Contents

Chapter 5

Anti-vibration mounting system

Coerenza tra Assi-Figure-Equazioni e TODO

1. quando il movimento avviene lungo l'asse verticale z, e.g. in Fig. (5.1), le x vanno in z e di conseguenza usare z anche nelle eq; nota: aggiungere in Fig, nella caption e in (5.0.1), anche $F_{1,2}(t)$
2. la Fig. (5.2) va modficata con un *taglio* che permette al lato vert di muoversi lungo x ed aggiungendo $x_g(t)$ (oltre ai $\dot{x}_1, ...$)
3. Fig. (5.3) e Fig. (5.4) inalterate; la Fig. (5.5) ed il testo che segue vogliono $y \to z$; nota: ora $kz_{st} = mg$ dopo Fig. (5.6)
4. tutte le x in section (5.2.3) si salvano mettendo un riferimento esplicito alla Fig. (5.2); ne segue però che, in section (5.2.3.2), dobbiamo $y \to x$, $Y \to x$; nella section (5.2.3.3), per non entrare in conflitto con la X, possiamo dire che cambiamo leggermente la notazione e lasciamo a secondo membro la y.
5. per quanto riguarda la section (5.3), per salvare il testo, ruotiamo Fig. (5.14), Fig. (5.15), Fig. (5.16) Fig. (5.17)
6. per le Fig. (5.20) e Fig. (5.21) $x \to z$; attenzione, ho preparato nuovi schizzi per essere coerenti con i valori nel nuovo nb (Chap5-Fig5-21- verifica.nb); anche il grafico nel nb relativo alla Fig. (5.22) va ritoccato
7. in Fig. (5.24) bisogna cambiare solo $y \to z$ nella figura; nella caption attenzione, ora ruota intorno a y etc...
8. Conclusione: a me pare la soluzione meno indolore, ma manca anche il *placet* di MD per la rotazione delle figure Fig. (5.14), Fig. (5.15), Fig. (5.16) e Fig. (5.17)
9. ============= TODO sparsi ==============
10. dire che in questo capitolo usiamo le { }
11. p195 OK NBB (+2): il rosso, tolto!
12. p195 OK NBB *In this case...* in rosso; togliere!

13. p195 OK NBB (-4): il rosso; togliere!

14. p196 nota 3, su frequency content; ora è un lusso; togliere!

15. p196 nota 4: richiesta NBB; OK, spiegato! *discrete*

16. p196 OK NBB, (-2) dopo section (5.1): modificato! e togliere! il rosso

17. p197 in Fig. (5.2): manca $x_g(t)$; $\dot{x}$ invece della barretta; decidere come indicare il *ground*, coerente con le altre figure (5.1), (5.3), (5.4), (5.5), (5.14), (5.15), (5.16), (5.17), (5.20), (5.21)

18. p197 nota 6, per Hooke NBB nota storica, indicato! rif e tolto! il rosso

19. p197 OK NBB (-6), *The work*; togliere! il rosso

20. p197 nota NBB (pdp), togliere! perché $x_g(t)$ comparirà in Fig. (5.2)

21. p199 nota 7 (pdp) ; GZ, aggiunto!

22. p201, aggiunto! rif a Puri in nota 8

23. p201 nota MD e NBB; è ideale: neanche nel vuoto siderale, in quanto trascura l'attrito intrinseco interno; lasciare inalterato od aggiungere una nota *in ideal conditions* ?

24. p202 la Fig. (5.6) forse andrebbe meglio metterla insieme a Fig. (5.6), Fig. (5.9), Fig. (5.10), Fig. (5.11) con un GridArray; in ogni modo sono bruttine; decidere!

25. p202 modificato! dopo *If we recall...*

26. p202 pdp, nota 9 NBB, molto incerto se fare un rimando alle Navier-Stokes; toglierei!

27. p202 in Fig. (5.7); 1D - one dimension, decidere!

28. p204 NBB, migliorare la Fig. (5.8) e rimpolpare?

29. p204 pdp, modificata la frase con rinvio a (5.2.19)

30. p205, nota MD e GZ: non ricordo, toglierei!

31. p206, nota 10 NBB: OK, aggiunti! due esempi (galva e cannon)

32. p207, nota 11, messi! due rif, ¿sceglierne uno?

33. p208, dubbio! se inserire nb (tipo per Balestra, Marocchi con uso di NonLinearFit); vedi Fig. (5.2.59); per esempio delcavendish-v7-21May2012-V804.nb in E-Balestra oppure cavendish-62.nb in E-Marocchi

34. p209, OK alle tre note NBB: i rossi, togliere!

35. p209, qui si ripresenta il problema degli assi indicato in 5 TODO

36. p209, nota 11; messa! una piccola nota ...da migliorare!
37. p210, dopo (eq:no11 5.2.59), togliere! il marginpar; il nb è quello a pdp Chap5-1-1DOF...(ma verificare!)
38. p211, (+2) solito problema degli assi
39. p211, nota 12, mettere somewhere una definizione di *harmonic*; $\sin(\pi t)$ è *harmonic*?; ma già prima a p209
40. p212, per nota MD inserito! un *the above ...*
41. p213, modificato! $r > \sqrt{2}$
42. p213, solito problema notazione assi in (complex 5.2.3.3)
43. p215 nel titolo (2m3k 5.14) k andato! in *springs*
44. p216 giusto! il segno in (Lagrange 5.3.4); Meiro p255
45. p217 ¿migliorare? l'impaginazione, troppi *bianchi*
46. p218 OK tre NBB, togliere! i rossi
47. p218, aggiunta appendice R sulle ω; ¿letta e commentata?
48. p219, modificato! in nota 15 il rif al nb ...4DOF-2m2k-...
49. p215, p222, p224 e p225, problema dei pedici + tratteggio del *ground* in (2m3kbis 5.15)e (firstm 5.16), (secondm 5.17)
50. p225, ...*The following graphs*... Check il **Check** delle figure
51. p225, rivedere il commento di NBB *Invece di radunare...*
52. p226, OK due NBB, togliere! i rossi
53. p227, la Fig. (plat2dof 5.20) è da rifare come scritto nella caption (vedi mio schizzo a matita); definire il senso positivo di θ; ancora, coerenza tra C.G, in figura e C_m nel testo
54. p227, check NBB *platform angular mass...*
55. p227, nota 16; si potrebbbe accennare ad latre coppie di coordinate; gz, da fare
56. p227, ho girato! (pdp) la frase di NBB; togliere! i rossi
57. p228, invertito! k_2 con l_2 in (V60_B 5.3.78)
58. p228, ho tolto! la nota 19 (inoltre era spostata (?) a p229; abbiamo gà un rimando alle ω

