

# Chapter 21 Probability Distributions

----->

2



- (a)  $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i,$

$n =$  number of values

28
- (b)  $\bar{x} = \frac{1}{n} \sum_{i=1}^n n_i x_i,$

$n_i =$  frequency of values  $x_i$
- (c)  $\bar{x} = \int_{x_1}^{x_2} x f(x) \mathrm{d} x,$

$f(x) =$  probability density function

When measured, a physical quantity was found to have the following values:

$x_i$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
	2.9	3.1	3.5	3.5	3.7	4.1

The mean value is  $\bar{x} =$  .....

----->

29

## 21.1 Discrete Probability Distributions

2

**Objective:** Concepts of random variable, discrete probability distributions, determination of the probability of a random variable from a random experiment.

This chapter requires knowledge of the preceding one.

**READ:** 21.1.1 Discrete probability distributions  
Textbook pages 531–534

3

$$\bar{x} = 3.47$$

29

Given 20 measured values (set A):

1.2	1.0	1.1	1.3
1.1	1.2	1.2	1.1
1.4	1.3	1.3	1.1
1.2	1.2	1.4	1.1
1.2	1.0	1.2	1.4

With these values draw up a frequency table.

Obtain the frequencies and the relative frequencies.

The first step is the preparation of the table.

30

A random experiment consists in tossing a coin. We choose as the random variable  $x$  the event ‘tail’. Give the probability distribution of the random variable  $x$  in the form of a table

3

Random variable $x$	Probability $P$

Solution

-----> 5

Explanation wanted

-----> 4



Measured value $x$	Frequency $f$	Relative frequency $f_r$

30

This is the usual form for a frequency table.  
Draw up this table and insert the values.

-----> 31

The sample space consists of the elements ‘head’ and ‘tail’. The values of the random variable  $x = 0$  and  $x = 1$  are assigned to these respectively. Both events occur with the probability

4

$$P = \frac{1}{2}$$

Hence the above table for the probability distribution reads

Random variable $x$	Probability $P$

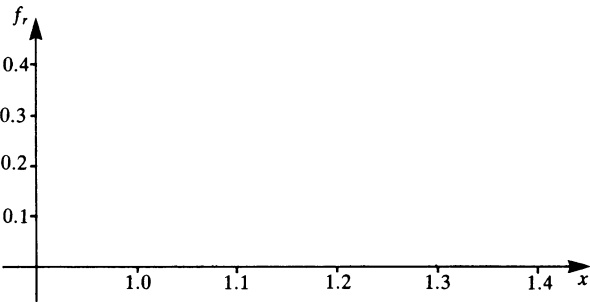
----->5



$x$	$f$	$f_r$
1.0	2	0.10
1.1	5	0.25
1.2	7	0.35
1.3	3	0.15
1.4	3	0.15

31

Fill in the frequency distribution on the diagram.



----->32

Random variable $x$	Probability $P$
0	$\frac{1}{2}$
1	$\frac{1}{2}$

5

The random experiment now consists of the simultaneous tossing of three coins. We choose (arbitrarily) for the random variable  $x$  the number of coins showing ‘head’ minus the number of coins showing ‘tail’.

We have to determine the probability distribution of the random variable  $x$ .

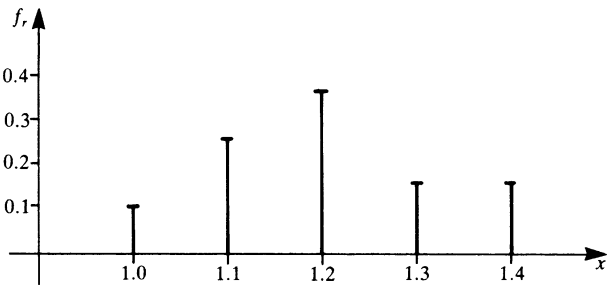
Random variable $x$	Probability $P$

Solution

-----> 8

Explanation wanted

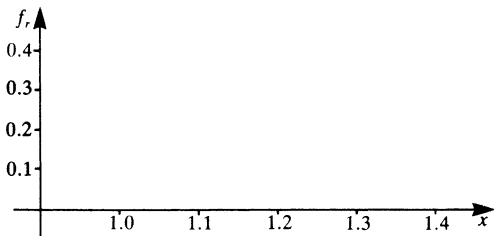
-----> 6



32

Set A

Given: 20 other measured values:  
We have already computed the frequency distribution. Plot it in the diagram below.



