u - \frac{3}{u^2} - \frac{7}{u^4} \right) \mathrm{d}u \\
&= \frac{u^2}{2} + \frac{3}{u} + \frac{7}{3u^3} + C \\
&= \frac{1}{2}\cosh^2 x + \frac{3}{\cosh x} + \frac{7}{3\cosh^3 x} + C
\end{aligned}$$

111

We now consider integrals of the type: 111

$$\int R(\sin x, \quad \cos x, \quad \tan x, \quad \cot x)\,\mathrm{d}x$$

where the integrand is a rational function of the trigonometric functions. What substitution would you make in this case? Consult the textbook.

$$u = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

--------------------▷ 112

$$u = \tan\frac{x}{2}$$

112

With this substitution the integral becomes

$$\int R_1(u)\,\mathrm{d}u$$

where R_1 is a rational function of u.

In the textbook we expressed $\sin x$, $\cos x$, $\tan x$, $\cot x$ in terms of this new function.

Evaluate

$$\int \frac{\mathrm{d}x}{1+\cos x} = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

using the substitution $u = \tan\frac{x}{2}$.

113

$$\int \frac{\mathrm{d}x}{1+\cos x} = \tan\frac{x}{2} + C$$ 113

If you have had some difficulties consult the textbook. There a similar integral is evaluated; go through its solution carefully.

Exercises are a kind of self-assessment, showing if we are able to apply the methods we have just learnt.

Unfortunately we tend to forget; repeating the methods helps to overcome forgetfulness. Hence exercises are particularly valuable when they are done the following day, a week later and a month later, say.

You will find a number of exercises in the textbook.

114

Now you may have a break. But before starting the break you ought to do two things:

114

(1) ....

(2) ....

------------------▷ 115

(1) Recall what you have just learnt, making sure you have understood the subject matter. 115
(2) Fix the duration of the break or the time when you intend to resume work.

These two things must become a habit, not only now but whenever you are studying a subject using texts.

(1) Give in your own words a résumé of what you have just learnt.
(2) Write down the time when you will continue working.

End of the break . . .

116

Let us now carry on. Check the time against the one you fixed at the start of the break.

116

117

6.6 Integration by Partial Fractions

117

Objective: Understanding the principle of partial fractions. Concept of partial fractions, proper and improper fractions.

The general procedures may seem cumbersome, but you will understand the procedures with the examples.

READ: 6.5.7 Integration by partial fractions
Real and unequal roots
Textbook pages 172–177

----------------▷ 118

It is important that you should understand the principle of decomposition into partial fractions. 118

Express the following in partial fractions:

$$\frac{1}{1-x^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 121

If you require a detailed explanation

--------------------▷ 119

119

Step 1: Find the roots of the denominator and then express it as the product of factors of the lowest possible degree:

$$1 - x^2 = 0$$

The roots are $x_1 = 1$, $x_2 = -1$.
The denominator can be expressed thus:

$$(1 - x^2) = (1 + x)(1 - x)$$

Step 2: Express as partial fractions:

$$\frac{1}{1 - x^2} = \frac{A}{1 + x} + \frac{B}{1 - x}$$

Step 3: Clear the fractions:

$$\frac{1}{1 - x^2} = \frac{A(1 - x)}{(1 + x^2)} + \frac{B(1 + x)}{(1 - x^2)}$$

$$1 = A(1 - x) + B(1 + x)$$

To calculate the values of A and B substitute the values of the roots x_1, x_2 successively; this yields

$$A = \text{..................................}, \quad B = \text{..................................}$$

--------------------▷ 120

120

$$A = \frac{1}{2}, \quad B = \frac{1}{2}$$

The fraction becomes: $\dfrac{1}{1-x^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

121

$$\frac{1}{1-x^2} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$ 121

Here is another problem (in case of difficulty try to solve the problem by using the textbook and following the steps described).

Evaluate $\int \frac{\mathrm{d}x}{x(2x+3)} =$ ....

--------------------▷ 123

Explanation required

--------------------▷ 122

122

Step 1: Express the integrand as partial fractions.
The integrand is already expressed in factors. Thus the roots of the denominator are $x_1 = 0$, $x_2 = -\frac{3}{2}$.

Step 2: Rewrite the integrand as the sum of partial fractions

$$\frac{1}{x\,(2x+3)} = \frac{A}{x} + \frac{B}{2x+3} = \frac{A\,(2x+3)}{x\,(2x+3)} + \frac{Bx}{x\,(2x+3)}$$

Step 3: Calculate the values of the constants by inserting successively the roots

$$A = \ldots\ldots\ldots\ldots\ldots\ldots, \qquad B = \ldots\ldots\ldots\ldots\ldots\ldots$$

Now evaluate the integral

$$\int \frac{\mathrm{d}x}{x\,(2x+3)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

123

$$A = \frac{1}{3}, \quad B = -\frac{1}{3}$$

$$\int \frac{\mathrm{d}x}{x\,(2x+3)} = \frac{1}{3}\int \frac{\mathrm{d}x}{x} - \frac{1}{3}\int \frac{\mathrm{d}x}{(2x+3)}$$

$$= \tfrac{1}{3}\ln|x| - \tfrac{1}{6}\ln|2x+3| + C$$

123

In case of difficulties try to solve the example using the textbook.

--------------------▷ 124

We now discuss an example where the integrand is an improper fractional rational function. You will recall that we have to transform the integrand into the sum of a polynomial and a proper fractional rational function. 124

I would like to do the example

--------------------▷ 125

I know the work and would like to skip the example

--------------------▷ 127

The example is: 125

$$\int \frac{x^3 - 1}{x\,(x+1)}\,\mathrm{d}x$$

The integrand is an improper fraction. We must divide the numerator by the denominator to ensure that the numerator is of lower degree than the denominator before we can express the integrand as partial fractions.

$$\frac{x^3 - 1}{x^2 + x} = \underbrace{x - 1}_{\substack{\text{integral}\\ \text{rational}\\ \text{function}}} + \underbrace{\frac{x - 1}{x\,(x+1)}}_{\substack{\text{proper fractional}\\ \text{rational function}}}$$

126

Now we can solve the integral 126

$$\int \frac{x^3-1}{x^2+x} = \int x\,\mathrm{d}x - \int \mathrm{d}x + \int \frac{x-1}{x\,(x+1)}\,\mathrm{d}x$$

The last integral can be solved by establishing partial fractions

$$\frac{x-1}{x\,(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

Evaluate A and B, solve the integral and check your result:

$$\int \frac{x^3-1}{x\,(x+1)} = \frac{x^2}{2} - x - \ln|x| + 2\ln|x+1| + C$$

Decomposition into partial fractions is carried out step by step as shown in the textbook.

--------------------▷ 127

6.7 Partial Fractions: Real and Repeated Roots, Complex Roots

127

The next two sections in the textbook show cases of partial fractions. The basic procedure is the same. Complex numbers are dealt with in Chapter 9. If you are not familiar with complex numbers skip that part of section 6.5.6 and return to it later.

READ: 6.5.6 Real and repeated roots
Complex roots
Textbook page 168–172

--------------------▷ 128

Evaluate $\int \frac{\mathrm{d}x}{x^3(x+1)}$ 128

To express the integrand in partial fractions we need the roots of the denominator. They are

$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$x_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad x_4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The roots of the denominator are .... but three are ...., therefore we have

$$\frac{1}{x^3(x+1)} = \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots$$

--------------------▷ 129

$$x_1 = x_2 = x_3 = 0, \quad x_4 = -1$$

129

The roots are *real* but three are *equal* (or *repeated*). Hence we must set up

$$\frac{1}{x^3(x+1)} = \frac{N(x)}{D(x)} = \frac{A_1}{x^3} + \frac{A_2}{x^2} + \frac{A_3}{x} + \frac{B}{x+1}$$

Explanation:

The roots $x_1 = x_2 = x_3 = 0$ in the denominator are repeated. If we set up

$$\frac{N(x)}{D(x)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x} + \frac{E}{x+1}$$

we see that after clearing the fractions on the right the common denominator does not agree with $N(x)$. Agreement will occur only if we set up the fractions shown above.

I found the solution ▷ 134

I require further explanation ▷ 130

130

Clearing the fractions

$$\frac{1}{x^3(x+1)} = \frac{A_1}{x^3} + \frac{A_2}{x^2} + \frac{A_3}{x} + \frac{B}{x+1}$$

Or multiplying by $D(x) = x^3(x+1)$ we have:

$1 =$ ....

131

$$1 = A_1(x+1) + A_2x(x+1) + A_3x^2(x+1) + Bx^3$$

131

Since this identity is true for all values of x we can substitute successively

$x = x_1 = 0$	hence	$1 = \ldots\ldots\ldots\ldots\ldots\ldots$
$x = x_4 = -1$		$1 = \ldots\ldots\ldots\ldots\ldots\ldots$
$x = 1$		$1 = \ldots\ldots\ldots\ldots\ldots\ldots$
$x = 2$		$1 = \ldots\ldots\ldots\ldots\ldots\ldots$

We can choose the values of x conveniently.

132

132

For $x = x_1 = 0$ we have $1 = A_1$

$x = x_4 = -1$ $\quad 1 = -B$

$x = 1$ $\quad 1 = 2A_1 + 2A_2 + 2A_3 + B$

$x = 2$ $\quad 1 = 3A_1 + 6A_2 + 12A_3 + 8B$

From the first two equations we find

$$A_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$B = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Inserting these values in the last two equations yields

$$A_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_3 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

133

$$A_1 = 1, \quad A_2 = -1$$
$$A_3 = 1, \quad B = -1$$

133

With these values we can evaluate the integral, hence

$$\int \frac{\mathrm{d}x}{x^3(x+1)} = \int \frac{\mathrm{d}x}{x^3} - \int \frac{\mathrm{d}x}{x^2} + \int \frac{\mathrm{d}x}{x} - \int \frac{\mathrm{d}x}{x+1}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 134

$$\int \frac{dx}{x^2(x+1)} = -\frac{1}{2x^2} + \frac{1}{x} + \ln|x| - \ln|x+1| + C$$

134

The following example uses complex numbers, which are dealt with in Chapter 9. If you are not familiar with these skip the example and go to

--------------------▷ 138

Otherwise

--------------------▷ 135

Evaluate $\int \frac{7x^2 - 10x + 37}{(x+1)(x^2 - 4x + 13)}\,\mathrm{d}x$ using partial fractions. 135

Proceed as before; first obtain the roots of the denominator.
The roots are
$x_1 = \ldots\ldots\ldots\ldots$, $x_2 = \ldots\ldots\ldots\ldots$, $x_3 = \ldots\ldots\ldots\ldots$

The partial fractions are

$$\frac{7x^2 - 10x + 37}{(x+1)(x^2 - 4x + 13)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

136

$$x_1 = -1, \quad x_2 = 2+3\mathrm{j}, \quad x_3 = 2-3\mathrm{j}$$

136

We express the integrand by

$$\frac{7x^2-10x+37}{(x+1)(x^2-4x+13)} = \frac{A}{x+1} + \frac{Px+Q}{x^2-4x+13}$$

Now calculate the constants A, P and Q in the same way as before and then evaluate the integral.

--------------------▷ 137

$$A = 3, \quad P = 4, \quad Q = -2$$

137

and using the table of integrals

$$\begin{aligned}\int \frac{7x^2 - 10x + 37}{(x+1)(x^2 - 4x + 13)}\,\mathrm{d}x &= 3\int \frac{\mathrm{d}x}{1+x} + \int \frac{4x-2}{x^2-4x+13}\,\mathrm{d}x \\ &= 3\ln|1+x| + 2\ln\left|x^2 - 4x + 13\right| + 2\tan^{-1}\left(\frac{x-2}{3}\right) + C\end{aligned}$$

If your solution does not agree, follow the worked example in section 6.5.6 in the textbook. It is similar to the one above and you should then be able to obtain the correct solution.

138

Integration by partial fractions follows a definite pattern. First make sure that the numerator is of lower degree than the denominator. 138

Step 1: Find the roots of the denominator of the integrand.
Step 2: Rewrite the original integrand as a sum of the partial fractions.
Step 3: Finally, calculate the values of the constants which appear with each partial fraction. Now you can evaluate the corresponding integrals, and hence the original integral.

Now it is time for a break.

--------------------▷ 139

6.8 Rules for Solving Definite Integrals Substitution

139

Objective: Practice of different methods of evaluating definite integrals.

READ: **6.6 Rules for solving definite integrals**
6.7 Mean value theorem
Textbook pages 177–180

----> 140

Evaluate $$\int_0^{1.76} x^3\,\mathrm{d}x + \int_{1.76}^{2} x^3\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

140

141

$$\int_0^{1.76} x^3\,\mathrm{d}x + \int_{1.76}^{2} x^3\,\mathrm{d}x = \int_0^2 x^3\,\mathrm{d}x = \left[\frac{1}{4}x^4\right]_0^2 = 4$$

141

The following generally holds true

$$\int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 142

$$\int_a^b f(x)\,\mathrm{d}x$$

142

$$\int_0^x \left(\frac{ax^3}{4} + bx^2 + c\right)\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

143

$$\frac{ax^4}{16} + \frac{bx^3}{3} + cx$$ 143

If $F(x) = \mathrm{e}^x$

then $F'(x) = f(x) = \mathrm{e}^x$

Evaluate

$$\int_0^1 \mathrm{e}^x \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\mathrm{e}^x + C$

144

$\mathrm{e} - 1$

145

Wrong; you are not paying attention. 144

A definite integral $\int_a^b f(x)\,\mathrm{d}x$ with fixed limits a and b is equal to the difference $F(b) - F(a)$ if $F(x)$ is a primitive function of $f(x)$; i.e.

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

The given function was $f(x) = \mathrm{e}^x$; its primitive function is $F(x) = \mathrm{e}^x$. In order to calculate the value of the integral you have to insert the limits. Hence

$$\int_0^1 \mathrm{e}^x\,\mathrm{d}x = F(1) - F(0)$$

but $F(1) = \mathrm{e}^1 = \mathrm{e}$
and $F(0) = \mathrm{e}^0 = 1$
Therefore the value of the integral is ....

$\dashrightarrow$ 145

e − 1 is correct

145

If $f(x) = x^2$, then the indefinite integral (general solution) is

$$\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + 2$$

This result is ☐ correct 147

☐ wrong 146

146

You are right!

The indefinite integral defines a *family* of primitive functions which differ from each other by arbitrary constants. Hence we should write:

$$\int x^2 \,\mathrm{d}x = \frac{1}{3}x^3 + C$$

or generally

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

Skip the next frame → 148

The arbitrary constant has been given the value 2. It is not the general solution.
$\frac{1}{3}x^3 + 2$ is only one possible primitive function of $f(x) = x^2$.

147

The indefinite integral defines a family of primitive functions; hence we must write

$$\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C$$

where C depends on a boundary condition.

Generally

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

148

Evaluate the integral $\int_1^2 (3x-4)^2\,\mathrm{d}x$ by means of the substitution $t = 3x - 4$. 148

In the textbook we used the letter u for the substitution but we can obviously use whatever letter we care to choose. A change in notation should not cause you any problems.

In terms of the new variable t the integral and its *limits* are given by:

$$\int_1^2 (3x-4)^2\,\mathrm{d}x = \int_1^2 t^2\,\mathrm{d}t$$

--------------------▷ 149

$$= \int_1^2 \frac{1}{3} t^2\,\mathrm{d}t$$

--------------------▷ 150

$$= \int_{-1}^2 \frac{1}{3} t^2\,\mathrm{d}t$$

--------------------▷ 153

I want a detailed explanation of the solution

--------------------▷ 151

You made a mistake, you forgot to substitute for dx. 149

Since $t = 3x - 4$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = 3, \quad \text{therefore} \quad \mathrm{d}x = \frac{1}{3}\,\mathrm{d}t$$

You also have to change the limits of integration. The new limits are calculated by substituting in the equation for t as a function of x.

$$\int_1^2 (3x-4)^2\,\mathrm{d}x = \int_1^2 \frac{1}{3} t^2\,\mathrm{d}t$$

--------------------▷ 150

$$= \int_{-1}^2 \frac{1}{3} t^2\,\mathrm{d}t$$

--------------------▷ 153

Detailed explanation

--------------------▷ 151

The limits of integration are wrong. 150

The new limits are obtained as follows:

since $t = 3x - 4$, then

when $x_1 = 1$ we have $t_1 = 3 \times 1 - 4 = -1$

when $x_2 = 2$ we have $t_2 = 3 \times 2 - 4 = 2$

The integral becomes

$$\int_{-1}^{2} \frac{1}{3} t^2 \, \mathrm{d}t$$

I want to go on

---------------------▷ 154

I want a detailed explanation of the method of substitution

---------------------▷ 151

151

To help you understand the method we will consider a different example and go through the solution step by step. As you follow the steps complete the right-hand side, which was the original problem.

Step	*Example*	*Exercise*
	$\int_0^2 (2x+3)\,\mathrm{d}x$	$\int_1^2 (3x-4)^2\,\mathrm{d}x$
(1) Select a substitution	$t = 2x+3$	$t = 3x-4$
(2) Differentiation with respect to the original variable	$\frac{\mathrm{d}t}{\mathrm{d}x} = 2$	$\frac{\mathrm{d}t}{\mathrm{d}x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
(3) Find the value of dx	$\mathrm{d}x = \frac{1}{2}\,\mathrm{d}t$	$\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
(4) Calculate thenew limits	$t_1 = 2\times 0+3 = 3$ $t_2 = 2\times 2+3 = 7$	$t_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $t_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$
(5) The new integral is:	$\int_3^7 \frac{1}{2}t\,\mathrm{d}t$	$\int \ldots\ldots\,\mathrm{d}t = \ldots..$

152

152

We were required to evaluate $\int_1^2 (3x-4)^2 \mathrm{d}x$

(1) Substitution: $t = 3x - 4$

(2) Differentiation: $\frac{\mathrm{d}t}{\mathrm{d}x} = 3$

(3) Differential: $\mathrm{d}x = \frac{1}{3}\mathrm{d}t$

(4) New limits: $t_1 = 3 \times 1 - 4 = -1$
$t_2 = 3 \times 2 - 4 = 2$

(5) New integral: $\frac{1}{3}\int_{-1}^{2} t^2 \,\mathrm{d}t = 1$

154

Correct!

The value of the integral is:

$$\int_1^2 (3x-4)^2 \mathrm{d}x = \frac{1}{3}\int_{-1}^2 t^2\,\mathrm{d}t = \left[\frac{1}{9}t^3\right]_{-1}^2 = \frac{8}{9}+\frac{1}{9} = 1$$

Note that we could have evaluated the same integral without a substitution:

$$\begin{aligned}\int_1^2 (3x-4)^2 \mathrm{d}x &= \int_1^2 (9x^2-24x+16)\mathrm{d}x \\ &= \left[3x^3-12x^2+16x\right]_1^2 \\ &= (24-48+32)-(3-12+16) = 8-7 = 1\end{aligned}$$

154

Evaluate the following definite integrals:

154

(1) $\int_0^{1/2} \sin \pi x \, \mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots$

(2) $\int_1^2 a v^2 \, \mathrm{d}v = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\int_0^3 \mathrm{e}^{-\gamma t} \, \mathrm{d}t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

155

$$(1)\ \frac{1}{\pi}, \quad (2)\ \frac{7a}{3}, \quad (3)\ \frac{1}{\gamma}(1-e^{-3\gamma})$$

155

I want to carry on

----------------------▷ 158

The notations are troubling me

----------------------▷ 156

As we have said before, physical quantities are denoted by certain convenient letters, e.g. time t, mass m, force F, velocity v, frequency ω, modulus of elasticity E, pressure p, density ρ, volume V, etc ... The list is as long as you like. 156

If you are not familiar with the notation of a variable it is advisable to replace it by a familiar one.

Replace the variable of integration by x and evaluate the integrals:

$$\int \frac{1}{2}\cos\phi \, d\phi = \dots\dots\dots\dots$$

and

$$\int (3\sin\alpha + \cos\alpha)\, d\alpha = \dots\dots\dots\dots$$

--------------------▷ 157

$$\int \frac{1}{2}\cos\phi\, d\phi \rightarrow \frac{1}{2}\int \cos x\, dx = \frac{1}{2}\sin x + C \rightarrow \frac{1}{2}\sin\phi + C$$

157

$$\int (3\sin\alpha + \cos\alpha)\, d\alpha \rightarrow \int (3\sin x + \cos x)\, dx = -3\cos x + \sin x + C$$
$$\longrightarrow -3\cos\alpha + \sin\alpha + C$$

The procedure should be clear. It often helps to facilitate the integration.

Step 1: Replace the unfamiliar variable by a familiar one, e.g. x.
Step 2: Integrate, use the table of integrals.
Step 3: Replace the familiar variable by the original one.

158

You will find more exercises in the textbook. Remember that doing a lot of examples immediately following the learning of a particular aspect of the work is not too helpful. It is more beneficial if you do some the following day, a week later or a month later, or even a year later! 158

159

6.9 Improper Integrals

159

Objective: Concept of improper integrals.

Task: Evaluation of convergent improper integrals

The concept of the definite integral is extended to include cases in which the limits of integration are infinite. One such integral with an infinite limit frequently encountered is:

$$\int_{x_0}^{\infty} \frac{\mathrm{d}x}{x^2}$$

Example: The work done while projecting a body outside the gravitational field of the Earth.

READ: 6.8 Improper integrals
Textbook pages 181–183

--------------------▷ 160

An integral in which at least one limit of integration tends to infinity is called an improper integral. 160

Can such an integral have a finite value?

☐ Yes → 162

☐ No → 161

You are wrong. An improper integral may have a finite value.

161

Perhaps you are getting tired. There was a lot of material to cover; a break might be a good thing.

The last section dealt with improper integrals and it was shown that such integrals can have finite values even if one of the limits tends to infinity. This does not hold true for all such integrals. An improper integral with a finite value is said to be convergent.

--------------------▷ 162

You are right. Improper integrals may have finite values!

162

The integral $\int_a^\infty \frac{dx}{x^2}$ is of special interest to physicists.

What is its value?

$$\int_a^\infty \frac{dx}{x^2} = \ldots\ldots\ldots\ldots\ldots\ldots$$

163

$$\int_a^{\infty} \frac{\mathrm{d}x}{x^2} = \left[-\frac{1}{x}\right]_a^{\infty} = -\frac{1}{\infty} + \frac{1}{a} = \frac{1}{a}$$

163

The improper integral $\int_a^{\infty} \frac{\mathrm{d}x}{x^2}$ occurs frequently in physics, for instance in the calculation of the energy required to:

(i) take a body 'outside' the gravitational field of the Earth (i.e. to infinity),
(ii) remove an electron from an atom.

The name improper integral should not disturb you! It is only a special case of the definite integral and simply means that one of the limits tends to infinity.

164

What is the value of the improper integral

164

$$\int_a^{\infty} \frac{1}{x}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots \qquad (a > 0)?$$

I have found the solution --------------------▷ 167

I want further explanation --------------------▷ 165

Further explanation: 165

Step 1: We start with a definite integral and evaluate it:

$$\int_a^b \frac{1}{x}\,\mathrm{d}x = \Big[\ln x\Big]_a^b, \quad a \text{ and } b \text{ positive}$$

Step 2: We insert the limits and carry out the limiting process $b \longrightarrow \infty$:

$$\int_a^\infty \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} \int_a^b \frac{1}{x}\,\mathrm{d}x = \lim_{b\to\infty} (\ln b - \ln a)$$

Does the term including b converge towards a finite value?

☐ Yes 166

☐ No 167

No! It does not, for the simple reason that as $b \longrightarrow \infty$ the value of In b grows beyond all bounds, i.e. In $b \longrightarrow \infty$. This improper integral is *not* convergent. 166

Hence $\int_a^\infty \frac{\mathrm{d}x}{x} = \Big[\ln x\Big]_a^\infty =$ ....

-------------------▷ 167

$$\int_a^{\infty} \frac{\mathrm{d}x}{x} = \infty$$

167

All clear

168

Detailed solution

165

If you need a little more practice try the following problems. They are similar to those we have discussed already. 168

(1) $\displaystyle\int_4^\infty \frac{\mathrm{d}s}{s^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(2) $\displaystyle\int_{10}^\infty \frac{\mathrm{d}x}{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{(2+x)^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(4) $\displaystyle\int_1^\infty \frac{\mathrm{d}\gamma}{\gamma^4} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 169

(1) $\frac{1}{4}$ (2) ∞ (3) $\frac{1}{3}$ (4) $\frac{1}{3}$

169

There are more problems in the textbook!

You should now know whether you can solve the problems easily, in which case you need not do any more, or whether you do not find the solutions easily, in which case more exercises are needed.

170

In previous chapters we have discussed three study techniques; what are they? 170
Write them down, using keywords:

(1) ....
(2) ....
(3) ....

171

171

(1) Balance between periods of study and breaks, keeping to an established timetable.

(2) Intensive reading:
Taking notes of new concepts, definitions and rules; inthe case of mathematical derivations, doing them yourself.

(3) Selective reading:
Searching quickly for new information; glancing over large sections of text looking for particular material.

Presumably you wrote these down in your own words. You should know these techniques by now. Remember that it is not sufficient to know them; the problem is to apply them!

--------------------▷ 172

6.10 Line Integrals

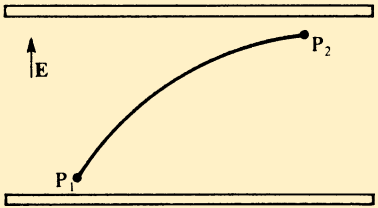
172

This section may be of more interest to the physicist than to the engineer.

The basic concept of the line integral is quite easy to understand, even if the notation of some expressions may seem cumbersome.

READ: 6.9 Line integrals
Textbook pages 183–186

173



The figure shows a capacitor. An electrical charge q is moved from point

$\mathbf{P}_1 = (x_1, \ y_1)$ to point

$\mathbf{P}_2 = (x_2, \ y_2)$. The force acting on the charge is

$$\mathbf{F} = \mathbf{E}q$$

173

We want to evaluate the work done.

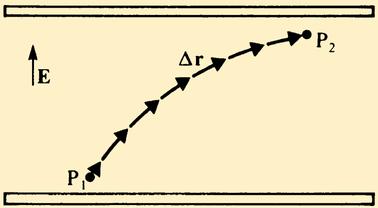
Write the general expression for the line integral

$$W = \ldots\ldots\ldots\ldots\ldots\ldots$$

174

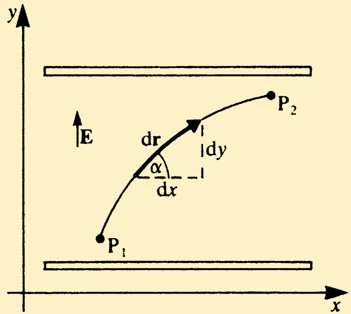
$$W = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \int_{P_1}^{P_1} \mathbf{E}q \cdot d\mathbf{r}$$

174



We can approximate the line integral by a sum. The work done could be calculated for each path element. Analytically the line integral can be solved if we succeed in transforming it into known and solvable integrals.

175



175

To calculate:

$$W = q\int_{P_1}^{P_2} \mathbf{E}\cdot \mathrm{d}\mathbf{r}$$

We assume a homogeneous electrical field in the y-direction:

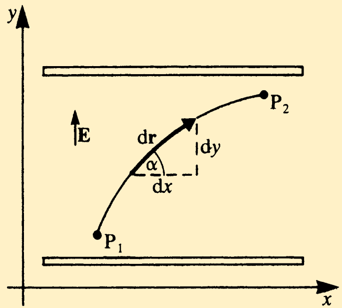
$$\mathbf{E} = (0, \quad E).$$

The path element is $\mathrm{d}\mathbf{r} = (\mathrm{d}x, \quad \mathrm{d}y)$

Thus

$$W = q\int \mathbf{E}\cdot \mathrm{d}\mathbf{r} = q\int_{P_1}^{P_2} \ldots\ldots\ldots\ldots\ldots$$

176



$$W = q\int_{P_1}^{P_2} \mathbf{E}\cdot \mathrm{d}\mathbf{r} = q\int_{y_1}^{y_2} E\,\mathrm{d}y$$ 176

Only the movement in the y-direction contributes to the work W.

Inserting the limits yields

$$\int_{y_1}^{y_2} E\,\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 177

$$\int_{y_1}^{y_2} E \, \mathrm{d}y = E(y_2 - y_1)$$

177

A satellite moves in a circular orbit. The work done during one rotation around the Earth is given by the line integral

$$W = \int_{\text{circle}} \mathbf{F} \cdot \mathbf{r} \, \mathrm{d}\mathbf{r}$$

For the direction of **F** and d**r** you can give the value of this line integral:

$$W = \int_{\text{circle}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots\ldots$$

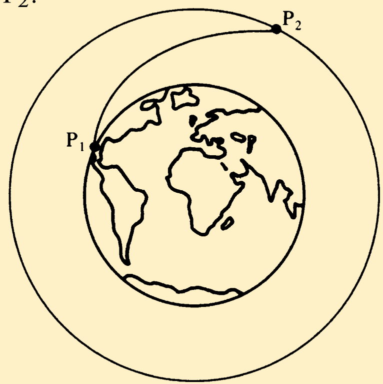
178

178

$$W = \int \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$$

Explanation: since **F** and d**r** are perpendicular the scalar product vanishes.

The figure shows the Earth and the path of another satellite. The satellite starts at P_1. It reaches its orbit at P_2.



Can you give the work to be done against the gravitational forces?

r_1 = radius of the Earth
r_2 = radius of the orbit

$$\mathbf{F} = \gamma \frac{mM}{r^2} \frac{\mathbf{r}}{r}.$$

Remember that $\frac{\mathbf{r}}{r}$ is a unit vector with a radial direction.

$W =$ 183

Explanation 179

We ask for the work to be done against the gravitational forces. We do not consider the work done to accelerate the satellite. 179
The work is given by the line integral

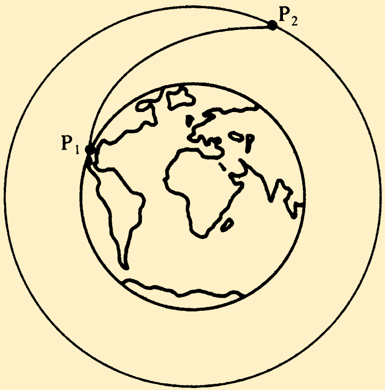
$$W = \int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r}$$

Now remember that the gravitational field is a conservative field. Thus for given points P_1 and P_2 the work

☐ depends on the geometric form of the path

☐ does not depend on the geometric form of the path

----------------▷ 180



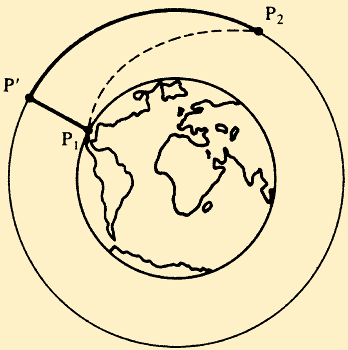
The work does not depend on the geometric form of the path. It depends only on the coordinates of P_1 and P_2.

180

In this case you may find a path for which the line integral is easy to evaluate.

Sketch it on the drawing.

----------------▷ 180



The path is composed of two elements. First, a path in radial direction from P_1 to P'; second, an arc of a circle from P' to P_2.

181

The work done can be calculated for both parts. Remember that

$$\mathbf{F} = \gamma \frac{mM}{r^2}\frac{\mathbf{r}}{r}$$

$$W_1 = \int_{P_1}^{P'} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$W_2 = \int_{P'}^{P_2} \mathbf{F}\cdot d\mathbf{r} = \ldots\ldots\ldots\ldots\ldots\ldots$$

182

$$W_1 = \gamma m M \left(\frac{1}{r_1} - \frac{1}{r'}\right)$$ 182

$W_2 = 0$ since d**r** is perpendicular to **F**.

183

$$U = \gamma m M \left(\frac{1}{r_1} - \frac{1}{r'} \right)$$

183

If you want a detailed explanation

179

Now you deserve a break! But remember:
It is important to recapitulate before you have a break.

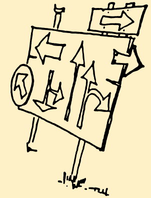
Chapter 7
Applications of Integration

0

1

This chapter demonstrates in detail the use of integral calculus to solve special problems in physics and mechanics. It may be skipped for the time being and can be used later on as a reference when problems are encountered. 1

Thus this study guide will be divided in separate units for each topic discussed.
Choose the topic you want to study:



Areas ----------------▷ 2

Lengths of curves ----------------▷ 30

Surface area and volume of a solid of revolution ----------------▷ 33

Applications to mechanics ----------------▷ 41

The Theorems of Pappus
Moment of inertia, second moment of area ----------------▷ 60

For the time being I choose to proceed with chapter 8,
"Taylor series and power" series and will skip this study guide proceed to chapter 8

7.1 Areas

2

The integral calculus has been introduced in chapter 6 by solving the area problem. Thus the first following example will be easy to read. In the second example the notation is changed, so you have to read quite carefully.

READ **7.1 Areas**
Textbook pages 191-194

-----------------▷ 3

Given a function $f(x) = \frac{C}{x}$.

$\boxed{3}$

Calculate the area between x = 1 and x = 5. First give the formal solution in form of a ………..integral. Let C be C = 2

A = ……….

--------------------▷ $\boxed{4}$

definite integral

$A = C[\ln x]_1^5$

4

Using your calculator evaluate

$A = C[\ln x]_1^5 =$ .... = ...

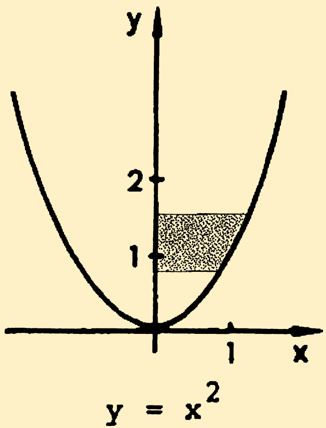
5

$$A = 2 \cdot [\ln x]_1^5 = 2(\ln 5 - \ln 1) = 2 \cdot \ln 5$$

5

$$= 2 \cdot 1.6094 = 3.2189$$

The figure below shows the well known parabola
$y = x^2$.
The area shaded is named ………… .



-----------------▷ 6

Complementary area

6

Calculate the complementary area for $y_1 = 1$ and $y_2 = 3$.

A = ...

Solution found ----------------▷ 8

Help wanted ----------------▷ 7

7

To solve $A_C = \int_{y_1}^{y_2} x\, dy$ Given $y = x^2$

First we must find $x = g(y)$

In this case

x = ……………

Now solve

$$A_C = \int_1^3 x dy$$

$$A_C = \ldots\ldots\ldots\ldots$$

-----------------▷ 8

8

$$x = +\sqrt{y}$$

$$A_C = \int_1^2 \sqrt{y}\,dy = \frac{2}{3}\left[y^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}\left[2^{\frac{3}{2}} - 1\right] = \frac{2}{3}1.828 = 1.219$$

If you had difficulties solving the example in the textbook regarding thermodynamics we suggest to calculate the example substituting p by y and V by x. Then try again to calculate the example following the text given in the textbook.

-----------------▷ 9

7.1.1 Areas for parametric functions

9

This section reguires knowledge of parametric functions which have been introduced in chapter 5 section 5.10.
It may be helpful to rehearse this section before proceeding.
In case of difficulties go back to section 5.10. The cycloid which is discussed in the following has been introduced at the end of section 5.10

Now study **7.1.1 Areas of parametric functions**
Textbook pages 194–195

-----------------▷ 10

Following the text given, calculate the area of a half circle. The circle of radius R is given in parametric form by 10

$$x = R \cdot \cos\varphi$$
$$y = R \cdot \sin\varphi$$

A = ...

Solution -----------------▷ 13

Help -----------------▷ 11

The parameter is φ.

Note: In the text the parameter is t.

The text shows that

$$A = \int_{\varphi_1}^{\varphi_2} g(\varphi)\frac{dx}{d\varphi}\,d\varphi$$

11

Given $y = R\sin\varphi$ $x = R\cos\varphi$

Remember that we start integrating at x = 0 which corresponds to $\varphi_1 = \dfrac{\pi}{2}$ and proceed to $x = R$ coresponding to $\varphi_2 = 0$

Thus we get A = ……………………..

Solution -----------------▷ 13

Further help wanted -----------------▷ 12

Given $A = \int_{\varphi_1}^{\varphi_2} g(\varphi) \cdot \frac{dx}{d\varphi} d\varphi$ 12

Since $y = R\sin\varphi$ and $x = R\cos\varphi$

$$\frac{dx}{d\varphi} = -R\sin\varphi$$

Boundaries: We integrate from $x = 0$ and $\varphi = \frac{\pi}{2}$ to $x = \mathrm{R}$ and $\varphi = 0$

Thus we get

$$A = \int_{\frac{\pi}{2}}^{0} R\sin\varphi(-R\sin\varphi)d\varphi = R^2 \int_{0}^{\frac{\pi}{2}} \sin^2\varphi d\varphi$$

$$\mathrm{A} = R^2 \frac{1}{2}[\varphi - \sin\varphi \cdot \cos\varphi]_0^{\frac{\pi}{2}} = \ldots\ldots\ldots\ldots\ldots$$

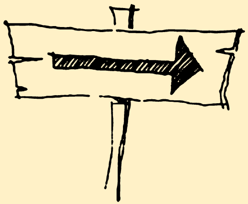
----------------▷ 13

13

$$A = \frac{R^2 \pi}{4}$$

A result well known for a quarter of a circle.

In case of difficulties try again to solve the last exercise on your own



----------------▷ 14

7.1.2 Areas in polar coordinates

14

Polar coordinates have not been introduced yet. They will be introduced in chapter 13, section 13.4. They are quite easy to understand. If you want to study the following section, you have to study section 13.4 before. It is a short section without greater difficulties.
Having done read

7.1.2 Areas in polar coordinates
Textbook pages 195–196

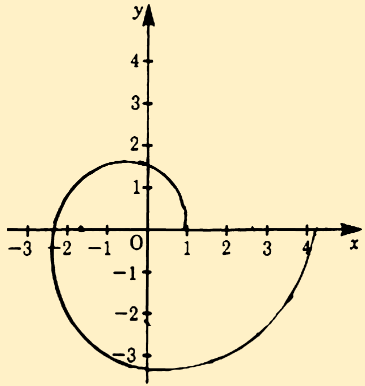
----------------▷ 15

Given a spiral

$r = (1 + 0.5\varphi)$

Calculate the area $0 \le \varphi \le 2\pi$

A = ...



15

Solution found -----------------▷ 17

Hint wanted -----------------▷ 16

It is fundamental to understand the formula. Go back to the textbook and read once more the derivation of the formula $A = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r^2 d\varphi$ 16

Given $r = (1 + 0.5\varphi)$
The boundaries are $\varphi_1 = 0$ $\varphi_2 = 2\pi$

Now insert *r* into the integral which is easy to solve.

A = …………

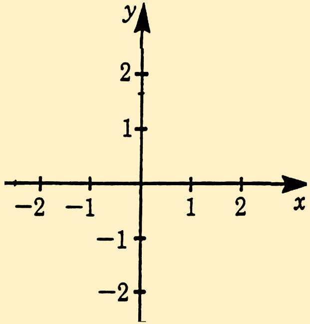
----------------▷ 17

$$A = \frac{1}{2}\int_0^{2\pi}(1+0.5\varphi)^2\,d\varphi = \frac{1}{2}\int_0^{2\pi}\left(1+\varphi+\frac{1}{4}\varphi^2\right)d\varphi = \frac{1}{2}\left[\varphi+\frac{\varphi^2}{2}+\frac{1}{4\cdot 3}\varphi^3\right]_0^{2\pi} = \pi+\pi^2+\frac{1}{3}\pi^3$$

17

Calculate the total area bounded by the curve $r = 1+\cos\varphi$ and the $x = axis$

First try to sketch the curve

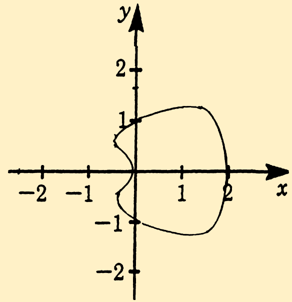


Now calculate the total area

A = ………………………..

Solution found -----------------▷ 20

Hint needed -----------------▷ 18



18

The area consists of two parts whose areas are equal.

Since the total area is asked for, you may calculate the first part and double the result.
Or you calculate both parts using the appropriate boundaries.
For the first part you get

$$A_1 = \frac{1}{2}\int_0^{\pi}(1+\cos\varphi)^2\,d\varphi = \frac{1}{2}\int_0^{\pi}\left(1+2\cos\varphi+\cos^2\varphi\right)d\varphi = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_1 = \frac{1}{2}\left[\varphi + 2\sin\varphi + \frac{1}{2}(\varphi + \sin\varphi\cdot\cos\varphi)\right]_0^{\pi}$$

Thus

$$A_1 = \ldots\ldots\ldots$$

----------------▷ 19

$$A_1 = \frac{1}{2}\pi + \frac{1}{4}\pi = \frac{3}{4}\pi$$ 19

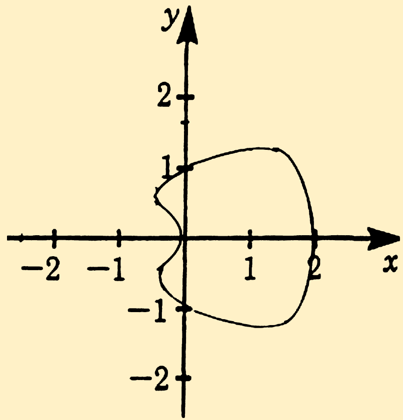
Thus the total area is A = …….

Now try the other way. The result must be the same.

A = …….

-----------------▷ 20

$$A = \frac{3}{2}\pi$$



20

21

7.1.3 Areas of closed curves

This section will be easy to understand since no new concepts are involved.

READ **7.1.3 Areas of closed curves**
Textbook pages 197–198

22

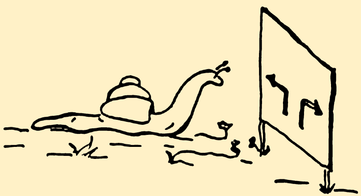
Calculate the area bounded by the following curves:

22

$y^2 = 4x$ and $x^2 = 6y$

This is an intriguing question at first sight.

A = ……



Solution found ----------------▷ 29

Hints wanted ----------------▷ 23

Given $y_1^2 = 4x$ and $x^2 = 6y_2$

23

First we transform these equations into a form we can integrate:

$y_1 = ...$ $y_2 = ...$

-----------------▷ 24

$$y_1 = 2\sqrt{x} \text{ and } y_2 = \frac{x^2}{6}$$

24

These forms can be integrated because they are standard.
Thus we get the expressions

$A_1 = \ldots\ldots.$ $A_2 = \ldots\ldots..$

25

$$A_1 = \int_{x_0}^{x_1} y_1 dx = 2\int_{x_0}^{x_1} \sqrt{x}dx = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_{x_0}^{x_1}$$

25

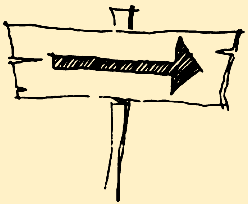
$$A_2 = \int_{x_0}^{x_1} y_2 dx = \int_{x_0}^{x_1} \frac{x^2}{6} dx = \left[\frac{x^3}{3\cdot 6}\right]_{x_0}^{x_1}$$

Now the boundaries x_0 and x_1 have to be determined. Given $y_1 = 2\sqrt{x}$ and $y_2 = \frac{x^2}{6}$.

The first intersection point is $x_0 = 0$.

For the second intersection point the y-values must coincide. This gives $y_1 = y_2$ and therefore the equation

……. = ……….



----------------▷ 26

$$2\sqrt{x_1} = \frac{x_1^{\,2}}{6}$$

26

Thus we get the second limit

$x_1 = \ldots\ldots\ldots$

Solution ----------------- ▷ 28

Further hint ----------------- ▷ 27

Given $2\sqrt{x_1} = \frac{x_1^{\;2}}{6}$ Wanted x_1

27

We transform

$$12 = \frac{x_1^{\;2}}{\sqrt{x_1}} = x_1^{\frac{3}{2}}$$

Next transformation:

$$12^2 = x_1^{\;3} = 144$$

$$x_1 = \ldots\ldots\ldots\ldots\ldots$$

---------------------▷ 28

$x_1 = \sqrt[3]{144}$

28

Now we can insert $x_0 = 0$ and $x_1 = \sqrt[3]{144}$ into the results obtained before (see frame 25)

$$A_1 = \left[\frac{4}{3}x^{\frac{3}{2}}\right]_0^{x_1} \text{ and } A_2 = \left[\frac{x^3}{3\cdot 6}\right]_0^{x_1}$$

$A = ...$

-----------------▷ 29

$A = 8$

29

Again it is up to you to study further sections of applications of integral calculus or to skip these for the time being and to return later on when this material may be needed.

Thus choose

7.2. Lengths of curves ----------------▷ 30

7.3 Surface area and volume of a solid of revolution ----------------▷ 33

7.4 Applications to mechanics ----------------▷ 41

7.4.1 The Theorems of Pappus ----------------▷ 50

7.4.2 Moment of inertia, second moments of area ----------------▷ 60

7.2 Lengths of curves

30

This section needs careful reading. Try to follow the transformations, executing them on a separate sheet. Basically we apply the theorem of Pythagoras to a small triangle.

READ **7.2. Lengths of curves**
Textbook pages 198–202

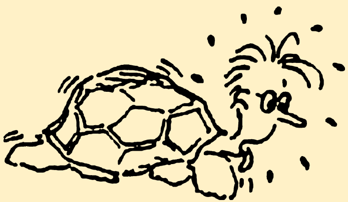
----------------▷ 31

Try to solve the first example given in the textbook again, this time on your own without using the book.
Calculate the length of a circle of radius R.

31

L = ………………….

In case of difficulties solve the problem using the textbook



-----------------▷ 32

$L = 2\pi R$ 32

Now again it is up to you to choose:

7.3 Surface area and volume of a solid of revolution ----------------▷ 33

7.4 Applications to mechanics ----------------▷ 41

7.4.3 The Theorems of Pappus ----------------▷ 50

7.4.4 Moment of inertia, second moment of area ----------------▷ 60

If you want to skip all sections for the time being Proceed to chapter 8

7.3 Surface area and volume of a solid of revolution

33

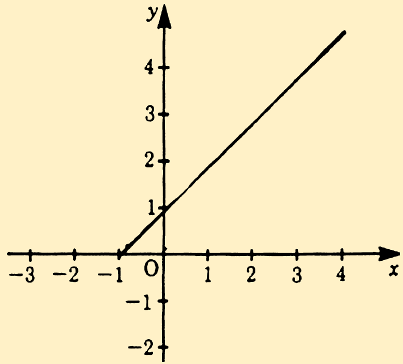
This section requires knowledge of the preceeding section i.e. lengths of curves.
It is of interest mainly to engineers and consists of worked out examples. It is worthwhile to follow the examples given in the textbook calculating on a separate sheet.

READ **7.3 Surface Area and Volume of a solid of Revolution**
Textbook pages 202–208

-----------------▷ 34

Given a straight line

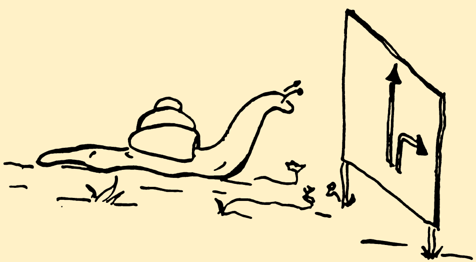
$y = 1 + x$



34

It is rotated about the x-axis.
Calculate the area of the surface thus generated between the boundaries $x_1 = a$ and $x_2 = b$.

A = ……………………



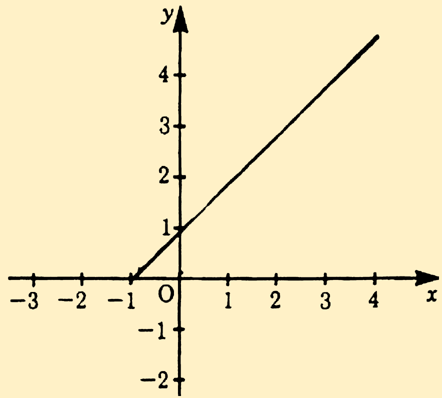
Solution found --------------------▷ 38

Hints wanted --------------------▷ 35

Given $y = 1 + x$

Look into the textbook and repeat the formula
for the surface generated:

A = ……………………..



35

36

36

$$A = 2\pi \int_a^b y\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} dx$$

Given $y = 1 + x$

Inserting you get

A = ……………………..

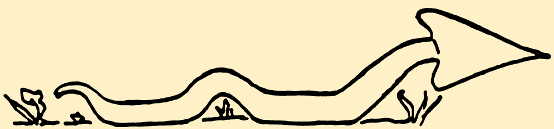
-----------------▷ 37

$$A = 2\pi \int_a^b (1+x)\cdot\left(2^{\frac{1}{2}}\right)dx$$

37

Now it is quite easy to solve the integral

A = ……………………



38

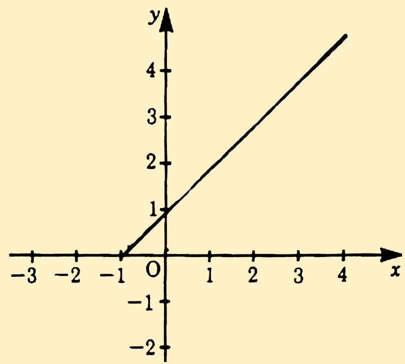
$$A = 2 \cdot \sqrt{2}\pi \left[x + \frac{x^2}{2} \right]_a^b = 2\sqrt{2}\pi \left[b + \frac{b^2}{2} - a - \frac{a^2}{2} \right]$$

38

Given the same straight line
$y = 1 + x$

It is rotated about the x-axis.
Give the volume of the solid generated. The boundaries are $x_1 = a$, $x_2 = b$

V = ...



Solution found ----------------▷ 40

Hint wanted ----------------▷ 39

Proceed as calculating the exercises before.
Look into the textbook for the formula, insert and integrate the standard integrals.

39

V = ………………………………..

-----------------▷ 40

$$V = \pi\left[x + x^2 + \frac{x^3}{3}\right]_a^b = \pi\left[b + b^2 + \frac{b^3}{3} - a - a^2 - \frac{a^3}{3}\right]$$

40

The integrals in the last example have been standard. But as a rule these integrals are quite cumbersome. Nowadays they are solved in pratice with the help of computer programs like EULER/MAXIMA or Mathematica or Mathlab.

-----------------▷ 41

7.4 Applications to mechanics

41

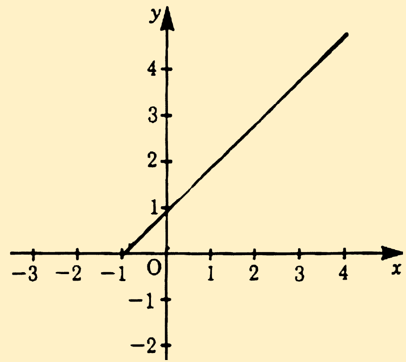
The basic physics concepts of the next sections will be known from physics lessons. However, the calculation of examples may be new.

READ **7.4.1 Basic concepts of mechanics**
7.4.2 Center of mass and centroid
Textbook pages 208–211

-----------------▷ 42

Given a straight line $y = 1 + x$
It will be rotated about the x-axis between
$x_1 = a$ and $x_2 = b$, thus generating a solid.
Calculate the solid´s center of mass which
may also be called centroid.

$x_C =$ ....



42

Solution found ----------------▷ 44

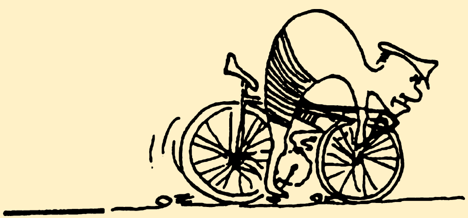
Hints wanted ----------------▷ 43

The center of mass or centroid is given by two values: $\bar{x}_C$ and $\bar{y}_C$.
For the given solid it follows from symmetry that $\bar{y}_C = 0$, since the solid is generated by rotating y = 1 + x around the x-axis.
There remains the task of calculating $\bar{x}_C$.
Write down the formula 7.11a.

Insert y = 1 + x and solve the definite standard integrals regarding the boundaries.

$\bar{x}_C =$ ....

The calculation of the area A is standard.
In case of further doubts study the textbook again.



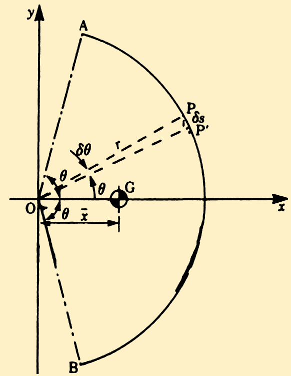
----------------▷ 44

$$\bar{x}_C = \frac{1}{A}\int_a^b (1+x)x\,dx = \frac{1}{A}\left[\frac{b^2}{2}+\frac{b^3}{3}-\frac{a^2}{2}-\frac{a^3}{3}\right]$$

44

A is given by $A = \int_a^b y\,dx = \int_a^b (1+x)\,dx = \left[b+\frac{b^2}{2}-a-\frac{a^2}{2}\right]$

A circular plate is cut into a circular sector of r = 1m and included angle of 2Θ. Find the position of the centroid or $\bar{x}_C$ along the axis of symmetry



$\bar{x}_C =$............

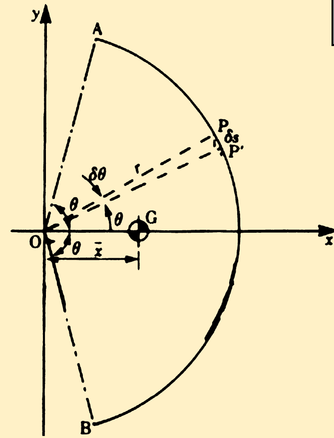
Solution found ----------------▷ 49

Hint wanted ----------------▷ 45

45

This problem is related to the example treated in the textbook. A thin strip was bent into a circular arc. The arc subtended an angle at the center. The position of the center of mass has been calculated to be

$$\bar{x}_C = \frac{r \sin \Theta}{\Theta}$$



Referring to the figure the circular sector is now given by the area ABO

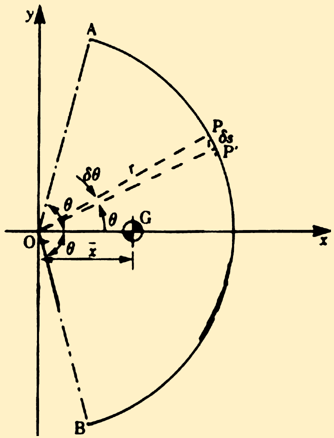
First let us calculate its area A:

A = ………………

Solution -----------------▷ 47

Hint -----------------▷ 46

46



We want to calculate the area of the circular sector ABC.
Using polar coordinates we get

$$A = \int_{-\Theta}^{+\Theta} \frac{R^2 d\Theta}{2} = \ldots\ldots\ldots\ldots\ldots$$

-----------------▷ 47

$A = R^2\Theta$

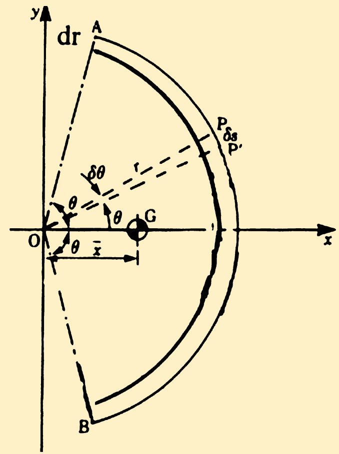
47

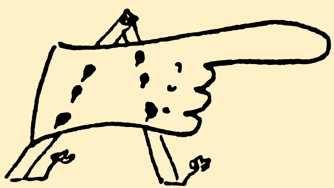
The first moment of the circular sector is $A \cdot \bar{x}_C$.
This must equal the sum of moments of all circular strips of width dr.
A strip has the first moment

$$\bar{x} \cdot = \frac{r \sin \Theta}{\Theta} \Delta A_s$$

The area ΔA_s of a strip is $\Delta A_s =$ ....

Hint: The length of the strip is $r \cdot 2\Theta$
Its width is dr.



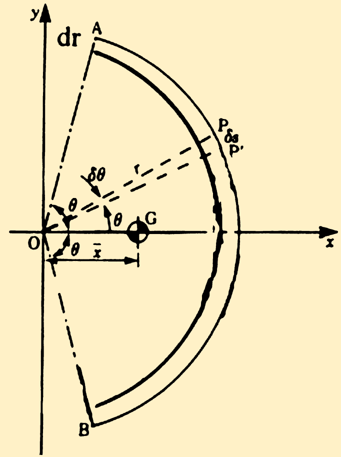


----------------▷ 48

$\Delta A_s = r \cdot 2\Theta \cdot \Delta r$ or $dA_s = r \cdot 2\Theta dr$

Now we sum up the moments of all strips from
r = 0 to r = R
This must equal $A \cdot \bar{x}$:

$$A \cdot \bar{x} = \int_0^R \frac{r \sin \Theta}{\Theta} \cdot r 2\Theta dr = \ldots\ldots\ldots\ldots\ldots$$



48

-----------------▷ 49

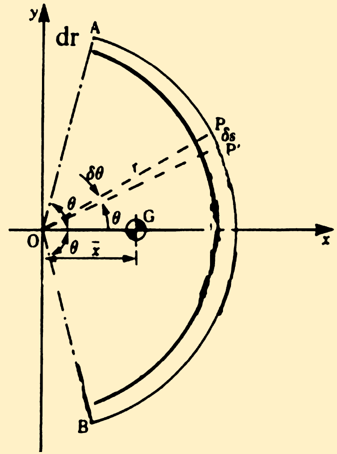
49

$$A\bar{x} = \frac{2R^3}{3} \cdot \sin\Theta$$

Since we calculated A in frame 47 to be $A = R^2\Theta$

We finally get

$$\bar{x} = \frac{2}{3} R \cdot \frac{\sin\Theta}{\Theta}$$



50

7.4.1 The Theorems of Pappus

50

The Theorems of Pappus show that knowledge of the center of mass often helps to solve problems.

READ **7.4.3 The Theorems of Pappus**
Textbook pages 211–213

-----------------▷ 51

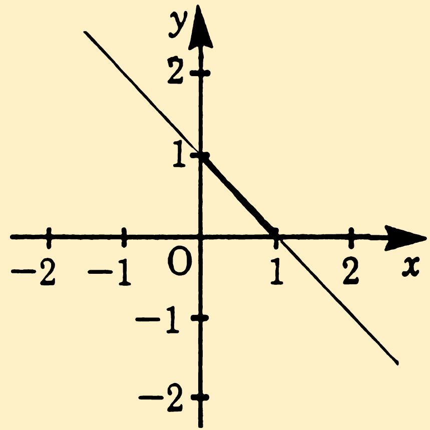
Given a straight line

$y = 1 - x$

We regard the domain $0 \le x \le 1$

Rotating the line around the x-axis generates a cone.
Calculate the surface of the cone.

S = ……………………..



51

Solution found -----------------▷ 54

Hint welcome -----------------▷ 52

You need the length L of the curve, which in our case is 52

L = ………………..

And you need the ordinate of the center of mass of the curve, which is obvious

$\bar{y}$ = ………………..

53

53

$L = \sqrt{2}$

$\bar{y} = \dfrac{1}{2}$

Now you apply the first theorem of Pappus to obtain the surface S of the cone

S = ...

------------------▷ 54

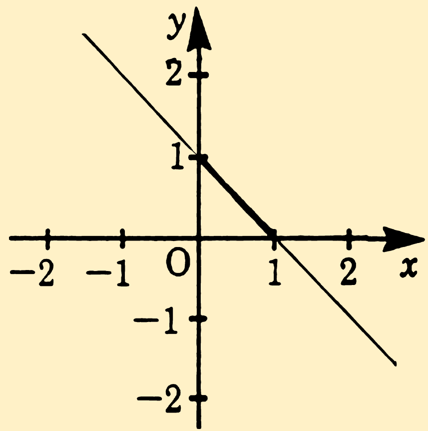
$S = \pi\sqrt{2}$

54

Now let us calculate the volume of the cone generated using the second theorem of Pappus.

Given $y = 1 - x$, domain $0 \le x \le 1$

V = ...



Solution found ----------------▷ 57

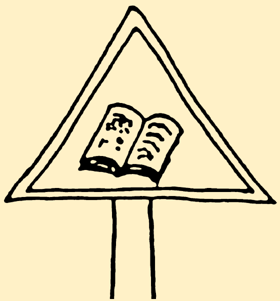
Help wanted ----------------▷ 55

From the textbook we know that

55

V = ...

Hint: In case of doubts try to understand the derivation of this formula reading the section in the textbook again.



-----------------▷ 56

$$V = 2\pi \bar{y} \cdot A = \pi \int_a^b y^2 dx$$

56

Given is y = 1 - x and a = 0 and b = 1

It is quite easy to insert the given values and to solve the integral.

V = ……………………

-----------------▷ 57

$V = \dfrac{\pi}{3}$ 57

Find the center of mass of a half circle of radius R.

$\bar{y} =$ ....

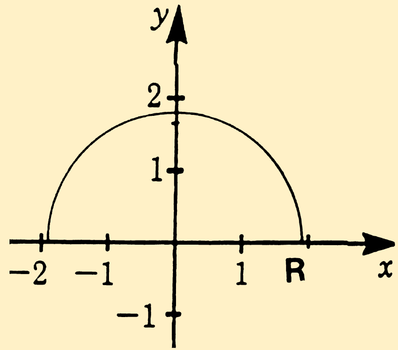
Solution found ----------------▷ 60

Help welcome ----------------▷ 58

58

Wanted is the center of mass of a half circle. Rotating about the x-axis it generates a full sphere.
Pappus´ second theorems states

$V = 2\pi \, \overline{y} \cdot A$



The volume V of a sphere is known to be $V = \dfrac{4}{3}\pi R^3$ and the area A of a half circle is known to be

$A = \dfrac{1}{2} R^2 \pi$.

Thus we insert and solve for $\overline{y}$

$\overline{y} =$ ....

-----------------▷ 59

$$\bar{y} = \frac{4}{3\pi} \cdot R$$

59

60

7.4.2 Moment of inertia; second moment of area

60

The basic concepts introduced in this section will be known from physics lectures. In case you are not familiar with these concepts take notes of all new concepts and theorems. You will use your notes while working with the exercises and examples.
Since the section is a bit extended, study the first four pages including perpendicular and parallel axis theorems.
If you have difficulties with the first example, you will be given hints in the study guide later on.

READ **7.4.4 Moment of inertia, second moment of area**
Moment of inertia
Perpendicular and parallel axis theorems
Textbook pages 213–218

----------------▷ 61

In the textbook the first example demonstrated the calculation of the moment of inertia of a disc. 61

Difficulties may have been due to the solution of the integral which is indeed tricky do solve

$$\int_0^{\frac{\pi}{2}} \cos^2 \Theta \sin^2 \Theta d\Theta = \frac{\pi}{16}$$

I know how to solve the integral ------------------▷ 65

Detailed solution wanted ------------------▷ 62

62

To solve:

$$I_x = 4\rho R^4 h \cdot \int_0^{\frac{\pi}{2}} \cos^2 \Theta \sin^2 \Theta d\Theta$$

We remember the addition theorem. See appendix of chapter 3:

$$\sin 2\Theta = 2\sin\Theta \cdot \cos\Theta$$

Inserting into the integral gives

$$I_x = 4\rho R^4 h \int_0^{\frac{\pi}{2}} \quad = \ldots\ldots\ldots\ldots\ldots\ldots$$

-----------------▷ 63

63

$$I_x = 4\rho R^4 h \int_0^{\frac{\pi}{2}} \left(\frac{\sin 2\Theta}{2} \right)^2 d\Theta$$

Now we substitute $2\Theta = \varphi \qquad 2d\Theta = d\varphi$

New upper boundary: π

$$I_x = 4\rho R^4 h \int_0^{\pi} \frac{\sin^2 \varphi}{4} \frac{d\varphi}{2}$$

You know how to solve this integral:

$$I_x = 4\rho R^4 h \cdot \frac{1}{8} \cdot [\ldots\ldots\ldots\ldots\ldots\ldots]_0^{\pi}$$

----------------▷ 64

$$I_x = 4pR^4h \cdot \frac{1}{8}\left[\frac{\varphi}{2} - \sin\varphi \cdot \cos\varphi\right]_0^{\pi} = 4\rho R^4 h \frac{1}{8} \cdot \frac{\pi}{2} = \frac{\rho R^4 h \cdot \pi}{4}$$

64

65

Given a rod of length 2m and $\rho = 0.5\dfrac{kg}{m}$. 65

Calculate its moment of inertia if rotated about the

a) center of gravity
b) one end

$I_{center\,of\,gravity} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$I_{end} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ----------------▷ 69

Help welcome ----------------▷ 66

The total mass of the rod is $M = \rho L = 1kg$.

66

Remember how to calculate the moment of inertia about an axis denoting by x the distance of a mass element dm from the axis.

I = …………………

----------------▷ 67

$$I = \int x^2 dm$$

67

Since $dm = \rho dx$ you can now calculate

$$I_{center\,of\,gravity} = \int_{-1}^{+1} \rho \cdot x^2 dx = \ldots\ldots\ldots\ldots\ldots$$

-----------------▷ 68

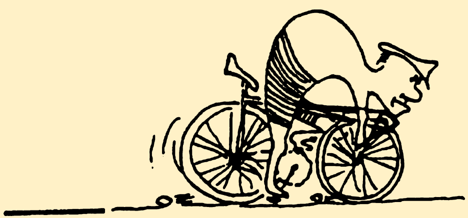
$$I_x = \rho \int_{-1}^{+1} x^2 dx = \rho \left[\frac{x^3}{3} \right]_{-1}^{+1} = \frac{2 \cdot \rho}{3} = \frac{1}{3} kgm^2$$

68

For a rotation about one end you follow same reasoning. But this time the axis of rotation is shifted to x = 0.

The boundaries of the integral have to be changed.

$I_{end} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$



-----------------▷ 69

a) $I_{center\ of\ gravity} = \frac{1}{3} kgm^2$

69

b) $I_{end} = \frac{8}{3}\rho = \frac{4}{3} kgm^2$

Let us now calculate the moment of inertia of the rod using the parallel axis theorem (Steiner′s Theorem).

Given the same rod of length 2m whose mass per meter is $\rho = 0.5 \frac{kg}{m}$

We just calculated its moment of inertia for an axis through its center of gravity to be $I_{center} = \frac{1}{3} kgm^2$

Calculate its moment of inertia if it is rotated around an end.

$I_{end} =$ ....

----------------▷ 70

$$I_{end} = \frac{4}{3} kgm^2$$

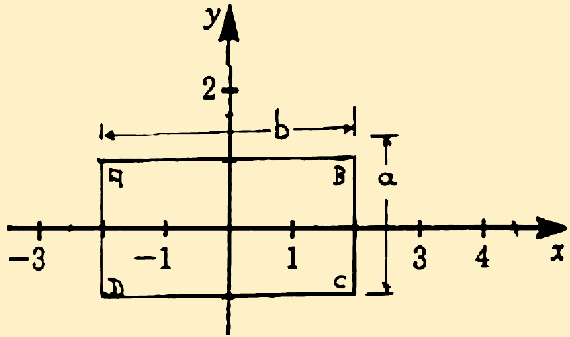
70

It is the same result calculated before.

Now proceed to the second part of section 7.4.4 which will be of interest to civil engineers.

READ **7.4.4 Moments of inertia; second moment of inertia**
Radius of gyration
Second moment of inertia
Center of pressure
Textbook pages 218–223

-----------------▷ 71



Given a rectangle ABCD with sides a = 2 and b = 4 and mass $\rho = 0.1kg$ per square unit.
Determine the moment of inertia about the z-axis.

$I_z =$ ....

Solution found ----------------▷ 75

Hints welcome. The problems seems intriguing to me ----------------▷ 72

The given problem can be solved using the perpendicular axis theorem.
Determining the moments of inertia about the x-axis and the y-axis is quite easy.

72

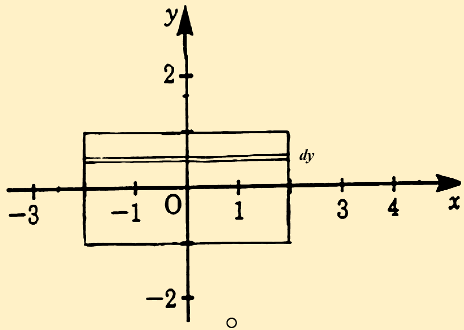
$I_x =$ ....

$I_y =$ ....

Solution found ----------------▷ 75

Detailed solution ----------------▷ 73

73



Regarding a strip parallel to the x-axis

$$dm = \rho \cdot 4 \cdot dy$$

$$dI = y^2 \cdot dm = y^2 4\rho dy$$

$$I_x = 4\rho \int_{-1}^{+1} y^2 dy = 4\rho \left[\frac{y^3}{3} \right]_{-1}^{+1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now determine the moments of inertia about the x-axis following the same reasoning regarding a strip parallel to the y-axis

$$dm = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$I_y = \ldots\ldots\ldots\ldots\ldots\ldots$$

----------------▷ 74

74

$$I_x = \frac{8}{3}\rho$$

$$dm = \rho \cdot 2 \cdot dy$$

$$I_y = \frac{32}{3}\rho$$

We remember the perpendicular axis theorem

$$I_z = I_x + I_y$$

Thus regarding our results we obtain

$I_z = ...$

----------------▷ 75

$$I_z = \frac{40}{3}\rho$$

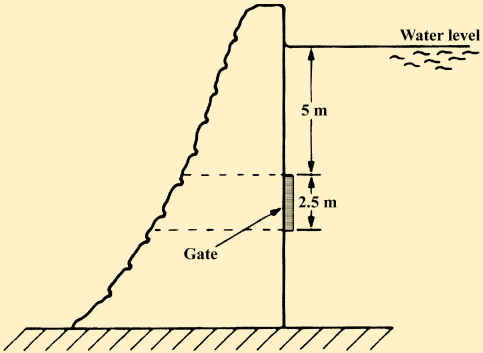
75

Now try to solve the example given on page 220 (fig. 7.26) in the textbook.
In case of difficulties look at the detailed calculation given there.

76

The exercise 22 at the end of this chapter reads as follows:

76



A rectangular plate of base 5m and height 8m is immersed in a lake. Calculate the total pressure on the plate and the depth of the centre of pressure if the plate is vertical.
Density of water = 1000 kg/m^3

Try to answer the first question. We denote the depth x and the width of the gate a
Total pressure on the gate

F = ……………………..

Solution found -----------------▷ 78

Hints wanted -----------------▷ 77

We regard a horizontal strip of width a = 5m and height dx in the depth x:
The force on the strip is:

77

$$dF = \rho \cdot g \cdot x \cdot dx \cdot a$$

The total force on the gate:

$$F = \rho \cdot g \int_{5}^{7.5} x dx \cdot a$$

Now solve the integral, insert the boundaries and calculate the result:

F = …………………

-----------------▷ 78

F = 230kN

78

Now solve the second question. Calculate the position x_C of the center of pressure.
We regard a horizontal strip of width a = 5m and height dx in the depth x: Its moment regarding the line of the water level is

$$dF \cdot x = a \cdot dx \cdot \rho \cdot gx^2 = a\rho gx^2 dx$$

Thus

$$F \cdot x_C = a\rho g \int_5^{7.5} x^2 dx$$

$$F \cdot x_C = \text{..........................}$$

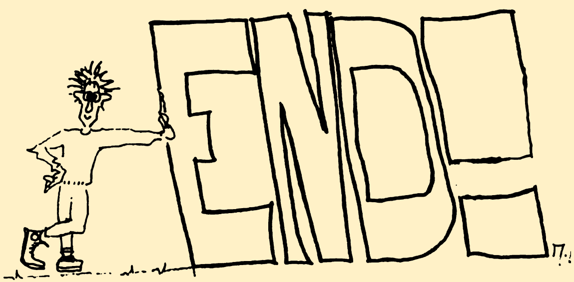
-----------------▷ 79

$$F \cdot x_C = a\rho g \left[\frac{x^3}{3} \right]_5^{7.5} \cong 1456kNm$$

Using this result we obtain

$$x_C = 6.33m$$

By now you have reached the end of this chapter which posed some tricky calculations.
But after all you mastered the stuff successfully: Congratulation!



of chapter 7

Chapter 8
Taylor Series and Power Series

8.1 Expansion of a Function in a Power Series

2

Objective: Concepts of power series, factorials, Maclaurin's series, Taylor's series, interval of convergence.

READ: **8.1 Introduction**
8.2 Expansion of a function in a power series
8.3 Interval of convergence of power series
Textbook pages 229–235

3

Name at least three concepts which were newly introduced in this section:

3

(1) . . . .
(2) . . . .
(3) . . . .

4

(1) Power series
(2) Maclaurin's series
(3) Interval of convergence

4

Give three reasons for expanding a function in a power series:

(1) ....
(2) ....
(3) ....

5

(1) The first terms of a power series are often suitable for obtaining an approximate value of the function.

(2) Power series can be differentiated and integrated term by term.

(3) We can use power series to calculate the values of many functions.

5

The expression $n!$ is read as ....

The expression $n!$ means ....

6

factorial n

$$n! = 1 \times 2 \times 3 \times \ldots \times (n-1)n$$

6

Work out the following, simplifying where possible:

$$\begin{aligned}
5! &= \ldots\ldots\ldots\ldots\ldots \\
\frac{7!}{5!} &= \ldots\ldots\ldots\ldots\ldots \\
\frac{(n+1)!}{n!} &= \ldots\ldots\ldots\ldots\ldots \\
\frac{9!}{11!} &= \ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

7

$$\begin{aligned}
5! &= 120 \\
\frac{7!}{5!} &= \frac{1\times 2\times 3\times 4\times 5\times 6\times 7}{1\times 2\times 3\times 4\times 5} = 6\times 7 = 42 \\
\frac{(n+1)!}{n!} &= \frac{1\times 2\times 3\times \ldots\times n\times (n+1)}{1\times 2\times 3\times \ldots\times n} = n+1 \\
\frac{9!}{11!} &= \frac{1\times 2\times 3\times ..\times 9}{1\times 2\times 3\times \ldots\times 10\times 11} = \frac{1}{110}
\end{aligned}$$

7

You can sometimes simplify an expression involving factorials by cancelling factors common to both numerator and denominator.

Did you make some mistakes?

Yes --------------------▷ 8

No --------------------▷ 11

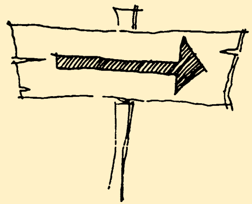
The symbol $n!$ (read as factorial n) is an abbreviation for the product of the first n natural numbers

8

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

What is $(n-2)!$?

$$(n-2)! = \ \ldots\ldots\ldots\ldots\ldots$$



9

$$(n-2)! = 1 \times 2 \times 3 \times \ldots \times (n-3)(n-2)$$

9

Here are some more examples for you! Remember to simplify if you can.

(1) $\dfrac{n!}{(n-2)!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(2) $\dfrac{3!5!}{6!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(3) $\dfrac{100!}{101!} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

10

10

(1) $$\frac{n!}{(n-2)!} = \frac{1 \times 2 \times 3 \times \ldots \times (n-2)(n-1)n}{1 \times 2 \times 3 \times \ldots \times (n-2)} = (n-1)n$$

(2) $$\frac{3!5!}{6!} = \frac{1 \times 2 \times 3 \times 1 \times 2 \times 3 \times 4 \times 5}{1 \times 2 \times 3 \times 4 \times 5 \times 6} = \frac{1 \times 2 \times 3}{6} = 1$$

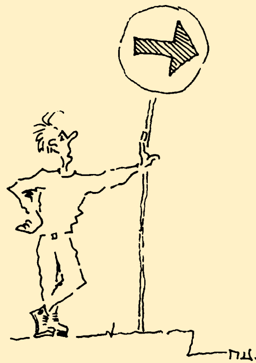
(3) $$\frac{100!}{101!} = \frac{1 \times 2 \times 3 \times \ldots \times 100}{1 \times 2 \times 3 \times \ldots \times 100 \times 101} = \frac{1}{101}$$

In case of difficulties consult the textbook.

11

Write down the general form of Maclaurin's series for a function $f(x)$. You may have to look at the textbook again. If you do, don't just look at it, write it down! This will help you to fix it in your mind. 11

$f(x) =$ ....



12

$$f(x) = f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots \qquad \boxed{12}$$

Use the series to expand $\cos x$ up to the term $n = 4$ in accordance with the steps below.

Step 1: Obtain derivatives $f', f'', f''', f^{(4)}$.
Step 2: Calculate the values of the function and its derivatives at $x = 0$.
Step 3: Substitute the values of $f(0),\ f'(0),\ \ldots, f^{(4)}(0)$ in Maclaurin's series.

$$\cos x \approx \ \ldots\ldots\ldots\ldots\ldots$$

$\dashrightarrow$ 13

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$ 13

Correct 15

Wrong, further explanation required 14

To express cos x in a power series according to Maclaurin's expansion we have to proceed as follows: 14

Step 1: Obtain the derivatives

$$\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \\ f^{(4)}(x) &= \cos x \end{aligned}$$

Step 2: Obtain the values of the derivatives at $x = 0$

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \\ f^{(4)}(0) &= 1 \end{aligned}$$

Step 3: Substitute the values of $f'(0), \ldots, f^{(4)}(0)$ in Maclaurin's series

$$f(x) \approx f(0) + f'(0)\frac{x}{1!} + f''(0)\frac{x^2}{2!} + f'''(0) + \frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!} + \ldots$$

$$\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

--------------------▷ 15

Use Maclaurin's series to expand $f(x) = \frac{1}{(1+x)^2}$ up to the third term, i.e. $n = 3$.

15

What steps should you follow?

Step 1: ....

Step 2: ....

Step 3: ....

- - - - - - - - - - - - - - - - - - ▷ 16

Step 1: Obtain the derivatives $f'(x), f''(x), f'''((x))$.
Step 2: Obtain the values of the function and its derivatives at $x = 0$.
Step 3: Substitute these values in Maclaurin's series.

16

$$f(x) \approx \sum_{n=0}^{n=3} \frac{f^{(n)}(0)}{n!} x^n$$

Now execute the steps

$$f(x) = \frac{1}{(1+x)^2}$$

Step 1:

$$f'(x) = \;.............$$
$$f''(x) = \;.............$$
$$f'''(x) = \;.............$$

17

$$f'(x) = \frac{-2}{(1+x)^3}$$
$$f''(x) = \frac{6}{(1+x)^4}$$
$$f'''(x) = \frac{-24}{(1+x)^5}$$

17

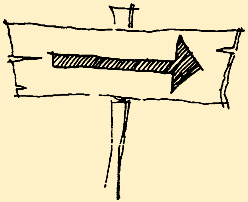
Correct → 23

Wrong → 18

Where did you go wrong? We must try to analyse your error, but we can only do this if we know the reason for your difficulties. 18

Never let errors rest; they have to be eliminated, because they cannot go away by themselves!

Even errors due to carelessness should not be allowed to increase.



Error found ------------------▷ 23

Explanation of calculation ------------------▷ 19

Your error may have occurred during differentiation of $f(x) = \dfrac{1}{(1+x)^2}$ 19

To differentiate this function we can use the quotient rule (look it up again if you have forgotten it), or we can write the function as $f(x) = (1+x)^{-2}$ and use the function of a function rule. The latter form is much easier to apply in this instance.

$$f(x) = (1+x)^{-2} = u^{-2};\ u = 1+x$$
$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = -2u^{-3} \times 1 = -2u^{-3}$$
$$= -2(1+x)^{-3} = \frac{-2}{(1+x)^3}$$

Similarly $f''(x) = +6(1+x)^{-4} = \dfrac{6}{(1+x)^4}$

and $f'''(x) = -24(1+x)^{-5} = -\dfrac{24}{(1+x)^5}$

20

If you used the quotient rule you should have obtained the same result.

20

$$f'(x) = \frac{0\times(1+x)^2 - 2\times(1+x)}{(1+x)^4} = \frac{-2}{(1+x)^3}$$

$$f''(x) = \frac{0\times(1+x)^3 - (-2)\times 3\times(1+x)^2}{(1+x)^6} = \frac{6}{(1+x)^4}$$

$$f'''(x) = \frac{0\times(1+x)^4 - 6\times 4\times(1+x)^3}{(1+x)^8} = \frac{-24}{(1+x)^5}$$

The differentiation is clear --------------------▷ 24

Still having difficulties in differentiating --------------------▷ 21

21

The nth term in Maclaurin's series is

$$a_n = \frac{f^{(n)}(0)}{n!}$$

$f^{(n)}(0)$ is the value of the nth derivative of the function at $x = 0$.

Hence to obtain the series for $f(x)$ we need the higher derivatives

$$f'(x), f''(x), \ldots, f^{(n)}(x).$$

Since you are experiencing difficulties with the process of differentiating you should interrupt this section of the work for the time being.

Read section 5.6 in the textbook to revise the concept of higher derivatives, and read again the section regarding the quotient rule and the derivative of a function of a function (chain rule).

22

The function in our case is a quotient; to differentiate it we can, on the one hand, apply the quotient rule. On the other hand, since the function can be written 22

$$f(x) = (1+x)^{-2} = u^{-2}$$

we can also use the function of a function rule.

You need some practice in differentiating! Try to apply both rules to obtain the first four derivatives of the function

$$\begin{aligned} f(x) &= \frac{1}{(1+x)^2} \\ f'(x) &= \ldots\ldots\ldots\ldots\ldots\ldots \\ f''(x) &= \ldots\ldots\ldots\ldots\ldots\ldots \\ f'''(x) &= \ldots\ldots\ldots\ldots\ldots\ldots \\ f^{(4)}(x) &= \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

---------> 23

$$f'(x) = \frac{-2}{(1+x)^3}$$
$$f''(x) = \frac{6}{(1+x)^4}$$
$$f'''(x) = \frac{-24}{(1+x)^5}$$
$$f^{(4)}(x) = \frac{120}{(1+x)^6}$$

23

In case of further difficulties you should revise the calculation of derivatives using the study guide for chapter 5.

--------------------▷ 24

The first three derivatives of the function $f(x) = \dfrac{1}{(1+x)^2}$ are

24

$$f'(x) = \frac{-2}{(1+x)^3}, \quad f''(x) = \frac{6}{(1+x)^4}, \quad f'''(x) = \frac{-24}{(1+x)^5}$$

Thus step 1 is completed.

Step 2: Substituting $x = 0$ in each yields

$$f(0) = 1; \quad f'(0) = -2; \quad f''(0) = 6; \quad f'''(0) = -24$$

Step 3: Inserting these values in Maclaurin's expansion we find

$$\frac{1}{(1+x)^2} \approx \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

-------------------▷ 25

$$\frac{1}{(1+x)^2} \approx 1 - 2x + \frac{6}{2!}x^2 - \frac{24}{3!}x^3 \text{ up to } n = 3$$

$$\approx 1 - 2x + 3x^2 - 4x^3$$

25

Note: In many cases it is sufficient to determine the first three or four terms of Maclaurin's series in order to infer the form of the complete series. In our case we will have

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 + 7x^6 - \dots$$

26

Consider the series $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

26

If we replace x by $-x$ we find

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \dots$$

Thus we find the series for

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

Now consider the power series for e^x.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Obtain the series for e^{-x}

$e^{-x} =$ ....

27

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

27

Express the function ln $(1 + x)$ as a Maclaurin's series up to $n = 3$.

What steps should you follow?

Step 1: ....

Step 2: ....

Step 3: ....

28

Step 1: Obtain the derivatives

$f'(x)$, $f''(x)$, $f'''(x)$

Step 2: Obtain the values of the function and its derivatives at $x = 0$, i.e.

$f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$

Step 3: Substitute in Maclaurin's series

28

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Now execute the steps:

Step 1:

$f(x) \quad = \ln(1+x)$
$f'(x) \quad = \dots\dots\dots\dots\dots\dots\dots\dots$
$f''(x) \quad = \dots\dots\dots\dots\dots\dots\dots\dots$
$f'''(x) = \dots\dots\dots\dots\dots\dots\dots\dots$

Step 2:

$f(0) \quad = \dots\dots\dots\dots\dots\dots\dots\dots$
$f'(0) \quad = \dots\dots\dots\dots\dots\dots\dots\dots$
$f''(0) \quad = \dots\dots\dots\dots\dots\dots\dots\dots$
$f'''(0) = \dots\dots\dots\dots\dots\dots\dots\dots$

Step 3:

$\ln(1+x) \quad \approx \dots\dots\dots\dots\dots$

29

29

$$f(x) = ln(1+x) \qquad f(0) = 0$$
$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$
$$f''(x) = \frac{-1}{(1+x)^2} \qquad f''(0) = -1$$
$$f'''(x) = \frac{2}{(1+x)^3} \qquad f'''(0) = 2$$
$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Correct --------------------▷ 34

Errors; explanation required --------------------▷ 30

Step 1: The first derivative of the function $\ln(1+x)$ can be obtained by the chain rule. 30

$$f(x) = \ln(1+x) = \ln g \quad \text{where } g = 1+x$$
$$\text{then } f(x) = \frac{1}{g} g' = \frac{1}{1+x} \quad \text{since } g' = 1$$

The next derivatives are obtained in the same way or by using the quotient rule;
hence $f''(x) = \dfrac{1}{(1+x)^2}$.

$f'''(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

31

$$f'''(x) = \frac{2}{(1+x)^3}$$

31

Step 2: Obtain the values of the function and its derivatives at $x = 0$.

$$f'(0) = \ln(1+0) = 0$$
$$f'(0) = \frac{1}{1+0} = 1$$
$$f''(0) = -\frac{1}{(1+0)^2} = -1$$
$$f'''(0) = \frac{2}{(1+0)^3} = 2$$

Step 3: Substitute these values in Maclaurin's series. (In case of difficulties return to the textbook to check the formula.)

$$f(x) = \ln(1+x) = 0 + 1 \times x + \frac{-1}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

32

32

A quite powerful expansion is the binomial expansion.
You know from the textbook:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots$$

The exponent n may be a fraction.
Obtain the following expansions by applying this formula:

(1) $\sqrt{1+x} = \cdots\cdots\cdots\cdots\cdots$

(2) $\sqrt{1+x^2} = \cdots\cdots\cdots\cdots\cdots$

(3) $\frac{1}{\sqrt{1-x^3}} = \cdots\cdots\cdots\cdots\cdots$

Solutions found $\dashrightarrow$ 34

Hints and detailed explanation required $\dashrightarrow$ 33

Each one of the given problems can be solved by applying the general form of the binomial series. 33

We compare the given problem with the general formula for $(a+b)^n$ in order to find the actual substitutions for a, b and n.

Problem (1): $\sqrt{1+x} = (1+x)^{1/2}$ yields $a=1,\ b=x,\ n=\frac{1}{2}$

Problem (2): $\sqrt{1+x^2} = (1+x^2)^{1/2}\ yields\ a=1,\ b=x^2,\ n=\frac{1}{2}$

Problem (3): $\frac{1}{\sqrt{1+x}^3} = (1+x^3)^{-1/2}$ yields $a=1,\ b=x^3,\ n=-\frac{1}{2}$

Now try again:

$$\sqrt{1+x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sqrt{1+x^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1}{\sqrt{1+x^3}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 34

$$(1)\quad \sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2!}x^2 + \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdot\frac{1}{3!}x^3 + \dots$$

34

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \dots$$

$$(2)\quad \sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \dots$$

$$(3)\quad \frac{1}{\sqrt{1-x^3}} = 1 + \left(-\frac{1}{2}\right)(-x^3) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{1}{2!}(-x^3)^2$$

$$+ \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{1}{3!}(-x^3)^3 + \dots$$

$$= 1 - \frac{x^3}{2} + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \dots$$

Note: By this method many functions can be reduced to a binomial series.

If $f(u) = \sin u$ then the general form of this function as a series is

$$f(u) = \sum_{n=0}^{\infty} \dots\dots\dots\dots\dots\dots$$

$$\sin u = \dots\dots\dots\dots\dots\dots$$

35

$$f(u) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)u^n}{n!}$$

$$\sin = u - \frac{u^3}{3!} + \frac{u^5}{5!}$$

35

A power series (such as Maclaurin's) does not always converge for all values of the variable. The interval of convergence can be determined in a way similar to that shown by the examples in the textbook.

Important series such as the exponential functions e^x, e^{-x} and the trigonometric functions $\sin x$, $\cos x$ are convergent for all values of x.

We will now consider the solution of an example on convergence in detail. You may skip it if you wish.

Example on convergence --------------------▷ 36

I will skip it --------------------▷ 38

Determine the interval of convergence for the following series

36

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \pm \frac{x^n}{n} \ldots$$

This series converges for all values of x if and only if

$$x < R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Obtain

$$\left|\frac{a_n}{a_{n+1}}\right| = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

37

$$\left|\frac{a_n}{a_{n+1}}\right| = \frac{n+1}{n} = 1 + \frac{1}{n}$$ 37

Now obtain the radius of convergence

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|1 + \frac{1}{n}\right| = \ldots\ldots\ldots\ldots\ldots$$

The answer is: $R = 1$

Hence the series for $\ln(1+x)$ will converge for $-1 < x < 1$. The end points are not included, of course. (ln(0) is not even defined.)

--------------------▷ 38

8.2 APPROXIMATE VALUES OF FUNCTIONS

Objectives: Concepts of approximate polynomials, remainder.

READ: **8.4 Approximate values of functions**
Textbook pages 235–237

39

Let the expansion of a function into a power series be broken off after n terms. What are the names of the two parts of this power series?

39

$$f(x) = \underbrace{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}_{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} + \underbrace{a_{n+1} x^{n+1} + \ldots}_{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}$$

----------▷ 40

approximate polynomial $P_n(x)$ of the nth degree
remainder: $R_n(x)$

40

We shall now deal with the approximate polynomial.

Given the series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

Write down the first four approximate polynomials:

$P_1(x) =$ ....

$P_2(x) =$ ....

$P_3(x) =$ ....

$P_4(x) =$ ....

----------------▷ 41

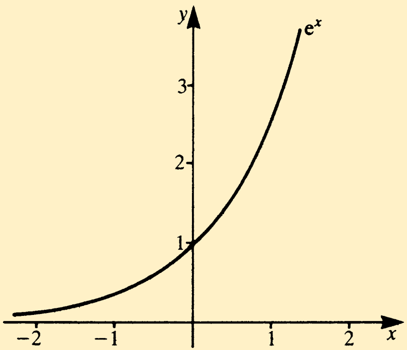
$$P_1(x) = 1 + x$$

$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

$$P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

41



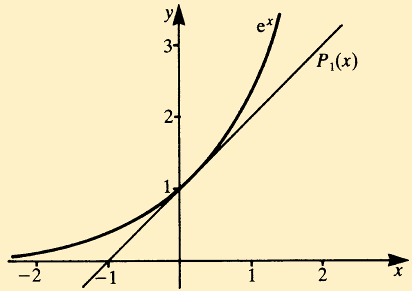
The figure shows the graph of the function

$$y = e^x$$

Draw on the diagram the first approximation

$$P_1(x) = 1 + x$$

42



42

The approximation $P_1(x) = 1 + x$ is a straight line which is tangential to the curve of $y = e^x$ at $x = 0$. At $x = 0$ the slopes of the function and of the polynomial are the same.

The coefficient a_1 of the approximate polynomial $P_1(x) = a_0 + a_1 x = 1 + x$ was chosen to satisfy the first derivative of the function e^x. A better approximation to the function in the neighbourhood of $x = 0$ is

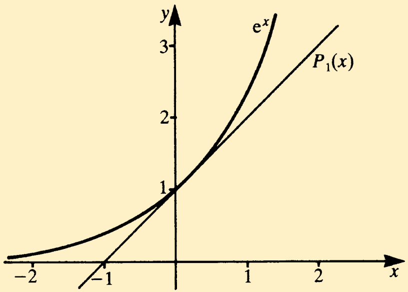
$$P_2(x) = 1 + x + \frac{x^2}{2!}$$

The graph of this function is a ....

--------------------▷ 43

parabola

43

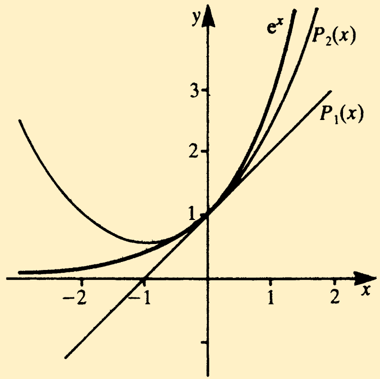


Draw on the diagram the approximation

$$P_2(x) = 1 + x + \frac{x^2}{2}$$

44

44



The parabola is a better approximation to e^x. It has the same slope at $x = 0$ as well as the same curvature, i.e. the second derivatives of the function and the polynomial are the same at $x = 0$.
$f''(0) =$ .... $P_2''(0) =$ ....

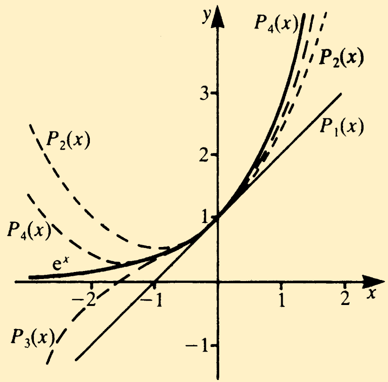
45

$$f''(0) = e^0 = 1, \; P_2''(x) = 1$$

45

The third approximation to e^x is given by $P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

It is a polynomial of the third degree, which is even better than the two previous ones.



The diagram shows the graph of the function

$$f(x) = e^x$$

with the four approximations

$$P_1(x), \; P_2(x), \; P_3(x) \; and \; P_4(x).$$

It demonstrates quite clearly that the higher the degree of the approximate polynomial the more closely the approximations fit the graph.

46

We cut off the series for e^x after $n = 4$ so that

46

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

There is an error whose value can be estimated by the expression

$$R_n = \frac{f^{(n+1)}(,)}{(n+1)!} x^{n+1}$$

It is called ....

In our case $f(x) = e^x$

$$R_4 = \text{..................}$$

47

The remainder (or Lagrange's form of the remainder) 47

$$R_4 = \frac{e^{\xi}x^5}{5!}, \ 0 < \xi < x$$

You should remember that when you cut off a series after n terms you automatically create an error. This error can be estimated. It will be as small as you like since you can yourself fix the order of the approximation in a practical situation.

If you would like to do an example on evaluating the error, one is coming up!