59. p229, la Fig. (5.21) è da rifare coerente con il nuovo schizzo; dai conti risulta che non è coerente con IMGC; check! nuovi conti in Chap5-Fig5-21-verifica.nb

60. p230, ora la nota NBB (a metà p circa) è a posto; tolti! i rossi

61. p230, dubbio! se mettere il nb della nota 19; check! i nb in nota 21

62. p231, mancano le caption di Fig. (5.22) e Fig. (5.23); (gz)

63. p231, controllare! le conclusioni (pdp, -2)

64. p232, non capisco nota NBB (+3)

65. p232, la Fig. (5.24) va ritoccata: con grigi diversi, anche per i CG; manca il vettore PP"; etc... dovrebbe ruotare intorno a y se accettiamo proposta al punto 7 (rosso)

66. p232, OK nota NBB (-2), toglere! il rosso

67. p233, attenzione a $\mathbf{R}(\mathbf{r})$ in (5.4.1) vs $R\mathbf{r}$ in (5.4.40)

68. p233, attenzione a $\wedge$ vs x; vedi anche nota a margine

69. p234, caption in Fig. (5.25) dopo averla rifinita; cercare modelli in Ward, credo

70. p235, OK per NBB (pdp), togliere! i rossi

71. p236, OK per NBB *In the same...*, togliere! i rossi

72. p237, nota in rosso NBB; è un problema di notazione, bisogna decidere tra $\alpha, \mathbf{U}, \boldsymbol{\Theta}, \ldots$; ¿perché non lasciamo α ?

73. p238, nota NBB (+3), ritorna il problema sui tensori; a me pare illuminante il caso del tensore di Inerzia (vedi section 14 in Brennon)

74. p238, OK per NBB (-2), togliere! i rossi

75. p241, è chiaro il collegamento con FEM?; inoltre, verificare! l'aggancio con Ansys (vedi Chap3 Ela)

76. p241 (-1), togliere! i rossi

In the Introduction, the small double arrow in Fig. (??) indicates that the movement of the analyzer has to satisfy two conditions. First, the ρ and θ angles (pitch and yaw, respectively) must be controlled accurately, i.e., the maximum $\Delta\rho$ and $\Delta\theta$ admissible must be kept within 1 nrad and 1 μrad, respectively[1]. Second, the parasitic vibrations caused by several sources of mechanical and/or acoustic noise, must not perturb the signals monitored by the four detectors. It is therefore mandatory to design an anti-vibration mounting ([?], [?]) to protect the kernel of the experiment and avoid the corrugations of the x-ray and optical fringes. NBB

To give an idea of a possible assembly forming an anti-vibration mounting, we show in Fig. (5.1) a simplified model of a two-degree-of-freedom system in which

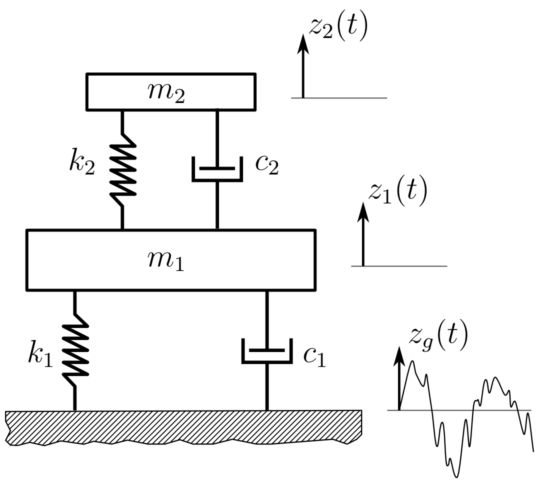


Figure 5.1: Two-degree of freedom anti-vibration mounting system. The elastic supports (springs) are characterized by the stiffness coefficient k and the dampers by the viscosity coefficient c. $x_1(t), x_2(t)$ denote the displacement of the masses from their condition of equilibrium, along the vertical axis x; the movement of the ground is described by $x_{\mathrm{g}}(t)$; Chap5-Fig-5-1-piattaforma.pdf

piattaforma

the mouvement of the masses $m_1(t), m_2(t)$, induced by the ground motion $x_{\mathrm{g}}(t)$, is confined along the vertical direction and no rotations are allowed. The system may therefore be described by the following equations

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} k_1 x_g + c_1 \dot{x}_g \\ 0 \end{Bmatrix} \tag{5.0.1}$$

eqsys

where $x_1(t), x_2(t)$ denote the displacements of the masses m_1, m_2 from their condition of equilibrium, the constants k_1, k_2 represent the spring (or stiffness) constants and c_1, c_2 are called coefficients of viscous damping[2]. The search of the solutions of the above equations aims to reduce the ratio $x_2(t)/x_{\mathrm{g}}(t)$ to the smallest (possible) value by choosing adequately the quantities m, k and c and taking into account NBB

[1] the remaining $\Delta\psi$ roll angle is easily kept within 1 mrad

[2] the mechanism by which the vibration energy is gradually converted into heat or sound can be modelled in different ways; here we consider only the damping arising when, for example, a fluid flows around a surface

the frequency content[3] of the perturbing signal $x_g(t)$. Consequently, we start by giving in section 5.1 some simple considerations on the elements constituting a discrete mechanical system[4]. Then, in section 5.2.1, we consider the response of a single degree of freedom (1-DOF) when no external excitation is present; for this reason, we speak about *free* response. In section 5.2.3 we examine the response of the same system when the excitation is in the form of initial displacements or initial velocities or both; the excitation can also depend on forces which persist for an extended period of time; we speak in this case of *forced* response. The study of the influence of the ground motion is particularly important because we derive a relationship between the natural frequency of the anti-vibration mounting and the lowest frequency of $x_g(t)$ in order to have a reduction of the amplitude of the disturbing signal. In section 5.3 we consider two-degree-of-freedom systems, either when two masses m_1, m_2 can translate only along the vertical axis, or when a single mass can translate along the vertical axis and rotate around its centre of mass. Two methods, based on Newton's second law and on Lagrange equations, will be used to derive the governing equations and to consider the relevant natural frequencies. Finally, in section 5.4 we examine the dynamic response of a rigid body (in practice, a 4200-kg concrete block) representing an anti-vibrating mounting characterized by six degrees of freedom; we use a matrix formulation and the solution of the corresponding differential equations are illustrated in a *Mathematica* notebook and compared with the results obtained through a finite element code.