$x$	Frequency $f$	Relative frequency $f_r$
1.18	2	0.10
1.19	5	0.25
1.20	7	0.35
1.21	3	0.15
1.22	3	0.15

-----> 33

Random experiment: tossing 3 coins

Random variable:  $x = x_{\text{head}} - x_{\text{tail}}$

Possible elementary occurrences (first coin, second coin, third coin):

(HHH), (HHT), (HTH), (THH), (HTT), (THT), (TTH), (TTT).

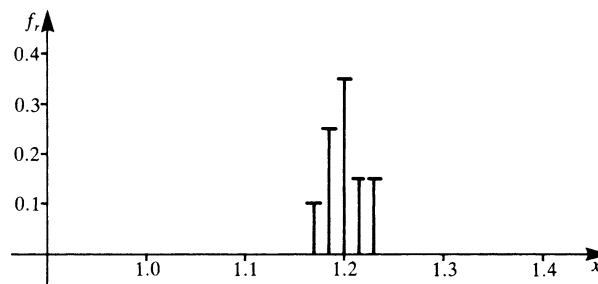
The random variable  $x$  for the outcome HTH is for example:  $x = 2 - 1 = 1$

Each of the elementary occurrences had the probability  $\frac{1}{8}$ .

6

7

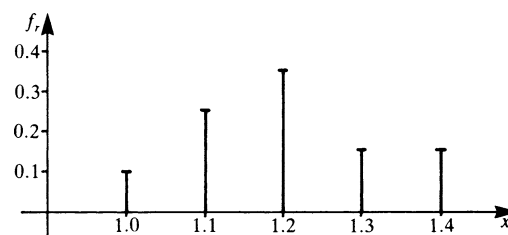
Set B



33

The frequency distribution for this Set B of measured values has the same mean value as the previous example Set A. The previous example is shown below.

Set A



Which do you think is the more reliable measurement? Set .....

34

Now give the probability distribution:

7

Outcome	Random variable $x$	Probability $P(x)$
HHH	.....	.....
HHT	.....	.....
HTH	.....	.....
THH	.....	.....
HTT	.....	.....
THT	.....	.....
TTH	.....	.....
TTT	.....	.....

-----> 8

Set B

34

Given the probability distribution  $P(1), \dots, P(k)$  for the set of values  $x_1, \dots, x_k$  of the random variable  $x$ , the mean value is defined as

$\bar{x} = \dots\dots\dots$

-----> 35

Random variable $x$	Probability $P(x)$
3	$\frac{1}{8}$
1	$\frac{3}{8}$
-1	$\frac{3}{8}$
-3	$\frac{1}{8}$

8

-----> 9

$$\bar{x} = \sum P(i)x_i \quad i = 1, 2, \dots, k$$

35

Here are the results of some measurements:

$x_i$	Relative frequency
4	0.1
5	0.3
6	0.4
7	0.2

The mean value is  $\bar{x} = \dots\dots$

-----> 36



In section 21.1.1 of the textbook we dealt with the case of two dice in which the random variable was the ‘sum of the number of spots’.

Now use as the random variable  $x$  the number of spots on the first die minus the number of spots on the second die, and determine the probability distribution.

Random variable $x$	Probability $P(x)$

Solution -----> 11

Explanation wanted -----> 10



$$\begin{aligned}\bar{x} &= P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 \\ &= 0.4 + 1.5 + 2.4 + 1.4 = 5.7\end{aligned}$$

36

What is the mean number (of spots) when throwing a die?

Mean number = ..... -----> 37

Random experiment: throwing two dice.  
Random variable

10

$$x = (\text{Number of spots on first die}) - (\text{Number of spots on second die}).$$

To help you we give below three values of the random variable with their outcomes and probabilities.