I would rather carry on

--------------------▷ 52

I would like to do the example

--------------------▷ 48

The series for $\cos x$ is

48

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

If we wish to compute the value of the cosine function at $x = 1$ (i.e. 1 radian = 57.3 degrees) then

$$\begin{aligned}\cos 1 &= 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \ldots \\ &= 1 - 0.5 + 0.0417 - 0.0014 + \ldots\end{aligned}$$

If we take $n = 2$ as a first approximation then

$$\cos 1 = 1 - 0.5 + R_2(1) = 0.5 + R_2(1)$$

Now we evaluate the error.

Since the general form of the remainder is

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1}$$

it follows that with $n = 2$ we have

$R_2(1) = \ldots\ldots\ldots\ldots\ldots$

--------------------▷ 49

$$R_2(1) = \frac{f^{(3)}(\xi)}{3!} = \frac{\sin\xi}{6}, \quad 0 < \xi < 1$$ 49

We do not know the exact value of ξ, but we can be certain that the error will not exceed the value of $R_2(1)$ with $\xi = 1$, since the sine function is monotonically increasing in the interval (0, 1).

Hence the error will not be greater than

$$|R_2(1)| = \left|\frac{\sin\xi}{6}\right| = \left|\frac{\sin(1)}{6}\right| = \frac{0.842}{6} \approx 0.14$$

Our approximation gave $\cos 1 = 0.5$

The exact value is 0.5403

What is the actual error E?

$$E = \dots\dots\dots\dots\dots\dots$$

----------▷ 50

$$E = 0.5403 - 0.5 = 0.0403$$

50

Note: This is less than 0.14, which we predicted as the largest possible value.

The approximation for cos 1 can be improved if we take $n = 4$.

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$
$$\cos 1 \approx 1 - \frac{1}{2!} + \frac{1}{4!} = 1 - 0.5 + 0.0417 = 0.5417$$

What is the actual error E now?

$$E = \ldots\ldots\ldots\ldots\ldots$$

51

$$E = 0.5403 - 0.5417 = -0.0014$$

51

Note: The error is estimated to be less than

$$|R_4(1)| = \left| \frac{f^{(5)}(\xi)}{5!} \right| = \left| -\frac{\sin(\xi)}{120} \right| \leq \frac{0.842}{120} \approx 0.0070$$

52

Some remarks on human memory will follow. Do you want to skip them?.

52

--------------------▷ 57

During an oral exam the examiner asks a student:

> Explain to me the relationship between differentiation and integration and write down the symbol for the indefinite integral.

The student hesitates, and hesitates ...

Finally the examiner states, ‘Integration is the inverse operation to differentiation. The general solution of the integration is the indefinite integral. Here are two alternatives. How should it be written?’

$$\text{A} \int f(x)\mathrm{d}x = F(x)$$

$$\text{B} \int f(x)\mathrm{d}x = F(x) + C$$

The student replies: ‘Yes, solution B is the correct one. I understood that well at the time.’ To this the examiner says: ‘But you didn’t know when I asked you just now.’
Who is right?

--------------------▷ 53

Both are right; things are a bit more complicated. 53

The student had understood the matter at the time of studying it. He quickly recognised the correct solution again.

The examiner emphasised, and rightly so, that the question wasn't answered without considerable help. The student was neither in a position to describe the relationship nor to write down the symbol *actively*.

Conclusion:

> Recognition is easier than reproduction. But reproduction and application are the objectives of our learning.

Anyone who believes that, a year from now, he will be able to reproduce everything he now understands, is greatly mistaken.

--------------------▷ 54

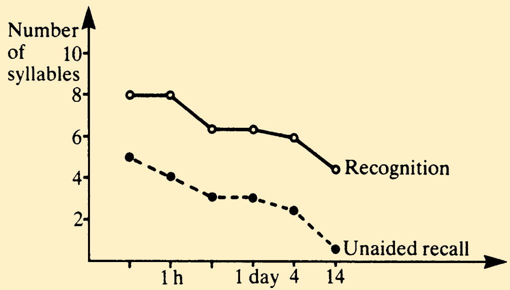
54

In an experiment A. Miles (1960) gave 60 people 10 syllables each to learn. This learning material was studied three times in succession. At two different intervals each person was examined according to two different methods.

(1) Unaided recall: number of syllables reproduced without aid.
(2) Recognition: number of learned syllables which could be recognised from an extensive list.

They were tested immediately after this learning period and then 1 hour, 6 hours, 1 day, 4 days and 14 days later.

The diagram shows the results



55

Similar dependencies are also found in the case of meaningful subject matter. Free reproduction is more difficult than recognition. 55

In exam situations facts concerning specific questions must be actively reproduced. Incidentally, this also goes for a large number of situations in which learned material has to be applied. As in the experiment with meaningless syllables, with meaningful subject matter too the difference between the skill shown in unaided recall and that shown in recognition is still great. Here there exists the possibility that we subjectively deceive ourselves: we often mistake those facts that we once studied, but thereafter only recognise, for facts which have been well memorised.

This is often self-deception.

--------------------▷ 56

Let us assume that, through intensive reading, you have understood a fact. That is to say that the terms can be actively reproduced and the operations which were learned can be carried out. A well-known safeguard against forgetting things is revision — a process you are now familiar with. At the end of every work section it is recommended that you go over the contents again and try to write down all the keywords from memory before you stop for a break. 56

The second phase of revision is to check, after an interval of perhaps a week, whether you can actively reproduce the most important contents of the previous lesson.

If you have difficulty here it is important that you repeat the lesson once more.

In order not to forget this, put a slip of paper into the textbook saying 'Lesson must be repeated'.

57

8.3 Expansion of a Function $f(x)$ at an Arbitrary Position. Applications of Series. Approximations

57

Objective: Evaluation of the first terms of a Taylor's series at $x_0 \neq 0$, application of series.

READ: **8.5 Expansion of a function $f(x)$ at an arbitrary position**
8.6 Applications of series
8.6.1 Polynomials as approximations
Textbook pages 237–241

58

Having read the relevant section in the textbook write down the formula for Taylor's series: 58

$$f(x) = \ldots\ldots\ldots\ldots\ldots$$

59

$$f(x) = f(x_0) + f'(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \ldots \qquad \boxed{59}$$

Suppose you needed to calculate sin 47° and you did not have tables or a scientific calculator at hand. You must expand the sine function at an appropriate position.

Which α (or x_0) will be suitable?

$$\alpha_0 = \ \ldots\ldots\ldots\ldots\ldots$$

$\boxed{60}$

$\alpha_0 = 45°$, i.e. $x_0 = \frac{\pi}{4}$, is a good choice because $\sin 45°$ is known from a simple triangle, i.e. $\sin 45° = \frac{1}{\sqrt{2}}$, and the values of the derivatives are known too. The differences $(\alpha - \alpha_0)$, or $(x - x_0)$, will be small. 60

What steps are needed to obtain the first four terms of the expansion?

Step 1: ....

Step 2: ....

Step 3: ....

- - - - - - - - - - - - - - - - - - -▷ 61

Step 1: Obtain the derivatives $f'(x)$, $f''(x)$, $f'''(x)$, etc...

Step 2: Calculate the values of the function and its derivatives at $x = x_0$.

Step 3: Substitute the values $f'(x_0)$, $f''(x_0)$, etc. in the Taylor's series.

61

Now proceed to compute the first four terms of the expansion for sin 47°. Remember that you must express the angles in radians: $x = \alpha \dfrac{\pi}{180}$

$$\sin 47^\circ = \dots\dots\dots$$

62

$$\sin 47^\circ \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(2\frac{\pi}{180}\right) - \frac{1}{2\sqrt{2}}\left(2\frac{\pi}{180}\right)^2 - \frac{1}{6\sqrt{2}}\left(2\frac{\pi}{180}\right)^3 \quad \text{up to } n = 3$$

$\boxed{62}$

Correct $\boxed{64}$

Error; detailed solution required $\boxed{63}$

Here is the solution in detail.

63

Step 1: The derivatives are

$$f(x) = \sin x \qquad f''(x) = -\sin x$$
$$f'(x) = \cos x \qquad f'''(x) = -\cos x$$

Step 2: The values of the function and its derivatives at $x_0 = 45\dfrac{\pi}{180}$

$$f(x_0) = \frac{1}{\sqrt{2}} \qquad f''(x_0) = \frac{-1}{\sqrt{2}}$$
$$f'(x_0) = \frac{1}{\sqrt{2}} \qquad f'''(x_0) = \frac{-1}{\sqrt{2}}$$

Step 3: Substitute these values in Taylor's series:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$$
$$\sin 47^\circ \approx \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\left(2\frac{\pi}{180}\right) - \frac{1}{2\sqrt{2}}\left(2\frac{\pi}{180}\right)^2 - \frac{1}{6\sqrt{2}}\left(2\frac{\pi}{180}\right)^3 \quad \text{up to } n = 3$$

64

64

We assume you have a simple (non-scientific) calculator. Calculate, as an approximation, the value of $\sin 47^\circ$ to five decimal places taking (a) the first two terms of the expansion and (b) the first three terms of the expansion.

What are the errors in each case, knowing that the exact value (to five decimal places) of $\sin 47^\circ = 0.73135$?

(a) $\sin\ 47^\circ =$ .... error = ...

(b) $\sin\ 47^\circ =$ .... error = ...

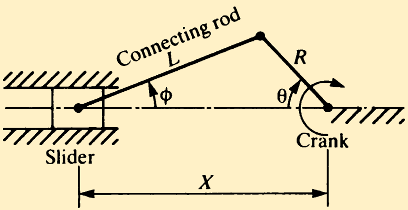
--------------------▷ 65

65

(a) 0.73179 error 0.00044
(b) 0.73136 error 0.00001

Now we give an example which is of special interest to mechanical engineers; other readers may skip it and go to

-------------------▷ 76



The figure shows the slider crank mechanism as used in the petrol and diesel engine as well as in reciprocating pumps and compressors. There are hundreds of millions of such mechanisms throughout the world. As it is a very important device we propose to examine its kinematics.

Using simple geometry express the displacement x of the slider (or piston) as a function of R, L, the crank angle θ and the connecting rod angle ϕ, as shown.

$$x = \dots\dots\dots\dots\dots\dots$$

-------------------▷ 66

$$x = R\cos\theta + L\cos\varphi$$ 66

It is more convenient in practice to express x as a function of R, L and θ only. Eliminate φ and obtain

$$x = \ldots\ldots\ldots\ldots\ldots$$

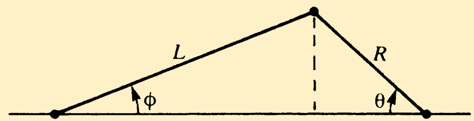
Solution ⟶ 68

Hints and explanation ⟶ 67

67

Given: $x = R\cos\theta + L\cos\varphi$

To eliminate φ use the fact that
$L\sin\varphi = R\sin\theta$.
Remembering that
$\cos^2\varphi + \sin^2\varphi = 1$, thus
we have



$$\cos\phi = \sqrt{1-\sin^2\phi}$$
$$= \sqrt{1-\left(\frac{R}{L}\right)^2 \sin^2\theta}$$

Hence

$$x = R\cos\theta + L\cos\varphi = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

68

$$x = R\cos\theta + L\sqrt{1 - \left(\frac{R}{L}\right)^2 \sin^2\theta}$$ 68

If your result does not agree with the one above go back to

67

The ratio $\frac{R}{L}$ is small, usually about $\frac{1}{3}$ or $\frac{1}{4}$.

Use the binomial expansion to obtain an approximate value for the square root term. (The first three terms will be sufficient.) Then complete the expression for the displacement x of the slider.

$$x = R\cos\theta + L \;\ldots\ldots\ldots\ldots\ldots$$

Solution

70

Hints and further explanations

69

We are asked to evaluate the first three terms of the binomial expansion of $\sqrt{1-\left(\frac{R}{L}\right)^2 \sin^2\theta}$ **69**

Step 1: The first three terms of the binomial expansion are

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2$$

Step 2: $\sqrt{1-\left(\frac{R}{L}\right)^2 \sin^2\theta} = \left(1-\left(\frac{R}{L}\right)^2 \sin^2\theta\right)^{1/2}$

$$\text{hence} \quad n = \frac{1}{2}, \quad x = -\left(\frac{R}{L}\right)^2 \sin^2\theta$$

Step 3: Substitute in the binomial expansion and obtain

$$\left(1-\left(\frac{R}{L}\right)^2 \sin^2\theta\right)^{1/2} = 1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{R}{L}\right)^4 \sin^4\theta$$

Step 4: The complete expression for the displacement x of the slider is

$$x = R\cos\theta + L(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

---------------------▷ 70

$$x = R\cos\theta + L\left(1 - \frac{1}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta - \frac{1}{8}\left(\frac{R}{L}\right)^4 \sin^4\theta \ldots\right)$$ 70

If your result is wrong

--------------------▷ 69

Note: in practice only the first two terms of the expansion are significant, except for very high performance engines.

Use the first two terms of the expansion to obtain an expression for the acceleration $\ddot{x}$ of the slider.

Notation: $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$ where t is the time. Remember that θ is also a function of time but $\dot{\theta}$ is supposed to be constant.

$$\ddot{x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found

--------------------▷ 73

Hints and detailed solution

--------------------▷ 71

With the first two terms of the expansion we have 71

$$x = R\cos\theta + L - \frac{L}{2}\left(\frac{R}{L}\right)^2 \sin^2\theta$$

We differentiate x twice to obtain the acceleration.

Obtain $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ first, remembering that

$\frac{\mathrm{d}}{\mathrm{d}t}\sin^2\theta = \left(\frac{\mathrm{d}}{\mathrm{d}\theta}\sin^2\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ (function of a function rule):

$$\dot{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

72

$$\dot{x} = -R\dot{\theta}\left(\sin\theta \frac{R}{2L}\sin 2\theta\right)$$ **72**

A detailed explanation of this differentiation is given below. Skip it if you obtained this result.

We require $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}\theta}\dfrac{\mathrm{d}\theta}{\mathrm{d}t}$

Now
$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = -R\sin\theta + 0 - \frac{L}{2}\left(\frac{R}{L}\right)^2 \frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2\theta)$$

but
$$\frac{\mathrm{d}}{\mathrm{d}\theta}(\sin^2\theta) = 2\sin\theta\cos\theta = \sin 2\theta$$

therefore
$$\dot{x} = -R\dot{\theta}\left(\sin\theta + \frac{R}{2L}\sin 2\theta\right)$$

Now differentiate once more, remembering that

$$\frac{\mathrm{d}\dot{\theta}}{\mathrm{d}t} = \ddot{\theta} = 0$$

$$\ddot{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ **73**

$$\ddot{x} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = \frac{\mathrm{d}\dot{x}}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = -R\dot{\theta}\left(\cos\theta + \frac{R}{L}\cos 2\theta\right)\dot{\theta}$$

$$= -R\dot{\theta}^2\left(\cos\theta + \frac{R}{L}\cos 2\theta\right)$$

73

In case of difficulties go through the explanation again, beginning with frame 68.

-------------------▷ 74

With the help of Maclaurin's and Taylor's series, as well as the binomial expansion, we can derive approximate expressions for some functions. This is particularly helpful when dealing with complicated expressions but we have to ensure that the error involved is kept within limits which are acceptable in practice, e.g. 1%, 5% or 10%. It depends very much on the nature of the problem and therefore no general rule can be given. 74

The table of approximations for typical functions at the end of section 8.6.3 in the textbook contains the errors for the first and second approximations.

Give the first and second approximations for cos x using this table.

First approximation: $\cos x \approx$........................

Second approximation: $\cos x \approx$........................

--------------------▷ 75

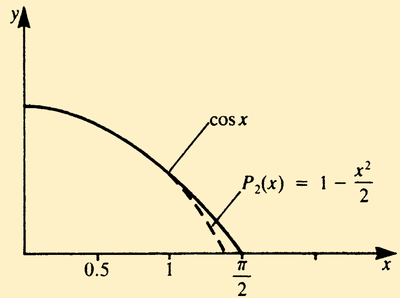
First approximation: $\cos x \approx 1 - \frac{x^2}{2!}$

75

Second approximation: $\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

For small angles, i.e. small x, the first approximation for the cosine is

$$\cos x \approx 1 - \frac{x^2}{2!}$$



Let us consider the error made by using this approximation for $x = 0.5$ (radians).

$$\cos 0.5 = 0.8776(\text{exact value})$$

$$P_2(0.5) = 1 - \frac{0.5^2}{2} = 0.8750$$

The difference is $\cos(0.5) - P_2(0.5) = 0.0026$ thus the error is smaller than 3%.

Now compute the error, using the same approximation, when $x = 0.75$ radians.
$\cos 0.75 = 0.7317 \quad E = cos 0.75 - P_2(0.75) =$ ....

76

$$E = 0.7317 - 0.7188 = 0.0129$$

76

The function $f(x) = \sqrt{1+x}$ is to be replaced by an approximate expression in the range $0 \leq x \leq 0.5$ with an error not greater than 1%. Which is the simplest approximation that can be used?

Use the table of approximations for typical functions given in the textbook.

☐ First approximation --------------------▷ 77

☐ Second approximation --------------------▷ 78

Wrong

77

The first approximation for $(1+x)^{1/2}$ is $1+\frac{1}{2}x$; it has an error not exceeding 1% in the range

$$0 \leq x \leq 0.32$$

The second approximation $1+\frac{1}{2}x-\frac{1}{8}x^2$ has an error not exceeding 1% in the range

$$0 \leq x \leq 0.66$$

which more than meets the requirement. Hence only the second approximation is acceptable.
Look back at the table of approximations for typical functions in the textbook to check this statement.

--------------------▷ 78

Correct

78

Approximations are frequently used to compute particular values of the exponential, logarithmic and trigonometrical functions when tables or scientific calculators are not available. To illustrate this point suppose that in a particular problem the value of $e^{0.2}$ is required, i.e. the value of e^{x_0} where $x_0 = 0.2$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

As a first approximation we have

$$e^{x_0} \approx 1 + x_0 = 1 + 0.2 = 1.2$$

As a second approximation we have

$$e^{x_0} \approx 1 + x_0 + \frac{{x_0}^2}{2} = \dots\dots\dots\dots\dots\dots$$

79

1.22

79

Fractions whose denominator does not differ greatly from unity can easily be calculated by means of an approximation.
Example:

$$\frac{1}{0.94} = \frac{1}{1-0.06}$$

Using the expansion for $\dfrac{1}{1-x}$ we have

$$\frac{1}{1-x} \approx \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \qquad \frac{1}{1-0.06} \approx \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

What is the percentage error E? Use the table!

☐ $E < 1\%$

☐ $E < 10\%$

--------------------▷ 80

$$\frac{1}{1-x} \approx 1+x; \frac{1}{1-0.06} \approx 1.06$$ 80

$E < 1\%$ for all x in the range $0 \leq x \leq 0.1$

Approximate: $\dfrac{1}{\sqrt{0.6}} = \dfrac{1}{\sqrt{1-0.4}}$

In the table you will find the approximation for the function

$$\frac{1}{\sqrt{1+x}}$$

You can use this expansion if you substitute x by $-x$. Thus

$$\frac{1}{\sqrt{1+(-x)}} \approx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

81

$$\frac{1}{\sqrt{1+(-x)}} \approx 1 + \frac{x}{2} + \frac{3}{8}x^2$$ 81

Now calculate

$$\frac{1}{\sqrt{1-0.4}} = \text{.........................}$$

☐ The error is less than 1%

☐ The error is less than 10%

☐ The error exceeds 10%

Use the table in the textbook!

82

$$\frac{1}{\sqrt{1-0.4}} = \approx 1.26; \ \text{truevalue} = 1.291$$

82

The error is less than 10%, but more than 1%.

Calculate $\sqrt{1.4}$ using an approximation with an error less than 1%.

Solution 84

Hints and detailed solution 83

We wish to compute $\sqrt{1.4}$.

We transform the number under the square root sign in order to obtain an expression which is included in the table of approximations. 83

$$\sqrt{1.4} = \sqrt{1+0.4}$$

In the table we find

$$\sqrt{1+x} \approx 1 + \frac{x}{2}; \text{ for } x = 0.4 \text{ the error exceeds } 1\%$$

$$\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}; \text{ for } x = 0.4 \text{ the error exceeds } 1\%$$

Now calculate

$$\sqrt{1+0.4} = \ldots\ldots\ldots\ldots\ldots\ldots$$

----------------▷ 84

$$\sqrt{1+0.4} \approx 1 + \frac{0.4}{2} - \frac{0.4^2}{8} = 1.18$$

84

Have a break!
You should by now be able to decide for yourself when to have a break and how long it should last. Furthermore, you should stick to the duration you have fixed for it.

Remember to do something quite different during a break; you must give your brain a rest!

After your break

85

INTEGRATION BY MEANS OF SERIES

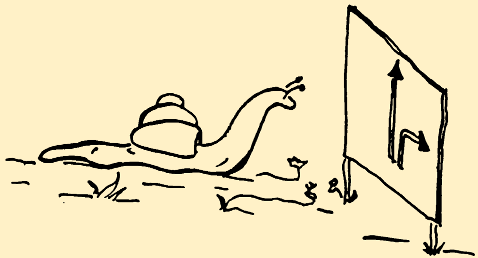
85

READ: **8.6.2 Integration of functions when expressed as power series**
8.6.3 Expansion in a series by integrating
Textbook pages 242–244

86

If an integral cannot be solved by any of the well-known methods, and provided that the series is convergent, it is useful to expand a function in a series and to integrate the series. 86

Calculate the value of $\int_0^{0.53} \sqrt{1+x^3}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



Solution ▷ 89

Hints and further explanation wanted ▷ 87

To solve $\int_0^{0.53} \sqrt{1+x^3}\mathrm{d}x$ proceed as follows:

87

We expand the integrand in a series. To do this we can use the binomial expansion which was derived in section 8.2 of the textbook.
Hence the series for $(1+x^3)^{1/2}$ is

$$(1+x^3)^{1/2} = \dots\dots\dots\dots\dots\dots \quad |x| < 1$$

Now we integrate term by term

$$\int_0^x (1+x^3)^{1/2}\mathrm{d}x = \dots\dots\dots\dots\dots\dots$$

88

$$(1+x^3)^{1/2} = 1+\frac{1}{2}x^3-\frac{1}{8}x^6+\frac{1}{16}x^9-\frac{5}{128}x^{12}+\ldots \qquad \text{provided } |x|<1 \qquad \boxed{88}$$

$$\int_0^x (1+x^3)^{1/2}\mathrm{d}x = x+\frac{x^4}{8}-\frac{x^7}{56}+\frac{x^{10}}{160}-\frac{5x^{13}}{1664}+\ldots \qquad \text{provided } |\mathrm{x}|<1$$

At this stage we introduce the limits for x, i.e. $x=0$ and $x=0.53$ so that

$$\int_0^{0.53} (1+x^3)^{1/2}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\dashrightarrow$ 89

$$\int_0^{0.53}(1+x^3)^{1/2}\mathrm{d}x = 0.53 + \frac{0.53^4}{8} - \frac{0.53^7}{56} + \ldots \approx 0.5398$$

89

Calculate the value of the integral

$$y = \int_0^{0.4} \sin x\sqrt{1+x^3}\mathrm{d}x$$

to four decimal places.

It cannot be solved by a well-known method. Thus it is useful to express the integrand as a power series and to integrate term by term:

$$y = \int_0^{0.4} \sin x\sqrt{1+x^3}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found

-------------------▷ 91

Further explantion wanted

-------------------▷ 90

90

The integrand is a product. Both factors are series which you have encountered already:

$$\sin x(1+x^3)^{1/2} = \underbrace{\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots\right)}_{\substack{\text{Series for } \sin x \text{ which}\\ \text{you can look up}}}\underbrace{\left(1+\tfrac{1}{2}x^3-\tfrac{1}{8}x^6+\ldots\right)}_{\substack{\text{Binomial series for}\\ \sqrt{1+x^3}=(1+x^3)^{1/2}}}$$

These series are both convergent for $|x|<1$. We now multiply the two series and obtain a new series:

$$\sin x(1+x^3)^{1/2} = x-\frac{x^3}{6}+\frac{x^4}{2}+\frac{x^5}{120}-\ldots$$

We integrate term by term, giving

$$\begin{aligned} y &= \int_0^{0.4}\left(x-\frac{x^3}{6}+\frac{x^4}{2}+\frac{x^5}{120}-\ldots\right)\mathrm{d}x = \left[\frac{x^2}{2}-\frac{1}{24}x^4+\frac{1}{10}x^5+\frac{1}{720}x^6-\ldots\right]_0^{0.4} \\ &= \frac{1}{2}(0.4)^2-\frac{1}{24}(0.4)^4+\frac{1}{10}(0.4)^5+\frac{1}{720}(0.4)^6-\ldots \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

91

$$y = 0.080$$

91

We conclude this section by using an integral to obtain the expansion in a series for a given function.

We use an example which has been solved previously in a different way:

Obtain a power series for $\ln(1+x)$, knowing that

$$\ln(1+x) = \int \frac{1}{1+x} \mathrm{d}x$$

$$\ln(1+x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found 93

Hints and detailed solution 92

The required series is obtained by expanding the integrand in a series using the binomial expansion. We have 92

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 - x^4 - \dots$$

Hence

$$\begin{aligned}\ln(1+x) &= \int_0^x (1+x)^{-1}\mathrm{d}x = \int_0^x (1 - x + x^2 - x^3 + x^4 - \dots)\mathrm{d}x \\ &= \dots\dots\dots\dots\dots\dots\end{aligned}$$

93

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \text{ provided } -1 < x \leq 1$$ 93

Finally, a few comments on how to plan your work. In an experiment school children were presented with a subject to study on their own. The children were divided into two groups.

Experimental group: A work plan was given to the pupils according to which the written subject matter had to be worked out.

Control group: These pupils were given the material with only general instructions.

Immediately after the lesson, as well as 10 days later, the children were tested to determine to what extent the subject matter could be reproduced:

Reproduction	With plan	Without plan
after the lesson	65%	61%
10 days later	46%	26%

94

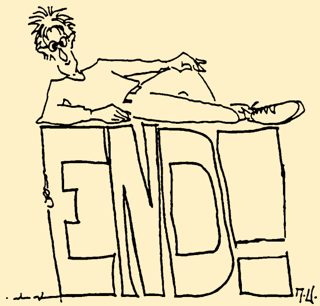
The children who had learned in accordance with the plan had learned more efficiently. The results lend themselves to generalisation. Study planning is

(a) situation analysis(how much time is available, what is your own personal capacity?);
(b) analysis of aim (what must be learned, how well must it be learned, which are the priorities and where do they lie?).

By means of a study plan, which is basically a time schedule and an analysis of objectives, a complex task can be divided up, and priorities can be established.

Such a form of work planning can already be made with the help of a simple pocket calendar.

This study guide aids you to plan and divide up your work. That is why it is so effective. In the long term, however, you should attempt to take on this job yourself. There are not always study guides around!



of Chapter 8

Chapter 9
Complex Numbers

First, a little revision of the previous chapter!

2

What are the three most important facts about Taylor's and Maclaurin's series?

(1) ....

(2) ....

(3) ....

3

(1) A function $f(x)$ can be expanded as a power series of the form $a_0 + a_1 x + a_2 x^2 + \ldots$ (or equivalent).

3

(2) The coefficients of the power series can be obtained if we know the derivatives of $f(x)$:

$$a_n = \frac{f^{(n)}}{n!}$$

(3) Power series enable us to derive approximations for many functions; this is helpful in calculations.

4

9.1 Definition and Properties of Complex Numbers

4

Objective: Concepts of imaginary numbers, complex numbers, real part, imaginary part; addition, subtraction, multiplication and division of complex numbers.

Complex numbers are helpful in the solution of differential equations, particularly in connection with problems involving oscillations. If you already know about complex numbers then read in the textbook those parts of the topic which are new to you.

READ: 9.1 Definition and properties of complex numbers
Textbook pages 249–252

5

Which of the following numbers are imaginary?

5

- ☐ j^2 6
- ☐ $4j$ 8
- ☐ $4 + 4j$ 7

Wrong! j is imaginary but $j^2 = -1$ is real.

6

Try again; which of the following numbers is imaginary?

- ☐ $4j$ 8
- ☐ $4 + 4j$ 7

Almost correct, but not quite: $4 + 4j$ is a complex number because it consists of a non-zero real part and a non-zero imaginary part.

7

8

Correct!

8

Now simplify j^4

$$j^4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found

10

Explanation wanted

9

$j^2 = -1$, by definition

Hence $j^4 = j^2 j^2 = (-1)(-1) = 1$

9

Now try

$$j^{12} = \text{.............................}$$

10

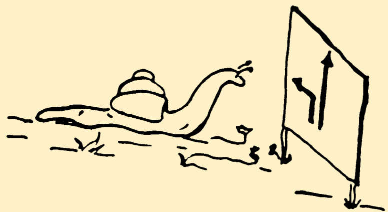
1

10

To simplify powers of j we always use the following procedure: We split the given power of j into a number of factors, each factor being $j^2 = -1$.

$j^n = (j \times j)(j \times j)..... = (-1)(-1)...$; possibly one factor j remains unmatched.

Aside: Such a method leads inevitably to the solution of the problem. It is referred to as an 'algorithm'. If we can find an algorithm then the solution of a particular problem is in our pocket.



Compute j^5

$$j^5 = \text{..............................}$$

12

Explanation required

11

Let us apply the algorithm:

11

We split the powers of j into a number of factors, each factor being $j^2 = -1$.

Example: $j^7 = (j \times j)(j \times j)(j \times j)j = (-1)(-1)(-1)j = -j$

$$j^5 = \dots\dots\dots\dots\dots\dots$$

12

$$j^5 = j$$

12

Calculate $\sqrt{-9} =$ ....

14

Explanation required

13

Here is the sequence of operations for $\sqrt{-16}$:

13

$$\sqrt{-16} = \sqrt{16(-1)} = \sqrt{16}\sqrt{-1} = 4\mathrm{j}$$

Now try again

$$\sqrt{-9} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

14

$$\sqrt{-9} = 3\mathrm{j}$$

14

Extracting the root of a negative number always follows the same algorithm.

$$\sqrt{-a^2} = \sqrt{a^2(-1)} = \sqrt{a^2}\sqrt{-1} = a\mathrm{j}$$

Evaluate

$$\sqrt{-b^2} = \dots\dots\dots\dots\dots\dots$$

$$\sqrt{-9} + \sqrt{-25} = \dots\dots\dots\dots\dots\dots$$

15

$$bj$$

$$3j + 5j = 8j$$

15

Simplify

$$\sqrt{-18} + \sqrt{-50} = \text{..............................}$$

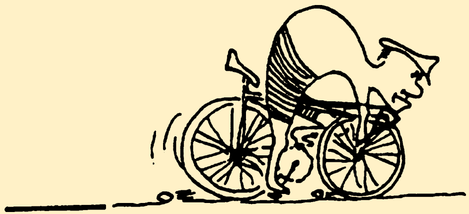
16

$$8\sqrt{2}j$$

16

Correct

17



Error: follow the solution below:

$$\begin{aligned}\sqrt{-18}+\sqrt{-50} &= \sqrt{(-1\times 9\times 2)}+\sqrt{(-1)25\times 2}\\ &= 3\sqrt{2}j+5\sqrt{2}j\\ &= 8\sqrt{2}j\end{aligned}$$

17

Simplify these expressions

17

(a) $\sqrt{-2}\sqrt{-8} =$ ....

(b) $\dfrac{\sqrt{-6}}{\sqrt{3}} =$ ....

(c) $\dfrac{1}{(-\mathrm{j})^3} =$ ....

18

(a) -4
(b) $\sqrt{2}j$
(c) $-j$

18

Correct

20

Errors; detailed solution required

19

19

(a)
$$\sqrt{-2}\sqrt{-8} = \sqrt{2(-1)}\sqrt{2\times 4(-1)}$$
$$= \sqrt{2}\mathrm{j}\times\sqrt{2}\times 2\mathrm{j}$$
$$= 2\times 2\mathrm{j}^2 = -4$$

(b)
$$\frac{\sqrt{-6}}{\sqrt{3}} = \frac{\sqrt{3\times 2(-1)}}{\sqrt{3}} = \frac{\sqrt{3\times 2}}{\sqrt{3}}\mathrm{j} = \sqrt{2}\mathrm{j}$$

(c)
$$\frac{1}{(-\mathrm{j})^3} = \frac{1}{(-1)^3\mathrm{j}^3} = \frac{1}{-1(-j)}$$
$$= \frac{1}{\mathrm{j}} = \frac{\mathrm{j}}{\mathrm{j}\times\mathrm{j}} = -\mathrm{j}$$

Alternatively,
$$\frac{1}{(-\mathrm{j})^3} = \frac{(-\mathrm{j})}{(-\mathrm{j})^3(-\mathrm{j})} = -\mathrm{j}\frac{1}{(-1)^4}(-\mathrm{j})^4 = -\mathrm{j}\times 1$$

If you still have problems go back to the textbook or ask one of your fellow students.

20

The general form of a complex number is

20

$$z = x + \mathrm{j}y$$

x is called: ....
y is called: ....

21

x is called the real part
y is called the imaginary part

21

Given the complex number

$$z = 3 + 4\mathrm{j}$$

what is the complex conjugate of z?

$$z^* = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

22

$$z^* = 3 - 4\text{j}$$

22

The complex conjugate is obtained by replacing j with $-$j.

23

Compute the sum of the following complex numbers:

23

$$z_1 = 1 + \mathrm{j}$$
$$z_2 = -3 - \mathrm{j}$$

$$z_1 + z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

24

24

-2

Here is how the solution is arrived at:

To obtain a sum of complex numbers we add the real parts and the imaginary parts separately:

$$z_1 = 1 + \mathrm{j}$$
$$z_2 = -3 - \mathrm{j}$$

$$z_1 + z_2 = (1-3) + (1-1)\mathrm{j} = -2$$

Now try

$$z_3 = 7 + 3\mathrm{j}$$

$$z_4 = -9 - 3\mathrm{j}$$

$$z_3 + z_4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

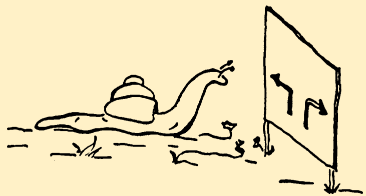
25

-2

25

Subtract $z_2 = 1 - \mathrm{j}$ from $z_1 = 3 + \mathrm{j}$

$$z_1 - z_2 = \ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found 27

Detailed explanation 26

The problem was:

26

$$\text{if } z_1 = 3 + \mathrm{j}$$
$$\text{and } z_2 = 1 - \mathrm{j}$$

$$\begin{aligned} \text{then } z_1 - z_2 &= \text{difference of real parts} + \text{difference of imaginary parts} \\ &= (3-1) \qquad\qquad\qquad\qquad + (+\mathrm{j} - (-\mathrm{j})) \\ &= 2 + 2\mathrm{j} \end{aligned}$$

Now try again

$$z_3 = 5 + 4\mathrm{j}$$
$$z_4 = 3 + 2\mathrm{j}$$

$$z_3 - z_4 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

27

$$2 + 2j$$

27

Multiplication of complex numbers.
Given:

$$z_1 = 3 + 5j$$
$$z_2 = 2 + 4j$$

$$z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution 29

Explanation wanted 28

The multiplication of two complex numbers is carried out in a very similar way to the multiplication of two binomial expressions. We must keep in mind that $j^2 = -1$. 28

Look at this new example:

$$z_1 = (2 + j)$$
$$z_2 = (1 - 2j)$$
$$z_1 z_2 = (2 + j)\,(1 - 2j)$$

Follow the arrows, first the top ones and then the bottom ones, and obtain

$$z_1 z_2 = 2 - 4j + j + 2$$
$$= 4 - 3j$$

Now try the original problem again:

$$z_1 = 3 + 5j$$
$$z_2 = 2 + 4j$$
$$z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 29

$$-14+22\mathrm{j}$$

29

Now for this problem:
given

$$z_1 = 1+\mathrm{j}$$
$$z_2 = 2+3\mathrm{j}$$
$$z_3 = 1-4\mathrm{j}$$

obtain the product $z_1 z_2 z_3 =$ ....

30

$$z_1 z_2 z_3 = 19 + 9\mathrm{j}$$ 30

Correct 32

Wrong, or explanation wanted 31

Since we have more than two factors we multiply in stages:

31

$$\begin{aligned} z_1 z_2 z_3 &= (1+\mathrm{j})(2+3\mathrm{j})(1-4\mathrm{j}) \\ &= (2+3\mathrm{j}+2\mathrm{j}+3\mathrm{j}^2)(1-4\mathrm{j}) \\ &= (-1+5\mathrm{j})(1-4\mathrm{j}) \\ &= -1+4\mathrm{j}+5\mathrm{j}-20\mathrm{j}^2 \\ &= 19+9\mathrm{j} \end{aligned}$$

32

Obtain the product $z_1 z_1{}^*$,
given $z_1 = 4 + 2\mathrm{j}$.
z^* is called ....
$z_1^* =$ ....
$z_1 z_1^* =$ ....

32

33

z^* is called the complex conjugate of z 33

$$z^* = 4 - 2\text{j}$$

$$z_1 z_1^* = 20$$

Note: The product of a complex number and its complex conjugate is always a real number.

Correct 35

Wrong; explanation required 34

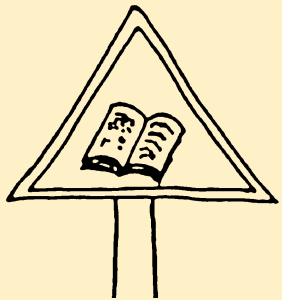
Follow the solution:

34

Required: $z_1 z_1^*$; given: $z_1 = 4 + 2j$.
First we need $z_1^* = 4 - 2j$, the complex conjugate.
The next step should be easy:

$$\begin{aligned} z_1 z_1^* &= (4+2j)(4-2j) \\ &= 16 - 8j + 8j + 4j^2 \\ &= 16 - 4(-1) = 16 + 4 = 20 \end{aligned}$$

A real number!



Remember that the product of a complex number and its conjugate is real.

35

Multiply $(2-3j)$ by a suitable factor in order to obtain a real number. 35
$(2-3j)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ a real number.

Hint: From the previous frames you should know that the product of conjugate complex numbers yields a real number. Hence the complex conjugate of $(2-3j)$ is $(2+3j)$; the rest is straightforward. You have done it before.

--------------------▷ 36

$$(2-3\mathrm{j})(2+3\mathrm{j}) = 13$$

36

Given: $z_1 = 27 + \sqrt{3}\,\mathrm{j}$ and $z_2 = 3\sqrt{3}$

What is z_1 divided by z_2?

$$\frac{z_1}{z_2} = \text{.............................}$$

37

$$\frac{z_1}{z_2} = \frac{27 + \sqrt{3}\mathrm{j}}{3\sqrt{3}} = 3\sqrt{3} + \frac{1}{3}\mathrm{j}$$ 37

Correct

-------------------▷ 38

Wrong? Here is the solution:

$$\begin{aligned}\frac{27 + \sqrt{3}\mathrm{j}}{3\sqrt{3}} &= \frac{27}{3\sqrt{3}} + \frac{\sqrt{3}\mathrm{j}}{3\sqrt{3}} = \frac{9}{\sqrt{3}} + \frac{1}{3}\mathrm{j} \\ &= \frac{9\sqrt{3}}{3} + \frac{1}{3}\mathrm{j} = 3\sqrt{3} + \frac{1}{3}\mathrm{j}\end{aligned}$$

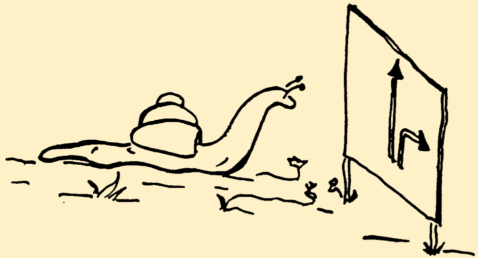
The result is obtained by dividing the real part and the imaginary part by the real number, separately.

-------------------▷ 38

Have a look at this:

38

$$\frac{(4-\sqrt{3}j)}{2j} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found

40

Detailed solution wanted

39

We transform the fraction to get a real denominator:

39

$$\frac{4-\sqrt{3}\mathrm{j}}{2\mathrm{j}} = \frac{4-\sqrt{3}\mathrm{j}}{2\mathrm{j}}\,\frac{\mathrm{j}}{\mathrm{j}}$$

$$= \frac{4\mathrm{j}-\sqrt{3}\mathrm{j}^2}{2\mathrm{j}^2} = \frac{4\mathrm{j}+\sqrt{3}}{-2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

40

$$-\frac{1}{2}\sqrt{3}-2j$$

40

Given: $z_1 = 8 + 7j$ and $z_2 = 3 + 4j$

Obtain:

$$\frac{z_1}{z_2} = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Solution 42

Explanation 41

To divide two complex numbers we first convert the denominator to a real number. This is achieved by multiplying the denominator by its conjugate, and of course the numerator is multiplied by the number also. 41

We then have

$$\frac{z_1}{z_2} = \frac{z_1 \times z_2^*}{z_2 \times z_2^*} = \frac{(8+7\mathrm{j})}{(3+4\mathrm{j})}\frac{(3-4\mathrm{j})}{(3-4\mathrm{j})} = \frac{(8+7\mathrm{j})(3-4\mathrm{j})}{9+16}$$

$$= \frac{24-28\mathrm{j}^2-32\mathrm{j}+21\mathrm{j}}{25}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

42

$2.08 - 0.44j$

42

Now it is time for a short break.

43

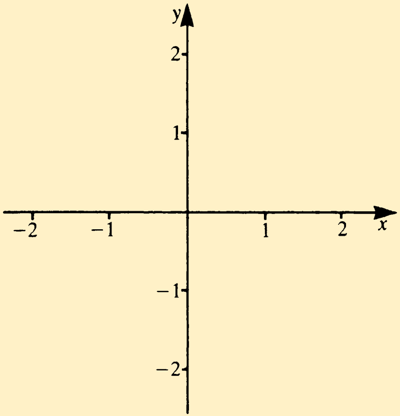
9.2 Graphical Representation of Complex Numbers

43

Objective: Concepts of the Argand diagram, modulus and argument of a complex number, polar form of a complex number.

READ: 9.2 Graphical representation of complex numbers
Textbook page 252–255

44



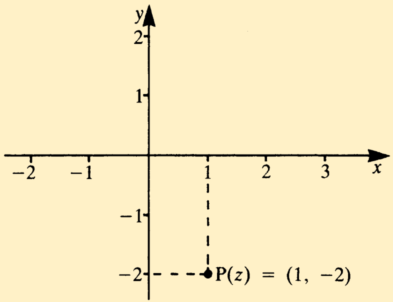
Indicate on the diagram the position of the point P(z) which belongs to the complex number

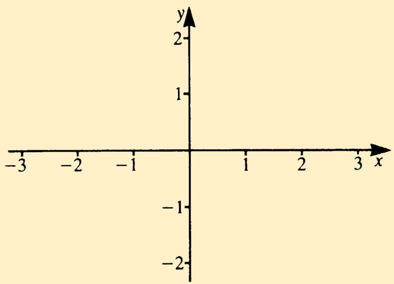
$$z = 1 - 2\mathrm{j}$$

44

45

45

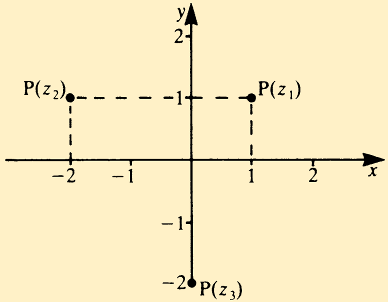




Indicate on the diagram the following complex numbers

$$z_1 = 1 + \mathrm{j}$$
$$z_2 = -2 + \mathrm{j}$$
$$z_3 = -2\mathrm{j}$$

46



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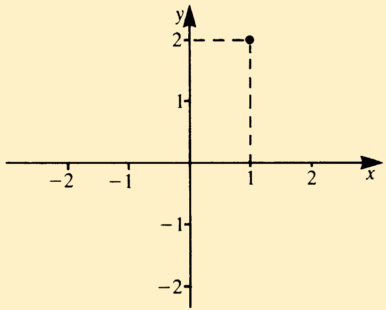


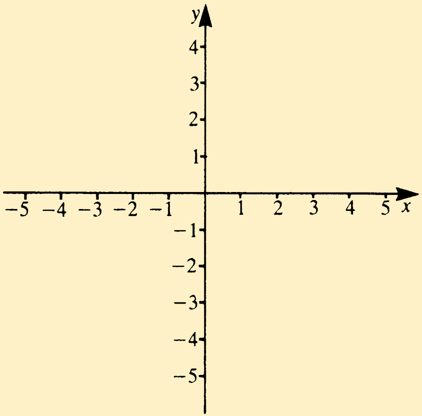
What are the real part x and the imaginary part y of the complex number z shown in the diagram?

$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

47

$$z = 1 + 2\text{j}$$

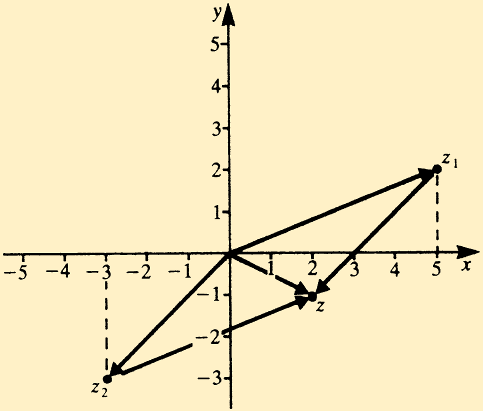
47



Obtain the sum of $z_1 = 5 + 2\text{j}$ and $z_2 = -3 - 3\text{j}$ graphically. Remember the rules of geometric addition and subtraction of vectors. What is the new complex number z?

$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

48



$z = 2 - j$

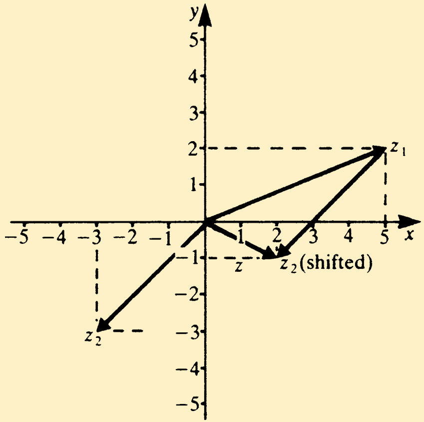
48

Correct

50

Wrong, I have difficulties

49



Given: $z_1 = 5 + 2\mathrm{j}$

$z_2 = -3 - 3\mathrm{j}$

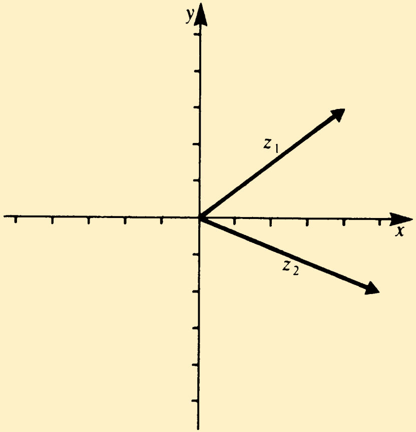
What is $z = z_1 + z_2$ graphically?

49

The solution is obtained by the following steps: We regard the complex numbers as vectors. We obtain a sum by adding $\mathbf{z}_1$ and $\mathbf{z}_2$ geometrically. Note that the addition of vectors means adding the components. Now the x-components are the real parts and the y-components are the imaginary parts.

Note: We could have obtained the same result by shifting vector $\mathbf{z}_1$ instead of $\mathbf{z}_2$. Check it for yourself!

--------------------▷ 50



Obtain graphically the difference

$$\mathbf{z} = \mathbf{z}_1 - \mathbf{z}_2$$

given that

$$\mathbf{z}_1 = 4 + 3\mathrm{j}$$

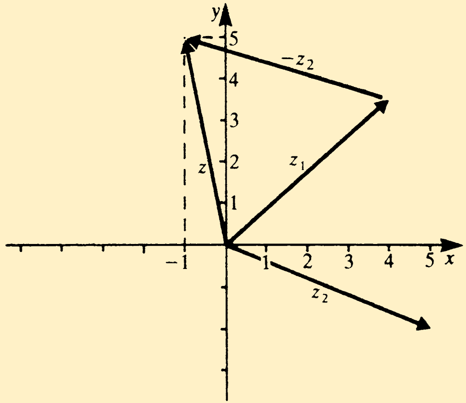
and

$$\mathbf{z}_2 = 5 - 2\mathrm{j}$$

Hence $\mathbf{z} =$ ....

50

51



$$\mathbf{z} = -1 + 5j$$

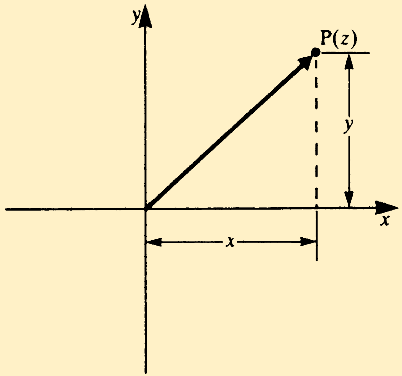
Correct 53

Wrong 54

You might find it useful to revise basic facts on vectors in a plane.
Read Chapter 3 of the textbook again, especially section 3.3.

52

53



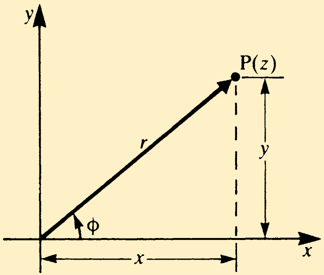
Given the complex number $z = x + jy$ as shown.

53

The point $P(z)$ can also be fixed using polar coordinates r and ϕ.

Fill in r and ϕ.

54



54

Let z be given in the forms (a) $z = x + jy$

and (b) $z = r(\cos\phi + j\sin\phi)$

What are the relationships between x and y, and r and ϕ?

$x =$ .... $y =$ ....

Can you answer from memory?

☐ Yes ------------------▷ 56

☐ No ------------------▷ 55

Look it up in the textbook in section 9.2.2 or in the table of formulae.

Now express x and y in terms of r and ϕ.

55

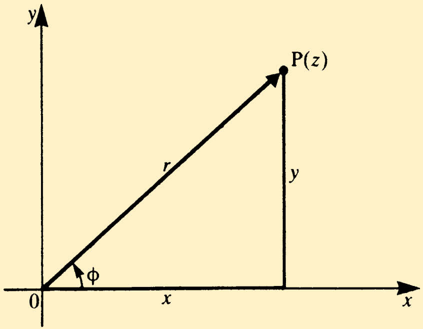
$x =$ ....

$y =$ ....

56

$$x = r\cos\phi$$
$$y = r\sin\phi$$

56



Express r and ϕ in terms of x and y.

$r =$ ....

$\phi =$ ....

57

$$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

57

Given the complex number

$$z = a^2 + (b + c)\mathrm{j}$$

a, b and c are real. Express z in polar form:

$r =$ ....

$\phi =$ ....

58

$$r = \sqrt{a^4 + (b+c)^2}$$

$$\phi = \tan^{-1}\left(\frac{b+c}{a^2}\right) \text{ or } \tan\phi = \frac{b+c}{a^2}$$

58

Correct 60

Wrong; detailed explanation 59

59

Given: $z = a^2 + (b+c)\mathrm{j}$

Compare with the standard form:

$$z = x + \mathrm{j}y$$

It follows that $$x = a^2 \text{ and } y = b + c;$$

therefore $$r = \sqrt{x^2 + y^2} = \sqrt{a^4 + (b+c)^2}$$

and $$\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{b+c}{a^2}\right)$$

or $$\tan\phi = \frac{b+c}{a^2}$$

60

Given: $z = 1 + \mathrm{j}$

Express z in the polar form $r(\cos\phi + \mathrm{j}\sin\phi)$ 60

☐ Can you do it straight away? 64

☐ Do you need help? 61

61

We want the expression $z = r(\cos\phi + \mathrm{j}\sin\phi)$
Since $z = 1 + \mathrm{j}$

then $x = 1$ and $y = 1$

Hence $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

and $\tan\phi = \dfrac{y}{x} = \dfrac{1}{1} = 1$

$\tan\phi = 1$ occurs for values of $\phi = \dfrac{\pi}{4}$ and $\dfrac{5}{4}\pi$

We must determine to which quadrant our complex number belongs.

$$\phi = \text{..............................}$$

62

$$\phi = \frac{\pi}{4}$$

62

Correct?

Good; write down the complex number in polar form, knowing that

$$r = \sqrt{2},\ \phi = \frac{\pi}{4}$$

$$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

64

Wrong, explanation wanted

63

To determine the correct value for the angle you should refer to the position of z in the complex plane. In our example, since $z = 1 + 1\mathrm{j}$, i.e. $x = +1$ and $y = +1$, it follows that it lies in the first quadrant. 63

Therefore $0 < \phi < \dfrac{\pi}{2}$; in our case $\phi = \dfrac{\pi}{4}$.

Hence $z = \sqrt{2}(\cos\dfrac{\pi}{4} + \mathrm{j}\sin\dfrac{\pi}{4})$

64

$$z = \sqrt{2}(\cos\frac{\pi}{4} + \mathrm{j}\sin\frac{\pi}{4})$$

64

65

You have just converted a complex number from its algebraic form $x + \mathrm{j}y$ into its polar form $r(\cos\phi + \mathrm{j}\sin\phi)$. We now try to formulate an algorithm for this conversion. 65

Conversion of a complex number of the form $z = x + \mathrm{j}y$ into the form $z = r(\cos\phi + \mathrm{j}\sin\phi)$:

Step 1: Conversion relationships:

$$r = \sqrt{x^2 + y^2}$$

$$\tan\phi = \frac{y}{x}$$

Insert the given values into these equations.

Step 2: Obtain the possible values of ϕ using tables or a calculator.

Step 3: Select the value of ϕ which corresponds to the given complex number by considering its position in the complex plane.

Step 4: Substitute the values of r and ϕ in the equation:

$$z = r(\cos\phi + \mathrm{j}\sin\phi)$$

--------------------▷ 66

66

A further example:

Put

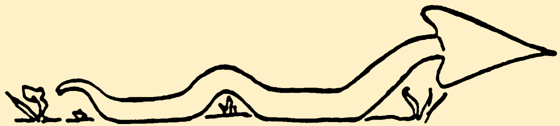
$$z = -1 + \mathrm{j} \text{ in the form}$$

$$z = r(\cos\phi + \mathrm{j}\sin\phi)$$

$$r = \dots\dots\dots\dots$$

$$\phi = \dots\dots\dots\dots$$

$$z = \dots\dots\dots\dots$$



67

$$r = \sqrt{2},\ \phi = \frac{3}{4}\pi$$

$$z = \sqrt{2}(\cos\frac{3}{4}\pi + \mathrm{j}\sin\frac{3}{4}\pi)$$

67

Correct 69

Wrong, detailed solution 68

68

$$z = -1 + \mathrm{j}$$

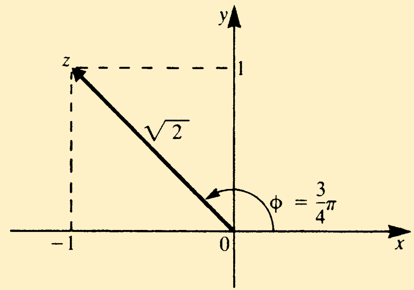
Given: $x = -1$, $y = 1$
Required: r and ϕ

Step 1: $r = \sqrt{x^2 + y^2} = \sqrt{2}$
$\tan\phi = -1$

Step 2: $\phi = \frac{3}{4}\pi$ or $\frac{7}{4}\pi$

Step 3: P(z) lies in the second quadrant, i.e. ϕ lies between $\frac{\pi}{2}$ and π, therefore $\phi = \frac{3}{4}\pi$.

Step 4: $z = \sqrt{2}(\cos\frac{3}{4}\pi + \mathrm{j}\sin\frac{3}{4}\pi)$



69

The conversion of a complex number z from the form $x + jy$ into the form $r(\cos\phi + j\sin\phi)$ (and the converse) is not difficult; the only subtlety is the correct determination of the angle ϕ.

69

Now some advice on study techniques for effective revision of study material and some basic results of experiments on memory.

Shortcut

-------------------▷ 75

Experiments on memory were first carried out by Ebbinghaus a hundred years ago. First some material is given to be memorised (meaningless syllables, poems, series of numbers, terms, definitions or mathematical statements).

After a certain period of time the degree of retention of the material is investigated. There are different methods:

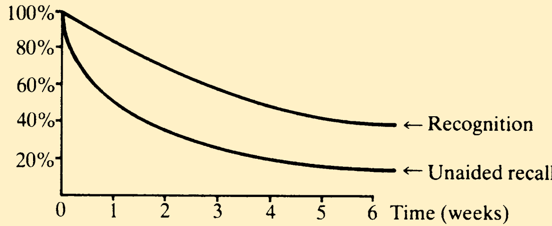
Unaided recall: The experimental subject must recall what he has remembered without assistance.

Recognition: Here we measure the percentage of the original material that is recognised.

-------------------▷ 70

Results of experiments on memory (meaningless syllables)

70



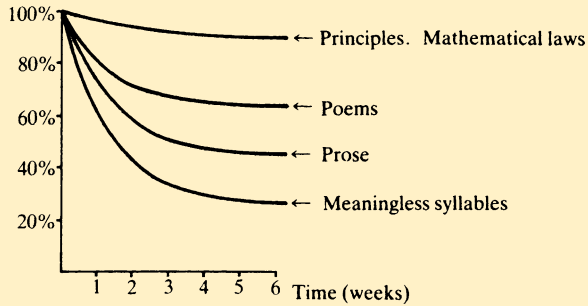
The diagram shows the so-called ‘retention curves’ which express as a percentage the temporal decrease of the original memory content.

Curves result which are roughly exponentially decreasing. Most facts that one recognises while reading cannot be actively reproduced. Recognition of facts gives one the false impression of possessing a higher degree of knowledge. This can be seen in all situations in which you are required to present facts without any help, and to apply your knowledge. This was mentioned previously in the study guide for Chapter 8.

--------------------▷ 71

The availability of memory content depends on the structuring of the subject content. Material that is well learned, and understood in its context, remains at your disposal for a longer time. 71

The diagram shows retention curves for different subject matter which has been learned thoroughly. It is clear that it is advantageous always to think of the learned material *in its context*.



72

Effect on retention of one's attitude towards learning: 72

Lewin (1963) reports the following experiment:

A student was asked to read mnemonic material to his fellow students until they were able to reproduce it. Afterwards, the student who had presented the material was invited to present the text unaided. Unlike the other students, he remembered almost nothing.

Thus, while studying, it is always advantageous to decide what is worth retaining in order to extract or at least to underline it.

73

Revision within the framework of the course: 73

We have already mentioned revision at the end of a chapter before a break, and revision after a week before starting a new chapter.

These forms of revision should be supplemented by additional systematic revisions after a longer time interval. Therefore you should make out a plan for your revision. This plan may imply that while you are busy working through an advanced chapter you are also revising a chapter you worked on four weeks previously.

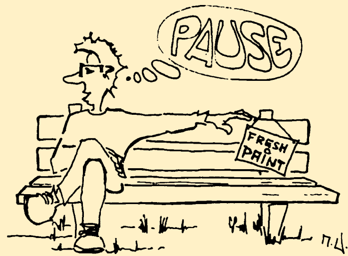
--------------------▷ 74

Each revision should be done in stages: 74

Step 1: You write down, from memory, terms introduced in the chapter. Compare this list with your notes or the textbook and expand it.

Step 2: You try to recall the meaning of the terms without help. Check these against the text and your notes. Incorrectly reproduced meanings, as well as those terms which in the beginning are not remembered, must be learned again.

Step 3: You work out the exercises of the chapter.



These revision techniques *are* useful — but only for those who apply them.
Now it is time for a break.

75

9.3 Exponential Form of the Complex Number

75

Objective: Concepts of Euler's formula, exponential form of cosine and sine functions.

READ: 9.3 Exponential form of complex numbers
Textbook pages 256–263

76

Express the complex number z in exponential form:

76

$$z = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

77

$$z = r\mathrm{e}^{\mathrm{j}\alpha}$$

77

There exists a mathematical relationship between the exponential function $\mathrm{e}^{\mathrm{j}\alpha}$ and the cosine and sine functions. Can you name it?
Can you give the formula?

$$\mathrm{e}^{\mathrm{j}\alpha} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

78

Euler's formula

$$e^{j\alpha} = \cos\alpha + j\sin\alpha$$

78

The expressions for cos α and sin α in exponential form are

$$\cos\alpha = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sin\alpha = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

These relationships are very important; for example, in the study of oscillations (Chapter 10).

79

$$\cos\alpha = \frac{1}{2}\left(e^{j\alpha} + e^{-j\alpha}\right)$$
$$\sin\alpha = \frac{1}{2}\left(e^{j\alpha} - e^{-j\alpha}\right)$$

79

What is the complex conjugate number of $z = re^{j\alpha}$?

$z^* = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

80

$$z^* = r\mathrm{e}^{-\mathrm{j}\alpha}$$

80

Correct

81

The complex conjugate is obtained, very simply, replacing j by −j. You saw this before; it is the definition of the complex conjugate.

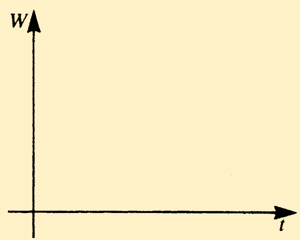
81

Have a look at the expression

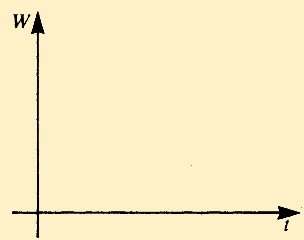
81

$$W(t) = e^{(\sigma t + j\omega t)}$$
$$= e^{\sigma t}(\cos \omega t + j \sin \omega t)$$

What do the graphs of the real part look like when
(a) $\sigma > 0$?
and (b) $\sigma < 0$?



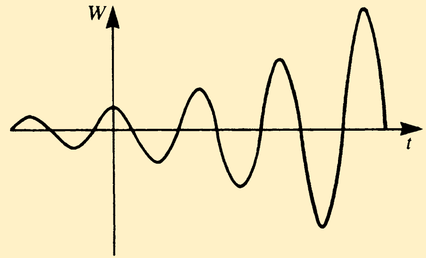
(a) $\sigma > 0$



(b) $\sigma < 0$

82

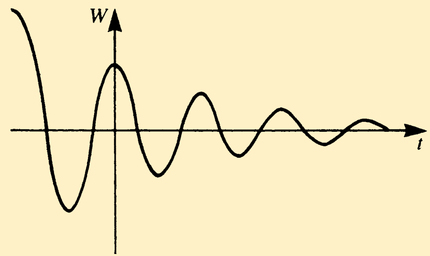
The real part is $e^{\sigma t} \cos \omega t$



82

When $\sigma > 0$ the graph represents an oscillation whose amplitude is increasing with time.

When $\sigma < 0$ the graph represents a damped oscillation.



83

Given: two complex numbers

83

$$z_1 = 2e^{j\pi}$$

$$z_2 = 2e^{j2\pi/3}$$

$$z_1 z_2 = \dots\dots\dots\dots\dots\dots$$

Solution 85

Hints 84

To multiply two complex numbers we multiply the moduli (the magnitudes), and add the arguments (the angles). 84

If

$$z_1 = 2e^{j\pi}$$
$$z_2 = 1e^{2j\pi}$$
$$z_1 z_2 = 2 \times 1\, e^{j\pi + j2\pi} = 2e^{j3\pi}$$

Now try

If

$$z_1 = 2e^{j\pi}$$
$$z_2 = 2e^{j2\pi/3}$$
$$z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

85

$$z_1 z_2 = 2 \times 2\mathrm{e}^{\mathrm{j}(2\pi/3+\pi)} = 4\mathrm{e}^{\mathrm{j}(5\pi/3)}$$

85

Remember: multiply the moduli and add the arguments.

Given: $z_1 = 2\mathrm{e}^{\pi\mathrm{j}/3}$ and $z_2 = 2\mathrm{e}^{-2\pi\mathrm{j}/3}$

Obtain $\dfrac{z_1}{z_2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

86

$$\frac{z_1}{z_2} = e^{\pi j}$$

86

Correct

87

Wrong: follow the solution given below.

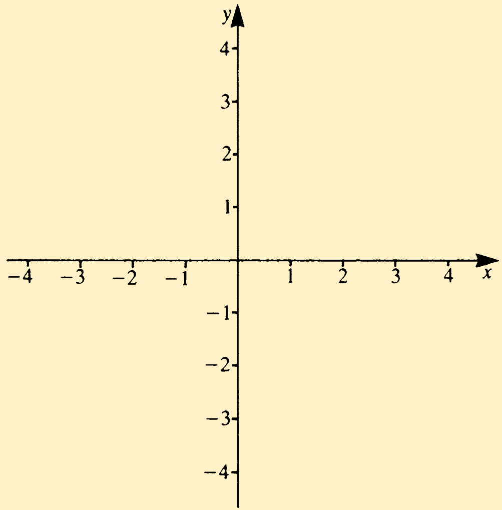
To divide two complex numbers you divide the moduli (the magnitudes) and subtract the arguments (the angle). Thus

$$\frac{z_1}{z_2} = \frac{2}{2}e^{(\pi/3+2\pi/3)j} = e^{\pi j}$$

87

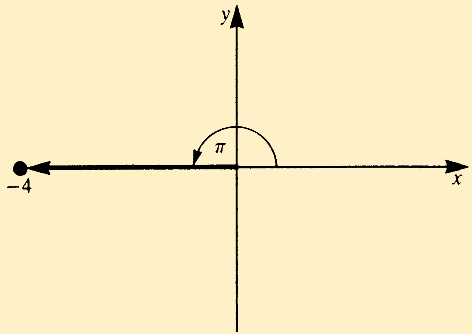
Simplify the expression $4e^{\pi j}$ and show its position in the complex plane.

87



88

88



$z = 4e^{\pi j} = -4$

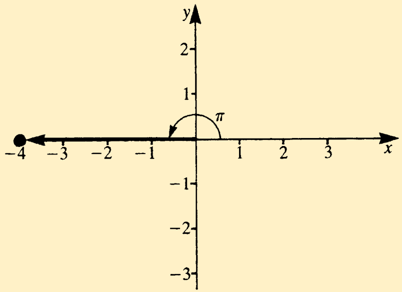
Correct 90

Wrong 89

89

$z = 4e^{\pi j}$ has $r = 4$ for magnitude
and $\phi = \pi = 180°$ for the angle.
Hence the point $P(z)$ in the complex plane is obtained from the fact that the coordinates are

$$x = -4 \text{ and } y = 0$$



Since $z = r\cos\alpha + jr\sin\alpha$

$$x = r\cos\alpha = 4\cos 180° = -4$$

and $y = r\sin\alpha = 4\sin 180° = 0$

Therefore $z = -4$.

90

Given $z = 4e^{\pi j}$
Calculate $z^3 =$ ....

90

In case of difficulty refer to the textbook.

91

$$z^3 = 64e^{3\pi j} = -64$$

91

Here is the detailed solution:

$$z = 4e^{\pi j}, \quad z^3 = 4^3 e^{3\pi j} = 64e^{3\pi j} = 64(-1)$$

Extract the square root of

$$z = 4e^{\pi j}$$
$$\sqrt{z} = \ldots\ldots\ldots\ldots\ldots\ldots$$

92

$$z = 2e^{\pi j/2}$$

92

Here is the detailed solution:

To extract the root of a complex number we find the root of the magnitude and divide the angle by the radical index. Thus:

$$\sqrt{z} = \sqrt{4}e^{j\pi/2} = 2e^{\pi j/2}$$

Given a complex number in the form

$$z = 1 + j$$

express it in the exponential form $z = re^{j\alpha}$

$$z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 93

$$z = \sqrt{2}e^{\pi j/4}$$

93

Correct

94

Wrong: follow the solution step by step:

Step 1: $x = 1$, $y = 1$

Step 2: $r = \sqrt{x^2 + y^2} = \sqrt{2}$

$\tan\alpha = \dfrac{y}{x} = 1$

Step 3: $\tan\alpha = 1$ gives $\alpha = \dfrac{\pi}{4}$ or $\dfrac{5}{4}\pi$

Step 4: $P(z)$ lies in the first quadrant, hence $\alpha = \dfrac{\pi}{4}$

Step 5: The complex number in exponential form is $z = \sqrt{2}e^{\pi j/4}$

94

Consider $z(\alpha) = r\mathrm{e}^{\mathrm{j}\alpha}$
as a function of the angle α, as indicated by $z(\alpha)$.
What is the period of $z(\alpha)$?....................

94

95

2π

95

$z(\alpha)$ has a period of 2π which means that

$$z(\alpha) = z(\alpha + 2\pi) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi)}$$

or
$$z(\alpha) = z(\alpha + 2\pi + 2\pi + \ldots) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi+2\pi+\ldots)}$$

and generally

$$z(\alpha) = r\mathrm{e}^{\mathrm{j}(\alpha+2\pi k)}$$

where $k = 1, 2, 3, \ldots$

96

96

You must be exhausted!

You deserve a break!

Before you start your break recapitulate the most important aspects of what you have just learned, i.e. concepts, definitions, formulae.

Resume your work at the end of your timed break.

97

9.4 Complex Numbers Expressed in Polar Form

97

These operations are equivalent to those using the exponential form. Both forms are used in practice.

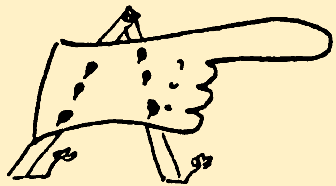
READ: 9.4 Operations with complex numbers expressed in polar form
Textbook pages 263–267

--------------------▷ 98

Multiplication of two complex numbers expressed in polar form.

98

Given: $z_1 = r_1(\cos\phi_1 + j\sin\phi_1)$
and $z_2 = r_2(\cos\phi_2 + j\sin\phi_2)$
then $z_1 z_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



If you have difficulties go back to the textbook and solve this example while consulting section 9.4.1

-------------------▷ 99

$$z_1 z_2 = r_1 r_2[\cos(\phi_1 + \phi_2) + \mathrm{j}\sin(\phi_1 + \phi_2)]$$

99

Now for a numerical example.

Calculate $z = z_1 z_2$ if

$$\begin{aligned} z_1 &= 4(\cos 30^\circ + \mathrm{j}\sin 30^\circ) \\ z_2 &= 5(\cos 55^\circ + \mathrm{j}\sin 55^\circ) \\ z &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

100

$$z = 20(\cos 85^\circ + j \sin 85^\circ)$$ 100

Correct 102

Wrong 101

To multiply two complex numbers expressed in polar form we multiply their moduli and add their arguments. 101

Since $z_1 = 4(\cos 30° + j\sin 30°) = r_1(\cos\phi_1 + j\sin\phi_1)$
and $z_2 = 5(\cos 55° + j\sin 55°) = r_2(\cos\phi_2 + j\sin\phi_2)$

Then

$$z_1 z_2 == r(\cos\phi + j\sin\phi)$$

where $r = r_1 r_2 = 4 \times 5 = 20$
and $\phi = \phi_1 + \phi_2 = 30° + 55° = 85°$

102

Write down the general expression for the quotient of two complex numbers, given 102

$$z_1 = r_1(\cos\phi_1 + \mathrm{j}\sin\phi_1)$$

and
$$z_2 = r_2(\cos\phi_2 + \mathrm{j}\sin\phi_2)$$

$$z_3 = \frac{z_1}{z_2} = \dots\dots\dots\dots\dots\dots = r_3\,(\cos\phi_3 + \mathrm{j}\sin\phi_3)$$

Where $r_3 = \dots\dots\dots\dots\dots$
and $\phi_3 = \dots\dots\dots\dots\dots$

In case of difficulty consult the textbook

103

$$z_3 = \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\phi_1 - \phi_2) + \mathrm{j}\sin(\phi_1 - \phi_2)\right]$$ 103

hence $$r_3 = \frac{r_1}{r_2},\ \phi_3 = \phi_1 - \phi_2$$

Let's have a numerical example.

Given: $$z_1 = 6(\cos 72^\circ + \mathrm{j}\sin 72^\circ)$$

$$z_2 = 3(\cos 20^\circ + \mathrm{j}\sin 20^\circ).$$

Obtain: $$z_3 = \frac{z_1}{z_2} = \text{..............................}$$

104

$$z = \frac{z_1}{z_2} = 2(\cos 52^\circ + j\sin 52^\circ)$$

104

Correct

105

Mistake:

To divide two complex numbers expressed in polar form we divide their moduli and subtract their arguments.

Now do it!

105

Let us now move on to the problem of raising a complex number to a power n. 105

Can you write the result down now without referring to any notes?

$$z^n = [r(\cos\phi + j\sin\phi)]^n = \text{..........................}$$

Write down the rule in you own words.

106

$$z^n = [r(\cos\phi + j\sin\phi)]^n = r^n(\cos n\phi + j\sin n\phi)$$ 106

Rule: To raise a complexnumber to a power n we raise the modulus to that power and multiply the argument by that power.

If you had any difficulty you should read section 9.4.2 of the textbook.

If we set $r = 1$, we obtain De Moivre's theorem, namely

$$(\cos\phi + j\sin\phi)^n = \ldots\ldots\ldots\ldots\ldots\ldots$$

107

$$(\cos\phi + \mathrm{j}\sin\phi)^n = \cos n\phi + \mathrm{j}\sin n\phi$$

107

Example:

$$z = (\cos 20^\circ + \mathrm{j}\sin 20^\circ)^6$$

then $z =$ ....

108

$$z = \cos 120^\circ + \mathrm{j} \sin 120^\circ$$

108

Calculate:

$$z = \left(1 - \mathrm{j}\sqrt{3}\right)^5 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

109

$$z = 32\left(\cos 60^\circ + \mathrm{j}\sin 60^\circ\right) = 16 + 16\sqrt{3}\,\mathrm{j}$$ 109

Correct 111

Wrong, or detailed solution wanted 110

The problem is to calculate $z = \left(1 - \mathrm{j}\sqrt{3}\right)^5$ 110

To solve it we must first convert it into a polar form, since the rules for raising a complex number to a power are based on their polar form. We have $z = x + \mathrm{j}y$; therefore $x = 1$, $y = -\sqrt{3}$.

In polar form: $r = \sqrt{1+3} = 2$

$\tan\phi = -\sqrt{3} = 300^\circ$ (fourth quadrant)

The given complex number in polar form is

$z = 2(\cos 300^\circ + \mathrm{j}\sin 300^\circ)$

When raised to the fifth power

$$\begin{aligned} z^5 &= 2^5(\cos(5\times 300^\circ) + \mathrm{j}\sin(5\times 300^\circ)) \\ &= 32(\cos 60^\circ + \mathrm{j}\sin 60^\circ) \\ &= 32\left(\frac{1}{2} + \mathrm{j}\frac{\sqrt{3}}{2}\right) = 16 + 16\sqrt{3}\mathrm{j} \end{aligned}$$

111

De Moivre's theorem holds true for positive, negative and fractional powers. 111

Give the general formula for extracting the n roots of a complex number:

$$\sqrt[n]{x + jy} = \text{.............................}$$

In case of difficulty look it up in your textbook (sections 9.4.2 and 9.4.3).

112

$$\sqrt[n]{x+\mathrm{j}y} = \sqrt[n]{r}\left[\cos\left(\frac{\phi}{n}+\frac{360^\circ k}{n}\right)+\mathrm{j}\sin\left(\frac{\phi}{n}+\frac{360^\circ k}{n}\right)\right]$$ 112

$k = 0,\ \pm 1,\ \pm 2,\ \pm 3$

k is a whole number in the formula.
What does it account for?

k accounts for the fact that ....
....

113

k accounts for the fact that we obtain n roots by considering the periodicity of the trigonometric functions. 113

A numerical example should clarify the matter.

Obtain the three solutions of

$$z = \sqrt[3]{\cos 135^\circ + \mathrm{j}\sin 135^\circ}$$

$z_1 = \dots\dots\dots\dots\dots\dots\dots\dots\dots$

$z_2 = \dots\dots\dots\dots\dots\dots\dots\dots\dots$

$z_3 = \dots\dots\dots\dots\dots\dots\dots\dots\dots$

114

$$z_1 = 0.707 + 0.707\mathrm{j}$$
$$z_2 = -0.966 + 0.259\mathrm{j}$$
$$z_3 = 0.259 - 0.966\mathrm{j} \qquad \text{to 3 decimal places}$$

114

Correct: splendid! 116

Wrong: too bad, don't despair 115

115

To calculate

$$z = \sqrt[3]{\cos 135^\circ + \mathrm{j}\sin 135^\circ}$$

we apply the general rule:

$$\begin{aligned}\sqrt[n]{x+\mathrm{j}y} &= \sqrt[n]{r(\cos\phi + \mathrm{j}\sin\phi)}\\ &= \sqrt[n]{r}\left[\cos\left(\frac{\phi}{n} + \frac{360^\circ k}{n}\right) + \mathrm{j}\sin\left(\frac{\phi}{n} + \frac{360^\circ k}{n}\right)\right]\end{aligned}$$

To get three roots we take $k = 0, 1, 2$.

Applying the formula to our example gives

$$z = \sqrt[3]{1}\,[\cos(45^\circ + 120^\circ k) + \mathrm{j}\sin(45^\circ + 120^\circ k)]$$

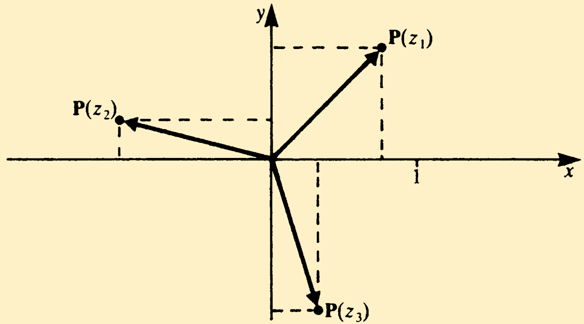
and the roots are

$$\begin{aligned}k &= 0, z_1 = \cos 45^\circ + \mathrm{j}\sin 45^\circ = 0.707 + 0.707\mathrm{j}\\ k &= 1, z_2 = \cos 165^\circ + \mathrm{j}\sin 165^\circ = -0.966 + 0.259\mathrm{j}\\ k &= 1, z_3 = \cos 285^\circ + \mathrm{j}\sin 285^\circ = 0.259 - 0.966\mathrm{j}\end{aligned}$$

116

The result of the previous example can be displayed on the Argand diagram as shown

116



You should note that the argument increases by $\dfrac{360°}{3} = 120°$.

The three roots represented by the three vectors $\mathbf{P}(z_1)$, $\mathbf{P}(z_2)$ and $\mathbf{P}(z_3)$ form an equilateral triangle and their tips lie on a circle of radius 1.

What is the principal root of

$$\sqrt[n]{x + \mathrm{j}y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

117

$$\sqrt[n]{x+\mathrm{j}y} = \sqrt[n]{r}\left(\cos\frac{\phi}{n} + \mathrm{j}\sin\frac{\phi}{n}\right)$$

117

One more example:

Obtain the three roots of unity i.e. $\sqrt[3]{1}$.

$$z_1 = \dots\dots\dots\dots\dots$$
$$z_2 = \dots\dots\dots\dots\dots$$
$$z_3 = \dots\dots\dots\dots\dots$$

Solution

--------------------▷ 119

Hints and detailed solution

--------------------▷ 118

To obtain the solution we proceed in the usual way, and don't be put off by the fact that our number is a real one! 118

In this case $x = 1$, $y = 0$
hence $r = 1$, $\tan\phi = 0°$, $\phi = 0°$

Applying the formula yields

$$\sqrt[3]{1} = \cos\left(\frac{0°}{3} + \frac{360°k}{3}\right) + \mathrm{j}\sin\left(\frac{0°}{3} + \frac{360°k}{3}\right)$$

$$= \cos 120°k + \mathrm{j}\sin 120°k$$

The roots are:

$k = 0$: $z_1 = \cos 0° + \mathrm{j}\sin 0° =$ .... (Principal value)
$k = 1$: $z_2 = \cos 120° + \mathrm{j}\sin 120° =$ ....
$k = 2$: $z_3 = \cos 240° + \mathrm{j}\sin 240° =$ ....

119

119

$$z_1 = 1$$

$$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{j} \qquad z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{j}$$

Now at the end of this chapter, some remarks on problem solving.

Problem solving requires

— knowledge acquired from previous chapters and sections

— the application of simple problem-solving methods

Here is a problem:

A satellite travels in a circular path round the Earth at a constant speed and at an altitude of 1700 km, measured from the Earth's surface.

The time needed to complete the orbit is $T = 2\,\mathrm{h}$.

The radius of the Earth is 6400 km.

For how long can an observer see this satellite above the horizon?

Before you attempt the problem, please read the general comments on the following pages.

--------------------▷ 120

Solving a problem can be attempted in the following way: 120

Step 1: Analysis of the situation:
The given facts are transferred either into a drawing or into formulae and relations. For our problem, you are advised to make a drawing and to consider carefully which quantities are known.

Step 2: Analysis of the aim:
You attempt to formulate exactly which quantities you would like to know.

Step 3: Solution of the problem:
You try to eliminate the discrepancy between the values you know and those you don't know. An important stage in the process is to look for intermediate values which can be calculated from the known ones, and with the help of which you can deduce the values sought.

Step-by-step calculation of the solution --------------------▷ 121

The solution — in case you need no help --------------------▷ 124

1. Analysis of the situation: 121

Very often a sketch reduces the given information to the essential data. Draw a section through the part of the satellite path visible to the observer. Joining of the visible part of the satellite's path with the centre of the Earth forms the angle α.

2. Analysis of the aim:

We are looking for the time T_s, which the satellite needs to pass through the section of the satellite path visible to an observer.

3. Solution of the problem:

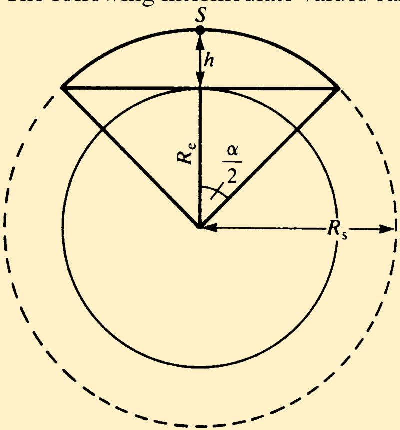
We can find intermediate values like travel speed, v and angle α of the known values, from which we can calculate the values we are looking for.

If this is already sufficient advice for you --------------------▷ 124

Now try to draw out a sketch --------------------▷ 122

In the sketch the visible part of the satellite path is S. 122

The following intermediate values can be directly determined:



Speed of travel: from the time needed for orbit and the radius of the satellite's path. The radius of the course is made up of the height above the Earth plus the Earth's radius.

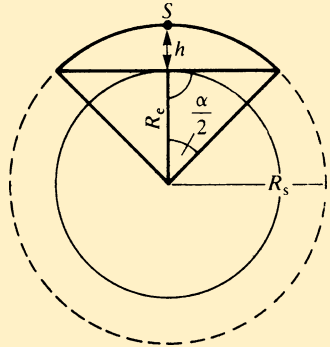
The length of the visible part of the satellite's course: from the radius of the satellite's course and angle α.

Angle α: from the Earth's radius and the radius of the satellite's path.

The problem is not completely explained here.

Detailed solution ------------------▷ 123

Solution ------------------▷ 124



123

Given values:

Radius of Earth $R_e = 6400\,\text{km}$

Altitude of satellite $h = 1700\,\text{km}$

Time of orbit $T = 2\,\text{h} = 120\text{ min}$

Intermediate values:

Speed of satellite: $v = \dfrac{2\pi R_s}{T}$

Length of visible part: $S = \dfrac{\alpha}{360^\circ} 2\pi R_s$ since $\dfrac{\alpha}{360^\circ} = \dfrac{S}{2\pi R_s}$

Angle α: $\cos\dfrac{\alpha}{2} = \dfrac{R_e}{R_s} = \dfrac{6400}{8100} = 0.7901$

$$\frac{\alpha}{2} = 38^\circ \quad \alpha = 76^\circ$$

Time of observation: $T_s = \dfrac{\alpha}{360^\circ} T$ or $T_s = \dfrac{S}{v}$

$T_s = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

124

Time of observation: $T_\mathrm{s} = 0.422\,\mathrm{h} = 25$ min

124

125

of Chapter 9

Chapter 10
Differential Equations

2

Before going on to the topic of differential equations you should recapitulate the content of the chapter on complex numbers. Write down the important keywords. Do not spend more than 5 minutes on it!

2

3

These keywords could be:

3

(i) The imaginary unit $\mathrm{j} = \sqrt{-1}$

(ii) A complex number consists of a real part and an imaginary part; i.e.

$z = x + \mathrm{j}y$

$x =$ real part

$\mathrm{j}y =$ imaginary part; y is a real number.

The modulus $|z| = \sqrt{x^2 + y^2}$ and the argument $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

(iii) Complex numbers can be represented in the Argand plane: x is the real axis and y the imaginary axis.

(iv) The complex number z may be expressed as $z = r\mathrm{e}^{\mathrm{j}\phi}$.

4

You could have added:

4

(v) The complex conjugate z^* of a complex number $z = x + \mathrm{j}y$ is defined as

$$z^* = x - \mathrm{j}y$$

(vi) Euler's formula:

$$r\mathrm{e}^{\mathrm{j}\phi} = r(\cos\phi + \mathrm{j}\sin\phi)$$

(vii) The exponential form of a complex number can be transformed thus:

$$\mathrm{e}^{(x+\mathrm{j}y)} = \mathrm{e}^{x}\mathrm{e}^{\mathrm{j}y} = \mathrm{e}^{x}(\cos y + \mathrm{j}\sin y)$$

Since differential equations require a knowledge of complex numbers you need to be sure that you understand the fundamental operations of complex algebra. If you are in doubt you should return to Chapter 9 in the textbook.

5

Differential equations are a branch of the differential and integral calculus. They play a vital role in many aspects of physics and engineering as well as in such topics as economics. 5

Before going any further make sure that you are confident of your ability to differentiate and integrate.

Prerequisites for this chapter are:

(i) differentiation and integration
(ii) algebra and trigonometry
(iii) exponential functions
(iv) complex numbers

6

6

On differentiation:

Test yourself with the following

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^4) = \ldots\ldots\ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos kx) = \ldots\ldots\ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(4\sin^2 3t) = \ldots\ldots\ldots$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{ax}) = \ldots\ldots\ldots$$

7

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^4) = 3 \times 4x^{(4-1)} = 12x^3$$
$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos ks) = -k \sin kx$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(4\sin^2 3t) = 4\frac{\mathrm{d}}{\mathrm{d}x}(\sin 3t)^2$$
$$= 4 \times 2\sin 3t \cos 3t \times 3 = 24\sin 3t \cos 3t = 12\sin 6t$$

7

(Note: Apply the function of a function rule twice: put $z = \sin 3t$ and $u = 3t$.) $\frac{\mathrm{d}}{\mathrm{d}x}(x\mathrm{e}^{ax}) = xa\mathrm{e}^{ax} + \mathrm{e}^{ax} = \mathrm{e}^{ax}(1+ax)$ (Note: Use the product rule.)

Differentiate:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(4\mathrm{e}^{7x}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(A\cos\omega t + B\sin\omega t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (A, B, \omega \text{ are constants})$$
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[a\sin(\omega t - \phi)] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (a,\ \omega,\ \phi \text{ are constants})$$

8

$$\frac{d^2}{dx^2}(4e^{7x}) = 4\frac{d}{dx}\left(\frac{d}{dx}e^{7x}\right) = 4\frac{d}{dx}(7e^{7x}) = 28\frac{d}{dx}(e^{7x}) = 196e^{7x}$$ 8

$$\begin{aligned}\frac{d^2}{dt^2}(A\cos\omega t + B\sin\omega t) &= \frac{d}{dt}\left[\frac{d}{dt}(A\cos\omega t + B\sin\omega t\right]\\ &= \frac{d}{dt}(-A\omega\sin\omega t + B\omega\cos\omega t)\\ &= -A\omega^2\cos\omega t - B\omega^2\sin\omega t\\ &= -\omega^2(A\cos\omega t + B\sin\omega t)\end{aligned}$$

$$\frac{d^2}{dt^2}[a\sin(\omega t - \phi)] = -a\omega^2(\omega t - \phi)$$

If you encountered difficulties you should revise Chapter 5, calculation of differential coefficients, now. This is a prerequisite for the present chapter and your attention must not be diverted by difficulties with elementary calculations.

--------------------▷ 9

On integration:

Test yourself with the following integrals:

9

$$\int a\mathrm{e}^{bt}\,\mathrm{d}t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \cos(\omega t - \phi)\,\mathrm{d}t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int \left(10x^3 + 7x^2 - \frac{5}{x}\right)\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int (15\cos 7x - 4\sin x)\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 10

$$\int a e^{bt}\,dt = a\int e^{bt}\,dt = \frac{a}{b}e^{bt} + C$$ 10

$$\int \cos(\omega t - \phi)\,dt = \int \cos u \frac{du}{\omega} = \frac{1}{\omega}\sin u + C = \frac{1}{\omega}\sin(\omega t - \phi) + C$$

$$\int \left(10x^3 + 7x^2 - \frac{5}{x}\right)dx = 10\int x^3\,dx + 7\int x^2\,dx - 5\int \frac{dx}{x}$$

$$= \frac{10}{4}x^4 + \frac{7x^3}{3} - 5\ln|x| + C$$

$$\int (15\cos 7x - 4\sin x)\,dx = 15\int \cos 7x\,dx - 4\int \sin x\,dx$$

$$= \frac{15}{7}\sin 7x + 4\cos x + C$$

Do not forget the constant of integration!

If you experienced difficulties you should reread Chapter 6, calculation of integrals. Integration, too, is a prerequisite for the present chapter.

--------------------▷ 11

You should commit to memory the derivatives and integrals of fundamental functions, i.e. 11

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1} + C\,,\ n \neq -1$$
$$\frac{d}{dx}\sin x = \cos x \qquad \int \cos x\,dx = \sin x + C$$
$$\frac{d}{dx}\cos x = -\sin x \qquad \int \sin x\,dx = -\cos x + C$$
$$\frac{d}{dx}e^x = e^x \qquad \int e^x = e^x + C$$
$$\frac{d}{dx}\ln x = \frac{1}{x} \qquad \int \frac{dx}{x} = \ln|x| + C\,,\ x \neq 0$$

The derivatives and integrals of other functions will be found in the tables in the textbook.

12

Finally, before starting on differential equations, do the following exercises: 12

(a) If $y = 24e^{-2t}\cos(3t-5)$
then $\dfrac{dy}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(b) If $x^2+4x-21=0$
the roots are $r_1 = \ldots\ldots\ldots\ldots\ldots\ldots$, $r_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

(c) If $x^2-x+2.5=0$
the roots are $r_1 = \ldots\ldots\ldots\ldots\ldots$, $r_2 = \ldots\ldots\ldots\ldots\ldots$

(d) Evaluate $\int 24\cos(15t+ß)\,dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

13

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -24\mathrm{e}^{-2t}\left[3\sin(3t-5)+2\cos(3t-5)\right]$ (product rule) 13

(b) $r_1 = 3,\ r_2 = -7$

(c) $r_1 = \dfrac{1}{2}+\dfrac{3}{2}\mathrm{j},\ r_2 = \dfrac{1}{2}-\dfrac{3}{2}\mathrm{j}$ (quadratic equation, complex numbers)

(d) $\dfrac{24}{15}\sin(15t+\pi)+C$

If you have doubts then you must return to the relevant chapters in the textbook and the study guide before going on to differential equations. You must overcome your difficulties first.

--------------------▷ 14

10.1 Differential Equations: Concepts and Classification

14

Objective: Concept of a differential equation, types of differential equation.

READ: 10.1 Concept and classification of differential equations
Textbook pages 275–279

It is especially useful to memorise the somewhat cumbersome classification of differential equations. Take notes.

15

Answer the following questions without any help and check your results using your notes. Which of the following equations are DEs?

15

(a) $x^n = y^3$
(b) $f(x) = 4x^{-1} + 3$
(c) $f(x) = f'(x)$
(d) $y = y^3$
(e) $y = (y'')^3 + 2xy + 17$
(f) $0 = y' + ay' + by + C$
(g) $y'' = C$

16

The DEs are: (c), (e), (f), (g).

16

Not more than one mistake

19

More than one mistake

17

Using the textbook, which are DEs?

17

(a) $y' + C = y'' + y^3$
(b) $f(x) = x^3 + 2x^2 + 3x + 5$
(c) $y'' = (y')^5 + (y'')^2$
(d) $y^3 = 2xy$
(e) $y'' = y$
(f) $y = y^2$

18

The DEs are: (a), (c), (e).

Which of the following are second-order DEs?

19

(a) $(y'')^3 + (y')^4 + y^5 = C$
(b) $y^2 + (y')^2 = x$
(c) $y'' = 0$
(d) $y''' + y'' = 0$

20

(a), (c) are second-order DEs.

20

All correct

23

Mistakes; further explanation wanted

21

A DE is said to be of the second order if the *highest* derivative in the equation is the second derivative. 21

Which of the following are second order DEs?

(a) $y'' + y''' = 0$
(b) $y'' + C = y^3$
(c) $y' = 2xy + y^2$
(d) $y' - y'' = 0$

22

Second-order DEs are (b) and (d).

22

23

Which of the following DEs are linear?

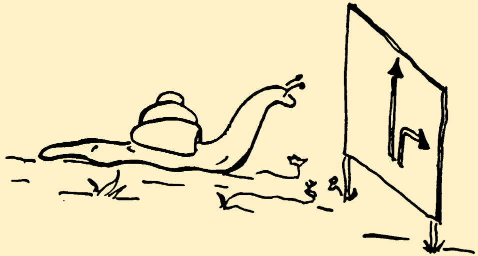
23

(a) $C_2 y'' + C_1 y' + C_0 y = f(x)$
(b) $xy'' + x^2 y' = y$
(c) $(y'')^2 + y' = y + C$
(d) $y' = y^3$

24

(a), (b).

24



All correct

27

Mistakes

25

A DE is linear if its derivatives (y', y'', etc.) and the function itself (y) occur to the first power, i.e. to the first degree, and there are no products like yy' etc. 25

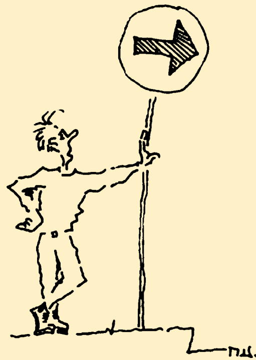
Which DEs are linear?