Several notebooks illustrate different methods either to find out the natural frequencies of the system or to investigate the dependence of the solution of the differential equations on the parameters which characterize the system. Although the given examples focus the attention on basic concepts which are preparatory to more complex situations, they can supply useful information and hints during the design phase.

5.1 Elements of a vibratory system

Element

NBB like that mi sembra pesante

NB

To predict the dynamical behaviour of a system governed by Eq .(5.0.1), it is convenient to start at the beginnings by introducing some elementary concepts. Therefore, let us consider in Fig. (5.2) the elements of one of the simplest vibratory systems.

[3] qui ci vorrebbe un rimando ad una appendice dove si dice come ricava lo spettro, pensiamoci se avremo tempo

[4] With *discrete* we refer to a large class of systems which can be described by lumping their masses and moments of inertia

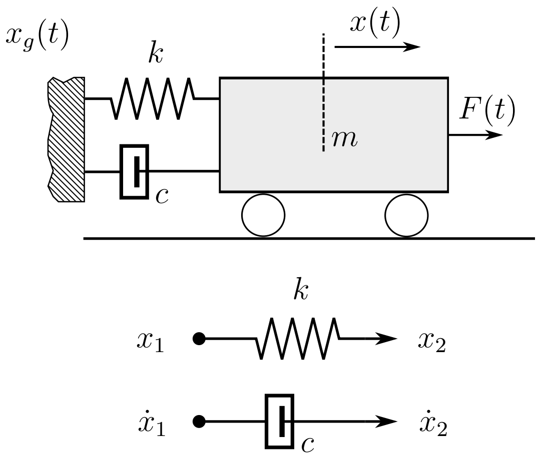


Figure 5.2: Elements of a mass-spring system in horizontal position; k, c, m denote the spring, the damper (dashpot) and the mass, respectively; $F(t)$ and $x_g(t)$ indicate excitations; Chap5-Fig-5-2-moriz-ab-Bozza.pdf

moriz

The three basic elements, the *mass*, the *spring* and the *damper* are represented in a rather idealized way. If the *mass* m is assumed to be a rigid body, it represents the coupling between force F_m and acceleration $\ddot{x}$ according to Newton's law of motion

$$F_m = m\ddot{x}. \tag{5.1.1}$$

The *spring* k is assumed to be elastic and of negligible mass[5]. A spring force F_s exists if the spring is extended or compressed, that is, when there is a relative displacement between the two ends x_1 and x_2 of the spring. If the spring deformation is proportional to the spring force, the spring obeys Hooke's law[6] NBB

$$F_s = k(x_2 - x_1), \tag{5.1.2}$$

spri

where the constant of proportionality k is called *stiffness* or *spring constant*. The units of k are newton per meter (N/m).

The element relating forces to velocities is assumed to be massless and is generally known as *viscous damper* or *dashpot*. In practice, it can consist of a piston fitting loosely in a cylinder filled with viscous fluid flowing around the piston. If the damping force F_d is proportional to the velocities $\dot{x}_1$ and $\dot{x}_2$ of the ends of the dashpot, we can write

$$F_d = c(\dot{x}_2 - \dot{x}_1), \tag{5.1.3}$$

dash

where the constant of proportionality is called *coefficient of viscous damping* c. The units of c are newton · second per meter (N s/m). The work done by moving the dashpot is converted into heat, so the damping element cannot be considered conservative. NBB

The excitations of the system from its rest position can be given in the form of initial displacements and velocities, or in the form of externally applied forces to the mass $F(t)$ and/or to the base $x_g(t)$.

per base intendiamo la parete di sinistra?

[5] also damping, due to internal friction, is negligible

[6] The Hooke's law is named after the British physicist Robert Hooke (1635-1703) and was published as the solution *Ut tensio sic vis* of an anagram. For further details, see section (??) stre-stra

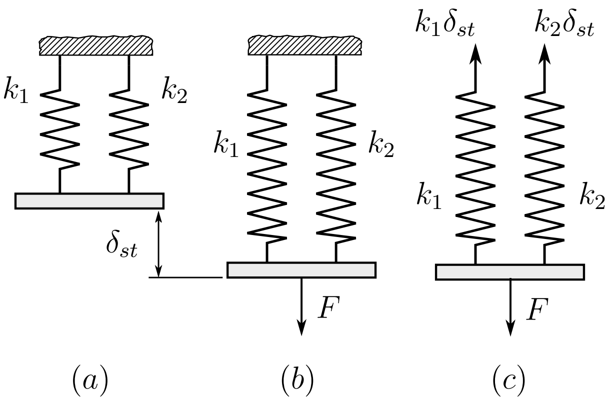
Even if the previous discussion deals with translational motion, analogous relationships can be derived for systems undergoing torsional vibrations.

5.1.1 Equivalence of Systems

Many other systems composed by spring and masses can be reduced to the simple system of Fig. (5.2), thus representing an equivalent system to be studied. The equivalence may be achieved by combining several springs into a single equivalent spring. We can distinguish two main cases.