Outcome		Random variable $x$	Probability $P(x)$
1st die	2nd die		
1	6	-5	$1 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
1	3	-2	$4 \times \frac{1}{6} \times \frac{1}{6} = \frac{4}{36}$
2	4		
3	5		
4	6		
4	1	3	$3 \times \frac{1}{36} = \frac{3}{36}$
5	2		
6	3		

Now complete the table of frame 9.

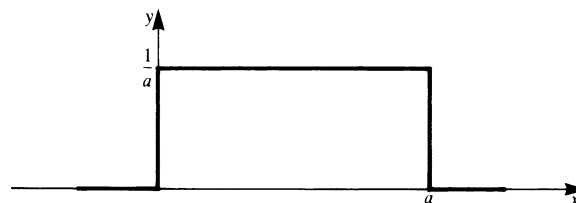
11

$$\text{Mean number} = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6 = 3.5$$

37

A random variable has the following probability density function:

$$f(x) = \begin{cases} \frac{1}{a} & \text{for } 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$



Calculate the mean value of the random variable  $x$ .

$$\bar{x} = \dots\dots\dots$$

38

Random variable $x$	Probability $P(x)$	11
-5	$\frac{1}{36}$	
-4	$\frac{2}{36}$	
-3	$\frac{3}{36}$	
-2	$\frac{4}{36}$	
-1	$\frac{5}{36}$	
0	$\frac{6}{36}$	
1	$\frac{5}{36}$	
2	$\frac{4}{36}$	
3	$\frac{3}{36}$	
4	$\frac{2}{36}$	
5	$\frac{1}{36}$	

-----> 12

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx = \int_0^a x \frac{1}{a} dx = \frac{a}{2}$$

38

Correct, and I want to carry on

-----> 42

Explanation, or another exercise wanted

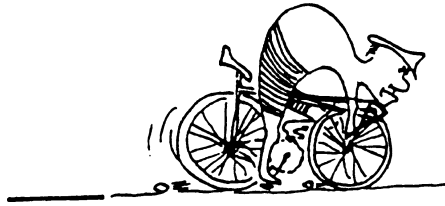
-----> 39

Here is a funny problem.

George maintains he can distinguish between two sorts of beer A and B by tasting them. Henry does not believe him.

Can you think of a possible experimental set-up which would be suitable for checking this statement?

12



Solution

----->

14

Hint

----->

13

The mean value  $\bar{x}$  of a continuous random variable with the probability density function  $f(x)$  is defined by

39

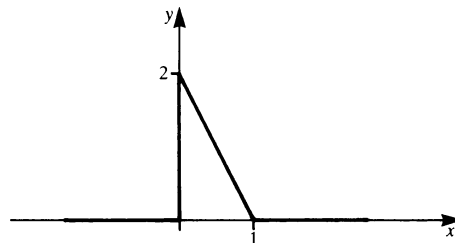
$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx$$

The limits of integration in a particular case are determined by the range of definition of the random variable  $x$ .

What is the mean value of the random variable  $x$  whose probability density function is defined below

$$f(x) = \begin{cases} 2(1-x) & \text{for } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$\bar{x} = \dots\dots\dots$

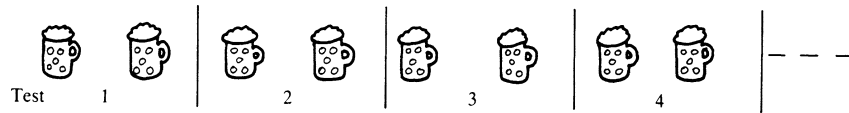


----->

40

A suitable experiment could look like this. George tries to identify the beer by carrying out the following tests:

13



For each test both sorts of beer are offered for identification.

How many tests are necessary to enable George to prove his statement and convince Henry? (The probability that the result could occur accidentally should be less than 0.01.) .....