(a) $y' + y'' + y^2 = 0$
(b) $y'' + 3xy + C = 0$
(c) $y' = C + x^2$
(d) $y' + y'' = 2x + 5$

26

The linear DEs are: (b), (c), (d).

26



27

Which of the following DEs are homogeneous?

27

(a) $y'' + y + C = 0$
(b) $y'' + y = x^3$
(c) $y'' + 5x = 0$
(d) $y' + y = 0$

28

(d) is homogeneous

28

Correct

31

Wrong

29

Read the definition in the textbook. Which of the following DEs are homogeneous? Check your answer using the textbook.

29

(a) $y'' + x = C$
(b) $xy = 0$
(c) $xy' = x$
(d) $y'' + y' = 2xy^2$

30

(b) and (d) are homogeneous

30

31

In section 10.1 it was shown that the general solution of a DE contains arbitrary constants. How many such constants are there in the general solution of a second-order DE?

31

Solution ----> 33

Further explanation wanted ----> 32

The number of arbitrary constants in a DE is easily memorised: the number corresponds to the number of integrations necessary to obtain a solution. In the case of an nth order DE we have to integrate n times and, therefore, there will be n arbitrary constants. 32

How many arbitrary constants are there in the general solution of a second-order DE? ....

--------------------▷ 33

Two constants

33

How many boundary conditions are required to obtain the values of the constants in a second order DE? ....

If the values of the constants are fixed we have a special solution called ....

Answers found

--------------------▷ 35

Further explanation wanted

--------------------▷ 34

Since an nth order DE contains n arbitrary constants it follows that these n constants can be determined if we specify n boundary conditions, i.e. one condition for each constant. 34

How many boundary conditions are required to obtain the values of the constants in a second-order DE? ....

Find in the textbook the name of the special solution if the constants are determined according to boundary conditions: ....

--------------------▷ 35

two boundary conditions

particular solution, or particular integral

35

Let us recapitulate:

The order of a DE is given by the order of the highest derivative in the equation.

A DE is linear if y and the derivatives of y are of the first power and no products of y and its derivatives occur.

The general solution of a DE of nth order contains n arbitrary constants. They can be determined if n additional conditions are given which are called boundary conditions.

If the constants in the general solution are determined according to certain boundary conditions we call this special solution the *particular solution* or *particular integral*.

36

10.2 Preliminary Remarks

36

In this section it is shown that the general solution of a non-homogeneous linear DE consists of two parts: a general solution of the homogeneous DE and a particular solution of the non-homogeneous DE.

READ: 10.2 Preliminary remarks
Textbook pages 279–280

37

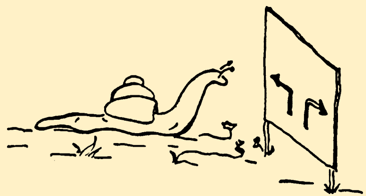
Given the non-homogeneous DE

$$y'' = x$$

obtain the general solution

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

37



Solution found ---------------------▷ 41

Further explanation wanted ---------------------▷ 38

$y'' = x$ is a non-homogeneous DE
$y'' = 0$ is the homogeneous DE

38

The general solution can be found in three steps, as is outlined in the textbook.

Step 1: Find the complementary function y_c of the homogeneous DE.
Note: complementary function means the general solution of the homogeneous DE.

In this case

$$y'' = 0$$
$$y_{\mathrm{c}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

39

Verification:

$$y_c = C_1 x + C_2$$
$$y_c{}' = C_1$$
$$y_c{}'' = 0$$

39

Step 2: Find a particular integral y_p of the non-homogeneous DE.
Note: particular integral means a solution of the non-homogeneous DE.

In this case

$$y'' = x$$
$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

40

$$y_p = \frac{x^3}{6}$$

40

Verification:

$$y_p'' = \frac{3 \times 2 \times x}{6} = x$$

Step 3: The general solution of the non-homogeneous DE is the sum of the complementary function y_c and a particular integral y_p.

In our case

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 41

$$y = \frac{x^3}{6} + C_1 x + C_2$$

41

Given the non-homogeneous DE

$$2y' = x + 1$$

Write down the homogeneous DE

....

The general solution of the homogeneous DE is called

....

Obtain it (step 1)

$$y_c = \text{...........................}$$

42

$$2y' = 0$$

42

the complementary function y_c:

$$y_c = C_1 \ (\text{since } y_c' = 0)$$

Given: $$2y' = x + 1$$

Obtain a solution of the non-homogeneous DE (step 2). It is called ....

$$y_p = \text{..........................}$$

43

a particular integral

$$y_p = \frac{x^2}{4} + \frac{x}{2}$$

43

Given: $2y' = x + 1$

Complementary function $y_c = C_1$, particular integral $y_p = \dfrac{x^2}{4} + \dfrac{x}{2}$

Now obtain the general solution (step 3)

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

44

$$y = y_c + y_P = \frac{x^2}{4} + \frac{x}{2} + C_1$$ 44

Note: To determine C_1 we need a supplementary (boundary) condition.

Suppose $y(0) = 1$

We obtain

$$y(0) = 1 = 0 + 0 + C_1$$

Thus $C_1 = 1$, in this case.

45

10.3 General Solution of First- and Second-Order DEs with Constant Coefficients

45

The following section is somewhat lengthy. You would be well advised to divide it into two or three parts, revising at the end of each part.

Take notes and follow the calculations separately.

READ: 10.3 General solution of first- and second-order DEs with constant coefficients
10.3.1 Homogeneous linear DE
Textbook pages 281–286

--------------------▷ 46

What is the auxiliary equation of this DE?

46

$$y'' + 2y' - y = 0$$

(Use the exponential solution $y = C\mathrm{e}^{rx}$.)

Answer found

48

Further explanation wanted

47

47

Let us discuss an example:

Given: $y'' - y = 0$

If $y = Ce^{rx}$ then $y' = Cre^{rx}$ and $y'' = Cr^2e^{rx}$.

Substituting in the DE yields

$$(r^2 - 1)Ce^{rx} = 0$$

The term in the brackets must be zero, since $Ce^{rx} \neq 0$.

Thus the auxiliary equation reads $r^2 \quad - 1 = 0$.

Using $y = Ce^{rx}$, obtain the auxiliary equation of the DE

$$y'' + 2y' - y = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

48

$$r^2 + 2r - 1 = 0$$

48

The auxiliary equation of the DE

$$3y'' + 2y' - 2y = 0$$

is ....

The roots of the auxiliary equation are:

$$r_1 = \text{.........................}, \quad r_2 = \text{.........................}$$

Answers found --------------------▷ 50

Difficulties --------------------▷ 49

Go back to section 10.3.1 in the textbook and study it more thoroughly, then try again. 49

Auxiliary equation of $3y'' + 2y' - 2y = 0$ (use $y = C\mathrm{e}^{rx}$): ....

Obtain the roots:

$$r_1 = \text{..........................}, \quad r_2 = \text{..........................}$$

Perhaps you ought to revise quadratic equations? (These were treated in Chapter 1.)

--------------------▷ 50

$$3r^2 + 2r - 2 = 0$$
$$r_1 = \frac{1}{3}\left(-1 + \sqrt{7}\right)$$
$$r_2 = \frac{1}{3}\left(-1 - \sqrt{7}\right)$$

50

What is the general solution of the DE

$$3y'' + 2y' - 2y = 0?$$

$y =$ ....

Solution 53

Further explanation wanted 51

51

Let's do an easier example.

Find the general solution of the DE

$$4y'' - y = 0$$

by following the scheme:

Step 1: Exponential solution: $y = C\mathrm{e}^{rx}$

Step 2: Establish the auxiliary equation: $ar^2 + br + c = 0$

....

Step 3: Find the roots of the auxiliary equation: $r_1 =$ ...., $r_2 =$ ....

Step 4: General solution of the DE: $y =$ ....

Note: Since the DE is of the second order we find two roots and thus two exponential solutions. The general solution is the sum of both.

--------------------▷ 52

Auxiliary equation: $4r^2 - 1 = 0$ 52

Roots of the auxiliary equations: $r_1 = 0.5$, $r_2 = -0.5$

General solution of the DE: $y(x) = C_1 e^{0.5x} + C_2 e^{-0.5x}$

Note: the general solution contains two arbitrary constants.

Now try again, following the same scheme.

Given:

$$3y'' + 2y' - 2y = 0$$
$$y = Ce^{rx}$$

Auxiliary equation $3r^2 + 2r - 2 = 0$

The roots of the quadratic equation are

$$r_1 = \frac{1}{3}(-1 + \sqrt{7}),\ r_2 = \frac{1}{3}(-1 - \sqrt{7})$$

Since both roots are real and distinct, the general solution is given by

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Remember the two arbitrary constants; the DE is of the second order!

53

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

or

$$y = C_1 e^{1/3(-1+\sqrt{7})x} + C_2 e^{1/3(-1-\sqrt{7})x}$$

53

Given the DE: $16y'' - 8y' + 26y = 0$
Auxiliary equation: $16r^2 - 8r + 26 = 0$

The roots are $$r_1 = \frac{1}{4} + \frac{5}{4}\mathrm{j}$$

and $$r_2 = \frac{1}{4} - \frac{5}{4}\mathrm{j}, \ (\mathrm{j} = \sqrt{-1})$$

The real form of the solution of the DE is $y =$ ....
Note: In case of difficulties solve the example step by step, using the textbook.

54

$$y = \mathrm{e}^{\frac{1}{4}x}(C_1 \cos\frac{5}{4}x + C_2 \sin\frac{5}{4}x)$$

54

What is the general real-valued solution of this DE?

$$3y'' + 5y' + 4y = 0$$

$$y = \text{..........................}$$

55

$$y = \mathrm{e}^{-\frac{5}{6}x}\left(C_1 \cos\frac{\sqrt{23}}{6}x + C_2 \sin\frac{\sqrt{23}}{6}x\right)$$

55

Hint: The roots of the auxiliary equation
$3r^2 + 5r + 4 = 0$ are

$$r_1 = -\frac{5}{6} + \frac{1}{6}\sqrt{23}\mathrm{j}, \qquad r_2 = -\frac{5}{6} - \frac{1}{6}\sqrt{23}\mathrm{j}$$

Find the real-valued solution of the DE

$$y'' + 2y' + 5y = 0$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

56

$$y = \mathrm{e}^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

56

Detailed solution:

Auxiliary equation: $r^2 + 2r + 5 = 0$

The roots are $r_1 = -1 + 2\mathrm{j}, \quad r_2 = -1 - 2\mathrm{j}$

According to the formula in the textbook the real-valued solution is

$$y = \mathrm{e}^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

Obtain the general solution of the DE

$$\frac{3}{2}y'' + \frac{1}{2}y' + \frac{1}{24}y = 0$$

$y = $ ....

-------------------▷ 57

$$y = C_1 e^{-\frac{1}{6}x} + C_2 x e^{-\frac{1}{6}x}$$
$$= e^{-\frac{1}{6}x}(C_1 + C_2 x)$$

57

Correct 59

Wrong, or detailed solution wanted 58

Given: $\frac{3}{2}y'' + \frac{1}{2}y' + \frac{1}{24}y = 0$

58

Auxiliary equation:

$$\frac{3}{2}r^2 + \frac{1}{2}r + \frac{1}{24} = 0$$

Roots: $r_1 = r_2 = -\frac{1}{6}$ i.e. equal roots.

The solution is:

$$y = C_1 e^{-\frac{1}{6}x} + C_2 x e^{-\frac{1}{6}x}$$

With the help of the scheme in the textbook find the general solution of the DE

$$y'' - 2y' + y = 0$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

Check your solution by yourself; you must obtain

$$y = C_1 e^x + C_2 x e^x = e^x(C_1 + xC_2)$$

--------------------▷ 59

The most important aim of this section is to learn how to solve homogeneous DEs using the method of exponential solution. 59

This method will enable you to solve many DEs encountered in physics and engineering.

One more exercise!

Find the general solution of the first order DE

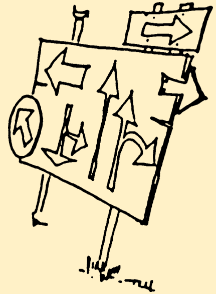
$$2y' = 3y$$

$$y = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

60

$$y = Ce^{\frac{3}{2}x}$$

60



Correct 64

Wrong, or further explanation wanted 61

The DE is

61

$$a_1 y' + a_0 y = 0$$

Its auxiliary equation is

$$a_1 r_1 + a_0 = 0$$

whose root is $r_1 =$ ....

62

$$r_1 = -\frac{a_0}{a_1}$$

62

The solution of the homogeneous first order DE is therefore

$$y = C\mathrm{e}^{r_1 x} = C\mathrm{e}^{-\frac{a_0}{a_1}x}$$

Now we calculate the solution of the DE $2y' = 3y$.

Rewriting the equation we have

$$2y' - 3y = 0$$

Hence the auxiliary equation is

$$2r_1 - 3 = 0$$

Therefore $r_1 = \dfrac{3}{2}$

and the solution is

$$y = C\mathrm{e}^{\frac{3}{2}x}$$

Find the solution of the DE: $\dot{N}(t) = -\lambda N(t)$

$$N(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

63

$$N(t) = C\mathrm{e}^{-\lambda t}$$

63

Note: this is a function of the variable t.

64

We finally summarize the procedure for the solution of homogeneous, linear first- and second-order DEs with constant coefficients. The general form of such a DE is 64

$$a_2 y'' + a_1 y' + a_0 y = 0$$

The solution is carried out in three steps:

Step 1: Establish the auxiliary equation; this means

(i) replacing y'' by r^2
(ii) replacing y' by r
(iii) replacing y by 1

Step 2: Determine the roots r_1 and r_2 of the auxiliary equation.

--------------------▷ 65

Step 3: The general solution of the DE depends on the nature of the roots r_1 and r_2. There are three possible cases: 65

(a) r_1 and r_2 are real and unequal, i.e. $r_1 \neq r_2$
(b) r_1 and r_2 are real and equal, i.e. $r_1 = r_2$
(c) r_1 and r_2 are complex, i.e. $r_1 = a + \mathrm{j}b$; $r_2 = a - \mathrm{j}b$

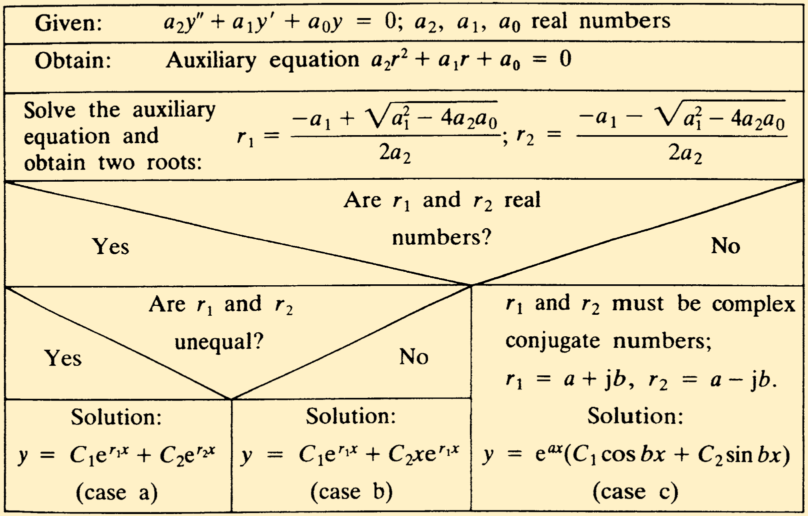
The corresponding solutions are:

(a) $y = C_1 \mathrm{e}^{r_1 x} + C_2 \mathrm{e}^{r_2 x}$
(b) $y = C_1 \mathrm{e}^{r_1 x} + C_2 x \mathrm{e}^{r_1 x} = \mathrm{e}_1^r x (C_1 + C_2 x)$
(c) $y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$

Such a procedure can be used to solve any problem of a given type. It is called an algorithm. Algorithms are used extensively in computer programming.

--------------------▷ 66

The algorithm for the solution of homogeneous second-order linear DEs with constant coefficients can be depicted graphically. (We here use a technique called Nassi-Shneiderman diagram.) 66



67

10.4 Solution of the Non-Homogeneous Second-Order DE with Constant Coefficients

67

Objective: General and particular solution of the non-homogeneous linear DE with constant coefficients.

READ: **10.3.2 Non-homogeneous linear DE**
Textbook pages 287–292

68

Given a non-homogeneous DE of the form

68

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Let y_c be the general solution of the accompanying homogeneous DE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

y_c is also called the complementary function, or CF for short.
Let a solution of the non-homogeneous DE be y_p.
y_p is also called a particular integral, or PI for short.
What is the general solution of the non-homogeneous DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)?$$

$$y = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

69

$$y = y_c + y_p$$ 69

The general solution of the non-homogeneous DE is the sum of the general solution of the homogeneous DE and a particular solution of the non-homogeneous DE.

This rule holds true in general for linear non-homogeneous DEs of any order, but we shall only consider first- and second-order DEs now.

The CF (complementary function) of the DE

$$y'' + 3y' = x + \frac{1}{3}$$

is $y_c = C_1 + C_2 e^{-3x}$.

A PI (particular integral) of that DE is

$$y_p = \frac{x^2}{6}$$

What is the general solution of $y'' + 3y' = x + \frac{1}{3}$?

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 70

$$y = C_1 + C_2 e^{-3x} + \frac{x^2}{6}$$ 70

If the given DE is

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

let the complementary function be denoted by y_c, and let a particular integral be denoted by y_p.

The general solution is

$$y = \text{..............................}$$

71

$$y = y_c + y_p$$

71

Given: $y'' + 23y' + 15y = 6$

Can you guess a PI (particular integral) of the DE?

$$y_p = \dots\dots\dots\dots\dots\dots$$

Solution found 74

Further explanation wanted 72

Let us try a simpler problem. Consider the DE

72

$$y'' + y = 2$$

Required: a PI (particular integral).

Try these functions:

$$y = 1$$
$$y = 2$$
$$y = x$$
$$y = 2x$$

Which one fits the DE?

$$y_{\mathrm{p}} = \text{..........................}$$

73

$$y_{\mathrm{p}} = 2$$

73

To guess a particular solution may be quite subtle a problem. Only experience will help in many cases, but sometimes it is quite easy. Given the DE

$$y'' + 23y' + 15y = 6$$

Try

$$y = 6$$
$$y = 6 + x$$
$$y = \frac{2}{5}$$
$$y = \frac{6}{15} + 23x$$
$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

74

$$y_{\mathrm{p}} = \frac{2}{5}$$ 74

There are special cases of non-homogeneous DEs where the variables can be separated.

Solve the DE

$$y'' = a \qquad \text{(a constant)}$$

Integrating twice yields

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found --------------------▷ 76

Detailed solution --------------------▷ 75

Detailed solution of the DE:

75

$$y'' = a \quad (a \text{ is a constant})$$

$$y'' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = a$$

In this case we can integrate directly.

The first step is to integrate once with respect to x; this yields

$$y\prime = ax + C_1$$

Integrating once more yields

$$y(x) = \text{..........................}$$

-------------------▷ 76

$$y(x) = \frac{ax^2}{2} + C_1 x + C_2$$ 76

This example illustrates a special case of a DE whose solution is obtained by direct integration because the variables can be separated.

A practical example is that of a body thrown vertically upwards, if we neglect air resistance.

The DE governing the motion has been mentioned at the beginning of Chapter 10 in the textbook.

$\ddot{y} = -g$ (g is acceleration due to gravity.)

(The dot notation refers to differentiation with respect to the time t.)

Solve this equation on your own:

$$y(t) = \text{..........................}$$

--------------------▷ 77

$$y(t) = -g\frac{t^2}{2} + C_1 t + C_2$$

77

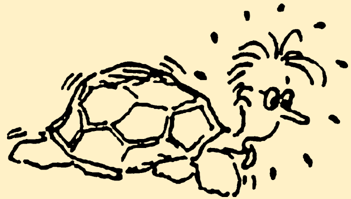
The subject matter we have been discussing required a fair amount of concentration, perhaps more than usual. If you study a certain amount each week, or within a given period, and recapitulate at the end of each section you will progress more rapidly than if you do not follow a definite study plan.

Before closing the study guide you should recall what you have just learnt. Write down the most important aspects of the subject matter and compare them with the textbook.

78

End of the first part of differential equations.

78



79

10.5 Solution by Substitution or by Trial

79

In this section we shall explain techniques to obtain or to guess particular solutions for frequent types of non-homogeneous DEs of the linear form.

READ: 10.3.2 Non-homogeneous linear DE
Solution by substitution or by trial
Textbook pages 287–292

80

The homogeneous linear DE of the second order can be solved algorithmically. On the other hand, certain types of non-homogeneous DEs are best solved by trial. 80

In the following frames we are concerned with finding the particular integral (PI) by trial, i.e. by assuming a function of the same type as $f(x)$ in the DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

Type 1: If $f(x)$ is a polynomial of degree n, i.e.

$$f(x) = a + bx + cx^2 + \cdots$$

then a trial solution for the PI is

$$y_{\mathrm{p}} = \text{..............................}$$

81

$$y_p = A + Bx + Cx^2 + \cdots$$

81

Obtain a PI of the DE

$$y'' - 5y' + 6y = x^2$$

$$y_p = \dots\dots\dots\dots\dots\dots\dots\dots$$

Solution found → 87

Further explanation and detailed solution wanted → 82

For the PI of $y'' - 5y' + 6y = x^2$ 82

we can try

$y_p = A + Bx + Cx^2$ as a solution,

i.e. a polynomial of the second degree.

Note: No intermediate power of x can be omitted, even if the right-hand side of the DE does not contain all powers.

Hence $y_p' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_p'' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

83

$$y_\mathrm{p}' = B + 2Cx$$
$$y_\mathrm{p}'' = 2C$$

83

Substituting in the DE $y'' - 5y' + 6y = x^2$ yields:

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = x^2$$

84

$$2C - 5B - 10Cx + 6A + 6Bx + 6Cx^2 = x^2$$ 84

To find the values of A, B and C that will satisfy the DE equate the coefficients of x^2, x and the constant terms.

Hence $C =$ ...., $B =$ ...., $A =$ ....

Solution found: 86

Further explanation needed: 85

85

Given: $2C - 5B - 10Cx + 6A + 6Bx + 6Cx^2 = x^2$

Equating coefficients of x^2: $6C = 1$,

therefore $C = \frac{1}{6}$

Equating coefficients of x: $-10C + 6B = 0$ (since there is no x on the RHS)

therefore $B = \frac{10}{6} \times \frac{1}{6} = \frac{5}{18}$

Constant term: $2C - 5B + 6A = 0$ (since there is no constant term)

Solving for A yields

$$A = \frac{1}{6}(5B - 2C) = \frac{1}{6}\left(\frac{25}{18} - \frac{1}{3}\right) = \frac{19}{108}$$

Thus we have $A =$ ...., $B =$ ...., $C =$ ....

86

$$C = \frac{1}{6}; \quad B = \frac{5}{18}; \quad A = \frac{19}{108}$$

86

Inserting into the trial solution

$$y_{\mathrm{p}} = A + Bx + Cx^2 \text{ yields}$$
$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

87

$$y_{\mathrm{p}} = \frac{19}{108} + \frac{5}{18}x + \frac{1}{6}x^2$$ 87

or

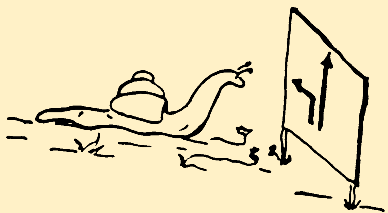
$$y_{\mathrm{p}} = \frac{1}{108}(18x^2 + 30x + 19)$$

Next problem:

Obtain a PI of the DE

$$y'' + 4y' + 5y = 3x - 2$$

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found ---------------------▷ 92

In difficulty; further explanation wanted ---------------------▷ 88

$f(x) = 3x - 2$ is a polynomial function.
What should your trial solution be, according to the textbook?

88

$$y_{\text{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

89

$$y_{\mathrm{p}} = A + Bx$$ 89

Note: If you try Bx alone as a solution it is not the general polynomial of the first degree.

Hence $y_{\mathrm{p}}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y_{\mathrm{p}}'' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

90

$$y_p' = B;\ y_p'' = 0$$

90

Substitute in the DE $y'' + 4y' + 5y = 3x - 2$

.... $= 3x - 2$

Equate coefficients; hence

$$A = \text{......................},\quad B = \text{......................}$$

Finally, the PI is

$$y_p = \text{......................}$$

PI found ▷ 92

Further hints ▷ 91

Substituting in the DE yields

91

$$4B + 5A + 5Bx = 3x - 2$$

Now we equate the coefficients of x and the constant terms:

$$B = \frac{3}{5} \quad (\text{since } 5Bx = 3x)$$
$$A = -\frac{22}{25} \quad (\text{since } 4B + 5A = -2)$$

Hence the PI is

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

92

$$y_p = \frac{1}{25}(15x - 22)$$ 92

Obtain a PI of the DE

$$y''' - y'' - 6y = x^2 - 3x - 2 \quad (Note\text{: } f(x) \text{ is a polynomial again.})$$

$$y_p = \text{..........................}$$

Solution found 97

Further explanation wanted 93

Given: $y''' - y'' - 6y = x^2 - 3x - 2$ 93

The fact that it is a third-order linear DE does not require any change of method in evaluating the PI, so find the trial solution first:

$$y_p = \text{..........................}$$

94

94

$$y_{\mathrm{p}} = A + Bx + Cx^2$$

Differentiate three times successively

$$y'_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$y''_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$y'''_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

95

$$y_p' = 2Cx + B, \quad y_p'' = 2C, \quad y_P''' = 0$$

95

We substitute in the DE: $y''' - y'' - 6y = x^2 - 3x - 2$ and get

$$0 - 2C - 6A - 6Bx - 6Cx^2 = x^2 - 3x - 2$$

Equate coefficients:
for x^2: ....
for x: ....
Constant term: ....

96

$-6C = 1$; therefore $C =$ ....

$-6B = -3$; therefore $B =$ ....

$-2C - 6A = -2$; therefore $A =$ ....

96

The trial solution was $y_p = A + Bx + Cx^2$

The PI is $y_p =$ ....

97

$$y_{\mathrm{p}} = \frac{7}{18} + \frac{1}{2}x - \frac{1}{6}x^2$$

97

Now we tackle the second type of function $f(x)$.

Given the DE

$$a_2 y'' + a_1 y' + a_0 y = f(x)$$

If $f(x) = C\mathrm{e}^{\lambda x}$ what would your trial solution be for the PI?

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

98

Trial solution: $y_p = A\,e^{\lambda x}$ 98

Since this function is to be a particular integral of the DE we have to find the values of A that will satisfy the equations.

Find the PI of the DE

$$y'' + 5\,y' - 14y = 2e^x$$

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found 100

Explanation wanted 99

Given the DE: $y'' + 5\,y' - 14y = 2\mathrm{e}^{x}$

We assume a solution of the exponential type, i.e. $y_{\mathrm{p}} = A\mathrm{e}^{x}$.

We need to find the value of the constant A that will satisfy the DE

99

$$y'' + 5\,y' - 14y = 2\mathrm{e}^{x}$$

If $y_{\mathrm{p}} = A\,\mathrm{e}^{x}$

then $y_{\mathrm{p}}' = A\,\mathrm{e}^{x}$ and $y_{\mathrm{p}}'' = A\,\mathrm{e}^{x}$

Substituting in the DE yields

$$A\,\mathrm{e}^{x} + 5\,A\mathrm{e}^{x} - 14\,A\mathrm{e}^{x} = 2\mathrm{e}^{x}$$

Dividing by e^{x} we have $A(1+5-14) = 2$

Therefore $A = -\dfrac{1}{4}$

The PI is $y_{\mathrm{p}} =$ ....

--------------------▷ 100

$$y_p = -\frac{1}{4}e^x$$

100

Find the PI of the DE

$$2y'' + 7y' - 15y = 3e^{2x}$$
$$y_p = \dots\dots\dots\dots\dots\dots\dots\dots$$

Solution found 102

Explanation wanted 101

Given: $2y'' + 7y' - 15y = 3e^{2x}$ 101

The DE is similar to the last one except that there is an e^{2x} instead of an e^{x}.

Hence for the PI we assume a solution

$$y_p = Ae^{2x}$$

Thus $y_p' = 2Ae^{2x}$ and $y_p'' = 2 \times 2Ae^{2x} = 4Ae^{2x}$

Substituting in the DE we find

$$(2 \times 4 + 7 \times 2(-15))Ae^{2x} = 3e^{2x}$$

Dividing by e^{2x} yields $(8 + 14 - 15)A = 3$

Therefore $A = \dfrac{3}{7}$

The PI is: $y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

- - - - - - - - - - - - - - - - - - - -▷ 102

$$y_p = \frac{3}{7}e^{2x}$$

102

The PI of the DE

$$a_2 y'' + a_1 y' + a_0 y = a e^{\lambda x}$$

is obtained by trying a solution

$$y_p = A e^{\lambda x}$$

This we have seen in the textbook and in the last two examples.

Can this method fail?

☐ Yes --------------------▷ 104

☐ No --------------------▷ 103

103

You are not right.

The method fails if λ is a root r of the auxiliary equation.

The auxiliary equation of the homogeneous DE is

$$a_2 r^2 + a_1 r + a_0 = 0$$

If λ is a root of the equation it follows that

$$a_2 \lambda^2 + a_1 \lambda + a_0 = 0$$

hence with substitution $y_\mathrm{p} = A\mathrm{e}^{\lambda x}$ we find that

$$(a_2 \lambda^2 + a_1 \lambda + a_0) A = a$$

i.e. $A = \dfrac{a}{a_2 \lambda^2 + a_1 \lambda + a_0} = \dfrac{a}{0}$ which is not defined.

We must, therefore, find a new trial solution.

--------------------▷ 104

Yes is the right answer. The method may fail. 104

If in the DE

$$a_2 y'' + a_1 y' + a_0 = a\mathrm{e}^{\lambda x}$$

λ is a root of the auxiliary equation, what would be a trial solution for the PI? (Consult the textbook, if necessary.)

$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

105

$$y_{\mathrm{p}} = Axe^{\lambda x}$$ 105

Find the PI of the DE

$$y'' + 2y' - 3y = 4e^{x}$$
$$y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found 107

Explanation wanted 106

The DE $y'' + 2y' - 3y = 4e^x$ 106

has the auxiliary equation $r^2 + 2r - 3 = 0$ whose roots are $r_1 = 1$, $r_2 = -3$.

The CF is $y_c = C_1 e^x + C_2 e^{-3x}$

The first term is the same as the RHS of the DE except for the constant, i.e. $4e^x$ is a part of the CF. We must try

$y_p = Axe^x$ for the PI.

Thus $y'_p = Axe^x + Ae^x$

and $y''_p = Axe^x + 2Ae^x$

Substituting in the DE we have

$$Ae^x(x + 2 + 2x + 2 - 3x) = 4e^x$$

Dividing by e^x and solving for A yields $A = 1$.

Therefore the PI is $y_p =$ ....

107

$$y_p = x e^x$$

107

The following DE occurs frequently in the study of forced oscillation with damping:

$$\ddot{x} + \xi \omega_n \dot{x} + \omega_n^2 x = F e^{j\omega t}$$

Find the PI and the value of its amplitude.

(Note the imaginary unit j. ξ, ω_n, ω are constants, t is the time.)

$x_p =$ ....

Amplitude = ...

Solution --------------------▷ 110

Explanation wanted --------------------▷ 108

The DE

$$\ddot{x} + \xi\omega_n\dot{x} + \omega_n^2 x = F\mathrm{e}^{\mathrm{j}\omega t}$$

108

is similar to the type we discussed at the beginning of this sequence, except that the independent variable is the time, hence the reason for using the dot notation.

$$f(x) = F\mathrm{e}^{j\omega t} \text{ is similar to } a\mathrm{e}^{bx} \text{ with } a = F \text{ and } bx = \mathrm{j}\omega t.$$

For the PI let

$$x_\mathrm{p} = A\mathrm{e}^{\mathrm{j}\omega t}$$

Thus

$$\dot{x}_\mathrm{p} = \mathrm{j}\omega A\mathrm{e}^{\mathrm{j}\omega t}$$

and $\ddot{x}_\mathrm{p} = -\omega^2 A\mathrm{e}^{\mathrm{j}\omega t}$, since $\mathrm{j}^2 = -1$

.

Substituting in the DE yields

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = F\mathrm{e}^{\mathrm{j}\omega t}$$

109

$$Ae^{j\omega t}(-\omega^2 + j\xi\omega_n\omega + \omega_n^2) = Fe^{j\omega t}$$

109

Rearranging yields

$$A = \frac{F}{\omega_n^2 - \omega^2 + j\xi\omega_n\omega}$$

Thus $x_p =$ ....

The fraction is the complex amplitude.

We remember that the modulus of a complex number

$$z = a + jb \text{ is } |z| = \sqrt{a^2 + b^2}$$

Obtain the modulus of the denominator:

$$\text{Amplitude} = \frac{|F|}{\dots\dots\dots\dots\dots\dots\dots\dots\dots}$$

110

$$x_{\mathrm{p}} = \frac{1}{\omega_n^2 - \omega^2 + \mathrm{j}\xi\omega_n\omega} F \mathrm{e}^{\mathrm{j}\omega t}$$

110

$$\text{Amplitude} = \frac{|F|}{\sqrt{(\omega_n^2 - \omega^2)^2 + \xi^2\omega_n^2\omega^2}}$$

In case of difficulty

108

Finally, we look at the third type of non-homogeneous DE.
Obtain the PI of the DE

$$y'' + 7y' + 10y = 20\cos 4x \qquad y_{\mathrm{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found

112

Detailed solution wanted

111

111

Given: $y'' + 7y' + 10y = 20\cos 4x$

If the RHS of a DE is of the form $R_1 \sin ax + R_2 \cos ax$

a trial solution is $A \sin ax + B \cos ax$.

Let the PI be $y_p = A \sin 4x + B \cos 4x$.

Thus $y_p' = 4A \cos 4x - 4B \sin 4x$ and $y_p'' = -16A \sin 4x - 16B \cos 4x$

Substituting in the DE we have

$-16B \cos 4x - 16A \sin 4x - 28B \sin 4x + 28A \cos 4x + 10B \cos 4x + 10A \sin 4x = 20 \cos 4x$

For this to be satisfied for all values of x, the coefficients of $\cos 4x$ must be the same on both sides of the equation; similarly those of $\sin 4x$. Hence we obtain two simultaneous equations:

$$-6B + 28A = 20$$

$$-28B - 6A = 0$$

Solving for A and B yields $A = \dfrac{28}{41}$ and $B = -\dfrac{6}{41}$

Hence the PI $y_p =$ ....

112

$$y_{\mathrm{p}} = -\frac{6}{41}\cos 4x + \frac{28}{41}\sin 4x$$ 112

Let us now consider another DE.

The charge Q in a particular electrical circuit is given by the DE

$$\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$$

Obtain a PI of this DE.

The current I in the circuit is $\dot{Q}$. It is the rate of change of the charge with time. Find the current:

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ------------------▷ 117

Explanation and detailed solution wanted ------------------▷ 113

To obtain a PI for the DE $\dot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$ 113

we assume a solution of the form $Q_{\mathrm{p}} = A\sin 2t + B\cos 2\mathrm{t}$

A and B must be chosen to satisfy the equation.

Differentiating Q we obtain

$$\dot{Q}_{\mathrm{p}} = 2A\cos 2t - 2B\sin 2t$$

and

$$\ddot{Q}_{\mathrm{p}} = -4A\sin 2t - 4B\cos 2t$$

By substituting in the DE and equating sine and cosine terms we have two equations:

....

....

Solution found for A and B ---> 115

Further explanation wanted ---> 114

114

The DE was

$$\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$$
$$Q_{\mathrm{p}} = A\sin 2t + B\cos 2t$$
$$\dot{Q}_{\mathrm{p}} = 2A\cos 2t - 2B\sin 2t$$
$$\ddot{Q}_{\mathrm{p}} = -4A\sin 2t - 4B\cos 2t$$

Substituting in the DE yields

$$\begin{aligned} &-4A\sin 2t - 4B\cos 2t + 4A\cos 2t - 4B\sin 2t \\ &+ 2A\sin 2t + 2B\cos 2t = 3\sin 2t \end{aligned}$$

Equating coefficients of sin $2t$ and cos $2t$ we find

$$\sin 2t(-4A - 4B + 2A) = 3\sin 2t$$
$$\cos 2t(-4B + 4A + 2B) = 0$$

The two equations to determine A and B are

....

....

115

$$-2A - 4B = 3$$
$$+4A - 2B = 0$$

115

Now we solve the equations, obtaining

$$A = -\frac{3}{10};\ B = -\frac{6}{10}$$

Therefore the PI for the DE is

$$Q_{\mathrm{p}} = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Remember that we tried a solution of the form

$$Q = A\sin 2t + B\cos 2t$$

116

$$Q_{\mathrm{p}} = -\frac{3}{10}\sin 2t - \frac{6}{10}\cos 2t$$ 116

It follows that for this particular solution the current I is

$$I = \frac{\mathrm{d}Q_{\mathrm{p}}}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots$$

117

$$\frac{\mathrm{d}Q_{\mathrm{p}}}{\mathrm{d}t} = I = \frac{6}{5}\sin 2t - \frac{3}{5}\cos 2t$$ 117

We have established a PI for the DE $\ddot{Q} + 2\dot{Q} + 2Q = 3\sin 2t$:

$$Q_{\mathrm{p}} = -\frac{3}{10}\sin 2t - \frac{6}{10}\cos 2t$$

Now we want to obtain the complete solution for the current I.

To do this we have to find the complementary function, i.e. I_{c}.

The complete solution consists of ....

118

The complete solution consists of the complementary function, and the particular integral. Since we have already found the PI we now require the CF. 118

The homogeneous DE is

$$\ddot{Q} + 2\dot{Q} + 2Q = 0$$

The auxiliary equation is

....

whose roots are

$$r_1 = \text{..........................}, \qquad r_2 = \text{..........................}$$

119

$$r^2 + 2r + 2 = 0$$
$$r_{1,2} = -\frac{2}{2} \pm \frac{1}{2}\sqrt{2^2 - 4 \times 2}$$

119

Therefore $r_1 = -1 + \mathrm{j}$, $r_2 = -1 - \mathrm{j}$

Hence Q_c = ...

The complete solution, I = ...

Solution found --------------------▷ 122

Further explanation wanted --------------------▷ 120

If r_1 and r_2 are the roots of the auxiliary equation then 120

$$
\begin{aligned}
Q_c &= A_1 e^{r_1 t} + A_2 e^{r_2 t} \\
&= A_1 e^{(-t+jt)} + A_2 e^{(-t-jt)} = e^{-t}(A_1 e^{jt} + A_2 e^{-jt})
\end{aligned}
$$

This is a complex quantity. We are interested in its real part:

$$Q_c = e^{-t}(C_1 \cos t + C_2 \sin t)$$

121

Since $I_c = \dfrac{dQ_c}{dt} = \dot{Q}_c$ 121

Differentiating Q_c we have

$$I_c = \dot{Q}_c = -e^{-t}(C_1 \cos t + C_2 \sin t) + e^{-t}(-C_1 \sin t + C_2 \cos t)$$
$$= e^{-t}[\cos t(C_2 - C_1) + \sin t(-C_1 - C_2)$$

or $I_c = e^{-t}(A \cos t + B \sin t)$
where $A = C_2 - C_1$, $B = -(C_1 + C_2)$
The complete solution is

$$I = I_c + I_p$$
$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

122

$$I = e^{-t}(A\cos t + B\sin t) + \frac{6}{5}\sin 2t - \frac{3}{5}\cos 2t$$ 122

Let us now discuss another difficult DE:

Obtain a PI for the DE

$$\ddot{x} + 4x = 3\cos 2t$$

$$x_p = \dots\dots\dots\dots\dots\dots\dots\dots$$

Solution found --------------------▷ 127

Hints and explanation wanted --------------------▷ 123

The auxiliary equation is $r^2 + 4 = 0$, i.e. $r = \pm 2\mathrm{j}$.

This is the breakdown situation, because of the factor 2 in cos $2t$.

In this case a trial solution is

$$x_\mathrm{p} = At\sin 2t + Bt\cos 2t$$

We need the second differential coefficient:

$$\ddot{x}_\mathrm{p} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

123

124

$$\ddot{x}_{\mathrm{p}} = 4A\cos 2t - 4B\sin 2t - 4At\sin 2t - 4Bt\cos 2t$$ 124

Substituting in the DE $\ddot{x} + 4x = 3\cos 2t$ for x_{p} and $\ddot{x}_{\mathrm{p}}$ leads to

.... $= 3\cos 2t$

125

$$4A\cos 2t - 4B\sin 2t = 3\cos 2t$$

125

Solving for A and B yields

$$A = \dots\dots\dots\dots, \quad B = \dots\dots\dots\dots$$

Thus the PI is

$$x_{\mathrm{p}} = \dots\dots\dots\dots\dots$$

Solution 127

Further hints wanted 126

We have $4A\cos 2t - 4B\sin 2t = 3\cos 2t$ 126

Comparing coefficients of sin $2t$ and cos $2t$ we find

$$4A = 3, \text{ hence } A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$-4B = 0, \text{ hence } B = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since we assumed $x_p = At\sin 2t + Bt\cos 2t$ we obtain the PI

$$x_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

127

$$A = \frac{3}{4};\; B = 0$$

$$x_{\mathrm{p}} = \frac{3}{4} t \sin 2t$$

127

You deserve a break!

TIME YOURSELF!

128

The break is over! 128

Before going any further you should recapitulate, if possible without reference to the text-book or the study guide.

If the $f(x)$ in the DE

$$a_2y'' + a_1y' + a_0y = f(x)$$

is of the form

(i) $a + bx + cx^2 + \cdots$
a trial solution for the PI is ....

(ii) Ce^{bx}
a trial solution for the PI is ....

(iii) $R_1 \sin ax + R_2 \cos ax$
a trial solution for the PI is ....

--------------------▷ 129

(i) $y_p = A + Bx + Cx^2 + \cdots$
(ii) $y_p = Ae^{\lambda x}$
(iii) $y_p = A\sin ax + B\cos ax$

129

Remember that in case (ii), if λ is a root of the auxiliary equation the trial solution should be

$$y_p = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

In case (iii), if a is a term of the CF the trial solution must be

$$y_p = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

130

(ii) $y_p = xAe^{\lambda x}$

(iii) $y_p = x(A\sin ax + B\cos ax)$

130

This section has been quite demanding. But DEs are important for understanding quite a number of basic topics in physics, economics and engineering.

Some remarks on motivation: 131

Straight on: 136

131

In a survey successful students were asked about the reasons for their success in studying.

Their report was as follows:

Good studying habits	38%
Interest	25%
Intelligence	15%
Other reasons	22%

Less successful students questioned in the survey gave the following reasons for their failure:

Lack of effort	25%
Lack of interest	35%
Personal problems	8%
Various reasons	32%

Lack of interest, lack of effort, inefficient studying habits are all connected.

There is a reason — a motive — behind every action. Motives also determine the intensity and the course of the learning process.

--------------------▷ 132

Two students are having an animated conversation about music and about the concerts they have been to recently. They exchange expert opinions on conductors and performances and compare the different interpreters' conceptions of Mozart's piano concertos. At the end they talk about their studies. Both deplore their memory. One reads chemistry, the other biology. 132

Which statement would you feel inclined to agree with?

The students have a selective memory. It is bad at chemistry and biology, but when it comes to music it is exceptionally good.

Both students have a normal memory. They are just more interested in music than in their studies.

133

That both students possess a good memory is probably an accurate claim. 133

Success in learning and motivation are connected.

The students' interest in music is a *primary motivation*. Spending time on music is the result of one's own personal desire. It is a satisfying activity.

Spending one's time on studying doesn't appear to be so much fun. Perhaps it is done only to enable one to earn a living later. In this case we are talking about *secondary motivation*.

As far as studying is concerned, secondary motivation is the more frequent of the two types. This can, however, be partly transformed into primary motivation when interest is awakened through study success. Many psychological studies have shown that the more time one spends on a subject and the more one understands about it, the more interesting it becomes.

Even studying mathematics can be interesting.

--------------------▷ 134

The proportion of primary or secondary motivation in the different fields of study differs from person to person. 134

The experience of success has a stimulating effect on both types of motivation.

Setting oneself attainable goals, while also keeping a check on oneself, can produce positive results. Quickly attainable goals like, for example, the mastering of easy exercises increase self-confidence and thus indirectly increase the chances of becoming interested.

Sometimes you have to decide between two activities. For example

(a) You must study a chapter.
(b) A friend calls and suggests you go swimming together.

Here, a conflict of motivation presents itself. Both activities are desirable but you can pursue only one.

What would you do?

135

Regardless of the answer you give, one thing can be assumed. For example, let's consider secondary-motivated activities. Here, in any conflict of motivation, the danger of opting for those activities which involve the least effort is far greater. 135

Someone who has to prepare a seminar paper may succumb to the temptation of washing the car, papering the living room or painting the furniture. Thus he avoids making the necessary effort and still retains the feeling of having done something that was necessary.

Difficult and unpleasant tasks which subjectively appear daunting can be changed into easier tasks if you break them down into smaller parts. That's what the study guide does for you. But you can do that for yourself, too. A work plan helps to break down difficult tasks into smaller steps.

Note the Chinese proverb:

Every long march begins with a first step.

--------------------▷ 136

10.6 Variation of Parameters

136

This section is not compulsory. It is supplementary and may be studied later on.

I would like to skip it.

--------------------▷ 141

I would like to work through it!

Variation of parameters is a systematic but possibly lengthy method to find a PI of a non-homogeneous DE.

READ: 10.3.2 Non-homogeneous linear DE
Method of variation of parameters
Textbook pages 287–292

--------------------▷ 137

Obtain the PI of the DE

$$y'' - 4y = x$$

using the method of variation of parameters.

Use the scheme given in the textbook.

137

$$y_{\mathrm{p}} = \dots\dots\dots\dots\dots\dots\dots\dots$$

Solution

140

Further explanation and detailed solution

138

Given the DE: $y'' - 4y = x$ 138

Two independent solutions of the CF are $y_1 = \mathrm{e}^{-2x}$ and $y_2 = \mathrm{e}^{2x}$

and the derivatives are $y_1' = -2\mathrm{e}^{-2x}$ and $y_2' = 2\mathrm{e}^{2x}$

The particular integral is assumed to be of the form $y_\mathrm{p} = V_1\mathrm{e}^{-2x} + V_2\mathrm{e}^{+2x}$.

Substituting in equations 3 and 6 of section 10.3.2 of the textbook yields the relations:

$$V_1'\mathrm{e}^{-2x} + V_2'\mathrm{e}^{2x} = 0 \qquad [1]$$

$$-2V_1'\mathrm{e}^{-2x} + 2V_2'\mathrm{e}^{2x} = x \qquad [2]$$

Solve [1] for V_1' and find $V_1' = -V_2'\mathrm{e}^{4x}$

Substitute in [2] and obtain

$$V_2' = \frac{x\mathrm{e}^{-2x}}{4} \qquad [3]$$

and

$$V_1' = -\frac{x\mathrm{e}^{2x}}{4} \qquad [4]$$

139

The functions V_1 and V_2 are obtained by integration of equations 3 and 4. 139

Hence

$$V_2(x) = \frac{1}{4}\int xe^{-2}\,dx \qquad (\text{let } u = x,\;\; v' = e^{-2x})$$
$$V_1(x) = -\frac{1}{4}\int xe^{2x}\,dx \qquad (\text{let } u = x,\;\; v' = e^{2x})$$

The solutions are

$$V_2(x) = \frac{e^{-2x}}{16}(-2x-1)$$
$$V_1(x) = -\frac{e^{2x}}{16}(2x-1)$$

Remembering that the PI is given by

$$y_p = V_1(x)y_1 + V_2(x)y_2$$

then by substituting y_1 and y_2 we have

$$y_p = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

140

$$y_p = -\frac{1}{4}x$$ 140

You should verify that $y_p = -\dfrac{1}{4}x$ is a particular integral of $y'' - 4y = x$.

You should always satisfy yourself of the correctness of your solution.

141

10.7 Boundary Value Problems

141

Objective: Evaluation of the arbitrary constants given certain boundary conditions.

READ: 10.4 Boundary value problems
Textbook pages 293–295

142

The DE $y' - 4y = 0$ has the general solution

$y = Ce^{4x}$

Obtain the particular solution of the DE given that when $x = \dfrac{1}{4}$, $y = 2e$.

142

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

143

$$y = 2e^{4x}$$ 143

Correct 145

Wrong, or explanation wanted 144

To obtain the particular solution of the DE $y' - 4y = 0$, we have to insert into the general solution $y = Ce^{4x}$ the boundary conditions given. 144

In this case the boundary conditions are those when $x = \frac{1}{4}$, $4y = 2e$. This means the curve of the solution must contain the point $\left(\frac{1}{4}, \quad 2e\right)$. Substituting in the general solution we have

$$2e = Ce^{4 \times 1/4} = Ce, \quad \text{hence } C = 2,$$

and the particular solution is $y = 2e^{4x}$.

145

The DE $\dot{v}(t) = -g$ has the general solution

$$v(t) = -gt + C$$

145

Obtain the value of the constant if $t = 0$ when $v(t) = v_0$.

$$C = \text{..........................}$$

146

$$C = v_0$$

Hence $v(t) = -gt + v_0$.

146

147

Now to boundary value problems for second-order DEs.

We know that we need two boundary conditions.

Obtain the particular solution of the DE

147

$$y'' - 3y' + \frac{9}{4}y = 0$$

given that (a) $x = \frac{2}{3}$ when $y = 3\mathrm{e}$

and (b) $x = \frac{2}{3}$ when $y' = \frac{15}{2}\mathrm{e}$

The general solution is

$$y = C_1 \mathrm{e}^{\frac{3}{2}x} + C_2 x \mathrm{e}^{\frac{3}{2}x}$$

$$C_1 = \ldots\ldots\ldots\ldots\ldots\ldots, \quad C_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Particular solution: $y = \ldots\ldots\ldots\ldots\ldots$

148

$$C_1 = 1,\ C_2 = 3$$
$$y = e^{3x/2} + 3xe^{3x/2}$$

148

Correct 150

Wrong, or further explanation wanted 149

We have two boundary conditions:

149

$$y\left(\frac{2}{3}\right) = 3\mathrm{e}$$
$$y'\left(\frac{2}{3}\right) = \frac{15}{2}\mathrm{e}$$

The general solution is:

$$y = C_1\mathrm{e}^{3x/2} + C_2 x\mathrm{e}^{3x/2}$$

The derivative is

$$y' = \frac{3}{2}C_1\mathrm{e}^{3x/2} + C_2\mathrm{e}^{3x/2} + \frac{3}{2}C_2 x\mathrm{e}^{3x/2}$$

Substituting the boundary conditions yields

$$3\mathrm{e} = C_1\mathrm{e} + \frac{2}{3}\mathrm{e}C_2 \quad \text{and} \quad \frac{15}{2}\mathrm{e} = \left(\frac{3}{2}C_1 + C_2(1+1)\right)$$

from which we get

$$C_1 = 1 \text{ and } C_2 = 3$$

150

The general solution of the DE $\ddot{x} = -g$ (motion of a freely falling body) is 150

$$x = -\frac{g}{2}t^2 + C_1 t + C_2$$

Obtain the values of the constants such that the equation satisfies the boundary conditions:
when $t = 0$, $x = 0$ and $\dot{x} = v_0$

$$x = \dots\dots\dots\dots\dots\dots$$

151

$$x = \frac{g}{2}t^2 + v_0 t$$

151

Solution: First condition: $x(0) = C_2 = 0$
Therefore $C_2 = 0$

Second condition: $\dot{x}(0) = C_1 = v_0$
Therefore $C_1 = v_0$

Further examples of using boundary conditions will be given in the following section on applications.

152

10.8 Application to Problems in Physics and Engineering

Objective: Solution of typical DEs encountered in practice.

This section shows typical applications of DEs in science. We will use the symbols generally used in practical applications.

READ: 10.5 Some applications of differential equations
Textbook pages 295–304

153

Growth and decay functions. 153

We frequently encounter in practice DEs whose solutions are of the form

$$y = \alpha e^{\beta t} \quad \text{or} \quad y = \alpha e^{-\beta t}$$

The first one shows that the quantity y increases exponentially with time whilst the second one shows that y decreases exponentially with time. They represent processes of 'growth' and 'decay' respectively, e.g. the growth of viruses and the decay of a radioactive substance. Other examples are to be found in electrical networks and in oscillations. The time t is not always the independent variable.

Develop the DE of a bacterial culture for which the rate of growth $\frac{d}{dt}N(t)$ is proportional to the actual number $N(t)$ of bacteria present. Call α the constant of proportionality.

....

-------------------▷ 154

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \alpha N(t)$$

154

or

$$\dot{N}(t) = \alpha N(t)$$

Now solve the equation $\dot{N}(t) = \alpha N(t)$

$$N(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

155

$$N(t) = C\mathrm{e}^{\alpha t}$$

155

Explanation of the solution:

$$\dot{N}(t) - \alpha N(t) = 0$$

This is a homogeneous equation of the first order with constant coefficients.

The auxiliary equation is $r - \alpha = 0$, root $r = \alpha$.

Hence the solution is

$$N = C\mathrm{e}^{\alpha t}$$

It may also be noted that in this case it is possible to separate the variables:

$$\frac{\mathrm{d}N}{N} = \alpha \mathrm{d}t$$

Integrating yields

$$\ln N = \alpha t + C$$

Solving for N we find $N = C\mathrm{e}^{\alpha t}$

--------------------▷ 156

Given: $N(t) = C\mathrm{e}^{\alpha t}$

If at $t = 0$, $N = 100$ bacteria are present, obtain the particular solution of the DE. 156

$$N(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: With $N = 100$ when $t = 0$ we have a boundary condition. Note that $\mathrm{e}^0 = 1$.

157

$$N(t) = 100e^{\alpha t}$$

157

Radioactive decay.

The decay of a sample of radium has been investigated.

If N is the total number of nuclei present in the sample at time t and $\mathrm{d}N$ is the number decaying in time $\mathrm{d}t$ then

$$\mathrm{d}N = -N\lambda \mathrm{d}t$$

or

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

which may be written as $\dot{N} + \lambda N = 0$, a first order DE with constant coefficients.

If N_0 is the number of nuclei when $t_0 = 0$, obtain the particular solution of the DE.

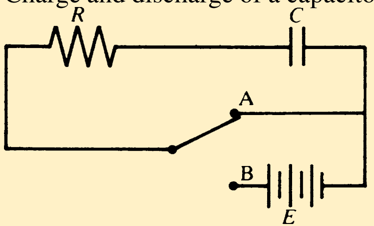
$$N = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

158

$$N = N_0 e^{-\lambda t}$$

158

Charge and discharge of a capacitor in an electrical circuit.



The figure shows a simple circuit consisting of a capacitor of C farads, a resistor of R ohms, two switches A and B and a battery having a constant voltage E.

Case (a): Initially the capacitor is not charged, i.e. the boundary condition is that when $t = 0$, $Q = 0$. (Q is the charge.)

When switch B is closed and A open a current flows in the circuit and the charge in the capacitor increases. The DE for the charge as a function of time is

$$R\dot{Q} + \frac{1}{C}Q = E$$

Solve the equation: i.e. obtain a particular solution

$$Q = \text{..................................}$$

--------------------▷ 159

$$Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$$

159

Correct --------------------▷ 162

Wrong, or detailed solution wanted --------------------▷ 160

The DE is $R\dot{Q} + \frac{1}{C}Q = E$. 160

We will solve this equation by using the method we have developed.

But it may be added that the method of separating the variables and direct integration is equally convenient in this case.

The solution will consist of two functions, namely the CF and the PI. The CF is the solution of

$$R\dot{Q} + \frac{1}{C}Q = 0$$

The CF is $Q_c = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: If you rewrite the equation using the substitutions $R = a_1$; $\frac{1}{C} = a_0$; $Q = y$ you will find the problem quite easy.

$\dashrightarrow$ 161

$$Q_{\mathrm{c}} = A\mathrm{e}^{-\frac{t}{RC}}$$

161

$\left(\text{You will have noted that the auxiliary equation is } r + \dfrac{1}{RC} = 0. \text{ The root is } r = -\dfrac{1}{RC}\right)$

The PI of $R\dot{Q} + \dfrac{1}{C}Q = E$

is $Q_{\mathrm{p}} =$ ....

162

$$Q_{\mathrm{p}} = EC$$

162

Explanation:

Let $Q_{\mathrm{p}} = B$, a constant, then $\dot{Q}_{\mathrm{p}} = 0$.

It follows that $0 + \dfrac{1}{C}B = E$.

Therefore $B = EC$ is a particular integral.

Now find the general solution using $Q_{\mathrm{c}} = A\mathrm{e}^{-\frac{t}{RC}}$; $\quad Q_{\mathrm{p}} = EC$:

$$Q = \text{..........................}$$

163

$$Q = A\mathrm{e}^{-\frac{t}{RC}} + CE$$

163

Now use the boundary condition ($t = 0$, $Q = 0$) to calculate A and hence the particular solution.

$$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

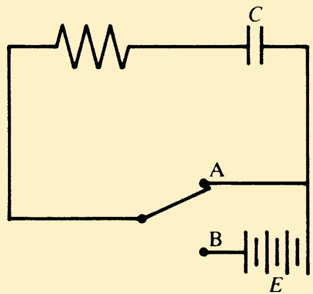
164

$$A = -EC$$

$$Q = EC\left(1 - e^{-\frac{t}{RC}}\right)$$

164

(Note that the charge Q tends to the value EC exponentially as $t \to \infty$.)



Case (b): Referring to the diagram, if the charge is Q_0 when $t = 0$ and switch A is closed (B open) a current will flow in the circuit and the capacitor will discharge.

The DE in this case is

$$R\dot{Q} + \frac{1}{C}Q = 0.$$

Obtain the particular solution

$$Q = \dots\dots\dots\dots\dots\dots$$

165

$$Q = Q_0 \mathrm{e}^{-\frac{t}{RC}}$$

165

When $t = RC$ by what percentage of its original value has the charge dropped?

166

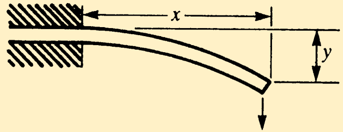
63%

(Explanation: When $t = RC$, $Q_1 = Q_0 e^{-1} \approx 0.37\,Q_0$. Hence the drop in the charge is $Q_0 - Q_1 = Q_0 - 0.37\,Q_0 = Q_0(1-0.37) = 0.63\,Q_0$.) 166

In your studies you will come across problems of this kind; the product RC is known as the 'time constant'.

Now for quite an interesting application.

The following DE occurred in a problem concerned with the deflection y of a rod.



$$y'' + 0.5y + 2.5\cos 0.8x = 0$$

Obtain the particular solution. The boundary conditions are

$$x = 0, \qquad y = 0, \qquad y' = 0.$$

$$y = \text{.........................}$$

Solution --------------------▷ 177

Further explanation wanted --------------------▷ 167

Rewriting the DE we have

$$y' + 0.5y = -2.5\cos 0.8x$$

The homogeneous equation is

....

167

168

$$y'' + 0.5y = 0$$

168

The auxiliary equation is ....

The roots of the auxiliary equations are

$$r_1 = \text{..........................}, \quad r_2 = \text{..........................}$$

Thus the complementary function is

$$y_c = \text{.................................}$$

169

The auxiliary equation is $r^2 = -0.5$.
The roots are $r = \pm\sqrt{-0.5} = \pm 0.707\mathrm{j}$.

169

CF: $y_\mathrm{c} = C_1 \mathrm{e}^{0.707\mathrm{j}x} + C_2 \mathrm{e}^{-0.707\mathrm{j}x}$
or $y_\mathrm{c} = C \sin(0.707x + \phi)$ (the real part)

For the PI of the DE $y'' + 0.5y = -2.5\cos 0.8x$
let $y_\mathrm{p} =$ ....

170

$$y_{\text{p}} = A \sin 0.8x + B \cos 0.8x$$ 170

The derivatives are

$$y'_{\text{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y''_{\text{p}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

171

$$y_p' = 0.8A\cos 0.8x - 0.8B\sin 0.8x$$
$$y_p'' = -0.64A\sin 0.8x - 0.64B\cos 0.8x$$

171

Substitute in the DE

$$y'' + 0.5y = -2.5\cos 0.8x$$

and obtain

$$\begin{aligned}&-0.64A\sin 0.8x - 0.64B\cos 0.8x\\&+0.5A\sin 0.8x + 0.5B\cos 0.8x = -2.5\cos 0.8x\end{aligned}$$

Solve for A and B:

$a =$ ...., $B =$ ....

Solution 173

Further explanation wanted 172

Given $-0.64A\sin 0.8x - 0.64B\cos 0.8x + 0.5A\sin 0.8x + 0.5B\cos 0.8x = -2.5\cos 0.8x$ 172

To solve for A and B equate the coefficients of the sine and cosine terms on both sides of the equation. You should obtain

$-0.64A + 0.5A = 0;$ therefore $A =$ ....

$-0.64B + 0.5B = -2.5;$ therefore $B =$ ....

173

$$A = 0, \qquad B = \frac{2.5}{0.14} = 17.86$$ 173

The general solution of the DE

$y'' + 0.5y' + 2.5\cos 0.8x = 0$ (using $y_c = C\sin(0.707x + \phi)$

and $y_p = 17.86\cos 0.8x$)

is $y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

174

$$y = y_c + y_p = C\sin(0.707x + \phi) + 17.86\cos 0.8x$$ 174

To calculate C and ϕ, the two arbitrary constants, substitute the boundary conditions in the general equation. The boundary conditions are:

when $x = 0$, y and y' are both zero

$$y' = \ldots\ldots\ldots\ldots\ldots\ldots$$

(i) When $x = 0$, $y = 0$; hence $0 = \ldots\ldots\ldots\ldots\ldots\ldots$

(ii) When $x = 0$, $y' = 0$; hence $0 = \ldots\ldots\ldots\ldots\ldots\ldots$

175

$$y' = 0.707C\cos(0.707x+\phi) - 17.86 \times 0.8\sin 0.8x$$

175

(i) $0 = C\sin\phi + 17.86$

(ii) $0 = 0.707C\cos\phi$

Solve for C and ϕ

$C = \ldots\ldots\ldots\ldots\ldots\ldots$, $\phi = \ldots\ldots\ldots\ldots\ldots\ldots$

176

$$\phi = \frac{\pi}{2},\ C = -17.86$$

176

Explanation: From (ii) $C \cos\phi = 0$, from (i) we know $C \neq 0$. Hence $\cos\phi = 0$. Therefore $\phi = \frac{\pi}{2}$ is a possible solution.

Substitution in (i) yields: $C = -17.86$

The particular solution is

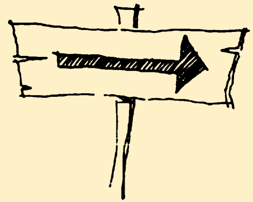
$$y = \text{..........................}$$

177

$$y = 17.86(\cos 0.8x - \sin(0.707x + \frac{\pi}{2}))$$

177

Have a 10 minute break to relax and collect your thoughts before you continue with the last example of applications. Time yourself!



178

The following DE arose while the performance of an anti-vibration mounting was being investigated: 178

$$\ddot{x} + 60\dot{x} + 3600x = 150\sin 65t$$

x is the displacement, t is the time.

The boundary conditions are:

when $t = 0$, $x = 0$ and $\dot{x} = 0$

Solve the equation.

Make sure that you carry out the solution step by step: above all, don't use short-cuts because they do not save time in the long run, and you are more likely to make mistakes. Follow a logical sequence by drawing up an algorithm, if you wish, and following it. Switch on your calculator, you will need it.

$$x = \text{..........................}$$

Solution --------------------▷ 183

Detailed solution wanted --------------------▷ 179

Here is the detailed solution to check against your own:

179

Step 1: The DE and boundary conditions: $\ddot{x}+60\dot{x}+3600x=150\sin 65t$

$$x=0,\quad \dot{x}=0 \text{ when } t=0$$

Step 2: Solution of the homogeneous DE: $\ddot{x}+60\dot{x}+3600x=0$
The auxiliary equation is: $r^2+60r+3600=0$

Its roots are $r_{1,2}=-\frac{60}{2}\pm\frac{1}{2}\sqrt{60^2-4\times 3600}=-30\pm\sqrt{3}\times 30\mathrm{j}$

i.e. $r_1=-30+51.96\mathrm{j}$, $\quad r_2=-30-51.96\mathrm{j}$.

The CF is $x_c=C\,\mathrm{e}^{-30t}\cos(51.96t-\phi)$.
C and ϕ are two arbitrary constants.

180

Step 3: Obtain the PI

180

$$\text{Let } x_p = A\sin 65t + B\cos 65t$$
$$\dot{x}_p = 65A\cos 65t - 65B\sin 65t$$
$$\ddot{x}_p = -65^2 A\sin 65t - 65^2 B\cos 65t$$

We substitute the expressions x_p, $\dot{x}_p$ and $\ddot{x}_p$ in the DE and get

$$\begin{aligned}&-4225A\sin 65t - 4225B\cos 65t\\ &+65\times 60A\cos 65t - 65\times 60B\sin 65t\\ &+3600A\sin 65t + 3600B\cos 65t = 150\sin 65t\end{aligned}$$

To calculate the constants A and B equate the coefficients of sin $65t$ and cos $65t$ on both sides of the equation.

This gives (1) ....

(2) ....

181

$$(1) \quad -625A - 3900B = 150$$
$$(2) \quad -625B + 3900A = 0$$

181

Determination of A and B:

$$A = -0.006$$
$$B = -0.0375$$

Hence the PI is $x_p =$ ....

182

$$x_p = -0.006\sin 65t - 0.0375\cos 65t$$ 182

Step 4: The general solution is $x = x_c + x_p$; i.e.

$$x = Ce^{-30t}\cos(51.96t - \phi) - 0.006\sin 65t - 0.0375\cos 65t$$

Step 5: Boundary conditions: At $t = 0$, $x = 0$, $\dot{x} = 0$.

This gives for the first condition $0 = C\cos\phi - 0.0375$. [1]

Differentiating the general solution yields

$$\dot{x} = -51.96Ce^{-30t}\sin(51.96t - \phi) - 30Ce^{-30t}\cos(51.96t - \phi) - 0.39\cos 65t + 2.4375\sin 65t$$

This gives for the second condition $\dot{x} = 0$, $t = 0$.

$$0 = 51.96C\sin\phi - 30C\cos\phi - 0.39 \quad [2]$$

(Remember $\sin(-\phi) = -\sin\phi$, $\cos(-\phi) = \cos\phi$.)
Solving equations 1 and 2 we find $C = 0.0475$, $\phi = 0.66$ radians.

Step 6: Write down the particular solution
$x =$ ....

→ 183

$$x = 0.0475e^{-30t}\cos(51.96t - 0.66) - 0.006\sin 65t - 0.0375\cos 65t$$ 183

Straight on

186

You may need a break now. But reflect for a moment on the aims of the study guide. There are two fundamental aims.

First aim: To impart mathematical knowledge for application to physics and engineering problems.
Second aim: To promote your study skills and your ability to use written texts.

The first aim need not be discussed. The second aim is worth commenting on. Study skills and the ability to use written texts are important for you.

The advanced student relies even more than the beginner on his or her ability to use written sources of information.

184

The promotion of your study skills by the study guide is important because once you have finished with the guide you will have to control your learning process by yourself. Thus you should understand the control techniques recommended by the study guide. 184

The study guide tries to build up confidence in your own ability by means of examples of varying difficulty and by giving you hints or part solutions where necessary, or by directing you back to sections in the textbook.

You therefore adopt the habit of reading the textbook carefully and of controlling your progress.

After each section you are asked simple questions about the new concepts, and even their names. Then you are encouraged to learn how to test yourself, beginning with learning by rote and continuing with active problem solving.

--------------------▷ 185

Some special techniques have been outlined and practised by the guide: 185

— how to break great learning tasks into bits which can be handled more easily,
— how to use different reading techniques; intensive reading combined with note taking versus selective or orientational reading,
— how to make the best use of breaks,
— how to control learning progress,
— how to make learning more active by using self-testing techniques.

186

10.9 General Linear First-Order DEs

186

The following sections offer in a more concise fashion some additional methods of solving certain types of DE. For beginners it may be advisable to skip the rest of this chapter during a first course and to return to it when the need arises.

Objective: Determination of the general solution of *any* linear first-order differential equation (i.e. the coefficients are not necessarily constants).

READ: 10.6 General linear first-order DEs
Textbook pages 304–308

-------------------▷ 187

We are now going to illustrate the straightforward method of solving a linear first-order differential equation using the integrating factor. In case of difficulties consult the textbook. 187

Given: $xy' + y - x^2 = 0$
First identify the coefficients:

$$p(x) = \ldots\ldots\ldots\ldots\ldots\ldots, \quad q(x) = \ldots\ldots\ldots\ldots\ldots\ldots, \quad f(x) = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 188

$$p(x) = x, \quad q(x) = 1, \quad f(x) = x^2$$

188

Write down the formula for the integrating factor in terms of $p(x)$, $q(x)$ and $f(x)$:

$$I(x) = \dots\dots\dots\dots\dots\dots$$

Now let us solve the differential equation under consideration:

$$xy' + y - x^2 = 0$$

We follow the steps described in section 10.6.2.

Step 1: Determine the integrating factor $I(x) = \dots\dots\dots\dots\dots\dots$

I need some help 189

Solution 190

The general formula for the integrating factor $I(x)$ is: 189

$$I(x) = e^{\int \frac{q(x)}{p(x)}\,dx}$$

It must be computed for $p(x) = x, \quad q(x) = 1.$

Compute $\displaystyle\int \frac{q(x)}{p(x)}\,dx = \ldots\ldots\ldots\ldots\ldots\ldots,\quad I(x) = \ldots\ldots\ldots\ldots\ldots\ldots$

190

$$I(x) = \mathrm{e}^{\int \frac{q(x)}{p(x)}\,\mathrm{d}x}$$
$$I(x) = \mathrm{e}^{\int \frac{1}{x}\,\mathrm{d}x} = \mathrm{e}^{\ln|x|} = |x|$$

190

(As an aside, the constant of integration is of no significance for our purposes.)

Step 2: The following must be evaluated:

$$y(x) = \frac{1}{I(x)} \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x$$

We know $p(x) = x, \quad q(x) = 1, \quad f(x) = x^2, \quad I(x) = |x|.$

$\frac{1}{I(x)} = \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x =$ ....

-------------------▷ 191

$$y(x) = \frac{1}{I(x)} \int \frac{I(x)}{p(x)} f(x)\,\mathrm{d}x = \frac{1}{x}\left(\frac{x^3}{3} + C\right)$$
$$= \frac{x^2}{3} + \frac{C}{x}$$

191

Let us tackle another first-order linear differential equation by the same method:

$$(x+1)y' + y = (x+1)^2$$

Preliminary step:

$$p(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad q(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Step 1:

$$\int \frac{q(x)}{p(x)}\,\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad I(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Step 2:

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

192

$$p(x) = x+1, \;\; q(x) = 1, \;\; f(x) = (x+1)^2$$
$$\int \frac{q(x)}{p(x)}\,\mathrm{d}x = \ln|x+1|, \;\; I(x) = |x+1|$$
$$y(x) = \frac{1}{x+1}\int (x+1)^2\,\mathrm{d}x = \frac{1}{3}(x+1)^2 + \frac{C}{x+1}$$

192

Can every linear first-order differential equation be solved using the integrating factor?

☐ Yes --------------------▷ 193

☐ No --------------------▷ 195

You are too optimistic! Try to solve:

193

$$\ln|x|\, y' + \frac{1}{x}\, y = \frac{1}{(\ln|x|)^2}$$

$p(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\quad q(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\quad f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,$

$$\int \frac{q(x)}{p(x)}\, dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots,\quad I(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(Hint\colon \int \frac{dx}{x \ln|x|} = \int \frac{(\ln|x|)'}{\ln|x|}\, dx \right)$$

194

$$p(x) = \ln|x|, \quad q(x) = \frac{1}{x}, \quad f(x) = \frac{1}{(\ln|x|)^2}$$

$$\int \frac{q(x)}{P(x)} \mathrm{d}x = \int \frac{\mathrm{d}x}{x \ln|x|} = \ln(\ln|x|), \; I(x) = \ln|x|$$

194

We have succeeded in completing the first step. But we encounter difficulties during the second step. We must evaluate:

$$y(x) = \frac{1}{I(x)} \int \frac{I(x)}{p(x)} f(x) \mathrm{d}x = \frac{1}{\ln|x|} \int \frac{\mathrm{d}x}{(\ln|x|)^2}$$

But this integral cannot be solved at all by elementary methods!

195

You are right. *In practice*, not all linear first-order differential equations can be solved by this method. The reason is that there is no guarantee that the necessary integrations can be performed. A solution does exist, but it may not be soluble otherwise than by numerical means. 195

Here is one last linear first-order differential equation for you to solve:

$$(\sin x)y' + (\cos x)y = \cos^2 x$$

Work through the necessary steps and write down the solution.

$$y(x) = \text{..........................}$$

Solution --------------------▷ 197

I need some help! --------------------▷ 196

Given: $(\sin x)y' + (\cos x)y = \cos^2 x$

196

Determine: $p(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $q(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,
$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Compute: $\int \frac{q(x)}{p(x)}\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$, $\int \frac{I(x)}{p(x)} f(x)\mathrm{d}x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(*Hints*: You can look up both integrals in the table of the textbook at the end of Chapter 6. Recall: $\frac{\cos x}{\sin x} = \cot x$.)

$$y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 197

$$p(x) = \sin x, \quad q(x) = +\cos x, \quad f(x) = \cos^2 x$$
$$\int \frac{q(x)}{p(x)}\mathrm{d}x = +\ln|\sin x|, \quad \int \frac{I(x)}{p(x)} f(x)\mathrm{d}x = \frac{x}{2} + \frac{\sin x \cos x}{2} + C$$
$$y(x) = \frac{x}{2\sin x} + \frac{\cos x}{2} + \frac{C}{\sin x}$$

197

Further examples of first-order linear differential equations are given in the exercises in the textbook.

-------------------▷ 198

10.10 Some Remarks on General First-Order DEs

198

Bernoulli type DEs are new but you have encountered the separation of variables before. It is a method which is quite straightforward — provided it can be applied. This is the case if the DE can be rearranged in such a way that one variable only is found on the RHS and the other on the LHS.

READ: 10.7.1 Bernoulli's equations
10.7.2 Separation of variables
Textbook pages 308–310

199

Which of the following differential equations can be written in the form of a Bernoulli differential equation? 199

(1) $y' + xy = (\cos x)y$
(2) $xy' + y = x^2y^2$
(3) $y' + x^2y = xe^y$

200

Only equation (2) is of the Bernoulli type. Equation (1) looks like a Bernoulli equation, but it is in fact linear: 200

$$y' + (x - \cos x)y = 0$$

The substitution $u = y^{1-n}$ would not work in that case ($n = 1$), and, in fact, it is not necessary.

Now try solving differential equation (2) written in normal form:

$$y' + \frac{y}{x} = xy^2$$

$$n = \text{..........................},\ u = \text{..........................},\ u' = \text{..........................}$$

Substitute into the given differential equation to obtain an equation for $u(x)$: ....

--------------------▷ 201

$$n = 2, \quad u = y^{-1}. \quad \text{Thus } uy^2 = y.$$
$$u' = -y'y^{-2}. \quad \text{Thus } -u'y^2 = y'.$$

201

Differential equation for $u(x)$:

$$u' - \frac{1}{x}u = -x$$

Solve the differential equation above for u: $u(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Resubstitute: $y(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

202

$$u = x\int \frac{1}{x}(-x)\,\mathrm{d}x = x\,(C - x) = Cx - x^2$$

$$y = \frac{1}{Cx - x^2}$$

202

In the general Bernoulli equation are fractional values for n admissible?

☐ Yes

☐ No

203

Yes, indeed!

203

Reduce the following differential equation to a linear one:

$$y' + 2xy = x\sqrt[4]{y}$$

$$n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad u(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now the linear differential equations reads: $\frac{4}{3}u' + \ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots$

204

$$n = \frac{1}{4}$$
$$u = y^{3/4}$$
$$\frac{4}{3}u' + 2xu = x$$

204

Solve this linear first order equation using the method described in 10.6.2 (integrating factor):

$$u(x) = \dots\dots\dots\dots\dots\dots\dots\dots$$

Then resubstitute to find $y(x) = \dots\dots\dots\dots\dots\dots\dots\dots$

205

$$u(x) = \frac{1}{2} + C\mathrm{e}^{-\frac{3}{4}x^2}$$ 205

$$y = \sqrt[3]{\left(\frac{1}{2} + C\mathrm{e}^{-\frac{3}{4}x^2}\right)^4}$$

Now on to the last Bernoulli differential equation of this chapter! Solve:

$$y' + 2xy + xy^4 = 0$$

$n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$u(y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 206

$$n = 4, \quad u = y^{-3}, \quad u' - 6xu = 3x$$
$$u(x) = -\frac{1}{2} + C\mathrm{e}^{3x^2}, \; y = \frac{1}{\sqrt[3]{C\mathrm{e}^{3x^2} - \frac{1}{2}}}$$

206

Let us now turn to another technique. Solve:

$$(1 + x^2)\, y' - xy^2 = 0$$

I need some help! 207

Solution 208

Try to solve the differential equation by separation of the variables:

207

$$(1+x^2)\,y' - xy^2 = 0$$

Rewrite it in the following form with the variables separated:

$$p(y)\mathrm{d}y = -q(x)\,\mathrm{d}x$$

Then integrate to solve the differential equation.

208

$$(1+x^2)y' = xy^2$$

$$\frac{\mathrm{d}y}{y^2} = \frac{x}{1+x^2}\,\mathrm{d}x$$

Integration:

$$-\frac{1}{y} = \frac{\ln(x^2+1)}{2} + C_1$$

$$y = \frac{-2}{\ln(C\,(x^2+1))}$$

208

Here is another differential equation which you should be able to solve by separating the variables:

$$y' + xy' + y = 1$$

$$y(x) = \dots\dots\dots\dots$$

209

$$y(x) = \frac{C}{x+1} + 1$$

209

(Note that the LHS of the given DE $= [(x+1)y]'$.)

This is the end of your work with Chapter 10 for the time being. The succeeding sections of the textbook require some knowledge of partial derivatives which are treated in Chapter 12. Having worked through Chapter 12 (Functions of several variables, partial derivatives) you should return to Chapter 10 on differential equations.

Then read: **10.7.3 Exact equations**
10.7.4 The integrating factor general case
10.8 Simultaneous DEs
Textbook pages 309–318
10.9 Higher order DEs
10.10 Some advice on intractable DEs

We shall not give exercises for sections 10.7.3 and 10.7.4 in the programmed study guide. By now you know well how to proceed on your own. Work through each section in the textbook and try to solve at least one problem posed in the corresponding exercises at the end of Chapter 10. In case of difficulties when solving differential equations try to copy exactly the procedure shown in the examples in the textbook.

210

Try to solve the following simultaneous differential equations using the first method discussed in section 10.8; t is the independent variable. 210

Solve for x:

$$2\dot{x} + 3x - y = 0 \qquad [1]$$

$$3\dot{y} + 10x - 4y = 0 \qquad [2]$$

First differentiate equation 1:

$$\dots\dots\dots\dots = 0 \qquad [3]$$

211

$$2\ddot{x} + 3\dot{x} - \dot{y} = 0 \qquad [3]$$

211

Given were:

$$2\dot{x} + 3x - y = 0 \qquad [1]$$
$$3\dot{y} + 10x - 4y = 0 \qquad [2]$$

Solve for y from [1] and for $\dot{y}$ using [3]:

$$y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\dot{y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Substitute for y and $\dot{y}$ in [2]. You should then obtain a DE for x.
....

212

$$y = 2\dot{x} + 3x$$
$$\dot{y} = 2\ddot{x} + 3\dot{x}$$
$$6\ddot{x} + \dot{x} - 2x = 0$$

212

Now we must solve the differential equation $6\ddot{x} + \dot{x} - 2x = 0$.

The auxiliary equation is ....

Its roots are $r_1 =$ ...., $r_2 =$ ....

The solution for x is

$$x = \text{..........................}$$

Substituting in the equation $y = 2\dot{x} + 3x$

we get the solution:

$$y = \text{..........................}$$

213

$$6r^2 + r - 2 = 0$$
$$r_1 = \frac{1}{2}, \quad r_2 = -\frac{2}{3}$$
$$x = Ae^{\frac{1}{2}t} + Be^{-\frac{2}{3}t}$$
$$y = 4Ae^{\frac{1}{2}t} + \frac{5}{3}Be^{-\frac{2}{3}t}$$

213

All correct

215

I need help and explanation

214

Go carefully through example 1 in section 10.8. To solve for x and y as functions of t from the equations we eliminate y first in order to obtain a differential equation in $x = x(t)$ only. Although the two equations are of the first order, the elimination process leads to a second order DE which we can solve by the exponential method we studied previously. Hence we get a solution for x. To obtain a solution for y we substitute for x and $\dot{x}$ in equation [1]: 214

$$2\dot{x} + 3y - y = 0 \qquad [1]$$

$$x = Ae^{\frac{1}{2}t} + Be^{-\frac{2}{3}t}$$
$$\dot{x} = \frac{1}{2}Ae^{\frac{1}{2}t} - \frac{2}{3}Be^{-\frac{2}{3}t}$$

The solution is

$$y = 4Ae^{\frac{1}{2}t} + \frac{5}{3}Be^{-\frac{2}{3}t}$$

215

As our final example:
Solve

215

$$\dot{x} = 2x + 3y$$
$$\dot{y} = 2x + y$$

by the second method of section 10.8.

Let $x = \dots\dots\dots\dots\dots\dots\dots\dots,\quad y = \dots\dots\dots\dots\dots\dots\dots\dots$
$\dot{x} = \dots\dots\dots\dots\dots\dots\dots\dots,\quad \dot{y} = \dots\dots\dots\dots\dots\dots\dots\dots$

216

$$x = a\mathrm{e}^{rt}, \quad y = b\mathrm{e}^{rt}$$
$$\dot{x} = ra\mathrm{e}^{rt}, \quad \dot{y} = rb\mathrm{e}^{rt}$$

216

Substitute in the DE:

....

....

217

$$
\begin{aligned}
&(r-2)a-3b=0, &&\text{i.e. } (r-2)a=3b \\
&-2a+(r-1)b=0, &&\text{i.e. } (r-1)b=2a
\end{aligned}
$$

217

Eliminate a and b : $(r-2)(r-1)ab=6ab$, i.e. $(r-2)(r-1)-6=0$

Then solve for r:

$$r_1 = \text{.........................}, \quad r_2 = \text{.........................}$$

218

$$r_1 = -1, \quad r_2 = 4$$

218

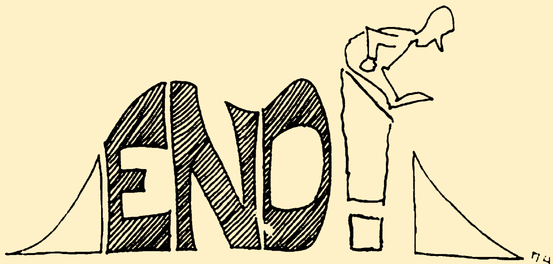
The solutions are

$$x = \text{.........................}$$
$$y = \text{.........................}$$

219

$$x = a_1 e^{-t} + b_1 e^{4t}$$
$$y = a_2 e^{-t} + b_2 e^{4t}$$

This chapter on differential equations has been demanding. But if you worked through it carefully you will have gained enough experience to solve many of the DEs which you will encounter later on.



END OF CHAPTER 10

Chapter 11
Laplace Transforms

0

$\dashrightarrow$ $\boxed{1}$

Laplace transforms

1

Objective: In the first section the definition of Laplace transforms will be explained and developed. It will prove useful, to copy all definitions and results into your notebook when going through this and subsequent sections. Then they will be readily accessible to you and you will not have to take recourse to the textbook for each and every single computation.

READ **11.1 Introduction**
11.2 The Laplace transform definition
Textbook pages 321–322

When done ------------------▷ 2

Write down the definition of the Laplace transform:

2

$\mathcal{L}[f(t)] =$ ....

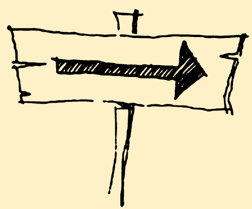
Being well versed in the notation greatly eases not only studying the textbook but to an even greater extent eases later practice.

Write down the same definition using different notations

$\mathcal{L}[y(t)] =$ ....

$\mathcal{L}[f(x)] =$ ....

$\mathcal{L}[y(x)] =$ ....



--------------------▷ 3

3

$$\mathcal{L}\left[f(t)\right] = \int_0^{\infty} f(t)\cdot e^{-s\cdot t}dt = \bar{f} \qquad \text{or } F(s)$$

$$\mathcal{L}\left[y(t)\right] = \int_0^{\infty} y(t)\cdot e^{-s\cdot t}dt = \bar{y} \qquad \text{or } Y(s)$$

$$\mathcal{L}\left[f(x)\right] = \int_0^{\infty} f(x)\cdot e^{-s\cdot x}dx = \bar{f} \qquad \text{or } F(s)$$

$$\mathcal{L}\left[y(x)\right] = \int_0^{\infty} y(x)\cdot e^{-s\cdot x}dx = \bar{y} \qquad \text{or } Y(s)$$

Complete the definition

$\mathcal{L}^{-1}\left[F(s)\right] =$..............................

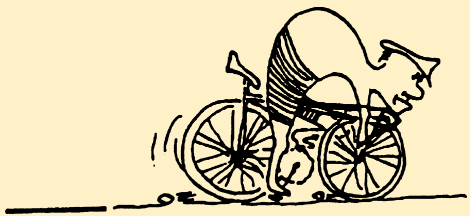
------------------▷ 4

The symbol $\mathcal{L}^{-1}\left[F(s)\right]$ stands for the inverse Laplace transform.

4

$\mathcal{L}^{-1}\ f(s) = F(t)$ or $F(x)$ in case we use the variable x.

Before going through examples that will show the extreme usefulness of Laplace transforms, you will have to stand a certain number of dry spells, while you take in the rules for the transforms and copy them into your notebook.



------------------ ▷ 5

11.1 Laplace transform of standard functions and general theorems

5

Objective: In this section you will acquaint yourself with the main facts on how to transform basic functions like e^{at}, $\sin \omega t$, $C \cdot t$, etc. Funnily enough, t^n will have to wait for a while.
Please follow all arguments, and copy all calculations and results into your notebook.
Since this section is quite long we suggest you study it in parts. In this section we will use the following notation: few denote the original function by $y(t)$ or $f(x)$.

Study **11.3 Laplace transform of standard functions**
Theorem I: The shift Theorem
Textbook pages 322–324

When done ------------------▷ 6

Given the constant function $y(t)$
Try to derive its Laplace transform $\bar{y}(s)$ on your own.
$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

6

Solution found ------------------▷ 8

Help needed ------------------▷ 7

7

Given the constant function $y(t) = C$
Wanted: Laplace transform $\bar{y}(s)$.
You have to solve

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot y(t) \cdot dt$$

For $y(t) = C$ we insert C into the integral:

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot C \cdot dt$$

Factor the constant C, solve the integral, and insert the limits of integration:

$$\bar{y}(s) = \int_0^\infty e^{-st} \cdot C \cdot dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 8

$$\bar{y}(s) = C \cdot \int_0^\infty e^{-st} dt$$

$$\bar{y}(s) = C\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{C}{s}$$

8

Likewise we can determine the Laplace transform of an exponential function.

Given: $y(t) = e^{at}$

To do: Compute $\bar{y}$:

$\bar{y} =$.......................................

Computation successfully done ------------------▷ 10

Help needed ------------------▷ 9

9

Given: $y(t) = e^{at}$
Wanted: Laplace transform $\vec{y}$.

Again, you must compute the Laplace transform by evaluating an integral.

$$\mathcal{L}[y(t)] = \vec{y} = \int_0^\infty e^{-st} \cdot y(t) \cdot dt$$

Insert $y(t)$ as given:

$$\mathcal{L}[y(t)] = \vec{y}(s) = \int_0^\infty e^{-st} \cdot e^{at} \cdot dt$$

By factoring t in the exponent, we get an integral, which we have already solved frequently.
Lastly, we insert the limits of integration.

$$\mathcal{L}[y(t)] = \vec{y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In case of remaining difficulties consult the textbook.

------------------▷ 10

$$\bar{y} = \int_0^\infty e^{-(s-a)} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty = \frac{1}{(s-a)}$$ 10

The Laplace transforms of trigonometric functions are obtained by a direct approach, too.
Let us start with the sine function:

$$y(t) = \sin \omega t$$
$$\mathcal{L}[\sin \omega t] = \ldots\ldots\ldots\ldots\ldots$$

Hint: Use Euler's formula to express $\sin \omega t$ by exponential functions.

Solution found ------------------▷ 13

Help needed ------------------▷ 11

The Laplace transform of the sine function $y(t) = \sin \omega t$ is to be found. 11
We already know the transformation of the exponential function

$$\mathcal{L}\left[e^{at}\right] = \bar{y}(s) = \frac{1}{s-a}$$

Hint: You should have taken a note of this result before, in order to quickly have access to it.

We recall Euler's formula

$$e^{i\omega t} = i \sin \omega t + \cos \omega t$$

$$e^{-i\omega t} = -i \sin \omega t + \cos \omega t$$

So, $\sin \omega t$ can be expressed as a difference of exponential functions.

$$e^{i\omega t} - e^{-i\omega t} = \ldots\ldots\ldots\ldots$$

------------------ ▷ 12

$$e^{i\omega t} - e^{-i\omega t} = 2i \sin \omega t$$

12

This implies $\sin \omega t = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

Since we know the transform of an exponential function, the problem is almost solved. A little bit of calculation is all that remains.

We know:

$$\mathcal{L}\left[e^{at}\right] = \bar{y}(s) = \frac{1}{s-a}$$

By inserting we obtain:

$$\mathcal{L}\left[\sin \omega t\right] = y(s) = \frac{1}{2i}\left[\frac{1}{s - i\omega} - \frac{1}{s + i\omega}\right]$$

Using the common denominator for the fractions we get:

$\bar{y}(s)$=…………………

-------------------▷ 13

$$\mathcal{L}[\sin \omega t] = F(s) = \frac{1}{2i}\left[\frac{s + i\omega - s + i\omega}{s^2 + \omega^2}\right]$$

13

Or simply

$$\mathcal{L}[\sin \omega t] = \bar{y}(s) = \left[\frac{\omega}{s^2 + \omega^2}\right]$$

The Laplace transform of the cosine function can be arrived at in quite a similar way.
Try to solve the problem on your own.

$\mathcal{L}[\cos \omega t] = \bar{y}(s) =$ ....

Solution found ------------------▷ 18

Help needed ------------------▷ 14

To be found: $\mathcal{L}[\cos\omega t]$. 14

Again, we start by expressing $\cos\omega t$ as a sum or a difference of exponential functions using Euler's Formula.

Recall:

$e^{i\omega t} =$ ……………………………………..

$e^{-i\omega t} =$ ……………………………………

Therefore, the cosine function can alternatively be expressed as

$\cos\omega t =$ ………………………………………

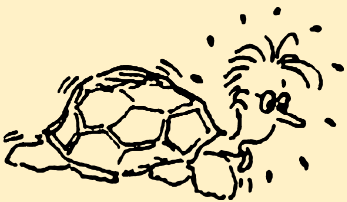
------------------▷ 15

$$e^{i\omega t} = i\sin\omega t + \cos\omega t$$
$$e^{-i\omega t} = -i\sin\omega t + \cos\omega t$$
$$\cos\omega t = \frac{1}{2}\left[e^{i\omega t} + e^{-i\omega t}\right]$$

15

Now we can determine:

$\mathcal{L}[\cos\omega t] = \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



Solution found ------------------▷ 18

One more hint needed? ------------------▷ 16

The task is to determine the Laplace transform of $f(t) = \cos \omega t$.

16

$$\mathcal{L}[\cos \omega t] = \int_0^\infty e^{-st} \cos \omega t \, dt$$

We know already

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

We also know

$$\cos \omega t = \frac{1}{2}\left[e^{i\omega t} + \mathrm{e}^{-i\omega t}\right]$$

By combining both results we obtain

$$\mathcal{L}[\cos \omega t] = \mathcal{L}\left[\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 17

$$\mathcal{L}\left[\cos\omega t\right] = \mathcal{L}\left[\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)\right] = \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s + i\omega}\right]$$ 17

After rearranging and using the common denominator we finally end up with:

$\mathcal{L}\left[\cos\omega t\right] = \vec{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 18

$$\mathcal{L}[\cos\omega t] = \bar{y}(s) = \frac{s}{s^2 + \omega^2}$$

18

Compute the Laplace transform

$$\bar{y}(s) = \mathcal{L}[5 \cdot \sin 4t] = \dots\dots\dots\dots\dots\dots$$

------------------- ▷ 19

$$\bar{y}(s) = \frac{5 \cdot 4}{s^2 + 16}$$

19

Compute the Laplace transform $\bar{y}(s)$ for $y(t) = 5 \cdot \cos 4t$

$\bar{y}(s)$ ....

-------------------▷ 20

$$\bar{y}(s) = \frac{5 \cdot s}{s^2 + 16}$$

20

Given a linear function $y(t) = C \cdot t$

To be found the Laplace transform: $\mathcal{L}[C \cdot t] = \bar{y}(s)$

$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found --------------------▷ 22

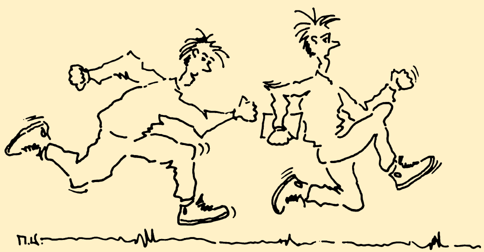
Hint needed --------------------▷ 21

21

How to determine the Laplace transform of a linear function is described in detail in the textbook. Please recap section 11.3 and follow all computations meticulously. If necessary, also look up the technique of integrating by parts.

In parallel to the procedure shown in the textbook compute

$\mathcal{L}[C \cdot t] = \ldots\ldots\ldots\ldots\ldots\ldots$



------------------▷ 22

$$\mathcal{L}[C \cdot t] = \overline{y} = \frac{C}{s^2}$$

22

Given $y(t) = 5 \cdot t$

Compute $\mathcal{L}[y(t)] = \overline{y}$ ....