Case 1: Springs in parallel

When a force F is applied to the two springs k_1 and k_2 in Fig. (5.3), the system



paral **Figure 5.3:** Springs in parallel. Rao p24; da Chap5-Fig-5-5-paral-ab-Bozza.pdf

undergoes a static deflection δ_{st} and two equivalent reactions $F_1 = k_1\delta_{st}$ and $F_2 = k_2\delta_{st}$ are generated. Hence, if we denote by k_{eq} the stiffness of an equivalent spring representing the combined effect of k_1 and k_2, the equilibrium equation

$$F = k_{eq}\delta_{st} = k_1\delta_{st} + k_2\delta_{st} \tag{5.1.4}$$

yields

$$k_{eq} = k_1 + k_2. \tag{5.1.5}$$

Case 2: Springs in series

Next we consider the two spring in Fig. (5.4). Under the action of the force F, springs k_1 and k_2 undergo elongations δ_1 and δ_2, respectively. Since both springs are subjected to the same force F, we have

$$F = k_1\delta_1 F = k_2\delta_2. \tag{5.1.6}$$

k1k2

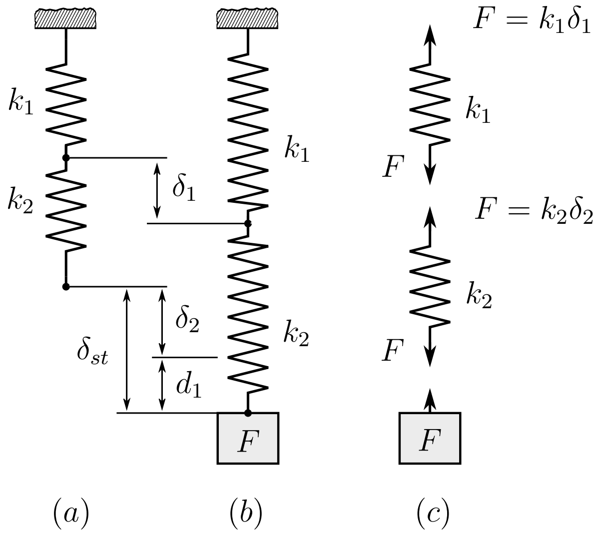
If k_{eq} represents the equivalent spring constant, then we must have

$$F = k_{eq}\delta_{st}, \tag{5.1.7}$$

keq

and, from Eq. (5.1.6) and Eq. (5.1.7), we get

$$k_{eq}\delta_{st} = k_1\delta_1 = k_2\delta_2 \tag{5.1.8}$$



series

Figure 5.4: Springs in series. Rao p25; da Chap5-Fig-5-6-series-ab-Bozza.pdf

or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2}. \tag{5.1.9}$$

If we substitute these values into the equation $\delta_{st} = \delta_1 + \delta_2$, we obtain

$$\delta_{st} = \frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} \tag{5.1.10}$$

and we can write

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{5.1.11}$$

5.2 Systems with a Single Degree of Freedom

1DOF

From a general point of view, the equation governing the motion of the system in Fig. (moriz 5.2), using Newton's second law applied to the forces acting on the mass m, can be written as

$$m\ddot{x}(t) = F(t) - F_s(t) - F_d(t) \tag{5.2.1}$$

and, using Eq. (spri 5.1.2) and Eq. (dash 5.1.3), we have

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t), \tag{5.2.2}$$

New1

having assumed that the spring is weightless.[7] The static equilibrium position of the system, in the absence of external forces, coincides with the position in which the spring is unstretched.

When the spring is stretched, the stored elastic potential energy can be found by calculating the work necessary to obtain the given stretching $x = x_2 - x_1$, that is

$$U_{ela} = -\int_0^x F_s\, dx. \tag{5.2.3}$$

[7]We shall keep this assumption in the following

In the case of a hookean linear spring, we get

$$U_{ela} = -\int_0^x (-kx)dx = \frac{1}{2}kx^2. \tag{5.2.4}$$

If we were dealing with the system of Fig. (5.5), where the force of gravity has to be considered, we would reach a slightly different conclusion.

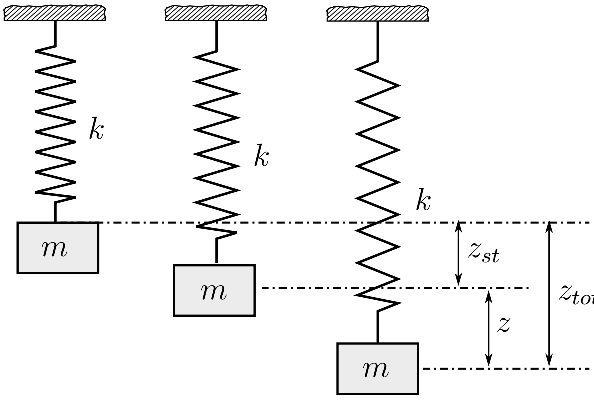


Figure 5.5: A mass-spring system in vertical position; appunti miei; da Chap5-Fig-5-3-mvert-ab-Bozza.pdf

On the left of Fig. (5.5) we have omitted for simplicity the viscous effects and the spring k, as already said, is assumed weightless. At the center, the mass m hangs at the lower end of the spring and reaches its static equilibrium position when $mg = ky_{st}$. On the right of Fig. (5.5), the mass is further deflected a distance y from its static equilibrium position; therefore, the application of the Newton's second law gives

$$m\ddot{y}_{tot} = mg - k(y_{st} + y). \tag{5.2.5}$$

If we remember that $y_{tot} = y_{st} + y$ so that $\ddot{y}_{tot} = \ddot{y}$, we can write

$$m\ddot{y} = mg - ky_{st} - ky = -ky. \tag{5.2.6}$$

We are now interested in evaluating the net potential energy of the system when the mass is extended from the equilibrium position y_{st} to the deflected position y_{tot}. The gain of elastic energy is given by

$$U_{ela} = \frac{1}{2}ky_{tot}^2 - \frac{1}{2}ky_{st}^2 = \frac{1}{2}ky_{st}^2 + \frac{1}{2}ky^2 + ky_{st}y - \frac{1}{2}ky_{st}^2 = \frac{1}{2}ky^2 + mgy. \tag{5.2.7}$$

At the same time, the gravitational potential energy due to the change in elevation of the mass amounts to $U_{grav} = -mgy$. Therefore,

$$U_{sis} = U_{ela} + U_{grav} = \frac{1}{2}ky^2 + mgy - mgy = \frac{1}{2}ky^2, \tag{5.2.8}$$

and the net potential energy of the entire system depends only upon the stretching of the spring from its equilibrium position. We conclude that when a mass oscillates along a vertical direction, we can ignore the effect of gravity, provided that we measure its displacement y from its static equilibrium position. When damping elements and external forces are taken into account, the equation of motion describing the vertical system will be the same as the one we found for the horizontal one. The class of systems whose motion can be studied by means of Eq. (5.2.2) has a single degree of freedom, namely, the variable $y(t)$ or $x(t)$. Systems with a single degree of freedom will be the main subject of this section.