14

$$\bar{x} = \frac{1}{3}$$

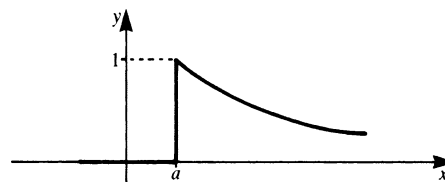
40

Solution:

$$\bar{x} = \int_{-\infty}^0 x \times 0 \times dx + \int_0^1 x 2(1-x) dx + \int_1^{\infty} x \times 0 \times dx = 2 \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 \times \frac{1}{6} = \frac{1}{3}$$

Given the probability density function

$$f(x) = \begin{cases} e^{-(x-a)} & \text{for } a \leq x \\ 0 & \text{otherwise} \end{cases}$$



Is the normalisation condition

$$\int_{-\infty}^{\infty} f(x) dx = 1 \text{ satisfied?}$$

No: compute the normalisation factor.

Yes: give the mean value.

$$\bar{x} = \dots\dots\dots$$

41

George tries to identify the kind of beer by a test. For each test both sorts of beer are offered for identification. The test is repeated.

Number of test and correct identification	1	2	3	4	5	6	7	...
	Yes	Yes	Yes	Yes	Yes	Yes	Yes	...
Random probability $P(n) = \left(\frac{1}{2}\right)^n$	0.5	0.25	0.13	0.06	0.03	0.016	0.0078	...

The probability that in seven consecutive tests George hits accidentally on the right beer is 0.0078. This is less than 0.01.

The probability density function  $f(x)$  is normalised since

$$\int_{-\infty}^{\infty} f(x) dx = \int_a^{\infty} e^{-(x-a)} dx = \left[ -e^{-(x-a)} \right]_a^{\infty} = 1$$

$$\bar{x} = \int_{-\infty}^{\infty} x f(x) dx = \int_a^{\infty} x e^{-(x-a)} dx$$

We integrate by parts and obtain

$$\bar{x} = \left[ -x e^{-(x-a)} \right]_a^{\infty} - \left[ e^{-(x-a)} \right]_a^{\infty} = a + 1$$

We met this type of probability density function in the textbook when we determined the probability of an air molecule in the atmosphere being in a particular region.

Assume you have a test on the content of the present study guide containing ten questions and for each question four solutions are given.

15

Your task is to find the right one, assuming that you have not yet studied the chapter on probability. You are nevertheless determined to do the test since there is a chance you may find the correct solution. The test is considered successful if you have scored at least 80%.

What is your chance of achieving this result by accident?  $P(\text{accident}) = \dots$

Hint

----->

16

Solution

----->

20

## 21.4 The Normal Distribution as the Limiting Value of the Binomial Distribution

42

**Objective:** Concepts of binomial distribution, normal distribution (Gaussian distribution), application of the binomial distribution.

**READ:**    21.3 The normal distribution as the limiting value of the binomial distribution  
Textbook pages 539–545

----->

43

The random variable  $x$  for one question can take on two values: 1 — correct; 0 — wrong.  
 The events (solution of the questions) take place independently of each other.  
 For one question there are given four solutions.  
 The probability of ticking the correct solution of one question by accident is

16

$$P(x = 1) = \dots$$

The probability of ticking the wrong solution of one question by accident is

$$P(x = 0) = \dots$$

----->

17

Tick the exercises that can be solved by applying the binomial distribution. If in doubt consult the textbook.

43

- (a) ☐ 5 dice are thrown. What is the probability that 3 dice show an even number of spots?
- (b) ☐ One die is thrown 6 times. What is the probability that an even number is thrown each time?
- (c) ☐ A box contains 1 white ball and 2 red balls. One ball is taken out and after that another one.  
What is the probability that both balls are red?
- (d) ☐ How many possibilities are there of drawing one red card out of a pack of cards?

----->

44



$$P(x = 1) = \frac{1}{4}$$

$$P(x = 0) = \frac{3}{4}$$

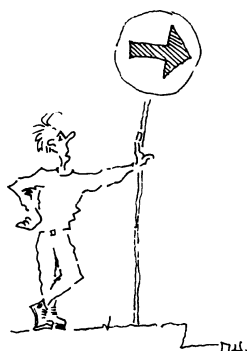
17

The probability of ticking 10 correct solutions accidentally is

$$P(z = 10) = \dots\dots$$

The probability of ticking 9 correct solutions accidentally is

$$P(z = 9) = \dots\dots$$



18

(a) and (b)

44

5 dice are thrown. What is the probability that 3 dice show an even number?