--------------------▷ 23

$$\bar{y}(s)=\frac{5}{s^2}$$

23

Horizontal shift: If a function is shifted to the right by a units, its transform is

$$\mathcal{L}[y(t-a)]=e^{-as}\cdot\bar{y}(s) \quad \int_0^\infty y(t-a)\cdot e^{-s\cdot t}dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: Substitute u for t-a and evaluate the integral with respect to u

------------------▷ 24

$$\mathcal{L}[y(t-a)] = e^{-as}\cdot\bar{y}(s) \text{ because } \int_0^\infty y(t-a)\cdot e^{-s\cdot t}dt = \int_{-a}^\infty y(u)\cdot e^{-s\cdot(u+a)}du = e^{-as}\cdot\int_{-a}^\infty y(u)\cdot e^{-s\cdot u}du$$ 24

Let us now start with the linear function $y(t) = 5t$, which is represented by a straight line through the origin.

We already know its transform to be $\bar{y} = \dfrac{5}{s^2}$

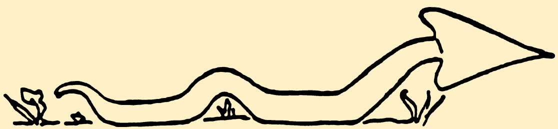
Let us now look at the slightly more general linear function $y(t) = 5t - 15$

As can easily be seen, it is a straight line which is shifted to the right by 3 units, since

$$y(t) = 5(t-3) = 5t - 15$$

Compute the transform of this function using the result on horizontal shifting:

$\mathcal{L}[5(t-3)] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



------------------▷ 25

$$\mathcal{L}[5(t-3)] = e^{-3s} \cdot \frac{5}{s^2}$$

25

Using the same argument we are in a position to transform any linear function.

Given $y(t) = a \cdot t - b$

Step 1: Rearrange the formula, so that you can apply the Laplace transform

$y(t) =$ ………………………………

------------------▷ 26

Step 1: The straight line is shifted to the right by $\frac{b}{a}$ units.

26

Step 2: The Laplace transform is $\bar{y}(s)=\frac{a}{s^2}\cdot e^{-\frac{b}{a}\cdot f}$.

Example: Given the straight line $y(t)=5t-50$

Rewrite the function so that the Laplace transform can be obtained

$y(t)=$…………………

Now compute the Laplace transform

$\bar{y}=$ …………………

--------------------▷ 27

$y(t) = 5t - 50 = 5(t - 10)$

27

$\dot{y}(s) = \dfrac{5}{s^2} \cdot e^{-10s}$

Look up the shift theorem in the textbook.

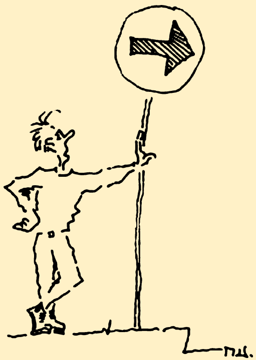
Given a function $y(t)$ and its transform $\bar{y}(s)$

We then know $\bar{y}(s + a) =$ …………………

--------------------▷ 28

$$\bar{y}(s+a) = \mathcal{L}\left[e^{-at} \cdot y(t)\right]$$ 28

Proof from the textbook understood -------------------▷ 30



Additional explanations for the proof -------------------▷ 29

Contents | First Page | Previous View | Next View | Print | Exit

To be shown: $y(s+a) = \mathcal{L}\left[e^{-at} \cdot y(t)\right]$ 29

We must determine the Laplace transform of the function
$e^{-at} \cdot y(t)$

Assuming that the transform of $y(t)$ is known to be $\bar{y}(s)$,
we recollect the definition of the Laplace transform

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \int_0^\infty e^{-at} \cdot e^{-st} \cdot y(t)dt$$

The integrand can be simplified as follows

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \int_0^\infty e^{-(s+a)t} \cdot y(t)dt$$

The integral evaluates to $\bar{y}(s+a)$, which is assumed to be known.

Thus, it is proved that $\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \bar{y}(s+a)$

------------------▷ 30

Given $y(t) = 3 \cdot \sin 2t \cdot e^{-4t}$
Find the transform $\bar{y}(s)$

$\bar{y}(s) =$ ....

30

Solution found --------------------▷ 33

Help needed --------------------▷ 31

Given $y(t) = 3\sin 2t \cdot e^{-4t}$ 31

Because of the damping factor e^{-4t} we can use the shift theorem.

We start by computing the transform of

$\mathcal{L}[3\sin 2t] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: This result was already obtained. Possibly you may wish to consult your notes or the textbook.

..

$\mathcal{L}[3\sin 2t] = \bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 32

$$\mathcal{L}[3\sin 2t] = \bar{y}(s) = \frac{3 \cdot 2}{s^2 + 4}$$ 32

However, we must determine $\mathcal{L}[\bar{y}(t)] = \mathcal{L}\left[3\sin 2t \cdot e^{-4t}\right]$

We now use the shift theorem

$$\mathcal{L}\left[e^{-at} \cdot 3\sin 2t\right] = \bar{y}(s+a)$$

$$\mathcal{L}\left[e^{-4t} \cdot 3\sin 2t\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 33

$$\mathcal{L}\left[e^{-4t} \cdot 3\sin 2t\right] = \frac{3 \cdot 2}{(s+4)^2 + 4}$$

33

Next task:

Given: $\bar{y}(t) = e^{-2t} \cdot \cos \pi t$

Wanted: $\mathcal{L}\left[e^{-2t} \cdot \cos \pi t\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 35

Help needed ------------------▷ 34

We use the same procedure as in the preceding exercise.
First, determine $\mathcal{L}[\cos \pi t]$ and, second, apply the shift theorem.

34

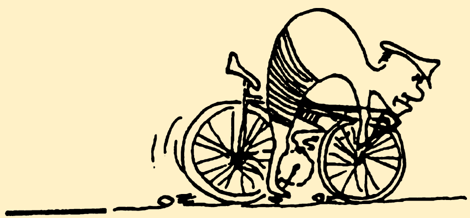
$$\mathcal{L}[\cos \pi t] = \frac{s}{s^2 + \pi^2}$$

The shift theorem tells us

$$\mathcal{L}\left[e^{-at} \cdot y(t)\right] = \bar{y}(s+a)$$

Thus

$$\mathcal{L}\left[e^{-2t} \cdot \cos \pi t\right] = \ldots\ldots\ldots\ldots\ldots\ldots$$



-------------------▷ 35

$$\mathcal{L}\left[e^{-2t}\cdot\cos\pi t\right]=\frac{s+2}{(s+2)^2+\pi^2}$$ 35

Understanding and applying the linearity theorem should be quite obvious.
Obtain the Laplace transform of

$y(t)=2\cos 2\pi t+\sin 2\pi t$:

$\mathcal{L}[2\cos 2\pi t+\sin 2\pi t]=$..................................

Hint: All transforms have already been introduced. If necessary consult your notes or the textbook.

------------------ ▷ 36

$$\mathcal{L}[2\cos 2\pi t + \sin 2\pi t] = \frac{2 \cdot s}{s^2 + (2\pi)^2} + \frac{2\pi}{s^2 + (2\pi)^2} = \frac{2s + 2\pi}{s^2 + (2\pi)^2}$$ 36

37

11.2 Laplace transform of derivatives

37

Objective: Obtain a good command of how to use two very useful theorems and, as a by-product, transform old friends like t^n .
This is the last section preparing you for highly useful applications of the Laplace transforms. Once again, we advise you to take notes of all results, rules, and theorems; this will aid your memory.

Read the second part of Section 11.3

Theorem II: Transforms of products *t y(t)*
Theorem III: Linearity
Theorem IV: Transforms of derivatives
Textbook pages 324–327

When done -------------------▷ 38

38

In this chapter we use the following abbreviations for denoting the values of the original function $y(t)$ or $f(t)$ and its derivatives at $t=0$:

$y(0)=$ ....

$y'(0)=$ ....

$y''(0)=$ ....

Equivalently we use the notation

$f(0)=$ ....

$f'(0)=$ ....

$f''(0)=$ ....

------------------- ▷ 39

39

$$y(0) = y_0 \qquad f(0) = f_0$$

$$y'(0) = y'_0 \qquad f'(0) = f'_0$$

$$y''(0) = y''_0 \qquad f''(0) = f''_0$$

The advantage of this shorthand notation will become apparent when we turn to solving differential equations.

Given: $y(t) = t \cdot e^{-at}$

To be found: $\mathcal{L}[y(t)] = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

Solution found ------------------▷ 43

Help needed ------------------▷ 40

In the textbook you find that the Laplace transform of a function $t \cdot y(t)$ is given by a derivative:

40

$\mathcal{L}[t \cdot y(t)] = -\frac{d}{ds}[\bar{y}(s)]$ if we denote $\mathcal{L}[y(t)]$ by $\bar{y}(s)$

In our case we already know the Laplace transform $\bar{y}(s)$ for $y(t) = e^{-at}$ = ...

If necessary, consult your notes or the textbook pages 332 and 334.

-------------------▷ 41

The Laplace transform of $y(t) = e^{-at}$ is given by $\bar{y}(s) = \frac{1}{s+a}$ 41

In order to determine the Laplace transform of $t \cdot y(t)$ the theorem tells us to find the derivative of $\bar{y}(s)$ with respect to s, and change the sign:

$$\frac{d}{ds}[\bar{y}(s)] = -\ \mathcal{L}[t \cdot y(t)]$$

In our case $\bar{y}(s) = \frac{1}{s+a}$

Thus, we get $\frac{d}{ds}[\bar{y}(s)] = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 42

Hint (redundant, we *do* hope): $\frac{1}{s+a} = (s+a)^{-1}$

$$\frac{d}{ds}\left[\bar{y}(s)\right]=\frac{-1}{(s+a)^2}$$ 42

So we now know $\frac{d}{ds}\left[\bar{y}(s)\right]=-\frac{1}{(s+a)^2}=-\mathcal{L}\left[t\cdot e^{-at}\right]$

Thus, we have identified the Laplace transform for $t\cdot e^{-at}$:

$\mathcal{L}\left[t\cdot e^{-at}\right]=$..............................

------------------▷ 43

$$\mathcal{L}\left[t \cdot e^{-at}\right] = \frac{1}{(s-a)^2}$$

43

Let us turn to another exercise. This time we use the notation $f(t)$ and for the Laplace transform $F(s)$

Given: $f(t) = t^3$

To be found: $F(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

44

$f(t)=t^3 \qquad F(s)=\frac{3!}{s^4}=\frac{6}{s^4}$ 44

Easy, wasn't it? $\mathcal{L}\left[t\cdot t^2\right]=-\frac{d}{ds}\mathrm{L}\left[t^2\right]=\frac{d^2}{(ds)^2}\mathrm{L}[t]=\frac{d^2}{(ds)^2}\frac{1}{s^2}$

Next task --------------------▷ 45

Now this exercise will be slightly more demanding:

45

Given: $f(t) = 3\sin^2(2t)$

Wanted: $F(s) = \ldots\ldots$..........................

Solution of the somewhat tricky task found ------------------▷ 51

Help needed ------------------▷ 46

The transform of $\sin^2(2t)$ can neither be found in the table nor, as yet, in our notes. 46

Therefore, let us try to rewrite the given expression $\sin^2(2t)$ so that we can find the Laplace transform with our given arsenal. We will use Pythagoras' theorem and the addition formulae for trigonometric functions so that the squared expression is eliminated.

$\sin^2(2t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution of this intermediate step found ------------------▷ 50

Further hints needed ------------------▷ 47

47

Recall Pythagoras' theorem $\cos^2 t + \sin^2 t = 1$ and
the addition formula $\cos(u+v) = \cos u \cdot \cos v - \sin u \cdot \sin v$

Now, in a seemingly round about way, let us first compute

$\cos(t+t) = \cos 2t =$……………………..

------------------▷ 48

$\cos 2t = \cos^2 t - \sin^2 t$

48

Because of $\cos^2 t = 1 - \sin^2 t$, we can eliminate $\cos^2 t$ from the expression above.

$\cos 2t = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

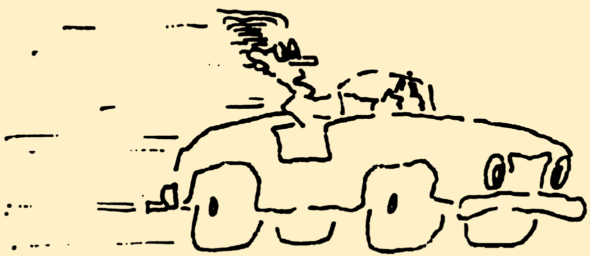
-------------------▷ 49

$\cos 2t = 1 - 2\sin^2 t$

49

So we are in a position to rewrite $\sin^2 t$ as follows

$\sin^2 t =$.............................



50

$$\sin^2 t = \frac{1}{2}[1 - \cos 2t]$$

50

The original task was to obtain the Laplace transform of the function $f(t) = 3\sin^2(2t)$.
Using the above result we get:

$$f(t) = 3 \cdot \frac{1}{2}(1 - \cos 4t)$$

This expression is now accessible to transformation: $\mathcal{L}\left[\frac{3}{2}(1 - \cos 4t)\right] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 51

$$\mathcal{L}\left[\frac{3}{2}(1-\cos 4t)\right]=\frac{3}{2}\cdot\frac{1}{s}-\frac{3}{2}\cdot\frac{s}{s^2+4^2}$$

51

$$=\frac{3}{2}\left[\frac{1}{s}-\frac{s}{s^2+4^2}\right]=\frac{3}{2}\left[\frac{s^2+16-s^2}{s(s^2+16)}\right]$$

$$=\frac{3\cdot 8}{s(s^2+16)}=\frac{24}{s(s^2+16)}$$

--------------------▷ 52

Let us do two more exercises using the table of inverse transformations. 52

Given the Laplace transform $F(s) = \frac{4}{s(s^2+4)}$

Find the inverse transform $f(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 54

Help needed ------------------▷ 53

53

We need to find the inverse transform of: $F(s)=\dfrac{4}{s\left(s^2+4\right)}$

In the table you find that

given $F(s)=\dfrac{1}{s\left(s^2+\omega^2\right)}$ the inverse transform is $f(t)=\dfrac{1}{\omega^2}(1-\cos\omega t)$

Constant factors stay unchanged. If you identify $\omega^2=4$ and $\omega=2$, then you obtain:

$f(t)=$ ....

------------------ ▷ 54

$$f(t)=\frac{4}{4}(1-\cos 2t)=1-\cos 2t$$ 54

One last task.

Given $F(s)=\dfrac{2}{\left(s^2-4s+3\right)}$

Then the inverse transform is $f(t)=$.........................

Solution found --------------------▷ 57

Help --------------------▷ 55

55

We must transform

$$F(s) = \frac{2}{(s^2 - 4s + 3)}$$

In the table you can find

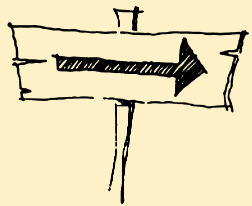
$F(s) = \frac{1}{(s-a)\cdot(s-b)}$ which leads to the inverse transform $f(t) = \frac{1}{a-b}\left[e^{at} - e^{bt}\right]$

In order to factor the denominator let us find the roots of the quadratic expression:

$s_1 = \ldots\ldots\ldots\ldots\ldots$ $s_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

Thus, the denominator can be written as a product:

$$s^2 - 4s + 3 = (\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)\cdot(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$



-------------------▷ 56

56

$s_1 = 3 \qquad s_2 = 1$

So we rewrite the quadratic expression as a product of linear factors:

$$s^2 - 4s + 3 = (s-3)\cdot(s-1)$$

Given

$F(s) = \dfrac{1}{(s-a)\cdot(s-b)}$ the inverse transform is $f(t) = \dfrac{1}{a-b}\left[e^{at} - e^{bt}\right]$

Given

$F(s) = \dfrac{2}{(s-3)\cdot(s-1)}$ the inverse transform is $f(t) =$..................................

------------------- ▷ 57

$$f(t)=\frac{2}{2}\left[e^{3t}-e^{t}\right]$$

57

More exercises and their solutions can be found in the textbook.

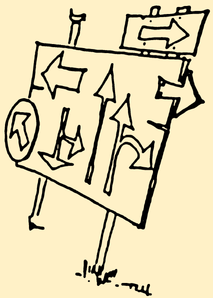
Please keep in mind: Easy exercises are fun. But the important thing is to tackle the hard ones.

--------------------▷ 58

In the next section we will finally encounter applications. Certain algebraic transformations of fractions, i.e. the decomposition into partial fractions, will prove especially useful.

58

This technique was already dealt with in the textbook, in a previous section on integration by partial fractions.
If you do not feel familiar enough with it, go back to section 6.5.7 in the textbook pages 170–174 and also to frames 117–138 in the accompanying chapter in the study guide.



After studying the textbook go to ------------------▷ 61

Partial fractions well known ------------------▷ 59

Just to make sure, split the following expressions into partial fractions: 59

1.) Roots real and unequal:

$$f_1(x) = \frac{5x+11}{x^2+6x+8}$$

$$f_1(x) = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

Roots real and repeated:

$$f_2(x) = \frac{1}{x-3x^2+4} = \frac{1}{(x+1)\cdot(x-2)^2} = \dots\dots\dots\dots\dots\dots\dots\dots$$

Roots real and complex

$$f_3(x) = \frac{2x^2-13x+20}{x(x^2-4x+5)} = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

-------------------▷ 60

$$f_1(x)=\frac{5x+11}{x^2+6x+8}=\frac{5x+11}{(x+2)\cdot(x+4)}=\frac{1}{2(x+2)}+\frac{9}{2(x+4)}$$

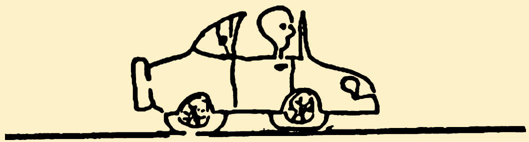
60

$$f_2(x)=\frac{1}{9(x+1)}+\frac{1}{3(x-2)}-\frac{1}{9(x-2)^2}$$

$$f_3(x)=\frac{4}{x}+\frac{-2x+3}{x^2+4x+5}$$

....

If you succeeded, you know the basic rules for finding partial fractions and you may proceed to the next section.



------------------▷ 61

If, however, you encountered difficulties, it makes sense to return to the textbook and study the relevant pages 170–174, work through the relevant frames in the study guide and do some examples. You will be happier knowing the basics well when entering subsequent sections.

Afterwards

------------------▷ 61

11.3 Solution of linear differential equations with constant coefficients

61

After having successfully gone through all preliminary steps, which might have been somewhat depletive in places, you now encounter applications and you will reap the benefits. Follow all the examples arduously and if necessary, consult the table on page 275.
You will learn to transform a set of linear differential equations into a set of algebraic equations and after solving these to use the inverse transform, thus obtaining a solution of the original DEs.
.

READ **11.4 Solution of linear differential equations with constant coefficients**
Textbook pages 328–329

When done ------------------▹ 62

62

All examples meticulously done and understood ------------------▷ 86

Hints and detailed calculations of the examples ------------------▷ 63

One slight irritation may arise from the notation used in this section: the function to be determined is denoted by $y(t)$ and, consequently, its derivatives by $y'(t)$ and $y''(t)$. This is in parallel to the notation $f(t)$, $f'(t)$, and $f''(t)$ that was used in some previous sections. 63

Difficulties with the first example --------------------▷ 64

Difficulties with the other example --------------------▷ 74

64

The first example deals with the differential equation $y'+4y = e^{-2t}$
The initial conditions are given as $t = 0,\ y_0 = 5$

In a first step we perform the Laplace transform of the differential equation and recall the rules of transformation

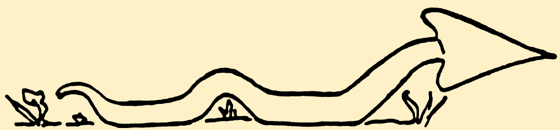
$$\mathcal{L}[y'] = s \cdot \bar{y}(s) - y_0$$

$$\mathcal{L}[y] = \bar{y}(s)$$

$$\mathcal{L}\left[e^{-at}\right] = \frac{1}{s+a}$$

The transform of the given differential equation is:…………………………………..

Please, check your results of Laplace transforming each member of the given equation using the tables on page 332 and 333.



------------------ ▷ 65

$$s \cdot \bar{y}(s) - y_0 + 4\bar{y}(s) = \frac{1}{s+2}$$ initial condition $y_0 = 5$ $\boxed{65}$

As a second step we solve the equation for $\bar{y}(s)$ and insert $y_0 = 5$.

$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\dashrightarrow$ $\boxed{66}$

$$\bar{y}(s) = \frac{5}{(s+4)} + \frac{1}{(s+2)\cdot(s+4)}$$

66

We are now in a position to perform the third step, finding the inverse transform. It is obtained by consecutively transforming both expressions, yielding y_1 and y_2. The complete inverse transform y then is given by the sum $y_1 + y_2$.

Consult the table on page 332 and 333 in the textbook and identify the inverse transforms of

$\bar{y}_1(s) = \frac{5}{s+4}$ $\qquad$ $y_1 = \dots\dots\dots\dots\dots\dots$

$\bar{y}_2(s) = \frac{1}{(s+2)\cdot(s+4)}$ $\qquad$ $y_2 = \dots\dots\dots\dots\dots\dots$

Solution found ------------------▷ 69

Help needed ------------------▷ 67

In the table on page 333 you see the inverse transforms for general classes of functions. 67

Rather than $y_1(s)=\dfrac{5}{s+4}$ the table shows $\bar{y}(s)=\dfrac{1}{s-a}$, its inverse transform being e^{at}.

Since the constant factor 5 remains unchanged by the transform. If we identify $a=-4$,

we can easily find the inverse transform for $\dfrac{5}{s+4}$ to be $y_1=$..........................

---------------------▷ 68

$y_1 = 5 \cdot e^{-4t}$

68

It now remains to determine the inverse transform y_2 for $\bar{y}_2(s) = \frac{1}{(s+2)\cdot(s+4)}$

From the table we know that for $\bar{y}(s) = \frac{1}{(s-a)\cdot(s-b)}$ the inverse transform is $y = \frac{1}{a-b}\left(e^{at} - e^{bt}\right)$

Inserting $a = -2$ and $b = -4$ we get:

$y_2 =$ ....

------------------▷ 69

$$y_2 = \frac{1}{2}\left[e^{-2t} - e^{-4t}\right]$$ 69

We already know: $y_1 = 5 \cdot e^{-4t}$

Now combine both expressions to obtain the complete inverse transform:

$y = y_1 + y_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 70

$$y = 5 \cdot e^{-4t} + \frac{1}{2}\left[e^{-2t} - e^{-4t}\right] = \frac{1}{2}e^{-2t} + \frac{9}{2}e^{-4t}$$ 70

In the textbook the inverse transform is arrived at in a slightly different way.

Show me how the expression from the textbook is arrived at ------------------▷ 71

Continue with the next exercise ------------------▷ 74

Given $\bar{y}(s)=\frac{5}{s+4}+\frac{1}{(s+2)\cdot(s+4)}$. Use the common denominator to get $\bar{y}(s)=\frac{5s+11}{(s+2)\cdot(s+4)}$ **71**

In order to arrive at possibly simpler expressions let us decompose into partial fractions:

$$\frac{5s+11}{(s+2)\cdot(s+4)}=\frac{A}{s+2}+\frac{B}{s+4}$$

We must determine A and B. First express everything with the common denominator:

$$\frac{5s+11}{(s+2)\cdot(s+4)}=\frac{As+4A+Bs+2B}{(s+2)\cdot(s+4)}$$

Clearing the fractions leads to: $5s+11=(A+B)\cdot s+4A+2B$

Determine A and B by comparing coefficients (equality holds for all different values of s).

$$5=A+B$$
$$11=4A+2B$$

$A=$ .... $B=$ ....

------------------▷ **72**

If decomposing into partial fractions proved difficult for you, go back to the textbook again to follow the solution of the exercise above.

$A = \frac{1}{2}$ $B = \frac{9}{2}$

72

Thus, the Laplace transform becomes:

$$\bar{y}(s) = \frac{5s+11}{(s+2)\cdot(s+4)} = \frac{1}{2(s+2)} + \frac{9}{2(s+4)}$$

Again, we quickly obtain the inverse transformation using the table in the textbook on page 332 and 333.

$y =$ ....

------------------ ▷ 73

$$y = \frac{1}{2} \cdot e^{-2t} + \frac{9}{2} e^{-4t}$$ 73

This coincides with the result previously obtained in this study guide.

There usually exist various different ways to rearrange the expression $\bar{y}(s)$ or $F(s)$. Experience, combined with some educated guessing, will guide you to finding expressions which are amenable to inverse transformations. One suitable approach often is the decomposition into partial fractions, which was dealt with in the textbook.

--------------------▷ 74

Now you decide how to proceed.

74

Difficulties with example 2 ------------------▷ 75

Difficulties with example 3 ------------------▷ 78

Difficulties with example 4 ------------------▷ 84

Straight on, no difficulties so far ------------------▷ 86

Example 2 deals with: $y''+5y'+4y=0$ initial conditions: $t=0$ $y_0=0$ $y'_0=3$ 75

Finding a solution always requires following these three steps:
Step 1: Apply the Laplace transform. Insert the initial conditions.
Step 2: Simplify the equation and solve for $\bar{y}(s)$ or in another notation $F(s)$.
Step 3: Apply the transform.
If needed, read again the first part of section 11.4 in the textbook.
. The following shows all details for example 2 from the textbook.
Difficulties may arise with regard to rewriting $\bar{y}(s)=\frac{3}{s^2+5s+4}$

Determine the roots in the denominator by solving the quadratic equation. That will enable us to represent the expression $\bar{y}(s)=\frac{3}{s^2+5s+4}$ as a product of linear factors.

$s_1=$....................
$s_2=$...................

------------------▷ 76

$s_1 = -4 \qquad s_2 = -1$

76

Therefore, the denominator can be written as a product of linear factors: $s^2 + 5s + 4 = (s+4)\cdot(s+1)$

Thus, the transform reads: $\bar{y}(s) = \dfrac{3}{(s+4)\cdot(s+1)}$

For this expression of $\bar{y}(s)$ the inverse transform can be found using the table: $y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

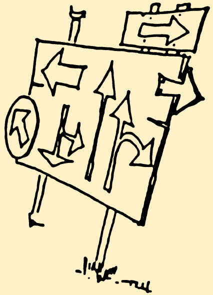
So we are done.
If, however, you prefer to use a more elementary approach, you may first split the expression for $\bar{y}(s)$ into partial fractions and then find the inverse transforms for elementary expressions of the type $\dfrac{1}{(s-a)}$, which are given by e^{at}, of course. For the similar case of $\bar{y}(s) = \dfrac{5s+11}{s(s+2)\cdot(s+4)}$ this has been demonstrated in frame 71, which you may like to refer to.

------------------ ▷ 77

$y = e^{-t} - e^{-4t}$

77

Again, please make a choice



Difficulties with example 3 ------------------▷ 78

Difficulties with example 4 ------------------▷ 84

Straight on ------------------▷ 86

Example 3

Solve the following differential equation: $y''+8y'+17y=0$
given the initial conditions: $t=0 \qquad y_0=0 \qquad y'_0=3$

Step 1: Apply the Laplace transform to the differential equation according to the rules:

…………………………..………………..= 0

-------------------▷ 79

$$\bar{y}(s)\cdot s^2 - y_0 - y'_0 - sy_0 + 8\bar{y}(s)\cdot s - 8y_0 + 17\bar{y}(s) = 0$$

79

Step 2: Insert the initial conditions $(t = 0, y_0 = 0, y'_0 = 2)$ and solve for $\bar{y}(s)$.

$\bar{y}(s) = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 80

$$\bar{y}(s) = \frac{3}{s^2 + 8s + 17}$$

80

In the table we find the inverse transform for the following function:

$$\bar{y}(s) = \frac{1}{(s-a)^2 + \omega^2} \qquad y = \frac{1}{\omega} \cdot e^{at} \cdot \sin \omega t$$

Thus, we face the task to rearrange the given denominator so that the inverse transform according to the rule above can be applied. We succeed by completing the square for $s^2 + 8s$ and obtain

$$\bar{y}(s) = \frac{3}{\left((s+4)^2 + 1\right)}$$

So, using the notation from the rule, $a = -4$ and $\omega^2 = 1$

$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------- ▷ 81

$y = 3 \cdot e^{-4t} \cdot \sin t$

81

We could have transformed the function $\bar{y}(s)$ differently. This is shown in example 2 in the textbook.

After determining the roots a and b of the denominator, we obtain the following expression:

$$\bar{y}(s) = \frac{3}{(s-a)\cdot(s-b)}$$

Show me this transformation also ------------------▷ 82

Difficulties with this example ------------------▷ 84

Ready to go on ------------------▷ 86

82

Given: $\bar{y}(s) = \dfrac{3}{s^2 + 8s + 17}$

Determine the roots of the denominator (solve the quadratic equation):

$s_1 = \ldots\ldots\ldots\ldots\ldots\ldots$

$s_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 83

$$s_1 = -4 + i$$
$$s_2 = -4 - i$$

83

Thus $\bar{y}(s) = \dfrac{3}{(s+4+i)\cdot(s+4-i)}$

So we have a type of expression that we know how to handle.

$$\bar{y}(s) = \frac{3}{(s-a)\cdot(s-b)}$$

$$y = \frac{1}{2i}\cdot\left(e^{-(4-i)t} - e^{-(4+i)t}\right)\cdot 3$$

Using Euler's formula we obtain the result already known

$$y = 3\cdot e^{-4t}\cdot \sin t$$

Difficulties with example 4 ------------------▷ 84

Ready to go straight on ------------------▷ 86

Example 4 asks us to solve: $y''+6y=t$ 84
with initial conditions: $t=0$ $y_0=0$ $y'_0=1$

Applying the Laplace transform and rearranging according to the known procedures results in

$$\bar{y}(s)=\frac{1}{s^2+6}+\frac{1}{s^2\left(s^2+6\right)}$$

Only the inverse transformations may pose a problem. But in the table we do find an inverse transform for both expressions. Putting $\omega^2=6$ we arrive at

$$y(t)=\frac{1}{\sqrt{6}}\sin\sqrt{6}\cdot t+\frac{1}{6\sqrt{6}}\left(\sqrt{6}\cdot t-\sin\sqrt{6}\cdot t\right)$$

Simplify $y(t)=$..

------------------▷ 85

$$y(t) = \frac{1}{6}\left(t + \frac{5}{\sqrt{6}}\sin\sqrt{6}t\right)$$ 85

Go on ----------------------▷ 86

You should now be in a position to solve the following DE. 86

$y''-6y'+8y=4$ Initial conditions: $t=0$ $y_0=0$ $y'_0=0$

Step 1: Obtain the Laplace transform of the differential equation

.................................=...............................

-------------------▷ 87

$$s^2\bar{y}(s) - sy_0 - y'_0 - 6s\bar{y}(s) + 6y_0 + 8\bar{y}(s) = \frac{4}{s}$$

87

Please note: The constant 4 on the right side of the original DE needed to be transformed also.

Step 2: Insert the initial conditions $(t = 0, y_0 = 0, y'_0 = 0)$ and solve for $\bar{y}(s)$:

$\bar{y}(s) = $ ....

----------------- ▷ 88

$$0\ \bar{y}(s) = \frac{4}{s\left(s^2 - 6s + 8\right)}$$

88

Decompose into partial fractions in order to find expressions that can easily be inverse transformed.

$\bar{y}(s) = \dots\dots\dots\dots\dots\dots\dots\dots$

Solution successfully found ------------------▷ 96

Help and explanation ------------------▷ 89

89

We must rewrite

$$\bar{y}(s) = \frac{4}{s(s^2 - 6s + 8)}$$

The bracketed expression in the denominator can be expressed as a product:

$$(s^2 - 6s + 8) = (s - a) \cdot (s - b)$$

Hint: In this special case you can either guess the correct values or, more generally, you must solve for the roots of the quadratic expression.

$$\bar{y}(s) = \frac{4}{s(s\text{...........}) \cdot (s\text{...........})}$$

-------------------▷ 90

$$\bar{y}(s) = \frac{4}{s(s-2)\cdot(s-4)}$$

90

Now the fraction can be split into partial fractions. In a later step the inverse transform will be applied.

$$\frac{4}{s(s-2)\cdot(s-4)} = \frac{A}{s} + \frac{B}{(s-2)} + \frac{C}{(s-4)}$$

$$\frac{4}{s(s-2)\cdot(s-4)} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ------------------▷ 95

Help ------------------▷ 91

Our approach for the decomposition into partial fractions is: 91

$$\frac{4}{s(s-2)\cdot(s-4)}=\frac{A}{s}+\frac{B}{(s-2)}+\frac{C}{(s-4)}$$

We must determine *A*, *B*, and *C*.

By using the common denominator of the partial fractions we obtain

$$\frac{4}{s(s-2)\cdot(s-4)}=\frac{(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)}{s(s-2)\cdot(s-4)}$$

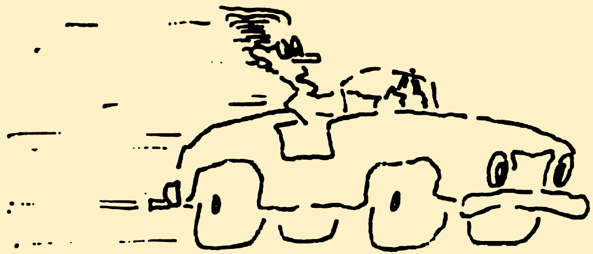
------------------▷ 92

$$\frac{4}{s(s-2)\cdot(s-4)}=\frac{A\cdot(s-2)\cdot(s-4)+B\cdot s(s-4)+C\cdot s(s-2)}{s(s-2)\cdot(s-4)}$$

92

Both numerators must be equal.
Expand the numerator on the right side and collect terms according to the different powers of s:

$4 =$ ....



93

$$4 = s^2[A + B + C] + s[-6A - 4B - 2C] + 8A$$

93

The equation can only be satisfied, if the right hand side is independent of both s and s^2. This amounts to both brackets being equal to 0.
From this we obtain the following equations to determine the unknown values of *A*, *B*, and *C*.

$$4 = 8A$$
$$0 = A + B + C$$
$$0 = -6A - 4B - 2C$$

Determine *A*, *B*, and *C*

$A =$ ....
$B =$ ....
$C =$ ....

Solution found -------------------▷ 95

One last hint -------------------▷ 94

94

The first equation implies $A = \frac{1}{2}$

By inserting into the other equations

$$A + B + C = 0 \qquad \text{and} \qquad -6A - 4B - 2C = 0$$

we obtain

$$\frac{1}{2} + B + C = 0 \qquad \text{and} \qquad -3 - 4B - 2C = 0$$

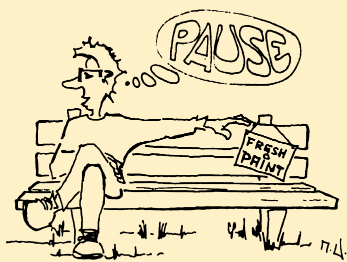
We rearrange and add

$$\begin{aligned} 2B + 2C &= -1 \\ -4B - 2C &= +3 \end{aligned}$$

This implies

$B =$ ....

$C =$ ....



--------------------▷ 95

$A = \frac{1}{2}$ $B = -1$ $C = \frac{1}{2}$

95

From this we obtain the desired form of the Laplace transform

$\ldots.\ \bar{y}(s) =$ ....

--------------------▷ 96

$$\bar{y}(s)=\frac{1}{2s}-\frac{1}{(s-2)}+\frac{1}{2}\frac{1}{(s-4)}$$ 96

Step 3 requires us to obtain the inverse transform using the given table.

$y(t)=$ ....

--------------------▷ 97

$$y(t)=\frac{1}{2}+\frac{1}{2}e^{4t}-e^{2t}$$

97

Let us turn to one last exercise which will be well known to you from physics -- the equation of motion in the gravitational field near the earth's surface. As is customary in physics, we denote time by t and the derivative with respect to time by $\dot{y}$. The direction of y is up. Then the acceleration is directed down, i.e. negative.

$$\ddot{y}=-g$$

Laplace transform

……………………………………

Solve for $\mathcal{L}(s)$

$\mathcal{L}(s)=$......................................

Inverse transform

$y(t)=$………………………………

-------------------▷ 98

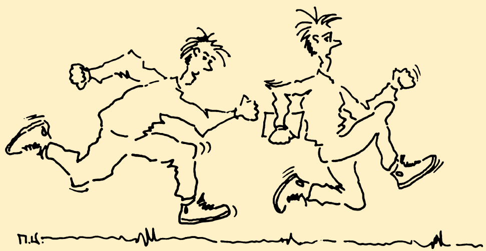
98

$$s^2 \cdot \mathcal{L}(s) - sy_0 - \dot{y}_0 = \frac{-g}{s}$$

$$\mathcal{L}(s) = \frac{-g}{s^3} + \frac{\dot{y}_0}{s^2} + \frac{y_0}{s}$$

$$y(t) = \frac{-g}{2} \cdot t^2 + \dot{y}_0 \cdot t + y_0$$

This is the famous and well-known equation for a freely falling object.



-------------------▷ 99

11.4 Solving simultaneous differential equations with constant coefficients

99

Read **11.5 Solving simultaneous differential equations with constant coefficients**
Textbook pages 330–331

------------------▷ 100

I have understood the principles of finding solutions and | 100

have followed all examples in the textbook ------------------▷ 138

Explanations for example 1 in the textbook ------------------▷ 101

Explanations for example 2 in the textbook ------------------▷ 121

The first example in the textbook was the following system of equations:

101

$3\dot{x} + 2x + \dot{y} = 1$ for $t = 0 : x_0 = y_0 = 0$

$\dot{x} + 4\dot{y} + 3y = 0$

Step 1: Apply the Laplace transform to both equations, observe the initial conditions, and simplify all expressions. Consult the textbook, if necessary.

....

....

-------------------▷ 102

$$(3s+2)\ \mathcal{L}[x] + s\ \mathcal{L}[y] = \frac{1}{s}$$
$$+s\cdot\ \mathcal{L}[x] + (4s+3)\ \mathcal{L}[y] = 0$$

102

Step 2: Now the system of equations must be solved for $\mathcal{L}[x]$ and $\mathcal{L}[y]$, consecutively. We start with $\mathcal{L}[x]$ and eliminate $\mathcal{L}[y]$. For this purpose let us multiply the first equation by $(4s+3)$ and the second equation by $(-s)$. That leads to

$$(3s+2)\mathcal{L}[x]\cdot(4s+3) + s(4s+3)\mathcal{L}[y] = \frac{4s+3}{s}$$
$$-s^2\mathcal{L}[x] \qquad\qquad -s(4s+3)\mathcal{L}[y] = 0$$

Add the equations to obtain

…………………………..=………………………..

------------------▷ 103

$$\mathcal{L}[x]\cdot\left[(3s+2)\cdot(4s+3)-s^2\right]=\frac{4s+3}{s}$$ 103

Now we must rearrange in order to obtain expressions that are amenable to the inverse transforms that are shown on page 327 of the textbook. Let us start with the expression in brackets on the left and expand the product:

$$\mathcal{L}[x]\cdot\left(12s^2+9s+8s+6-s^2\right)=\ \mathcal{L}[x\cdot]\cdot\left(11s^2+17s+6\right)$$

In order to obtain an expression of the type $A(s+a)\cdot(s+b)$, we factor the constant 11 and solve the quadratic equation. Thus we obtain

$$\mathcal{L}[x]\cdot 11\cdot(.......................)\cdot(.........................)=\frac{4s+3}{s}$$

------------------▷ 105

Further help ------------------▷ 104

104

The task is to solve $\mathcal{L}[x]\cdot\left(11s^2+17s+6\right)=\frac{4s+3}{s}$

By factoring 11 we obtain

$$\mathcal{L}\ [x]\cdot 11\cdot\left(s^2+\frac{17}{11}s+\frac{6}{11}\right)=\frac{4s+3}{s}$$

Let us now solve the quadratic equation

$$\left(s^2+\frac{17}{11}s+\frac{6}{11}\right)=0$$

$$s_1=-\frac{17}{2\cdot 11}+\sqrt{\frac{17^2}{(2\cdot 11)^2}-\frac{6}{11}}=-\frac{17}{2\cdot 11}+\sqrt{\frac{289-264}{(2\cdot 11)^2}}=-\frac{17}{2\cdot 11}+\sqrt{\frac{25}{(2\cdot 11)^2}}$$

$$=-\frac{17}{2\cdot 11}+\frac{5}{2\cdot 11}=-\frac{12}{2\cdot 11}=-\frac{6}{11}$$

$$s_2=-\frac{17}{2\cdot 11}-\frac{5}{2\cdot 11}=-\frac{22}{2\cdot 11}=-1$$

Thus we can express

$$\mathcal{L}[x]\cdot 11\cdot\left(s^2+\frac{17}{11}s+\frac{6}{11}\right)=\mathcal{L}[x]\cdot 11\cdot(\ldots\ldots\ldots\ldots\ldots)\cdot(\ldots\ldots\ldots\ldots\ldots)=\frac{4s+3}{s}$$

--------------------▷ 105

$$\mathcal{L}[x]\cdot 11\cdot(s+1)\cdot\left(s+\frac{6}{11}\right)=\mathcal{L}[x]\cdot(s+1)\cdot(11s+6)=\frac{4s+3}{s}$$ 105

Solving for $\mathcal{L}[x]$ this yields:

$$\mathcal{L}[x]=\frac{4s+3}{s(s+1)\cdot(11s+6)}=\frac{1}{11}\cdot\frac{(4s+3)}{s(s+1)\cdot\left(s+\frac{6}{11}\right)}$$

The fraction can be split into three partial fractions, whose denominators are given by the three linear factors. After performing a slightly tedious calculation, which is prone to errors by miscalculation, we obtain:

$\mathcal{L}[x]=$ ....

Calculation happily done ------------------▷ 112

Help and explanation ------------------▷ 106

Please note: The calculation could also be performed for the equivalent expression

$$\mathcal{L}[x]=\frac{(4s+3)}{11\cdot\left(s(s+1)\cdot\left(s+\frac{6}{11}\right)\right)}$$

The result, of course, would be unchanged.

The following fraction must be decomposed: $\dfrac{4s+3}{s(s+1)\cdot(11s+6)}$ 106

We expect to decompose into partial fractions with the three linear factors as the denominators:

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$$

Now let us add the partial fractions using the common denominator, in order to gain equations for determining *A*, *B*, and *C*.

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{\text{..........................}}{s(s+1)\cdot(11s+6)}$$

-------------------▷ 107

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{A\cdot(s+1)\cdot(11s+6)+B\cdot s(11s+6)+C\cdot s(s+1)}{s(s+1)\cdot(11s+6)}$$ 107

We multiply the expressions in the numerator and rearrange according to the powers of *s*:

$$\frac{4s+3}{s(s+1)\cdot(11+6)} = \frac{s^2(\dots\dots\dots\dots)+s\cdot(\dots\dots\dots\dots)+\dots\dots}{s(s+1)\cdot(11s+6)}$$

------------------ ▷ 108

$$\frac{4s+3}{s(s+1)\cdot(11s+6)} = \frac{s^2(11A+11B+C)+s(17A+6B+C)+6A}{s(s+1)\cdot(11s+6)}$$ 108

Comparing coefficients we obtain three equations for *A*, *B*, and *C*.

constant terms: $3 = 6A$
for *s*: $4 = 17A + 6B + C$
for s^2: $0 = 11A + 11B + C$

From this we easily determine

$A = \ldots\ldots\ldots\ldots$

$B = \ldots\ldots\ldots\ldots$

$C = \ldots\ldots\ldots\ldots$

Solution ------------------▷ 111

Explicit calculation ------------------▷ 109

We obtained three equations containing three variables

109

1) $3 = 6A$

2) $4 = 17A + 6B + C$

3) $0 = 11A + 11B + C$

Equation 1): $A = \frac{1}{2}$

Equation 2) minus Equation 3): $4 = -\frac{11}{2} - 11B - C + \frac{17}{2} + 6B + C$

$4 = 3 - 5B$

Hence $B = \ldots\ldots\ldots\ldots\ldots\ldots$

Insert B into equation 3): $0 = \frac{11}{2} - \frac{11}{5} + C$

$C = \frac{22 - 55}{10}$

$C = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

110

$A = \frac{1}{2}$ $B = -\frac{1}{5}$ $C = -\frac{33}{10}$ 110

Thus, the decomposition into partial fractions is complete:

$$\mathcal{L}[x] = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$$

$\mathcal{L}[x] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 111

$$\mathcal{L}[x]=\frac{1}{2s}-\frac{1}{5(s+1)}-\frac{33}{10}\cdot\frac{1}{(11s+6)}$$, or, equivalently $$\mathcal{L}[x]=\frac{1}{2s}-\frac{1}{5}\cdot\frac{1}{(s+1)}-\frac{3}{10}\cdot\frac{1}{\left(s+\frac{6}{11}\right)}$$ 111

Step 3:

Now we are in a comfortable position to do the inverse transform using the table on page 333 of the textbook. We obtain

$x=$ ....

------------------▷ 112

$$x = \frac{1}{2} - \frac{1}{5} \cdot e^{-t} - \frac{3}{10} \cdot e^{-\frac{6}{11} \cdot t}$$ 112

Now we must find y. In the given system of equations we eliminate $\mathcal{L}[x]$.
We have

$$\begin{aligned} (3s+2)\ \mathcal{L}[x] & \qquad + s\ \mathcal{L}[y] = \frac{1}{s} \\ + s\mathcal{L}[x] & \qquad + (4s+3)\ \mathcal{L}\ [y] = 0 \end{aligned}$$

A convenient way to eliminate $\mathcal{L}[x]$ is to multiply the first equation by $-s$ and the second one by $(3s+2)$, and then add the resulting equations.

That yields: ……………………..=……………………..

and ……………………..=……………………..

--------------------▷ 113

$$
\begin{aligned}
-s(3s+2)\cdot \mathcal{L}[x] \qquad\qquad\qquad & -s^2\cdot \mathcal{L}[y] = -1 \\
s(3s+2)\cdot \mathcal{L}[x] + (4s+3)\cdot(3s+2)\cdot & \mathcal{L}[y] = 0
\end{aligned}
$$

113

We now add and obtain

$$\mathcal{L}[y]\cdot\left[(4s+3)\cdot(3s+2)-s^2\right]=-1$$

Multiplying and collecting powers this leads to

$$\mathcal{L}[y]\cdot\left(11s^2+17s+6\right)=-1$$

The quadratic expression has already been factored in step 104 with the roots

$$s_1=-\frac{6}{11},\quad s_2=-1$$

Therefore $\mathcal{L}[y]=\dfrac{-1}{\dots\dots\dots\dots\dots\dots}$

-------------------▷ 114

$$\mathcal{L}[y] = \frac{-1}{(s+1)\cdot(11s+6)}$$ 114

We decompose the fraction into two partial fractions and obtain

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 118

Help and detailed computation ------------------▷ 115

115

To decompose into partial fractions: $\frac{-1}{(s+1)\cdot(11s+6)}$

Assumption:

$$\frac{-1}{(s+1)\cdot(11s+6)}=\frac{A}{(s+1)}+\frac{B}{(11s+6)}$$

We use the common denominator

$$\frac{-1}{(s+1)\cdot(11s+6)}=\frac{A(11s+6)+B(s+1)}{(s+1)\cdot(11s+6)}$$

After multiplying and ordering according to powers of s we obtain

$$\frac{-1}{(s+1)\cdot(11s+6)}=\frac{\dots\dots\dots\dots\dots\dots}{(s+1)\cdot(11s+6)}$$

------------------ ▷ 116

$$\frac{-1}{(s+1)\cdot(11s+6)}=\frac{s[11A+B]+6A+B}{(s+1)\cdot(11s+6)}$$

116

This leads us to the equations for determining A and B

$$\begin{aligned}-1&=6A+B\\0&=11A+B\end{aligned}$$

From this we obtain

$A =$............................ $B =$............................

----------------------▷ 117

$A = \frac{1}{5}$ $B = -\frac{11}{5}$ 117

Thus we obtain

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 118

$$\mathcal{L}[y]=\frac{1}{5(s+1)}-\frac{11}{5}\frac{1}{(11s+6)}$$, or, equivalently $$\mathcal{L}[y]=\frac{1}{5(s+1)}-\frac{1}{5}\frac{1}{\left(s+\frac{6}{11}\right)}$$

118

Step 3: Inverse transform

$y(t)=$..............................

-------------------▷ 119

$$y(t) = \frac{1}{5} \cdot e^{-t} - \frac{1}{5} \cdot e^{-\frac{6}{11} \cdot t} = \frac{1}{5}\left(e^{-t} - e^{-\frac{6}{11} \cdot t}\right)$$ 119

Most difficulties in algebraic operations arise by oversight or slips of the pen which afterwards can hardly be discovered.

--------------------▷ 120

No difficulties with example 2 from the textbook ------------------▷ 138

Show me a detailed walk through example 2 from the textbook ------------------▷ 121

Example 2: The following system of equations is to be solved

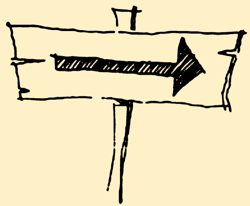
121

$$\ddot{x} + 2x - \dot{y} = 1 \qquad \text{for } t = 0: \quad x_0 = 1, \quad \dot{x}_0 = y_0 = \dot{y}_0 = 0$$
$$\dot{x} + \ddot{y} + 2y = 0$$

Step 1: Applying the Laplace transform and inserting the initial conditions results in:

......................................=....................................

......................................=.....................................



------------------- ▷ 122

$$\left(s^2+2\right)\mathcal{L}[x] \qquad -5\ \mathcal{L}[y]=\frac{1}{s}+s$$

122

$$s\ \mathcal{L}[x]+\left(s^2+2\right)\mathcal{L}[y]=x_0=1$$

Step 2: Solve for $\mathcal{L}[x]$ by eliminating $\mathcal{L}[y]$.
For this we multiply the first equation by $\left(s^2+2\right)$ and the second by s. Then by adding $\mathcal{L}[y]$ is eliminated.

We obtain

$\mathcal{L}[x]=$...........................=..............................

-------------------▷ 123

$$\mathcal{L}[x]\cdot\left[\left(s^2+2\right)^2+s^2\right]=\frac{s^2+2}{s}+s\left(s^2+2\right)+s$$

123

We rearrange the right hand side and use the common denominator *s*. On the left side we expand the square:

$$\mathcal{L}[x]\cdot[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]=\frac{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots}{s}$$

------------------▷ 124

$$\mathcal{L}[x]\cdot\left[s^4+5s^2+4\right]=\frac{s^4+4s^2+2}{s}$$ 124

This leads to

$$\mathcal{L}[x]=\frac{s^4+4s^2+2}{s\left(s^4+5s^2+4\right)}$$

We wish to decompose into partial fractions, which will yield expressions that can then be inversely transformed. For this we must expand the bracketed expression into a product of the type

$\left(s^2+a\right)\cdot\left(s^2+b\right)$.

For finding a and b we solve the equation s^4+5s^2+4 for s^2. With regard to s^2 it is just a quadratic equation. The zeroes are not difficult to find:

$s_1{}^2=\ldots\ldots\ldots$

$s_2{}^2=\ldots\ldots\ldots$

Hence

$s^4+5s^2+4=\left(s^2-\ldots\ldots\ldots\ldots\right)\cdot\left(s^2-\ldots\ldots\ldots\ldots\right)$

Solution found ------------------▷ 126

Show me the solution of the quadratic equation ------------------▷ 125

125

To be solved for s^2: $s^4 + 5s^2 + 4 = 0$

We solve the quadratic equation:

$$s^2 = -\frac{5}{2} \pm \sqrt{\frac{25}{4} - 4} \quad = -\frac{5}{2} \pm \sqrt{\frac{9}{4}} = -\frac{5}{2} \pm \frac{3}{2}$$

$s_1{}^2 = \ldots\ldots\ldots\ldots$

$s_2{}^2 = \ldots\ldots\ldots\ldots$

$s^4 + 5s^2 + 4 = \left(s^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right) \cdot \left(s^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right)$

------------------▷ 126

126

$$s_1^{\;2} = -1 \qquad s_2^{\;2} = -4$$

$$s^4 + 5s^2 + 4 = \left(s^2 + 1\right) \cdot \left(s^2 + 4\right)$$

Now the Laplace transform can be expressed as $\mathcal{L}[x] = \dfrac{s^4 + 4s^2 + 2}{s\left(s^2 + 1\right) \cdot \left(s^2 + 4\right)}$

Next, we decompose into partial fractions:

$$\frac{s^4 + 4s^2 + 2}{s\left(s^2 + 1\right) \cdot \left(s^2 + 4\right)} = \frac{A}{s} + \frac{B_1 + B_2 s}{\left(s^2 + 1\right)} + \frac{C_1 + C_2 s}{\left(s^2 + 4\right)}$$

Express the right side as a single fraction with the original common denominator

$$\frac{s^4 + 4s^2 + 2}{s\left(s^2 + 1\right) \cdot \left(s^2 + 4\right)} = \frac{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}{s\left(s^2 + 1\right) \cdot \left(s^2 + 4\right)}$$

----------------- ▷ 127

$$\frac{s^4+4s^2+2}{s(s^2+1)\cdot(s^2+4)}=\frac{A(s^4+5s^2+4)+B_1\cdot s(s^2+4)+B_2s^2(s^2+4)+C_1s(s^2+1)+C_2s^2(s^2+1)}{s(s^2+1)\cdot(s^2+4)}$$ 127

Clear the fractions. On the right side expand all products and order them according to powers of s.

$s^4+4s^2+2=$ ....

------------------ ▷ 128

$$s^4 + 4s^2 + 2 = s^4\left(A + B_2 + C_2\right) + s^3\left(B_1 + C_1\right) + s^2\left(5A + 4B_2 + C_2\right) + s\left(4B_1 + C_1\right) + 4A$$ 128

Since the coefficients of like powers of s must be equal, we obtain five equations for determining A, B_1, B_2, C_1, and C_2.

Solving for each unknown, consecutively, we get:

$A = \ldots\ldots$ $B_1 = \ldots\ldots$ $B_2 = \ldots\ldots$ $C_1 = \ldots\ldots$ $C_2 = \ldots\ldots$

Solution found ------------------▷ 130

Help and explicit calculation ------------------▷ 129

129

Compare coefficients for like powers of s

$$s^4+4s^2+2=s^4\left(A+B_2+C_2\right)+s^3\left(B_1+C_1\right)+s^2\left(5A+4B_2+C_1\right)+s\left(4B_1+C_1\right)+4A$$

Coefficients of s^4: $1=\left(A+B_2+C_2\right)$

Coefficients of s^3: $0=B_1+C_1$

Coefficients of s^2: $4=\left(5A+4B_2+C\right)$

Coefficients of s^1: $0=4B_1+C_1$

Coefficients of s^0: $2=4A$

Solve this set of linear equations, starting with A from the last equation.

$A=\ldots\ldots$ $B_1=\ldots\ldots$ $B_2=\ldots\ldots$ $C_1=\ldots\ldots$ $C_2=\ldots\ldots$

------------------▷ 130

$A = \frac{1}{2}$ $B_1 = 0$ $B_2 = \frac{1}{3}$ $C_1 = 0$ $C_2 = \frac{1}{6}$

130

Thus $\mathcal{L}[x] = \frac{s^4 + 4s^2 + 2}{s(s^4 + 5s^2 + 4)}$ can be written as a sum of three fractions

$\mathcal{L}[x] = \ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$

131

$$\mathcal{L}[x] = \frac{1}{2s} + \frac{1}{3}\frac{s}{(s^2+1)} + \frac{1}{6}\frac{s}{(s^2+4)}$$ 131

Step 3: In order to determine the inverse transform we use the table in the textbook on page 332 and 333 and obtain

$x(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

-------------------▷ 132

$$x(t) = \frac{1}{2} + \frac{1}{3}\cos t + \frac{1}{6}\cos 2t$$

132

So the first part of example 2 is done. It still remains to determine $\mathcal{L}[y]$ and $y(t)$.
Recall that the transformed system of equations was:

$$\left(s^2+2\right)\ \mathcal{L}[x] - s\ \mathcal{L}[y] = \frac{1}{s} + s$$
$$s\,\mathcal{L}[x] + \left(s^2+2\right)\ \mathcal{L}[y] = 1$$

In order to eliminate $\mathcal{L}[x]$, multiply the first equation by $(-s)$ and the second by $\left(s^2+2\right)$ and add them. That leads to

$\mathcal{L}[y] =$..........................

------------------▷ 133

$$\mathcal{L}[y]\cdot\left(\left(s^2+2\right)^2+s^2\right)=1$$

133

We expand the square and solve for $\mathcal{L}[y]$:

$\mathcal{L}[y]=$...........................

--------------------▷ 134

$$\mathcal{L}[y] = \frac{1}{\left(s^4 + 5s^2 + 4\right)}$$

134

The bracketed expression has already been factored in the preceding frames 125–127 resulting in $\left(s^2+1\right)\cdot\left(s^2+4\right)$. Next on the agenda is the decomposition into partial fractions:

$$\frac{1}{\left(s^2+1\right)\cdot\left(s^2+4\right)} = \frac{A}{s^2+1} + \frac{B}{s^2+4}$$

$$\frac{1}{\left(s^2+1\right)\cdot\left(s^2+4\right)} = \frac{A\left(s^2+4\right)+B\left(s^2+1\right)}{\left(s^2+1\right)\cdot\left(s^2+4\right)} = \frac{s^2(A+B)+4A+B}{\left(s^2+1\right)\cdot\left(s^2+4\right)}$$

From this we deduce in the usual way

$$\begin{aligned} A+B &= 0 \\ 4A+B &= 1 \end{aligned}$$

$A = \ldots\ldots\ldots\ldots$ $B = \ldots\ldots\ldots\ldots$

------------------ ▷ 135

$A = \frac{1}{3}$ $\qquad$ $B = -\frac{1}{3}$ $\boxed{135}$

By inserting we obtain $\mathcal{L}[y] = \frac{1}{3(s^2+1)} - \frac{1}{3(s^2+4)}$

Step 3: Find the inverse transform using the table on page 333 of the textbook:

$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------ ▷ $\boxed{136}$

$$y(t)=\frac{1}{3}\sin t-\frac{1}{6}\sin 2t$$ 136

Just one more exercise. The solution should not be difficult for you after working through the preceding examples.
The solution follows the now customary scheme consisting of three steps:
- Laplace Transformation
- Solve for $\mathcal{L}[x]$ and $\mathcal{L}[y]$ and rearrange into a convenient expression for applying the inverse transform.
- Perform the inverse transformation

------------------▷ 137

Given two simultaneous differential equations with constant coefficients:

137

$$4\dot{x} - \dot{y} + x = 1 \qquad \text{for } t = 0: \quad x_0 = y_0 = 0$$
$$4\dot{x} - 4\dot{y} - y = 1$$

$x(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 142

Stepwise solution showing intermediate results ------------------▷ 138

Given

$$4\dot{x} - \dot{y} + x = 1$$
$$4\dot{x} - 4\dot{y} - y = 1$$

for $t = 0: \ x_0 = y_0 = 0$

138

Step 1: Determine the Laplace transforms and insert the initial conditions:

....

....

-------------------▷ 139

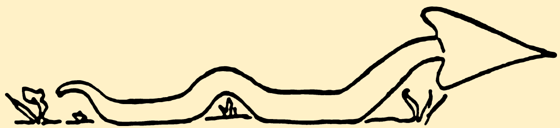
$$4s\,\mathcal{L}[x]-s\,\mathcal{L}[y]+\mathcal{L}[x]=\frac{1}{s}$$

$$4s\,\mathcal{L}[x]-4s\,\mathcal{L}[y]-\mathcal{L}[y]=\frac{1}{s}$$

139

Step 2: Solve for $\mathcal{L}[x]$:

$\mathcal{L}[x]=$............................



-------------------▷ 140

$$\mathcal{L}[x] = \frac{3s+1}{s \cdot 12\left(s+\frac{1}{6}\right) \cdot \left(s+\frac{1}{2}\right)}$$ 140

Decompose into partial fractions in order to obtain expressions which are easily inverse transformed:

$\mathcal{L}[x] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 141

$$\mathcal{L}[x] = \frac{1}{s} - \frac{3}{4} \cdot \frac{1}{\left(s + \frac{1}{6}\right)} - \frac{1}{4} \cdot \frac{1}{\left(s + \frac{1}{2}\right)}$$ 141

Step 3: The inverse transformation results in

$x(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 142

$$x(t) = 1 - \frac{3}{4} e^{-\frac{t}{6}} - \frac{1}{4} \cdot e^{-\frac{t}{2}}$$

142

Now we must determine $\mathcal{L}[y]$ and y.

The Laplace transform has already been performed in frame 139 with the following result:

$$4s\,\mathcal{L}[x] - s\,\mathcal{L}[y] + \mathcal{L}[x] = \frac{1}{s}$$

$$4s\,\mathcal{L}[x] - 4s\,\mathcal{L}[y] - \mathcal{L}[y] = \frac{1}{s}$$

Now compute

$y(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution happily found ------------------▷ 147

Stepwise solution showing intermediate results ------------------▷ 143

The following must be solved for $\mathcal{L}[y]$

143

$$4s\ \mathcal{L}[x] - s\,\mathcal{L}[y] + \ \mathcal{L}\ [x] = \frac{1}{s}$$

$$4s\,\mathcal{L}[x] - 4s\,\mathcal{L}[y] - \ \mathcal{L}[y] = \frac{1}{s}$$

By eliminating $\mathcal{L}[x]$ in the usual fashion we obtain

$\mathcal{L}[y] = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

----------------------▷ 144

$$\mathcal{L}[y] = \frac{-1}{s \cdot 12\left(s + \frac{1}{6}\right) \cdot \left(s + \frac{1}{2}\right)}$$

144

Decomposition into partial fractions produces expressions that are easily inversely transformed:

$$\mathcal{L}[y] = \frac{-1}{s \cdot 12\left(s + \frac{1}{6}\right) \cdot \left(s + \frac{1}{2}\right)} = \frac{A}{12s} + \frac{B}{\left(s + \frac{1}{6}\right)} + \frac{C}{\left(s + \frac{1}{2}\right)}$$

Determine the unknown values

$A =$…………….. $B =$…………….. $C =$…………

-------------------▷ 145

$A = -12$ $\qquad B = \frac{3}{2}$ $\qquad C = -\frac{1}{2}$ 145

Thus

$$\mathcal{L}[y] = \frac{-1}{s \cdot 12\left(s + \frac{1}{6}\right) \cdot \left(s + \frac{1}{2}\right)} = \dots\dots\dots\dots\dots\dots$$

-------------------▷ 146

$$\mathcal{L}[y] = -\frac{1}{s} + \frac{3}{2}\frac{1}{\left(s+\frac{1}{6}\right)} - \frac{1}{2}\frac{1}{\left(s+\frac{1}{2}\right)}$$ 146

Now we can determine the inverse transform with a little help from the table, as nobody can keep all transforms in memory. The result is:

$y(t) =$ ....

-------------------▷ 147

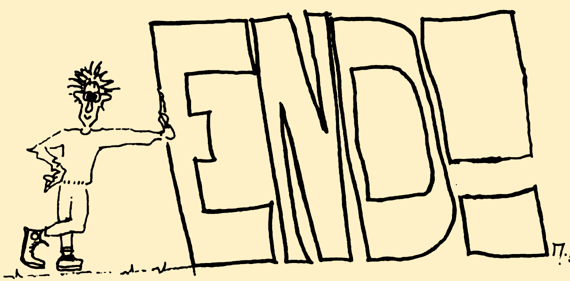
$$y(t) = -1 + \frac{3}{2} \cdot e^{-\frac{t}{6}} - \frac{1}{2} e^{-\frac{t}{2}}$$

Thus, we have determined the complete solution of the system of equations as follows

$$x(t) = 1 - \frac{1}{2}\left(e^{-\frac{t}{6}} + e^{-\frac{t}{2}}\right)$$

$$y(t) = -1 + e^{-\frac{t}{6}} - e^{-\frac{t}{2}}$$

You have successfully completed this somewhat demanding chapter, and you may be proud of your stamina!



of this chapter.

Chapter 12

Functions of Several Variables; Partial Differentiation; Total Differentiation

2

Before starting a new chapter quickly recapitulate the previous one, writing down all important facts from memory. Then check them with the text and any notes you made when you were studying that particular chapter.

This shouldn't take you more than 5 minutes, but don't skip it.

3

12.1 The Concept of Functions of Several Variables

3

Objective: Concept of a function of several variables, determination of surfaces of functions of two independent variables.

A very effective way of studying mathematical and physical concepts is to work with similar problems to those in the text, in parallel with it.

As you study the text carry out all the necessary operations using the function

$$z = f(x, \ y) = \mathrm{e}^{-(x^2+y^2)}$$

READ: **12.1 Introduction**
12.2 Functions of several variables
Textbook pages 339–340

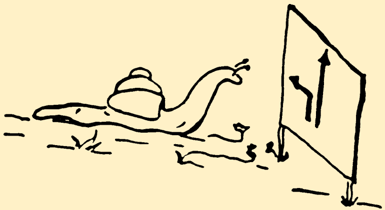
----> 4

Have you been working with the function

4

$$z = f(x, \; y) = e^{-(x^2+y^2)}$$

in parallel with the text?



☐ Yes ⇢ 6

☐ No ⇢ 5

That's a pity. 5

To solve a problem in parallel with the one in the text is cumbersome. But it does lead to a better understanding of the subject matter; in the long run it saves time.

So try to sketch the surface given by the equation

$$z = e^{-(x^2+y^2)}$$

following steps analogous to those in section 12.2.

6

Well done!

6

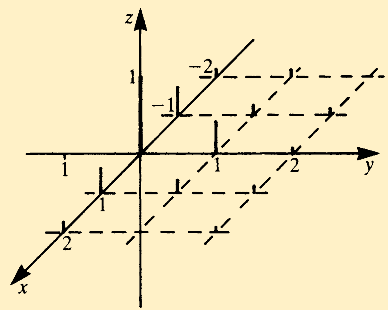
To carry out a calculation in parallel with the text instead of reading quickly may seem tiresome but if you do it you will reap the benefit.

Here are some hints for the solution of the problem to sketch

$$z = e^{-(x^2+y^2)}$$

The values have been rounded, e.g. $e^{-1} \approx 0.4$; $e^{-4} \approx 0.02$.

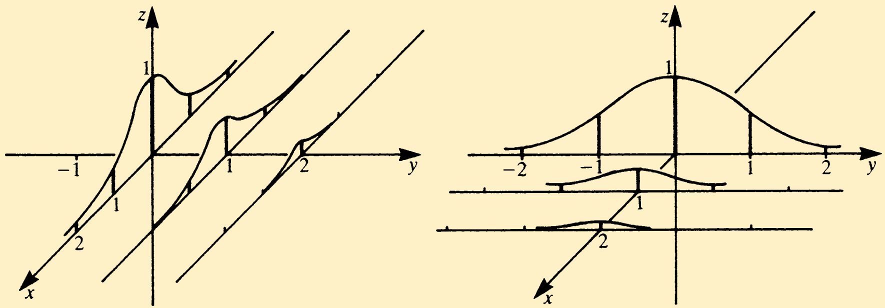
y \ x	0	1	2
0	1	0.4	0.02
1	0.4	0.1	0.007
2	0.02	0.007	0.0003



7

Your sketches might be similar to these

7



The surface is similar to the one in the text. However, it approaches zero very quickly for large values of x and y.

The function is a Gauss surface. This type of surface occurs in probability theory.

8

8

We shall now have a look at the sketching of functions of two variables. We saw in Chapter 5 how to do this in the case of a function of one variable, where we looked for key features.

Before proceeding, go back to the section on curve sketching in that chapter for a quick revision if you are not sure of how to proceed.

9

A function of two variables represents a surface in space and to sketch it we proceed in a manner similar to that for a single variable. The process is, however, more lengthy since a surface in space is a more complex geometrical figure than a plane curve. 9

To sketch the surface we can use either of two methods:

Method (a)

Draw up a table of values, a matrix, thus:

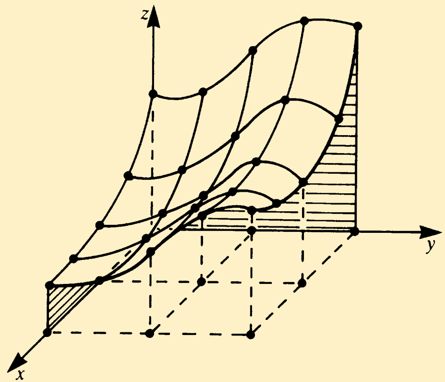
x \ y	0	1	2	…
0				
1	$z = f(x = 1, y = 0)$		$z = f(x = 1, y = 2)$	
2				
⋮				

For each pair of values (x, y) there corresponds a value for z in accordance with the equation $z = f(x, y)$. The points x, y, z are then placed in the coordinate system and connected to each other in the x-direction and in the y-direction. In this way we build up the surface.

10

The figure shows such a sketch

10



11

Method (b)

11

We look for particular features such as:

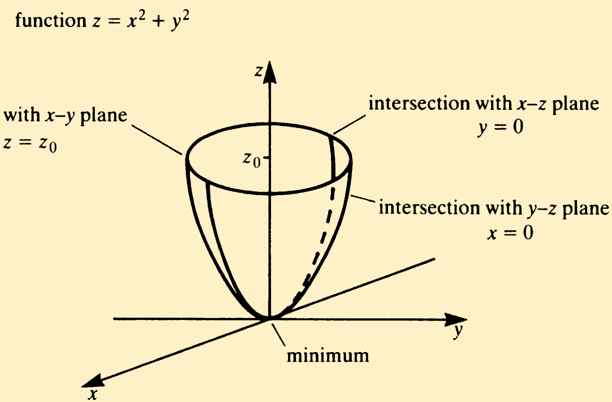
intersection with the x–z plane, by setting $y = 0$
intersection with the y–z plane, by setting $x = 0$
intersection with the x–y plane, by setting $z = 0$
and the intersections with planes parallel to:

the x–y plane, by setting $z = z_0$, a particular value
the x–z plane, by setting $y = y_0$
the y–z plane, by setting $x = x_0$

the behaviour of the function as $x \to \pm\infty$ and $y \to \pm\infty$.

These intersection curves serve to sketch the surface. Sometimes it is even possible to guess where the surface has a maximum and/or minimum. The calculation of minima and maxima is shown later in section 12.6.

--------------------▷ 12


function $z = x^2 + y^2$
z
intersection with x–y plane
$z = z_0$
intersection with x–z plane
$y = 0$
z_0
intersection with y–z plane
$x = 0$
y
minimum
x

12

13

With a pocket calculator it is quite easy to establish large tables of functional values. Nevertheless method (b) is most important. Often we only need a rough picture of the surface. Furthermore, the salient points are often of theoretical and practical importance. 13

14

Given the function

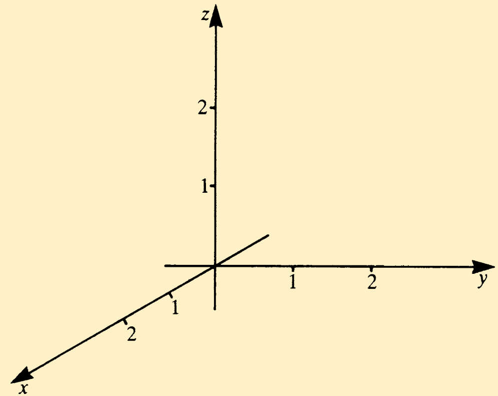
$$z = x^2 + y^2$$

14

sketch the intersections with

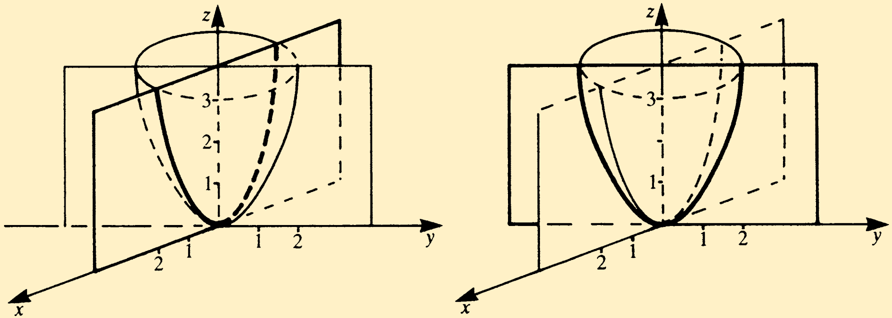
(a) the x–z plane $y = 0$

(b) the y–z plane $x = 0$



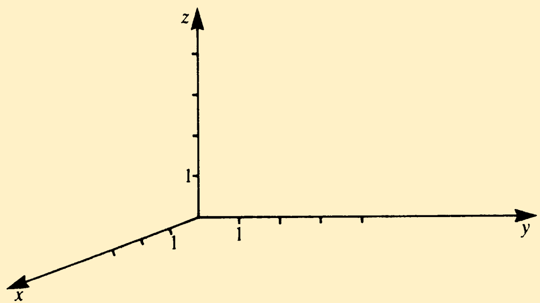
15

15



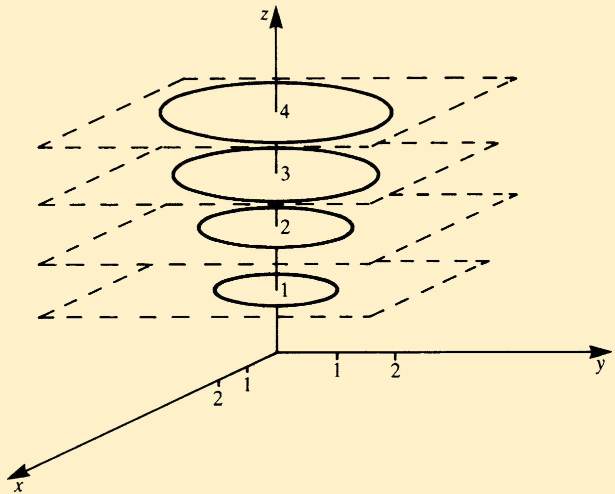
Explanation: The intersection with the x–z plane is obtained by setting $y = 0$, giving $z = x^2$. This is the equation of a parabola. The intersection with the y–z plane is obtained by setting $x = 0$, giving $z = y^2$.

Now sketch a few intersections with planes parallel to the x–y plane at $z = 1, 2, 3$ and 4: $z = x^2 + y^2$.



16

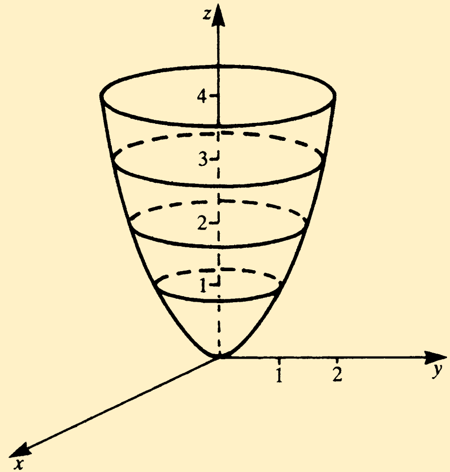
16



The intersecting curves with z = constant are circles.

17

17



By examining the intersecting curves we conclude that the equation $z = x^2 + y^2$ represents a paraboloid.

18

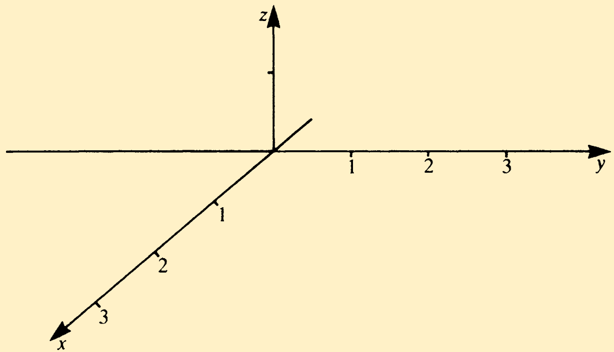
Sketch the surface whose equation is

18

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

First plot the intersecting curve in the y–z plane.

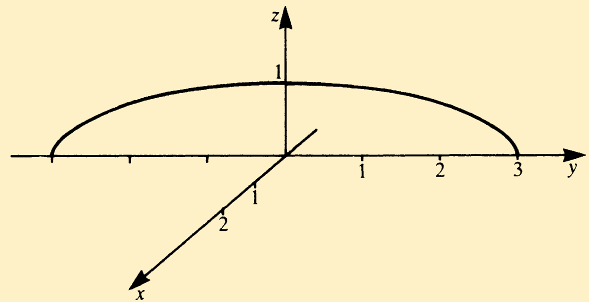
$$z(0, y) = \text{..............................}$$



19

$$z(0, y) = \sqrt{1 - \frac{y^2}{9}}$$ 19

This is an ellipse.



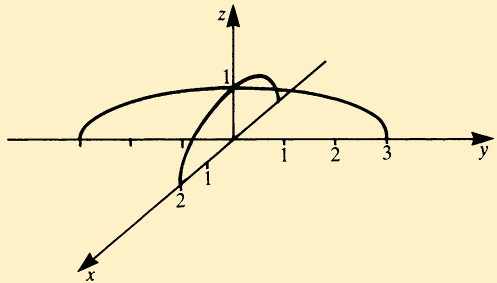
Now plot the intersecting curve in the x–z plane in the drawing above.

$$z(x, \ 0) = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 20

$z(x,0) = \sqrt{1 - \dfrac{x^2}{4}}$. This is an ellipse too.

20



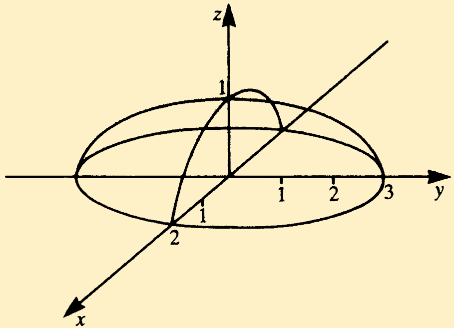
Now add the intersecting curve in the $x - y$ plane i.e.

$$0 = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

Plot it in the figure.

Write the equation $y = y(x) =$ ....

21



$y = 3\sqrt{1 - \frac{x^2}{4}}$ an ellipse in the x–y plane. 21

The function

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

represents the top half of an ellipsoid, above the x–y plane.

22

In case of errors try to identify their cause by retracing your steps. Remember that if $x = 0$ we have a plane coincident with the y–z plane but if $x = x_0$, some constant value, we have a plane parallel with the y–z plane but at a distance x_0 along the x-axis from the origin of the coordinate axes. Similarly if $y = 0$, $y = y_0$, and if $z = 0$, $z = z_0$. These planes will cut the surface along the curves defined by the equation for z. If $x = 0$ then $z = f(y)$ only and we have a plane curve in the y–z plane and so on with the other variables. 22

As a further exercise find the intersection curves for the given example

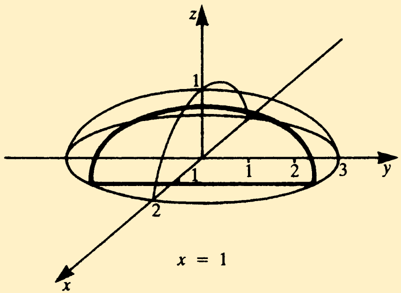
$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \text{ for } x = 1, \quad y = 1$$

--------------------▷ 23

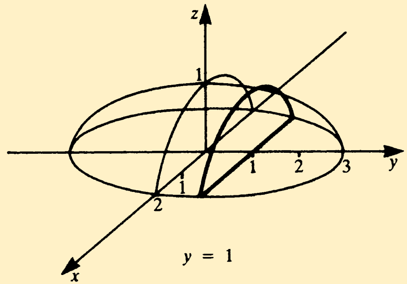
$$z(1,\ y) = \sqrt{\frac{3}{4} - \frac{y^2}{9}}; \qquad z(x,\ 1) = \sqrt{\frac{8}{9} - \frac{x^2}{4}}$$

23

Both equations represent an ellipse



$x = 1$



$y = 1$

You should now have a reasonable grasp of how to sketch a function of two variables. It is a surface in three-dimensional space.

We cannot sketch a function of three variables because we would need a four-dimensional space.

--------------------▷ 24

12.2 Partial Differentiation
Higher Partial Derivatives

24

Objective: Concepts of partial differentiation and higher partial derivatives.

READ: 12.3 Partial differentiation
12.3.1 Higher partial derivatives
Textbook pages 346–351

While reading the textbook do all calculations with the functions

$$z = \frac{1}{1 + x^2 + y^2} \text{ and } u = \frac{x}{y} + 2z$$

parallel with the text

--------------------▷ 25

The symbols for the partial derivative of a function $f(x, y)$ with respect to x are .... and ....

The symbols for the partial derivative with respect to y are .... and ....

25

--------------------▷ 26

$$\frac{\partial f}{\partial x}, f_x \qquad \frac{\partial f}{\partial y}, f_y$$ 26

Remember that the partial derivative of a function of several variables such as

$$z = f(x, \ y)$$

means that you differentiate the function with respect to a particular variable, x for instance, treating the others as if they were constants.

Now for some practice: $z = f(x, \ y) = x^2 + y^2$

Differentiating with respect to x yields

$$\frac{\partial f}{\partial x} = 2x + y^2$$

--------------------▷ 27

$$\frac{\partial f}{\partial x} = 2x + 2y$$

--------------------▷ 28

$$\frac{\partial f}{\partial x} = 2x$$

--------------------▷ 30

You have made a mistake! 27

Partial differentiation with respect to x, i.e. $\frac{\partial f}{\partial x}$ or f_x, means that we regard *all* variables as constants *except* x.

In this case y^2 is treated as a constant and since the derivative of a constant is zero it follows that

$$\frac{\partial}{\partial x}(x^2 + \text{constant}) = 2x$$

Hence

$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

Try these!

$$f(x, \ y) = x + y$$
$$f_x = \text{.........................}$$
$$g(x, \ y) = x + xy$$
$$g_x = \text{.........................}$$

29

28

You have made a mistake!

You differentiated x^2 correctly but you appear to have differentiated y^2 with respect to y as well.

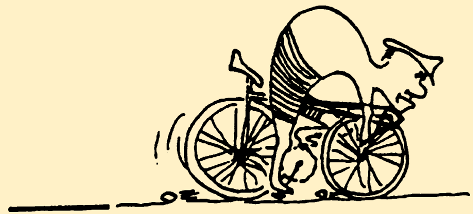
Partial differentiation with respect to x means that we regard all other variables as constants. Thus regard y as a constant and differentiate with respect to x only.

$$\frac{\partial}{\partial x}(x^2 + \text{constant}) = 2x$$

Hence

$$\frac{\partial}{\partial x}(x^2 + y^2) = 2x$$

Now try



$f(x, \ y) = x + y \qquad f_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

and

$g(x, \ y) = x + xy \qquad g_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

29

$$f_x = 1$$
$$g_x = 1 + y$$

29

Hints (in case you need them):
y is regarded as constant. Thus

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, \ y) = \frac{\partial}{\partial x}(x+y) = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y = 1 + 0$$
$$\text{and } \frac{\partial g}{\partial x} = gx = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}(xy) = 1 + y\frac{\partial}{\partial x}x = 1 + y$$

Now calculate again:

$$f(x, \ y) = x^2 + y^2 \qquad f_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 30

$2x$ is the correct solution

30

If $z = f(x,\ y) = x^2 + y^2 + 5$

$$\frac{\partial z}{\partial y} = f_y = \dots\dots\dots\dots\dots\dots\dots\dots$$

31

$$\frac{\partial f}{\partial y} = fy = 2y$$

31

Correct

32

Wrong: here is the solution for you to follow:
In this case we regard x as a constant and differentiate with respect to y, so that

$$\frac{\partial}{\partial y}(x^2 + y^2 + 5) \equiv \frac{\partial}{\partial y}(y^2 + \text{two constants}) = 2y$$

32

Differentiate partially the function $f(x, \ y) = 2x + 4y^3$

32

$$\frac{\partial f}{\partial x} \text{ or } f_x = \text{..........................}$$

$$\frac{\partial f}{\partial y} \text{ or } f_y = \text{..........................}$$

33

$$f_x = 2, \quad f_y = 12y^2$$

33

Correct

----------------------▷ 34

Wrong: In this case go back to the textbook, section 12.3. Solve the example consulting the textbook. Find the cause of your error.

$$f(x,\ y) = 2x + 4y^3$$

$f_x =$ ....

----------------------▷ 34

Obtain the partial derivatives of the function

34

$$f(x,\ \ y) = x^3 + 5xy - \tfrac{1}{2}y^2 + 3$$

$f_x =$ ....
$f_y =$ ....

35

$$f_x = 3x^2 + 5y, \qquad f_y = 5x - y$$

35

If $f(x,\ y) = 2x^3 \sin 2y$

$f_x =$ ....

$f_y =$ ....

36

$$f_x = 6x^2 \sin 2y$$
$$f_y = 4x^3 \cos 2y$$

36

One final example!

If $u = x^2 - \sin y \cos z$

$$\frac{\partial u}{\partial x} = \dots\dots\dots\dots\dots\dots$$
$$\frac{\partial u}{\partial y} = \dots\dots\dots\dots\dots\dots$$
$$\frac{\partial u}{\partial z} = \dots\dots\dots\dots\dots\dots$$

37

$$\frac{\partial u}{\partial x} = 2x$$
$$\frac{\partial u}{\partial y} = -\cos y \cos z$$
$$\frac{\partial u}{\partial z} = \sin y \sin z$$

37

If you had any difficulties you might revise the rules of differentiation of functions of one variable in Chapter 5.

--------------------▷ 38

A few remarks concerning the necessity of doing exercises. 38

In the study guide you are asked to solve a number of exercises and this helps to sort out your difficulties. Your aim should be to become independent of the study guide as you progress with the subject matter.

You should gradually work independently by selecting the exercises according to your needs.

Exercises act like red lights to warn you when you are experiencing difficulties. You must not ignore them; they are telling you that you missed something when you were studying the text and the worked exercises, perhaps owing to lack of concentration or because you did not work in parallel with the text as we explained earlier.

39

What is the geometrical meaning of

39

(1) the partial derivative $\frac{\partial f}{\partial x}$?

(2) the partial derivative $\frac{\partial f}{\partial y}$?

(3) What are the symbols for the partial derivatives of a function $f(x, y)$ with respect to x and y?

(4) What are the rules for partial differentiation with respect to x and with respect to y?

--------------------▷ 40

40

(1) $\dfrac{\partial f}{\partial x}$ gives the slope of the tangent to the surface $z = f(x, y)$ in the x-direction.

(2) $\dfrac{\partial f}{\partial y}$ gives the slope of the tangent to the surface $z = f(x, y)$ in the y-direction.

(3) $\dfrac{\partial f}{\partial x}$ or f_x $\dfrac{\partial f}{\partial y}$ or f_y.

(4) If $z = f(x, y)$ is differentiated partially with respect to x it means that we regard y as a constant and carry out the process of differentiation in the usual way.
Similarly when differentiating partially with respect to y, x is considered as a constant. We carry out the process of differentiation regarding y as variable.

Example: If $z = f(x,\ \ y) = \sin x + \cos y$

$$\text{then } \frac{\partial z}{\partial x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \quad \frac{\partial z}{\partial y} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 41

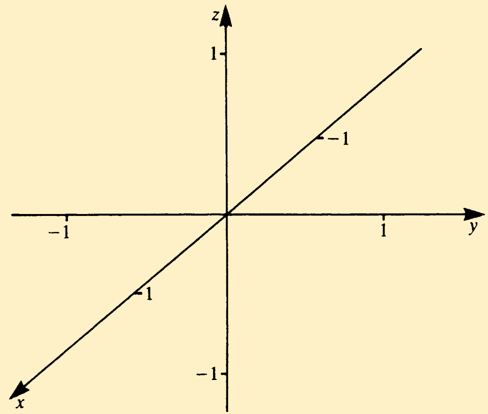
$$\frac{\partial z}{\partial x} = \cos x, \quad \frac{\partial z}{\partial y} = -\sin y$$

41

In what follows we shall look once more at the geometrical meaning of the partial derivatives f_x and f_y, using the sphere of unit radius as an example.

Many people find the geometrical illustration of mathematics very useful in gaining an understanding of a particular topic.

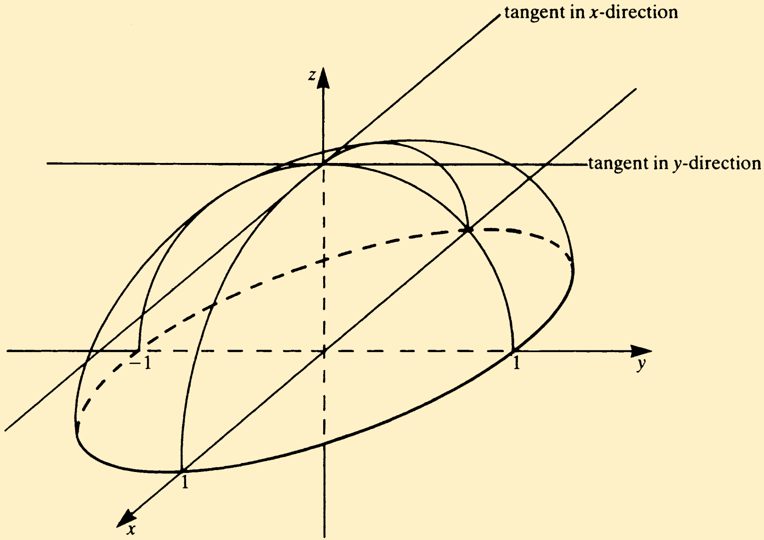
First sketch the upper half of the sphere of unit radius and then place tangents at the north pole P(0, 0, 1) in the x and y directions.



42

Correct your own sketch if necessary, or copy this diagram.

42



43

Now calculate the slope of the tangent at the point P(0, 0, 1) in the x-direction. For this we

43

(a) determine the partial derivative with respect to x, f_x, and
(b) insert the values for x and y at P(0, 0, 1) in f_x.

Note: The equation of the upper half of the unit sphere is

$$z = \sqrt{1 - x^2 - y^2}$$
$$f_x(0,\ 0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 45

Difficulties or explanations wanted --------------------▷ 44

Given: $z = \sqrt{1 - x^2 - y^2}$ 44

Required: $\dfrac{\partial z}{\partial x} = -\dfrac{x}{\sqrt{1 - x^2 - y^2}}$; y is regarded as a constant.

Now we insert the values: $x = 0$, $y = 0$; i.e.

$$\frac{\partial z}{\partial x}(0,\ 0) = \frac{0}{1 - 0 - 0} = 0$$

This means that the tangent is horizontal at that point.

45

$$f_x(0,\ 0) = 0$$

45

The tangent runs horizontally since its slope is zero.

It was $z = \sqrt{1 - x^2 - y^2}$

Now calculate $f_y(0,\ 0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution

--------------------▷ 47

Further explanation wanted

--------------------▷ 46

In this case x is treated as a constant. We must differentiate with respect to y. 46

$$z = \sqrt{1 - x^2 - y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}$$

Now we insert $x = 0$, $y = 0$. Thus the slope $\dfrac{\partial z}{\partial y}$ is zero, which means that the tangent is horizontal.

Given: $z = (1 - x)^2 + (1 - y)^2$

Obtain $\dfrac{\partial z}{\partial y}$ for $x = 1,\ \ y = 1$.

$$\frac{\partial z}{\partial y}(1,\ \ 1) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 47

-2

47

Now let us discuss higher partial derivatives.

In the textbook, section 12.3, we obtained the second partial derivative

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

of the function

$$f\,(x,\ \ y,\ \ z) = \frac{x}{y} + 2z$$

We obtained $f_{yx} = -\frac{1}{y^2}$

Now obtain the derivative

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$ of that same function and compare it with f_{yx}

$f_{xy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

48

$$f_{xy} = -\frac{1}{y^2}, \; f_{xy} = f_{yx}$$ 48

Explanation: Since $f(x, y, z) = \frac{x}{y} + 2z$

$\frac{\partial f}{\partial x} = \frac{1}{y}$; y and z are considered as constants.

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = -\frac{1}{y^2}$$

We observe that $f_{xy} = f_{yx}$.

This statement holds true for most functions encountered in physics and engineering provided that their partial derivatives are continuous.

Does $f_{zx} = f_{xz}$ hold true for that function?

$f_{zx} =$ ....

$f_{xz} =$ ....

--------------------▷ 49

Yes: $f_{zx} = 0$, $f_{xz} = 0$

49

The second derivative is independent of the order of differentiation provided that the first derivative is continuous and that the second derivative exists.

If you want another exercise ▷ 50

If you wish to carry on ▷ 52

If $f(x,\ y,\ z) = x^2y + y^2 + z^2x$

$f_{zx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\quad f_{xx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f_{xz} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\quad f_{yy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f_{yx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\quad f_{zz} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f_{xy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

50

51

$$\begin{array}{ll} f_{zx} = f_{xz} = 2z & f_{xx} = 2y \\ f_{yx} = f_{xy} = 2x & f_{yy} = 2 \\ & f_{zz} = 2x \end{array}$$

51

In case you do not agree with these answers the solution is given below for f_{zx} and f_{xx}.
Given: $f(x, y, z) = x^2y + y^2 + z^2x$
Partial derivative f_{zx}:

Step 1: $f_z = 2zx$ (x and y are regarded as constants)
Step 2: $f_{zx} = 2z$ (z is regarded as a constant)

Partial derivative f_{xx}:

Step 1: $f_x = 2xy + z^2$ (y and z are regarded as constants)
Step 2: $f_{xx} = 2y$ (y and z are regarded as constants)

52

52

Straight on

--------------------▷ 54

About the necessity of exercises:

In this study guide examples are worked out. Some of the foreseeable difficulties you may have are dealt with in detail in the text.

However, with regard to study techniques, your aim should be to try to work without the help of the given examples, using the techniques involved independently.

This means you choose and attempt the exercises according to necessity. Doing exercises is particularly important when they continue to prove difficult for you.

Exercises are indicators which objectively reveal existing problems of understanding that are not recognised subjectively. It is also important that you carry out corrections promptly.

--------------------▷ 53

If you do have difficulty with some exercises this may be due to one of two things: 53

(1) You have not mastered the contents of the textbook.
(2) The exercise requires the use of additional operations — knowledge of which we assume you already have but which you do not in fact possess.

Try to analyse your difficulties and to eliminate them with the aid of the relevant sections of the textbook.

At the end of the chapters in the textbook further exercises are given. Work out a sufficiently large number of these exercises.

--------------------▷ 54

12.3 Total Differential

54

Objective: Concepts of total differential of a function of two or three variables, contour lines, gradient.