5.2.1 Free vibrations without damping

sec:vibnoc

If we set c and $F(t)$ equal to zero in Eq. 5.2.2, the equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0\,. \tag{5.2.9}$$

eq:freendeq

In order to solve the above equation we postulate solutions of the form

$$x(t) = C\,e^{st}\,, \tag{5.2.10}$$

eq:freendsol

where C and s are constants to be determined. If we substitute Eq. (5.2.11) into Eq. (5.2.9) we shall find

$$s_{1,2} = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\,\omega_n\,, \tag{5.2.11}$$

eq:freendsol

where

$$\omega_n = 2\pi f = \left(\frac{k}{m}\right)^{1/2} \tag{5.2.12}$$

is known as the *natural angular frequency* of the system, that is the frequency at which the system naturally oscillates when perturbed from its static equilibrium position[8] . The general solution of the differential equation is obtained by combining the two particular solutions we just found (the *eigenfunctions* of the system)

MD: NBB corretto? GZ eanche nel vuoto siderale, rimane l'attrito interno

$$x(t) = C_1 e^{i\omega_n} + C_2 e^{-i\omega_n}\,. \tag{5.2.13}$$

eqC1C2c

By using the identities

$$e^{\pm i\omega_n t} = \cos\omega_n t \pm i\sin\omega_n t, \tag{5.2.14}$$

Eq. (5.2.13) becomes

$$x(t) = K_1\cos\omega_n t + K_2\sin\omega_n t, \tag{5.2.15}$$

ICO

where the values of the constants K_1 and K_2 depend on the initial displacement $x(0)$ and initial velocity $\dot{x}(0)$. For example, if $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, we have from Eq. (5.2.15)

$$x(0) = x_0 = K_1 \tag{5.2.16}$$

$$\dot{x}(0) = \dot{x}_0 = \omega_n K_2 \tag{5.2.17}$$

solIC

and the solution of Eq. (5.2.9), subjected to the initial conditions (5.2.17), is given by

$$x(t) = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t \tag{5.2.18}$$

solEQ

or, equivalently, by

$$x(t) = \sqrt{x_0^2 + (\frac{\dot{x}_0}{\omega_n})^2}\cos(\omega_n t - \arctan\frac{\dot{x}_0}{x_0\omega_n}). \tag{5.2.19}$$

solEQ2

An example of solutions of this kind is given in Fig. (5.6)

[8] If we do assume that the spring has a mass M, it can be shown that $\omega = \sqrt{k/(m+M/3)}$, ([?])

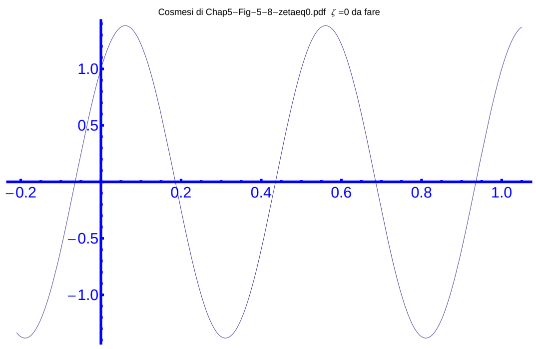


Figure 5.6: Undamped oscillations; da Book-Chap5-AntiVib-Fig&NB/Chap5-Fig-5-8-zetaeq0.pdf

subcase1

If we recall that for the vertical system of Fig. (5.5) $ky_{st} = mg$, we obtain, for the natural frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{mg}{\delta_{st} m}} = \frac{1}{2\pi}\sqrt{\frac{9.805}{\delta_{st[mm]}\cdot 10^{-3}}} \approx \frac{15.76}{\Delta_{mm}}. \qquad (5.2.20)$$

deltamm

The simple graphic of Fig. (5.7) shows the dependence of the natural frequency of the system on the static equilibrium deflection y_{st} expressed in millimetres and permits to estimate what elongation a spring has to have in its static equilibrium position to let the perturbed system oscillate with a certain frequency f. For example, to have a natural frequency of 0.5 Hz (i.e., a natural period of 2 seconds), the elongation of the spring, when the system is in static equilibrium, must be about 1 meter.

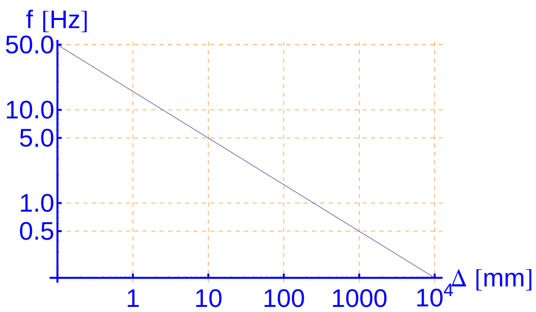


Figure 5.7: How the natural frequency of a 1D system depends on the extension of the spring; Book-Chap5-AntiVib-Fig&NB/Chap5-Fig-5-4-deltastat.nb

deltast

5.2.2 Free vibrations with damping

1DOFDamping

To study the effect of the viscous[9] damping c on the solution of Eq. (5.2.2), that we repeat here for convenience, NBB

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t), \qquad (5.2.21)$$

New2

[9] To calculate c we could refer to the Kelvin-Voigt model using Hagen-Poisuille equation. Actually the viscoelasticity interpretation is poor because we miss an analytical solution of Navier-Stokes equation... io l'ho inserita cosi può avere senso un richiamo simile?

we consider the general case of free vibrations. In absence of external forces, $F(t) = 0$, the equation of motion reduces to a homogeneous ordinary differential equation with constant coefficients. By analogy with the undamped case, we assume a solution in the form

$$x(t) = Ce^{st} \tag{5.2.22}$$

where C and s are constants to be determined. If we insert this function into Eq. (5.2.21) we obtain

$$ms^2 + cs + k = 0, \tag{5.2.23}$$

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}. \tag{5.2.24}$$

These roots give the following solutions of Eq. (5.2.21)

$$x_1(t) = C_1 e^{s_1 t} \text{ and } x_2(t) = C_2 e^{s_2 t}, \tag{5.2.25}$$

and therefore the general solution of Eq. (5.2.21) is obtained by a combination of the two solutions $x_1(t), x_2(t)$,

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

more explicitly,

$$x(t) = C_1\, e^{\left[\frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right] t} + C_2\, e^{\left[\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right] t},$$

where C_1, C_2 are constants to be determined from the initial conditions.