It is important that you substitute the correct values in the binomial formula. You can find it in the textbook. First calculate

$$n = \dots\dots$$

$$k = \dots\dots$$

$$P = \dots\dots$$

45

$$P(z = 10) = \left(\frac{1}{4}\right)^{10} = 0.000001$$

$$P(z = 9) = \left(\frac{1}{4}\right)^9 \times \frac{3}{4} \times 10 = 0.00003$$

18

Reason for the factor 10: 9 correct and 1 wrong solution can be achieved in 10 different ways.

---

Generally, the probability of obtaining  $z = a$  correct solutions is

$$P(z = a) = \binom{N}{a} P(x = 1)^a P(x = 0)^{N-a}$$

$$P(z = 8) = \dots\dots$$

This expression is identical with the binomial distribution, whose derivation you will find in the text-book.

19

$$n = 5$$

$$k = 3$$

$$P = \frac{1}{2}$$

45

---

Now we insert these values into the formula:

$$P(5, 3) = \dots\dots$$

46

$$P(z = 8) = \binom{10}{8} \left(\frac{1}{4}\right)^8 \left(\frac{3}{4}\right)^2 = 0.0004$$

19

The probability of obtaining at least 80% correct solutions when 10 problems are given is

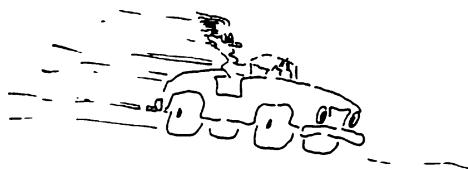
$$P(z \geq 8) = P(z = 10) + P(z = 9) + P(z = 8) = \dots\dots$$

----->

20

$$P(5, 3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{5}{16} \approx 0.3$$

46



----->

47

$$P(z \geq 8) = 0.0004$$

20

When you have worked it out you will discover that the probability of achieving 80% right solutions by accident is deplorably small; it is 0.0004. It is therefore better to study the subject first!

-----> 21

---

### 21.5 Properties of the Normal Distribution

47

**Objective:** Classification of normal distributions according to the parameters  $\sigma$  and  $\mu$ .

**READ:**    21.3.1 Properties of the normal distribution  
              21.3.2 Derivation of the binomial Distribution  
              Textbook pages 542–545

-----> 48

## 21.2 Continuous Probability Distributions

21

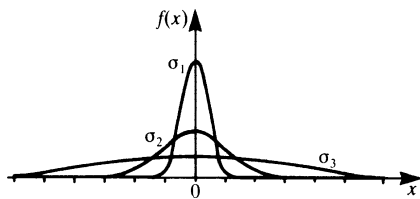
**Objective:** Concepts of continuous probability distributions, probability density function.

**READ:** 21.1.2 Continuous probability distributions  
Textbook pages 534–537

22

The following sketch shows three normal distributions. What parameter defines the different shapes of the curves?

48



$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

Parameter: .....

Arrange the parameters in descending order of magnitude:

..... > ..... > .....

49

Given: the probability density function

22

$$\phi(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}$$

What is the probability for this function when  $x = 2$ ? Attention please! Note the definition of the probability density function:

$$P(x = 2) = \dots$$

Solution

25

Explanation wanted

23

The parameter is  $\sigma$  (standard deviation) in  $f(x)$ .

49

$$\sigma_3 > \sigma_2 > \sigma_1$$

Given the normal distribution  $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$

What is the mean value of the random variable  $x$ ?

$$\bar{x} = \dots$$

*Hint:* you do not have to calculate it, but think!

50

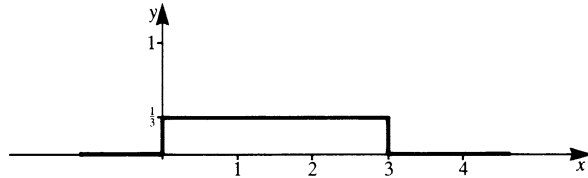
For a continuous random variable the probability is always zero if the random variable assumes a particular value. A probability which differs from zero can only be given for a finite interval of the random variable.