READ: **12.4.1 Total differential of functions**
12.4.2 Application: Small tolerances
12.4.3 Gradient
Textbook pages 352–360

Remember to be active as you read the textbook. Work in parallel with the text by doing the examples on a piece of paper.

--------------------▷ 55

The total differential of a function $f(x, \ y, \ z)$ is defined as:

55

$$\mathrm{d}f = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

For small values of $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ the following approximation holds good in practice

$$\Delta f \approx \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 56

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

$$\Delta f \approx \mathrm{d}f$$

56

(Δf being the actual change in the function.)

Obtain an expression for the total differential of the function

$$f(x,\ y,\ z) = \frac{x}{y} + z$$

$$\mathrm{d}f = \dots\dots\dots\dots\dots\dots$$

Solution ▷ 58

Explanation and help wanted ▷ 57

The total differential of a function is defined to be

57

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$$

If $f = \dfrac{x}{y} + z$, the derivatives are

$$\frac{\partial f}{\partial x} = \frac{1}{y}, \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial f}{\partial z} = 1$$

Substituting in the above equation yields

$$\mathrm{d}f = \ldots\ldots\ldots\ldots\ldots\ldots$$

58

$$\mathrm{d}f = \frac{\mathrm{d}x}{y} - \frac{x}{y^2}\mathrm{d}y + \mathrm{d}z$$

58

In case of errors go back to the explanation given in frame 57.

The projection of lines of constant z-value on to the x–y plane is called ....

Given a surface, which we know already to be a paraboloid:

$$z = x^2 + y^2$$

Calculate the projection of the line $z = 9$ on to the x–y plane.

$$y = \dots\dots\dots\dots$$

59

a contour line

$$y = \pm\sqrt{9 - x^2}$$

59

Hint: We had to solve $z = 9 = x^2 + y^2$ for y.

The function $f(x,\ y) = z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$ represents the top half of an ellipsoid. You know it already.

Obtain the contour lines for

$z = 0$ $y =$ ....
$z = \frac{1}{4}$ $y =$ ....
$z = \frac{1}{2}$ $y =$ ....
$z = \frac{3}{4}$ $y =$ ....
$z = 1$ $y =$ ....

60

60

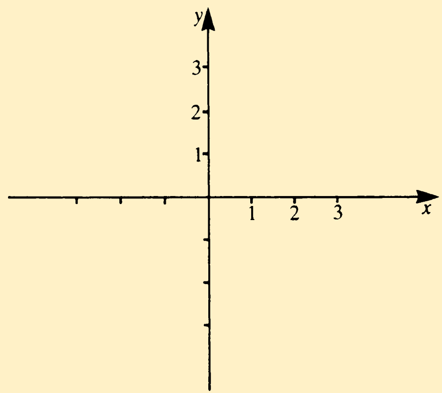
$z = 0,\;\; y = 3\sqrt{1 - \frac{x^2}{4}};\qquad z = \frac{1}{4},\;\; y = 3\sqrt{\frac{15}{16} - \frac{x^2}{4}}$

$z = \frac{1}{2},\;\; y = 3\sqrt{\frac{3}{4} - \frac{x^2}{4}};\qquad z = \frac{3}{4},\;\; y = 3\sqrt{\frac{7}{16} - \frac{x^2}{4}}$

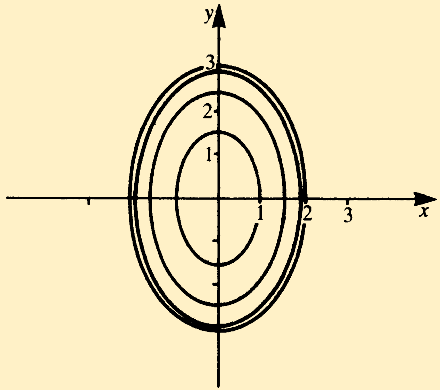
$z = 1,\;\; y = 0$ since $0 = -\frac{x^2}{4} - \frac{y^2}{9}$ is valid for $x = 0$ and $y = 0$ only

---

Sketch these contour lines.



61



61

If $z = 2x^2 + 5xy + 3y^2$

calculate the percentage error in taking the differential $\mathrm{d}z$ as an approximation to Δz when $x = 2.5$, $y = 5.75$, $\mathrm{d}x = 0.25$, $\mathrm{d}y = 0.15$.

$$\text{Percentage error} = \frac{\text{True value} - \text{Approximate value}}{\text{True value}} = \frac{\Delta z - \mathrm{d}z}{\Delta z}$$

Solution --------------------▷ 64

Further explanation wanted --------------------▷ 62

$$z = 2x^2 + 5xy + 3y^2$$

62

To calculate the percentage error, if we take the total differential $\mathrm{d}z$ as an approximation for Δz when $x = 2.5$; $y = 5.75$, $\mathrm{d}x = 0.25$, $\mathrm{d}y = 0.15$:

$$\text{Percentage error} = \frac{\text{True value} - \text{Approximate value}}{\text{True value}} = \frac{\Delta z - \mathrm{d}z}{\Delta z}$$

Step 1: Calculation of the approximate value. This is the approximate change in z if we use the total differential and insert the given values.

$$\mathrm{d}z = (4x + 5y)\mathrm{d}x + (5x + 6y)\mathrm{d}y$$
$$\mathrm{d}z = (4 \times 2.5 + 5 \times 5.75)0.25 + (5 \times 2.5 + 6 \times 5.75)\,0.15$$
$$\mathrm{d}z = 16.7375$$

--------------------▷ 63

Step 2: Calculation of the true change in z (true value).

63

$$\Delta z = f(x + \Delta x,\ \ y + \Delta y) - f(x,\ \ y)$$

$$f(2.5, 5.75) = 2(2.5)^2 + 5 \times 2.5 \times 5.75 + 3(5.75)^2 = 183.5625$$

$$f(2.5 + 0.25, 5.75 + 0.15) = 2(2.75)^2 + 5 \times 2.75 \times 5.9 + 3(5.9)^2 = 200.68$$

Thus

$$\begin{aligned}\Delta z &= 200.68 - 183.5625 \\ &= 17.1175\end{aligned}$$

Step 3: Calculation of the error by using dz as an approximation for the true change of the function Δz.

$$\Delta z - \mathrm{d}z = 17.1175 - 16.7375$$

Now calculate the percentage error:

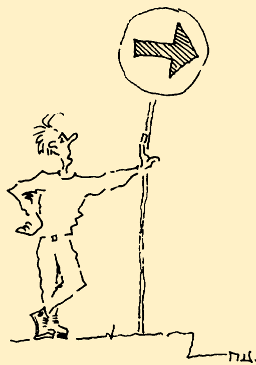
$$\begin{aligned}\text{Percentage error} &= \frac{\text{True value} - \text{Approximate value}}{\text{True value}} \times 100 \\ &= \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

--------------------▷ 64

$$\text{Percentage error} = \frac{\Delta z - \mathrm{d}z}{\Delta z} \times 100 = \frac{0.38 \times 100}{17.1175} = 2.22\%$$

64

If $u = f(x_1, \; x_2, \; x_3, \ldots, x_n)$ is a function of n independent variables x_i then provided that the change in their values, i.e. Δx_i, is small, the error in taking the total differential du as an approximation for the actual change Δu in the function is acceptable in many practical situations.



65

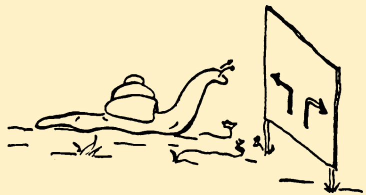
The length and diameter of a small cylinder are found to be 120 mm and 80 mm respectively when measured. If there is a probable tolerance of 0.15 mm in each measurement what is the approximate percentage tolerance in the value of the volume? 65

$$\frac{\delta V}{V} \times 100 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 66

$$\frac{\delta V}{V} \times 100 = 0.5\%$$

66



Correct 68

Wrong, or explanation wanted 67

67

Let d = diameter of the cylinder
l = length of the cylinder
Then the volume V is $V = \frac{\pi}{4}d^2 l$, i.e. $V = f(d, l)$
V is a function of the two independent variables d and l.
The tolerance of the volume is given by

$$\delta V = \sum \frac{\partial V}{\partial x_j} \mathrm{d}x_j$$

$$\delta V = V_d \delta d + V_l \delta l$$

We evaluate the partial derivatives and insert the given values.

$$V_d = \frac{\pi}{2}dl,\ V_d(80,\ 120) = 15\,079.65$$

$$V_l = \frac{\pi}{4}d^2,\ V_l(80,\ 120) = 5026.55$$

$$\delta V = (15079.65 + 5026.55)0.15 = 3015.93\,\mathrm{mm}^3$$

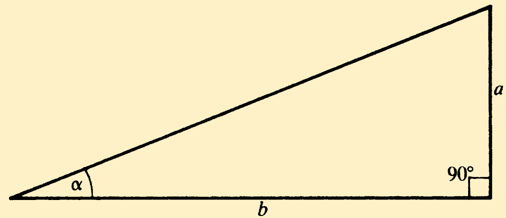
$$V(80,\ 120) = \frac{\pi}{4} \times 80^2 \times 120 = 603\,185.79\,\mathrm{mm}^3$$

Percentage error in $V = \dfrac{\delta V}{V} \times 100 = \dfrac{301\,593}{603\,185.79} = 0.5\%$

68

Now try the following problem.

68



A field in the shape of a right-angled triangle is measured: b is found to be 75.5 m and $\alpha = 25.5°$ with maximum errors of 0.5 m and 0.75° respectively. Calculate the percentage error in calculating the length a and the area A of the field.

Hint: Calculate in radians and convert $\delta\alpha$ into radians first.

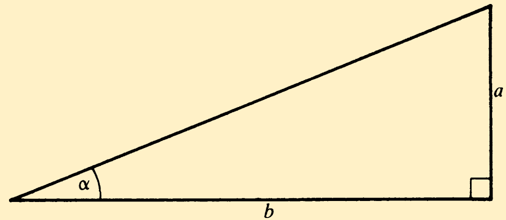
Solution 72

Further hints wanted 69

If you are in difficulty remember that from our knowledge of trigonometry we have: 69

Length: $a = b \tan \alpha$

Area: $A = \frac{ab}{2} = \frac{b^2 \tan \alpha}{2}$



Now try to evaluate $\frac{\delta a}{a} = \ldots\ldots\ldots\ldots\ldots\ldots \qquad \frac{\delta A}{A} = \ldots\ldots\ldots\ldots\ldots\ldots$

The relative error is given by $\frac{\delta a}{a}$ and $\frac{\delta A}{A}$.

The percentage error is given by $\frac{\delta a}{a} \times 100$ and $\frac{\delta A}{A} \times 100$.

Given $\qquad b = 75.5\,\text{m} \quad \alpha = 25.5^\circ$

$$\delta b = 0.5\,\text{m} \quad \delta\alpha = 0.75^\circ$$

Calculate the relative errors and hence the percentage errors.

Solution --------------------▷ 72

Detailed solution --------------------▷ 70

Consider the error in the length a of the triangle. By differentiating partially, first with respect to b and then with respect to α, we find 70

$\frac{\partial a}{\partial b} = \tan\alpha = \tan 25.5^\circ = 0.48$ to two decimal places

$$\frac{\partial a}{\partial \alpha} = \frac{b}{\cos^2\alpha} = \frac{75.5\,\text{m}}{\cos^2 25.5^\circ} = 92.68$$

Hence the error δa in a due to errors $\delta b = 0.5$, $\delta\alpha = 0.75^\circ$ is

$$\delta a = \frac{\partial a}{\partial b}\delta b + \frac{\partial a}{\partial \alpha}\delta\alpha = 0.48 \times 0.5 + \frac{92.68}{57.3} \times 0.75 = 1.45\,\text{m}$$

(Remember the factor 57.3 to convert 0.75° to radians!)

Percentage error in the length a

$$\frac{\delta a}{a} \times 100 = \frac{1.45 \times 100}{75.5\tan 25.5^\circ} = 4.03\%$$

----------------▷ 71

Now for the error in the area. 71

$$\frac{\partial A}{\partial b} = b\tan\alpha = 75.5\tan 25.5 = 36.01$$
$$\frac{\partial A}{\partial \alpha} = \frac{b^2}{2\cos^2\alpha} = \frac{75.5^2}{2\cos^2 25.5^\circ} = 3498.54$$

Hence $\delta A = \dfrac{\partial A}{\partial b}\mathrm{d}b + \dfrac{\partial A}{\partial \alpha}\mathrm{d}\alpha = 36.01 \times 0.5 + \dfrac{3498.54 \times 0.75}{57.3} = 63.80\,\mathrm{m}^2$

The percentage error in the measurement of the area is

$$\frac{\delta A}{A} \times 100 = \frac{6380}{\frac{1}{2} \times 75.5^2 \tan 25.5^\circ} = 4.69\%$$

72

Percentage error in the length a: 4.03%

72

Percentage error in the area A: 4.69%

In case of difficulties work through the detailed solution given in frames 69, 70 and 71 again.

Write down the two properties of the gradient:

Given the function $z = f(x, y)$

— The gradient is a .... vector to the contour lines. Thus it points in the direction of the ....

— The absolute value of the gradient is proportional to ....

73

— The gradient is a *normal* vector to the contour line. Thus it points in the direction of the greatest change in z. 73

— The absolute value of the gradient is proportional to the change in z per unit length in its direction.

Given: $z = f(x, y)$

Obtain

grad $f(x, y) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

74

$$\text{grad } f(x, y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

74

or

$$\text{grad } f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

If $f(x, y) = x^2 + y^2$

grad $f(x, y) =$ ....

If $u = x + y$

grad $u =$ ....

75

grad $f(x, y) = (2x, 2y)$ or grad $f(x, y) = 2x\mathbf{i}, 2y\mathbf{j}$
grad $u = (1, 1)$ *Note:* grad u is constant.

75

Obtain the gradient of the function

$$z = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$$

grad $z = \ldots\ldots\ldots\ldots\ldots\ldots$

Find the equation of the contour lines of the given function for $z = C$.

$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$

----------▷ 76

76

$$\operatorname{grad} z = \left(-\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}, -\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}} \right)$$

Contour line: $y = 3\sqrt{(1-c^2)-\frac{x^2}{4}}$

Obtain the direction of a tangent to the contour line with $z = C$.

$$\mathbf{t} = \left(1,\ \frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\mathbf{t} = (\text{..................................})$$

Direction of $\mathbf{t}$ and grad z: they are .... to each other.

77

$$\mathbf{t} = \left(1, -\frac{3x}{4\sqrt{(1-c^2) - \frac{x^2}{4}}}\right).$$

77

t and grad z are perpendicular to each other.

Now obtain

$\mathbf{t} \cdot \text{grad}\, z =$ ....

Remember: $\text{grad}\, z = \left(-\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}, -\frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\right)$

78

$$\mathbf{t} \cdot \text{grad} z = 0$$

78

The following type of function describes the potential energy in gravitational and electrical fields. Obtain the gradient.

$$P = \frac{P_0}{\sqrt{x^2 + y^2 + z^2}}$$

grad $P =$ ....

79

$$\operatorname{grad} P = \left(-\frac{P_0 x}{(x^2+y^2+z^2)^{3/2}}, -\frac{P_0 y}{(x^2+y^2+z^2)^{3/2}}, -\frac{P_0 z}{(x^2+y^2+z^2)^{3/2}}\right)$$

79

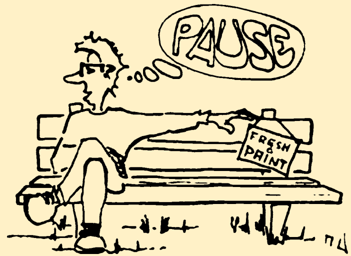
Obtain the magnitude of grad P.

$$|\operatorname{grad} P| = \ldots\ldots\ldots\ldots\ldots\ldots$$

----------▷ 80

$$|\mathrm{grad}\, P| = P_0 \frac{1}{x^2 + y^2 + z^2}$$ 80

Note: The gradient of a potential is equivalent to the acting force **F**. This relation is often used in mechanics and electricity.



81

12.4 Total Derivative

81

Objective: Concept of total derivative.

This section extends the concept of total differential of a function of two variables to two special cases.

In the first case it is assumed that the two variables both depend on a third. In the second case implicit functions are considered.

READ: 12.5 Total derivative
12.5.1 Explicit functions
12.5.2 Implicit functions
Textbook pages 360–363

82

If $z = f(x,\ \ y)$ and $x = x(t)$
$y = y(t)$

Obtain $\dfrac{\mathrm{d}z}{\mathrm{d}t} =$ ....

This expression is called ....

82

83

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ 83

total derivative or total differential coefficient

Let $z = x + y$ and $x = \sin t$
$y = \sin^2 t$

Obtain:
The total differential
....
The total derivative
....

84

Total differential

$$dz = dx + dy$$

84

Total derivative

$$\frac{dz}{dt} = \cos t + 2 \sin t \cos t$$

The function $z = x + y$ represents geometrically a ....
The function $z = x + y$, with further stipulations $x = \sin t$

$$y = \sin^2 t$$

represents a curve on the surface. It is a function of one variable only. The total derivative is the differential coefficient with respect to ....

This function can be represented by a graph in the z–t coordinate system.

85

surface in space; in this case it is a plane.

The total derivative is the differential coefficient with respect to t.

85

Consider an oil tank, the form of which is a cylinder of radius r and height h. Its volume is, of course,

$$V = \pi r^2 h$$

For arbitrary values of r and h this is a function of two variables. The size of the metal construction depends on the temperature t.

Let $h = h_0(1 + 16 \times 10^{-6} t)$

$r = r_0(1 + 16 \times 10^{-6} t)$

Obtain the total derivative of V.

$$\frac{dV}{dt} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 86

$$\frac{dV}{dt} = \pi 2rhr_0 \times 16 \times 10^{-6} + \pi r^2 h_0 \times 16 \times 10^{-6}$$ 86

Since $r \approx r_0$ and $h \approx h_0$ we can write

$$\begin{aligned}\frac{dV}{dt} &\approx \pi r^2 h(2 \times 16 \times 10^{-6} + 16 \times 10^{-6}) \approx 3\pi r^2 h \times 16 \times 10^{-6} \\ &\approx 3V \times 16 \times 10^{-6}\end{aligned}$$

Note: The same result could have been obtained by inserting the equation for $h(t)$ and $r(t)$ into $V = \pi r^2 h$ and differentiating with respect to t.

Given the result above

$$\frac{dV}{dt} \approx 3V \times 16 \times 10^{-6}$$

what is the percentage increase in volume if the temperature rises from 0° to 30°C?

Percentage increase: $\frac{\Delta V}{V}100 \approx$ ....

--------------------▷ 87

87

Percentage increase: $\frac{\Delta V}{V} 100 \approx \frac{\mathrm{d}V}{\mathrm{d}t} \Delta t \frac{1}{V} \times 100$

$$\frac{\Delta V}{V} \times 100 \approx 3 \times 16 \times 10^{-6} \times 30 \times 100 = 0.144\%$$

The percentage increase is small. It can be neglected in most practical cases.

Let $z = x^2 + y^2$
$x = e^t$
$y = e^{-t}$

Obtain the total differential

$$\mathrm{d}z = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Obtain the total derivative

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

88

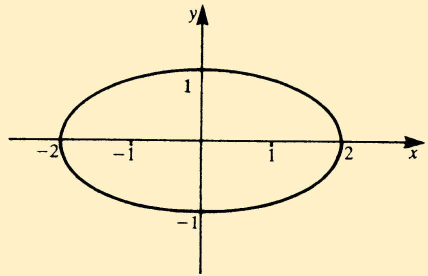
$$dz = 2x\,dx + 2y\,dy$$

$$\frac{dz}{dt} = 2e^{2t} - 2e^{-2t}$$

88

Let us now turn our attention to implicit functions.

Given: $x^2 + 4y^2 = 4$. It is the equation of an ellipse.



Obtain $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

89

$$\frac{dy}{dx} = -\frac{2x}{8y}$$

89

Hint: We have used the formula $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

for $f(x, \ y) = x^2 + 4y^2 - 4$, $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 8y$

To obtain $\frac{dy}{dx}$ as a function of x only we must use the given equation $x^2 + 4y^2 = 4$, solve for y and insert into the expression $\frac{dy}{dx}$.

Obtain $\frac{dy}{dx} = -\frac{2x}{8y}$ as a function of x only:

$$\frac{dy}{dx} = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

----------------▷ 90

$$\frac{dy}{dx} = -\frac{2x}{8\sqrt{1-\frac{x^2}{4}}}$$ 90

This result could also have been obtained by using the procedure outlined in Chapter 5, section 5.9.1 (implicit functions and their derivatives).

Now we give some hints on revision techniques and on preparing for examinations. Decide for yourself:

I'd like to find out a bit about exams and how to revise for them --------------------▷ 91

I don't have an exam for a few terms; I'd prefer to continue --------------------▷ 96

91

Examinations and their preparation.

Points of view about exams vary from suggestions for their total abolition to requests for stricter control and higher performance standards. We will not discuss the pros and cons of examinations but we must certainly tackle the problems that examinations pose.

Preparation for exams almost always takes place under pressure. This circumstance is conditioned partly by the individual, partly by the institution.

Here we would like to give a few, perhaps trivial, pieces of advice which may help to reduce the stress experienced under exam conditions.

92

(1) Analyse the demands and your competence.

92

Success in examinations depends largely on careful *analysis* and *planning*. In order to achieve this, you should first consider the following:

(a) What demands are made of you in the exam?
(b) Which of those demands can you already meet?
(c) Which knowledge do you still lack?

Then you should try to estimate how much time you need to acquire the desired knowledge. We recommend that you double the estimated time allowance since one normally underestimates the work load considerably and, besides that, you need to set aside a little extra time as a safety measure. You never know what may crop up!

--------------------▷ 93

(2) Plan your revision.

93

From the estimated time allowance, a *rough written study plan* for your exam preparation can be drawn up.

In the case of mathematics, for example, it is sufficient to link chapters with weeks. This helps you to distribute the material to be studied evenly over the available time. Of course it is much more difficult to keep to the study plan than it is to draw it up. This is because the further away an event (e.g. an examination) lies, the less seriously one takes it. With the help of a plan you can check to what extent the 'actual state' and 'desired state' truly correspond.

94

(3) Have a schedule for your revision. 94

Let us assume that we have notes from a textbook or course. They can be revised chapter by chapter.

First step: working through the chapter;

Second step: active recall and check with the aid of notes;

Third step: solution of exercises and problems;

Fourth step: deepening of your knowledge in certain specific fields.

A rather good work technique is to do the general preparation in a small group. Verbal communication of the meaning of terms and interrelations consolidates active knowledge. The next step is the solution of exercises and questions.

Group work enables you, among other things, to estimate your personal standard of knowledge by comparing it with that of your fellow students in the group.

95

Minimum demand for an exam: knowledge of the given terms and interrelations. This is a prerequisite for any deepening of understanding in the given field of study. 95

Skill in presenting and solving problems, and in the drawing of parallels between the different special fields of study within the subject, give above-average results — for you yourself and in the assessment of your performance.

--------------------▷ 96

12.5 Maxima and Minima of Functions of Two or More Variables

96

Objective: Concepts of maxima and minima of a function of two or more variables, calculation of the maxima and minima of given functions.

READ: 12.6 Maxima and minima of functions of two or more variables
Textbook pages 363–396

97

97

The function $z = f(x, y)$ depends on the independent variables x and y; it represents a surface in space whose shape may be quite complex. There may be 'hills' and 'valleys', i.e. points where the surface has a maximum or a minimum value.

Our task is to determine where these maxima and minima occur.

Suppose that x is kept constant at some value x_0; then $z = f(x_0, \ y)$ becomes a function of the single variable y, say $z = F(y)$. What is the first condition that z should have a maximum or a minimum and what does it mean geometrically?

The condition is that $\frac{dF}{dy} =$ ....

Geometrically it means that ....

....

--------------------▷ 98

$$\frac{\mathrm{d}F}{\mathrm{d}y} = 0$$ 98

Geometrically, it means that the slope of the tangent to the curve is zero; the tangent is horizontal, i.e. parallel to the y-axis.

Similarly, if y is kept constant at $y = y_0$, then $z = (x, \ y_0)$ becomes a function of the single variable x, say $z = G(x)$. The first condition to be satisfied for a maximum or a minimum is:

$$\frac{\mathrm{d}G}{\mathrm{d}x} = \text{............................}$$

Geometrically, this means ....

....

--------------------▷ 99

$$\frac{\mathrm{d}G}{\mathrm{d}x} = 0$$ 99

The slope of the tangent to the curve is zero; the tangent is horizontal, i.e. parallel to the x-axis.

Now consider the function $z = f(x, y)$. For this function to have a maximum or a minimum we must have:

....

....

------------------▷ 100

$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

100

Further explanation wanted on why both derivatives have to be zero?

☐ No

102

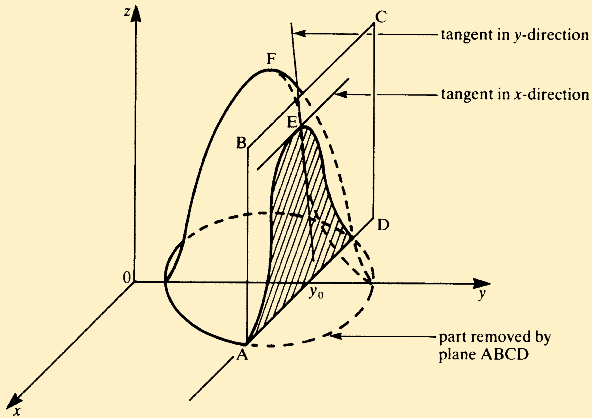
☐ Yes

101

With $z = f(x, y)$ we have a surface in space perhaps like the one shown below.

101

Suppose we cut through the surface with a plane $y = y_0$, ABCD. The intersection curve is a function of x. The slope of the tangent to it at E is zero. But the surface does not possess a maximum at E because the tangent in the y-direction is not zero. At F the tangents in the x and y directions are both zero. At F the surface has a maximum. For a maximum and a minimum we need:



$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

102

The surface $z = f(x, y)$ possesses maxima and minima if $f_x = 0$ *and* $f_y = 0$ simultaneously. 102

For what values of x and y will the function $z = x^2 + xy + y^2 - 6x + 2$ have a maximum or a minimum?

$x = \ldots\ldots\ldots\ldots\ldots\ldots,\quad y = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution --------------------▷ 104

Explanation and detailed solution wanted --------------------▷ 103

The conditions for a maximum or a minimum are 103

$$\frac{\partial f}{\partial x} = 0 \; and \; \frac{\partial f}{\partial y} = 0$$

Since $z = f(x, y) = x^2 + xy + y^2 - 6x + 2$ we have

$$f_x = 2x + y - 6 = 0 \qquad [2]$$

and

$$f_y = x + 2y = 0 \qquad [3]$$

To calculate the values of x and y we must solve equations 2 and 3 simultaneously.

What are these values?

$x =$ ...., $y =$ ....

104

$$x = 4, \quad y = -2$$

104

In case of difficulty work through the detailed solution given in frame 103.

We must now determine the conditions for a maximum, and those for a minimum.
Do you remember these conditions?

Yes --------------------▷ 107

I want a detailed explanation --------------------▷ 105

Suppose that at $x = x_1$ and $y = y_1$ the slopes of the tangent along the x and y axes are zero, i.e. the tangent plane formed by these two tangents is horizontal. To discover whether the function $z = f(x, \ y)$ at x_1 and y_1 has a maximum or a minimum we could proceed as follows. Let x_1 be increased by a small amount h, positive or negative, and y_1 be increased by an equally small positive or negative amount k. 105

We now examine the value

$$\Delta f = f(x_1 + h, \ \ y_1 + k) - f(x_1, \ \ y_1)$$

To do this we expand the function into a power series. The expansion is given without proof:

$$f(x_1 + h, \ \ y_1 + k) = f(x_1, \ \ y_1) + f_x(x_1, \ \ y_1)h + f_y(x_1, \ \ y_1)k$$
$$+ \frac{1}{2!} f_{xx}(x_1 y_1)h^2 + \frac{2}{2!} f_{xy}(x_1 y_1)hk + \frac{1}{2!} f_{yy}(x_1, \ \ y_2)k^2 + \ldots$$

If we insert into

$$\Delta f = f(x_1 + h, y_1 + k) - f(x_1, y_1)$$

we obtain

$$\Delta f = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

----------------------▷ 106

$$\Delta f = f_x(x_1 y_1) + f_y(x_1 y_1)k + \frac{1}{2!}f_{xx}(x_1,\ y_1)h^2 + \frac{2}{2!}f_{xy}(x_1,\ y_1)hk + \frac{1}{2!}f_{xx}(x_1,\ y_1)k^2 + \ldots$$

106

Compare this equation with the equivalent one in the textbook. They are the same. Only the notation is different. Now work through section 12.6 of the textbook again.

107

The mathematical condition to determine whether a function $z = f(x, \ y)$ of two independent variables possesses a maximum or a minimum are:
for a maximum ....
for a minimum ....
and in both cases ....

107

108

108

$f_{xx} < 0$ and $f_{yy} < 0$ for a maximum
$f_{xx} > 0$ and $f_{yy} > 0$ for a minimum
and $f_{xy}^2 - f_{xx} f_{yy} < 0$ in both cases
If you did not agree, or couldn't recall it, go back to section 12.6 or your notes.

We established earlier that the function

$$z = x^2 + xy + y^2 - 6x + 2$$

has a maximum or a minimum at $x = 4$, $y = -2$.

We want to decide whether it is a maximum or a minimum.

First we establish the derivatives at the point $(4, \; -2)$:

$$f_{xx}(4, \; -2) = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_{yy}(4, \; -2) = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_{xy}(4, \; -2) = \ldots\ldots\ldots\ldots\ldots\ldots$$

109

$$f_{xx} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$ 109

Solution correct

----------------▷ 110

If you had difficulties with the calculation of second derivatives the detailed solution is given here.

$$z = f(x, \; y) = x^2 + xy + y^2 - 6x + 2$$

We obtain the second derivatives.

$$\begin{aligned} f_x &= 2x + y - 6, & f_y &= x + 2y \\ f_{xx} &= 2, & f_{yy} &= 2 \end{aligned}$$

The second derivatives are constant. Thus they do not depend on the point (4, −2).

----------------▷ 110

Knowing the second derivatives $f_{xx} = 2$; $f_{xy} = 1$; $f_{yy} = 2$, decide whether the function has a maximum or a minimum at $x = 4$, $y = -2$. 110

☐ Maximum
--------------------▷ 111

☐ Minimum
--------------------▷ 112

Wrong!

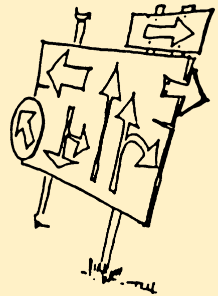
For a maximum $f_{xx} < 0$ and $f_{yy} < 0$. In our case $f_{xx} > 0$ and $f_{yy} > 0$, so the function has a minimum at $x = 4, \quad y = -2$. 111

112

Minimum is correct since $f_{xx} > 0$ and $f_{yy} > 0$. 112

But now the crucial question.

Are *all* the conditions satisfied? Have we established a minimum?



☐ No → 114

☐ Yes → 113

Wrong, unfortunately.

113

We have not yet examined the sign of

$$f_{xy}^2 - f_{xx} f_{yy} \quad \text{at} \quad x = 4, \;\; y = -2$$

114

To establish a minimum, finally, we check the last condition that must be satisfied. 114

Remember: $f_{xx}(4, \ -2) = 2$

$f_{yy}(4, \ -2) = 2$

$f_{xy}(4, \ -2) = 1$

$$f_{xy}^2 - f_{xx} f_{yy} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

115

$$f_{xy}^2 - f_{xx} f_{yy} = -3 < 0$$

115

The condition is satisfied; a minimum is established.

Now for another example. Given $z = x^3 y^2 (6 - x - y)$, determine whether this function has a maximum or a minimum.

Solution ▷ 119

Further explanation and detailed solution wanted ▷ 116

Let us go through the solution step by step.

116

Step 1: The function is $z = f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$. The first condition for a maximum or a minium is that the partial derivatives should be equal to zero, hence we have

$$f_x = \frac{\partial f}{\partial x} = 18x^2y^2 - 4x^3y^2 - 3x^2y^3 = x^2y^2(18 - 4x - 3y) = 0$$
$$f_y = \frac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3x^3y^2 = x^3y(12 - 2x - 3y) = 0$$

Step 2: Solve the two simultaneous equations to get the points of extreme value.
One solution is $x = y = 0$.
The other solution is found by solving

$$18 - 4x - 3y = 0$$
$$12 - 2x - 3y = 0$$

Subtracting yields $6 - 2x = 0$; hence $x = 3$, and substituting in one of the equations gives $y = 2$. Thus the second solution is $x = 3, \;\; y = 2$

--------------------▷ 117

Step 3: To examine the condition we need f_{xx}, f_{yy}, f_{xy} at the points (0, 0) and (3, 2).

$f_{xx}(0,0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad f_{xx} = (3,2) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f_{yy}(0,0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad f_{yy} = (3,2) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$f_{xy}(0,0) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad f_{xy} = (3,2) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

117

118

$$
\begin{array}{lll}
f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3 & f_{xx}(0,\ 0) = 0 & f_{xx} = (3,\ 2) = -144 \\
f_{yy} = 12x^3 - 2x^4 - 6x^3y & f_{yy}(0,\ 0) = 0 & f_{yy} = (3,\ 2) = -162 \\
f_{xy} = 36x^2y - 8x^3y - 9x^2y^2 & f_{xy}(0,\ 0) = 0 & f_{xy} = (3,\ 2) = -108
\end{array}
$$

118

Step 4: For a maximum, $f_{xx} < 0$ and $f_{yy} < 0$,
for a minimum, $f_{xx} > 0$ and $f_{yy} > 0$,
and $f_{xy}^2 - f_{xx}f_{yy} < 0$ in both cases.
Let us check first for the point (3, 2):

$$f_{xx} = -144, \quad f_{yy} = -162, \quad f_{xy} = -108$$

and $f_{xy}^2 - f_{xx}f_{yy} = -1164$

We therefore conclude that at the point $x = 3$, $y = 2$ the function possesses a ....

--------------------▷ 119

maximum

119

Now we check for the point $x = 0$, $y = 0$. We found $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 0$.
Thus the condition $f_{xy}^2 - f_{xx} f_{yy} < 0$ is not fulfilled.
The test gives no information.

120

Maximum at (3, 2).

Maximum, minimum or saddle point at (0, 0)?

No decision yet.

120

Let us recapitulate.

The condition for extreme values $f_x = 0$ and $f_y = 0$ gave two solutions. We established a maximum at (3, 2). But at the point $x = 0$, $y = 0$ the necessary condition for a maximum or a minimum is not fulfilled. We obtained

$$f_{xy}^2 - f_{xx} f_{yy} = 0$$

This does not agree with the condition

$$f_{xy}^2 - f_{xx} f_{yy} < 0$$

So far we cannot decide the question. The exercises might have been a bit exhaustive, and in the end disappointing. But after all you succeeded with this quite demanding section.

121

12.6 Wave Functions

121

The one-dimensional wave function is a function of two variables, a length and a time. It is a special but quite important case of a function of two variables.

READ: 12.7 Applications: Wave function and wave equation
Textbook pages 369–375

122

Write down the wave function for a one-dimensional wave travelling on a string in the x direction with amplitude A, velocity v and wavelength λ: 122

$$f(x,\ t) = \text{......................}$$

What is the circular frequency in terms of v or v and λ?

$$\omega = \text{.............................}$$

123

$$f(x,\ t) = A\sin(2\pi\frac{x}{\lambda} - \omega t)$$

123

or

$$f(x,\ t) = A\sin(\frac{2\pi}{\lambda}(x - vt))$$

$$\omega = 2\pi v = 2\pi\frac{v}{\lambda}$$

Given a wave function

$$f(x,\ t) = A\sin(\frac{2\pi x}{\lambda} - \omega t)$$

Can it also be written using the cosine function?

If yes, write it down

$$f(x,\ t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 124

Yes

$$f(x,\ t) = A\cos\left(\frac{2\pi x}{\lambda} - \omega t + \frac{\pi}{2}\right)$$

124

Given a wave function

$f(x,\ t) = 0.1\cos(4\pi x - 8\pi t)$, ($x$ in metres, t in seconds)

What is the wavelength? $\lambda =$ ....

What is the frequency? $\nu =$ ....

What is the phase velocity? $v =$ ....

What is the time of one oscillation of a given point x_0? $T =$ ....

125

$$\lambda = \frac{1}{2}\,\mathrm{m},\quad \nu = 4\,\mathrm{s}^{-1},\quad v = 2\,\mathrm{m/s},\quad T = \frac{1}{4}\,\mathrm{s}$$ 125

Given the same wave function.
$f(x,t) = 0.1\cos(4\pi x - 8\pi t)$

It is to describe the displacement of a fixed point on a string or cable. Obtain the displacement and velocity v of the point $x = 2$:

$f(2,\ t)$ = ...
$v(2,\ t)$ = ...
Maximum velocity = ...

Solution --------------------▷ 127

Further hints --------------------▷ 126

Given $f(x, \ t) = 0.1\cos(4\pi x - 8\pi t)$

126

The given function $f(x, \ t)$ represents the displacement of a point of the string depending on position x and time t. The function describes a wave with the phase velocity $v = 2\,\text{m/s}$.

Yet what is wanted is not the phase velocity but the velocity **v** of a point of the string at $x = 2$. The function $f(2, \ t)$ describes the oscillation of that given point.

$$f = (2, \ t) = 0.1\cos(8\pi - 8\pi t) = 0.1\cos(-8\pi t)$$

$$\text{v} = \frac{\mathrm{d}}{\mathrm{d}t} f(2, \ t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{v}_{\text{max}} = \ldots\ldots\ldots\ldots\ldots$$

→ 127

$$\mathbf{v} = -0.1(8\pi)\sin(-8\pi t)$$
$$\mathbf{v}_{\max} = 0.1 \times 8\pi = 2.512\,\text{m/s}$$

127

Given the one-dimensional wave equation

$$c^2 \frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial^2 f(x,t)}{\partial t^2}$$

show by verification that the following wave function is a solution:

$$f(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Solution 129

Detailed explanation wanted 128

Given: $f(x,\ t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$ 128

We want to show that it satisfies the relation

$$c^2\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial t^2}$$

We differentiate partially

$$\frac{\partial^2 f}{\partial x^2} = -A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$$\frac{\partial^2 f}{\partial t^2} = -A\left(\frac{2\pi}{\lambda}v\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

Inserting yields

$$-c^2 A\left(\frac{2\pi}{\lambda}\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right] = -A\left(\frac{2\pi}{\lambda}v\right)^2 \sin\left[\frac{2\pi}{\lambda}(x - vt)\right]$$

$c^2 = v^2$ and thus

$v = c$

-------------------▷ 129

The wave function is a solution of the wave equation with $v = c$. 129

Obtain the superposition of two waves with opposite velocities and equal frequencies:

$$f_1(x,\ t) = \sin\left(\frac{2\pi}{\lambda}x - \omega t\right)$$

$$f_2(x,\ t) = \sin\left(\frac{2\pi}{\lambda}x + \omega t\right)$$

$$f(x,\ t) = f_1(x,\ t) + f_2(x,\ t) = \text{...............................}$$

Solution 131

Further explanation wanted 130

Superposition means addition. We simply add both waves.

130

$$f = \sin\left(\frac{2\pi}{\lambda}x - \omega t\right) + \sin\left(\frac{2\pi}{\lambda}x + \omega t\right)$$

Using the addition formulae we obtain

$$\begin{aligned} f &= \sin\frac{2\pi}{\lambda}x\cos\omega t - \cos\frac{2\pi}{\lambda}x\sin\omega t + \sin\frac{2\pi}{\lambda}x\cos\omega t + \cos\frac{2\pi}{\lambda}x\sin\omega t \\ &= 2\sin\frac{2\pi}{\lambda}x\cos\omega t \end{aligned}$$

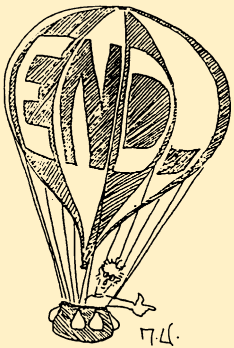
131

$$f(x,t) = 2\sin\frac{2\pi}{\lambda}x\cos\omega t$$

The resulting wave is called a standing wave.

The wave oscillates with the frequency ω at any position $x = x_0$. The amplitude of the wave is a function of x. The wave is at rest if $\sin\frac{2\pi}{\lambda}x$ becomes zero. These positions are called nodes.

For a node $\frac{2\pi}{\lambda}x = n2\pi$. Thus nodes occur at $x = n\lambda$ ($n = 1, 2, 3, \ldots$).



Standing waves can be interpreted as the superposition of two waves travelling in opposite directions.
In this book only rudimentary but interesting and basic aspects of waves could be included. More advanced textbooks must be consulted when this subject is needed, e.g. in your studies of mechanics, acoustics or electricity.

END OF CHAPTER 12

Chapter 13
Multiple Integrals; Coordinate Systems

2

Before starting a new topic we should recall the important concepts of the last chapter.

Name at least four of the most important concepts met within Chapter 12.

2

(1) ....

(2) ....

(3) ....

(4) ....

3

(1) function of several variables
(2) partial derivative
(3) total differential
(4) total derivative

3

---

(1) Given the function $f(x, y, z)$, give two of the notations used for the partial derivative with repect to y:
....
....

(2) Given the function $z = x^2 + y^2$, the total differential is:
....

4

(1) $\dfrac{\partial f}{\partial y}$, f_y

(2) $\mathrm{d}z = 2x\,\mathrm{d}x + 2y\,\mathrm{d}y$

4

The meaning of the differential is most important.

Complete the following sentence:

The total differential is a measure of the change in the function $z = f(x,\ y)$ if

....

5

Contents | First Page | Previous View | Next View | Print | Exit

The total differential is a measure of the change in the function $z = f(x, \ y)$ if x is increased by $\mathrm{d}x$ and y by $\mathrm{d}y$.

5

The total differential can be computed for functions of more than two variables. For example, the temperature T can be a function of three space coordinates. The total differential of the function $T = T(x, \ y, \ z)$ is a measure of the change in the temperature T if x, y and z are increased by $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$, respectively. Evaluate $\mathrm{d}T$ for $T(x, \ y, \ z)$

$$\mathrm{d}T = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

6

$$\mathrm{d}T = \frac{\partial T}{\partial x}\,\mathrm{d}x + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z}\,\mathrm{d}z$$

6

The last frames should have told you whether you ought to revise Chapter 12.

7

13.1 Multiple Integrals

7

The concept of multiple integrals generalises the concept of integrals of a single variable to several variables.

READ: 13.1 Multiple integrals
Textbook pages 379–381

8

The concept of multiple integrals in the textbook is developed following the considerations in Chapter 6. Instead of calculating areas we now calculate volumes. For example, the volume of a cube is the sum of all partial volumes, i.e. 8

$$V = \sum \Delta V_i$$

Each partial volume ΔV_i is the product of the lengths of the edges Δx_i, Δy_i and Δz_i.

To obtain the actual volume V we proceed to the limit by letting $\Delta V_i \rightarrow 0$, thus taking a larger and larger number of partial volumes N. Hence

$$V = \lim_{N \to \infty} \sum \Delta V_i = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum \Delta x \Delta y \Delta z$$

so that $V = \int \mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

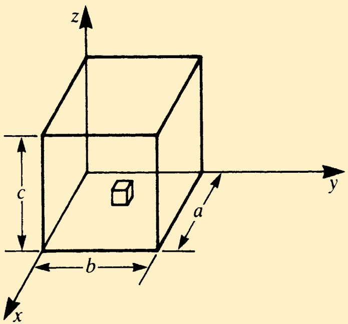
--------------------▷ 9

$$V = \int\int\int_v \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

9

This is a .... integral. It refers to the calculation of the volume of a cube.

Insert the limits of integration for each variable:



$$V = \int_{z=\ldots}^{\ldots}\int_{y=\ldots}^{\ldots}\int_{x=\ldots}^{\ldots} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

10

multiple integral

$$V = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

10

Exercises in the practical evaluation of integrals with constant limits will be given in the next section. There are no fundamentally new operations to learn.

11

13.2 Multiple Integrals with Constant Limits

11

Objective: Rules for the solution of one type of multiple integrals.

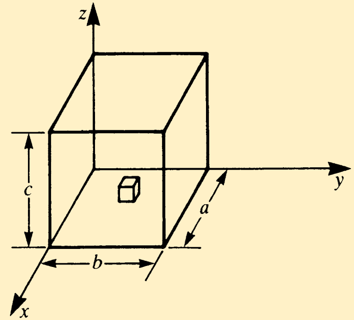
READ: **13.2 Multiple integrals with constant limits**
13.2.1 Decomposition of a multiple integral into a product of integrals
Textbook pages 381–383

12

The volume of a cube is a triple integral with constant limits of integration, i.e. 12

$$V = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

For the lower limit we explicitly denote the relevant variable (later on this can be dropped when there is no ambiguity) to ensure that no mistakes are made. Integration is carried out for one variable at a time; the other variables are assumed to be constant. The order of integration is immaterial. First integrate with respect to x:

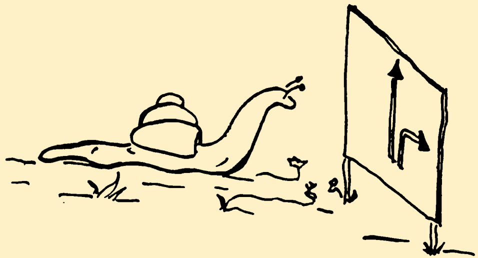


$V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

13

$$V = a \int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\,\mathrm{d}z$$

13



Correct ------------------▷ 15

Wrong, further explanation needed ------------------▷ 14

You were asked to evaluate the triple integral

14

$$V = \int_{x=0}^{c} \int_{y=0}^{b} \int_{z=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

with respect to x.

The rule is:

Except for x, all variables are treated as constants. We place in brackets what is considered constant, thus

$$V = \left[\int_{z=0}^{c} \int_{y=0}^{b} \right] \int_{x=0}^{a} \mathrm{d}x\,[\mathrm{d}y\,\mathrm{d}z]\,.$$

Now rearrange the equation so that all quantities considered constant for the time being are bracketed.

$$V = \int_{x=0}^{a} \ldots\ldots\ldots\ldots\ldots\ldots \left[\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

------------------▷ 15

$$V = \int_{x=0}^{a} \mathrm{d}x \left[\int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\, \mathrm{d}z \right]$$

15

The integral with respect to x can now be evaluated. The brackets are untouched during this operation.

$$V = \int_{x=0}^{a} \mathrm{d}x \left[\int \int \mathrm{d}y\, \mathrm{d}z \right] = (a-0) \left[\int \int \mathrm{d}y\, \mathrm{d}z \right]$$

Now we can do away with the brackets and evaluate

$$V = a \int_{z=0}^{c} \int_{y=0}^{b} \mathrm{d}y\, \mathrm{d}z$$

16

Evaluate the double integral

$$V = a\int_{z=0}^{c}\int_{y=0}^{b} \mathrm{d}y\,\mathrm{d}z$$

with respect to y:

$$V = \text{............................}$$

16

17

$$V = ab \int_0^c \mathrm{d}z$$

17

Now evaluate the final integral.

$$V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Errors or difficulties

--------------------▷ 18

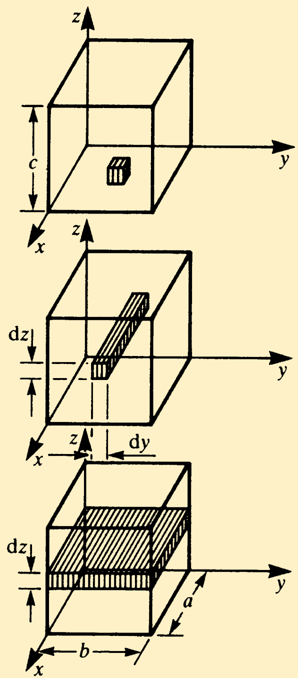
I want to go on

--------------------▷ 21

While integrating with respect to y we assume that all other variables are regarded as constants. You already know of something similar, namely partial differentiation. There we differentiated with respect to one variable at a time, regarding all others as constants. 18

Perhaps it will become clearer in the next frame when we consider the geometric meaning.

19



The volume integral

19

$$\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

had to be evaluated.

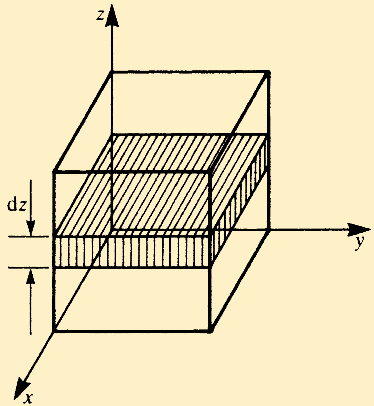
The geometric meaning of integration with respect to x is that elemental volumes are added together in the x-direction forming a column having a base equal to $\mathrm{d}y\,\mathrm{d}z$. This column is now the new integrand

$$\int_{z=0}^{c}\int_{y=0}^{b} a\,\mathrm{d}y\,\mathrm{d}z$$

The geometric meaning of integration with respect to y is that all the elemental columns are added together in the y-direction. A slice is then generated having a base equal to ab and thickness $\mathrm{d}z$.

Hence $\int_{z=0}^{c} ab\,\mathrm{d}z$ is left to be evaluated.

20



If we integrate with respect to z we are effectively adding up all the slices in the z-direction. The result is a cube of volume

20

$$V = abc$$

Let us recapitulate. The first integration corresponds to adding up the elemental volumes in one direction, giving rise to a column. The second integration is a summation of all the columns in the second direction, giving rise to a slice. The third integration is a summation in the third direction of all the slices; with this final step we have generated a cube and obtained its volume.

21

$$V = abc$$

21

Given the double integral

$$\int_{x=0}^{1} \int_{y=0}^{2} x^2 \, \mathrm{d}x \, \mathrm{d}y$$

to evaluate the inner integral we rearrange and place the inner integral in brackets. Carry out this step; which of the following is correct?

☐ $\displaystyle\int_{y=0}^{2} \left[\int_{x=0}^{1} x^2 \, \mathrm{d}x \right] \mathrm{d}y$ --------------------▷ 22

☐ $\displaystyle\int_{x=0}^{1} \left[\int_{y=0}^{2} x^2 \, \mathrm{d}x \right] \mathrm{d}y$ --------------------▷ 23

Correct!

22

You have realised that to integrate with respect to x we must first place the integrals in the right order.

Evaluate the integral.

25

You made a mistake!

If we integrate with respect to a particular variable we have to make sure that we take the limits appropriate to that variable.

23

The integral was

$$\int_{x=0}^{1}\int_{y=0}^{2} x^2\,\mathrm{d}x\,\mathrm{d}y$$

If we integrate with respect to x we have to place the limits in such a way that we can evaluate the inner integral first. Enter the appropriate limits

$$\int \left[\int x^2\,\mathrm{d}x\right]\mathrm{d}y$$

--------------------▷ 24

$$\int_{y=0}^{2}\left[\int_{x=0}^{1} x^2\,\mathrm{d}x\right]\mathrm{d}y$$

24

Now you should be able to solve the inner integral and substitute the correct limits.

$$\int_{y=0}^{2}\left[\dots\dots\dots\dots\dots\dots\right]\mathrm{d}y$$

25

$$\int_{y=0}^{2} \left[\frac{1}{3}\right] \mathrm{d}y = \frac{2}{3}$$

25

Note:

(1) A multiple integral with constant limits can be reduced to the successive evaluation of definite integrals.

(2) When evaluating an integral with respect to a particular variable we have to ensure that the limits inserted belong to that variable.

--------------------▷ 26

It is sometimes possible to decompose an integrand into a product of functions if each function depends on a single variable. In this case integration is particularly easy. 26

Which of the following integrands can be expressed as a product of independent functions?

(A) $\displaystyle\int_{x=0}^{2}\int_{y=1}^{2} \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y$

(B) $\displaystyle\int_{x=0}^{2}\int_{y=1}^{2} x\left(x+\frac{1}{y^2}\right)\mathrm{d}x\,\mathrm{d}y$

☐ A 27

☐ B 28

☐ A and B 29

Correct!

27

Now decompose the integrand into the product of independent functions.

$$\int_{x=0}^{2}\int_{y=1}^{2}\frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots$$

31

Wrong; the integrand was $x\left(x+\dfrac{1}{y^2}\right)$. The bracket contains both x and y. Try again! Which of the following functions can be decomposed into the product of functions which depend on one variable only?

28

- ☐ $f_1 = (x+2y)y$
- ☐ $f_2 = (x+x^2)(y+y^2)$
- ☐ $f_3 = \sin x \cos y$
- ☐ $f_4 = (\sin x + \sin y)y$

30

Wrong, unfortunately. You are right with the first integrand. 29

$\frac{x^2}{y^2}$ is a product of two functions each of which depends on one variable only.

But you are wrong with the second integrand: $x\left(x+\frac{1}{y^2}\right)$. The bracket contains both x and y and therefore depends on two variables.

Which of the following functions can be decomposed into the product of functions which depend on one variable only?

- ☐ $f_1 = (x+2y)y$
- ☐ $f_2 = (x+x^2)(y+y^2)$
- ☐ $f_3 = \sin x \cos y$
- ☐ $f_4 = (\sin x + \sin y)y$

→ 30

$$f_2 = (x + x^2)(y + y^2) = g(x)h(y)$$
$$f_3 = (\sin x)(\cos y) = g(x)h(y)$$

30

Decompose the integral into a product:

$$\int_{x=0}^{2}\int_{y=1}^{2} \frac{x^2}{y^2}\,\mathrm{d}x\,\mathrm{d}y = \ldots\ldots\ldots\ldots\ldots$$

31

$$\left(\int_0^2 x^2\,\mathrm{d}x\right)\left(\int_1^2 \frac{1}{y^2}\,\mathrm{d}y\right)$$

31

Now evaluate this integral!

32

$$\left(\int_{x=0}^{2} x^2\,\mathrm{d}x\right)\left(\int_{y=1}^{2} \frac{1}{y^2}\,\mathrm{d}y\right) = \frac{4}{3}$$ 32

Correct 34

Wrong, or further explanation wanted 33

$$\text{Let} \quad I = \left(\int_{x=0}^{2} x^2 \, \mathrm{d}x \right) \left(\int_{y=1}^{2} \frac{\mathrm{d}y}{y^2} \right)$$ 33

Since the functions and hence the integrals are independent we can solve them separately

(1) $I_1 = \int_0^2 x^2 \, \mathrm{d}x = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$

(2) $I_2 = \int_1^2 \frac{\mathrm{d}y}{y^2} = \left[-\frac{1}{y} \right]_1^2 = -\frac{1}{2} - (-1) = \frac{1}{2}$

Hence $I = I_1 I_2 = \frac{8}{3} \times \frac{1}{2} = \frac{4}{3}$

34

34

Evaluate the following integral, either by successive integration or by decomposing the integrand into a product.

$$A = \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/2} \sin\phi \cos\theta \, d\phi \, d\theta = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

35

$$A = 2$$

35

Further exercises are given in the textbook.

Exercises are most useful if you are not quite 100 per cent correct.

36

13.3 Multiple Integrals with Variable Limits

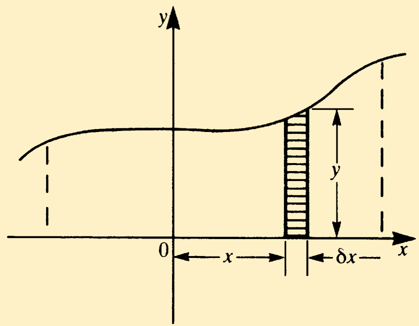
36

In general, multiple integrals have variable limits of integration, in which case the order of integration is important.

Objective: Evaluation of multiple integrals with variable limits.

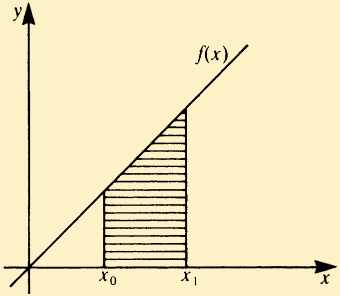
READ: 13.3 Multiple integrals with variable limits
Textbook pages 384–388

37



One aspect of this section is theoretically very interesting: the evaluation of an area leads to a double integral. The area problem was discussed in Chapter 6. It now turns out to be a special case where one integration has already been carried out without out being aware of it. We considered strips in the y-direction of width dx and height y; this is, in fact, the result of first integrating along the y-direction. 37

38



Let us investigate the practical aspects of the evaluation of multiple integrals. 38

Example: Area under the curve $y = x$ in the interval $x_0 \le x \le x_1$.

The area integral

$$A = \int \mathrm{d}A$$

becomes in terms of cartesian coordinates the integral

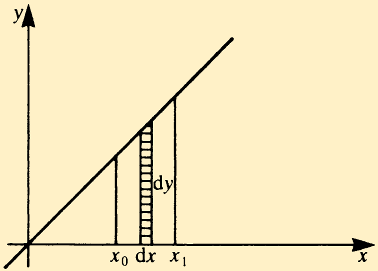
$$A = \int \int \mathrm{d}x\,\mathrm{d}y$$

Insert the limits for both variables:

$$A = \int_{y=\ldots}^{\ldots} \int_{x=\ldots}^{\ldots} \mathrm{d}x\,\mathrm{d}y$$

39

$$A = \int_{y=0}^{f(x)} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y = \int_{y=0}^{x} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y$$ 39



The limits of integration for x are obvious, but the limits for y are from 0 to the value $y = f(x)$. The functional value in this case is $y = x$.

In this example the order of integration is no longer arbitrary. The following rule must be observed:

We take for the inner integral the one whose limits of integration contain variables, and for the outer integral the one that has constant limits. Rearrange the given integrals and write the inner integral in brackets:

$A = \int_{y=0}^{x} \int_{x=x_0}^{x_1} \mathrm{d}x\,\mathrm{d}y$ $\qquad$ $A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

40

$$A = \int_{x=x_0}^{x_1} \left[\int_{y=0}^{x} \mathrm{d}y \right] \mathrm{d}x$$

40

Given the following integral:

$$I = \int_{u=a}^{v} \int_{v=1}^{2} uv \, \mathrm{d}u \, \mathrm{d}v$$

Rearrange the integral in such a way that the bracketed one is the first one to evaluate.

$$I = \int \left[\int \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]$$

41

$$I = \int_{v=1}^{2} \left[\int_{u=a}^{v} uv \, \mathrm{d}u \right] \mathrm{d}v$$

41

We have to integrate with respect to u first since its upper limit is a function of the variable v.

Rearrange the following integral so that it can be evaluated in the correct order.

$$I = \int_{x=0}^{y^2} \int_{y=0}^{z} \int_{z=0}^{1} xyz \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$I = \int \left[\int \left[\int \ldots\ldots\ldots \right] \right]$$

42

$$I = \int_{z=0}^{1} \left[\int_{y=0}^{z} \left[\int_{x=0}^{y^2} xyz \, \mathrm{d}x \right] \mathrm{d}y \right] \mathrm{d}z$$

42

Some careful rearranging was necessary in this case.

Now solve the integral!

$$I = \dots\dots\dots\dots\dots\dots$$

Solution --------------------▷ 46

Explanation and/or help is needed --------------------▷ 43

43

We rearrange the integral into a form suitable for evaluation. We have to choose as the inner integrals those whose limits are variable. The outermost integral must have constant limits.

In our example we have

$$I = \underbrace{\int_{z=0}^{1} \underbrace{\int_{y=0}^{z} \underbrace{\int_{x=0}^{y^2} xyz\,\mathrm{d}x}_{\text{Inner integral}} \;\mathrm{d}y}_{\text{Intermediate integral}} \;\mathrm{d}z}_{\text{Outer integral with constant limits}}$$

For the time being evaluate the inner integral

$$I = \int\int\left[\int_{x=0}^{y^2} xyz\,\mathrm{d}x\right]\mathrm{d}y\,\mathrm{d}z$$

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

----------------▷ 44

$$I = \int_{z=0}^{1} \int_{y=0}^{z} yz \, \frac{y^4}{2} \, \mathrm{d}y \, \mathrm{d}z$$

44

Remember that, while integrating with respect to x, the variables y and z are considered constant. The integral of x is $\frac{x^2}{2}$ substituting the limits $x = 0$ and $x = y^2$ gives $\frac{y^4}{2}$.

Now evaluate the next one (it was the intermediate one):

$$I = \int_{z=0}^{1} \Big[\ldots\ldots\ldots\ldots \Big] \, \mathrm{d}z$$

45

$$I = \int_{z=0}^{1} \frac{z^6}{12} z \; \mathrm{d}z = \int_0^1 \frac{z^7}{12} \mathrm{d}z$$

45

Finally, evaluate this last integral (which was the outer one):

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

46

$$I = \frac{1}{12 \times 8} = \frac{1}{96}$$ 46

In future it should be reasonably obvious which integral is to be the innermost and which is to be the outermost. The intermediate integrals should fall into place.

Have a break!

47

13.4 Coordinates: Polar, Cylindrical, Spherical

47

Objective: Concepts of polar, cylindrical and spherical coordinates, transformation of coordinates, applications to problems.

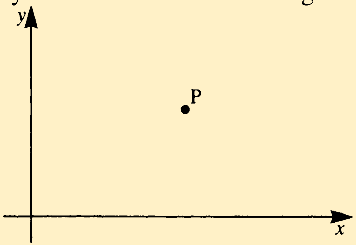
We have already met polar coordinates; cylindrical and spherical coordinates are new. The choice depends on the nature of the problem; it should be based on the ease with which the particular problem can be solved.

You know that working parallel with the textbook, pencil in hand, makes sense!

READ: **13.4 Coordinate systems**
13.4.1 Polar coordinates
13.4.2 Cylindrical coordinates
Textbook pages 383–393

48

Do you remember the following?



48

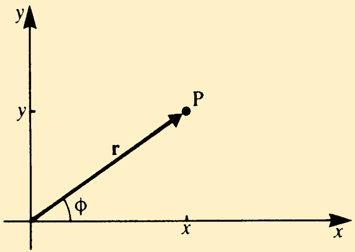
A point P is to be described in polar coordinates.

Name the two variables:

.... and ....

and place them on the drawing.

49



Length of the position vector **r**: r.
Angle with the x-axis: ϕ

49

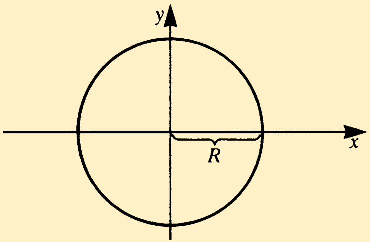
Write down the transformation equations

$$x = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$y = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$r = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\tan\phi = \ldots\ldots\ldots\ldots\ldots\ldots$$

50

$x = r\cos\phi$
$y = r\sin\phi$
$r = \sqrt{x^2 + y^2}$
$\tan\phi = \dfrac{y}{x}$

50



What is the equation of the central circle of radius R in polar coordinates?

....

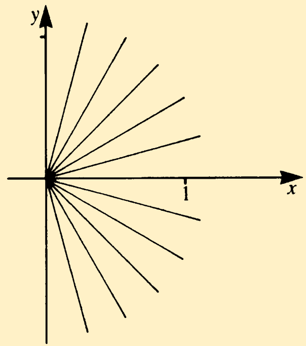
51

$$r = R$$

51

Note: In polar coordinates the equation of the central circle is just as simple as the equation of a straight line parallel to the x-axis (i.e. $y = a$) in Cartesian coordinates.

Sketch the function $r = \cos\phi$ by calculating the value of r for each value of ϕ and mark it on the sketch.



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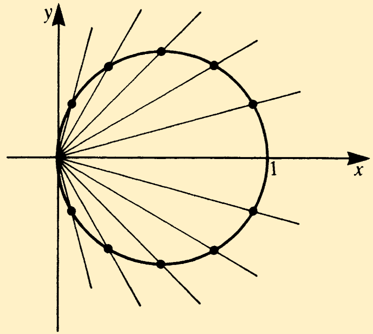
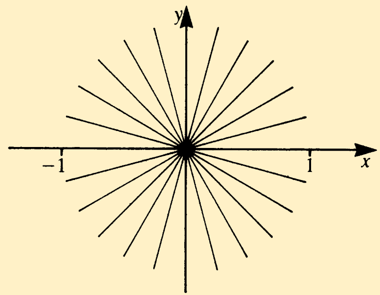


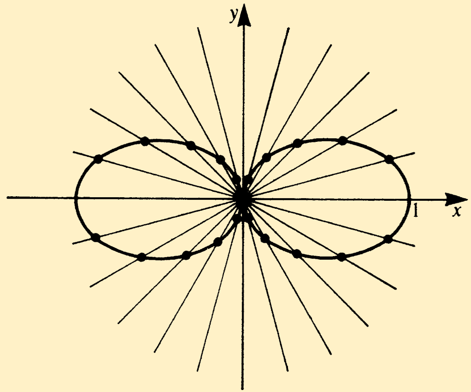
52

Sketch the function $r^2 = \cos 2\phi$.

Note: The function is defined for positive and negative values of ϕ



53



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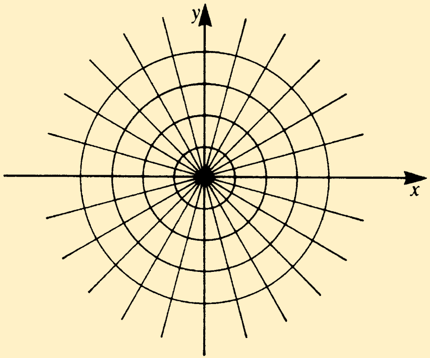


54

In Cartesian coordinates the lines

$x =$ constant and

$y =$ constant

are perpendicular to each other.

Similarly, in polar coordinates the lines

$r =$ constant and

$\phi =$ constant

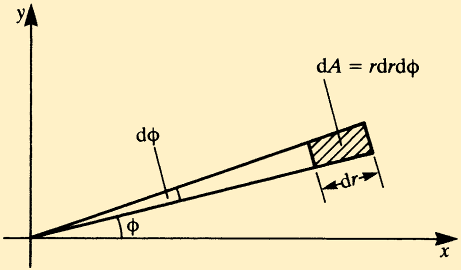
are perpendicular to each other.

Can you write down the element of area dA in polar coordinates?

$$\mathrm{d}A = \ldots\ldots\ldots\ldots\ldots\ldots$$

Can you give a drawing of dA?

--------------------▷ 55



$$dA = r\,dr\,d\phi$$

55

The area of a circle, for example, can now be computed easily.

$$A = \int dA$$

The integral can be written as a double integral, bearing in mind that we have two variables r and ϕ. To cover the entire circle r ranges from 0 to R and ϕ from 0 to 2π radians.

Hence, in polar coordinates

$$A = \dots\dots\dots\dots\dots\dots\dots\dots$$

56

$$A = \int_0^{2\pi} \int_0^R r \, \mathrm{d}r \, \mathrm{d}\phi$$

56

The limits are constant in this case.

What is the value of the integral?

$$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

57

$$A = \pi R^2$$

57

The drawing shows the graph of a curve known as the cardioid whose equation is

$$r = 1 + \cos\phi$$

Using the knowledge you have gathered so far calculate the area of the cardioid.

$$A = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

58

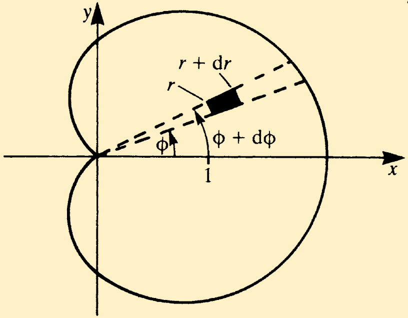
$$A = \int_{\phi=0}^{2\pi} \int_{r=0}^{R=1+\cos 2\phi} r\,\mathrm{d}r\,\mathrm{d}\phi = \frac{3}{2}\pi$$ 58

Did you obtain this result?

Yes 61

No 59

Never mind! Follow the solution carefully:



59

Consider the graph of the curve $r = 1 + \cos\phi$ and the elemental area shown in the figure.

Its area is $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\phi$.

To obtain the total area we have to integrate along r and along ϕ. To do this we must determine the limits of integration.

Since r is a function of ϕ we must integrate with respect to r first, whereas the limits for ϕ are constant.

Hence the area is given by

$$A = \int_0^{2\pi} \int_0^{1+\cos\phi} r\,\mathrm{d}r\,\mathrm{d}\phi$$

60

Now we must solve the integral: $A = \int_{\phi=0}^{2\pi} \int_{r=0}^{1+\cos\phi} r \, \mathrm{d}r \, \mathrm{d}\phi$ 60

Evaluating the inner one:

$$A = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{1+\cos\phi} \mathrm{d}\phi = \frac{1}{2}\int_0^{2\pi} (1+\cos\phi)^2 \, \mathrm{d}\phi$$
$$A = \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\phi + \cos^2\phi) \, \mathrm{d}\phi$$

You should be able to solve this integral with the knowledge acquired in Chapter 6. If necessary consult the table of integrals.

$$A = \frac{1}{2}\left[\phi - 2\sin\phi + \frac{\phi}{2} + \frac{\sin\phi\cos\phi}{2}\right]_0^{2\pi}$$

Hence $A = \frac{3}{2}\pi$.

61

Now let us extend this exercise and obtain the position of the centroid of the cardioid. This is a harder exercise. 61

If A = total area

$\bar{x}$ = position of the centroid, then

$$A\bar{x} = \text{..........................}$$

62

$$A\bar{x} = \int_A x \, \mathrm{d}A$$ 62

Since $x = r \cos\phi$ and $\mathrm{d}A = r \mathrm{d}r \; \mathrm{d}\phi$:

$$A\bar{x} = \int_0^{2\pi} \int_0^{1+\cos\phi} r \cos\phi \, r \, \mathrm{d}r \, \mathrm{d}\phi$$

Now evaluate the integral and determine $\bar{x} =$ ....

After that, determine $\bar{y} =$ .... (Think!)

Solution → 63

Help or explanation wanted → 64

$$\bar{x} = \frac{5}{6}, \quad \bar{y} = 0$$ 63

Correct 66

Wrong, or explanation wanted 64

Follow the solution carefully. Use a piece of paper and work in parallel; don't just read the explanation, you might miss something:

64

$$A\bar{x} = \int_0^{2\pi} \int_0^{1+\cos\phi} r^2 \cos\phi \, dr \, d\phi$$

We must integrate with respect to r first, since its upper limit is variable.

This yields

$$A\bar{x} = \int_0^{2\pi} \left[\frac{r^3}{3}\right]_0^{1+\cos\phi} \cos\phi \, d\phi$$
$$= \frac{1}{3} \int_0^{2\pi} (1+\cos\phi)^3 \cos\phi \, d\phi$$

Now we integrate with respect to ϕ, but first we must expand the integrand.

$$A\bar{x} = \frac{1}{3} \int_0^{2\pi} (\cos\phi + 3\cos^2\phi + 3\cos^3\phi + \cos^4\phi) \, d\phi$$

The result is ....

--------------------▷ 65

$$A\bar{x} = \frac{1}{3}\left[\sin\phi + 3\left(\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right) + 3\left(\frac{\sin\phi\cos^2\phi}{3} + \frac{2}{3}\sin\phi\right) + \frac{1}{4}\sin\phi\cos^3\phi + \frac{3}{4}\left(\frac{1}{2}\phi + \frac{1}{4}\sin 2\phi\right)\right]_0^{2\pi}$$

65

Substituting the limits yields

$$A\bar{x} = \frac{5}{4}\pi$$

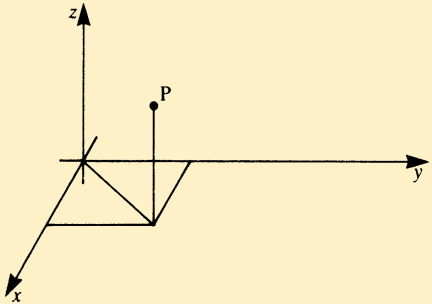
Since $A = \frac{3}{2}\pi$ it follows that

$$\bar{x} = \frac{5}{6}$$

Because of symmetry $\bar{y} = 0$, i.e. the centroid lies on the x-axis.

This was quite a hard exercise in integration!

66



Indicate the position of a general point P whose cartesian coordinates are (x, y, z) in cylindrical coordinates:

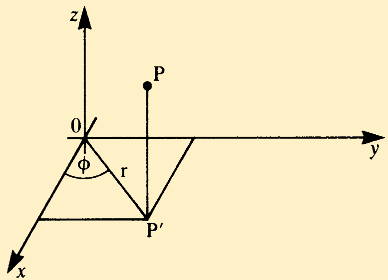
$x =$ ....

$y =$ ....

$z =$ ....

66

67



$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

67

What are the cylindrical coordinates of the point P(−1, 1, 3)?

$r =$ ....

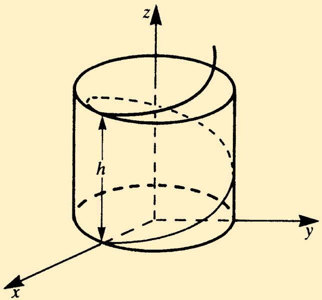
$\tan\phi =$ ....

$z =$ ....

68

$$r = \sqrt{2},\ \tan\phi = -1,\ z = 3$$

68



The figure shows a sketch of a helix of radius R and pitch h.

Can you write down the equation for it?

....

....

69

$$r = R,\ z = \frac{h}{2\pi}\phi$$

69

Wrong, or explanation wanted

70

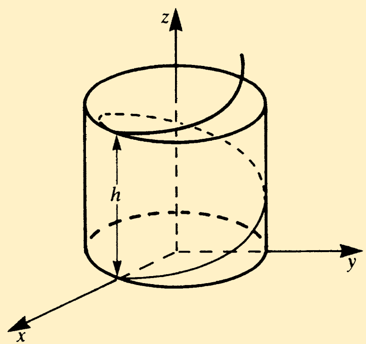
Correct

71

The helix is such that the distance r from the z-axis (central axis) is constant and equal to its radius R. 70

The projection of the helix on the $x-y$ plane is a circle for which we know the equation. The height z depends on the angle ϕ. In one revolution the height increases by an amount h (known as the pitch of the helix). It follows that

$$z = \frac{h}{2\pi}\phi.$$



71

Now derive the elemental volume in cylindrical coordinates by yourself. Check it with the textbook. 71

$$\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

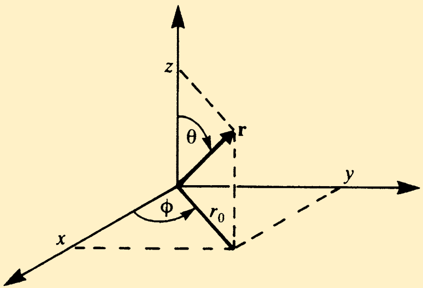
72

13.5 Spherical Coordinates

72

READ: **13.4.3 Spherical coordinates**
Textbook pages 393–396

73



Express the cartesian coordinates in terms of the spherical ones, referring only to the figure.

73

$x =$ ....

$y =$ ....

$z =$ ....

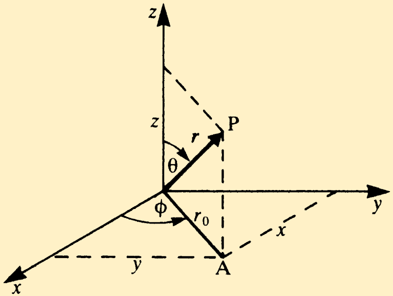
74

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

74

Correct 77

Wrong 75



Let's have a look at the diagram.

First we calculate the projection of r (the length of the position vector from O to P) on to the $x-y$ plane:

75

$$r_0 = r\sin\theta.$$

Remaining in the $x-y$ plane, we have by simple trigonometry

$$x = r_0\cos\phi = r\sin\theta\cos\phi$$
$$y = r_0\sin\phi = r\sin\theta\sin\phi$$

Now we consider the triangle OAP and get $z = r\cos\theta$.

Thus the transformation from spherical to cartesian coordinates is

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

------------------▷ 76

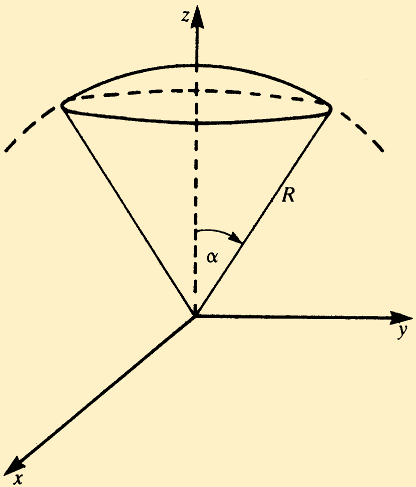
There is no particular need to memorise the expression for the element of volume $\mathrm{d}V$ in the case of spherical coordinates. What is important at this stage is: 76

(1) that you understand the derivation, and
(2) that you know where to find it when required, either in a table or in the textbook.

77

Let us try an example using spherical coordinates.

77



The figure shows a spherical sector of radius R, angle α.

We require its volume.

The elemental volume dV is

$$\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots$$

78

$$dV = r^2 \sin\theta dr \; d\theta \; d\phi$$

78

The volume is given by the integral. Insert the limits:

$$V = \int_{\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots\ldots}^{\ldots\ldots} \int_{\ldots\ldots}^{\ldots\ldots} r^2 \sin\theta \; dr \; d\theta \; d\phi$$

79

$$V = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\alpha} \int_{r=0}^{R} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

79

Note that all the limits of integration are constant.

Now evaluate the volume:

$$V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

80

$$V = \frac{2}{3}\pi R^3(1 - \cos\alpha)$$

80

Correct ⇢ 82

Wrong ⇢ 81

81

The integral was $V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{R} r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$.

The expression consists of simple functions, namely r^2 and $\sin\theta$. Let us integrate with respect to r first.

$$V = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\left[\frac{r^3}{3}\right]_0^R \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

Remember that while you integrate with respect to r, θ and ϕ are treated as constants.

$$V = \frac{R^3}{3}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$

We can now integrate with respect to θ; this gives

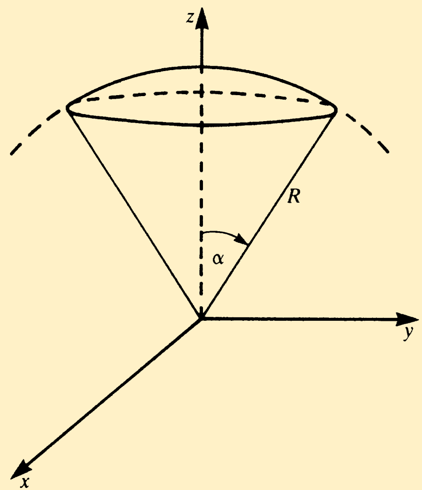
$$V = \frac{R^3}{3}\int_{\phi=0}^{2\pi}\Big[-\cos\theta\Big]_0^{\alpha}\mathrm{d}\phi = \frac{R^3}{3}(1-\cos\alpha)\int_{\phi=0}^{2\pi}\mathrm{d}\phi$$

Finally, we integrate with respect to ϕ. We find

$$V = \frac{2}{3}\pi R^3(1-\cos\alpha)$$

82

82



What is the position $\bar{z}$ of the centroid of the spherical sector?

$$\bar{z} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

83

Explanation wanted

84

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha)$$

83

Correct 91

Wrong, or detailed solution wanted 84

By definition, the z-component of the position of the centroid is

84

$$V\bar{z} = \int_V z\,\mathrm{d}V$$

Now we establish the integral using spherical coordinates. First find the limits and express $\mathrm{d}V$:

$$\int_V z\,\mathrm{d}V = \dots\dots\dots\dots\dots\dots\dots\dots\dots$$

85

$$\int_V z\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{R} z r^2 \sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$ 85

We must express z in terms of r, θ and ϕ before evaluating the integral:

$$z = \text{.........................}$$

86

$$z = r\cos\theta$$

86

If you didn't get this result look up the formula in the textbook (Table 13.1).

Substituting for z in the expression we have:

$$\int_V z\,\mathrm{d}V = \text{..........................}$$

87

$$\int_V z\,\mathrm{d}V = \int_0^{2\pi}\int_0^{\alpha}\int_0^{R} r^3 \sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

87

Now evaluate the integral.

$$V\bar{z} = \int_V z\,\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

88

$$\frac{1}{4}\pi R^4 \sin^2 \alpha$$

88

Correct 90

Wrong, or further explanation wanted 89

Perhaps your difficulty is evaluating the integral

89

$$\int_0^{\alpha} \sin\theta \cos\theta \,\mathrm{d}\theta?$$

It is an integral which we can evaluate using the substitution method.
Let

$$u = \sin\theta$$

$$\mathrm{d}u = \cos\theta \,\mathrm{d}\theta$$

hence

$$\int u\,\mathrm{d}u = \tfrac{1}{2}u^2 = \tfrac{1}{2}\sin^2\theta$$

Now verify that

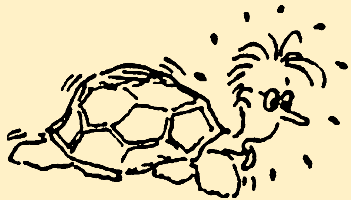
$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\alpha}\int_{r=0}^{R} r^3 \sin\theta\cos\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{1}{4}\pi R^4 \sin^2\alpha$$

--------------------▷ 90

Now complete the solution:

$$\bar{z} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

90



91

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha)$$ 91

Well done!

You now have two expressions for a spherical sector:

(a) its volume, $V = \dfrac{2}{3}\pi R^3(1-\cos\alpha)$, and

(b) the position of its centroid, $\bar{z} = \dfrac{3}{8}R(1+\cos\alpha)$.

As a final exercise, what are the volume and the centroid of a hemisphere?

$$V = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\bar{z} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Solution - - - - - - - - - - - - - - - - - - -▷ 93

Explanation wanted - - - - - - - - - - - - - - - - - - -▷ 92

In the case of a hemisphere the angle α in the expression is equal to $90°$, hence $\cos 90° = 0$. The expressions derived in this example are true for all values of α.

92

Now evaluate

$$V = \frac{2}{3}\pi R^3(1-\cos\alpha) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\bar{z} = \frac{3}{8}R(1+\cos\alpha) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

93

$$V = \frac{2}{3}\pi R^2$$
$$\bar{z} = \frac{3}{8}R$$

93

Straight on:

--------------------▷ 95

Didactical remarks:
Keeping a check on oneself is best done in the following way:

First phase: Attempt the calculation alone with as little help as possible. Compare your solution with the correct one given.

Second phase: If the result is correct this success can have a positive effect on study motivation.
If the result is incorrect you should set out to look for
(a) careless mistakes or slips,
(b) systematic mistakes.

Third phase: If you find a systematic mistake, jot it down on a piece of paper and on finishing the lesson eliminate its cause. This normally means repeating the relevant section of a textbook, along with the accompanying exercises.

--------------------▷ 94

The correction of systematic mistakes is cumbersome but it is the most effective way of improving one's competence. Therefore it is not wrong to deduce that we learn particularly effectively from our mistakes — assuming, of course, that we have *identified*, *analysed*, and *eliminated* the cause of the mistake. 94

95

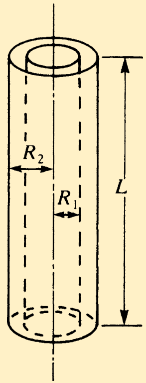
13.6 Application: Moment of Inertia of A Solid

95

The evaluation of areas and volumes has been used to introduce the concept of multiple integrals. As a further example the evaluation of moments of inertia will be shown.

READ: 13.5 Application: Moments of inertia of a solid
Textbook pages 397–400

96



We wish to find the moment of inertia of the hollow cylinder of length L having an inner radius R_1 and an outer radius R_2.

96

The constant density of the material is ρ. The moment of inertia with respect to the central axis is defined as

$$I = \int_V r^2 \, \mathrm{d}m$$

r is the distance from the axis and dm an element of mass.

$$\mathrm{d}m = \rho \, \mathrm{d}V$$

Hence

$$I = \rho \int r^2 \, \mathrm{d}V$$

The problem can be solved most easily using .... coordinates with

$$\mathrm{d}V = \ldots\ldots\ldots\ldots\ldots\ldots$$

97

cylindrical coordinates

97

$$dV = r\, d\phi\, dr\, dz$$

With these coordinates the integral becomes

$$I = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Don't forget the limits!

98

$$I = \rho \int_0^{2\pi} \int_0^L \int_{R_1}^{R_2} r^3 \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\phi = \frac{\pi}{2} \rho L \left(R_2^4 - R_1^4\right)$$ 98

Correct --------------------▷ 100

Wrong, or detailed explanation wanted --------------------▷ 99

99

In this example the limits are all constant, $r = R_1$ to R_2, $z = 0$ to L and $\phi = 0$ to 2π.
Therefore:

$$I = \rho \int r^2 \, \mathrm{d}V$$
$$= \rho \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \int_{r=R_1}^{R_2} r^3 \, \mathrm{d}r \, \mathrm{d}z \, \mathrm{d}\phi$$

Since all the limits are constant the order of integration is not important. Furthermore, this integral may be decomposed into a product of three integrals.

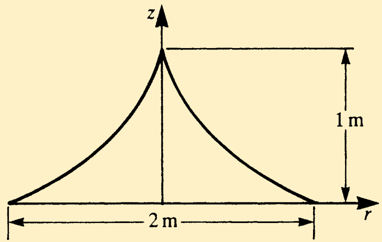
$$I = \rho \int_{\phi=0}^{2\pi} \mathrm{d}\phi \int_{z=0}^{L} \mathrm{d}z \int_{r=R_1}^{R_2} r^3 \, \mathrm{d}r$$
$$= \frac{\rho}{2} \pi \, L \left({R_2}^4 - {R_1}^4 \right)$$

100

Here is a little problem for you to solve. It is the last one of this chapter.

100

A stone cutter has just completed the solid shape, shown in the figure, out of sandstone. As you can see, it is a shape with axial symmetry. The density of sandstone is between 2400 and 2700 kg/m^3. The stone cutter has a van with a maximum capacity of 2.5 t. Can he use it to transport his workpiece?



You have to consider:

(1) the volume of the body,
(2) the weight of the body, remembering that the density is not known exactly, only as a range of values.

The equation of the boundary which defines the shape is

$$z = 1 - 1.5r + 0.5r^2$$

Can you do the problem?

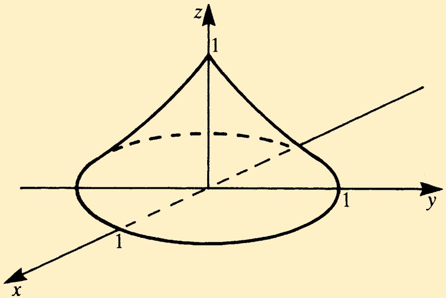
Without help --------------------▷ 104

With help --------------------▷ 101

First we need the volume of the solid. The cross-section is shown in the figure and the equation of the boundary line is

101

$$z = 1 - 1.5r + 0.5r^2$$



We have to decide which coordinate system to use. Since it is a solid of revolution, there is axial symmetry. Cylindrical coordinates seem the most appropriate.

The volume is then given by

$$V = \int\int\int r\, d\phi\, dr\, dz$$

Write down the expression for the volume with limits:

$$V = \int_{\dots}^{\dots}\int_{\dots}^{\dots}\int_{\dots}^{\dots} \dots\dots\dots\dots\dots\dots$$

--------------------▷ 103

Explanation wanted

--------------------▷ 102

Now for the limits:

ϕ goes from 0 to 2π

r goes from 0 to 1, as given

z must be given a variable upper limit since it is a function of r, hence

z goes from 0 to $1 - 1.5r + 0.5r^2$

Hence the expression for the volume becomes

$$V = \int_{\phi=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{1-1.5r+0.5r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi$$

102

103

$$V = \int_{\phi=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{1-1.5r+0.5r^2} r \, dz \, dr \, d\phi$$

103

Now evaluate the integral above.

$V = \ldots\ldots\ldots\ldots\ldots\ldots$

Solution --------------------▷ 105

Explanation wanted --------------------▷ 104

$$V=\int_{\phi=0}^{2\pi}\int_{r=0}^{1}\int_{z=0}^{1-1.5r+0.5r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\phi$$

104

The integral with variable limits is the integral with respect to z. It must be done first:

$$V=\int_0^{2\pi}\int_0^1 r\left(1-1.5r+0.5r^2\right)\mathrm{d}r\,\mathrm{d}\phi=2\pi\int_0^1\left(r-1.5r^2+0.5r^3\right)\mathrm{d}r\,\mathrm{d}\phi$$

$$V=\frac{\pi}{4}\,\mathrm{m}^3$$

105

$$V = \frac{\pi}{4}\,\mathrm{m}^3$$

105

What is the mass of the solid to be considered? ρ lies between 2400 and 2700 $\mathrm{kg/m^3}$.

$$M = \ldots\ldots\ldots\ldots\ldots\ldots$$

106

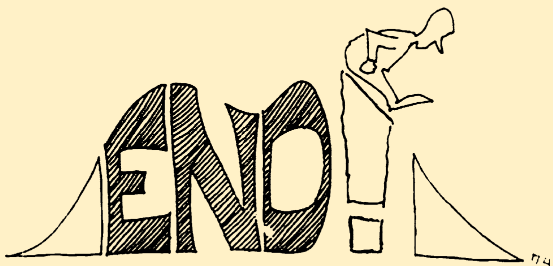
$$M = 2.12\,\mathrm{t}$$

$$M = \text{Volume} \times \text{Density}$$

To be on the safe side it is wiser to take the highest value for the density, hence

$$M = \frac{\pi}{4} \times 2700 = 2120\,\mathrm{kg} = 2.12\,\mathrm{t}$$

This is less than 2.5 tons, and so the van can transport the solid.



END OF CHAPTER 13

Chapter 14
Transformation of Coordinates; Matrices

2

14.1 Transformation of Coordinates; Matrices

Objective: Concepts of transformation, rotation.

In the introduction the importance of the choice of a suitable coordinate system is pointed out. It can save a considerable amount of work or even make possible the solution of a problem.

READ: 14.1 Introduction
Textbook pages 403–406

3

What kinds of transformations are mentioned?
Can you write them down from memory?

3

(1) ....
(2) ....

4

Contents | First Page | Previous View | Next View | Print | Exit

(1) Shift or translation
(2) Rotation

4

In textbooks usually a suitable coordinate system has already been chosen by the authors when solving a problem. Part of the work has already been done.

However, if you have to solve a problem yourself from scratch, you must decide upon a suitable coordinate system. You may also have to transform one coordinate system into another, hence the importance of coordinate transformations which we will consider in what follows.

--------------------▷ 5

14.2 Parallel Shift of Coordinates: Translation

5

Objective: Concepts of shift of coordinates and of transformed vectors.

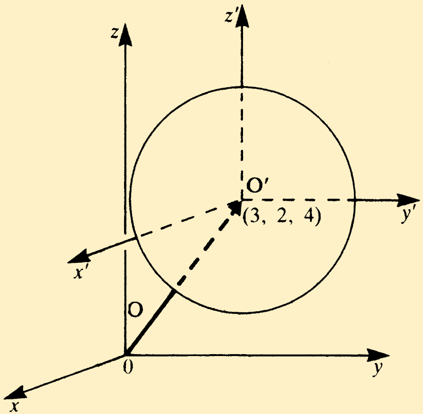
READ: 14.2 Parallel shift of coordinates: translation
Textbook pages 406–409

You should aim to follow the reasoning in your own words by writing the arguments down in your note book. It will help you to sort out your difficulties.

6

A sphere of radius $R = 2$ units has its center at the point O′ = (3, 2, 4). Its equation is

6



$$4 = (x-3)^2 + (y-2)^2 + (z-4)^2$$

The following transformation represents a shift of the origin O to the point O′:

$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

The equation becomes

$$4 = x'^2 + y'^2 + z'^2$$

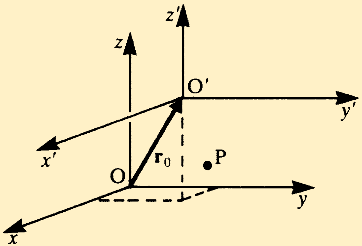
The coordinates of the original position vector to the center of the sphere were (3, 2, 4).

After the transformation the coordinates of the new position vector to the center of the sphere are:.........................

7

(0, 0, 0)

7



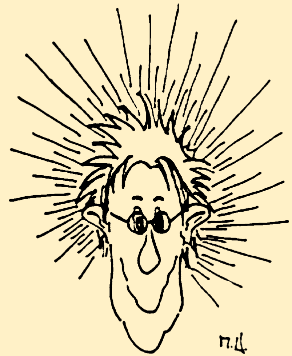
What are the new coordinates of the position vector to the point P = (5, 7, 2) after the coordinate transformation:

$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

$\mathbf{r}' = (2, \ 5, \ -2)$ ---> 8

$\mathbf{r}' = (8, \ 9, \ 6)$ ---> 9

Correct!
If we insert the transformation



8

$$x' = x - 3$$
$$y' = y - 2$$
$$z' = z - 4$$

in the equation for the coordinates x, y and z of the point P(5, 7, 2), we obtain $\mathbf{r}' = (2,\ 5,\ -2)$

I would like another exercise --------------------▷ 11

I want to carry on --------------------▷ 13

Wrong, unfortunately!

9

In section 14.1 we derived transformation formulae for the shift of a position vector $(x, \ y, \ z)$ to the position vector $(x', \ y', \ z')$ in the new coordinate system.

For a shift by a vector $\mathbf{r_0} = (x_0, \ y_0, \ z_0)$ the transformation equations are:

$$\begin{aligned} x' &= x - x_0 \\ y' &= y - y_0 \quad \text{or} \quad \mathbf{r}' = \mathbf{r} - \mathbf{r_0} \\ z' &= z - z_0 \end{aligned}$$

In our problem we had $x_0 = 3, \ \ y_0 = 2, \ \ z_0 = 4$, or a $\mathbf{r_0} = (3, \ 2, \ 4)$.

The point P in the $x{-}y{-}z$ system was given by $x = 5, \ \ y = 7$, and $z = 2$, or $\mathbf{r} = (5, \ 7, \ 2)$.

Now obtain the position vector $\mathbf{r}'$ of the point P in the $x' - y' - z'$ coordinate system.

$$\mathbf{r}' = (x', \ y', \ z') = \text{.......................}$$

-------------------▷ 10

$$\mathbf{r}' = (x',\ y',\ z') = (2,\ 5,\ -2)$$

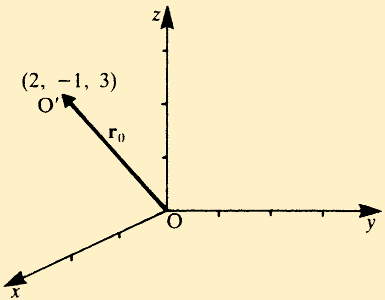
10

Error, or help required --> 11

All correct --> 13

Read section 14.1 again, and then do the following exercise:
The x–y–z– coordinate system is shifted by the vector

11



$$\mathbf{r}_0 = (2,\ \ -1,\ \ 3)$$

(a) Fill in on the sketch the new coordinate system.