It is convenient to express the above solutions in terms of a dimensionless parameter. To this end we define the critical damping C_c as the value of the damping constant c for which the radical in Eq. (5.2.24) becomes zero

$$\left(\frac{C_c}{2m}\right)^2 - \frac{k}{m} = 0, \tag{5.2.26}$$

that is,

$$C_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n = 2\sqrt{km}, \tag{5.2.27}$$

where ω_n is the natural frequency of the undamped oscillation, defined in the previous section. For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant

$$\zeta = \frac{c}{C_c} \tag{5.2.28}$$

If we use Eq. (5.2.28) and Eq. (5.2.27) we can write

$$\frac{c}{2m} = \frac{c}{C_c}\frac{C_c}{2m} = \zeta\,\omega_n \tag{5.2.29}$$

and, hence

$$s_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n = (-\zeta \pm \sqrt{\zeta^2 - 1})\omega_n \tag{5.2.30}$$

rad12

Then, the solution of the Eq. (5.2.21), divided by m and with $F(t) = 0$, takes the form

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0, \tag{5.2.31}$$

New3

with the corresponding solutions

$$x(t) = C_1 e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + C_2 e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{5.2.32a}$$

eqC1C2aa

$$= \left(C_1 e^{\sqrt{\zeta^2-1}\omega_n t} + C_2 e^{-\sqrt{\zeta^2-1}\omega_n t}\right) e^{-\zeta\omega_n t}. \tag{5.2.32b}$$

eqC1C2bb

The nature of the solutions s_1 and s_2 depends on the value of ζ which can be represented in the complex plane. In Fig. (5.8) the horizontal and vertical axes

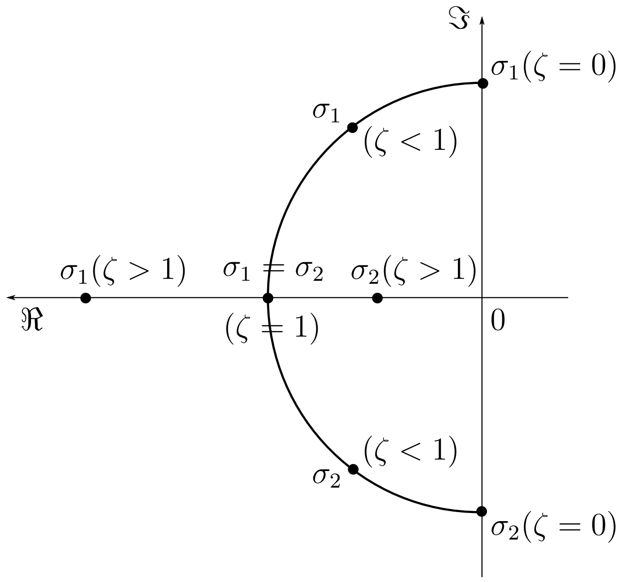


Figure 5.8: Rao p134 The semicircle represents the locus of the roots s_1 and s_2 when $0 < \zeta < 1$; rimpolpare !; da Chap5-Fig-5-7-radicis1s2.pdf

radici

are chosen as the real and imaginary axes and we see immediately the effect of the parameter ζ on the behaviour of the system. We can consider quantitatively how the response of the system depends on the damping ratio.

5.2.2.1 Undamped system ($\zeta = 0$ or $c = 0$)

In this case, the solutions of the characteristic equation Eq. (5.2.23) are

$$s_{1,2} = \pm\left(\frac{k}{m}\right)^{1/2} = \pm i\omega_n$$

and $x(t)$ is represented by (5.2.19).

5.2.2.2 Underdamped system ($\zeta < 1$ or $c < C_c$)

As the solutions of the characteristic equation are

MD: GZ: nb Mathematica ?

$$s_{1,2} = \left(-\zeta \pm i\sqrt{1-\zeta^2}\right)\omega_n\,,$$

we can write (eqC1C2bb 5.2.32b) more conveniently as

$$\begin{aligned}
x(t) &= e^{-\zeta\omega_n t}\left(C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t}\right)\\
&= e^{-\zeta\omega_n t}\left((C_1+C_2)\cos\sqrt{1-\zeta^2}\omega_n t + i(C_1-C_2)\sin\sqrt{1-\zeta^2}\omega_n t\right)\\
&= e^{-\zeta\omega_n t}\left(\tilde{C}_1\cos\sqrt{1-\zeta^2}\omega_n t + \tilde{C}_2\sin\sqrt{1-\zeta^2}\omega_n t\right)\\
&= e^{-\zeta\omega_n t}\left(\tilde{C}_1\cos\omega_d t + \tilde{C}_2\sin\omega_d t\right),
\end{aligned}$$

where $\omega_d = \sqrt{1-\zeta^2}\omega_n$ is called the *angular frequency of damped vibration* and is always less than the undamped angular frequency ω_n. The constants $\tilde{C}_1$ and $\tilde{C}_2$ can be found by imposing the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$; we obtain

$$\tilde{C}_1 = x_0 \ \text{ and } \ \tilde{C}_2 = \frac{\dot{x}_0 + \zeta x_0\omega_n}{\omega_d}. \tag{5.2.33}$$

Hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left(x_0\cos\omega_d t + \frac{\dot{x}_0 + \zeta x_0\omega_n}{\omega_d}\sin\omega_d t\right), \tag{5.2.34}$$ eqpiuIC

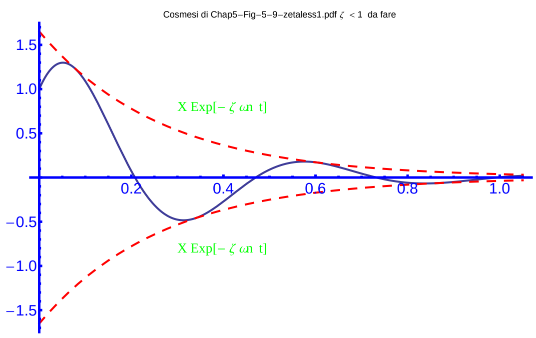
or, equivalently

$$x(t) = Ae^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\omega_n t - \phi_0\right) \tag{5.2.35}$$

where

$$A = \sqrt{\tilde{C}_1^2 + \tilde{C}_2^2} \ \text{ and } \ \phi_0 = \arctan\left(\tilde{C}_2/\tilde{C}_1\right). \tag{5.2.36}$$