23

Solve the following problem.

Given: the probability density function

$$f(x) = \begin{cases} \frac{1}{3} & \text{for } 0 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$



What is the probability when  $x = 1$ ?

$P = \dots\dots$

What is the probability  $P(2.0 \leq x \leq 2.5)$  in the range  $2 \leq x \leq 2.5$ ?

$P = \dots\dots$

24

$$\bar{x} = 0$$

50

The normal distribution has its maximum at  $x = 0$  and it is symmetrical about that point. Therefore  $\bar{x} = 0$ .

Any probability distribution symmetrical about  $x = 0$  has a mean value  $\bar{x} = 0$ .

The random variable  $x$  has the normal distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

What is the mean value of the random variable  $x$ ?

$\bar{x} = \dots\dots$

Think!

51

$$P(x = 1) = 0$$

$$P(2 \leq x \leq 2.5) = \frac{1}{6}$$

24

Detailed solution:

$$P = \int_2^{2.5} f(x) dx = \int_2^{2.5} \frac{1}{3} dx = \frac{1}{3}(2.5 - 2.0) = \frac{1}{6}$$

Given: the probability density function  $\phi(x) = \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{2}}$

What is the probability when  $x = 2$ ?

$$P(x = 2) = \dots\dots\dots$$

25

$$\bar{x} = \mu$$

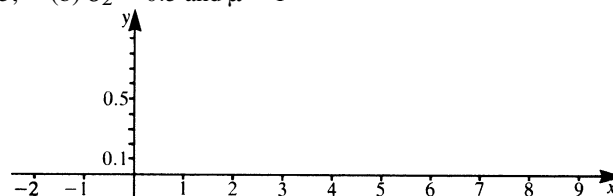
51

(See the textbook if in doubt.)

Familiarity with the normal distribution can only be acquired through exercises. Sketch two normal distributions

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(a)  $\sigma_1 = 1$  and  $\mu = 5$ ; (b)  $\sigma_2 = 0.5$  and  $\mu = 1$



Hint: use the following estimate:  $\frac{1}{2\sqrt{\pi}} \approx \frac{1}{2.5} = 0.4$

52

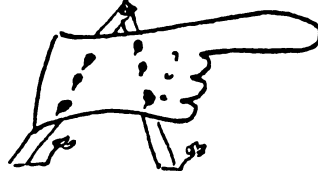


$$P(x = 2) = 0$$

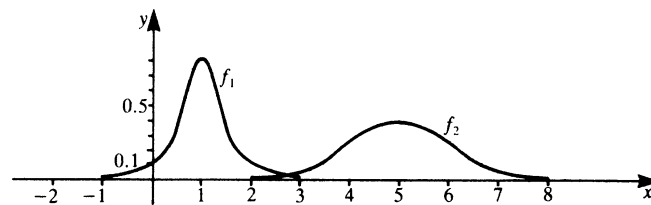
25

Remember: a probability which differs from zero can only be given for a *finite interval* of the continuous random variable.

In case of doubt read the explanation given in frame 23.



26



52

It was important for you to sketch the curves using just a few values. The differences between the two curves are the positions of the mean values and the variances.

53

### 21.3 Mean Value

26

**Objective:** Concept of arithmetical mean value for discrete and continuous variables.

**READ:** 21.2 Mean values of discrete and continuous variables  
Textbook pages 537–539

27

---

If you are not an enthusiastic mathematician — these are very rare — then you have worked very hard throughout this somewhat difficult topic, but your persistence will pay off!

53



OF CHAPTER 20

What are the different forms of the arithmetic mean value?

27

- (a)  $\bar{x} = \dots\dots\dots$  (finite number of values)
- (b)  $\bar{x} = \dots\dots\dots$  (discrete random variable)
- (c)  $\bar{x} = \dots\dots\dots$  (continuous random variable)

28

Please continue on page 1  
(bottom half)