(b) Determine the new coordinates of the position vector $\mathbf{r}_1 = (2,\ \ 1,\ \ 2)$:

$$\mathbf{r}'_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

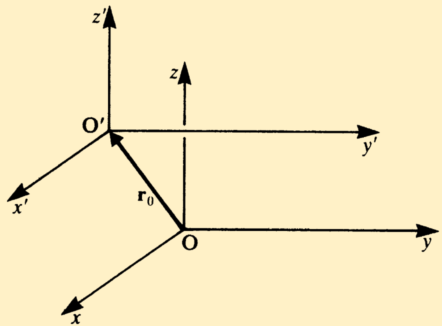
(c) A point P with position vector

$$\mathbf{r}_2 = (2,\ \ -2,\ \ 4)$$

will have as its new coordinates:

$$x'_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y'_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$z'_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 12



(b) **r** becomes

12

$$\mathbf{r}'_1 = (0, \ 2, \ -1)$$

(c)

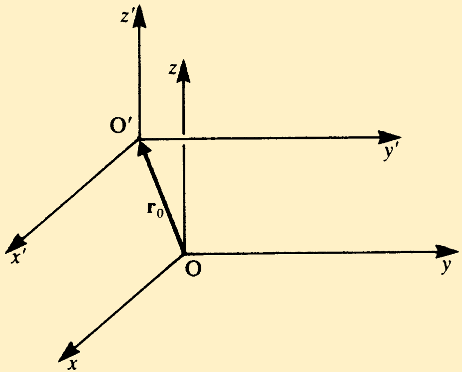
$$x'_2 = 0$$
$$y'_2 = -1$$
$$z'_2 = 1$$

13

Another example!

The transformation from an $x{-}y{-}z$ coordinate system to an $x' - y' - z'$ system is accomplished by a shift, a translation, of the origin O to the new origin O$\prime$ by the vector $\mathbf{r}_0 = (0,\ \ 1,\ \ 3)$.

13



A position vector

$$\mathbf{r} = (1,\ \ 13,\ \ -4)$$

becomes, as a result of this transformation, the vector $\mathbf{r}\prime$ given by:

$$\mathbf{r}' = (-1,\ \ -12,\ \ 7)$$

14

$$\mathbf{r}' = (1,\ \ 12,\ \ -7)$$

15

$$\mathbf{r}' = (1,\ \ 14,\ \ -1)$$

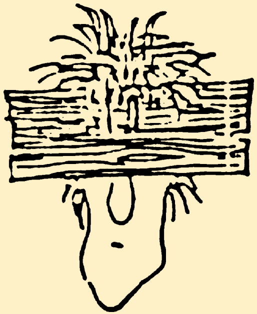
16

No!

You have found $\mathbf{r}' = \mathbf{r}_0 - \mathbf{r}$.

The correct relation is $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ whose coordinates are

14



$$x' = x - x_0$$
$$y' = y - y_0$$
$$z' = z - z_0$$

13

Correct!

Since $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$

15

17

Wrong!

Instead of $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$ you have found the vector $\mathbf{r}' = \mathbf{r} + \mathbf{r}_0$.

16

Was it an oversight?

--------------------▷ 13

Do you have difficulties in understanding?

--------------------▷ 11

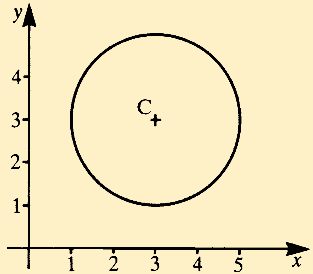
The next exercise is optional. Decide for yourself if you need it. One *needs* exercises even if they appear to be difficult! 17

I'll skip the next exercise

--------------------▷ 23

Next exercise

--------------------▷ 18



The circle shown has a radius $R = 2$ units and its center C has coordinates (3, 3).

18

(a) What is the equation of the circle? ....

(b) By what vector $\mathbf{r}_0$ must the coordinate system be shifted in order to obtain the following equation for the circle:

$$x'^2 + y'^2 = 4?$$

$$\mathbf{r}_0 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 19

(a) $(x-3)^2+(y-3)^2=4$

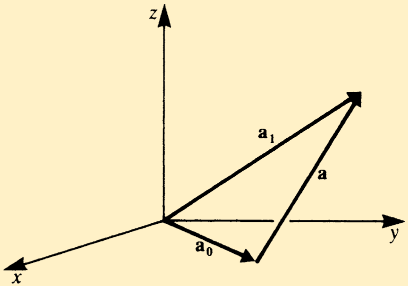
or

$x^2-6x+y^2-6y=-14$

(b) $\mathbf{r}_0=(3,\ \ 3)$

19

20



The vector **a** starts at

$$\mathbf{a}_0 = (1,\ \ 1,\ \ 0)$$

and ends at

$$\mathbf{a}_1 = (1,\ \ 3,\ \ 2)$$

Give

(a) the components of
$\mathbf{a} =$ ....

(b) the magnitude of **a**,
$a =$ ....

20

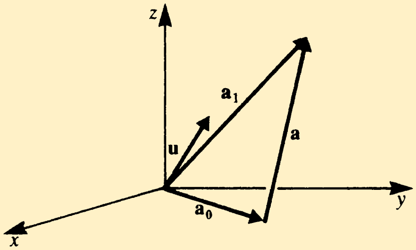
The coordinate system is now shifted by the vector $\mathbf{u} = (1,\ \ 1,\ \ 1)$.

(1) The starting point of the vector **a** is now $\mathbf{a}'_0 =$ ....

(2) The end point of **a** is now $\mathbf{a}'_1 =$ ....

(3) Components of $\mathbf{a} =$ ....

21



(a) $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_0 = (0,\ 2,\ 2)$

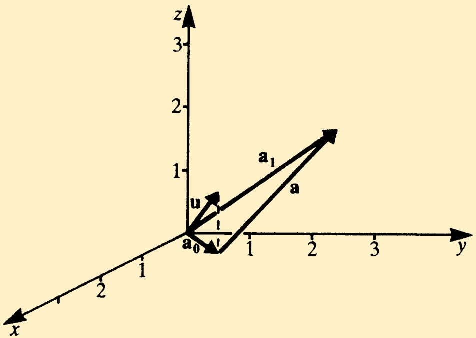
(b) $a = \sqrt{0 + 2^2 + 2^2} = 2\sqrt{2}$

(1) $\mathbf{a}'_0 = (0,\ 0,\ -1)$

(2) $\mathbf{a}'_1 = (0,\ 2,\ 1)$

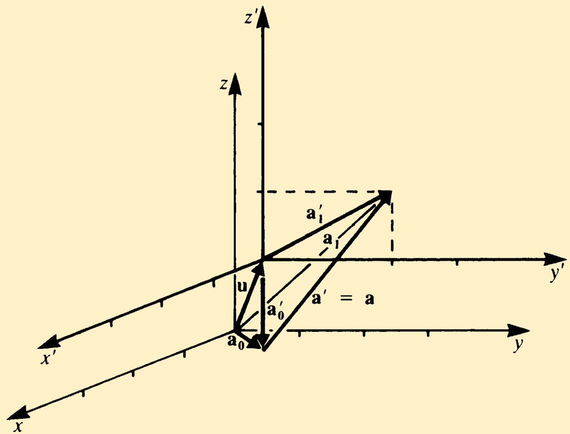
(3) $\mathbf{a} = (0,\ 2,\ 2)$

21



Fill in the sketch of the new coordinate system as a result of a shift by the vector **u** = (1, 1, 1) and check the numerical results.

22



22

23

14.3 Rotation in a Plane

23

Objective: Concepts of rotation and successive rotations; rotations about one axis of the coordinate system for the three-dimensional case.

READ: 14.3.1 Rotation in a plane
Textbook pages 409–412

Follow the text carefully, reading without pen and paper is day-dreaming! Write down the transformation equations for the rotation of a two-dimensional coordinate system through a given angle.

--------------------▷ 24

Let the rectangular $x - y$ coordinate system be rotated through an angle $\phi = \dfrac{\pi}{2}$. 24

What are the components of the vector $\mathbf{r} = (1,\ 2)$ in the new coordinate system?

Do not hesitate to use the equations you wrote down in your notebook. You will use them again later on.

$$\mathbf{r}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

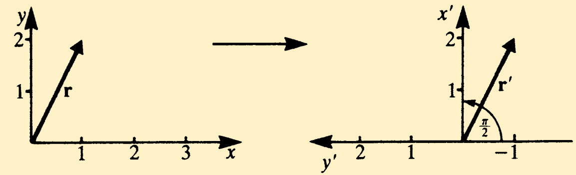
-------------------▷ 25

$$\mathbf{r}' = (2, \quad -1)$$

25

This exercise can be solved in two ways:

(a) Sketch the system of coordinates before and after the rotation thus:



Notice that $\mathbf{r}' = (2, \quad -1)$

(b) Use the transformation equations

$x' = x\cos\phi + y\sin\phi$

$y' = -x\sin\phi + y\cos\phi$

and insert the values $x = 1, \quad y = 2, \quad \phi = \dfrac{\pi}{2}$ to obtain $x' = 2$ and $y' = -1$

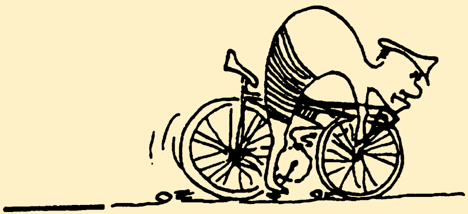
I'd like to carry on ▷ 28

I'd like another exercise ▷ 26

The coordinate system is rotated through an angle $\phi = \frac{\pi}{3}$. Obtain the vector $\mathbf{r}'$ given that $\mathbf{r} = (-2, 1)$.

26

$$\mathbf{r}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



27

$$\mathbf{r}' = (-1 + \tfrac{1}{2}\sqrt{3}, \quad \sqrt{3} + \tfrac{1}{2})$$

27

Solution:

It is achieved more quickly by using the transformation equations:

We were given $\phi = \frac{\pi}{3}$, $\mathbf{r} = (x, \quad y) = (-2, \quad 1)$. Substituting in the equations yields:

$$x' = -2\cos\frac{\pi}{3} + \sin\frac{\pi}{3}$$

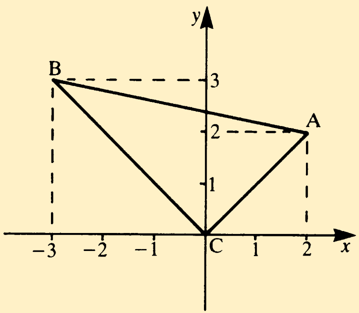
$$y' = 2\sin\frac{\pi}{3} + \cos\frac{\pi}{3}$$

Since $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ we get

$$\mathbf{r}' = \left(-1 + \tfrac{1}{2}\sqrt{3}, \quad \sqrt{3} + \tfrac{1}{2}\right)$$

Difficulties? If so, read section 14.3.1 once more and do the examples without the help of the textbook.

-------------------▷ 28



Given the right-angled triangle whose corners are:

28

$$A = (2,\ 2)$$
$$B = (-3,\ 3)$$
$$C = (0,\ 0)$$

Rotate the system of coordinates in such a way that the points A and B coincide with the axes.

(a) Determine the angle of rotation: $\tan\phi =$ ....

(b) Carry out the transformation for A and B.

$A' =$ ....

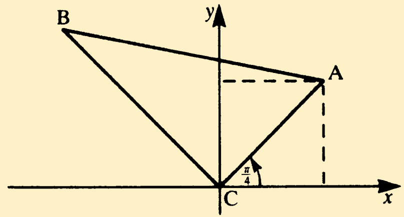
$B' =$ ....

(c) Fill in the new position on the drawing.

--------------------▷ 29

29

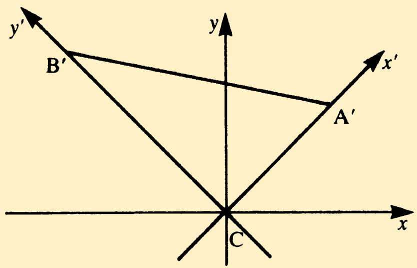
(a) $\tan \phi = 1$



(b) $A' = \left(2\sqrt{2},\ 0\right)$

$B' = \left(0,\ 3\sqrt{2}\right)$

(c)



30

14.4 Successive Rotations

30

We have just been considering the essential steps concerning a rotation in two dimensions.

The result of two rotations carried out successively through angles ϕ and ψ is equivalent to a single rotation through an angle $(\phi + \psi)$.

This statement is proved in section 14.3.2. You now have a choice:

Skip the proof and carry on immediately

--------------------▷ 31

Read the proof

READ: 14.3.2 Successive rotations
Textbook pages 412–413

--------------------▷ 31

14.5 Rotation in Three-Dimensional Space

31

Objective: Concepts of spatial rotations, transformation of a vector when the coordinate system is rotated about the z- or x-axis.

READ: 14.3.3 Rotations in three-dimensional space
Textbook pages 418–420

32

The three-dimensional system is rotated about the z-axis. 32

Let the angle of rotation be $\frac{\pi}{2}$. Find the components of the position vector $\mathbf{r} = (2,\ 3,\ 1)$ in the transformed system.

$$\mathbf{r}' = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 33

$$\mathbf{r}' = (3,\ \ -2,\ \ 1)$$

33

Explanation: There are two ways of proceeding:

(a) For a rotation about the z-axis through an angle $\phi = \frac{\pi}{2}$ the z-axis does not change, the x-axis is rotated into the y-axis and the y-axis into the negative x-axis. It therefore follows that

$$\begin{aligned} x' &= y = 3 \\ y' &= -x = -2 \\ z' &= z = 1 \end{aligned}$$

hence $\mathbf{r}' = (3,\ \ -2,\ \ 1)$

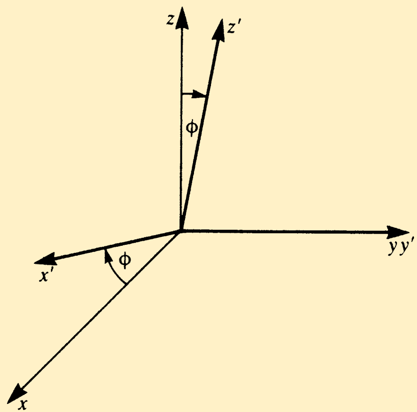
(b) We can use the transformation equations

$$\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \\ z' &= z \end{aligned}$$

and substitute the numerical values for x, y and ϕ, and obtain

$$\mathbf{r}' = (3,\ \ -2,\ \ 1)$$

34



Derive the transformation equations for a rotation about the y-axis through an angle ϕ:

$x' =$ ....

$y' =$ ....

$z' =$ ....

34

----------------------▷ 35

$$x' = x\cos\phi + z\sin\phi$$
$$y' = y$$
$$z' = -x\sin\phi + z\cos\phi$$

35

Explanation: During a rotation about the y-axis the y component of a vector $\mathbf{r} = (x, \ y, \ z)$ is unchanged. The projection $\mathbf{r}_{xz} = (x, \ y)$ of the vector $\mathbf{r}$ in the $x - z$ plane is transformed in accordance with the equations given in section 14.3.1 by replacing y by z.

While working through the mathematical programme, phases may arise where, in spite of your efforts to pick up the subject matter, you find you are making no progress.
Comments on learning curves and learning plateaus

--------------------▷ 36

Otherwise

--------------------▷ 39

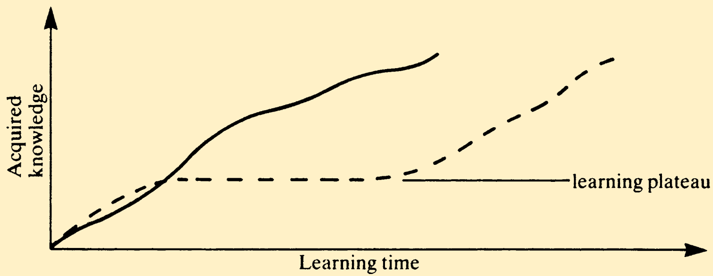
(1) Learning curves

36

Learning processes can be plotted on graphs, the so-called learning curves. In principle they show the quantitative increase in knowledge, dependent on learning time.

The form these learning curves take can be very varied. It is, for example, determined by individual peculiarities, previous knowledge and by the nature of the subject matter.

The diagram shows two typical learning curves. One of them shows a learning plateau:



37

37

(2) Learning plateaus

'Learning plateau' is the description given to a time phase in which no progress in learning can be subjectively determined. It frequently corresponds to a state of discontent. A learning plateau of this kind does not mean that the utmost limit of your learning capacity has already been reached — to determine such a limit is hardly possible, anyway.

This phase of what appears to be ineffectiveness sometimes indicates a transition between two levels of competence. This means that the knowledge and all its possible applications, acquired during the preceding period of study, must first be integrated before any further progress begins.

38

(3) Learning plateau — something can be done about it! 38

Within this book it is possible to mention just a few points about the complex structure of learning difficulties.

— The division of the field of study into different sections very often leads to overall coherence being lost. Try to structure the study material when you are revising! Terms and essential statements should be seen in relation to each other. Revision from the notes you have written yourself is especially effective.

— Lack of work planning is frequently conductive to uneconomical learning. A change in certain working and living habits may lead to greater subjective satisfaction in one's personal capacity. In some passages we have tried to give some advice on this topic.

→ 39

14.6 Matrix Algebra

39

Objective: Concepts of matrices, columns and rows of a matrix, square matrices, addition, subtraction and multiplication of matrices, multiplication of matrices by vectors and scalars.

READ: 14.4 Matrix algebra
Textbook pages 415–420

Don't forget to work in parallel with the text, taking notes and doing the examples yourself. This section is quite long and contains new concepts and rules. You may therefore wish to split it into a number of subsections with short breaks in between.

40

Write down the columns and rows of the matrix

40

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 4 \\ -3 & 6 & 2 \end{pmatrix}$$

Columns: ....
Rows: ....

41

Columns: $\begin{pmatrix} 1 \\ -3 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

41

Rows: $(1 \quad 2 \quad 4)$, $(-3 \quad 6 \quad 2)$

When we consider a single row or column of a matrix we speak of a *row vector* or a *column vector*.

Specify the types and the orders of the following two matrices:

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 4 & 8 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 5 & 2 & 1 \\ 7 & 3 & 9 \end{pmatrix}$$

A is a .... matrix of order ....

B is a .... matrix of order ....

42

A is a square matrix of order 2×2, $(n \times n)$
B is a rectangular matrix of order 2×3, $(m \times n)$

42

Given: the four matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \\ 2 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \\ 2 & 0 \end{pmatrix}, \mathbf{D} = \begin{pmatrix} 1 & 4 & 2 \\ 1 & 3 & 0 \end{pmatrix}$$

Which matrices are equal?

43

Matrices **A** and **C** are equal since

43

(i) they are of the same order, 3×2 in this case,
(ii) the elements of **A** are equal to the corresponding elements of **C**,

Generally speaking

$$a_{ik} = c_{ik} \qquad \begin{array}{l} i = 1,\ 2,\ 3,\ \ldots,\ m \\ k = 1,\ 2,\ 3,\ \ldots,\ n \end{array}$$

Add the two matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} -3 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

What are the necessary conditions for adding (or subtracting) matrices?

....

C = A + B =

....

-------------------▷ 44

The conditions for adding (or subtracting) matrices are that they must be of the same order, $m \times n$.

44

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \begin{pmatrix} 4-3 & 0+1 & 2-1 \\ 2+0 & 0+1 & 4-2 \\ 0+1 & 1+0 & 0+1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

Now for another exercise:

Given the matrix $\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 & 1 \\ 2 & 0 & -4 & 2 \\ 0 & 11 & 0 & 10 \end{pmatrix}$

Multiply $\mathbf{A}$ by the number 3, a scalar:

$$3\mathbf{A} =$$

....

-------------------▷ 45

$$3\mathbf{A} = \begin{pmatrix} 12 & 0 & 6 & 3 \\ 6 & 0 & -12 & 6 \\ 0 & 33 & 0 & 30 \end{pmatrix}$$

The result is obtained by multiplying every element of the matrix by the number 3.

45

Now suppose we wish to multiply a matrix $\mathbf{A}$ by a vector $\mathbf{r}$ given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}$ is a matrix of order 2×2, and $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is a 2×1 column vector.

Then we obtain:

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ -3 & 6 \end{pmatrix}$$

46

$$\begin{pmatrix} 4+8 \\ -3+6 \end{pmatrix} = \begin{pmatrix} 12 \\ 3 \end{pmatrix}$$

48

$$\begin{pmatrix} 4 & 2 \\ -12 & 6 \end{pmatrix}$$

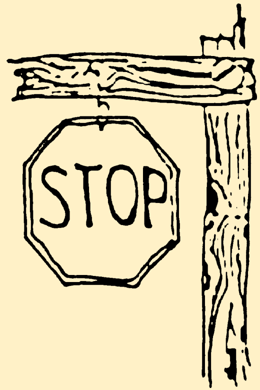
46

$$\begin{pmatrix} 4+2 \\ -12+6 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

49

Wrong!

46



The expression **Ar** is also a vector!

Read once more the definition of the product **Ar** in section 14.4, and try again with the following example:

$$\mathbf{Ar} = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: It is useful to follow the scheme given in the textbook for multiplication of matrices, even if one matrix is a vector.

47

Following the scheme we obtain

47

$$\begin{matrix} & \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ \mathbf{Ar} = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} & \begin{pmatrix} 0+2 \\ 3+8 \end{pmatrix}; \quad \mathbf{Ar} = \begin{pmatrix} 2 \\ 11 \end{pmatrix} \end{matrix}$$

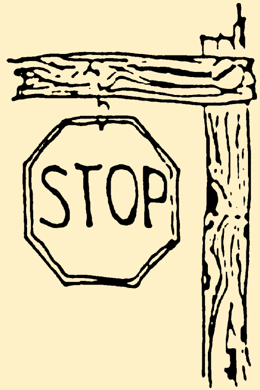
Now compute

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

49

48

Wrong, unfortunately!
The definition of the product **Ar** is:



$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Remember the scheme:

$$\begin{array}{cc} & \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & \begin{pmatrix} x' \\ y' \end{pmatrix} \end{array}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}$$

Using this expression compute again

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \text{..............................}$$

49

49

You found the correct solution:

$$\begin{pmatrix} 1 & 2 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -6 \end{pmatrix}$$

Evaluate the product

$$\begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & -3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} =$$

....

-------------------▷ 50

$$\begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & -2 \\ 2 & -3 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 3\times 4+0\times 6+2\times 5 \\ -1\times 4+1\times 6-2\times 5 \\ 2\times 4-3\times 6+0\times 5 \end{pmatrix} = \begin{pmatrix} 22 \\ -8 \\ -10 \end{pmatrix}$$

50

Multiplication of two matrices.
Which products of the following matrices are possible?

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 3 & 8 \end{pmatrix}$$

☐ **AB**

☐ **BC**

☐ **AC**

----------------▷ 51

The products **AC** and **BC** are possible. 51

Multiplication of two matrices is only possible if the number of columns of the first matrix (**A** and **B** in this case) is equal to the number of rows of the second matrix (**C** in this case).

It follows that **AB** is not possible.

Multiplying two matrices: First obtain an expression for the element c_{11} in

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} c_{11} & - & - \\ - & - & - \\ - & - & - \end{pmatrix}$$

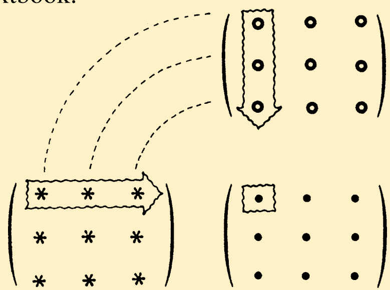
$$c_{11} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

52

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

52

The multiplication of two matrices can be remembered by using the following scheme, given in the textbook.



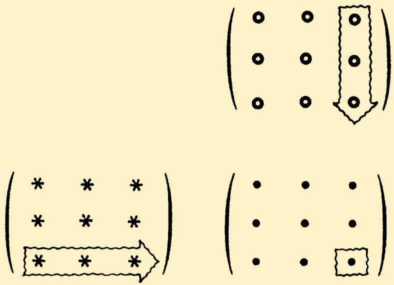
c_{11} can be considered as the product of the row vector **a** and the column vector **b**, and similarly for finding each element c_{ik}. Using this scheme obtain c_{33} and indicate it on a diagram similar to the one in this frame.

$c_{33} = \ldots\ldots\ldots\ldots\ldots\ldots$

53

53

$c_{33} = a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$



Because multiplication of matrices may be a little difficult at first it is helpful to adopt a scheme such as the one above, in which the correspondence between column vectors and row vectors is obvious.

54

Evaluate **AB** given

$$\mathbf{A} = \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix}$$

$\mathbf{AB} =$ ....

54

55

55

$$\begin{array}{cc} & \begin{pmatrix} 3 & 0 \\ -1 & 4 \end{pmatrix} \\ \begin{pmatrix} 0 & 2 \\ 6 & 1 \end{pmatrix} & \begin{pmatrix} 0-2 & 0+8 \\ 18-1 & 0+4 \end{pmatrix} \end{array} = \begin{pmatrix} -2 & 8 \\ 17 & 4 \end{pmatrix}$$

Evaluate

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix} =$$

....

Hint: Use the scheme!

-------------------▷ 56

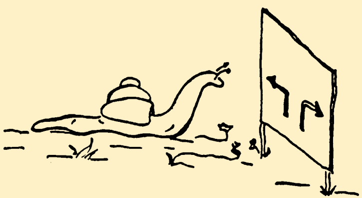
$$\begin{pmatrix} 2 & 3 & -2 \\ 2 & 8 & 2 \\ 4 & 7 & 0 \end{pmatrix}$$

56

Here is the detailed solution:

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & 3 \\ 1 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0\times 2+1\times 0+1\times 1+1\times 1 & 0\times 1-1\times 1+1\times 1+1\times 3 & 0\times 0+1\times 0-1\times 2+1\times 0 \\ 0\times 2+0\times 0-1\times 1+3\times 1 & 0\times 1-0\times 1-1\times 1+3\times 3 & 0\times 0+0\times 0+1\times 2+3\times 0 \\ 1\times 2+0\times 0+0\times 1+2\times 1 & 1\times 1-0\times 1+0\times 1+2\times 3 & 1\times 0+0\times 0-0\times 2+2\times 0 \end{pmatrix}$$



Correct → 59

Wrong → 57

You have either made a computational error or you have still not grasped the rule for multiplying matrices. If the latter is the case you should study once more section 14.4 in the textbook and then do the following exercises. Remember the scheme given in the textbook. 57

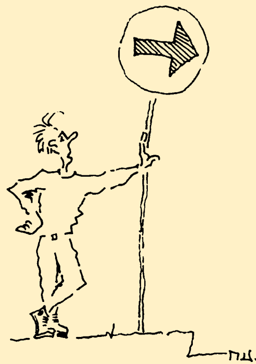
(a) $\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(b) $\begin{pmatrix} 3 & 2 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 58

58

(a) $\begin{pmatrix} 2 & 6 & 2 \\ 1 & 3 & 2 \\ 2 & 5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$



59

14.7 Rotations Expressed in Matrix Form

59

Objective: Concepts of matrix representation of the transformation equations, rotation of coordinates.

READ: 14.5 Rotations expressed in matrix form
Textbook pages 421–423

--------------------▷ 60

Set up the transformation matrix for a rotation of the two-dimensional coordinate system through an angle of 180°. You may use the formulae you wrote down as you studied the text. 60

....

--------------------▷ 61

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

61

Here is the solution:

The transformation matrix is

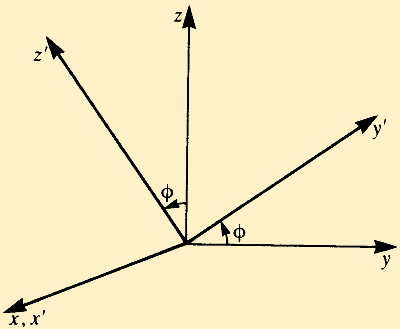
$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

Inserting the values for $\phi = 180^\circ = \pi$ radians yields

$$\begin{pmatrix} \cos\pi & \sin\pi \\ -\sin\pi & \cos\pi \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

The result could also have been obtained pictorially.

62



Set up the transformation matrix for a rotation in three-dimensional space about the x-axis, the angle of rotation being ϕ.

62

Transformation matrix

$\mathbf{A} =$

....

63

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

63

It is obtained as follows:

The transformation equations for rotation about the x-axis are

$$\begin{aligned} x' &= x \\ y' &= y\cos\phi + z\sin\phi \\ z' &= -y\sin\phi + z\cos\phi \end{aligned}$$

In matrix notation these are

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}}_{\text{Transformation matrix}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

--------------------▷ 64

14.8 Special Matrices

64

Objective: Concepts of the transpose of a matrix, null matrix, diagonal matrix, unit matrix, symmetric and skew-symmetric matrices, inverse matrix.

READ: **14.6 Special matrices**
14.7 Inverse matrix
Textbook pages 423–427

65

Obtain the transpose of the matrix

65

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix}$$

$$\mathbf{A}^{\mathrm{T}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

and

$$(\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \ldots\ldots\ldots\ldots\ldots\ldots$$

66

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix}, \quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & 2 \end{pmatrix}, \quad (\mathbf{A}^{\mathrm{T}})^{\mathbf{T}} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 2 \end{pmatrix} = \mathbf{A}$$

66

Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 4 & 3 & 4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 3 & -1 \end{pmatrix}$$

Obtain

$\mathbf{AB} =$ .... and $(\mathbf{AB})^{\mathrm{T}} =$ ....

67

$$\mathbf{AB} = \begin{pmatrix} 3 & 4 \\ 19 & 2 \end{pmatrix} \quad (\mathbf{AB})^{\mathrm{T}} = \begin{pmatrix} 3 & 19 \\ 4 & 2 \end{pmatrix}$$

67

Given: the following square matrices

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Give the name of each matrix.

A : ....

B : ....

C : ....

-------------------▷ 68

A: null matrix
B: diagonal matrix
C: unit matrix

68

Correct

69

Wrong: Read the definitions in the textbook again. Take notes!

69

If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

69

Obtain

$$\mathbf{AI} =$$

....

and

$$\mathbf{IA} =$$

....

70

$$\mathbf{AI} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

$$\mathbf{IA} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

70

i.e. the matrix **A** is unchanged.

Given: the two matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{pmatrix}$$

Which matrix is

(a) symmetric?
(b) antisymmetric, i.e. skew-symmetric?

A : ....
B : ....

→ 71

A: symmetric, since $a_{ij} = a_{ji}$
B: skew-symmetric, since $b_{ij} = -b_{ji}$

71

Given the matrix **A**, then
$\mathbf{A}^{-1}$ is called .... matrix of **A**.

$$\text{If} \quad \mathbf{A} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \quad \mathbf{A}^{-1} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$$

obtain

$$\mathbf{A}\mathbf{A}^{-1} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\mathbf{A}^{-1}\mathbf{A} = \ldots\ldots\ldots\ldots\ldots\ldots$$

72

$\mathbf{A}^{-1}$ is called the *inverse* matrix of $\mathbf{A}$.

72

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The calculation of the inverse matrix is not shown in this chapter. It will be shown in Chapter 15.

73

73

Before closing your book, notes, etc., you should recapitulate the concepts and rules you have just been learning. You should then have a break and do something totally different; we have said this to you on a number of occasions. It does make good sense!

Don't forget to do the exercises given in the textbook; remember that 'practice makes perfect'!

END OF CHAPTER 14

Chapter 15
Sets of Linear Equations; Determinants

2

15.1 Gauss Elimination, Successive Elimination of Variables, Gauss-Jordan Elimination

2

In this chapter several numerical examples are shown in the textbook. It is advisable to try to solve the examples by yourself and to check your solution afterwards. Since the algorithms are explained explicitly you should be able to find the solutions in most cases.

READ: 15.1 Introduction
15.2.1 Gauss elimination: successive elimination of variables
15.2.2 Gauss-Jordan elimination
Textbook pages 431–434

3

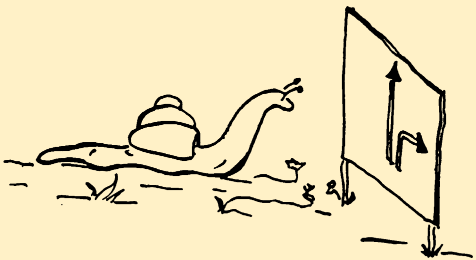
Given a system of two linear equations

3

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

find the solution using the Gauss-Jordan elimination.

$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution ---------------------▷ 7

Detailed calculation ---------------------▷ 4

4

Given:

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

First step: Dividing the first equation by a_{11} and eliminating x_1 in the second equation yields

$$x_1 + \frac{a_{12}}{a_{11}}x_2 = \frac{b_1}{a_{11}}$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 = b_2 - b_1\frac{a_{21}}{a_{11}}$$

Now execute the second step and write down the result:

$$x_1 + \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right)x_2 = b_2 - b_1\frac{a_{21}}{a_{11}}$$

Solution --------------------▷ 6

Further explanation --------------------▷ 5

Further explanation: Given the result of the first step

5

$$x_1 + \frac{a_{12}}{a_{11}} x_2 = \frac{b_1}{a_{11}}$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) x_2 = b_2 - b_1 \frac{a_{21}}{a_{11}}$$

Second step: Elimination of the coefficient of x_2 in the first equation. This step requires division of the second equation by the actual coefficient of x_2 which is bracketed. Then we can eliminate x_2 in the first equation.

$$x_1 + 0 = \dots\dots\dots\dots\dots\dots\dots\dots$$
$$0 + \left(a_{22} - \frac{a_{12}a_{21}}{a_{11}}\right) x_2 = b_2 - b_1 \frac{a_{21}}{a_{11}}$$

------------------▷ 6

$$x_1 + 0 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} \times \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$0 + x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

6

The first equation can be rearranged to a form similar to the second one, giving:

$$x_1 = \text{.......................}$$
$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}$$

7

$$x_1 = \frac{b_1 a_{22} - b_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$
$$x_2 = \frac{b_2 a_{11} - b_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

7

In case of difficulties repeat the transformations, beginning with frame 4.

Now solve the numerical example

$$x_1 + x_2 = \frac{7}{10}$$
$$2x_1 + 5x_2 = 2$$
$$x_1 = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$x_2 = \ldots\ldots\ldots\ldots\ldots\ldots$$

8

$$x_1 = \frac{1}{2}$$
$$x_2 = \frac{1}{5}$$

8

The explicit formula for systems of two linear equations for two unknowns is the only formula derived in detail. The formulae for other systems are formidable. In practice one would not use explicit formulae; rather the algorithm for the solution would be followed. You should, therefore, understand the logical basis of the elimination procedure.

--------------------▷ 9

15.2 Matrix Notation of Systems of Equations and Calculation of the Inverse Matrix

9

Solving systems of linear equations is facilitated by the matrix notation. We will use Gauss-Jordan elimination.

Also the calculation of the inverse matrix is shown to be possible using Gauss-Jordan elimination.

READ: 15.2.3 Matrix notation of sets of equations and determination of the inverse matrix
Textbook pages 434–437

--------------------▷ 10

Given a system of linear algebraic equations **Ax = b**

10

$$\begin{aligned} x_1 + x_2 + 0 + 3x_4 &= 16 \\ x_1 + 2x_2 - x_3 + 5x_4 &= 25 \\ 0 + x_2 + 0 + x_4 &= 8 \\ 3x_1 + 5x_2 - 2x_3 + 12x_4 &= 64 \end{aligned}$$

The augmented matrix **A** | **I** is a .... matrix.

Write down the augmented matrix **A** | **I**

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{c|c} \quad & \quad \end{array} \right)$$

....

11

$\mathbf{A} \mid \mathbf{I}$ is a 4×8 matrix

11

$$\mathbf{A}|\mathbf{I} = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & -2 & 12 & 0 & 0 & 0 & 1 \end{array}\right)$$

Evaluate the inverse $\mathbf{A}^{-1}$. To do this transform the matrix so that the first part is a unit matrix. Then the second part will be $\mathbf{A}^{-1}$.

Solution

--------------------▷ 16

Detailed solution with further explanation

--------------------▷ 12

12

Consider the given matrix.
First step: Elimination of the elements in the first column beneath a_{11}.

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 1 & 2 & -1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 3 & 5 & -2 & 12 & 0 & 0 & 0 & 1 \end{array}\right) \quad \begin{array}{l} \text{Row 2 : subract row 1} \\ \text{Row 3 : } a_{13} \text{ is already zero} \\ \text{Row 4 : substract } 3 \times \text{ row 1} \end{array}$$

Execute the subtractions for the whole rows:

$$\left(\begin{array}{c|c} \quad & \quad \\ \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{array}\right)$$

13

13

The changed elements are circled

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 3 & -3 & 0 & 0 & 1 \end{array}\right)$$

Second step: Elimination of the elements beneath and above a_{22}:
Row 1: subtract row 2
Row 3: subtract row 2
Row 4: subtract $2 \times$ row 2

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

14

The changed elements are circled

14

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 2 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right)$$

Third step: Elimination of the elements beneath and above a_{33}:
Row 1: subtract row 3
Row 2: add row 3
Row 4: a_{34} is already zero

$$\left(\begin{array}{c|c} & \\ & \\ & \\ & \end{array}\right)$$

....

15

The changed elements are circled

15

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 2 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 \end{array}\right)$$

Fourth step: Division of row 4 by $a_{44} = -1$ and elimination of the elements above a_{44}.
After division by a_{44}:
Row 1: subtract $2 \times$ row 4
Row 2: subtract row 4
Row 3: add row 4
Result:

$$\left(\begin{array}{c|c} \quad & \quad \\ & \\ & \end{array}\right)$$

....

16

$$\mathbf{I}|\mathbf{A}^{-1} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & -4 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -2 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & -1 \end{array}\right)$$

16

Now write down $\mathbf{A}^{-1}$, which we obtained by this transformation of $\mathbf{A}|\mathbf{I}$ into $\mathbf{I}|\mathbf{A}^{-1}$.

$$\mathbf{A}^{-1} = \left(\qquad \right)$$

....

Check your result by calculating
$\mathbf{A}^{-1}\mathbf{A}$ and $\mathbf{A}\mathbf{A}^{-1}$

17

$$\mathbf{A}^{-1} = \begin{pmatrix} -1 & -4 & -1 & 2 \\ -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & 2 & 0 & -1 \end{pmatrix}$$

17

We have thus calculated the inverse of $\mathbf{A}$. And your check should result in $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$.

Now for the solution of the original system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 16 \\ 25 \\ 8 \\ 64 \end{pmatrix}$$

Write down the augmented matrix $\mathbf{A}|\mathbf{b}$

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

------------------▷ 18

$$\mathbf{A}|\mathbf{b} = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 16 \\ 1 & 2 & -1 & 5 & 25 \\ 0 & 1 & 0 & 1 & 8 \\ 3 & 5 & -2 & 12 & 64 \end{array}\right)$$

18

Solve the system of linear equations by executing the Gauss-Jordan elimination for the augmented matrix.

The steps are exactly the same as in the previous example. The difference is that the matrix **A** has been augmented by **b** instead of **I**. Thus all transformations are applied to **b**. All steps are explained in detail in frames 12 to 15, and in case of difficulties repeat these frames. Result:

$$\left(\begin{array}{c|c} \quad & \quad \end{array}\right)$$

....

$x_1 =$ ....

$x_2 =$ ....

$x_3 =$ ....

$x_4 =$ ....

------------------▷ 19

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \qquad \begin{aligned} x_1 &= 4 \\ x_2 &= 6 \\ x_3 &= 1 \\ x_4 &= 2 \end{aligned}$$

19

The execution of the transformations requires attention but they are not difficult in principle. The matrix notation and the execution of all transformations with the augmented matrices helps to avoid errors. It may be useful, too, to write down all transformations explicitly, as has been done in frames 12 to 15.

20

15.3 Existence of Solutions

20

In the textbook we again have worked examples at the end of the section. Within the examples matrix notation is used. In case of difficulties it may be advisable to write down the equations explicitly.

READ: 15.2.4 Existence of solutions
Textbook pages 437–439

--------------------▷ 21

(A) If you have to solve 4 non-homogeneous equations and 6 variables:
Then at most .... variables can be determined.
At least .... variables are free to be chosen.

21

(B) Suppose you have 4 homogeneous linear equations. State the form of the solution.

Trivial solution: ....
If a non-trivial solution exists it ....

-------------------▷ 22

(A) At most 4 variables can be determined.

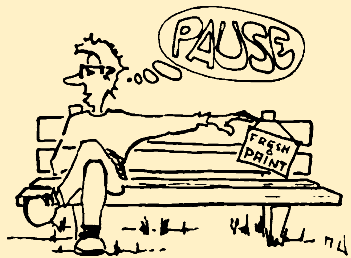
(B) At least 2 variables are free to be chosen.

22

Trivial solution: $x_j = 0, \quad j = 1, 2, 3, 4.$

If a non-trivial solution exists it is not unique and has at least one variable free to be chosen.

In practical applications it is useful first to check whether solutions exist and whether they are unique. By applying Gauss-Jordan elimination the solution clearly shows its structure.



23

15.4 Determinants

23

Objective: Concept of a determinant, competence to evaluate the determinants of 2×2 and 3×3 matrices.

READ: **15.3.1 Preliminary remarks on determinants**
15.3.2 Definition and properties of an *n*-row determinant
Textbook pages 440–446

24

Given: the determinant of the matrix which we used in frames 11 to 15

24

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix}$$

Determine the minor of a_{12} : $\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$

....

Determine the cofactor A_{12} : $\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}$

....

In case of difficulties refer to the textbook.

25

25

minor of $a_{12} = \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

cofactor $A_{12} = (-1)^{1+2} \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix} = - \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

Evaluate the cofactor: $A_{12} = - \begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix}$

Try both methods, Sarrus' rule and evaluating using cofactors.

$$A_{12} = \ldots\ldots\ldots\ldots\ldots$$

26

Using cofactors, expanding for the first row:

26

$$-\begin{vmatrix} 1 & -1 & 5 \\ 0 & 0 & 1 \\ 3 & -2 & 12 \end{vmatrix} = [1 \times (-2) - (-1)(-3)] = 1$$

Sarrus' rule

$$-\begin{pmatrix} 1 & -1 & 5 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & -2 & 12 & 3 & -2 \end{pmatrix} = -(-3-(-2)) = 1$$

27

The properties of determinants are useful if the determinants of a large matrix have to be evaluated since the effort of calculation can be considerably reduced. 27

Given:

$$\det \mathbf{A} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix}$$

Try to use property 5 (adding multiples of a row to another) to simplify the task of evaluation.

$$\det \mathbf{A} = \begin{vmatrix} \\ \\ \\ \\ \end{vmatrix}$$

.....

----------------▷ 28

There are several simplifications possible. In this case one could do the following:

28

$$\begin{vmatrix} 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 5 \\ 0 & 1 & 0 & 1 \\ 3 & 5 & -2 & 12 \end{vmatrix} \begin{matrix} \\ \text{(row 2: subtract row 1)} \\ \text{(row 4: subtract } 3 \times \text{row 1)} \\ \\ \end{matrix} = \begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -2 & 3 \end{vmatrix}$$

Incidentally, this simplification equals the first step of the Gauss-Jordan elimination (see frame 12).

Now only the cofactor A_{11} has to be evaluated.

Thus det **A** =

....

29

$$\det \mathbf{A} = a_{11} \times A_{11} = 1 \times \begin{vmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 2 & -2 & 3 \end{vmatrix} = -1$$

29

We end the evaluation of determinants now since it is sufficient to know the principles.

--------------------▷ 30

15.5 Rank of a Determinant Applications of Determinants

30

You should know the concept of rank of a matrix and its determinant, since the structure of solutions of systems of linear equations is determined by this rank. This is shown in connection with Cramer's rule which is introduced as an application of determinants.

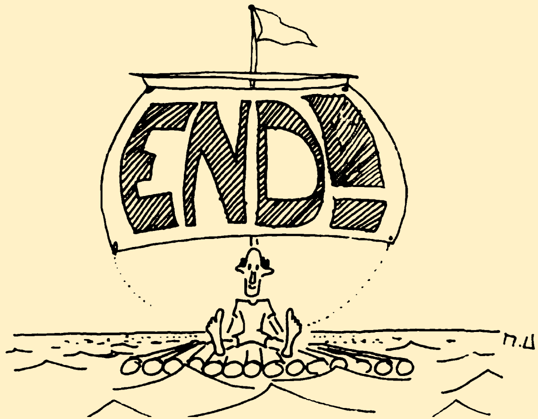
READ: 15.3.3 Rank of a determinant and rank of a matrix
15.3.4 Applications of determinants
Textbook pages 446–451

Follow the examples given in the textbook.

31

In section 15.3.4 three examples have been given for the application of Cramer's rule. We will not discuss any more examples for this rule in the study guide, since for practical applications we suggest the use of the Gauss or the Gauss-Jordan elimination to solve systems of linear equations. Doing this, the rank of the coefficients matrix will be obtained automatically, too. 31

END OF CHAPTER 15



Chapter 16
Eigenvalues and Eigenvectors of Real Matrices

16.1 Eigenvalues and Eigenvectors

Objective: Concepts of real eigenvalue of a square matrix, eigenvector corresponding to an eigenvalue, characteristic polynomial, geometric significance of eigenvectors, determination of all the real eigenvalues and corresponding eigenvectors of any given 2×2 or 3×3 matrix.

READ: 16 Eigenvalues and eigenvectors of real matrices
Textbook pages 453–461

3

Suppose a square matrix $\mathbf{A}$ is applied to a vector $\mathbf{r}$ yielding $\mathbf{r}'$ such that $\mathbf{r}' = \mathbf{A}\mathbf{r} = \lambda\mathbf{r}$

Then $\mathbf{r}$ is said to be an .... of $\mathbf{A}$ and λ is an ...., provided $\lambda \neq$ ...... and $\mathbf{r} \neq$ ......

3

4

eigenvector
eigenvalue
$\lambda \neq 0, \ \mathbf{r} \neq \mathbf{0}$

4

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

Can you be certain that a real eigenvalue exists?

Yes

6

No

5

In principle you are right: a 2×2 matrix need not have any real eigenvalue. But take a closer look at the matrix $\mathbf{A}$. It is symmetric and it is not singular. Now, there is a theorem saying that such a symmetric matrix always has the maximum number of real eigenvalues. Thus for a symmetric 2×2 matrix there must be two real eigenvalues.

5

6

Since $\mathbf{A}$ is symmetric, two real eigenvalues must exist. For an arbitrary 2×2 matrix no real eigenvalue need exist.

6

In order to find the eigenvalues we look for solutions of the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ with $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

What is the name of that equation? ....

Write down the equation for the matrix under consideration.

Find its roots. $\lambda_1 =$ .... $\lambda_2 =$ ....

7

characteristic equation of **A** 7

$$\det\begin{pmatrix} 2-\lambda & 2 \\ 2 & 5-\lambda \end{pmatrix} = (2-\lambda)(5-\lambda)-4 = \lambda^2-7\lambda+6=0$$

Roots: $\lambda_1 = 1,\ \lambda_2 = 6$

There are two eigenvalues of **A**. In order to find corresponding eigenvectors, one must solve two systems of homogeneous linear equations.

Do you remember how these arise? Yes:

--------------------▷ 9

No: Read on!

For the sake of clarity, let us consider the first eigenvalue $\lambda_1 = 1$. For a corresponding eigenvector $\mathbf{r}_1$ the following matrix equation must hold:

$$\mathbf{A}\mathbf{r}_1 = \lambda_1 \mathbf{r}_1$$

Equivalently: $(\mathbf{A}-\lambda_1\mathbf{I})\mathbf{r}_1 = \mathbf{0}$

Write this down explicitly, using $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$: ....

Recall that $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

--------------------▷ 8

$$\begin{pmatrix} 2-1 & 2 \\ 2 & 5-1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

8

Multiplying out yields the system of two homogeneous linear equations for the first eigenvector:

$$\begin{aligned} x_1 + 2y_1 &= 0 \\ 2x_1 + 4y_1 &= 0 \end{aligned}$$

A solution of these equations gives the components of an eigenvector.

How can you be certain that a non-zero solution exists?

- - - - - - - - - - ▷ 9

Given: $\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = 6.$

9

In order to find an eigenvector for $\lambda_1 = 1$ the following system of equations must be solved non-trivially:

$$\begin{aligned} x_1 + 2y_1 &= 0 \\ 2x_1 + 4y_1 &= 0 \end{aligned}$$

A non-zero solution must exist, since the determinant of the coefficients, which is det $(\mathbf{A} - \lambda_1 \mathbf{I})$, vanishes.

Find a solution to the system of equations given above.

--------------------▷ 10

We have $x_1 = -2y_1$. Choosing $x_1 = 2$, a particular solution reads $x_1 = 2, \quad y_1 = -1$. 10
Written differently:

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(Other choices for x_1 leading to other particular solutions are equally valid.)

This vector $\mathbf{r}_1$ must satisfy the equation

$$\mathbf{A}\mathbf{r}_1 = \lambda_1 \mathbf{r}_1 \quad \text{with} \quad \mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$$

or, equivalently,

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{r}_1 = \mathbf{0},$$

Check this numerically.

--------------------▷ 11

$$\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}2\\ -1\end{pmatrix} = \begin{pmatrix}2-2\\ 4-4\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

11

The other eigenvalue of the matrix was found to be $\lambda_2 = 6$. Write down the corresponding matrix equation for an eigenvector $\mathbf{r}_2$.

$$\mathbf{A} = \begin{pmatrix}2 & 2\\ 2 & 5\end{pmatrix}$$

--------------------▷ 12

$$\mathbf{A}\mathbf{r}_2 = 6\mathbf{r}_2 \qquad \text{i.e.} \quad (\mathbf{A} - 6\mathbf{I})\mathbf{r}_2 = \mathbf{0}$$ 12

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Write down the two homogeneous linear equations embodied by the last matrix equation. Find a particular non-trivial solution, i.e. an eigenvector $\mathbf{r}_2$ to λ_2.

--------------------▷ 13

$$\begin{aligned} -4x_2 + 2y_2 &= 0 \\ 2x_2 - y_2 &= 0 \end{aligned}$$

13

If we choose $y_2 = 2$ then $x_2 = 1$ and

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(Other choices of y_2 leading to other particular solutions are equally valid.)

Verify that $\mathbf{r}_2$ is an eigenvector corresponding to $\lambda_2 = 6$.

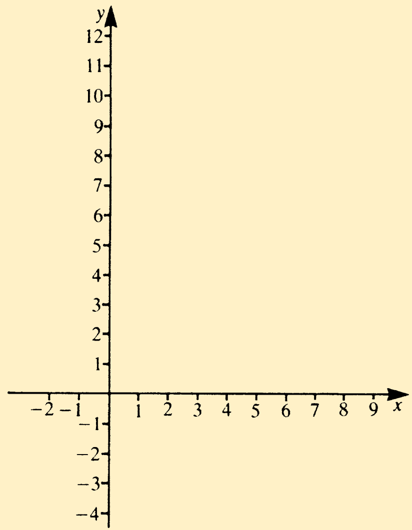
14

14

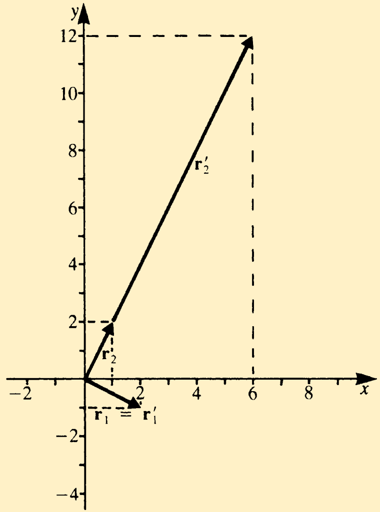
$$\mathbf{A}\mathbf{r}_2 = 6\mathbf{r}_2 :$$

$$\mathbf{r}_2' = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+4 \\ 2+10 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = \mathbf{6} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Draw $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_1'$ and $\mathbf{r}_2'$.



15



15

What do you notice about the angle between $\mathbf{r}_1$ and $\mathbf{r}_2$? Prove your claim.

16

The two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are orthogonal.

16

$$\text{Proof}: \mathbf{r}_1\mathbf{r}_2 = (2, \quad -1)\begin{pmatrix}1\\2\end{pmatrix} = 2\times 1 - 1\times 2 = 0$$

More abstract reasoning: The matrix $\mathbf{A}$ under consideration is symmetric. A theorem says that eigenvectors in that case are orthogonal.

$$\text{For } \mathbf{A} = \begin{pmatrix}-1 & -1 & 2\\ -1 & -1 & -2\\ 2 & -2 & -2\end{pmatrix}$$

Prove that $\lambda_1 = -2$ is an eigenvalue of $\mathbf{A}$.
Find the corresponding eigenvector.

I need some help --------------------▷ 17

I can do it --------------------▷ 20

In order to verify that $\lambda_1 = -2$ is an eigenvalue one may compute $\det(\mathbf{A} - \lambda_1 \mathbf{I})$ which has to vanish. 17

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix}, \quad (\mathbf{A} + 2\mathbf{I}) = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

Compute its determinant.

18

One can, for example, use Sarrus' rule: 18

$$0+4+4-4-4-0=0$$

The first part of the problem is done. For the second part, finding an eigenvector $\mathbf{r}_1$, we must solve a system of equations. As a matrix equation we must have

$$\mathbf{A}\mathbf{r}_1 = -2\mathbf{r}_1 \text{ or equivalently } (\mathbf{A}+2\mathbf{I})\mathbf{r}_1 = 0$$

Write down the system of equations with $\mathbf{r}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$.

19

$$\begin{aligned} x_1 - y_1 + 2z_1 &= 0 \\ -x_1 + y_1 - 2z_1 &= 0 \\ 2x_1 - 2y_1 \qquad &= 0 \end{aligned}$$

19

The determinant of the matrix of coefficients has been shown to vanish. Therefore, one of the equations depends linearly on the others. In fact:

$$\text{first equation} = -\text{second equation}$$

We can thus disregard the second equation. Now compute a particular solution $\mathbf{r}_1$ of the remaining two equations.

20

The determinant of $(\mathbf{A} + 2\mathbf{I})$ vanishes, so $\lambda_1 = -2$ is an eigenvalue of $\mathbf{A}$. The system of linear equations determining a corresponding eigenvector $\mathbf{r}_1$ reduces to

20

$$x_1 = y_1 \quad \text{and} \quad z_1 = 0$$

A particular solution is given by choosing $x_1 = 1$:

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Any other non-zero choice of x_1 leading to another particular solution is equally valid.

Check that $\mathbf{r}_1$ is indeed an eigenvector of

$$\mathbf{A} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix}$$

21

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & +0 \\ -1 & -1 & -0 \\ 2 & -2 & -0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} = -2\mathbf{r}_1$$

21

Do you feel in a position to compute the characteristic polynomial of **A**?

Yes: Splendid! Please do it.

--------------------▷ 22

No: Reread section 16.2 in the textbook. Then try to compute it.

--------------------▷ 22

The characteristic polynomial of **A** reads

$$\lambda^3 + 4\lambda^2 - 4\lambda - 16 = 0$$

22

Splitting off the linear factor $(\lambda - (-2))$ yields:

$$\lambda^3 + 4\lambda^2 - 4\lambda - 16 = (\lambda + 2)(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

The roots of the quadratic term are

$$\lambda_2 = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\lambda_3 = \ldots\ldots\ldots\ldots\ldots\ldots$$

23

$$(\lambda+2)(\lambda^2+2\lambda-8)$$
$$\lambda_2 = 2, \quad \lambda_3 = -4$$

23

An eigenvector corresponding to $\lambda_2 = 2$ is asserted to be

$$\mathbf{r}_2^* = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}$$

Check if this assertion is true.

24

$\mathbf{r}_2^*$ is *not* an eigenvector of $\mathbf{A}$: 24

$$\begin{pmatrix} -1 & -1 & 2 \\ -1 & -1 & -2 \\ 2 & -2 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -7 \\ 5 \\ 24 \end{pmatrix} \neq \lambda_2 \mathbf{r}_2^*$$

Find an eigenvector corresponding to $\lambda_2 = 2$.

25

To solve: $(\mathbf{A} - 2\mathbf{I})\mathbf{r}_2 = 0$.

25

$$\begin{aligned} -3x_2 - y_2 + 2z_2 &= 0 \\ -x_2 - 3y_2 - 2z_2 &= 0 \\ 2x_2 - 2y_2 - 4z_2 &= 0 \end{aligned}$$

Thus: $x_2 = -y_2$ and $z_2 = x_2$.

A particular solution is obtained by letting $x_2 = 1$:

$$\mathbf{r}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Find an eigenvector corresponding to $\lambda_3 = -4$.

26

$\mathbf{r}_3$ must be orthogonal to both $\mathbf{r}_1$ and $\mathbf{r}_2$, since $\mathbf{A}$ is symmetric. Therefore, $x_3 = -y_3$ and $z_3 = -2x_3$. A solution reads: 26

$$\mathbf{r}_3 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

(a) What is the maximum number of real eigenvalues an $n \times n$ matrix can have?
(b) Are there matrices which do not possess the maximum number of real eigenvalues?
(c) Give an example if $n = 2$.

-------------------▷ 27

27

(a) n
(b) Yes
(c) Any rotation of the plane about an angle $\alpha \neq 0$ or π does not possess any real eigenvalue.

So far we have dealt exclusively with symmetric matrices. Now let us consider a matrix which is non-symmetric:

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

Find its eigenvalues and eigenvectors.

28

(a) Characteristic equation: $\lambda^2 - \lambda - 6 = 0$

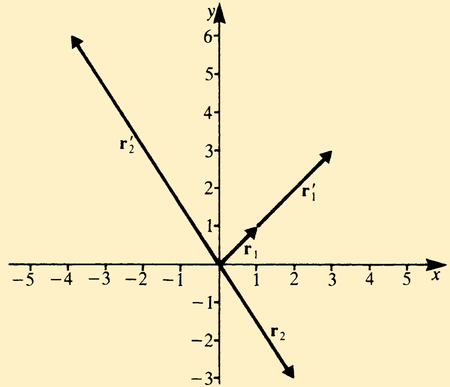
28

(b) Eigenvalues: $\lambda_1 = 3, \ \lambda_2 = -2$

(c) Eigenvectors: $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

Draw $\mathbf{r}_1, \ \mathbf{r}_2, \ \mathbf{r}'_1$ and $\mathbf{r}'_2$ in a diagram.
Are $\mathbf{r}_1$ and $\mathbf{r}_2$ orthogonal?

- - - - - - - - - - ▷ 29



They are not orthogonal:

$$\mathbf{r}_1\mathbf{r}_2 = (1, \ \ 1)\begin{pmatrix} 2 \\ -3 \end{pmatrix} = 2 - 3 \neq 0.$$

Well done!
You have worked well, and now you deserve a break.
For further study you should solve the problems given in the textbook.

END OF CHAPTER 16

Chapter 17
Vector analysis: Surface integrals, Divergence, Curl, and Potential

0

1

17.1 Flow of a vector field through a surface element

1

The prerequisite to study this chapter is a certain familiarity with functions of several variables, partial differentiation, and total differentiation. These topics you have studied in chapter 12. Perhaps a short repetition may help you.

Study in the textbook **17. 1 Flow of a vector field through a surface element**
Textbook pages 461–464

Go to ------------------▷ 2

2

In the section you studied some new concepts have been introduced and defined.
Write down at least three of them:

1. ……………………………………………………..

2. ………………………………………..

3.……………………………………………

--------------------▷ 3

Flow density $\vec{j}$

Surface element vector $\vec{A}$

Flow of a vector field $\vec{F}$ through a surface $\vec{A}$

3

Try to write down on a separate sheet the definitions and meanings first using your memory but in case of difficulties by consulting your notes and if you do not succeed the textbook.

------------------ ▷ 4

The flow density $\vec{j}$ is the quantity which passes through a unit area per time unit. The area is assumed to be perpendicular to the flow.

4

The surface element vector is a vector $\vec{A}$ whose direction is perpendicular to the surface and whose magnitude is equal to the area A.

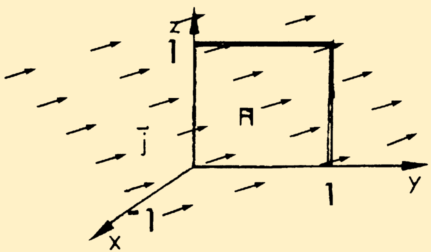
Flow of a vector field through a surface element is given by the dot product of the two vectors $\vec{F} \cdot \vec{A}$.

-------------------▷ 5

We want the flow of a vector field through an even square surface whose magnitude is A. | 5 |

The surface belongs to the y-z plane. The flow hits the surface at an angle $\beta \quad \beta \leq \frac{\pi}{2}$

The flow vector is constant and given by $\vec{j} = (-j_x, j_y, 0)$



We decompose the task:

1. We determine $\vec{A}$
2. We calculate the flow $\vec{j} \cdot \vec{A}$

The flow is: …………………

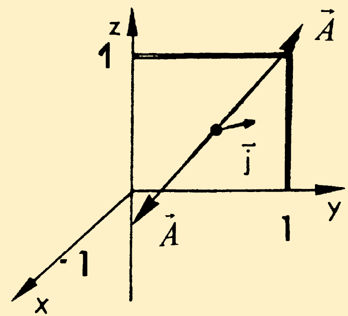
Further explanations wanted --------------------▷ 6

Solution found --------------------▷ 11

6

We first determine the surface element vector $\vec{A}$. Since the surface element belongs to the y-z plane its direction is parallel to the x-axis. Thus, $\vec{A}$ also has the direction of the x-axis.

$\vec{A} = (+1,0,0)$ or $\vec{A}(-1,0,0)$ Since the flow vector is to hit the surface element at an angle less than $\frac{\pi}{2}$ we obtain $\vec{A} = \ldots\ldots\ldots\ldots\ldots\ldots$



Hint: remember the definition of the given $\vec{j}$.

-------------------▷ 7

$\vec{A} = A(-1,0,0)$ 7

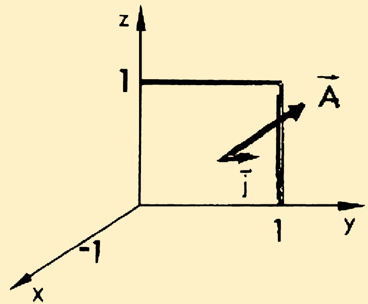
No more problem --------------------▷ 9

More explication wanted --------------------▷ 8

The surface element belongs to the y-z plane. Thus, its surface element vector is directed in the x-direction. But there are two possibilities. It may point in the positive or in the negative x-direction. We have to decide which of these directions is the right one. In our exercise is given the condition that the angle between both given vectors is to be less than 90° or $\frac{\pi}{2}$.

8

In our exercise the x-component of $\vec{j}$ is negative. Thus, $\vec{A}$ has to point in the negative direction of the x-axis. From this we obtain: $\vec{A} = (-1,0,0,)$



-------------------▷ 9

We determined the surface element vector $\vec{A}$ to be $\vec{A} = A(-1,0,0)$

9

Now we can determine the flow I of $\vec{j}$ through the surface:

$$\vec{j} = (-j_x, j_y, 0)$$

$$I = \vec{j} \cdot \vec{A} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found -------------------▷ 11

Further help wanted -------------------▷ 10

We remember the dot product of two vectors $\vec{a} = (a_x, a_y,, a_{z)}$ and $\vec{b} = (b_x, b_y, b_z)$ is defined by **10**

$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$

Now you should be able to calculate using the results obtained before:

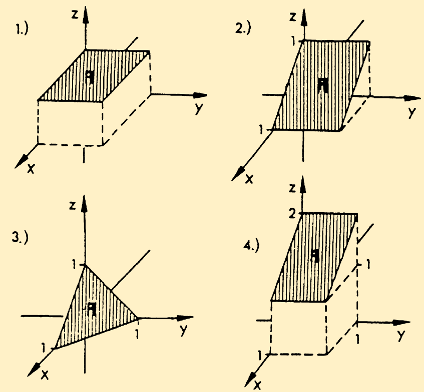
$\vec{j} \cdot \vec{A} =$ ……………………………………………………..

--------------------▷ 11

$I = j_x \cdot A$

11

Give the surface element vector for the four given surfaces the magnitude of which is always A



12

In each case we have two answers. The difference is the sign.
This time a vector field that determines the direction is not given.

12

1. $\vec{A} = A(0,0,1)$. or $\vec{A} = A(0,0-,1)$

2. $\vec{A} = \frac{A}{\sqrt{2}}(1,0,1)$) or $\vec{A} = \frac{A}{\sqrt{2}}(-1,0-,1)$

3. $\vec{A} = \frac{A}{\sqrt{3}}(1,1,1)$ or $\vec{A} = \frac{A}{3}(-1,-1,-1)$

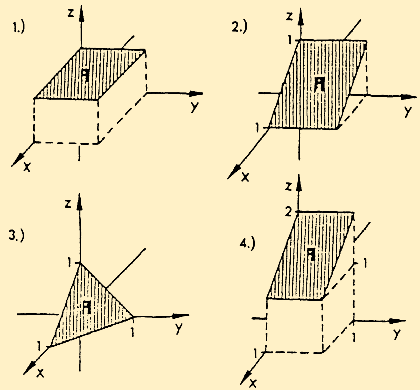
4. $\vec{A} = \frac{A}{\sqrt{2}}(1,0,1)$ or $\vec{A} = \frac{A}{\sqrt{2}}(-1,0,-1)$

All correct ------------------ ▷ 16

Errors or explanation wanted ------------------ ▷ 13

13

The surface element vector $\vec{A}$ is perpendicular to the surface. Thus, we look at first for a vector perpendicular to the surface, the magnitude of which does not matter for the time being. We give the first two solutions then obtain the next two solutions.
a has not been defined yet.
a will be defined later.



1. $\vec{A} = a(0,0,1)$

2 . $\vec{A} = a(1,0,0)$

3. $\vec{A} = a(..............)$

4, $\vec{A} = a(.................)$

----------------------▷ 14

14

1. $\vec{A} = a(0,0,1)$

2. $\vec{A} = a(1,0,1)$

3. $\vec{A} = a(1,1,1)$

4. $\vec{A} = a(1,0,1)$

Now we will determine a. $\vec{A}$ must have the magnitude A. To obtain this we have to choose a.
For the first surface it is evident that:

$\vec{A} = A(0,0,1)$ In this case $a = A$

For the second exercise we have

$\vec{A} = \frac{A}{\sqrt{2}}(1,0,1)$ We may prove it: $A^2 = \frac{A^2}{2}(1+1)$

Now you may try to calculate a for the third exercise $\vec{A} = ...(1,1,1)$

-------------------▷ 15

. $\vec{A} = \frac{A}{\sqrt{3}}(1,1,1)$ Prove: $(\vec{A})^2 = \frac{A^2}{3}(1+1+1) = A^2$

15

We give the systematic solution. Given: $\vec{A} = a(a_x, a_y, a_z)$.

If $\vec{A}$ is to be of magnitude A this results in $(\vec{A})^2 = A^2$ this results in

$$A^2 = a^2(a_x^2 + a_y^2 + a_z^2)$$

Thus, finally we obtain

$$a = \frac{A}{\sqrt{a_x^2 + a_y^2 + a_z^2}}$$

In our case no direction of the vector field is given.
Thus, the sign of the surface element vector may be inverted.

------------------▷ 16

17.2 Surface integral

16

Study in the textbook 17.2 Surface integral

Textbook pages 464–466

This done go to

-------------------▷ 17

The following integral is named ………

17

$$I = \oint \vec{F} \cdot d\vec{A}$$

The circle in the integral $I = \oint \vec{F} \cdot d\vec{A}$

significates the integral has to be taken over a ………………………surface.

-------------------▷ 18

Surface integral

18

Closed surface

Give at least three examples for a closed surface ….

1.: ………………………………………………………..

2: ………………………………………………

3:…………………………………………………..……………………………………………...…….
……………………………………..

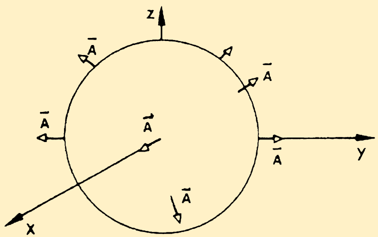
-----------------▷ 19

Control your examples using the definitions given in the textbook

19

This control is necessary if there are no right answers given.
In your further study and in practice this is the normal case.

--------------------▷ 20



The direction of the surface element vectors for closed surfaces is defined

20

unambiguous --------------------▷ 21

ambiguous --------------------▷ 22

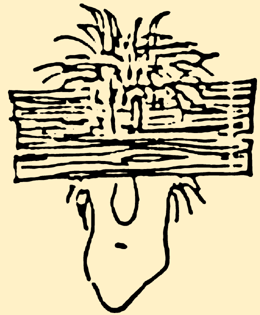
You are right. With closed surfaces the sign is defined unambiguously. 21

With closed surfaces the surface element vector always points to ....

....

--------------------▷ 23

Unfortunately you are wrong.



22

Look again into the textbook and read again the definition or better the convention of the direction of surface elements.

The surface element vectors are

a) perpendicular to the surface
b) they point with closed surfaces always to…………………………..

------------------▷ 23

The surface element vector for closed surfaces always point to the outer space. 23

This is a convention, but it is worth memorizing.

24

17.3 Special cases of surface integrals

24

Study in the textbook **17.3.1 Flow of a homogeneous vector field through a cuboid**

Pages 466–468

-------------------▷ 25

Decide whether the following vector fields are homogeneous or not

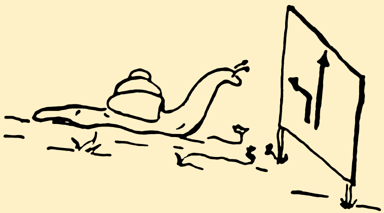
25

	Vector field is homogeneous	
	yes	no
1. $\vec{F} = \frac{(x, y, z)}{x^2 + y^2 + z^2}$	☐	☐
2. $\vec{F} = (1, 0, x)$	☐	☐
3. $\vec{F} = (y, x, z)$	☐	☐
4. $\vec{F} = (6, 3, 5)$	☐	☐
5. $\vec{F} = (2, 0, 0)$	☐	☐

------------------▷ 26

Homogeneous vector fields are 4. 5.

26



All correct → 29

Not all answers correct → 27

27

In this case we suggest you read again the section you just studied.
Theses questions did not imply calculations. Thus, you have problems with the understanding of the definition of a homogeneous vector field.
Read again to try and find the correct answers.

Vector field	homogeneous	not homogeneous
1. $\vec{F} = \dfrac{(1,2,3)}{x^2 + y^2 + z^2}$	☐	☐
2. $\vec{F} = (\dfrac{1}{x}, \dfrac{1}{y}; \dfrac{1}{z})$	☐	☐
3. $\vec{F} = (1;0;0)$	☐	☐
4. $\vec{F} = (x;0;0)$	☐	☐
5. $\vec{F} = (21;1;)$	☐	☐

------------------▷ 28

28

1. $\vec{F} = \dfrac{(1,2,3)}{x^2 + y^2 + z^2}$ not homogeneous

2. $\vec{F} = (\dfrac{1}{x}, \dfrac{1}{y}; \dfrac{1}{z})$ not homogeneous

3. $\vec{F} = (1;0;0)$ homogeneous

4. $\vec{F} = (x;0;0)$ not homogeneous

5. $\vec{F} = (2\,1;1;\,)$ homogeneous

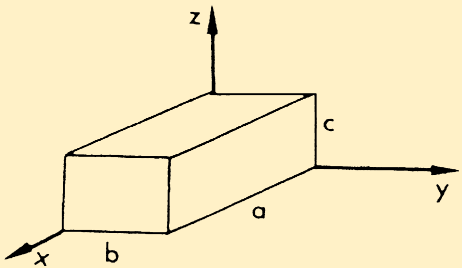
------------------ ▷ 29

Determine the flow of the field $\vec{F}(x,y,z)=(1,4,3)$ through a cube whose faces are parallel to the axis of the coordinate system.

29

The flow I of the vector field $\vec{F}$ is

$I =$..........................

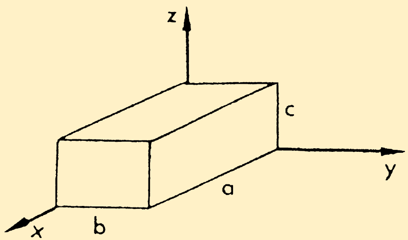


Solution found ------------------▷ 31

One more hint wanted ------------------▷ 30

Hint: $\vec{F} = (1,4,3)$ is a homogeneous vector field.
Thus, you can apply rule 17.7 given in the textbook.

30



Give the flow of the vector field $\vec{F} = (1;4;3)$ through a cube whose faces are not parallel to the axes of the coordinate system.

$I =$..................................

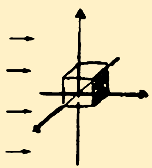
------------------▷ 31

$I = 0$

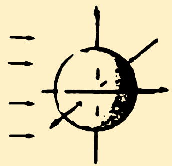
31

For which of the given surfaces below does the flow of the homogeneous field $\vec{F} = (0,2,0)$ not vanish? The surfaces are closed.

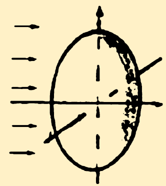
cuboid ☐



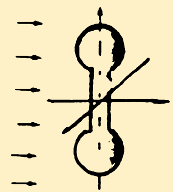
sphere ☐



ellipsoid ☐



dumbbell-shaped ☐



------------------▷ 32

None of these. All are closed. A homogeneous vector field vanishes for all closed surfaces. 32

Given the vector field $\vec{F} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$. Does its flow through a sphere vanish if the sphere's center coincides with the origin of the coordinate system?

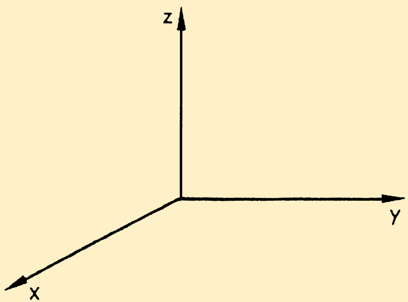
Answer found ------------------▷ 35

Help or more explanation wanted ------------------▷ 33

The surface of the sphere is closed.

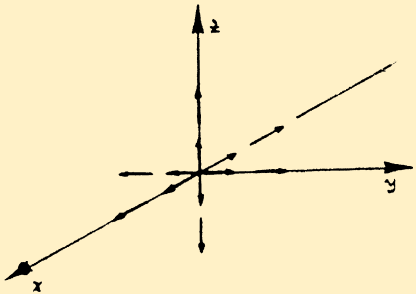
33

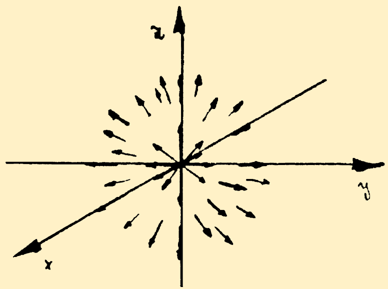
Given the vector field $\vec{F} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$. This vector field is not homogeneous.



Sketch some vectors of the field along the axes of the coordinate system

------------------▷ 34



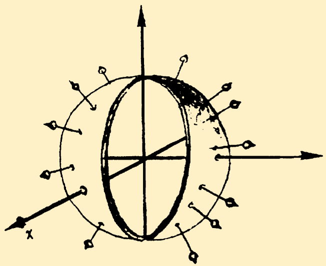


34

If vectors in all directions are sketched, the field is represented as shown to the left.
Imagine a sphere with its center at the origin of the coordinate system. The field passes through the surface always from the inner side to the outer side.

Does the flow through the surface vanish?

-------------------▷ 35



No. The flow of $\vec{F} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}$ does not vanish.

35

The field passes at all points of the surface from the inner space to the outer space.

A vector field $\vec{F}$ has radial symmetry if

1…………………………………………………

2……………………………………….

In case of doubt look again at textbook chapter 13 (section "coordinate systems").

------------------▷ 36

A vector field has radial symmetry if
1. All vectors point in a radial direction and;
2. Its amount depends only on the radius r.

36

Decide whether the following field possesses radial symmetry:

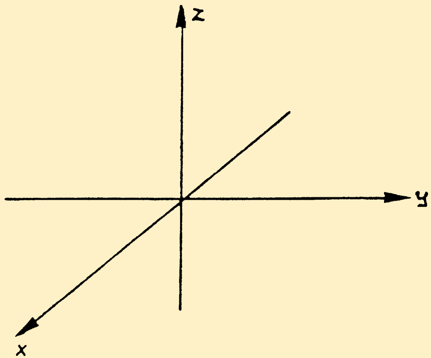
2. Its amount depends only on r. $\vec{F} = \dfrac{(x, y, z)}{(\sqrt{x^2 + y^2 + z^2})^3} = \dfrac{\vec{r}}{r^3}$

Solution found ------------------▷ 39

Further explanation or help wanted ------------------▷ 37

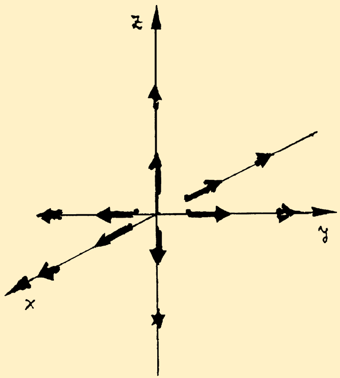
37

Firstly sketch some of the vectors of the field $\vec{F} = \dfrac{\vec{r}}{r^3}$



------------------▷ 38

38



Your sketch may be similar to the sketch to the left.
The vectors point outwards. Because of r^3 in the denominator the amount of the vectors decreases with the distance from the origin.
The vectors at other positions point in a radial direction as well.
Since $|(x, y, z)| = \mathrm{r}$ we obtain:

$$F = \frac{(x, y, z)}{(\sqrt{x^2 + y^2 + z^2})^3} = \frac{r}{r^3} = \frac{1}{r^2}$$

F only depends on r.
Thus, this vector field has ....

--------------------▷ 39

The vector field $\vec{F}$ has radial symmetry because

$\vec{F} = \dfrac{(x,y,z)}{(\sqrt{x^2+y^2+z^2})^3}$ points in a radial direction and its amount depends only on r.

39

40

17.3.1 Flow of a spherically symmetrical field through a sphere

40

In this section we show the calculation of surface integrals through a surface of a sphere. In physics this is an important special case.

Read in the textbook **17.3.2 Flow of a spherically symmetrical field through a sphere**
Textbook pages 468–469

-------------------▷ 41

41

Calculate the surface integral $\oint \vec{F} \cdot d\vec{A}$ of the field $\vec{F} = \frac{\vec{e}_r}{r^2}$

with $\vec{e}_r = \frac{\vec{r}}{r}$ for a surface of a sphere with radius R.

Wanted is the flow of $\vec{F}$ through the surface of the sphere.

$\oint \vec{F} \cdot d\vec{A} =$ ....

------------------ ▷ 42

$$\oint \vec{F} \cdot d\vec{A} = \oint \frac{dA}{r^2} = 4\pi R^2 \cdot \frac{1}{R^2} = 4\pi$$ 42

Given a field of a force with radial symmetry: $\vec{F}(r) = \frac{a}{r^3}\vec{e}_r$ with $\vec{e}_r \frac{\vec{r}}{r}$

Compute the flow of the field of the force $\vec{F}(r)$ through the surface of a sphere which has a distance R from the origin of the field. The center of the field is defined by $r = 0$.

Solution found --------------------▷ 44

Explanation or help wanted --------------------▷ 43

The field of the force is given by $\vec{F}(r) = \frac{a}{r^3}\vec{e}_r$ with $\vec{e}_r \frac{\vec{r}}{r}$

43

Wanted: the flow given by $\oint \vec{F} \cdot d\vec{A}$

First hint: The field has radial symmetry of the following form: $\overline{F} = f(r) \cdot \vec{e}_r$

Second hint: $\oint \vec{F} \cdot d\vec{A}$ is calculated for the general case in the section of the textbook you just studied.

Repeat your study and try again:

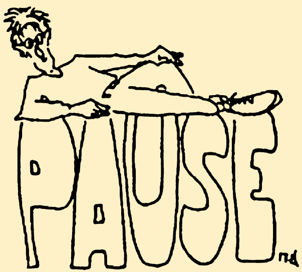
$\oint \vec{F} \cdot d\vec{A}$ = ...

------------------▷ 44

$$\oint \frac{a}{r^3}\vec{e}_r d\vec{A} = \oint f(r)\vec{e}_r d\vec{A} = 4\pi R^2 \cdot f(R) = 4\pi R^2 \cdot \frac{a}{R^3} = \frac{4\pi \cdot a}{R}$$

44

Have a short break



------------------▷ 45

17.3.2 Application: The electrical field of a point charge.

45

Here we apply the new knowledge

Study in the textbook **17.3.3 Application: The electrical field of a point charge**
Textbook page 470

------------------ ▷ 46

The calculations in the foregoing section are obvious. In this case using physical units facilitates the calculations.

46

Thus, we can proceed to the next section without further exercises.

Go to ------------------▷ 47

17.4 General case of computing surface integrals

47

In the following section we compute the surface integral for a general case. This section is slightly formal and more difficult. It is worth studying this section if you are not in a hurry and did not have difficulty with the preceding subject matter. Decide according to your preferences.

I prefer to skip section 17.4 for the time being and want to proceed

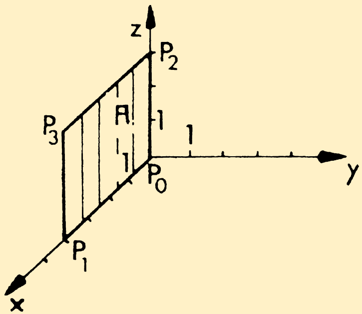
------------------▷ 55

I prefer to study section 17.4.

Then study in the textbook

17.4 General case of computing surface integrals
Textbook pages 470–474

------------------▷ 48



A rectangular plane A is defined by the points $P_0 = (0,0,0)$, $P_1 = (4,0,0)$, and $P_3 = (4,0,0)$. Thus it lies in the x-z plane.

48

An inhomogeneous vector field is given by $\vec{F} = (0, 2x, 0)$.

Compute $\int \vec{F} d\vec{A} = \ldots\ldots..$

Solution found --------------------▷ 54

Explanation or hints wanted --------------------▷ 49

In this case we have an inhomogeneous field. The field is quite simple, because it has only one component in the y-direction. Remember the field was given by $\vec{F} = (0, 2x, 0)$ 49

To obtain $I = \int_A \vec{F} \cdot d\vec{A}$ we determine $\vec{F}$ and $d\vec{A}$

$\overline{F} = \ldots\ldots\ldots\ldots$

$d\vec{A} = \ldots\ldots\ldots$

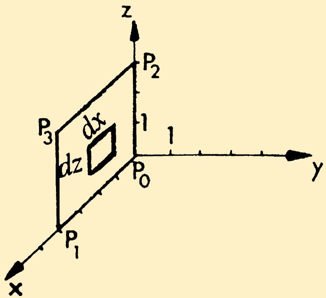
$I = \ldots\ldots\ldots\ldots\ldots$

Solutions found --------------------▷ 52

Further help and explanations --------------------▷ 50

We have to determine $\vec{F}$ and $d\vec{A}$. 50

Difficulties may arise in determining $d\vec{A}$. Plane A lies in the x-z plane, as was pointed out in frame 48. Thus, the surface element vector points in the y-direction.



The amount of a differential surface element for this plane is given by dA $= dx \cdot dz$.
The surface element vector in the y-direction with amount dxdz is given by

$d\vec{A} = (.............)$

Remember: $\vec{F}$ is given to be $\vec{F} = (....................)$

-------------------▷ 51

$$d\vec{A} = (0, dx \cdot dz, 0)$$
$$\vec{F} = (0, 2x, 0)$$

51

We want to determine the flow $I = \int_A \vec{F} \cdot d\vec{A}$ a

Now we can calculate the dot product in the integral using the results obtained:

$$I = \int_A \vec{F} \cdot d\vec{A} \quad = \int_A (0, 2x, 0) \cdot (0, dxdz, 0) = \int_A \ldots \ldots\ldots\ldots\ldots\ldots\ldots$$

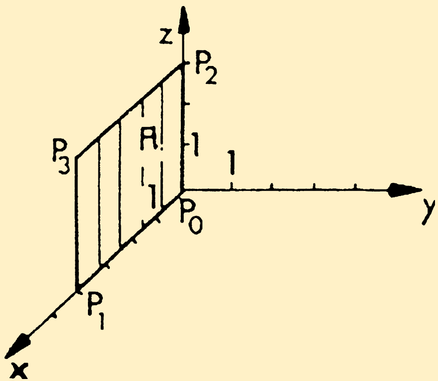
------------------ ▷ 52

$$\int_A \vec{F} d\vec{A} = \int_A 2x dx dz$$

52

This integral must be calculated for plane A.
Written correctly this is a double integral. Write this integral as a double integral and insert the limits given by our plane A:

$$A\int 2x \cdot dx \cdot dz = \int_{x=\ldots\ldots}^{\ldots\ldots\ldots} \int_{z=\ldots\ldots}^{\ldots\ldots\ldots} 2x \cdot dx \cdot dz$$



-------------------▷ 53

$$I = \int_A 2x dx \cdot dz = \int_{x=0}^{4} \int_{z=0}^{3} 2x dx \cdot dz$$

53

You can calculate the double integral. It was explained in chapter 13 "Multiple integrals."
In case of difficulties you should repeat that chapter, at least section 13.2 "multiple integrals with constant limits."

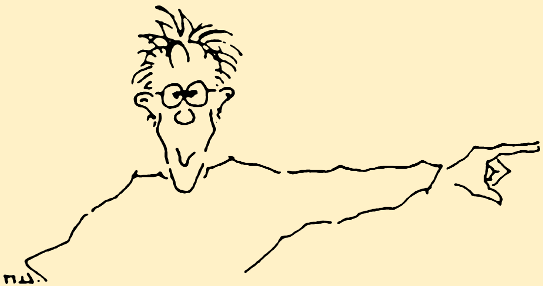
$$I = \int_A 2x dx \cdot dz = \int_{x=0}^{4} \int_{z=0}^{3} 2x dx \cdot dz = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------ ▷ 54

$$I = \int_A \vec{F} \cdot d\vec{A} = 48$$

54

Congratulations. You have mastered some difficult reasoning.

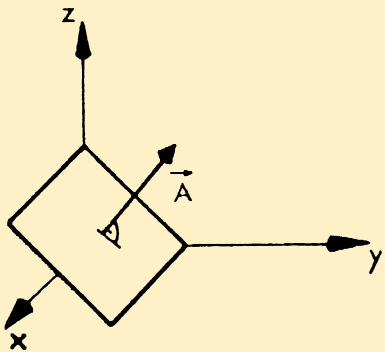


55

Before we finish this section we should recapitulate.
In a vector field $\vec{F}$ we have a quadratic plane with an area of 2.
The position is sketched below.

Give the surface element vector $\vec{A}$. $\vec{A}=$.........

55

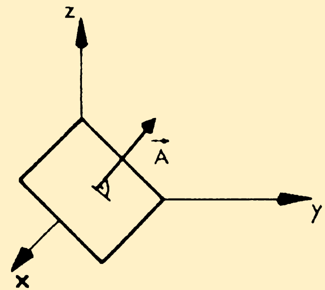


56

$\vec{A} = (0, \sqrt{2}, \sqrt{2})$ You may have factorized the root: $\vec{A} = \sqrt{2}(0,1,1)$

56

Calculate for this plane A the flow of three vector fields:



$F_1 = (0,6,0)$ $I_1 = \ldots\ldots\ldots\ldots\ldots$

$F_2 = (0,2,1)$ $I_2 = \ldots\ldots\ldots\ldots$

$F_3 = (6,0,0)$ $I_3 = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 57

$I_1 = 6 \cdot \sqrt{2}$ $I_2 = 3 \cdot \sqrt{2}$ $I_3 = 0$

57

Given a vector field with radial symmetry $\vec{j}$. The origin coincides with the origin of the coordinate system.

The amount of $\vec{j}$ is constant.

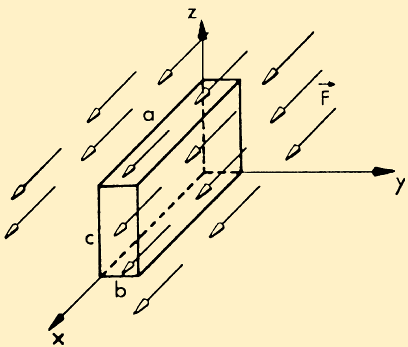
Calculate the flow I of $\vec{j}$ through a sphere with radius R. The center of the sphere lies in the origin of the coordinate system.

$I =$……………

------------------▷ 58

$$I = \int \vec{j} \cdot d\vec{A} = 4\pi R^2 \left| \vec{j} \right|$$

58



Calculate the flow Φ of a vector field

$\vec{F} = (1,0,0)$

Through the sketched cuboid with the edges

a = 6, b = 1, c = 3

$\Phi =$ ....

------------------▷ 59

$\Phi = 0$

59

Given $\vec{F} = (0.5, 0, 0.5)$

Calculate the flow for three planes:

$\vec{A}_1 = (1, 1, 0)$ $I_1 = \ldots\ldots\ldots\ldots$

$\overline{A}_2 = (1, 0, 1)$ $I_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

$\vec{A}_3 = (1, 1, 1,)$ $I_3 = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 60

$I_1 = 0.5$ $I_2 = 0$ $I_3 = 0$

60

Hint to improve your study skills.

All textbooks have an index. It should be a habit to use the index. Nobody remembers all he should have learned. Stop your time to find the definition of Bernoulli`s equation in the textbook using the index.

………………………

------------------▷ 61

As a rule you will need 20 to 30 seconds to find such a definition. That is not much time. 61

If when reading a section of a textbook you encounter a concept the meaning of which you have forgotten, it is a normal tendency to overlook this and to hope that this meaning will be explained later or that you will remember it later.
Unfortunately this may cause learning difficulties, sometimes even serious difficulties which will cost you a lot of time. The new subject matter is explained in textbooks using concepts and words the reader is supposed to be familiar with. Unfortunately if a new concept is explained with concepts and words the reader is not familiar with, the probability that he understands the new subject matter is small. To avoid this it is worthwhile to develop two skills which seem quite simple but which are not that simple to apply.
First you should develop a competence in noticing concepts or words you do not understand. Most of us have a strong tendency to overlook them.
Second you should develop a tendency to use the index and read again the meaning of these concepts or words you are not familiar with.
You may think this wastes your time, however, on the whole this saves a lot of time.

------------------ ▷ 62

The tendency to overlook things we do not understand is natural and even necessary to survive. Nobody can understand everything. But if we even do not notice that we do not understand something, this may be dangerous. And may have consequences. Try to develop competence in noticing things you do not understand and develop a tendency to use indexes and encyclopedia. At least a few times per week.

62

63

17.5 Divergence of a vector field and Gauss's theorem

63

The introduction of the concept of divergence is closely related to the last section on surface integrals.

Study in the textbook

17.5 Divergence of a vector field
17.6 Gauss's theorem
Pages 475–479

Having concluded go to ------------------▷ 64

$\vec{F}$ represents a vector field. Complete the definition

64

$div \quad \vec{F} =$ ....

$div \quad \vec{F}$ is a ....

-------------------▷ 65

$$div \quad \vec{F} = \frac{\delta F_x}{\delta x} + \frac{\delta F_y}{\delta y} + \frac{\delta F_z}{\delta z}$$

65

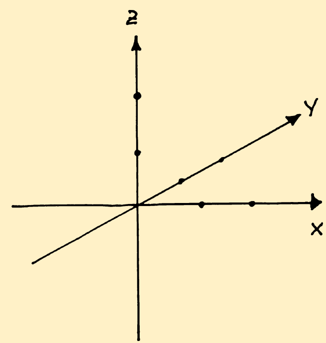
$div \quad \vec{F}$ is a scalar field.

Calculate the divergence for the given vector field:

$$\vec{F} = (x, \quad y + b, \quad -z^2)$$

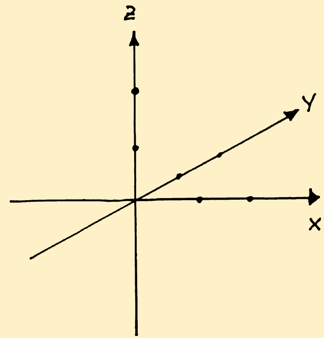
$$div \quad \vec{F} = \dots$$

Determine the location of sources and sinks.



66

$div \quad \vec{F} = (1+1-2z) = (2-2z) = 2(1-z)$



66

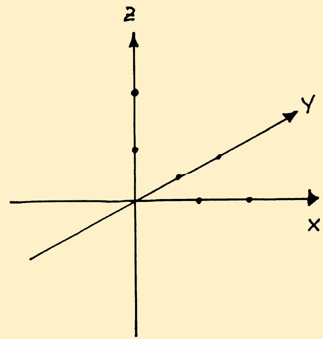
For the plane $z=1$ there are no sources or sinks.
The space above this plane, $z>1$, consists of sinks.
The space beneath this plane, $z \le 1$, consists of sources

All correct, go ▷ 69

Another exercise wanted ▷ 67

67

We have no sink or sources if $div \quad \vec{F} = 0$.
With the preceding example this was the case for $div\vec{F} = 2(1 - z) = 0$.
The equation $1 - z = 0$ or $z = 1$ represents a plane which is parallel to the x-y-plane. For it $div \quad \vec{F}$ is negative. This means the space above the plane $z = 1$ consists of sinks.
Beneath the plane $z = 1$ $div\vec{F}$ is positive and thus the space consists of sources.



Calculate the divergence for the following vector field:

$\vec{F} = (x^2 + 1, \quad y, \quad z + 5)$:

$div \quad \vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Find the spaces where we have sources or sinks and where the field is free of sources and sinks.

------------------▷ 68

68

$div\vec{F} = 2x + 1 + 1 = 2(x+1)$

For the plane $x = -1$ the field is free of sinks and sources.
The space to the left of this plane, defined by $x < -1$, consists of sinks.
The space to the right of this plane, defined by $x > -1$, consists of sources.