The combined result of a decreasing exponential and a sine wave is a *damped sine wave* oscillating in the space between the exponential curve and its mirrored image, as shown in Fig. (subcase2 5.9)



subcase2

Figure 5.9: Free vibrations with $\zeta < 1$ Rao 132; Book-Chap5-AntiVib-Fig&NB/Chap5-Fig-5-9-zetaless1.pdf

5.2.2.3 Critically damped system ($\zeta = 1$ or $c = C_c$)

In this case the two roots s_1 and s_2 in Eq. (5.2.30) are equal

$$s_1 = s_2 = -\omega_n = -\frac{c}{2m} = -\frac{C_c}{2m} \tag{5.2.37}$$

If we let ζ approach unity in the limit of Eq. (5.2.30), we have $\omega_d \to 0$, $\cos\omega_d t \to 1$, $\sin\omega_d t \to \omega_d t$. Hence, the last of Eqs. (5.2.33) yields

$$x(t) = e^{-\omega_n t}\left(\tilde{C}_1 + \tilde{C}_2\omega_d t\right). \tag{5.2.38}$$

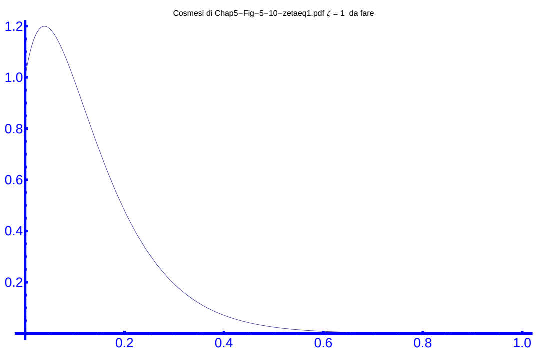
If we apply the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, we have

$$\begin{aligned} \tilde{C}_1 &= x_0 \\ \tilde{C}_2 &= (\dot{x}_0 + x_0\omega_n)/\omega_d \end{aligned} \tag{5.2.39}$$

and, finally

$$x(t) = e^{-\omega_n t}\left[x_0 + (\dot{x}_0 + x_0\omega_n)t\right] \tag{5.2.40}$$

which represents an aperiodic response as shown in Fig. (5.10). It is interesting to note that, for a given initial excitation a critically damped system reaches the equilibrium position without oscillating in the fastest way[10].



subcase3 **Figure 5.10:** Free vibrations with $\zeta = 1$, Rao 132; Book-Chap5-AntiVib-Fig&NB/Chap5-Fig-5-10-zetaeq1.pdf

5.2.2.4 Overdamped system ($\zeta > 1$ or $c > C_c$)

The solutions of the characteristic equation Eq. (5.2.23) are

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n < 0 \tag{5.2.41}$$ s22

with $s_2 \ll s_1$. Then the solutions take the form

$$x(t) = \left(C_1 e^{\sqrt{\zeta^2-1}\omega_n t} + C_2 e^{-\sqrt{\zeta^2-1}\omega_n t}\right) e^{-\zeta\omega_n t} \tag{5.2.42}$$

[10] For example, in a balistic galavanometer or in a barrel of a cannon the recoil mechanisms are critically damped

with

$$C_1 = \frac{x_0\,\omega_n\left(\sqrt{\zeta^2-1}+\zeta\right)+\dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{5.2.43}$$

$$C_2 = \frac{x_0\,\omega_n\left(\sqrt{\zeta^2-1}-\zeta\right)-\dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{5.2.44}$$

An example of a typical solution of this kind is given in Fig. (5.11)

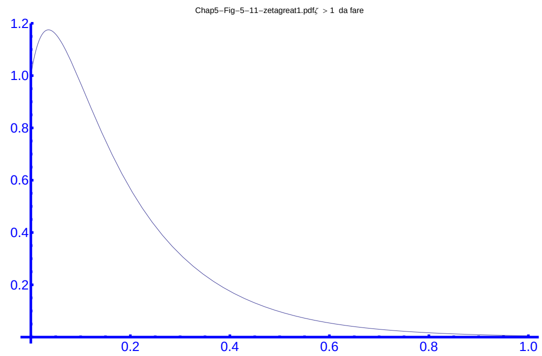


Figure 5.11: Free vibrations with $\zeta > 1$ Rao 133; Book-Chap5-AntiVib-Fig&NB/Chap5-Fig-5-11-zetagreat1.pdf

subcase4

5.2.2.5 Logarithmic decrement

Very often the amount of damping in a given system cannot be evaluated analytically[11]. However, in many practical cases in which damping is viscous and the system is underdamped, we can experimentally observe the rate at which the amplitude of the free damped vibrations decreases.

If we indicate with t_1 and t_2 the times corresponding to two consecutive displacements x_1 and x_2 measured one period apart (e.g., in correspondence of the first two maxima in Fig. (5.12)),

we can form the ratio

$$\frac{x_1}{x_2} = \frac{Ae^{-\zeta\omega_n t_1}\cos(\omega_d t_1-\phi_0)}{Ae^{-\zeta\omega_n t_2}\cos(\omega_d t_2-\phi_0)}. \tag{5.2.45}$$

eqlog

Because $t_2 = t_1 + T_d$, where $T_d = 2\pi/\omega_d$ is the period of the damped vibration, we have

$$\cos(\omega_d t_2-\phi_0) = \cos(\omega_d t_1 + 2\pi - \phi_0) = \cos(\omega_d t_1 - \phi_0), \tag{5.2.46}$$

so that Eq. (5.2.45) reduces to

$$\frac{x_1}{x_2} = \frac{Ae^{-\zeta\omega_n t_1}}{Ae^{-\zeta\omega_n (t_1+T_d)}} = e^{\zeta\omega_n T_d}. \tag{5.2.47}$$

eqlog3

[11] In the case of a piston of diameter d and length L, with two holes of diameter D, assuming that the oil has a viscosity η and density ρ, we have a damping constant $c = 4\pi L\eta(d/D)^4$ ([?],[?])