Go on to ------------------▷ 69

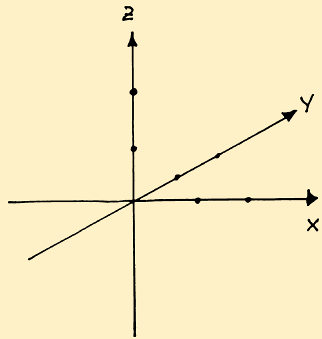
69

Calculate the divergence for the vector field

$\vec{F} = (x^3, \ y^3, \ -3z)$

$div\vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Where do you find sinks and sources?



…………………………………………………………………………………………………

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…………………………………………………………………………………………………

------------------▷ 70

70

$div \quad \vec{F} = 3x^2 + 3y^2 - 3 = 3(x^2 + y^2 - 1)$

Sources and sinks vanish for $x^2 + y^2 = 1$.

This is a circle with radius 1 which does not depend on the value of z.

Thus it is a cylinder.

The space inside the cylinder consists of sinks since $div \quad \vec{F}$ is negative.

The space outside the cylinder consists of sources since $div \quad \vec{F}$ is positive.

------------------ ▷ 71

In the textbook the Nabla operator has been introduced. It is a new notation which seems to be merely formal. But it will prove quite useful because it is a short-hand notation that simplifies notations once you have familiarity with it.

71

$$\vec{\nabla} = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z} \right)$$

$\vec{\nabla} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 72

$$\vec{\nabla} = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z} \right)$$

72

Using the Nabla operator we obtain a short-hand notation to represent
the gradient of a scalar field $f(x, y, z)$ and
the divergence of a vector field $\vec{F}(x, y, z)$.
We start with the calculation of the gradient:

$grad\ f(x, y, z) = \vec{\nabla} \cdot f(x, y, z) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 73

$$grad f(x,y,z)=\vec{\nabla}\cdot f(x,y,z)=\left(\frac{\delta f}{\delta x}\cdot\vec{e}_x+\frac{\delta f}{\delta y}\cdot\vec{e}_y+\frac{\delta f}{\delta z}\cdot\vec{e}_z\right)$$

73

$\vec{\nabla}$ is a vector. In this case we compute the product of a vector with a scalar since the function $f(x,y,z)$ is a scalar.
The result is a vector since the product of a vector with a scalar is a vector.

Now we calculate the divergence of a vector field

$$div \quad \vec{F}=\vec{\nabla}\cdot\vec{F}(x,y,z)=\frac{\delta F_x}{\delta x}\vec{e}_x+\frac{\delta F_y}{\delta y}\vec{e}_y+\frac{\delta F_z}{\delta z}\cdot\vec{e}_z$$

Again $\vec{\nabla}$ is a vector. But this time we compute
the dot product of the vector Nabla with the vector $\vec{F}$. The result is a scalar. We remember well from vector algebra the dot product of two vectors results in a scalar.
Calculate $\vec{\nabla}\cdot\varphi(x,y,z)$ for $\varphi=(x^2+y^2+z^2)$:
$grad \quad \varphi=\vec{\nabla}\cdot\varphi=$..................................

Calculate $\vec{\nabla}\cdot\vec{F}$ for $\vec{F}=(x^2,y^2,z^2)$:
$div \quad \vec{F}=\vec{\nabla}\cdot\vec{F}=$.......................................

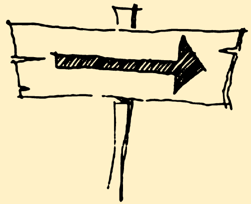
--------------------▷ 74

$$grad \quad \varphi = \vec{\nabla} \cdot \varphi = (2x, 2y, 2z) = (2x\vec{e}_x + 2y\vec{e}_y + 2z\vec{e}_z)$$

74

(The vector *grad* φ may be written down in short-hand or extensively)

$$div \quad \vec{F} = \vec{\nabla} \cdot \vec{F} = (2x + 2y + 2z)$$



75

In the textbook page 470 the electrical field of a sphere is discussed. 75
Given a sphere with homogeneous charge distribution ρ , the total charge Q ,
and radius *R*. Then outside the surface of the sphere the electrical field is given by

$$\vec{E}(x,y,z)=\frac{Q}{4\pi\varepsilon_0}\frac{(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

Now calculate the divergence $\vec{E}$ outside the surface of the sphere.

$div \quad \vec{E}$ =...

------------------▷ 76

$div \quad \vec{E} = 0$

76

In case of difficulties consult the textbook again.

....

Inside the sphere the electrical field is given by

$$\vec{E}(x, y, z) = \frac{Q}{4\pi\varepsilon_0 R^3}(x, y, z)$$

Calculate the divergence inside the sphere:

:

$div \quad \vec{E} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 77

$$div \quad \vec{E} = \frac{3Q}{4\pi\varepsilon_0 R^3} = \frac{\rho}{\varepsilon_0}$$

77

In case of difficulties remember $\frac{3}{4}\pi R^3 = V$ and $\frac{Q}{V} = \rho$.

....

Write down from memory the Gauss theorem. If possible do not consult the textbook. You should memorize this theorem.

....

....

....

--------------------▷ 78

$$\int_V div \ \vec{F} \cdot dV = \oint_{A(V)} \vec{F} \cdot \overrightarrow{dA}$$

78

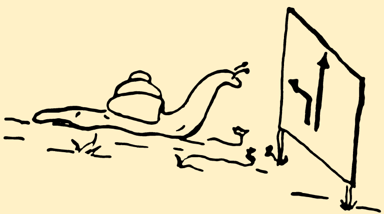
....

In the following we will use Gauss's theorem to calculate the given electrical field inside and outside of the surface of the sphere.

First calculate the electrical field outside the surface of the sphere:

$\vec{E} =$ ....

Solution found ------------------▷ 82



Help and explanations wanted ------------------▷ 79

In the textbook page 470 the electric field of a sphere is given with radius R and charge density ρ. Outside of the sphere the electrical field is: 79

$$\vec{E}(x,y,z)=\frac{Q}{4\pi\varepsilon_0}\frac{(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

We can obtain this result using Gauss's theorem. The center of the sphere is to coincide with the origin of the coordinate system.

First we calculate the total charge Q of the sphere. Given radius R and charge density inside the sphere ρ:

$Q = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 80

$$Q = \int_{V_{Kugel}} \rho \ \ dv = \frac{4\pi}{3}\rho R^3$$

80

Secondly we calculate the electrical field outside the sphere using Gauss's theorem.

$$\int_V div\vec{F}\cdot dV = \oint_{A(V)} \vec{F}\cdot\overrightarrow{dA}$$

From electrodynamics we know that the flow of the electrical field through a closed surface is proportional to the enclosed total charge Q.

$$\oint_{Surfase} \vec{E}\cdot\overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

Calculate the flow through the surfaces outside the sphere with the given radius $R_{outside}$:

$$\oint \vec{E}\overrightarrow{dA} = \dots\dots\dots\dots\dots\dots = \frac{Q}{\varepsilon_0}$$

------------------▷ 81

$$\int \vec{E}\cdot\overrightarrow{dA} = |E|\cdot 4\pi R^2_{außen} = \frac{Q}{\varepsilon_0}$$ 81

From this result we obtain:

$$|E| = \frac{Q}{\varepsilon_0 \cdot 4\pi R^2_{außen}}$$

The field has spherical symmetry. The field vector points to the outside and its direction is given by the unit vector:

$$\frac{\vec{r}}{|r|} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}$$

Thus we obtain: $|r| = R = \sqrt{x^2 + y^2 + z^2}$

$$\vec{E} = \frac{Q}{\varepsilon_0 4\pi R^2}\cdot\frac{\vec{r}}{|r|} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 82

$$\vec{E}(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\frac{(x,y,z)}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

82

Now we are to calculate the electrical field inside the sphere.
Let us regard a surface of a sphere inside with radius R_{inside}.
We are to calculate the enclosed charge. Use Gauss's theorem again and use the total charge Q of the original sphere.

$\vec{E}_{innen} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Further help and detailed calculation ------------------▷ 83

Solution happily found ------------------▷ 86

The surface of an inner sphere encloses a part of the total charge given by:

83

$Q_{innen} = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 84

$$Q_{innen} = \int \rho dV = \frac{4\pi}{3} \cdot R_{innen}^3 \cdot \rho$$

84

Now we apply Gauss's theorem. Given: $div \quad \vec{E} = \frac{\rho}{\varepsilon_0}$.

$$\oint_{surface} \vec{E} \cdot \overrightarrow{dA} = \int_{volume} div \quad \vec{E} \quad dV$$

We calculate both integrals:

a) $\oint_{Surface} \vec{E} \cdot \overrightarrow{dA} \quad =$.................

b) $\int_{Volume} div \quad \vec{E} dV =$..................

Inserting into Gauss's theorem we obtain:..............=.....................

------------------▷ 85

85

a) $\int \vec{E} \cdot \overrightarrow{dA} = E \cdot 4\pi R_{innen}^2$

b) $\int div\vec{E}dV = \frac{\rho}{\varepsilon_0} \cdot \frac{4\pi}{3} \cdot R_{innen}^3$

Inserting into Gauss's theorem we obtain

$$\left|\vec{E}\right| \cdot 4\pi R_{innen}^2 = \frac{\rho}{\varepsilon_0} \cdot \frac{4\pi}{3} \cdot R_{innen}^3$$

$$\left|\vec{E}\right| = \frac{\rho}{\varepsilon_0} \frac{R_{innen}}{3}$$

To obtain $\vec{E}$ we have to regard the direction of $\vec{E}$:

$$\vec{E} = |E| \cdot \frac{\vec{r}}{|r|} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Remember: $|\vec{r}| = R_{inside}$

-------------------▷ 86

$$\vec{E}_{innen} = \frac{\rho}{\varepsilon_0} \cdot \frac{(x, y, z)}{3}$$

86

Finally, we substitute ρ by the total charge Q of the original sphere using the known equation

$$Q = \rho \cdot \frac{4\pi}{3} R^3$$

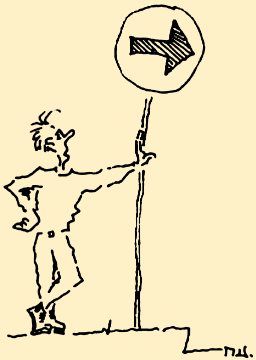
$\vec{E}_{innen} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 87

$$\vec{E}_{innen} = \frac{Q}{4\pi\varepsilon_0 \cdot R^3} \cdot (x, y, z)$$

87

In case of remaining difficulties repeat this section of the study guide and consult the textbook again.



Go on to 88

17.6 Curl of a vector field and Stoke's theorem

88

The concept of curl will be introduced. After completing section 17.7 it is suggested you have a break and a cup of coffee or tea. But do not forget to take notes of new definitions on a separate sheet.

Study **17.7 Curl of a vector field.**
17.8 Stoke's theorem
Textbook pages 480–485

Having studied --------------------▷ 89

If a vector field $\vec{F}_1$ is curl free we have: …………=………..

If a vector field $\vec{F}_2$ has curl we have gilt……..…..=………..

89

----------------------> 90

Field is curl free $\oint \vec{F}_1 \cdot \overrightarrow{ds} = 0$

Field has curl $\oint \vec{F}_2 \cdot \overrightarrow{ds} \neq 0$

90

It is not that easy, but it is worthwhile to memorize the definition of curl or to reconstruct it using the formula $rot \quad \vec{F} = \vec{\nabla} \times \vec{F}$.

$rot \quad \vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 91

91

$$rot \quad \vec{F} = \begin{pmatrix} \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \\ \frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x} \\ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \end{pmatrix}$$

Write down as a determinant:

$rot \quad \vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 92

92

$$rot\vec{F} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ F_x & F_x & F_z \end{vmatrix}$$

Expand the determinant and give:

$rot\vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 93

93

$$rot \quad \vec{F} = \begin{pmatrix} \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \\ \frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x} \\ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \end{pmatrix}$$

Regard a flow of water. The field of velocities may be given by:

$\vec{v} = (1, \quad \ln z, \quad 0)$

Calculate:

$div \quad \vec{v} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$rot \quad \vec{v} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

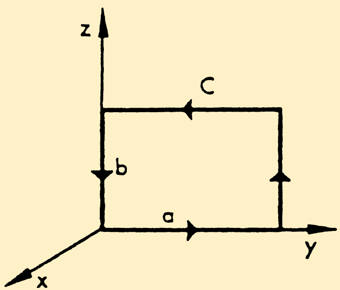
------------------ ▷ 94

$div\vec{v} = 0$

$rot\vec{v} = (-\frac{1}{z}, 0, 0)$

94

Calculate the line integral for the rectangle with sides a and b for a complete circulation.
Given $\vec{F}$ to be $\vec{F}(x,y,z) = (5, 0, z^2)$



$\oint \vec{F}\vec{ds} =$ ....

------------------▷ 95

The field of velocities is free of curl. Thus:

95

$$\oint \vec{F}\,\overrightarrow{ds} = 0$$

--------------------▷ 96

17.7 Potential of a vector field

96

Study **17.9 Potential of a vector field**
Textbook pages 485–487

97

In the textbook we treated the example of the gravitational field of a sphere, the mass of which is M. The case most familiar to us is the earth. At the surface of the earth, however, we simplify the situation by calculating with a homogeneous field of gravitation. In this simplification the x-y plane is parallel to the earth's surface. The z-axis points upwards and the origin of the coordinate system coincides with the earth's surface. This approximation holds for a lot of calculations. In this case the gravitational force acting on a mass m is given: 97

$\vec{F} =$ ....

------------------▷ 98

$\vec{F} = (0, \quad 0, \quad -m \cdot g)$

98

In the case of a gravitational field the force $\vec{F}$ is the product of the mass m of the body in question and the gravitational field vector $\vec{F}_g$ of the gravitational field. This field vector is in this case

$\vec{F}_g = -g \cdot \vec{e}_z = (0, 0, -g)$

In the following we discuss fields and their field vectors. This holds as well for electrical fields and field vectors. In the case of a static electrical field the force acting on a charge Q is the product of Q with the electrical field vector $\vec{E}$:

$\vec{F} = Q \cdot \vec{E}$

The field is completely represented by the electrical field vector $\vec{E}$.

Check gravitational field $\vec{F}_g$.

Is it free of curl?

$rot \quad \vec{F}_g = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\vec{F}_g$ is……………………..

--------------------▷ 99

$rot(0,0,-g) = 0$

99

$\vec{F}_g$ is curl-free, it is a conservative field.

Calculate the potential of $\vec{F} = m \cdot \vec{F}_g$.

Remember: The following convention holds in physics. For a given force field $\vec{F}_g$ the potential φ is the work done against the force field

$\varphi(x, y, z) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 100

$$\varphi(x,y,z) = m \cdot g \cdot z + C$$

100

The potential is thus determined except the integration constant C:
Determine the potential given above for three different situations:

1) $\varphi_1 = 0$ for the ground floor with $z_0 = 0$

$\varphi_1 = \ldots\ldots\ldots\ldots\ldots\ldots$

2) $\varphi_2 = 0$ for the underground of a high building with $z_0 = -10$

$\varphi_2 = \ldots\ldots\ldots\ldots\ldots\ldots$

3) $\varphi_3 = 0$ for the roof of a high building with $z_0 = 90$

$\varphi_3 = \ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 101

$\varphi_1 = (0,0,mgz)$

101

$\varphi_2 = (0,0,mg(z+10))$

$\varphi_3 = (0,0,mg(z-90))$

Give the gravitational field $\vec{F}_g$ for these three cases:

$\vec{F}_{g1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\vec{F}_{g2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\vec{F}_{g3} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 102

$\varphi_1 = \varphi_2 = \varphi_3 = (0, 0, mgz)$

102

In the textbook you studied the gravitational field of a mass M. This mass is assumed to be distributed homogeneously in the inner space of a sphere with radius R. Outside of the sphere the gravitational field is:

$$\vec{F}_g(x, y, z) = -\gamma \cdot M \frac{x, y, z}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3}$$

Simplify using $\vec{r} = (x, y, z)$

$\vec{F}_g(x, y, z) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

-------------------▷ 103

$$\vec{F}_g = -\gamma M \cdot \frac{\vec{r}}{r^3} = -\gamma M \frac{1}{r^2} \cdot \frac{\vec{r}}{r}$$

103

Now we calculate the potential $\varphi = -\int \vec{F}\,\overrightarrow{dr}$ for a path of integration in the direction of a radius.

Then we have: $\frac{\vec{r}}{r}\overrightarrow{dr} = dr$.

$$\varphi = \gamma M \int_{r_0}^{r} \frac{1}{r_2} \cdot \frac{r}{r} dr = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 104

104

$$\varphi(r) = \gamma M \cdot \int_{r_0}^{r} \frac{dr}{r^2} = -\gamma M \left[\frac{1}{r} \right]_{r_0}^{r}$$

$$\varphi(r) = \gamma M \left[\frac{1}{r_0} - \frac{1}{r} \right]$$

We can scale the potential of the gravitational field of the mass M in a way that it vanishes for $r \to \infty$, but the potential may be scaled as well to vanish for the surface for which holds $r = r_{surface}$

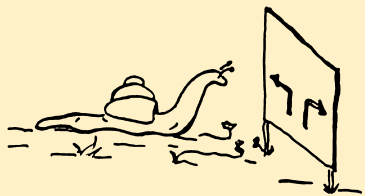
In the textbook we explained the first case.
In the study guide we calculated the second case.
Calculate the potential for the first case in order to vanish for infinity.

$\varphi_1 =$ ……………………..

-------------------▷ 105

$\varphi_1 = -\gamma M \frac{1}{r}$

105



All correct ----------------- ▷ 108

Further explanation wanted ----------------- ▷ 106

The potential of the gravitational field was represented by: 106

$$\varphi(r) = \gamma M \int_{r_0}^{r} \frac{dr}{r^2} = \gamma M \left[\frac{1}{r_0} - \frac{1}{r} \right]$$

The condition was $\varphi(r = \infty) = 0$

In this case the bracket must be zero. This is obtained by letting $\frac{1}{r_0} = 0$ or $r_0 = \infty$.

A difficulty in understanding this may arise because r_0 is the lower limit of integration and the lower limit can not be ∞ while the upper limit of integration is finite. But this problem is solved; we invert the direction of integration. We integrate from r to r_0.

Thus we obtain $\varphi(r) = \gamma M \int_{r_0}^{r} \frac{dr}{r^2} = -\gamma M \int_{r}^{r_0} \frac{dr}{r^2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 107

107

$$\varphi(r) = \gamma M\left[\frac{1}{r_0} - \frac{1}{r}\right]$$

Now we let r_0 grow to infinity to obtain the result of the foregoing frame.

$$\varphi_1 = -\gamma M \frac{1}{r}$$

Further problems may arise if we change signs. Do not underestimate the problems related to signs. It is always advisable to calculate meticulously.

--------------------▷ 108

Now we will treat the second case and let the potential of the gravitational field vanish for the surface of the earth. 108

We calculated before: $\varphi(r) = \gamma M\left[\frac{1}{r_0} - \frac{1}{r}\right]$

The height z above the surface is given by $r = r_0 + z$

Thus: $\varphi(z)$

$\varphi(z)=$..........................

-------------------▷ 109

$$\varphi(r) = \gamma M\left[\frac{1}{r_0} - \frac{1}{r_0 + z}\right]$$

109

For places near the surface we state $z << r_0$. Applying this approximation we write:

$$\frac{1}{r_0 + z} = \frac{1}{r_0\left(1 + \frac{z}{r_0}\right)} \approx \frac{1}{r_0}(\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

We insert this into the equation above and obtain:

$$\varphi(z) = \gamma M[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

------------------▷ 110

110

Approximation: $\dfrac{1}{r_0\left(1+\dfrac{z}{r_0}\right)} \approx \dfrac{1}{r_0}\left[1-\dfrac{z}{r_0}\right]$

$\varphi(z) = \gamma M \dfrac{z}{r_0{}^2}$

For the case of the earth with mass M and radius r_E this corresponds to the potential we used in previous frames. There we used the expression

$\varphi(z) = (0,0 \quad g \cdot z) = g \cdot z$

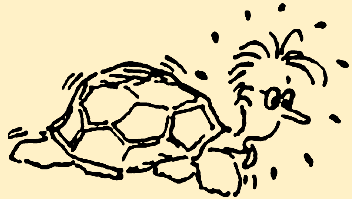
Both expressions are identical if we define g appropriately. Calculate and define:

$g = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 111

$$g = \frac{\gamma M}{r_0^{\,2}}$$

111



112

At the end of this chapter we recapitulate:

112

Definition of divergence of a vector field $\vec{F}$:

$div \quad \vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$

------------------ ▷ 113

$$div \quad \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \vec{\nabla} \cdot \vec{F}$$ 113

Calculate the divergence of the given vector field $\vec{F}$:

$$\vec{F} = \left(\frac{x^3}{3}, \frac{y^3}{3}, -\frac{z^2}{2} \right)$$

$div \quad \vec{F} =$…………………

Distribution of sources and sinks:

……………………………………………………………………………………………

……………………………………………………………………………………………

……………………………………………………………………………………………

------------------▷ 114

114

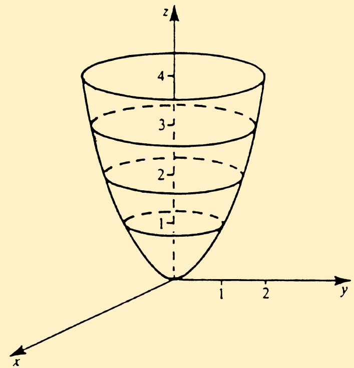
$div \quad \vec{F} = x^2 + y^2 - z$

No sinks or sources for:

$div \quad \vec{F} = 0$:

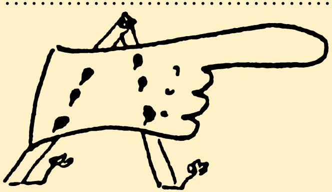
Thus, we obtain for this condition $0 = x^2 + y^2 - z$ or $z = x^2 + y^2$. This represents a paraboloid of revolution around the z-axis.

The inner space consists of sinks. The outer space consists of sources.



Give Gauss's theorem:

....=..................................



----------------------▷ 115

$$\int_V div \quad \vec{F} dV \qquad = \int \vec{F} \cdot d\vec{A}$$

Volume integral Surface integral. l

115

Give the definition of the curl of a vector field $\vec{F}$:

$rot \quad \vec{F} = \vec{\nabla} \times \vec{F} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 116

116

$$rot \quad \vec{F} = \begin{pmatrix} \frac{\delta F_z}{\delta y} - \frac{\delta F_y}{\delta z} \\ \frac{\delta F_x}{\delta z} - \frac{\delta F_z}{\delta x} \\ \frac{\delta F_y}{\delta x} - \frac{\delta F_x}{\delta y} \end{pmatrix}$$

Given a flow of water

$$\vec{V} = \left(z^2, \quad 0, \quad 0\right)$$

$$rot \quad \vec{V} = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 117

$rot\left(z^2,0,0\right)=\left(0,2z,0\right)$

117

Give Stoke's theorem:

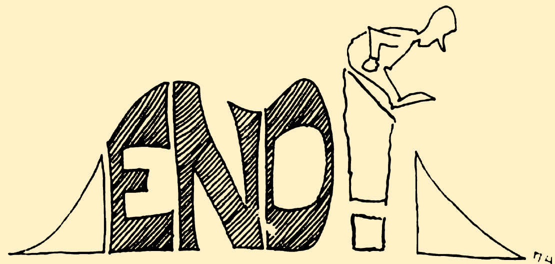
…………………………..=…………………………

------------------▷ 118

$\int\limits_V div\vec{F}dV =$	$\oint \vec{F} \cdot d\vec{A}$	$\int\limits_A rot\vec{F} \cdot \overrightarrow{dA} =$	$\oint\limits_{C(A)} \vec{F} \cdot \overrightarrow{ds}$
Volume integral	Surface integral	Surface integral	Line integral

The theorems of Gauss and Stokes are worth understanding and memorizing in order to avoid difficulties when applying them in physics.

You have reached the end of chapter 17.
You have now made considerable progress.



Chapter 18
Fourierseries

0

1

18.1 Expansion of a Periodic Function into a Fourierseries

1

In this section we show that every periodic function can be expanded into a sum of trigonometric functions. This is important for physics and electrical engineering.

2

In the following we will frequently use the addition theorems of trigonometric functions. Your understanding of the following will be much easier if you recapitulate some of the relations treated in chapters 1, 5, and 6 as a preparation.

2

Solve the following integrals:

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = \ldots\ldots\ldots\ldots$$

$$\int_{-\pi}^{+\pi} \cos nx \quad dx = \ldots\ldots\ldots\ldots$$

------------------▷ 3

3

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \quad dx = 0$$

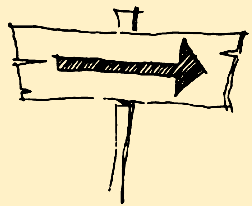
Remember the following relationships.
In case of difficulties you may read again section 1.5.6 (addition formulae) in chapter 1.

$\sin(n+m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\sin(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\cos(n+m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\cos(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



------------------▷ 4

$$\sin(n+m)x = \sin nx \cdot \cos mx + \sin mx \cdot \cos nx$$
$$\sin(n-m)x = \sin nx \cdot \cos mx - \sin mx \cdot \cos nx$$
$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

4

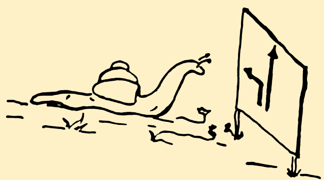
Using the addition formulae you can derive the following relationships which will be used repeatedly:

$\sin nx \cdot \cos mx =$ ....

$\cos nx \cdot \cos mx =$ ....

$\sin nx \cdot \sin mx =$ ....

These tasks may seem tedious but they can be solved by you.



Solution found: ------------------▷ 11

Hints wanted ------------------▷ 5

5

Let us start with the first task:

$\sin nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

We use the addition formula

$$\sin(n+m)x = \sin nx \cdot \cos mx + \sin mx \cdot \cos nx$$

$$\sin(n-m)x = \sin nx \cdot \cos mx - \sin mx \cdot \cos nx$$

If we add both lines we obtain:

$\sin(n+m)x + \sin(n-m)x = \ldots\ldots\ldots\ldots\ldots..$

------------------▷ 6

$$\sin(n+m)x + \sin(n-m)x = 2\sin nx \cdot \cos mx$$

6

From this follows directly the result of the first task:

$$\sin nx \cdot \cos mx = \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x$$

In the same manner you can solve the second task:

$$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found: ------------------▷ 8

More hints wanted ------------------▷ 7

7

The second task:

$$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

This time we use the addition formulae in which appear products of cos-functions.

$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

We add both lines and obtain:

$$\cos(n+m)x + \cos(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots$$

Thus, the solution is easily obtained:

$$\cos nx \cdot \cos mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 8

$$\cos nx \cdot \cos mx = \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$ 8

In the same manner we can solve the third task:

$\sin nx \cdot \sin mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: Use the addition formulae in which you find products of sin-functions

Solution found --------------------▷ 10

You want more hints: --------------------▷ 9

To solve: $\sin nx \cdot \sin mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ 9

We use:

$$\cos(n+m)x = \cos nx \cdot \cos mx - \sin nx \cdot \sin mx$$
$$\cos(n-m)x = \cos nx \cdot \cos mx + \sin nx \cdot \sin mx$$

We subtract the lines and obtain:

$$\cos(n+m)x - \cos(n-m)x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Thus we obtain the solution:

$$simnx \cdot \sin mx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------ ▷ 10

$$\cos(n+m)x - \cos(n-m)x = -2\cos nx \cdot \sin mx$$

$$\sin nx \cdot \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x$$

10

Through these repetitions we found the relationships we need to get the Fourierseries.

$\sin nx \cdot \cos mx =$ ....

$\cos nx \cdot \cos mx =$ ....

$\sin nx \cdot \sin mx =$ ....

------------------- ▷ 11

$$\sin nx \cdot \cos mx = \frac{1}{2}\sin(n+m)x + \frac{1}{2}\sin(n-m)x$$ 11

$$\cos nx \cdot \cos mx = \frac{1}{2}\cos(n+m)x + \frac{1}{2}\cos(n-m)x$$

$$\sin nx \cdot \sin mx = \frac{1}{2}\cos(n-m)x - \frac{1}{2}\cos(n+m)x$$

Take notes on a separate sheet of paper of these relationships and the following ones:

$$\int_{-\pi}^{+\pi} \sin nx \quad dx = \int_{-\pi}^{+\pi} \cos nx \quad dx = 0$$

Now we have gained all prerequisites to resolve our main objective.
Read carefully and control all transformations parallel to the text

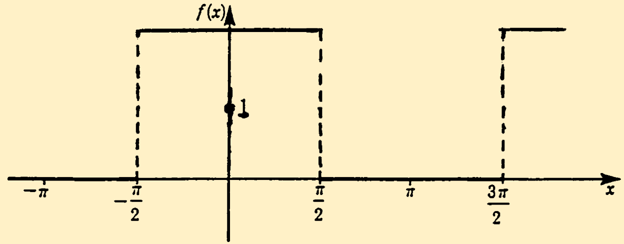
Study **18.1 Expansion of a Periodic Function into a Fourierseries**
Textbook pages 491–496

Then go to ------------------▷ 12

You might have had difficulties. This may be because the relationships we repeated beforehand are not familiar enough to you. In this case it is useful to calculate the transformations on a separate sheet and look up the required relationships in the textbook. **12**

With the summary in the textbook on page 494 we can obtain the Fourierseries for the rectangular function $f(x)$, which is defined in the interval $-\pi < x < \pi$.

$$f(x)=\begin{cases} 0 & \textit{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \textit{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \textit{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$f(x) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

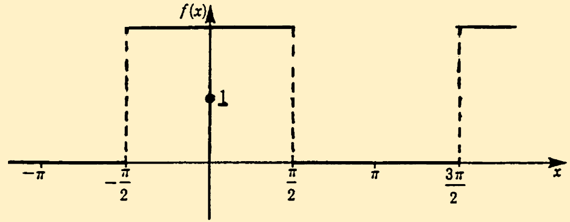
Solution found: ----------------------▷ 21

Help and further explanation ----------------------▷ 13

Wanted is the Fourierseries for the rectangular function defined by:

13

$$f(x)=\begin{cases} 0 & \textit{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \textit{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \textit{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



This rectangular function is defined by separate parts or successive separate sections. This implies that the coefficients in the summary (18.2) on page 494 have to be computed separately for each part or section; in other words the integrals must be computed for each part separately.

Let us start with a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution found ------------------▷ 15

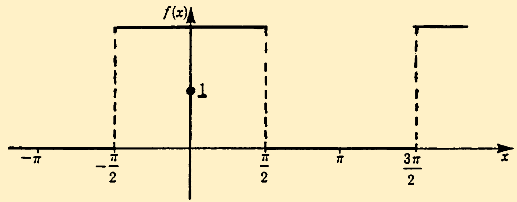
Help and explanation wanted ------------------▷ 14

14

To compute: $a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx =$

We had

$$f(x)=\begin{cases} 0 & \text{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



The integral has to be computed for the whole interval $-\pi \le x \le \pi$. For the whole the function is defined in three separate parts. Thus, the whole integral is the sum of three separate integrals for each part or section of the function:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} f(x)dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(x)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)dx$$

What remains is for us to insert the function for each part and to compute the integrals:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------ ▷ 15

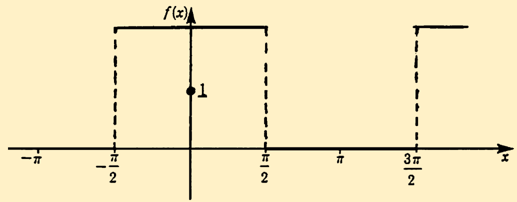
15

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} 0\cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0\cdot dx = \frac{2}{\pi}\left[\frac{\pi}{2}+\frac{\pi}{2}\right] = 2$$

Now we compute partwise $a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx$

The function was defined:

$$f(x) = \begin{cases} 0 & \text{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

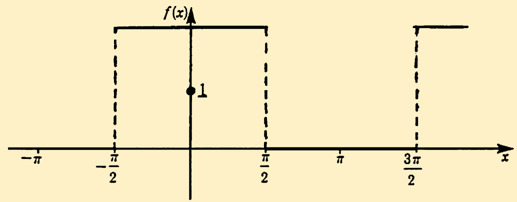
Solution found ------------------▷ 18

Help and more explanation ------------------▷ 16

To compute: 16

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx dx \text{ for}$$

$$f(x) = \begin{cases} 0 & \textit{für} \quad -\pi \leq x \leq -\frac{\pi}{2} \\ 2 & \textit{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \textit{für} \quad +\frac{\pi}{2} \leq x \leq +\pi \end{cases}$$



Again we have to compute the integrals for the three parts separately.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx \cdot dx = \ldots\ldots\ldots\ldots\ldots\ldots$$

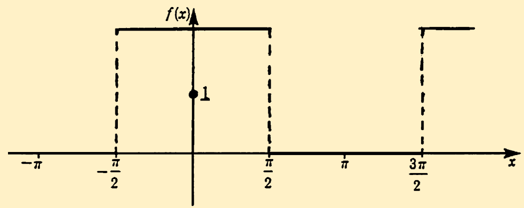
------------------▷ 17

17

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos nx \cdot dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} f(x)\cos nx \cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} f(x)\cos nx \cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx \cdot dx$$

The function has been

$$f(x) = \begin{cases} 0 & \text{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



Inserted we obtain three integrals to solve:

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------ ▷ 18

18

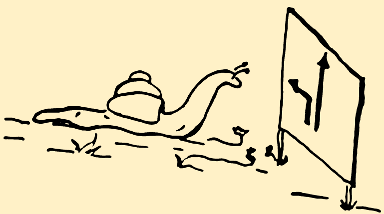
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\frac{\pi}{2}} 0 \cdot \cos nx dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2 \cdot \cos nx dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0 \cdot \cos nx dx$$

$$a_n = 0 + \frac{1}{\pi} \cdot \frac{2}{n}\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right] + 0$$

$$a_n = \frac{4}{\pi \cdot n}\sin n\frac{\pi}{2}$$

What remains is the task of calculating b_n:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin nx dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



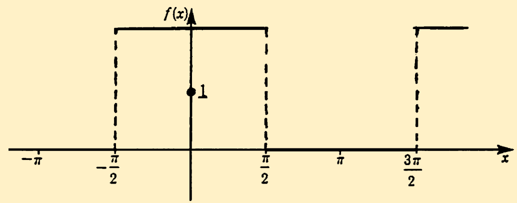
Solution found ------------------- ▷ 21

Help and more explanation ------------------- ▷ 19

Again we must decompose the whole integral into three parts, which have to be solved separately. 19

The function was:

$$f(x)=\begin{cases} 0 & \text{für} \quad -\pi \le x \le -\frac{\pi}{2} \\ 2 & \text{für} \quad -\frac{\pi}{2} < x < +\frac{\pi}{2} \\ 0 & \text{für} \quad +\frac{\pi}{2} \le x \le +\pi \end{cases}$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 20

20

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \cdot dx = \frac{1}{\pi}\int_{-\pi}^{-\frac{\pi}{2}} 0 \cdot \sin nx \cdot dx + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} 2 \cdot \sin nx \cdot dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} 0 \cdot \sin nx \cdot dx = \frac{1}{\pi} 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nx \cdot dx$$

This time we do not expect more difficulties:

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin nxdx = \ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 21

21

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \sin nx dx = \frac{2}{\pi}\left[-\frac{\cos n\frac{\pi}{2}}{n} + \frac{\cos n\frac{\pi}{2}}{n}\right] = 0$$

Thus our problem is solved. We obtained the Fourierseries for the rectangular function.

The series contains with exception of $a_0 = 1$ only cos-functions, the amplitudes of which decrease if n increases. Furthermore, the coefficients b vanish for n = even.

------------------- ▷ 22

18.2 Examples of Fourierseries

22

In the following section we do not expect fundamental difficulties. But within the calculations writing errors or computing errors are likely to occur. Thus, you have to concentrate on the accuracy of your calculations. Do not forget to control all transformations using a separate sheet of paper.

Study **18.2 Examples of Fourierseries**
Textbook pages 496–501

After your study go to ------------------▷ 23

The rectangular function treated in the last section of the study guide beginning with frame 12 is a

23

☐ Even function

☐ Odd function

24

Even function

24

For even functions we have:

☐ $a_n = 0$

☐ $b_n = 0$

For even functions the Fourierseries is composed of …………… functions.

25

For even functions $b_n = 0$ holds. 25

The Fourierseries is composed of cos-functions.

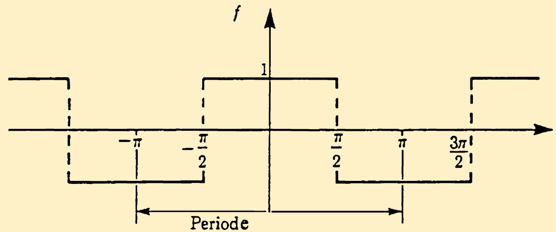
In the examples in the textbook the variable x might have been substituted by the variable t which denotes time. In physics and engineering we mostly deal with time. So we will use the variable t in the next example.

Here follows another exercise regarding a rectangular function --------------------▷ 26

Given the rectangular function:

26

$$f(t)=\begin{cases} -1 & \text{für} \quad -\pi \le t \le -\frac{\pi}{2} \\ 1 & \text{für} \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \\ -1 & \text{für} \quad \frac{\pi}{2} \le t \le \pi \end{cases}$$



Shift the rectangular function to obtain an odd function: Is there only one solution?

$$f(t)=\begin{cases} \\ \dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

- - - - - - - - - - ▷ 27

There exist two solutions that are of equal value:

27

Solution1: $f(t)=\begin{cases}-1 & \text{für} \quad -\pi < t < 0 \\ +1 & \text{für} \quad 0 < t < \pi\end{cases}$

Solution 2: $f(t)=\begin{cases}+1 & \text{für} \quad -\pi < t < 0 \\ -1 & \text{für} \quad 0 < t < \pi\end{cases}$

Compute the Fourierseries for the rectangular function of solution 1:

$f(t)=$ ....

Solution found ------------------▷ 33

Help wanted ------------------▷ 28

Given the rectangular waveform:

28

$$f(t) = \begin{cases} -1 & \textit{für} \quad -\pi < t < 0 \\ +1 & \textit{für} \quad 0 < t < \pi \end{cases}$$

It is a ……… function. In this case which coefficients vanish? ……

………….

-------------------▷ 29

29

Odd function

The coefficients a_n vanish.

Given again

$$f(t)=\begin{cases}-1 & \text{für} \quad -\pi < t < 0 \\ +1 & \text{für} \quad 0 < t < \pi\end{cases}$$

Please show that a_0 vanishes as well.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt == \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 30

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot dt + \frac{1}{\pi}\int_{0}^{\pi}(1)\cdot dt = -1+1=0$$

30

Compute now the coefficients b_n for the given rectangular waveform.

$$f(t)=\begin{cases}-1 & \textit{für} \quad -\pi < t < 0\\ +1 & \textit{für} \quad 0 < t < \pi\end{cases}$$

Split the integral below into two segments.:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot \sin nt \quad dt = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

-------------------▷ 31

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot \sin nt \quad dt + \frac{1}{\pi}\int_{0}^{\pi}\sin nt \cdot dt$$ 31

Solve the integrals regarding the given limits:…………………………

-------------------▷ 32

$$b_n = \frac{1}{\pi}\left[(-1)\cdot\left(-\frac{\cos 0}{n}\right)-(-1)\cdot\left(-\frac{\cos(-n\pi)}{n}\right)\right]+\frac{1}{\pi}\left[\left(-\frac{\cos n\pi}{n}\right)-\left(-\frac{\cos 0}{n}\right)\right]$$ 32

$$= \frac{1}{n\pi}\left[1-\cos(-n\pi)+1-\cos n\pi\right]$$

$$b_n = \frac{2}{n\pi}\left[1-\cos n\pi\right]$$

Now you can obtain: $f(t)=\sum_{n=1}^{\infty}$ ....

$\dashrightarrow$ 33

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[1 - \cos n\pi] \cdot \sin nx$$

33

Write down the first three terms of the series. Do not forget $\cos n\pi = (-1)^n$

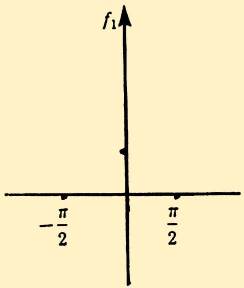
$$f(t) \approx \frac{2}{\pi}[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

------------------▷ 34

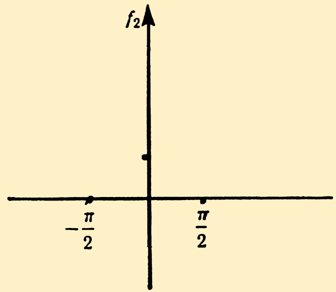
34

$$f(t) \approx \frac{2}{\pi}\left[\frac{2}{1}\sin t + \frac{2}{3}\sin 3t + \frac{2}{5}\sin 5t\right]$$

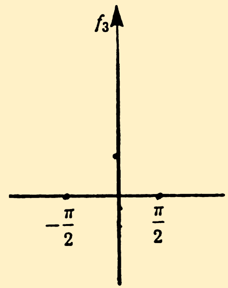
Sketch each of these three terms:



First term



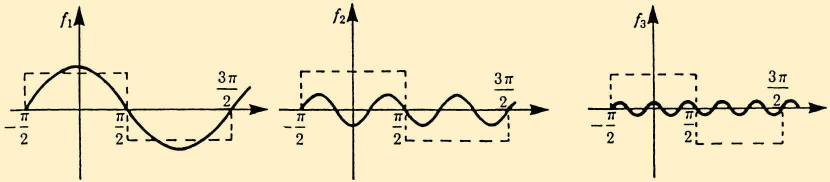
second term



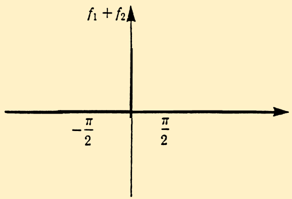
third term

35

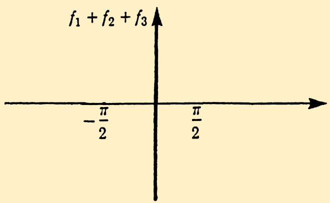
35



Try to sketch the sum of the first two terms $(f_1 + f_2)$



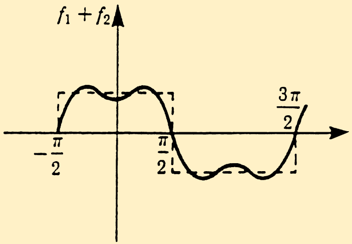
Now try to add the third term f_3.



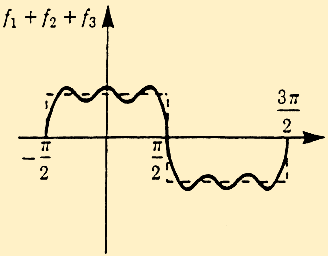
36

Superposition of the first two terms:

36



Superposition of the first three terms:

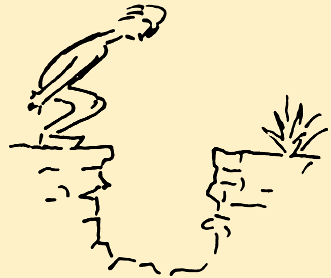


We observe the stepwise formation of the rectangular waveform.

------------------▷ 37

If you followed the calculations in the textbook and in this study guide without difficulties you may skip the next exercise.

37



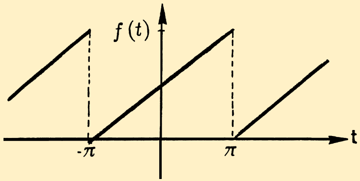
-------------------▷ 56

If you felt you had difficulties you should do the next exercise because all transformations will be explained in detail.

-------------------▷ 38

38

In the following we will calculate an example in detail.
Given a sawtooth-like function shown below.



In this case we do not need to progress stepwise. The function is defined for the entire period by only one function.

$$f(t)=\left(\frac{t}{\pi}+1\right) \quad für \quad -\pi<t<\pi$$

Calculation of a_0:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt=\frac{1}{\pi}\int_{-\pi}^{+\pi}\left(\frac{t}{\pi}+1\right)dt$$

There are only simple integrals to solve.

$$a_0=\frac{1}{\pi}\Big[\ldots\ldots\ldots\ldots\ldots\ldots\Big]_{-\pi}^{\pi}$$

-------------------▷ 39

$$a_0 = \frac{1}{\pi}\left[\frac{t^2}{2\pi} + t\right]_{-\pi}^{\pi} = \frac{1}{\pi}[\pi - (-\pi)] = 2$$

39

The sawtooth shown here can be transformed into the sawtooth curve treated in the textbook. To obtain this you may shift the function in the direction of the y-axis.

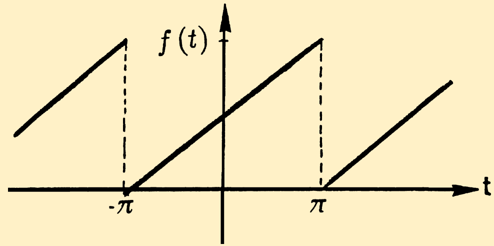
Go to --------------------▷ 40

Calculation of a_n

40

Given the sawtooth curve:

$$f(t) = \left(\frac{t}{\pi} + 1\right) \quad for \quad -\pi < t < \pi$$



To compute:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cdot \cos(nt) dt$$

We insert $f(t)$ into the integral and remember: The integral of a sum is the sum of the integrals.

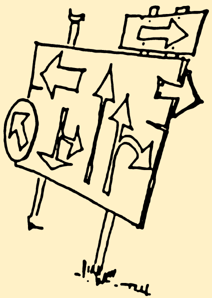
Go to ------------------▷ 41

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{t}{\pi}+1\right)\cos(nt)dt = \frac{1}{\pi}\left[\underbrace{\int_{-\pi}^{\pi}\frac{t}{\pi}\cos nt \quad dt}_{\text{Integral 1}} + \underbrace{\int_{-\pi}^{\pi}\cos nt \quad dt}_{\text{Integral 2}}\right]$$

41

Both integrals in the brackets have to be solved. We start with integral 1. We can solve it using integration by parts

$$\int_{-\pi}^{\pi}\frac{t}{\pi}\cos ntdt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$



Solution found ------------------▷ 45

Repetition of integration by parts ------------------▷ 42

Detailed solution ------------------▷ 44

Integration by parts is often useful when products have to be integrated. This method is explained in the textbook section 6.5.4. Perhaps it may help to read that section again. 42
Here is a short recapitulation.
Given two different functions $u(t)$ and $v(t)$. The limits of integration may be a and b. Then:

$$\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$$

I
In our case we have:

Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt dt$

We let : $u = \frac{t}{\pi} \quad v' = \cos nt$

Thus we obtain:

$u' = \ldots\ldots\ldots\ldots\ldots\ldots$

$v = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 43

43

$$u' = \frac{1}{\pi} \qquad v = \frac{1}{n}\sin nt$$

We had from before:

$$u = \frac{t}{\pi} \qquad v' = \cos nt$$

If you are not sure you may verify this by differentiating. Having done this we insert our results into the formula of the integration by parts.

$$\int_{-\pi}^{\pi} u \cdot v' = [u \cdot v]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u' \cdot v$$

Thus the first integral can be solved:

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt \quad dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 44

44

Integral 1 is: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos ntdt = \left[\frac{t}{\pi} \cdot \frac{1}{n} \sin nt \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{1}{\pi} \cdot \frac{1}{n} \sin ntdt$

The bracket vanishes because $\sin n\pi = \sin(-n\pi) = 0$
The remaining integral is solved easily:

$$\int_{-\pi}^{\pi} \sin ntdt = \left[-\frac{1}{n} \cos nt \right]_{-\pi}^{\pi}$$

We remember:

$$\cos n\pi = \cos(-n\pi)$$

Thus: $\int_{-\pi}^{\pi} \sin nt \quad dt = 0$

By this we calculated integral 1. The result is

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \cos ntdt = \ldots\ldots\ldots\ldots\ldots$$

------------------- ▷ 45

Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt dt = 0$

45

We had to compute

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{t}{\pi} + 1 \right) \cos nt dt = \frac{1}{\pi} \left[\underbrace{\int_{-\pi}^{\pi} \frac{t}{\pi} \cos nt \cdot dt}_{\text{Integral 1}} + \underbrace{\int_{-\pi}^{\pi} \cos nt \cdot dt}_{\text{Integral 2}} \right]$$

Integral 1 vanishes.

Integral 2 can be solved easily and does not pose any difficulty:

$$\int_{-\pi}^{\pi} \cos nt \cdot dt = \Big[\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{-\pi}^{\pi} = \ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 46

46

$$\int_{-\pi}^{\pi} \cos nt \cdot dt = \left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi} = 0$$

Thus, we obtained: $a_n = 0$

Now we have to calculate the coefficients b_n :

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt \cdot dt$$

We insert: $f(t) = \frac{t}{\pi} + 1$

By this we get: $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}$ ....

We summarize: The integral of a sum of functions is the sum of the integrals of these functions n:

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} \text{......................} + \int_{-\pi}^{\pi} \text{..........................}\right]$$

-------------------▷ 47

47

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\frac{t}{\pi}+1)\sin nt \quad dt$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi}\frac{t}{\pi}\sin nt dt + \int_{-\pi}^{\pi}\sin nt dt\right]$$

Integral 1 Integral 2

We start with integral 1.

Integral 1: $\int_{-\pi}^{\pi}\frac{t}{\pi}\sin nt dt = \ldots\ldots\ldots$

Solution found ------------------▷ 54

Recapitulation of integration by parts ------------------▷ 48

Detailed solution ------------------▷ 50

48

Integration by parts is often useful if products have to be integrated. This method is explained in the textbook section 6.5.4. Perhaps it may help to read that section again. Here is a short recapitulation.

Given two different functions $u(t)$ and $v(t)$. The limits of integration may be a and b.

Then holds the formula for integration by parts: $\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$

$$\int_a^b u \cdot v' = [u \cdot v]_a^b - \int_a^b u' \cdot v$$

To calculate Integral 1: $\int_{-\pi}^{\pi} \frac{t}{\pi} \sin nt \quad dt$

We let

$$u = \frac{t}{\pi} \qquad v' = \sin nt$$

From this we obtain

$u' = \ldots\ldots\ldots\ldots \quad v = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 49

$$u' = \frac{1}{\pi} \qquad v = -\frac{\cos nt}{n}$$

49

In case you are unsure you may verificate by deriving v. As we know the formula for integration by parts reads:

$$\int_a^b u \cdot v' = [u \cdot v]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} u' \cdot v$$

Using the functions above we get:

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \sin nt \quad dt = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

------------------▷ 50

$$\int_{-\pi}^{\pi} \frac{t}{\pi} \sin nt \, dt = \left[\frac{t}{\pi} \left(-\frac{\cos nt}{n} \right) \right]_{-\pi}^{\pi} - \int_{-\pi}^{+\pi} \frac{1}{\pi} \left(-\frac{\cos nt}{n} \right) dt$$

50

First we solve the terms in brackets. This can be done easily, but difficulties may arise if the signs are interchanged. This happens often in calculations.

We remember:

$$\cos n\pi = \cos(-n\pi)$$
$$\cos n\pi = (-1)^n$$

$$\left[\frac{t}{\pi} \left(-\frac{\cos nt}{n} \right) \right]_{-\pi}^{\pi} = \text{........................}$$

Answer found ------------------▷ 53

Detailed solution ------------------▷ 51

51

To solve: $\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi}$

We remember: $\cos n\pi = \cos(-n\pi) = (-1)^n$

Inserting the given limits we obtain:

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = [\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots]$$

------------------▷ 52

52

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi}=\left[\frac{\pi}{\pi}\left(-\frac{\cos n\pi}{n}\right)-\frac{-\pi}{\pi}\left(-\frac{\cos(-n\pi)}{n}\right)\right]$$

We simplify and get now:

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi}=\left[-\frac{\cos n\pi}{n}-\frac{\cos(-n\pi)}{n}\right]=-2\frac{\cos n\pi}{n}$$

Using $\cos n\pi=(-1)^{n}$ we can write:

$$-\cos n\pi=(-1)^{n+1}$$

Finally we obtain:

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi}=\dots\dots\dots\dots$$

-------------------▷ 53

53

$$\left[\frac{t}{\pi}\left(-\frac{\cos nt}{n}\right)\right]_{-\pi}^{\pi} = \frac{2}{n}(-1)^{n+1}$$

Thus our task is to solve:

$$\int_{-\pi}^{\pi} \frac{t}{\pi}\sin nt dt = \frac{2}{n}(-1)^{n+1} - \int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt$$

The remaining integral may be solved easily because it is a fundamental integral

$$:\int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt = \ldots\ldots\ldots\ldots\ldots\ldots$$

....

------------------▷ 54

54

$$\int_{-\pi}^{\pi} \frac{1}{\pi}\left(-\frac{\cos nt}{n}\right)dt = \left[-\frac{1}{\pi}\frac{\sin n\pi}{n^2}\right]_{-\pi}^{\pi} = 0$$

The integral vanishes because: $\sin n\pi = \sin(-n\pi) = 0$

Integral 2 vanishes

$$\int_{-\pi}^{\pi} \sin nt \quad dt = \left[\frac{1}{n}(-\cos nt)\right]_{-\pi}^{\pi} = 0$$

Because: $\cos(n\pi) = \cos(-n\pi)$

Finally, we determine b_n to be

$$b_n = \frac{2}{\pi \cdot n}(-1)^{n+1}$$

The sawtooth curve was defined by:

$$f(t) = \left(\frac{t}{\pi} + 1\right) \quad for \quad -\pi < t < \pi$$

The Fourierseries is thus given by

$f(t) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Hint: Do not forget : a_0 ..

------------------▷ 55

$$f(t) = 1 + \sum_{n=1}^{\infty} \frac{2}{\pi n} \cdot (-1)^{n+1} \sin nt$$

55

This Fourierseries differs from the Fourierseries calculated in the textbook only by the value of a_0. Both curves can be shifted in the y-direction to coincide.

--------------------▷ 56

18.3 Expansion of Functions of Period *L*

56

L may be any period depending on the given situation. Reading the section in the textbook you should control all substitutions using a separate sheet.

Study in the textbook: **18.3 Expansion of functions of Period *L***
Textbook page 501–502

57

57

No problems regarding the arbitrary period L ---------------------▷ 60

More and detailed explanations wanted ---------------------▷ 58

We started with the Fourierseries for the variable x and the period 2π. The Fourierseries is summarized in the table on page xxxx of the textbook. Use this table. 58

Now let us substitute x by $\frac{2\pi}{T}t$ and dx by $\frac{2\pi}{T}dt$.

For $t = T$ we get $x =$……………

Now try this substation for a_n: $x = \frac{2\pi}{T}t$ and $dx = \frac{2\pi}{T}dt$

We obtain

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 59

59

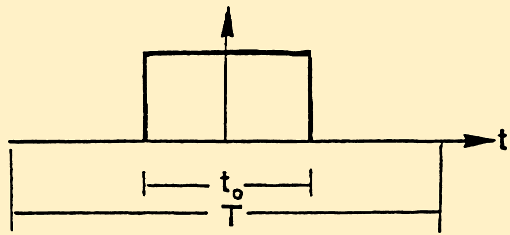
For $t = T$ we get $x = 2\pi$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos\left(\frac{n2\pi}{T}t\right)\cdot\frac{2\pi}{T}dt \quad = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cdot\cos\left(\frac{n2\pi}{T}t\right)dt$$

-------------------▷ 60

Calculate the Fourierseries for a rectangular curve. If the variable is time we have a rectangular waveform. Calculate the rectangular waveform using the variable t instead of x. 60

$$f(t)=\begin{cases} 0 & for \quad -\frac{T}{2} \le t \le -\frac{t_0}{2} \\ 1 & for \quad -\frac{t_0}{2} \le t \le \frac{t_0}{2} \\ 0 & for \quad \frac{t_0}{2} \le t \le \frac{T}{2} \end{cases}$$



Because the function is even $b_n = 0$

We have three sections of the function. For the first and for the third section we have $f(t)=0$. Thus, we do not need to calculate the integrals for the first and the third section. It remains to calculate the integral for the section in the middle:

We start with the calculation of a_0.

$$a_0 = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$$

$a_0 = \ldots\ldots\ldots\ldots\ldots$

------------------▷ 61

$$a_0 = \frac{2}{T}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t)dt = 2\frac{t_0}{T}$$ 61

Now we calculate a_n:

$$a_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t)\cdot\cos\left(\frac{2\pi n}{T}t\right)dt$$

$a_n = \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$

Solution found ------------------▷ 63

Further explanation wanted ------------------▷ 62

62

To calculate.

$$a_n = \frac{2}{T} \int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t) \cdot \cos\left(\frac{2\pi n}{T} t\right) dt$$

For the section in the middle holds:

$$f(t) = 1 \qquad \text{for} \qquad -\frac{t_0}{2} < t < \frac{t_0}{2}$$

We remember:

$$\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} \cos\left(\frac{2\pi n}{T} t\right) dt = \left[\frac{T}{2\pi n} \cdot \sin \frac{2\pi n \cdot t}{T}\right]_{-\frac{t_0}{2}}^{\frac{t_0}{2}}$$

Now we insert regarding the limits:

$a_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

------------------▷ 63

$$a_n = \frac{2}{T} \cdot \frac{T}{n2\pi} \cdot 2\sin\frac{n \cdot \pi \cdot t_0}{T} = \frac{2}{n\pi}\sin\frac{n\pi t_0}{T}$$ 63

Regarding $a_0 = \frac{2t_0}{T}$ we obtain finally the Fourierseries for the rectangular waveform of duration t_0 and the period T.

$f(t)=$………………………………..

------------------▷ 64

$$f(t)=\frac{t_0}{T}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi t_0}{T}\cdot\cos\frac{n2\pi t}{T}$$

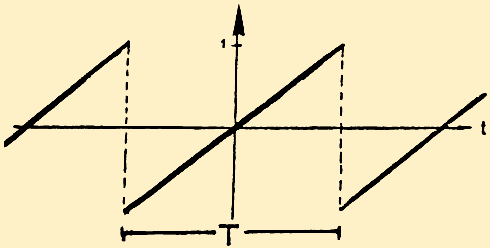
64

If you would like one more exercise --------------------▷ 65

If you prefer to go on --------------------▷ 70

65

The example of the sawtooth waveform has been treated exhaustively But up to now only for the period 2π . Now we will calculate an example for the period T.



The sawtooth waveform is defined for the period T by:

$f(t)=$..............

Hint: for $t=\frac{T}{2}$ holds: $f(t)=1$.

66

$$f(t)=\frac{2}{T}\cdot t \quad \text{for} \quad -\frac{T}{2}<t<\frac{T}{2}$$

66

We state: $f(t)$ is odd. Then all $a_n = 0$. We have to calculate only the coefficients b_n.

The integral can be solved by integration by parts:

$b_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Solution found ------------------▷ 68

Help and explanations wanted ------------------▷ 67

67

To calculate:

$$b_n = \left(\frac{2}{T}\right)^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} t \sin\left(\frac{n2\pi}{T}t\right)dt$$

We remember integration by parts: $: \int u \cdot v' = [u \cdot v] - \int u' \cdot v$

If $u = t$ and $v' = \sin\left(\frac{n2\pi}{T}t\right)dt$ we obtain:

$$u' = 1 \text{ and } v = -\cos\left(\frac{n2\pi}{T}t\right) \cdot \frac{T}{2\pi n}$$

You have to insert this into the integral shown above and calculate with a lot of patience:

$$b_n = \left(\frac{2}{T}\right)^2 \left[\Big[\ldots\ldots\ldots\ldots\ldots\ldots\ldots \Big]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \right]$$

--------------------▷ 68

$$b_n = \left(\frac{2}{T}\right)^2 \left[\left[t\left(-\cos\left(\frac{n2\pi}{T}t\right) \cdot \frac{T}{n2\pi} \right) \right]_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-\frac{T}{2}}^{\frac{T}{2}} 1 \cdot \left(-\cos\left(\frac{n2\pi}{T}t\right) \cdot \frac{T}{n2\pi} \right) \right]$$ 68

The remaining integral is a fundamental integral and may be solved easily.

Because of $\sin\left(\frac{n2\pi}{T} \cdot \frac{T}{2}\right) = \sin n2\pi = 0$ the integral vanishes.

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \cos\left(\frac{n2\pi}{T}t\right) = \frac{T}{n2\pi}\left[\sin\left(\frac{n2\pi}{T}t\right)\right]_{-\frac{T}{2}}^{\frac{T}{2}} = 0$$

Thus we obtain:

$$b_n = \left(\frac{2}{T}\right)^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$$

Hint: $-\cos n\pi = (-1)^{n+1}$

-------------------▷ 69

69

$$b_n = \left(\frac{2}{T}\right)^2 \left[2\left(\frac{T}{2}\right)^2 \cdot \frac{1}{n \cdot \pi} (-1)^{n+1} \right]$$

$$b_n = \frac{2}{n\pi} (-1)^{n+1}$$

The Fourierseries of the sawtooth waveform with the period T reads:

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot (-1)^{n+1} \cdot \sin\left(\frac{n2\pi}{T} t\right)$$

This is in agreement with the result on page 497 of the textbook.

------------------▷ 70

18.4 Fourier Spectrum

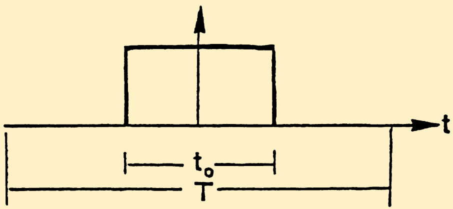
70

We remember a notation which we use frequently: $\sin\frac{n2\pi}{T}t = \sin\omega t$, and $\omega = \frac{n2\pi}{T}$

Study **18.4 Fourier Spectrum**
Textbook page 502–503

Go to ------------------▷ 71

You have already calculated the Fourierseries for the rectangular waveform. From the result you can construct the Fourier spectrum. 71



We just got: $f(t)=\frac{t_0}{T}+\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi t_0}{T}\cdot\cos\frac{n2\pi t}{T}$

Now calculate the coefficients a_n for $n=1$ up to $n=8$, for $T=2$ and $t_0=1$

$a_1=$	$a_2=$ ....
$a_3=$	$a_2=$ ....
$a_5=$	$a_6=$ ....
$a_7=$	$a_8=$ ....

Solution found ------------------▷ 73

Help wanted ------------------▷ 72

In our case the coefficients a_n are given by: 72

$$a_n = \frac{2}{n\pi} \cdot \sin \frac{n\pi t_0}{T}$$

We transform and insert the values: $T = 2$, $t_0 = 1$ to obtain:

$$a_n = \frac{2}{n\pi} \cdot \sin\left(n \cdot \frac{\pi}{2} \right)$$

Thus, we get rounded to two decimal places:

$a_1 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_3 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_5 = \ldots\ldots$ $a_6 = \ldots\ldots$

$a_7 = \ldots\ldots$ $a_8 = \ldots\ldots$

-------------------▷ 73

73

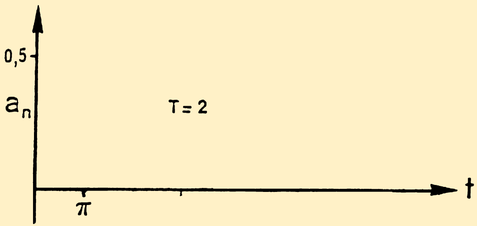
$a_1 = \ \ 0{,}64$ $\qquad a_2 = 0$

$a_3 = -0{,}21$ $\qquad a_2 = 0$

$a_5 = \ \ 0{,}13$ $\qquad a_6 = 0$

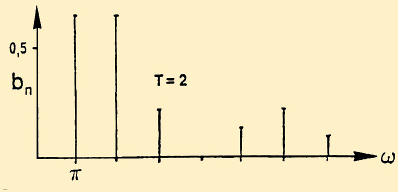
$a_7 = -0{,}09$ $\qquad a_8 = 0$

Sketch the Fourier spectrum of the rectangular function:



Hint:: $\omega_0 = \dfrac{2\pi}{T}, \quad \omega_n = \dfrac{n2\pi}{T} = n \cdot \pi$

--------------------▷ 74



Again we regard the Fourierseries of the rectangular waveform:

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi t_0}{T} \cdot \cos \frac{n2\pi t}{T}$$

Calculate the coefficients a_n given different values: $T = 8$ and $t_0 = 1$. Calculate again rounding to two decimal places:

$a_1 = \ldots\ldots$	$a_2 = \ldots\ldots$
$a_3 = \ldots\ldots$	$a_2 = \ldots\ldots$
$a_5 = \ldots\ldots$	$a_6 = \ldots\ldots$
$a_7 = \ldots\ldots$	$a_8 = \ldots\ldots$

I want further help ------------------▷ 75

Solution ------------------▷ 76

75

The coefficients a_n are given by:

$$a_n = \frac{2}{n\pi} \cdot \sin\left(\frac{n \cdot \pi}{T} \cdot t_0\right)$$

But this time the value of T has changed. It is now $T = 8$. The period has increased significantly. Insert and calculate again rounding to two decimal places starting with:

$\sin\frac{\pi}{8} = \ldots\ldots\ldots$ $\sin\frac{2\pi}{4} = \ldots\ldots\ldots\ldots$

$\sin\frac{3\pi}{8} = \ldots\ldots\ldots$ $\sin\frac{4\pi}{8} = \ldots\ldots\ldots\ldots$

$\sin\frac{5\pi}{8} = \ldots\ldots\ldots$ $\sin\frac{6\pi}{8} = \ldots\ldots\ldots\ldots$

$\sin\frac{7\pi}{8} = \ldots\ldots\ldots$ $\sin\frac{8\pi}{8} = \ldots\ldots\ldots\ldots$

Using these values we obtain finally:

$a_1 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_3 = \ldots\ldots$ $a_2 = \ldots\ldots$

$a_5 = \ldots\ldots$ $a_6 = \ldots\ldots$

$a_7 = \ldots\ldots$ $a_8 = \ldots\ldots$

-------------------▷ 76

76

$\sin\frac{\pi}{8} = 0{,}38 \qquad \sin\frac{2\pi}{8} = 0{,}71$

$\sin\frac{3\pi}{8} = 0{,}92 \qquad \sin\frac{4\pi}{8} = 1$

$\sin\frac{5\pi}{8} = 0{,}92 \qquad \sin\frac{6\pi}{8} = 0{,}71$

$\sin\frac{7\pi}{8} = 0{,}38 \qquad \sin\frac{8\pi}{8} = 0$

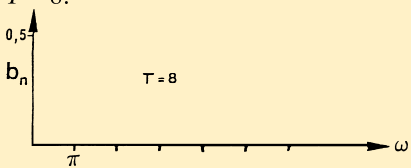
We obtain finally: $a_1 = 0{,}24 \qquad a_2 = 0{,}23$

$a_3 = 0{,}20 \qquad a_2 = 0{,}16$

$a_5 = 0{,}12 \qquad a_6 = 0{,}08$

$a_7 = 0{,}03 \qquad a_8 = 0$

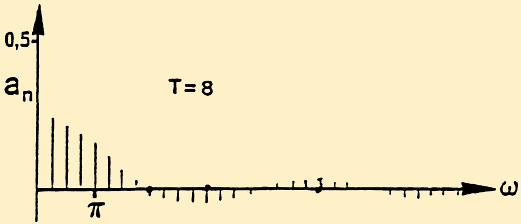
Using these values you can sketch the Fourier spectrum of the new rectangular waveform with period $T = 8$:



Remember: $\omega_n = n \cdot \omega_0 = n \cdot \frac{2\pi}{8} = n \cdot \frac{\pi}{4}$

---------------------▷ 77

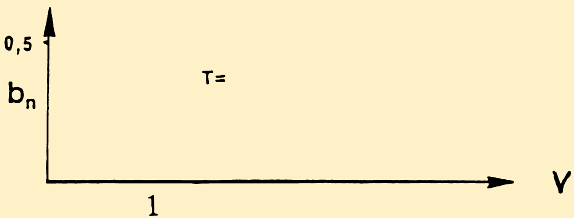
77



In theory we use most frequently the angular frequency ω. But you can as well use the frequency z $\nu = \frac{1}{T}$.

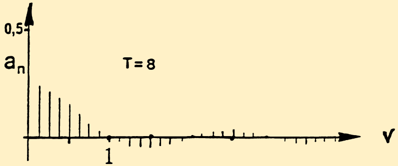
$\omega = 2\pi\nu$ $\nu = \frac{\omega}{2\pi}$

Sketch the Fourier spectrum for the frequency:



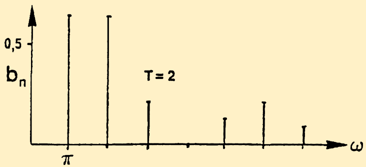
--------------------▷ 78

78



The Fourier spectrum is the same. Only the calibration of the *x*-axis has changed.

Now compare the Fourier spectrum of the same rectangular waveform for $t_0 = 1$ and two different periods $T = 2$ and $T = 8$:



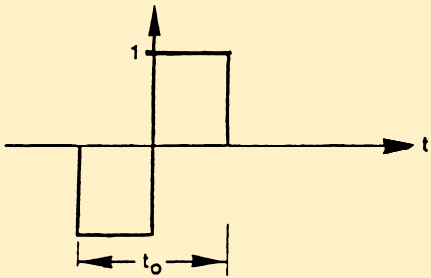
If the period T increases the distances between the components of the series decrease.

More exercises wanted -------------------▷ 79

Terminate this chapter -------------------▷ 86

Given an alternating rectangular waveform with a period t_0

79



$$f(t)=\begin{cases} -1 & \textit{für} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \textit{für} \quad 0 < t < \frac{t_0}{2} \end{cases}$$

This function is ☐ even

☐ odd

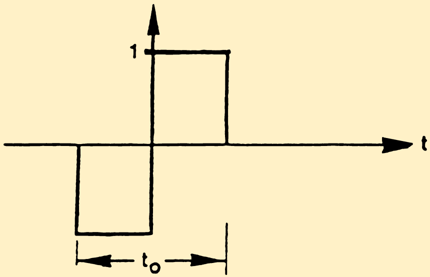
Thus the coefficients: ……………………… vanish.

------------------▷ 80

80

The function is odd.

Thus the coefficients a_n vanish.



Now calculate the remaining coefficients

$b_n = \ldots\ldots\ldots\ldots\ldots$

If you are unsure consult the formulae in the textbook.

Solution found ------------------▷ 83

Help needed ------------------▷ 81

81

To calculate: $b_n = \frac{2}{t_0} \int\limits_{-\frac{t_0}{2}}^{\frac{t_0}{2}} f(t) \sin\left(\frac{n2\pi}{t_0} \cdot t\right) dt$

We divide into two parts because $f(t)$ is defined for two parts:

$$f(t) = \begin{cases} -1 & \textit{für} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \textit{für} \quad 0 < t < \frac{t_0}{2} \end{cases}$$

Thus we obtain:

$$b_n = \frac{2}{t_0} \int\limits_{-\frac{t_0}{2}}^{0} (-1) \sin\left(\frac{n2\pi}{t_0} t\right) dt + \frac{2}{t_0} \int\limits_{0}^{\frac{t_0}{2}} \sin\left(\frac{n2\pi}{t_0} t\right) dt$$

It remains to solve the integrals and to regard the limits of integration.

$b_n = \ldots\ldots\ldots\ldots\ldots\ldots..$

Solution found --------------------▷ 83

Stepwise solution --------------------▷ 82

82

To calculate:

$$b_n = \frac{2}{t_0}\int_{-\frac{t_0}{2}}^{0}(-1)\sin\left(\frac{n2\pi}{t_0}t\right)dt + \frac{2}{t_0}\int_{0}^{\frac{t_0}{2}}\sin\left(\frac{n2\pi}{t_0}t\right)dt$$

Hint:

$$\int_{-\frac{t_0}{2}}^{0}\sin\left(\frac{n2\pi}{t_0}t\right)dt = \left[-\frac{t_0}{n2\pi}\cdot\cos\left(\frac{n2\pi}{t_0}t\right)\right]_{-\frac{t_0}{2}}^{0}$$

If you insert you will obtain:

$$b_n = \frac{2}{t_0}\left[\ (-1)\cdot\frac{t_0}{n2\pi}\left(-\cos\frac{n2\pi}{t_0}\cdot t\right)\ \right]_{-\frac{t_0}{2}}^{0} + \frac{2}{t_0}\left[\frac{t_0}{n2\pi}\left(-\cos\frac{n2\pi}{t_0}\cdot t\right)\right]_{0}^{\frac{t_0}{2}}$$

Now insert the limits, simplify, and obtain:

$b_n = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..$

-------------------▷ 83

$$b_n = \frac{1}{n\pi}\left[1 - \cos\left(\frac{n2\pi}{t_0}\cdot\frac{t_0}{2}\right) + 1 - \cos\left(\frac{n2\pi}{t_0}\cdot\frac{t_0}{2}\right)\right] = \frac{2}{n\pi}[1 - \cos n\pi]$$ 83

Since $\cos n\pi = (-1)^n$ we obtain:

$$b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

Now you can calculate:

$b_1 = \ldots\ldots\ldots\ldots$	$b_2 = \ldots\ldots\ldots\ldots$
$b_3 = \ldots\ldots\ldots\ldots$	$b_4 = \ldots\ldots\ldots\ldots$
$b_5 = \ldots\ldots\ldots\ldots$	$b_6 = \ldots\ldots\ldots\ldots$
$b_7 = \ldots\ldots\ldots\ldots$	$b_8 = \ldots\ldots\ldots\ldots$

-------------------▷ 84

84

$b_1 = 1,3$ $\quad$ $b_2 = 0$

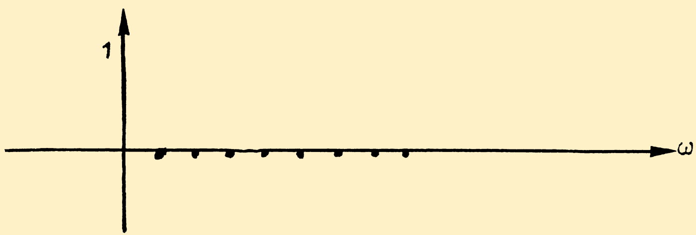
$b_3 = 0,42$ $\quad$ $b_4 = 0$

$b_5 = 0,25$ $\quad$ $b_6 = 0$

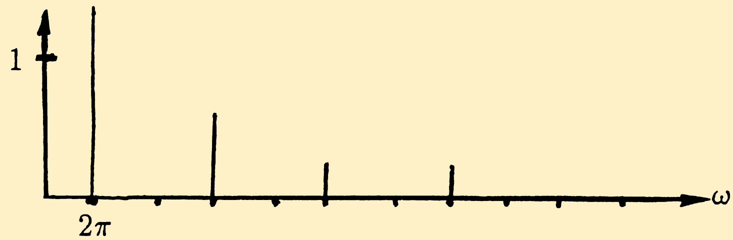
$b_7 = 0,18$ $\quad$ $b_8 = 0$

Sketch the Fourier spectrum for $t_0 = 1$.

Remember: $\omega_n = n \cdot \omega_0 = n \cdot \dfrac{2\pi}{t_0}$



85

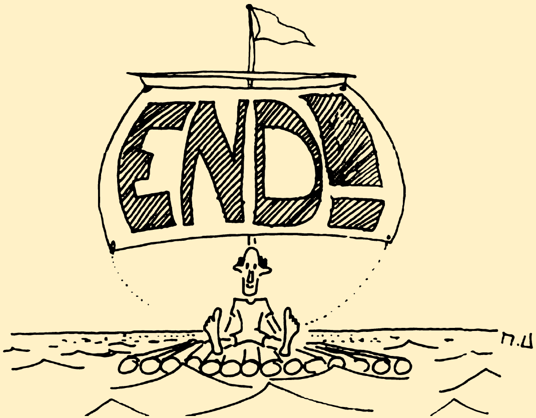


85

86

86

You have successfully finished the chapter Fourierseries. During this chapter you might have had some trouble with mere calculations and regarding limits of integration and regarding signs. But you overcame all difficulties. You should enjoy your success. Congratulations



End of chapter 18

Chapter 19
Fourier Integrals and Fourier Transforms

19.1 Transition from Fourier series to Fourier integral

2

In the preceding chapter it has been shown that any given periodic function can be represented by a sum of discrete sine and cosine functions. It is an important fact in physics and especially in information technology that periodic signals can be seen and produced by the superposition of sine and cosine oscillations or waves.

In this chapter we will show that even a single signal which is limited in duration (and hence non-periodic) can be represented as a sum of sine and cosine functions. The sum will turn into an integral and discrete functions will turn into a continuous distribution of oscillations or waves.

READ: 19.1 Transition from Fourier series to Fourier integral
Textbook pages 509–511

--------------------▷ 3

No difficulties in understanding the transition from the Fourier series to the Fourier integral

10

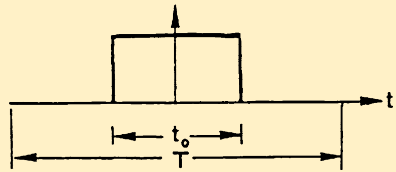
I want further explanations

4

Consider a Fourier series for a rectangular function. The rectangular function is a periodic signal of duration t_0 and period T.

4

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin \frac{n\pi t_0}{T} \cdot \cos \frac{2\pi n t}{T}$$



To obtain a single and isolated signal of duration t the period's duration T has to increase up to infinity. The duration of the period T corresponds to an oscillation whose fundamental harmonic frequency is $\omega_o = \frac{2\pi}{T}$

Hence the n^{th} harmonic frequency is $\omega =$ ....

--------------------▷ 5

5

$$\omega = n\omega_0 = n\frac{2\pi}{T}$$

This result can be transformed into $T = \frac{n \cdot 2\pi}{n \cdot \omega_o} = \frac{2\pi}{\omega_o}$

The given Fourier series of a rectangular periodic signal was

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin\frac{n\pi t_0}{T} \cdot \cos\frac{2\pi n t}{T}$$

The iteration number n increases from term to term by 1

We denote this by $\Delta n = 1$

We now insert $\Delta n = 1$ to obtain $f(t) =$ ....

6

$$f(t) = \frac{t_0}{T} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \cdot \sin \omega \frac{t_0}{T} \cdot \cos \omega t \cdot \Delta n$$ 6

We obtain a single signal of duration t_0 if we increase the distance between the signals, the period T, to infinity. Then the fundamental harmonic $\omega_0 = \frac{2\pi}{T}$ vanishes.

In the formula above the variable is n. If we want to transform the sum into an integral we must first substitute n by ω and Δn by $\Delta\omega$.

$$\omega = n \cdot \frac{2\pi}{T} \qquad \Delta\omega = \Delta n \cdot \frac{2\pi}{T}$$

$$n = \omega \cdot \frac{T}{2\pi} \qquad \Delta n = \frac{T}{2\pi}\Delta\omega$$

Now insert n and Δn

7

7

$$f(t) = \frac{t_0}{T} + \sum_{\omega=0}^{\omega=\infty} \frac{2 \cdot 2\pi}{\omega \cdot T\pi} \cdot \sin\omega \frac{t_0}{2} \cdot \cos\omega t \cdot \frac{T}{2\pi} \Delta\omega = \frac{t_0}{T} + \sum_{\omega=0}^{\omega=\infty} \frac{2}{\omega\pi} \sin\omega \frac{t_0}{2} \cdot \cos\omega t \cdot \Delta\omega$$

Now we can transform the sum into an integral.

T increases to infinity: $T \rightarrow \propto$ $f(t) =$ ....

- - - - - - - - - - - - - - - - - - - -▷ 8

$$f(t) = \int_0^{\infty} \frac{2}{\pi\omega} \cdot \sin\omega\frac{t_0}{2} \cdot \cos\omega t \cdot \mathrm{d}\omega$$ 8

Using $A(\omega) = \frac{2}{\pi\omega} \cdot \sin\omega\frac{t_0}{2}$ we can write the integral as $f(t) =$ ....

$\dashrightarrow$ 9

$$f(t) = \int_0^{\infty} A(\omega) \cos \omega t \cdot d\omega$$ 9

Now we proceed to the next section

10

19.2 Fourier transforms

10

READ: 19.2.1 Fourier cosine transform
19.2.2 Fourier sine transform
Textbook pages 511–513

11

The formulae to obtain the amplitude spectrum $A(\omega)$ and $B(\omega)$ are given in the textbook. They are verified but not derived. **11**

Verify that $A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cdot\cos\omega t\,\mathrm{d}t$ equals the $A(\omega)$ which we obtained in the preceding section from the transition from the sum to the Fourier integral: $A(\omega) = \frac{2}{\pi\omega}\cdot\sin\omega\frac{t_0}{2}$

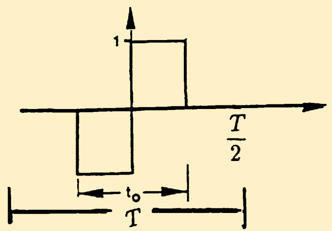
$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\cdot\cos\omega t\cdot\mathrm{d}t = \int \text{.............................} = [\text{.............................}]$$

--------------------▷ 12

$$A(\omega) = \frac{1}{\pi}\int_{-\frac{t_0}{2}}^{+\frac{t_2}{0}} \cos\omega t \cdot \mathrm{d}t = \frac{1}{\pi\omega}\left[\sin\omega t\right]_{-\frac{t_0}{2}}^{+\frac{t_0}{2}} = \frac{2}{\pi\omega}\cdot\sin\omega\frac{t_0}{2}$$ 12

We will also verify the formula for the amplitude spectrum of the Fourier sine transform. First we calculate the Fourier sequence of a periodic alternating rectangular wave form of duration t_0 and period T

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



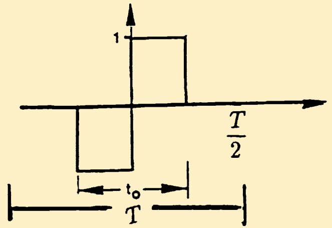
The function is ☐ even ☐ odd. Hence the .... vanish

13

The function is odd. Hence $a_n = 0$

13

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We calculate the b_n. Because we have two branches we have to split the integration.

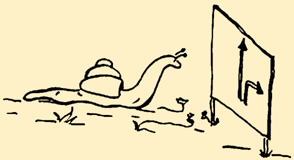
Obtain the two integrals. b_n ....

14

14

$$b_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{0}(-1)\cdot\sin\frac{n2\pi}{T}t\,\mathrm{d}t + \frac{2}{T}\int_{0}^{\frac{t_0}{2}} 1\cdot\sin\frac{n2\pi}{T}t\cdot\mathrm{d}t$$

The integral above can be resolved. You have to regard limits and may rearrange b_n ....



Integrals solved --------------------▷ 17

Hints and step-by-step calculation --------------------▷ 15

To calculate $b_n = \frac{2}{T}\int_{-\frac{t_0}{2}}^{0}(-1)\cdot\sin\frac{n2\pi}{T}t\cdot\mathrm{d}t + \frac{2}{T}\int_{0}^{\frac{t_0}{2}} 1\cdot\sin\frac{n2\pi}{T}t\cdot\mathrm{d}t$ 15

We remember $\int_{t_1}^{t_2}\sin\frac{n2\pi}{T}t\cdot\mathrm{d}t = \frac{T}{n2\pi}\left[-\cos\frac{n2\pi}{T}t\right]_{t_1}^{t_2}$

We calculate the integrals and obtain

$$b_n = \frac{2}{T}\left[(-1)\cdot\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{-\frac{t_0}{2}}^{0} + \frac{2}{T}\left[\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{0}^{\frac{t_0}{2}}$$

We rearrange and regard the limits to obtain b_n ....

Solution found --------------------▷ 17

Hint and help --------------------▷ 16

Given $b_n = \frac{2}{T}\left[(-1)\cdot\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{-\frac{t_0}{2}}^{0} + \frac{2}{T}\left[\left(-\cos\frac{n2\pi}{T}t\right)\cdot\frac{T}{n2\pi}\right]_{0}^{\frac{t_0}{2}}$ 16

We isolate the term $\frac{T}{n2\pi}$ and insert the limits. Because of $\cos(0) = 1$ we obtain

$$b_n = \frac{2}{T}\cdot\frac{T}{n2\pi}\left[\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)+\left(-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)+1\right)\right]$$

Now we rearrange and summarize b_n ....

17

$$b_n = \frac{2}{n\pi}\left[1 - \cos\left(\frac{n\pi}{T}\cdot t_0\right)\right]$$ 17

Thus the Fourier series is

$f(t) = $ ....

18

$$f(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\left(\frac{n2\pi}{T} \cdot \frac{t_0}{2}\right)\right) \cdot \left(\sin \frac{n2\pi}{T} t\right)$$

18

In the textbook, the Fourier coefficients are shown for $t_0 = 1$ and $T = 2t_0$, $T = 4t_0$ and $T = 8t_0$.

It is a useful exercise to calculate the b_n at least for one case numerically.

Calculation of b_n for $t_0 = 1$ and $T = 2$ --------------------▷ 19

I will skip this exercise --------------------▷ 21

Given the Fourier series $f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)\cdot\left(\frac{n2\pi}{T}t\right)$

19

To calculate $b_n = \frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{T}\cdot t_0\right)\right]$

We insert $t_0 = 1$ and $T = 2$

Fill in the calculated values.

Use a pocket calculator.

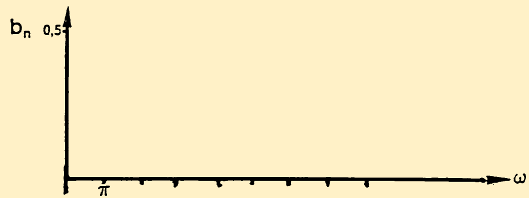
n	$\frac{2}{n\pi}$	$\cos n\frac{\pi}{2}$	b_n
1			
2			
3			
4			
5			
6			
7			
8			

20

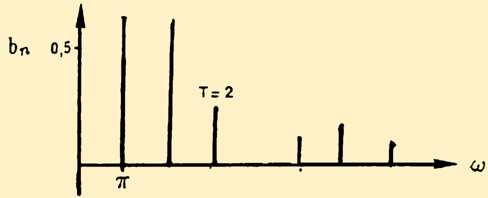
n	$\dfrac{2}{n\pi}$	$\cos n\dfrac{\pi}{2}$	b_n
1	0,64	0	0,64
2	0,32	-1	0,64
3	0,22	0	0,22
4	0,16	$+1$	0
5	0,13	0	0,13
6	0,11	-1	0,22
7	0,09	0	0,09
8	0,08	$+1$	0

Draw the amplitude spectrum for $t_0 = 1$ and $T = 2$

20



21



21

The b_n for different values of t_0 and T can be calculated in the same way.

In the textbook the results for $T = 4$ and $T = 8$ are shown.

22

The Fourier series for the periodic alternating rectangular wave form of duration t_0 and period T was $f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\left(\frac{n2\pi}{T}\cdot\frac{t_0}{2}\right)\right)\cdot\sin\left(\frac{n2\pi}{T}t\right)$ 22

Now follows the transition to the Fourier integral.

The sum increases step by step by $\Delta n = 1$. By the transition from the sum to the integral we must regard the relation between Δn and $\Delta\omega$ since $\frac{n2\pi}{T} = \omega$ and $n = \omega\frac{T}{2\pi}$ we get $\Delta n = \Delta\omega\frac{T}{2\pi}$ and $\Delta\omega = \Delta n\cdot\frac{2\pi}{T}$.

Substitute in the sum above T by ω $f(t) = \frac{2}{\pi}\sum_{n=1}^{n=\infty}$ ....

--------------------▷ 23

$$f(t)=\frac{2}{\pi}\sum_{n=1}^{n=\infty}\frac{2\pi}{\omega T}\left(1-\cos\omega\frac{t_0}{2}\right)\cdot\sin\omega t\cdot\Delta\omega\frac{T}{2\pi}$$ 23

Now we can reduce and perform the limiting process.

$$f(t)=\frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega}\left(1-\cos\omega\frac{t_0}{2}\right)\cdot\sin\omega t\cdot d\omega$$

The continuous amplitude spectrum will be $B(\omega)=$ ....

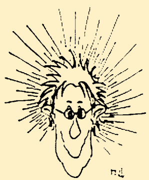
24

24

$$B(\omega) = \frac{2}{\pi \cdot \omega} \cdot \left(1 - \cos\omega \frac{t_0}{2}\right)$$

We can also obtain the continuous amplitude spectrum with the formula for the sine transform given in the textbook.

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{+\infty} f(t) \cdot \sin \omega t \cdot \mathrm{d}t \qquad B(\omega) = \text{............................}$$



Solution found ------------------▷ 28

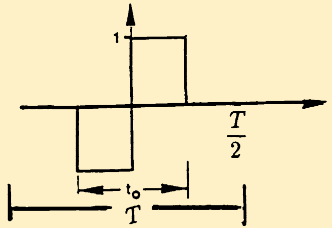
Hint and step-by-step solution ------------------▷ 25

To solve: $B(\omega) = \dfrac{1}{\pi} \displaystyle\int_{-\infty}^{+\infty} f(t) \sin \omega t \cdot \mathrm{d}t$

25

The function is defined in parts.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We calculate the integral for the two parts which do not vanish

$$B(\omega) = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots$$

26

26

$$B(\omega) = \frac{1}{\pi} \int_{-\frac{t_0}{2}}^{0} (-1) \sin \omega t \cdot \mathrm{d}t + \int_{0}^{\frac{t_0}{2}} \sin \omega t \cdot \mathrm{d}t$$

We solve the integrals and obtain

$$B(\omega) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

- ▷ 27

27

$$B(\omega) = \frac{1}{\pi}\left[(-1)\cdot(-\cos\omega t)\cdot\frac{1}{\omega}\right]_{-\frac{t_0}{2}}^{0} + \frac{1}{\pi}\left[\left(-\cos\omega t\cdot\frac{1}{\omega}\right)\right]_{0}^{\frac{t_0}{2}}$$

Now we regard the limits to obtain

$$B(\omega) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

28

$$B(\omega) = \frac{2}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right]$$

28

Thus we have again verified that by the limiting process $T \to \infty$ and $n \to \infty$ we obtain the same continuous amplitude spectrum as has been given in the textbook.

In the next section we will use complex numbers.

Please remember:

$$\mathrm{e}^{\mathrm{j}a} = \text{............................}$$
$$\mathrm{e}^{\mathrm{j}a+b} = \text{............................}$$

29

$$
\begin{aligned}
e^{ja} &= j\sin a + \cos a \\
e^{ja+b} &= e^{ja} \cdot e^{b} = e^{b}(j\sin a + \cos a)
\end{aligned}
$$

29

Do you remember the following expressions?

$$
\begin{aligned}
e^{j\omega t} &= \ldots\ldots\ldots\ldots\ldots\ldots \\
e^{-j\omega t} &= \ldots\ldots\ldots\ldots\ldots\ldots \\
e^{j\omega t} + e^{-j\omega t} &= \ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

30

$$\begin{aligned} e^{j\omega t} &= j \cdot \sin\omega t + \cos\omega t \\ e^{-j\omega t} &= -j \cdot \sin\omega t + \cos\omega t \\ e^{j\omega t} + e^{-j\omega t} &= 2\cos\omega t \end{aligned}$$

30

In case of difficulties recapitulate chapter 9 "Complex numbers". In the next section complex numbers will be used extensively.

Solve the two integrals using Euler's formula explained in the textbook Sect. 9.3.1

$$\int_{t_1}^{t_2} e^{-j\omega t}\,dt = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots$$

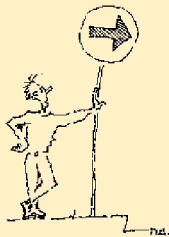
$$\int_{t_1}^{t_2} e^{j\omega t}\,dt = \ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 31

31

$$\int_{t_1}^{t_2} \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t = \frac{1}{-\mathrm{j}\omega}\cdot\left[\mathrm{e}^{-\mathrm{j}\omega t_2} - \mathrm{e}^{-\mathrm{j}\omega t_1}\right] \quad = \frac{1}{\omega}(\sin\omega t_2 - \sin\omega t_1) + \frac{\mathrm{j}}{\omega}(\cos\omega t_2 - \cos\omega t_1)$$

$$\int_{t_{\mathrm{j}1}}^{t_2} \mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}t = \frac{1}{\mathrm{j}\omega}\cdot\left[\mathrm{e}^{\mathrm{j}\omega t_2} - \mathrm{e}^{\mathrm{j}\omega t_1}\right] \quad = \frac{1}{\omega}(\sin\omega t_2 - \sin\omega t_1) + \frac{\mathrm{j}}{\omega}(\cos\omega t_1 - \cos\omega t_2)$$



32

19.2.3 Complex Representation of the Fourier Transform

32

19.3 Shift Theorem

Useful hint: calculate the derivations in the textbook separately on a sheet of paper

READ: 19.2.3 Complex Representation of the Fourier Transform
19.3 Shift Theorem
Textbook pages 514–515

33

In the textbook the continuous amplitude spectrum has been calculated for a rectangular function of duration t_0. Calculate it on your own.

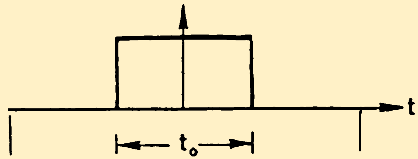
33

$$F(\omega) = \frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{\frac{t_0}{2}} 1 \cdot e^{-j\omega t}\,dt \ldots\ldots\ldots\ldots\ldots\ldots$$

Hint: regard the signs

The rectangular function has been defined

$$f(t) = \begin{cases} 0 & \text{for} \quad -\infty < t < -\frac{t_0}{2} \\ 1 & \text{for} \quad -\frac{t_0}{2} < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < \infty \end{cases}$$



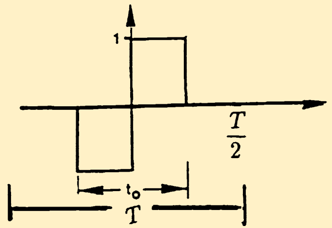
-------------------▷ 34

$$\mathrm{F}(\omega) = \frac{1}{\omega\pi} \sin\left(\omega \frac{t_0}{2}\right)$$

34

Given the function for the alternating rectangular signal of duration t_0:

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



The signal in complex representation by the Fourier integral

$f(t) =$ ....

The amplitude spectrum is $F(\omega) =$ ....

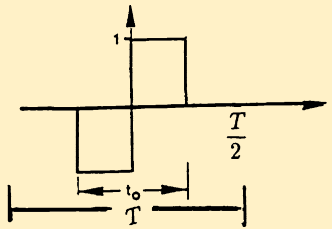
35

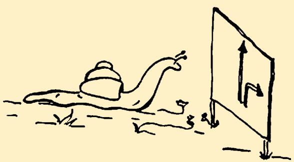
$$f(t) = \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} \, dt$$
$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \, dt$$

35

Calculate the amplitude function of the alternating rectangular signal of duration t_0

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$





Solution obtained ----------------▷ 40

Help and detailed solution ----------------▷ 36

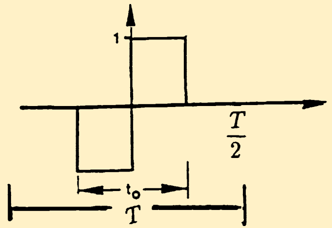
To be obtained: amplitude function of the alternating rectangular signal

36

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)\cdot \mathrm{e}^{-\omega t}\,\mathrm{d}t$$

The alternating rectangular signal has been defined

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < +\frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < +\frac{T}{2} \end{cases}$$



We have to calculate the integral of the parts where $f(t) =$ .... is not zero.

Write down the remaining integrals $F(\omega) =$ ....

37

$$F(\omega) = \frac{1}{2\pi}\int_{-\frac{t_0}{2}}^{0} (-1)\cdot e^{-j\omega t}\cdot dt + \frac{1}{2\pi}\int_{0}^{\frac{t_0}{2}} e^{-j\omega t}\cdot dt$$ 37

You can solve these integrals above regarding the limits

$F(\omega) = $ ....

Solution → 40

Further help and detailed solution → 38

To solve

38

$$F(\omega) = \frac{1}{2\pi} \int_{-\frac{t_0}{2}}^{0} (-1) \cdot e^{-j\omega t} \cdot dt + \frac{1}{2\pi} \int_{0}^{\frac{t_0}{2}} e^{-j\omega t} \cdot dt$$

Remember

$$\int_{t_1}^{t_2} e^{-j\omega t} dt = \frac{1}{-j\omega} \cdot \left[e^{-j\omega t_2} - e^{-j\omega t_1} \right]$$

Using the last form you should be able to solve the integrals.

You can solve these integrals above regarding the limits

$F(\omega) =$ ....

39

$$F(\omega) = \frac{1}{2\pi}\left(\frac{1}{-\mathrm{j}\omega}\right)\cdot\left(\left[(-1)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\right]_{-\frac{t_0}{2}}^{0} + \left[\mathrm{e}^{-\mathrm{j}\omega t}\right]_{0}^{\frac{t_0}{2}}\right)$$ 39

We insert the limits to obtain

$$F(\omega) = \frac{1}{2\pi(-\mathrm{j}\omega)}\cdot\left[\ldots\ldots\ldots\ldots\ldots\ldots\ldots\right]$$

Now go back to frame 39 at the lowest part of the page

40

$$F(\omega) = \frac{1}{2\pi(-\mathrm{j}\omega)} \cdot \left[-1 + \mathrm{e}^{\mathrm{j}\omega\frac{t_0}{2}} + \mathrm{e}^{-\mathrm{j}\omega\frac{t_0}{2}} - 1\right]$$

40

We carry out the sum and rearrange

$F(\omega) =$ ....

Remember $\frac{1}{\mathrm{j}} = \frac{i}{\mathrm{j}^2} = -\mathrm{j} = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$

41

41

$$\begin{aligned} F(\omega) &= \frac{1}{\pi\omega}\left[1-\cos\omega\frac{t_0}{2}\right]\cdot(-\mathrm{j}) \\ &= \frac{1}{\pi\omega}\left[1-\cos\omega\frac{t_0}{2}\right]\cdot \mathrm{e}^{-\frac{i\pi}{2}} \end{aligned}$$

Using the amplitude function we obtain the continuous amplitude spectrum

$A(\omega) =$ ....

The phase spectrum in our case is

$\phi(\omega) =$ ....

42

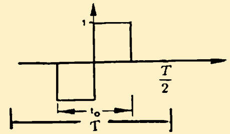
42

$$A(\omega) = \frac{1}{\pi\omega} \cdot \left[1 - \cos\omega\frac{t_0}{2}\right] \qquad \text{Phasespectrum}: \varphi(\omega) = \mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}$$

Exercise for shift theorem.

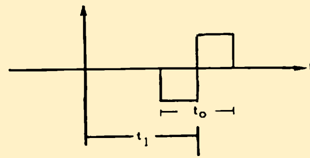
Given the alternating rectangular signal.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} < t < -\frac{t_0}{2} \\ -1 & \text{for} \quad -\frac{t_0}{2} < t < 0 \\ +1 & \text{for} \quad 0 < t < \frac{t_0}{2} \\ 0 & \text{for} \quad +\frac{t_0}{2} < t < \frac{T}{2} \end{cases}$$



We shift the Signal by the time t_1.

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} + t_1 < t < -\frac{t_0}{2} + t_1 \\ -1 & \text{for} \quad -\frac{t_0}{2} + t_1 < t < t_1 \\ +1 & \text{for} \quad t_1 < t < \frac{t_0}{2} + t_1 \\ 0 & \text{for} \quad \frac{t_0}{2} + t_1 < t < \frac{T}{2} + t_1 \end{cases}$$



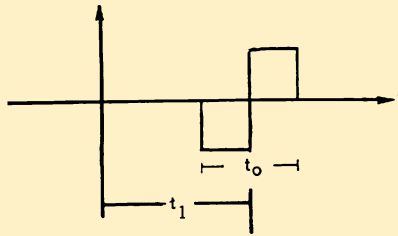
Calculate $F(\omega) = \ldots\ldots\ldots\ldots\ldots\ldots$ 46

Help and detailed solution 43

43

The given signal

$$f(t) = \begin{cases} 0 & \text{for} \quad -\frac{T}{2} + t_1 < t < -\frac{t_0}{2} + t_1 \\ -1 & \text{for} \quad -\frac{t_0}{2} + t_1 < t < t_1 \\ +1 & \text{for} \quad t_1 < t < \frac{t_0}{2} + t_1 \\ 0 & \text{for} \quad \frac{t_0}{2} + t_1 < t < \frac{T}{2} + t_1 \end{cases}$$



The amplitude function was

$$F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt$$

We write down the integral for the parts where $f(t)$ is not equal to zero

$F(\omega) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

44

44

$$F(\omega) = \frac{1}{2\pi} \int_{-\frac{t_0}{2}+t_1}^{t_1} (-1) \cdot e^{-j\omega t} \cdot dt + \frac{1}{2\pi} \int_{t_1}^{\frac{t_0}{2}+t_1} e^{-j\omega t} \cdot dt$$

We solve both integrals

$$F(\omega) = \frac{1}{2\pi} \left[\dots\dots\dots\dots\dots\dots \right] + \frac{1}{2\pi} \left[\dots\dots\dots\dots\dots\dots \right]$$

45

$$F(\omega) = \frac{1}{2\pi}\left[\frac{-1}{-\mathrm{j}\omega}\cdot \mathrm{e}^{-\mathrm{j}\omega t}\right]_{-\frac{t_0}{2}+t_1}^{t_1} + \frac{1}{2\pi}\left[\frac{1}{-\mathrm{j}\omega}\cdot \mathrm{e}^{-\mathrm{j}\omega t}\right]_{t_1}^{\frac{t_0}{2}+t_1}$$

45

Now your task is to insert the limits and to rearrange the remaining terms.

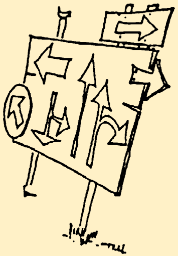
Hint: the term $\mathrm{e}^{-\mathrm{j}\omega t_1}$ can be isolated and put before the brackets

$F(\omega) = $ ....

46

$$F(\omega) = \frac{1}{\pi\omega}\left[1 - \cos\omega\frac{t_0}{2}\right] e^{-j\omega t_1 - j\frac{\pi}{2}}$$ 46

Main result: The continuous amplitude spectrum does not depend on a shift of the signal by the time t_1. Only the phase spectrum will be affected.



Remarks concerning the notation of Fourier transforms which are not handled consistently

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If you want to skip these remarks

--------------------▷ 50

In a note in the textbook we said that the notation of Fourier transform is not handled consistently. This may disturb you as the reader if you come across different notations in the literature such as: 47

a) $f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

c) $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

The differences refer to the constant factor $\frac{1}{2\pi}$

48

The different notations are:

a) $f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

c) $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$

48

To see that all notations are equivalent you may insert the second integral into the first. In the case a) of our notation we obtain

$$a) \quad f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \quad = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt \cdot e^{j\omega t} \cdot d\omega$$

Repeat this procedure for b) and c)

b) $f(t) =$ ....

c) $f(t) =$ ....

Solution ------------------▷ 50

Help ------------------▷ 49

Given

a) $f(t) = \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

49

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

c) $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) \cdot e^{+j\omega t} \cdot d\omega \qquad F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} \cdot dt$

We consider b). We insert $F(\omega) = \int_{-\infty}^{+\infty} f(t) \cdot e^{-j\omega t} dt$ into the first integral and obtain

b) $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \ldots\ldots\ldots\ldots\ldots \cdot e^{j\omega t} \cdot d\omega$

Now do the same with c)

c) $f(t) = \ldots\ldots\ldots\ldots\ldots$

→ 50

b) = c) $f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)\cdot e^{-j\omega t}\cdot dt\cdot e^{j\omega t}\cdot d\omega$ 50

Result: all three notations result in the same expression if we perform the Fourier transform and the reverse transform.

These remarks may help you to accept the different notations. Important for you is to use one of these notations consequently and exclusively.

--------------------▷ 51

19.4 Discrete Fourier transform, sampling theorem

51

19.5 Fourier transform of the Gaussian function

READ: 19.4 Discrete Fourier transform, sampling theorem
19.5 Fourier transform of the Gaussian function
Textbook pages 516–517

52

The following items are aimed to give a general orientation. The sampling theorem is fundamental for modern information technology which deals with the transformation of analog signals into discrete signals and the reconstruction of analog signals from discrete sample values. 52

A full reconstruction of a function from its sampling values is only possible if the sampling frequency captures the highest frequency in the amplitude spectrum.

The sampling frequency has to bee the highest frequency of the amplitud spectrum.

--------------------▷ 53

The sampling frequency has to be *twice* the highest frequency of the amplitude spectrum. 53

Given the bell-shaped Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{a}{2}t^2}$

It corresponds to the likewise bell-shaped amplitude function $F(\omega) = \frac{1}{\sqrt{a}} \cdot \mathrm{e}^{-\frac{\omega^2}{2a}}$

Both functions have a maximum for $t =$ .... and $\omega =$ ....

If t and ω increase, both exponential terms decrease.

Both terms are $\frac{1}{\mathrm{e}}$ if the exponent is -1: $\mathrm{e}^{-\frac{\omega^2}{2a}} = \mathrm{e}^{-\frac{a}{2}t^2} = \frac{1}{e}$

The exponents .... $=$ .... $= -1$

Exponents found --------------------▷ 55

Further explanation --------------------▷ 54

54

Given Gaussian function $f(t) = \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{a}{2}t^2}$.

Its amplitude spectrum $F(\omega) = \frac{1}{\sqrt{a}} \cdot \mathrm{e}^{-\frac{\omega^2}{2a}}$.

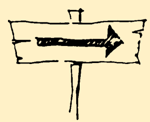
Both functions have its maximum at $\mathrm{e}^0 = 1$

Hence for the maximum holds $t =$ .... and $\omega =$ ....

If t and ω increase the absolute values of the exponents increase as well. Since the exponents are negative the values of the functions decrease up to $\frac{1}{\mathrm{e}}$ if the exponents are -1

In this case $\mathrm{e}^{-\frac{\omega^2}{2a}} = \mathrm{e}^{-\frac{a}{2}t^2} = \mathrm{e}^{-1}$.

Give the exponents .... $=$ .... $= -1$



- - - - - - - - - - - - - - - - - -▷ 55

55

Maxima for $t = 0$ and $\omega = 0$.

Decreases to $\frac{1}{e}$ if $-\frac{\omega^2}{2a} = -\frac{a}{2}t^2 = -1$.

By this we get for the decrease to $\frac{1}{\mathrm{e}}$

$$\frac{a}{2}t^2 = 1 \text{ and } t = \sqrt{\frac{2}{a}}$$

$$\frac{\omega^2}{2a} = 1 \text{ and } \omega = \sqrt{2a}$$

We repeat: If the parameter a is large

We get a narrow signal in the time domain and a wide amplitude spectrum in the frequency domain.

--------------------▷ 56

You have successfully reached the end of this chapter. You made an important step forward.

56

Chapter 20
Probability Calculus

2

20.1 Introduction

2

Objective: Concepts of macroscopic and microscopic properties of physical systems.

READ: **20.1 Introduction**
Textbook page 519

3

Without looking in the book name three

3

(a) macroscopic quantities

....

....

....

(b) microscopic quantities

....

....

....

4

Macroscopic quantities; they describe a total system, e.g.

4

pressure
volume
temperature
electrical and thermal conductivity
magnetism

Microscopic quantities: they describe the properties of individual elements of a system, e.g.

position of a particle
momentum of a particle
velocity of a particle
potential energy of a particle
kinetic energy of a particle

The resistivity of a conductor is a .... quantity.
The vibrational energy of a molecule is a .... quantity.

5

macroscopic
microscopic

5

The next section in the textbook contains many new concepts. Divide it into two or three parts and check your understanding after each part using your notes.

By the way, reading without pen and paper is day-dreaming! This should not be new to you since we have repeated it many times.

We do not propose to give you many more instructions about how to study since you should by now have developed a good technique, one which is the most beneficial for your needs.

Remember then to work with the text and have a break when you feel you need one, quite apart from what we may suggest. When you have a break, make it a real break by doing something very different. Finally, time your break and stick to your timing.

6

20.2 Concept of Probability

6

Objective: Concepts of classical and statistical definition of probability, elementary event, event, evaluation of probabilities.

READ: **20.2.1 Random experiment, outcome space and events**
20.2.2 The classical definition of probability
20.2.3 The statistical definition of probability
Textbook pages 520–523

7

A student can choose 3 books out of 5 (A, B, C, D, E) arbitrarily. What is the outcome space?

....

7

8

The outcome space consists of all possible groups of 3 books (denoted by A, B, C, D, E)
{ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE}

8

Write down the classic definition of probability for the occurrence of an event A:

$$P(A) = \dots\dots\dots\dots\dots\dots$$

9

$$P(A) = \frac{N_A}{N} = \frac{\text{number of elementary events contained in event } A}{\text{total number of possible elementary events}}$$ 9

The classic definition of probability refers to the following situation:

An experiment has N equally possible outcomes and event A consists of N_A of them.
You will need this definition throughout this chapter.

A box contains six balls:

3 black
2 green
1 yellow

If we take out one ball it is
an .... or ....

10

elementary event or a random experiment

10

A box contains six balls:

3 black
2 green
1 white

One ball is taken. How many elementary events are there?

Solution

--------------------▷ 12

Further explanation wanted

--------------------▷ 11

We have to distinguish between 'elementary event' and 'event'. 11

Consider 8 balls in a box: 3 black, 2 green and 3 white. Let us place them side by side thus:

Each ball can be chosen. This represents an *elementary event*. Thus we have 8 elementary events. Now let us consider only the colours. Taking out a ball with a distinct colour constitutes an '*event*', i.e. 'black'. Thus we have 3 events.

A box contains 3 black and 3 white balls. One ball is taken. We have .... elementary events.

--------------------▷ 12

6 elementary events

12

An urn contains 12 balls

6 red
3 green
2 white
1 black

One ball is taken out.

Number of elementary events ....

Number of events if colours are considered ....

13

12 elementary events
4 events

13

A deck of cards consists of 52 cards. They include 4 different kings: clubs, spades, hearts, diamonds. What is the probability of drawing the king of clubs, the deck having been properly shuffled?

$$P(\text{king of clubs}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution 15

Further explanation wanted 14

In the deck of 52 cards there is only 1 king of clubs.

14

Number of 'favourable elementary events' = 1

Number of 'possible elementary events' = 52

Hence $P(\text{king of clubs}) = \frac{1}{52}$

i.e. there is one chance in 52 of getting the king of clubs.

Do you know the exact definition of classic probability?

No? Then read the definition once more in the textbook, with pen and paper at hand.

Yes? Then do the following example:

What is the probability of drawing the king of spades?

$$P(\text{king of spades}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

15

$\frac{1}{52}$

15

Eight cards are lying on the table face down. We know there are 4 different jacks and 4 different queens. What is the probability of drawing the queen of hearts?

$$P = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

16

$\frac{1}{8}$

16

Steps required to compute the classic probability:
Step 1: Determine the number of favourable elementary events N_A.
Step 2: Determine the number of all possible elementary events N.

Then $P(A) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

17

$$P(A) = \frac{N_A}{N}$$ 17

In a drawer there are 10 shirts. Three shirts have a button missing. In the morning we hurriedly grab a shirt.

What is the probability of taking a shirt with all its buttons?

$$P = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

18

$$P = \frac{7}{10}$$

18

It helps to wear a necktie, anyway!

Solve the following problem:

When throwing a die what is the probability that the side with 3 spots will appear?

$P =$ 19

$$P = \frac{1}{6}$$

19

If you are still having difficulties then further exercises will be helpful. Do you wish to do some more exercises?

No, I want to carry on

--------------------▷ 21

Yes; see below

(1) A deck of cards has 26 red cards and 26 black cards. What is the probability of drawing a black card? ....

(2) When throwing a die what is the probability of obtaining a prime number? ....

--------------------▷ 20

20

(1) $P = \frac{26}{52} = \frac{1}{2}$

(2) $P = \frac{3}{6} = \frac{1}{2}$ (prime numbers are 2, 3 and 5)

Hopefully, both your answers were correct!

21

A particular experiment has been carried out 530 times. The result A has been obtained 50 times. The quantity 21

$h_A = \frac{50}{530}$ is called ....

For a very large N the ratio becomes the ....

If you cannot answer straight away refer back to your notes where you should have a list of keywords and their meanings. If that does not help go back to the textbook.

22

relative frequency
statistical probability

22

Which probability can be determined experimentally?
.... probability

To relieve the boredom of a car journey a girl starts counting the numbers of a particular model of car, e.g. the Escort, coming in the opposite direction. She found that out of 144 oncoming cars 8 were Escorts. The relative frequency of Escorts is

$$h_{\text{Escort}} = \text{..........................}$$

23

statistical probability

23

$$h_{\text{Escort}} = \frac{8}{144} = \frac{1}{18}$$

If you have been concentrating hard for some time you should have a few minutes' rest.

24

20.3 General Properties of Probabilities

24

Objective: Concepts of normalisation condition, addition theorem, certain event, impossible event.

READ: 20.2.4 General properties of probabilities
Textbook pages 523–525

25

A box contains 9 white balls. What is the probability of taking out 1 white ball? 25

$$P = \text{..................}$$

Solution 28

Further explanation required 26

The box contains white balls only. If we take out a ball it can only be a white one. This is a *certain event*. A certain event has the probability 1. 26

There are two extreme cases:

A certainty has the probability $P = \dots\dots\dots$

An impossible event has the probability $P = \dots\dots\dots$

27

$$P_{\text{certain}} = 1$$
$$P_{\text{impossible}} = 0$$

27

If a box contains a number of green balls and if you take one the probability that you get a green one is

$$P_{\text{green}} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

28

$$P = 1$$

28

In this case we have a certain event, which has the probability 1.

A box contains 3 white balls and 3 black balls. The probability of taking out a green ball is

$$P_{\text{green}} = \ldots\ldots\ldots$$

29

$$P_{\text{green}} = 0$$

29

An impossible event has the probability 0.

Write down in keywords the meaning of the normalisation condition:

....

....

30

The normalisation condition is:

The sum of the probabilities is 1 when referring to all the events of a defined sample space.

30

What is the probability of drawing out the jack of clubs *or* the king of diamonds *or* the queen of spades in a pack of 52 cards?

$$P = \dots\dots\dots\dots$$

--------------------▷ 31

$$P = \frac{3}{52}$$

31

An urn contains 12 balls:

6 red
4 white
1 green
1 black

The probability of drawing either a white or a green ball is

$$P(\text{white or green}) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution --------------------▷ 33

Further explanation wanted --------------------▷ 32

Consider this example. 32

Given: 6 lottery tickets with 2 first prizes, 2 consolation prizes and 2 blanks.

Probability of a first prize $P\left(\mathrm{F}\right)=\frac{2}{6}=\frac{1}{3}$

Probability of a consolation prize $P\left(\mathrm{C}\right)=\frac{2}{6}=\frac{1}{3}$

Probability of a first prize *or* a consolation prize is $P\left(\mathrm{F}\right)+P\left(\mathrm{C}\right)=\frac{2}{3}$

This was an example of the application of the *addition theorem*.

An urn contains 12 balls:

6 red
4 white
1 green
1 black

The probability of drawing either a white or a green ball is

$$P\,(\text{white or green}) = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 33

$$P\,(\text{white or green}) = \frac{5}{12}$$ 33

This was one of the applications of the addition theorem. It is applicable if the required probability is for one of a number of events, the events being independent.

34

20.4 Probability of Compound Events Which are Statistically Independent

34

Objective: Concepts of statistically independent events, compound probability, evaluation of compound probability of statistically independent events.

READ: 20.2.5 Probability of statistically independent events; compound probability
Textbook pages 525–527

35

Using keywords, write down the definitions for

35

(a) compound probability

....

....

and (b) statistically independent events

....

....

36

The definitions are: 36

(a) Probabilities of the simultaneous occurrence of two (or more) events are called compound events.
(b) Consider two groups of events, A and B. If the occurrence of events in group A is not influenced by the occurrence or non-occurrence of events in group B then the events of group A are statistically independent of the events in group B.

What is the probability that when throwing two dice the number 12 will show?
This means that each die must show the number 6.

$$P = \ldots\ldots$$

Solution --------------------▷ 38

Further explanation required --------------------▷ 37

You appear to be unsure whether you should apply the addition theorem or the compound probability theorem. 37

(1) The addition theorem is valid for questions concerning the occurrence of one event *or* the other. Each event excludes the other; the two are disjoint.
(2) Compound probabilities are valid for questions concerning the occurrence of one event *and* the other.

The problem was: what is the probability, when throwing two dice, that the number 12 will show? This event can only occur if the first die shows a 6 *and* the second die shows a 6.

The probability for this to occur is $P = \ldots\ldots$

38

$$\frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$$

38

Do the following exercise:

A coin is tossed twice. What is the probability that on each occasion it will be a tail?

$$P = \ldots\ldots$$

Solution --------------------▷ 40

Further explanation wanted --------------------▷ 39

The probability of obtaining a tail when tossing a coin is $\frac{1}{2}$. The probability of getting a tail twice in succession must be less than $\frac{1}{2}$. 39

The compound probability of statistically independent events A and B is

$$P_{\mathrm{AB}} = P_{\mathrm{A}} \times P_{\mathrm{B}}$$

Therefore the probability of two tails in succession is $P_{\mathrm{TT}} = \ldots\ldots$

40

$$\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$$

40

A box contains 18 balls

5 yellow
4 black
7 green
2 white

If we take out one ball, there are
.... elementary events
.... events with colours

41

18 elementary events
4 events with colours

41

The box with 18 balls is made up of:

5 yellow
4 black
7 green
2 white

We now take out 3 balls in succession, and each time we put the ball back in the box. The probability of the compound event

1 black
1 green
1 white is $P(\text{bgw}) =$ ....

The probability of the compound event

1 yellow
1 black
1 green is $P(\text{ybg}) =$ ....

42

$$P\,(\text{bgw}) = \frac{4}{18} \times \frac{7}{18} \times \frac{2}{18} = \frac{7}{729} \approx 0.01$$
$$P\,(\text{ybg}) = \frac{5}{18} \times \frac{4}{18} \times \frac{7}{18} = \frac{35}{1458} \approx 0.024$$

42

If you got the wrong results for both, then a revision will help.

--------------------▷ 43

20.5 Permutations and Combinations

Objectives: Concepts of permutations, factorials, evaluating the number of permutations, computing factorials, evaluating the number of permutations of elements some of which are identical.

READ: **20.3 Permutations and combinations**
20.3.1 Permutations
Textbook pages 527–528

--------------------▷ 44

44

Five friends are sitting on a bench in the following order:

Alice, Betty, Caroline, Doreen, Evelyn.

This is obviously a possible arrangement of the five, referred to as elements for short by mathematicians.

One possible arrangement is called ....

45

a permutation 45

Write down all the permutations of the three elements of x, y, z. 46

$$\begin{array}{ccc} xyz & yzx & zxy \\ xzy & yxz & zyx \end{array}$$

46

A permutation is a .... arrangement of ....
For three elements there are ...... permutations.

47

A permutation is a possible arrangement of a number of elements.
For three elements there are $3! = 6$ permutations

47

The symbol $N\,!$ is called ....
and it means, $N\,! =$ ....

48

$N!$ is called 'factorial N' or 'N factorial'.

48

$$N! = 1 \times 2 \times 3 \times \ldots \times (N-1) \times N$$

Compute

0! = ...

1! = ...

2! = ...

3! = ...

4! = ...

5! = ...

6! = ...

49

49

$$0! = 1$$
$$1! = 1$$
$$2! = 1 \times 2 = 2$$
$$3! = 1 \times 2 \times 3 = 6$$
$$4! = 1 \times 2 \times 3 \times 4 = 24$$
$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$
$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$$

Let us go back to our five friends Alice, Betty, Caroline, Doreen, Evelyn.

They wish to sit on a bench in a park.

In how many ways can they sit on the bench, i.e. how many permutations are there of five different elements?

There are .... permutations.

--------------------▷ 50

120 50

The number of permutations of five different elements is 5!

$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

How many different arrangements are possible with 5 equal elements? ....

51

Exactly ONE,
since, if we have *AAAAA*, interchanging always results in *AAAAA*. 51

How many different arrangements are there with five elements *aabbc*?

Arrangements which result in interchanging an element a with another a, or b with another b, are considered to be equal.

Hence there are .... different arrangements.

Solution ---> 53

Further explanation wanted ---> 52

The total number of arrangements of 5 distinct elements is given by the number of permutations of 5 elements: $5! = 120$. 52

In our case among the 120 permutations there exist some that differ only in that a pair of elements a or b has been interchanged. Thus the number of different arrangements is smaller.

Read section 20.3.1 once more and then try again.

How many different permutations are there with the 5 elements *aabbc*? ......

--------------------▷ 53

$$\frac{5!}{2!2!} = 30$$

53

54

20.6 Combinations

54

Objective: Concepts of combinations, binomial coefficient, determination of the number of combinations and evaluation of binomial coefficients.

READ: 20.3.2 Combinations
Textbook pages 528–530

55

How is the binomial coefficient defined?

55

$$_nC_k = \binom{n}{k} =$$

....

56

$$_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$ 56

Calculate:

(a) $\binom{3}{2} = \dots\dots\dots\dots\dots\dots$ (b) $\binom{5}{3} = \dots\dots\dots\dots\dots\dots$

(c) $\binom{5}{5} = \dots\dots\dots\dots\dots\dots$ (d) $\binom{4}{1} = \dots\dots\dots\dots\dots\dots$

Solutions

----------------▷ 59

Explanation wanted

----------------▷ 57

Take one more look at the definition of the binomial coefficient

57

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Example: $\binom{6}{4} = \frac{6!}{4!2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1) \times (2 \times 1)} = \frac{6 \times 5}{2} = 15$

Compute:

(a) $\binom{6}{5} =$ ....

(b) $\binom{6}{1} =$ ....

(c) $\binom{6}{2} =$ ....

58

(a) $\binom{6}{5} = \frac{6!}{5!\,(6-5)!} = \frac{6!}{5!\,1!} = 6$

58

(b) $\binom{6}{1} = \frac{6!}{1!(6-1)!} = 6$

(c) $\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15$

Now compute $\binom{3}{2} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\binom{5}{3} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$\binom{5}{5} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ $\binom{4}{1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

--------------------▷ 59

$$\binom{3}{2} = 3; \qquad \binom{5}{3} = 10; \qquad \binom{5}{5} = 1; \qquad \binom{4}{1} = 4$$

59

A group of 3 elements is to be formed out of 5 different elements.
How many ways to form a group are there? ....

60

There are $\binom{5}{3}$ different ways to form a group of three out of 5 different elements: 60

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = 10$$

A club has 20 members. A committee is to be formed consisting of 5 members.
How many possibilities are there of forming such a committee?

61

There are $\binom{20}{5}$ possibilities to form a committee of 5 members out of 20 members; i.e. it is a combination of 20 elements, taking 5 at a time. 61

Hence $$\binom{20}{5} = \frac{20!}{5!\,15!} = \frac{20 \times 19 \times 18 \times 17 \times 16}{1 \times 2 \times 3 \times 4 \times 5} \approx 15,000$$

We have come to the end of Chapter 20. But before leaving it altogether would you like to solve one final problem, on clairvoyance?

Yes --------------------▷ 62

No --------------------▷ 68

Here is the problem:

A researcher investigates whether some people are clairvoyant. For this purpose he sets 500 people the task of guessing the result of an experiment. 62

A coin is tossed 10 times behind a screen. Each person has to guess whether the outcomes are heads or tails.

It is assumed that people who make one mistake only in their forecasts are gifted with clairvoyance.

Do you think that such an experiment is suitable in order to discover whether a person is clairvoyant?

☐ Can you answer the question?

66

☐ Do you want a hint?

63

The problem can be solved with the help of probability calculus. 63

First hint: Determine the probability that no one amongst the 500 people satisfies the condition.

A further hint is required --------------------▷ 64

I can carry on --------------------▷ 66

Second hint: In order to determine the probability that no one amongst the 500 people satisfies the condition we have to determine the probability that a particular person will be correct at least 9 times. 64

Use the classic definition of probability

$$P = \ldots\ldots\ldots\ldots$$

I still have difficulties 65

I want to check my solution 66

For a particular person the number of possible forecasts is $2 \times 2 \ldots 2 \times = 2^{10} = 1024$. Number of favourable outcomes $= 10 + 1 = 11$. 65

Reasons:

A favourable outcome exists if there is at most one mistake in 10 forecasts. This mistake can occur in the first, second, ..., tenth number. There are 10 cases $= \binom{10}{1}$. A favourable case also exists if there is no mistake. This one case $= \binom{10}{0}$.

It therefore follows that $P = \frac{11}{2^{10}} \approx 10^{-2}$

The probability that one person will make more than one mistake is then $(1 - P) = (1 - 0.01) = 0.99$. The probability that all 500 people make more than one mistake is $(1 - 0.01)^{500} = 0.006$. This means that it is most likely that at least one person satisfies, by chance $(1 - 0.006) = 0.994$, the condition of '1 error at best'.

------------------▷ 66

The experiment is not suitable for testing the hypothesis. 66

According to the random law it is almost certain that at least one person in the sample will satisfy the condition.

The probability that at least one person satisfies by chance the given condition is 0.994.

To develop the problem a little further, let us assume that the researcher investigates a particular person. He starts with the assumption that at least 8 correct scores are sufficient in order to prove that a person has clairvoyant abilities. Is this assumption justified?

(Assume that the random probability of the occurrence of such an event has to be smaller than 0.01, i.e. smaller than 1%.)

--------------------▷ 67

He is also mistaken in this case.

67

There are $\binom{10}{2} + \binom{10}{1} + \binom{10}{0} = 56$ favourable events (8 or 9 or 10 correct scores).

Hence $P\,(x \geq 8) = \dfrac{56}{2^{10}} \approx 0.055$.

Thus the random probability is greater than 0.01.

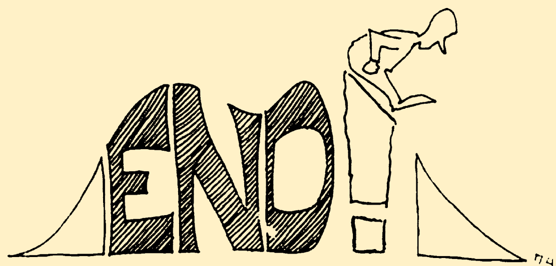
68

68

For the solution of this problem we have used

— the classic definition of probability
— the addition theorem of probability
— the compound probability for independent events
— binomial coefficients

In the next chapter you will encounter probability distributions, which will enable you to deal with the solution of similar problems.



OF CHAPTER 19

Chapter 21
Probability Distributions

21.1 Discrete Probability Distributions

2

Objective: Concepts of random variable, discrete probability distributions, determination of the probability of a random variable from a random experiment.

This chapter requires knowledge of the preceding one.

READ: 21.1.1 Discrete probability distributions
Textbook pages 531–534

3

A random experiment consists in tossing a coin. We choose as the random variable x the event 'tail'. Give the probability distribution of the random variable x in the form of a table

3

Random variable x	Probability P

Solution

5

Explanation wanted

4

The sample space consists of the elements ‘head’ and ‘tail’. The values of the random variable $x = 0$ and $x = 1$ are assigned to these respectively. Both events occur with the probability 4

$$P = \frac{1}{2}$$

Hence the above table for the probability distribution reads

Random variable x	Probability P

5

Random variable x	Probability P
0	$\frac{1}{2}$
1	$\frac{1}{2}$

5

The random experiment now consists of the simultaneous tossing of three coins. We choose (arbitrarily) for the random variable x the number of coins showing 'head' minus the number of coins showing 'tail'.

We have to determine the probability distribution of the random variable x.

Random variable x	Probability P

Solution ----------------▷ 8

Explanation wanted ----------------▷ 6

Random experiment: tossing 3 coins
Random variable: $x = x_{\text{head}} - x_{\text{tail}}$
Possible elementary occurrences (first coin, second coin, third coin):

(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT).

The random variable x for the outcome HTH is for example: $x = 2 - 1 = 1$

Each of the elementary occurrences had the probability $\frac{1}{8}$.

6

7

Now give the probability distribution:

7

Outcome	Random variable x	Probability $P(x)$
HHH	......	......
HHT	......	......
HTH	......	......
THH	......	......
HTT	......	......
THT	......	......
TTH	......	......
TTT	......	......

8

Random variable x	Probability $P(x)$
3	$\frac{1}{8}$
1	$\frac{3}{8}$
-1	$\frac{3}{8}$
-3	$\frac{1}{8}$

8

9

In section 21.1.1 of the textbook we dealt with the case of two dice in which the random variable was the 'sum of the number of spots'. 9

Now use as the random variable x the number of spots on the first die minus the number of spots on the second die, and determine the probability distribution.

Random variable x	Probability $P(x)$

Solution 11

Explanation wanted 10

10

Random experiment: throwing two dice.
Random variable

$$x = (\text{Number of spots on first die}) - (\text{Number of spots on second die}).$$

To help you we give below three values of the random variable with their outcomes and probabilities.

<table>
<tr><th colspan="2">Outcome</th><th>Random</th><th>Probability</th></tr>
<tr><th>1st die</th><th>2nd die</th><th>variable x</th><th>$P(x)$</th></tr>
<tr><td>1</td><td>6</td><td>-5</td><td>$1 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$</td></tr>
<tr><td>1</td><td>3</td><td rowspan="4">-2</td><td rowspan="4">$4 \times \frac{1}{6} \times \frac{1}{6} = \frac{4}{36}$</td></tr>
<tr><td>2</td><td>4</td></tr>
<tr><td>3</td><td>5</td></tr>
<tr><td>4</td><td>6</td></tr>
<tr><td>4</td><td>1</td><td rowspan="3">3</td><td rowspan="3">$3 \times \frac{1}{36} = \frac{3}{36}$</td></tr>
<tr><td>5</td><td>2</td></tr>
<tr><td>6</td><td>3</td></tr>
</table>

Now complete the table of frame 9.

11

11

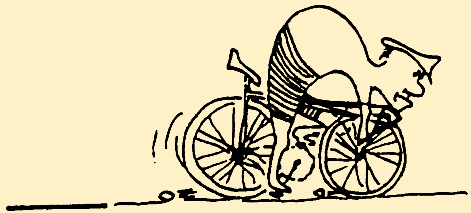
Random variable x	Probability $P(x)$
-5	$\frac{1}{36}$
-4	$\frac{2}{36}$
-3	$\frac{3}{36}$
-2	$\frac{4}{36}$
-1	$\frac{5}{36}$
0	$\frac{6}{36}$
1	$\frac{5}{36}$
2	$\frac{4}{36}$
3	$\frac{3}{36}$
4	$\frac{2}{36}$
5	$\frac{1}{36}$

12

Here is a funny problem. 12

George maintains he can distinguish between two sorts of beer A and B by tasting them. Henry does not believe him.

Can you think of a possible experimental set-up which would be suitable for checking this statement?

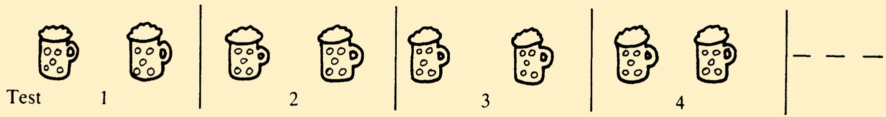


Solution --------------------▷ 14

Hint --------------------▷ 13

13

A suitable experiment could look like this. George tries to identify the beer by carrying out the following tests:



For each test both sorts of beer are offered for identification.

How many tests are necessary to enable George to prove his statement and convince Henry? (The probability that the result could occur accidentally should be less than 0.01.) ......

14

7

14

George tries to identify the kind of beer by a test. For each test both sorts of beer are offered for identification. The test is repeated.

Number of test and correct identification	1 Yes	2 Yes	3 Yes	4 Yes	5 Yes	6 Yes	7 Yes	... ...
Random probability $P(n) = \left(\frac{1}{2}\right)^n$	0.5	0.25	0.13	0.06	0.03	0.016	0.0078	...

The probability that in seven consecutive tests George hits accidentally on the right beer is 0.0078. This is less than 0.01.

--------------------▷ 15

Assume you have a test on the content of the present study guide containing ten questions and for each question four solutions are given. 15

Your task is to find the right one, assuming that you have not yet studied the chapter on probability. You are nevertheless determined to do the test since there is a chance you may find the correct solution. The test is considered successful if you have scored at least 80%.

What is your chance of achieving this result by accident? $P\,(\text{accident}) = \ldots\ldots$

Hint

--------------------▷ 16

Solution

--------------------▷ 20

The random variable x for one question can take on two values: 1 — correct; 0 — wrong. 16

The events (solution of the questions) take place independently of each other.

For one question there are given four solutions.

The probablity of ticking the correct solution of one question by accident is

$$P(x = 1) = \ldots\ldots$$

The probability of ticking the wrong solution of one question by accident is

$$P(x = 0) = \ldots\ldots$$

--------------------▷ 17

17

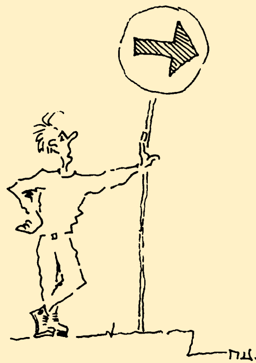
$$P(x = 1) = \frac{1}{4}$$
$$P(x = 0) = \frac{3}{4}$$

The probability of ticking 10 correct solutions accidentally is

$$P(z = 10) = \ldots\ldots$$

The probability of ticking 9 correct solutions accidentally is

$$P(z = 9) = \ldots\ldots$$



18

$$P(z=10) = \left(\tfrac{1}{4}\right)^{10} = 0.000\,001$$

18

$$P(z=9) = \left(\tfrac{1}{4}\right)^{9} \times \tfrac{3}{4} \times 10 = 0.000\,03$$

Reason for the factor 10: 9 correct and 1 wrong solution can be achieved in 10 different ways.

Generally, the probability of obtaining $z = a$ correct solutions is

$$P(z=a) = \binom{N}{a} P(x=1)^{a} P(x=0)^{N-a}$$

$$P(z=8) = \ldots\ldots$$

This expression is identical with the binomial distribution, whose derivation you will find in the textbook.

19

$$P(z=8)=\binom{10}{8}\left(\tfrac{1}{4}\right)^{8}\left(\tfrac{3}{4}\right)^{2}=0.0004$$

19

The probability of obtaining at least 80% correct solutions when 10 problems are given is

$$P(z\geq 8)=P(z=10)+P(z=9)+P(z=8)=\ldots\ldots$$

20

$$P(z \geq 8) = 0.0004$$

20

When you have worked it out you will discover that the probability of achieving 80% right solutions by accident is deplorably small; it is 0.0004. It is therefore better to study the subject first!

21

21.2 Continuous Probability Distributions

21

Objective: Concepts of continuous probability distributions, probability density function.

READ: **21.1.2 Continuous probability distributions**
Textbook pages 534–537

22

Given: the probability density function

22

$$\phi(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}$$

What is the probability for this function when $x = 2$? Attention please! Note the definition of the probability density function:

$$P(x = 2) = \ldots\ldots$$

Solution

--------------------▷ 25

Explanation wanted

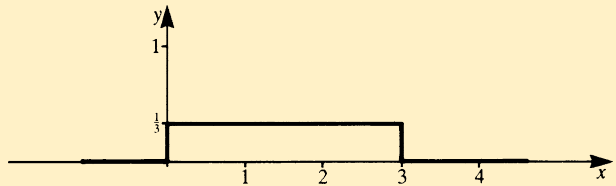
--------------------▷ 23

For a continuous random variable the probability is always zero if the random variable assumes a particular value. A probability which differs from zero can only be given for a finite interval of the random variable. 23

Solve the following problem.

Given: the probability density function

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



What is the probability when $x = 1$?

$$P = \ldots\ldots$$

What is the probability $P(2.0 \leq x \leq 2.5)$ in the range $2 \leq x \leq 2.5$?

$$P = \ldots\ldots$$

24

$$P(x=1)=0$$
$$P(2\le x\le 2.5)=\frac{1}{6}$$

24

Detailed solution:

$$P=\int_2^{2.5} f(x)\,\mathrm{d}x=\int_2^{2.5}\frac{1}{3}\,\mathrm{d}x=\frac{1}{3}(2.5-2.0)=\frac{1}{6}$$

Given: the probability density function $\phi(x)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\frac{x^2}{2}}$

What is the probability when $x=2$?

$$P(x=2)=\ldots\ldots..$$

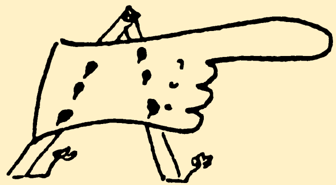
25

$$P(x = 2) = 0$$

25

Remember: a probability which differs from zero can only be given for a *finite interval* of the continuous random variable.

In case of doubt read the explanation given in frame 23.



26

21.3 Mean Value

26

Objective: Concept of arithmetical mean value for discrete and continuous variables.

READ: 21.2 Mean values of discrete and continuous variables
Textbook pages 537–539

27

What are the different forms of the arithmetic mean value?

27

(a) $\bar{x} =$ .... (finite number of values)
(b) $\bar{x} =$ .... (discrete random variable)
(c) $\bar{x} =$ .... (continous random variable)

28

(a) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$ $n =$ number of values

28

(b) $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} n_i x_i,$ $n_i =$ frequency of values x_i

(c) $\bar{x} = \int_{x_1}^{x_2} x f(x)\,dx,$ $f(x) =$ probability density function

---

When measured, a physical quantity was found to have the following values:

x_i	x_1	x_2	x_3	x_4	x_5	x_6
	2.9	3.1	3.5	3.5	3.7	4.1

The mean value is $\bar{x} =$ ....

29

$$\bar{x} = 3.47$$

29

Given 20 measured values (set A):

1.2	1.0	1.1	1.3
1.1	1.2	1.2	1.1
1.4	1.3	1.3	1.1
1.2	1.2	1.4	1.1
1.2	1.0	1.2	1.4

With these values draw up a frequency table.

Obtain the frequencies and the relative frequencies.

The first step is the preparation of the table.

--------------------▷ 30

30

Measured value x	Frequency f	Relative frequency f_r

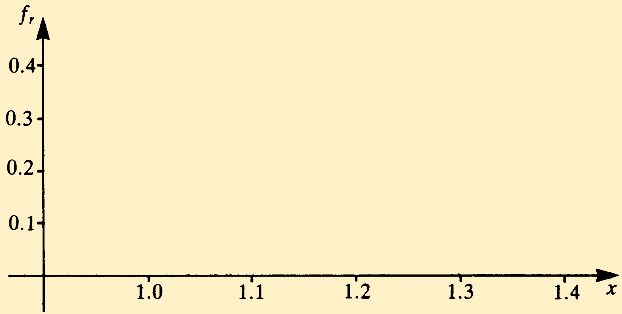
This is the usual form for a frequency table.
Draw up this table and insert the values.

31

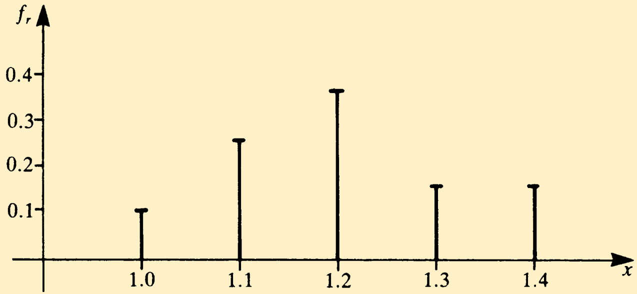
31

x	f	f_r
1.0	2	0.10
1.1	5	0.25
1.2	7	0.35
1.3	3	0.15
1.4	3	0.15

Fill in the frequency distribution on the diagram.



32



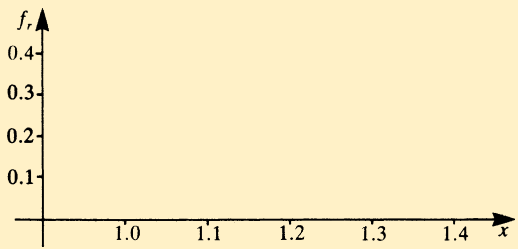
32

Set A

Given: 20 other measured values:

We have already computed the frequency distribution. Plot it in the diagram below.

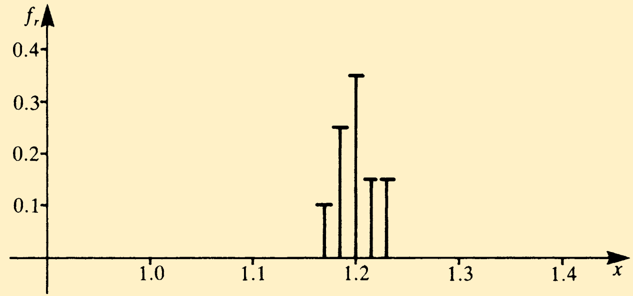
x	Frequency f	Relative frequency f_r
1.18	2	0.10
1.19	5	0.25
1.20	7	0.35
1.21	3	0.15
1.22	3	0.15



33

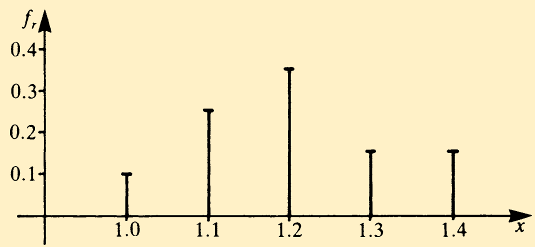
33

Set B



The frequency distribution for this Set B of measured values has the same mean value as the previous example Set A. The previous example is shown below.

Set A



Which do you think is the more reliable measurement? Set ......

34

Set B

34

Given the probability distribution $P(1), \ldots, P(k)$ for the set of values $x_1, \ldots, x_k$ of the random variable x, the mean value is defined as

$$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

35

$$\bar{x} = \sum P(i)\, x_i \quad i = 1, 2, \ldots, k$$

35

Here are the results of some measurements:

x_i	Relative frequency
4	0.1
5	0.3
6	0.4
7	0.2

The mean value is $\bar{x} = \ldots\ldots\ldots$

36

$$\begin{aligned}\bar{x} &= P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 \\ &= 0.4 + 1.5 + 2.4 + 1.4 = 5.7\end{aligned}$$

36

What is the mean number (of spots) when throwing a die?

Mean number $= \ldots\ldots\ldots$

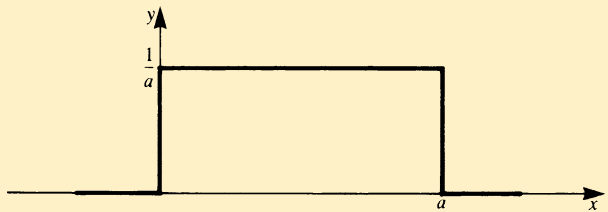
37

$$\text{Mean number} = \tfrac{1}{6} \times 1 + \tfrac{1}{6} \times 2 + \tfrac{1}{6} \times 3 + \tfrac{1}{6} \times 4 + \tfrac{1}{6} \times 5 + \tfrac{1}{6} \times 6 = 3.5$$

37

A random variable has the following probability density function:

$$f(x) = \begin{cases} \dfrac{1}{a} & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$



Calculate the mean value of the random variable x.

$$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

38

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x = \int_{0}^{a} x \frac{1}{a} \mathrm{d}x = \frac{a}{2}$$ 38

Correct, and I want to carry on → 42

Explanation, or another exercise wanted → 39

The mean value $\bar{x}$ of a continuous random variable with the probability density function $f(x)$ is defined by

39

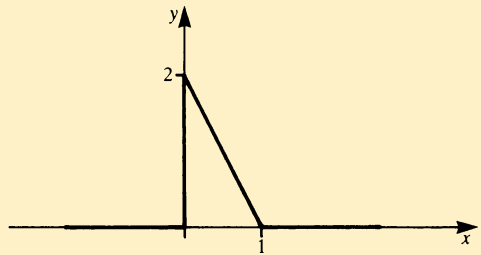
$$\bar{x} = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

The limits of integration in a particular case are determined by the range of definition of the random variable x.

What is the mean value of the random variable x whose probability density function is defined below

$$f(x) = \begin{cases} 2(1-x) & \text{for} \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$



40

$$\bar{x} = \frac{1}{3}$$

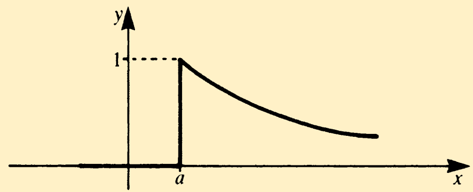
40

Solution:

$$\bar{x} = \int_{-\infty}^{0} x \times 0 \times \mathrm{d}x + \int_{0}^{1} x2(1-x)\mathrm{d}x + \int_{1}^{\infty} x \times 0 \times \mathrm{d}x = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2 \times \frac{1}{6} = \frac{1}{3}$$

Given the probability density function

$$f(x) = \begin{cases} \mathrm{e}^{-(x-a)} & \text{for} \quad a \leq x \\ 0 & \text{otherwise} \end{cases}$$



Is the normalisation condition

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1 \text{ satisfied?}$$

No: compute the normalisation factor.

Yes: give the mean value.

$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

41

The probability density function $f(x)$ is normalised since

41

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{a}^{\infty} \mathrm{e}^{-(x-a)}\mathrm{d}x = \left[-\mathrm{e}^{-(x-a)}\right]_a^{\infty} = 1$$

$$\bar{x} = \int_{-\infty}^{\infty} xf(x)\mathrm{d}x = \int_{a}^{\infty} x\mathrm{e}^{-(x-a)}\,\mathrm{d}x$$

We integrate by parts and obtain

$$\bar{x} = \left[-x\mathrm{e}^{-(x-a)}\right]_a^{\infty} - \left[\mathrm{e}^{-(x-a)}\right]_a^{\infty} = a+1$$

We met this type of probability density function in the textbook when we determined the probability of an air molecule in the atmosphere being in a particular region.

42

21.4 The Normal Distribution as the Limiting Value of the Binomial Distribution

42

Objective: Concepts of binomial distribution, normal distribution (Gaussian distribution), application of the binomial distribution.

READ: **21.3 The normal distribution as the limiting value of the binomial distribution**
Textbook pages 539–545

-------------------▷ 43

Tick the exercises that can be solved by applying the binomial distribution. If in doubt consult the textbook.

43

(a) ☐ 5 dice are thrown. What is the probability that 3 dice show an even number of spots?
(b) ☐ One die is thrown 6 times. What is the probability that an even number is thrown each time?
(c) ☐ A box contains 1 white ball and 2 red balls. One ball is taken out and after that another one. What is the probability that both balls are red?
(d) ☐ How many possibilities are there of drawing one red card out of a pack of cards?

--------------------▷ 44

(a) and (b)

44

---

5 dice are thrown. What is the probability that 3 dice show an even number?

It is important that you substitute the correct values in the binomial formula. You can find it in the textbook. First calculate

$$n = \ldots\ldots.$$

$$k = \ldots\ldots.$$

$$P = \ldots\ldots.$$

45

$$n = 5$$
$$k = 3$$
$$P = \frac{1}{2}$$

45

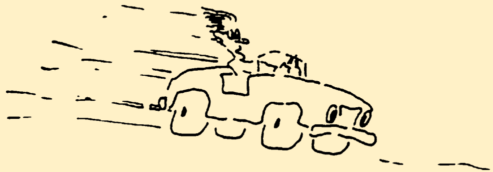
Now we insert these values into the formula:

$$P(5,\ \ 3) = \ldots\ldots$$

46

$$P(5,3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16} \approx 0.3$$

46



47

21.5 Properties of the Normal Distribution

47

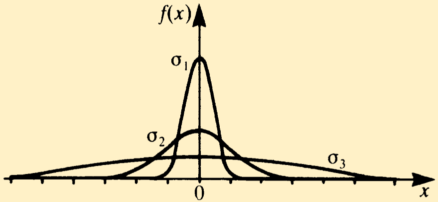
Objective: Classification of normal distributions according to the parameters σ and μ.

READ: **21.3.1 Properties of the normal distribution**
21.3.2 Derivation of the binomial Distribution
Textbook pages 542–545

48

The following sketch shows three normal distributions. What parameter defines the different shapes of the curves?

48



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Parameter: ......

Arrange the parameters in descending order of magnitude:

...... > ...... > ......

49

The parameter is σ (standard deviation) in $f(x)$.

$$\sigma_3 > \sigma_2 > \sigma_1$$

49

Given the normal distribution $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$

What is the mean value of the random variable x?

$$\bar{x} = \ldots\ldots$$

Hint: you do not have to calculate it, but think!

50

$$\bar{x} = 0$$

50

The normal distribution has its maximum at $x = 0$ and it is symmetrical about that point. Therefore $\bar{x} = 0$.

Any probability distribution symmetrical about $x = 0$ has a mean value $\bar{x} = 0$.

The random variable x has the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

What is the mean value of the random variable x?

$$\bar{x} = \ldots\ldots$$

Think!

51

$$\bar{x} = \mu$$

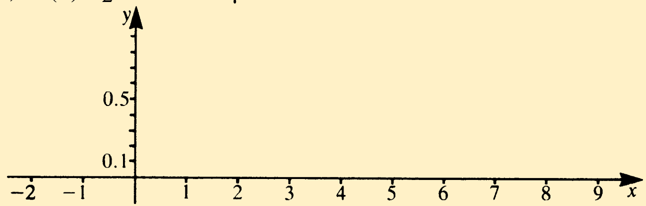
51

(See the textbook if in doubt.)

Familiarity with the normal distribution can only be acquired through exercises. Sketch two normal distributions

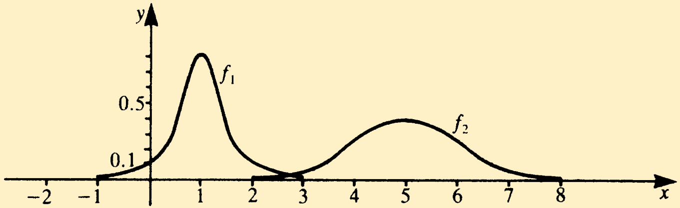
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

(a) $\sigma_1 = 1$ and $\mu = 5$; (b) $\sigma_2 = 0.5$ and $\mu = 1$



Hint: use the following estimate: $\frac{1}{2\sqrt{\pi}} \approx \frac{1}{2.5} = 0.4$

52



52

It was important for you to sketch the curves using just a few values. The differences between the two curves are the positions of the mean values and the variances.

53

If you are not an enthusistic mathematician — these are very rare — then you have worked very hard throughout this somewhat difficult topic, but your persistence will pay off! 53

OF CHAPTER 20

Chapter 22
Theory of Errors

Contents First Page Previous View Next View Print Exit

The theory of errors requires the knowledge gained from Chapter 20. A brief recapitulation is advisable. Name at least three concepts from Chapter 20.

....

....

....

2

3

Discrete probability distribution
Continuous probability distribution
Binomial distribution
Normal distribution
Mean value
Standard deviation

3

A coin is tossed 5 times. What is the probability $P(5;3)$ that a head will appear 3 times? (Head and tail are equally probable events.)

$$P(5;3) = \ldots\ldots$$

Hint: you can solve this exercise with the help of the binomial distribution.

4

$$P(5;3) = \binom{5}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^2 = 0.3125$$

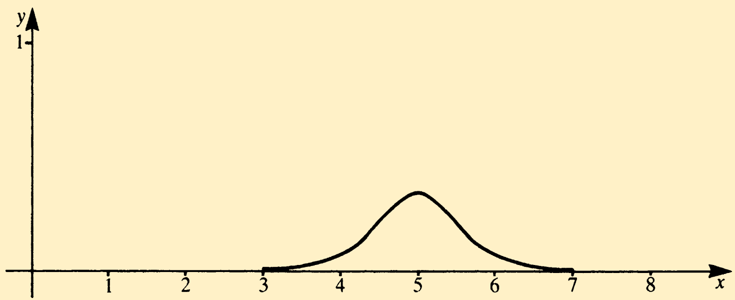
4

Given the probability density function of the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Its graph is shown below for $\sigma = 1$ and $\mu = 5$.

Sketch the graph of this function for $\sigma = \frac{1}{2}$ and $\mu = 5$ in the same coordinate system.



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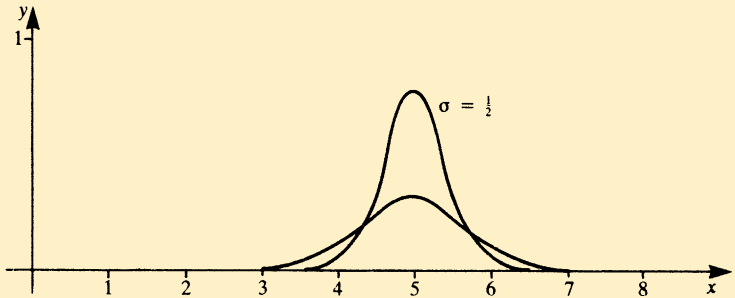


5

This exercise should have made it clear that the dispersion of the normal distribution is fixed by the parameter ....

and the mean value is given by ......

6

standard deviation σ
μ

6

And now let us go on to the theory of errors.

7

22.1 Purpose of the Theory of Errors, Mean Value, Variance

7

Objective: Concepts of systematic errors, random errors, variance, standard deviation, random sample, parent population, calculation of the standard deviation and variance for a series of readings.

READ: 22.1 Purpose of the theory of errors
22.2 Mean value and variance
Textbook pages 549–552

Suggestion: It will be beneficial to do the exercises in 22.1 and 22.2 in parallel with your studies.

--------------------▷ 8

All measurements are subject to random errors. In addition, systematic (constant) errors can also occur. 8

A tape measure is used to obtain the length of a room.

(a) A tape measure is elongated by heavy use and its true length is now 1004 mm instead of 1000 mm. The error in measurement is ....

(b) A measurement is carried out using a flexible steel tape which has to be set several times. The datum lines are marked on the floor with a pencil. Wrong setting will result in .... errors.

(c) At the wall the steel tape has to be bent, making it difficult to read; this will result in .... errors.

9

(a) systematic or constant
(b) random
(c) random

9

Correct 13

Help required 10

Let us make clear once more the difference between systematic (constant) and random errors: 10

Systematic errors arise as a result of, for example, inaccurate calibration or inherent errors in measuring instruments or faulty measuring techniques.

Examples: If the diameter of a hose is measured using a pair of callipers then the hose will be slightly deformed each time because of the pressure exerted. The result is an error in the measured diameter. If a tape measure is elongated from heavy use then the measurements will be too small.

Systematic or constant errors usually falsify the measurement in one direction only giving either too high or two low a value.

The characteristic of random errors is that they are subject to unpredictable statistical fluctuations. Hence the results will turn out to be too high on one occasion and too low on another.

11

What types of error will occur in the following situations? 11

(a) Unless an instrument with a pointer is read full face on a parallax error results. It is a .... error.
(b) The divisions of the scale of an ammeter are wide (thick) so that a reading has to be estimated. For the same position of the pointer different people will give different readings. The errors are ....
(c) The temperature of a very small mass of liquid is measured using a mercury thermometer. The thermometer absorbs heat from the liquid, hence its temperature decreases; the error is ....
(d) Scales not positioned horizontally give rise to a .... error.
(e) When weighing very small objects on a precision balance a draught will upset the reading because the pan will not always settle in the same position, giving rise to .... errors.

--------------------▷ 12

(a) systematic
(b) random
(c) systematic
(d) systematic
(e) random

12

13

The volume of a necklace with a pendant is determined by the overflow method. A container is filled with a liquid and the necklace is immersed in it. The liquid overflows into a groove and is collected in a measuring cylinder. 13

The test is repeated 10 times. We want to calculate the mean value, variance and standard deviation for this series of measurements.

Data:

Volume (unit of measurement: $10^3\,\mathrm{mm}^3$)

2.4, 2.7, 2.6, 2.5, 2.4, 2.6, 2.7, 2.6, 2.8, 2.7

First calculate the mean value

$$\bar{x} = \ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 14

$$\bar{x} = 2.6 \times 10^3\,\mathrm{mm}^3$$

14

To calculate the variance and standard deviation we can draw up a table and compute the deviations of the measurements about the mean and the squares of these deviations.

15

15

Complete the table:

Measured values x_i $[10^3\,\mathrm{mm}^3]$	Deviation from the mean $(x-\bar{x})$ $[10^3\,\mathrm{mm}^3]$	$(x-\bar{x})^2$ $[10^6\,\mathrm{mm}^6]$
2.4		
2.7		
2.6		
2.5		
2.4		
2.6		
2.7		
2.6		
2.8		
2.7		

It is advisable to repeat the calculations. Use your calculator.

16

Here is the completed table. We shall use it again.

16

x_i $[10^3\,\mathrm{mm}^3]$	$(x_i - \bar{x})$ $[10^3\,\mathrm{mm}^3]$	$(x_i - \bar{x})^2$ $[10^6\,\mathrm{mm}^6]$
2.4	−0.2	0.04
2.7	0.1	0.01
2.6	0.0	0.00
2.5	−0.1	0.01
2.4	−0.2	0.04
2.6	0.0	0.00
2.7	0.1	0.01
2.6	0.0	0.00
2.8	0.2	0.04
2.7	0.1	0.01

17

We must be careful when calculating the variance because we have to distinguish between variance of the random sample and the estimated variance for the parent population. 17

Variance of the random sample: $s^2 =$ ....

Estimated variance of the parent population: $\sigma^2 =$ ....

18

$$s^2 = \frac{0.16 \times 10^6}{10} = 0.016 \times 10^6\,\mathrm{mm}^6$$

$$\sigma^2 = \frac{0.16 \times 10^6}{9} = 0.018 \times 10^6\,\mathrm{mm}^6$$

18

Finally, calculate the best estimate of the standard deviation of the test data $\sigma =$ ....

19

$$\sigma = 0.13 \times 10^3\,\text{mm}^3$$ 19

Try to formulate in your own words (keywords) the significance of the standard deviation.

....
....
....

20

20

You could have written:

The individual values are dispersed about the mean value; the standard deviation is a measure of the dispersion of the values from the mean. The smaller the value of the standard deviation the closer the measured values are to the mean, and vice versa.

68% of the test values have a deviation less than $\pm\sigma$ (1 standard deviation) and about 32% have a deviation greater than $\pm\sigma$. These values are valid for random errors which are normally distributed as explained in section 22.5.

Calculation of the mean value and standard deviation is a routine task but we have to pay attention to the units used. It is advisable to use some standard method for carrying out these calculations.

--------------------▷ 21

The diameter of a wire has been measured 5 times. The results are given in the table: 21

d_i $[10^{-2}\mathrm{mm}]$	$(d_i - \bar{d})$	$(d_i - \bar{d})$
4		
3		
4		
5		
6		

Calculate the mean value and estimate the standard deviation

$$\bar{d} = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sigma = \ldots\ldots\ldots\ldots\ldots\ldots$$

----------▷ 22

$$\bar{d} = 4.4 \times 10^{-2}\,\text{mm}$$
$$\sigma = 1.14 \times 10^{-2}\,\text{mm}$$

22

You will find the details of the computation in the table:

d_i in mm	$(d_i - \bar{d})$ in mm	$(d_i - \bar{d})$ in mm^2
4×10^{-2}	-0.4×10^{-2}	0.16×10^{-4}
3×10^{-2}	-1.4×10^{-2}	1.96×10^{-4}
4×10^{-2}	-0.4×10^{-2}	0.16×10^{-4}
5×10^{-2}	0.6×10^{-2}	0.36×10^{-4}
6×10^{-2}	1.6×10^{-2}	2.56×10^{-4}
sum: 22×10^{-2}	0	5.20×10^{-4}

$$\bar{d} = \frac{22 \times 10^{-2}\,\text{mm}}{5} = 4.4 \times 10^{-2}\,\text{mm}$$
$$\sigma^2 = \frac{1}{(5-1)} \times 5.20 \times 10^{-4} = 1.30 \times 10^{-4}\,\text{mm}^2$$
$$\sigma = 1.14 \times 10^{-2}\,\text{mm}$$

23

If you experienced difficulties with the solution of the last example you should read section 22.2 in the textbook again. 23

It is important that you understand the following:

(1) A series of readings is a random sample of all possible test data.
(2) A series of readings has a mean value, a variance and a standard deviation.
(3) The parent population of all test data also has a mean value, a variance and a standard deviation. These values are estimated on the basis of the values obtained for the random sample.

For all calculations we use the formulae on page 541 in the textbook.

If you have not yet done exercise 22.2 on page 541 you should do so now!

--------------------▷ 24

22.2 Mean Value and Variance of Continuous Distributions Errors of the Mean Value

24

Objective: Concepts of deviation of the mean value, sampling error of the mean value, confidence intervals, evaluation of the standard deviation of the mean value.

Note: The short section ‘Mean value and variance of continuous distributions’ is an extension of the concepts derived for discrete test data. The most important section in the theory of errors is the ‘error in mean value’. It determines the reliability of the mean value of the test data.

READ: **22.3 Mean value and variance of continuous distributions**
22.4 Error in mean value
Textbook pages 554–557

25

Our goal is to determine the error of the mean value of a series of measurements. 25

We use a different notation for the standard deviation of the mean value; it is σ_M (as opposed to σ, which is used for a sample). We should become familiar with this distinction. In other texts these concepts are often used synonymously.

The standard deviation of the mean value of a series of measurements is the smaller the larger the number of measurements n. The following relation is valid:

$$\sigma_M = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is called ....

--------------------▷ 26

$$\sigma_M = \frac{\sigma}{\sqrt{n}}$$ 26

the sampling error, or mean error of the mean value.

Without the assistance of the example in the textbook, calculate the error of the mean value given the following data:

A series of 11 measurements (wire diameter) has a mean value $\bar{d} = 0.142\,\text{mm}$ and a variance $\sigma^2 = 0.046 \times 10^{-4}\,\text{mm}^2$.

$$\sigma_M = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

--------------------▷ 27

$$\sigma_M = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{4.6 \times 10^{-6}}{11}} \approx 0.00065\,\text{mm}$$ 27

You may find such calculations a little boring, but without doing them you will not get the feel of the problem and you will probably deceive yourself as to your understanding of the subject. It is only with practice that you will acquire this understanding.

In practice it is usual to quote the mean value and the sampling error of a series of readings in the following way:

wire diameter $d = \mu \pm \sigma_M$

in our case, $d =$ ....

28

$$d = 0.142 \pm 0.00065\,\text{mm}$$ 28

The standard deviation of the mean value can often be rounded.

Reason: The standard deviation of the mean value is a measure of the accuracy of the mean value, and this measure is itself the result of an estimation. It is therefore senseless to give too many digits.

Describe in your own words the significance of the standard deviation of the mean value in our example.

....

....

29

You could have written:

The true value is, with a 68% probability, in the interval $(0.142 - 0.00065)$ mm to $(0.142 + 0.00065)$ mm. This interval is called the confidence interval.

29

Let us compute the standard deviation of the mean value in the example concerning the necklace (cf. frame 13).

After 10 measurements the volume V of the necklace was found to be

$$\bar{V} = 2.60 \times 10^3\,\text{mm}^3$$

Standard deviation of the individual measurements

$$\sigma = 0.13 \times 10^3\,\text{mm}^3$$

Standard deviation of the mean value: $\sigma_M =$ ....

Hence the volume is: $V =$ ....

30

$$\sigma_M = \frac{\sigma}{\sqrt{10}}$$

$$\sigma_M = 0.04 \times 10^3\,\text{mm}^3$$

$$V = (2.60 \pm 0.04) \times 10^3\,\text{mm}^3$$

30

We can therefore expect the true value to lie in the range from $2.56 \times 10^3\,\text{mm}^3$ to $2.64 \times 10^3\,\text{mm}^3$ with a probability of 68%.

This implies that the true value lies outside this interval with a probability of 32%. Such uncertainty is often too large in practice.

Instead, one frequently gives the interval which contains the true value with 95% certainty.

Upper limit: ....

Lower limit: ....

--------------------▷ 31

Upper limit: $2.68 \times 10^3\,\text{mm}^3$
Lower limit: $2.52 \times 10^3\,\text{mm}^3$

31

The accuracy of the true value can be enhanced by increasing the number of individual measurements. Given 10 individual measurements in a particular case, how many measurements do we need to bring down the standard deviation of the mean value to

(a) one-half? $n_a = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

(b) one-third? $n_b = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

32

(a) $n_a = 40$
(b) $n_b = 90$

32

Correct

36

Wrong, or explanation required

33

33

In this case the starting point was 10 readings from which the standard deviation of the mean value was obtained.

We now ask ourselves: how many measurements should we carry out in order to halve this standard deviation?

The formula $\sigma_M = \dfrac{\sigma}{\sqrt{n}}$ contains the answer!

If we increase n the deviation σ_M decreases. If we want to double the denominator we must multiply n by 4, and if we want to treble the denominator we must multiply n by ....

34

9 (a ninefold increase in n)

34

An increase in the number of measurements reduces the standard deviation of the mean value; this means that the difference between the true value and our mean value will probably decrease.

How many measurements are necessary to bring down the sampling error from $0.04 \times 10^3\,\text{mm}^3$ to $0.01 \times 10^3\,\text{mm}^3$?

We originally had $n = 10$ measurements.

Now we need $n =$ .... measurements.

--------------------▷ 35

We need approximately $n = 160 = 10 \times 4^2$ measurements. 35

Now compute the standard deviation of the mean value for exercises 22.2(a) and (b) in the textbook.

36

22.3 Normal Distribution, Distribution of Random Errors

36

Objective: Concept of elementary error.

This section on the normal distribution borders on aspects of Chapter 20, where we defined the normal distribution as the limiting case of the binomial distribution.

READ: 22.5 Normal distribution Distribution of random errors
Textbook pages 557–558

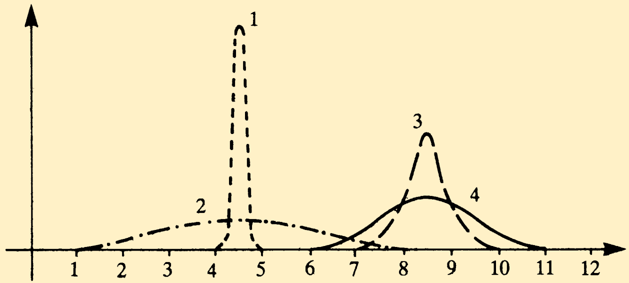
--------------------▷ 37

The judgement as to the accuracy of the mean value of a series of readings is based on the assumption that the dispersion of the test data about this value is a normal distribution. 37

This assumption might appear at first glance to be quite arbitrary; it has however been confirmed by observation. If we carry out a large number of measurements under identical conditions and draw a graph of the results we find that, in very many cases, the distributions follow the Gaussian bell-shaped curve.

38

38



The figure shows four different Gaussian distributions which differ in the position of the mean value and in the value of the standard deviation. Arrange the distributions 1, 2, 3 and 4 in the order of increasing σ from the smallest to the largest value.

39

1, 3, 4, 2

39

We can improve our retention of things through visualisation. This is a technique which depends very much on the individual but is particularly valuable for people who have a good imagination. Such a person often uses it without thinking that it is something special.

I would like to skip the following instructions --------------------▷ 44

I would like to learn more about imagination --------------------▷ 40

40

The rule is a simple one: we create a mental picture of the facts presented to us. Example: we can imagine the Gaussian distribution as a pointed bell-shaped curve which changes from being pointed to being flatter as the value of the standard deviation increases. (At the same time the maximum value decreases.) The reverse process can also be imagined. You saw this in frame 39; close your eyes for a minute and imagine these curves and you will virtually 'see' them being recreated for you by your memory. You can achieve this by concentrating hard!

41

Many people — not all — succeed in their first attempt to create such a mental picture of the facts presented to them. If you did not achieve this in your first attempt, try again; don't give up! 41

The person who succeeds will probably not forget what he has learned.

Bower, an American psychologist, found on the basis of empirical investigation that only 30% to 50% of facts are retained without a mental picture, whereas people using this technique retained 50% to 80% of the facts. Hence the benefits of this technique justify the effort required.

As an exercise, try to imagine how the Gaussian distribution shifts to the right or left as the mean value μ changes.

42

Visualisation activates your brain!
Many facts can be visualised:

42

— All (perhaps not really all) curves in which one parameter varies.
— You can also create mental pictures of vectors, for instance.

Visualisation or the creation of mental pictures can become a useful habit.

43

It is often useful to sketch facts on a sheet of paper and then to visualise them by creating a mental picture. 43

These pictures and their connections with the facts are coded and stored by the brain, making it easier to recall the facts when required later.

Using this technique, the same facts are coded within the brain in different ways and, so neurophysiologists say, in different places.

Furthermore, there are connections between these and consequently the same facts can be better retrieved afterwards.

--------------------▷ 44

You should be able to answer the next questions without the assistance of the textbook. For a normal distribution:

44

(1) ......% of all test data are to be expected within the interval $\mu \pm \sigma$, and
(2) ......% of all test data are to be expected within the interval $\mu \pm 2\sigma$.

--------------------▷ 45

(1) 68% (2) 95%

45

If we assume that the dispersion of test data is normally distributed about the mean value then it can be proved that:

The mean values of a series of readings are also normally distributed. The standard deviation of the mean values is given by

$$\sigma_M = \frac{\sigma}{\sqrt{n}}$$

The standard deviation of the mean value leads to the evaluation of the confidence intervals.

The diameter of a wire has been measured:

$$d = (0.1420 \pm 0.00065)\,\mathrm{mm}$$

Within what limits will the diameter lie with a probability of 95%?

Upper limit: ....

Lower limit: ....

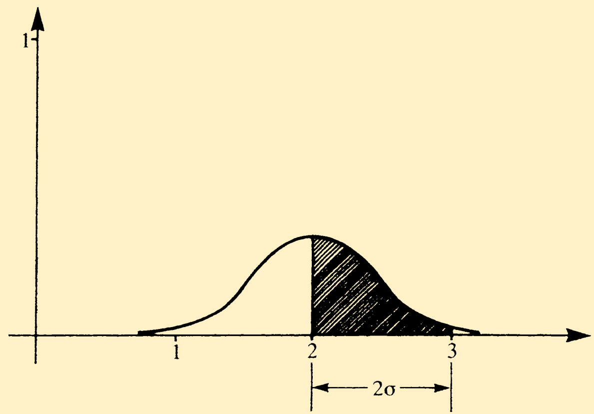
46

Upper limit: 0.1433 mm
Lower limit: 0.1407 mm

46

In case of difficulty read section 22.5 again.

For a normal distribution the shaded area contains ......% of all test data.

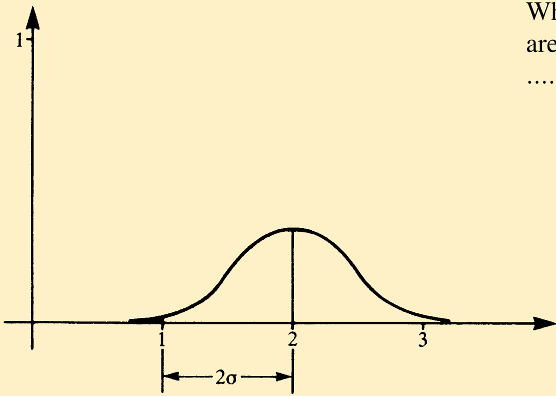


47

47.5%

47

Explanation: The Gaussian distribution is symmetrical about the mean value μ. Since 95% of all data lie in the range $\mu - 2\sigma$ and $\mu + 2\sigma$ then half will lie inside the interval, i.e. 47.5%.



What percentage of the test data lies in the shaded area?

......% of all test data.

48

2.5%
Explanation: 5% of all test data lie outside the 2σ-interval (2 standard deviations), hence the area shown represents 2.5% of all test data.

48

--------------------▷ 49

22.4 Error Propagation Law

49

Objective: Concept of error propagation law; calculation of the standard deviation of combined independent measurements.

Many physical quantities are computed from two or more measured quantities, e.g. $g = f(x,\ y,\ z)$. Therefore, the mean error of a quantity g depends on the error in the measurements of the individual quantities $x,\ y,\ z$, etc...

READ: **22.6 Law of error propagation**
22.7 Weighted average
Textbook pages 558–560

50

The Gaussian error propagation law states how the standard deviation of quantities that are not directly measurable can be determined by a combination of directly measurable quantities. 50

The formula in the textbook has not been derived. The application of this formula depends on your knowledge of partial derivatives which were treated in Chapter 12.

Do you know how to obtain the partial derivatives of a function?

Yes --------------------▷ 54

No --------------------▷ 51

Partial derivatives are a prerequisite for this section. They are explained in Chapter 12.
You should revise or study Chapter 12 now.

51

52

(1) Given: $f(x,\ y) = ax^2y + \frac{3}{4}xy^3$

52

Obtain

$$\frac{\partial f}{\partial y} = f_y = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial x} = f_x = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

(2) Given: another function with different variables: $f(D,\ G) = \dfrac{G}{\frac{4}{3}\left(\frac{D}{2}\right)^2}$

Obtain

$$\frac{\partial f}{\partial G} = f_G = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial D} = f_D = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

53

(1) $f_y = ax^2 + \frac{9}{4}xy^2, \quad f_x = 2axy + \frac{3}{4}y^3$ 53

(2) $f_G = \dfrac{3}{D^2} \quad f_D = \dfrac{-6G}{D^3}$

If you didn't get these answers then you ought to read the relevant section in Chapter 12 again, and do the accompanying exercises.

54

Two electrical resistors R_1 and R_2 have been measured several times and the following resistance values were obtained:

54

$$R_1 = (150 \pm 0.9)\,\text{ohms}(\Omega)$$
$$R_2 = (220 \pm 1.1)\,\text{ohms}(\Omega)$$

If these two resistors are connected in parallel what is the value and the standard deviation of the resultant resistance R?

The resultant of total resistance for a parallel connection is given by the equation:

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

hence

$$R = \dots\dots\dots\dots\dots\dots$$

55

$R = 89.19\Omega$ to 2 d.p.

55

We now apply the law of error propagation to calculate the mean error in R.

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

The expression for R corresponds to $g = f(x, \ y)$ in the textbook. We then obtain

$$\sigma_{MR} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

I can solve the exercise --------------------▷ 59

I need advice --------------------▷ 56

The resistance R corresponds to the quantity g in the textbook, with R_1 corresponding to x and R_2 to y. **56**

The function is

$$g = f(x, \ y) = \frac{xy}{x+y}$$

or

$$R = f(R_1, \ R_2) = \frac{R_1 R_2}{R_1 + R_2}$$

The formula for the law of propagation of the error is

$$\sigma_{Mg} = \sqrt{(f_x \sigma_x)^2 + (f_y \sigma_y)^2}$$

Hence in the case of the resistances

$$\sigma_{MR} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

-------------------▷ 57

$$\sigma_{MR} = \sqrt{\left(\frac{\partial f}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial f}{\partial R_2}\right)^2 \sigma_{R_2}^2}$$

57

We know $\sigma_{R1} = 0.9\Omega$ and $\sigma_{R2} = 1.1\Omega$.

We must obtain the partial derivatives:

$$\frac{\partial f}{\partial R_1} = \frac{\partial}{\partial R_1}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \frac{R_2^2}{(R_1 + R_2)^2} = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\partial f}{\partial R_2} = \frac{\partial}{\partial R_2}\left(\frac{R_1 R_2}{R_1 + R_2}\right) = \frac{R_1^2}{(R_1 + R_2)^2} = \ldots\ldots\ldots\ldots\ldots\ldots$$

Now insert the numerical values of R_1 and R_2.

58

$$\frac{\partial f}{\partial R_1} = \frac{(220)^2}{(150+220)^2} = 0.35\Omega$$
$$\frac{\partial f}{\partial R_2} = \frac{(150)^2}{(150+220)^2} = 0.16\Omega$$

58

We now recall the general equation:

$$\sigma_{MR} = \sqrt{\left(\frac{\partial f}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial f}{\partial R_2}\right)^2 \sigma_{R_2}^2}$$

Substituting numerical values yields:

$$\sigma_{MR} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and $$R = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

59

$$\sigma_{MR} = 0.36\,\Omega$$

59

Hence $$R = (89.19 \pm 0.36)\,\Omega$$

What is the total resistance R and the standard deviation if the resistors $R_1 = (150 \pm 0.9)\,\Omega$ and $R_2 = (220 \pm 1.1)\,\Omega$ are connected in series?

$$R = R_1 + R_2$$

$$R = \ldots\ldots\ldots\ldots$$

$$\sigma_{MR} = \ldots\ldots\ldots\ldots$$

------------------▷ 60

$$R = 370\,\Omega$$
$$\sigma_{MR} = 1.42\,\Omega$$
$$R = (370 \pm 1.42)\,\Omega$$

60

Correct → 62

Wrong → 61

Here is the solution:

61

$$R_1 = (150 \pm 0.9)\,\Omega, \quad R_2 = (220 \pm 1.1)\,\Omega$$
$$R = R_1 + R_2 = 370\,\Omega$$
$$\frac{\partial R}{\partial R_1} = 1, \quad \frac{\partial R}{\partial R_2} = 1$$
$$\sigma_{MR} = \sqrt{\left(\frac{\partial R}{\partial R_1}\right)^2 \sigma_{R_1}^2 + \left(\frac{\partial R}{\partial R_2}\right)^2 \sigma_{R_2}^2} = \sqrt{1 \times 0.9^2 + 1 \times 1.1^2}$$
$$= \sqrt{(0.81 + 1.21)\,\Omega^2} = \sqrt{2.02\,\Omega^2}$$
$$\sigma_{MR} = 1.42\,\Omega$$
$$R = (370 \pm 1.42)\,\Omega$$

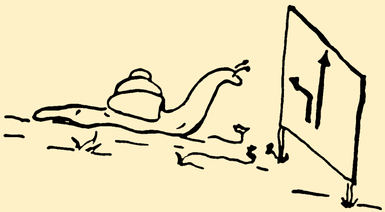
62

Interpretation of the error propagation law, in words:

If we determine a quantity from several individual quantities then the quality of the final result is determined by the quality of the individual values.

Do you think you would benefit from another exercise?

62



☐ No → 65

☐ Yes → 63

Example:
The sides of a cube have been measured; they are

63

$$x = (22 \pm 0.1)\,\text{mm}$$
$$y = (16 \pm 0.08)\,\text{mm}$$
$$z = (10 \pm 0.08)\,\text{mm}$$

Calculate the volume $V = xyz$ of the cube and the standard deviation σ_M.

$$V = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\sigma_M = \ldots\ldots\ldots\ldots\ldots\ldots$$

Final result, $V = \ldots\ldots\ldots\ldots\ldots\ldots$

-------------------▷ 64

$$V = 3520\,\text{mm}^3$$
$$\sigma_{MV} = 36.9\,\text{mm}^3$$
$$V = (3520 \pm 36.9)\,\text{mm}^3$$

64

$$x = (22 \pm 0.1)\,\text{mm}, y = (16 \pm 0.08)\,\text{mm}, z = (10 \pm 0.08)\,\text{mm}$$
$$V = xyz = 3520\text{mm}^3$$
$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(xyz) = yz = 160\,\text{mm}^2$$
$$\frac{\partial V}{\partial y} = xz = 220\,\text{mm}^2, \quad \frac{\partial V}{\partial z} = xy = 352\,\text{mm}^2$$
$$\sigma_{MV} = \sqrt{160^2\text{mm}^4 \times 0.1^2\text{mm}^2 + 220^2\,\text{mm}^4 \times 0.08^2\,\text{mm}^2 + 352^2\,\text{mm}^4 \times 0.08^2\,\text{mm}^2}$$
$$= \sqrt{256\,\text{mm}^6 + 4.84 \times 64\,\text{mm}^6 + 12.39 \times 64\,\text{mm}^6}$$
$$= \sqrt{(256 + 310 + 793)\,\text{mm}^6} = \sqrt{1359\,\text{mm}^6}$$
$$\sigma_{MV} = 36.9\,\text{mm}^3$$
$$V = (3520 \pm 36.9)\,\text{mm}^3$$

65

22.5 Curve Fitting: Method of Least Squares, Regression Line

65

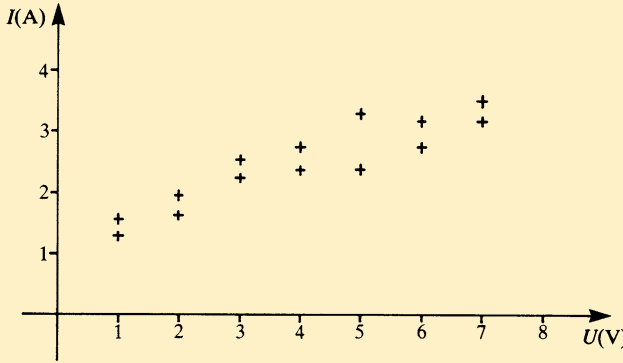
Objective: Concept of curve fitting, regression line.

We have seen that the mean value of a series of readings is more reliable than the individual measurements and that the sum of the squares of the deviations or errors is a minimum. In this section this fundamental idea is used to fit a curve through a series of data points.

However, we illustrate the technique with a straight line only as the best fit to a series of data points.

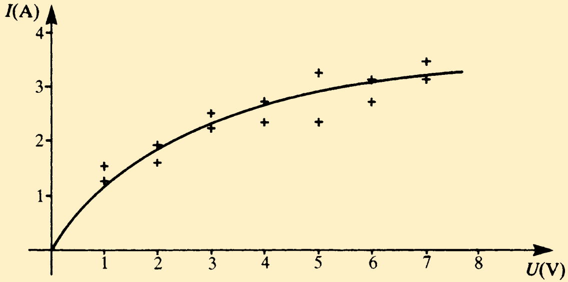
READ: 22.8 Curve fitting: method of least squares, regression line
22.9 Correlation and correlation coefficient
Textbook pages 561–567

66



The figure shows a number of data points representing the current I flowing in a light bulb as a function of the voltage. By eye, sketch what you consider to be the best curve through these data points. Draw a curve free hand.

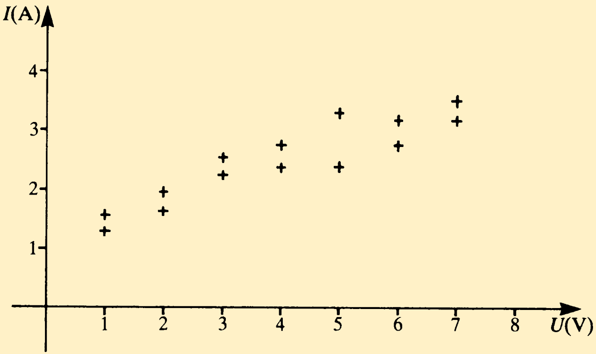
67



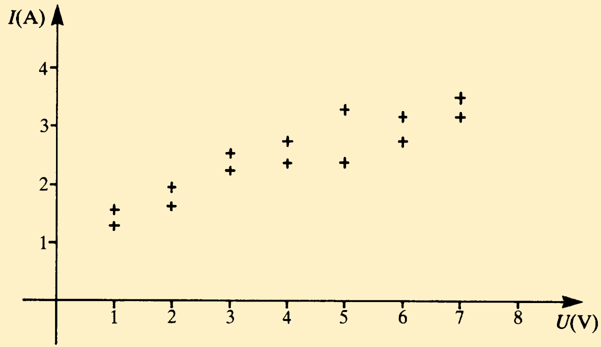
67

The figure shows a free-hand curve through the data points. We consider this curve to represent the mean values through these values.

Now replace this curve by the best straight line through the points.



68



68

Depending on the measurements and on theoretical consideration the curve which may best fit the data could be, for example, a parabola, an exponential or a logarithmic function or even a sine function. However, we often do not have a definite idea about the shape of the curve. In such situations the simplest curve is a straight line.

The technical name for it is: ....

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a regression line

69

Have you done and understood the example in the textbook?

☐ Yes

76

☐ No

70

It may be useful to carry out a simple example step by step. 70

The following values of current and voltage have been measured:

$U_i\,(\mathrm{V})$	2	3	4	5	6
$I_i\,(\mathrm{A})$	1.3	1.7	2.1	2.1	2.9

We want to find the regression line.

First, which products do we have to compute and sum? Set up a table!

In the textbook we obtained the regression line for an $x-y$ system of coordinates. Now the variables are U and I.

71

You should have drawn up a table like this:

71

	U_i (V)	U_i^2 (V^2)	I_i (A)	$U_i I_i$ (VA)
	2		1.3	
	3		1.7	
	4		2.1	
	5		2.4	
	6		2.9	
Σ				

Complete the table.

72

72

	$U_i\,(\mathrm{V})$	$U_i^2\,(\mathrm{V}^2)$	$I_i\,(\mathrm{A})$	$U_i I_i\,(\mathrm{VA})$
	2	4	1.3	2.6
	3	9	1.7	5.1
	4	16	2.1	8.4
	5	25	2.4	12.0
	6	36	2.9	17.4
Σ	20	90	10.4	45.5

The mean values of the voltage and the current can be computed:

$$\bar{U} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\bar{I} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

73

$$\bar{U} = 4\,\text{V} (\text{corresponds to } \bar{x})$$
$$\bar{I} = 2.08\,\text{A} (\text{corresponds to } \bar{y})$$

73

Here is the table once more:

	$U_i\,(\text{V})$	$U_i^2\,(\text{V}^2)$	$I_i\,(\text{A})$	$U_i I_i\,(\text{VA})$
	2	4	1.3	2.6
	3	9	1.7	5.1
	4	16	2.1	8.4
	5	25	2.4	12.0
	6	36	2.9	17.4
Σ	20	90	10.4	45.5

Inserting the sum in the equation $a = \dfrac{\sum x_i y_i - n\bar{x}\bar{y}}{\sum x_i^2 - n\bar{x}^2}$

$$a = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$b = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

74

$$a = 0.39\,\text{amps/volt}$$
$$b = 0.52\,\text{amps}$$

74

Correct 76

I need assistance 75

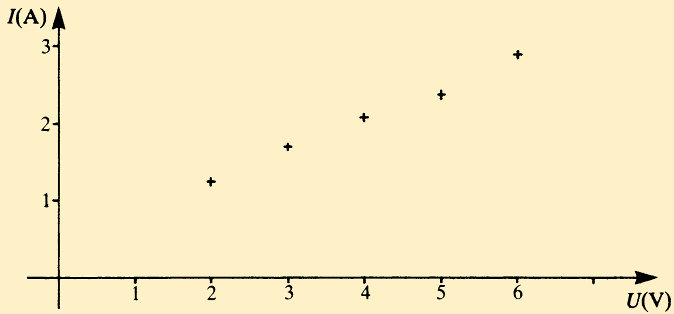
Perhaps your difficulty lies in understanding the equation for calculating a: a slight confusion with symbols possibly? In this case it is advisable to replace U by x and I by y in the equation. 75

At the end of the computation you can substitute back for U and I.

76

Here are the measured values in a coordinate system:

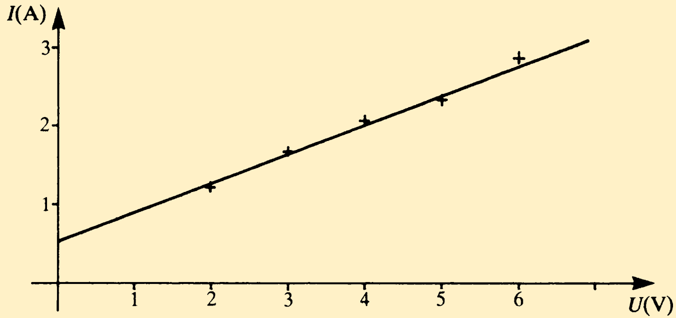
76



Try to fit a straight line through the points by eye. Then plot the straight line obtained by the method of least squares, namely

$$I = 0.39U + 0.52$$

77



77

In the textbook there follows a section on correlation and the correlation coefficient. These concepts are developed to measure the quality of the approximation of data sets by a regression line. But since this problem is of special interest only the study guide ends here.

This chapter on the theory of errors is quite important. We therefore repeat the content very briefly, but you should have made your own brief notes as you read the textbook!

78

(1) With the help of the theory of errors the magnitude of .... can be estimated. 78

(2) The quantity which is a measure of the dispersion of the individual values about the mean is called ....

(3) The dispersion of the test data can be estimated on the basis of random samples.

(4) The square root of the variance is also a measure of the dispersion; it is called ....

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(1) random errors 79

It is important to realise that the theory of errors does not make any allowance for systematic (constant) errors.

(2) variance

A series of readings is a random sample out of a parent population of all possible test data.

(4) standard deviation

68% of all test data deviate by less than one standard deviation about the mean.

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The most important conclusions are:

80

(5) The dispersion of the mean value is less than that of the individual values; hence the mean value is .... than the individual values. For this reason we carry out a number of series of measurements in science and engineering.

(6) The variance and the dispersion of the mean value can be determined by using the individual test data, and the following relation then holds true $\sigma_M =$ ....

(7) A quantity is computed from a series of measurements of several other quantities. These quantities are subject to errors; hence the error in the required quantity is obtained in accordance with the ....

(8) The method of least squares leads us to a deeper understanding of curve fitting. We have only dealt with the simplest case by fitting a straight line through the data points.
Such a straight line is often referred to in the literature as a ....

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(5) more reliable

(6) $\sigma_M = \dfrac{\sigma}{\sqrt{n}}$

(7) error propagation law

(8) regression line