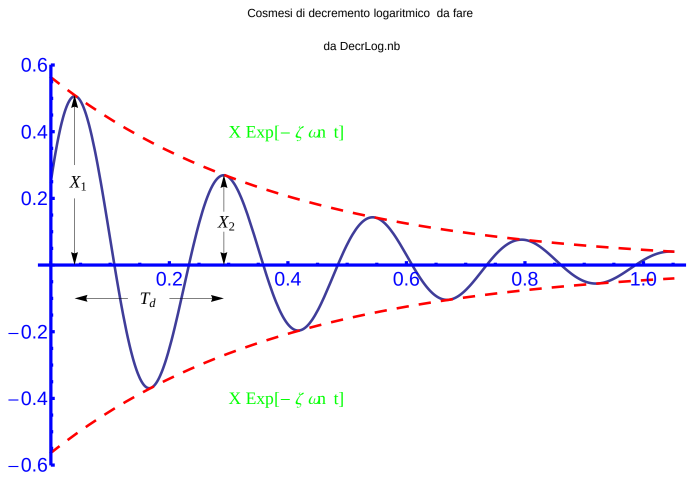


Figure 5.12: Two consecutive maxima of an underdamped system; Meiro p30 a mano; da Chap5-Fig-5-12-DecrLog.pdf; ? mio nb Mathematica per Balestra-Marocchi ? DecrLog

If we now introduce the notation

$$\delta = ln\frac{x_1}{x_2} = \zeta\omega_n T_d = \zeta\omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{5.2.48}$$ deltaandzeta

with δ known as the *logaritmic decrement*, we can obtain ζ by measuring two consecutive displacements x_1 and x_2; in fact,

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}. \tag{5.2.49}$$ delta

When ζ is small, Eq. (5.2.48) can be approximated by

$$\delta \approx 2\pi\zeta, \tag{5.2.50}$$

as illustrated in Fig. (5.13)

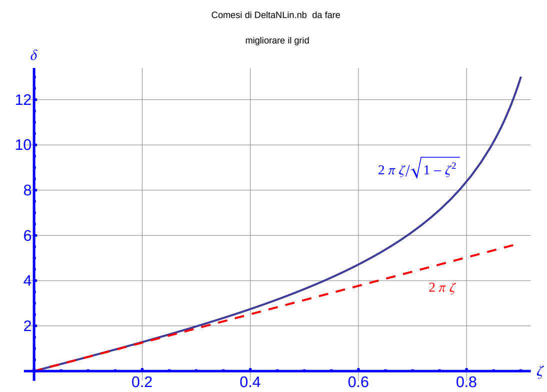


Figure 5.13: Dependence of logarithmic decrement with damping; Rao 136 p30; da Chap5-Fig-5-13-deltaNLin.pdf deltaNLin

5.2.3 Forced vibrations

The amplitude of the damped vibrations considered in the previous section decays with time and after some time the system comes to rest, because there is a continuous dissipation of energy. However, it is possible to keep up these vibrations applying a force or imposing a displacement to the system. When the frequencies of the driving and driven system are not the same, the amplitude corresponding to the natural frequency of the oscillator dies out and it begins to oscillate with the frequency of the impressed excitation. These are called forced vibrations. When the frequency of the driving force is near or coincides with the natural frequency of the driven oscillator, it appears the phenomenon of resonance. In this section we begin the discussion with simple harmonic excitation[12] due to their fundamental nature and practical applications. The case of a periodic excitation can be reduced to that of a harmonic excitation. We shall consider two cases: when the excitation is applied directly to the mass and when the base on which the mass rests is subjected to a shacking action.

5.2.3.1 Driving force applied directly to the mass m

MD: qui siamo orizzontali

If the driving function[13] is given by $F(t) = F_0 \sin \omega_f t$, the equation of motion (5.2.21) becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega_f t, \tag{5.2.51}$$

where ω_f is the angular driving frequency. If we divide Eq. (5.2.51) by m and introduce the damping ratio ζ we obtain

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{F_0}{m} \sin \omega_f t. \tag{5.2.52}$$

If we neglect the transient solution, i.e., we limit ourselves to the search of the steady-state solution, we let the solution of Eq. (5.2.52) have the form

$$x = X \sin(\omega_f t - \varphi),$$

which leads to

$$\dot{x} = X\omega_f \cos(\omega_f t - \varphi)$$
$$\ddot{x} = -X\omega_f^2 \sin(\omega_f t - \varphi).$$

From Eq. (5.2.52) we get

$$-X\omega_f^2 \sin(\omega_f t - \varphi) + 2\zeta\omega_n X\omega_f \cos(\omega_f t - \varphi) + \omega_n^2 X \sin(\omega_f t - \varphi) = \frac{F_0}{m} \sin \omega_f t, \tag{5.2.53}$$

and, exploiting the properties of the trigonometric functions,

$$-X\omega_f^2(\sin \omega_f t \cos \varphi - \cos \omega_f t \sin \varphi) + \tag{5.2.54}$$

[12] A possible definition of *simple harmonic motion* is: a type of periodic motion where the restoring force is directly proportional to the displacement of the object, but in the opposite direction.

[13] Loosely speaking, since Fourier analysis tells us that any function can be written in terms of sinusoidal functions, we can limit our discussion with a single term and exploit the principle of superposition

$$+2X\zeta\omega_n\omega_f(\cos\omega_f t\cos\varphi+\sin\omega_f t\sin\varphi)+$$
$$+X\omega_n^2(\sin\omega_f t\cos\varphi-\cos\omega_f t\sin\varphi)=\frac{F_0}{m}\sin\omega_f t,$$

and grouping the common terms we have

$$\sin\omega_f t[-X\omega_f^2\cos\varphi+X\omega_n^2\cos\varphi+2X\zeta\omega_n\omega_f\sin\varphi]+ \tag{5.2.55}$$
$$\cos\omega_f t[X\omega_f^2\sin\varphi-X\omega_n^2\sin\varphi+2X\zeta\omega_n\omega_f\cos\varphi]=\frac{F_0}{m}\sin\omega_f t.$$

eq:no7

Equating the coefficients of $\cos\omega_f t$ and $\sin\omega_f t$

$$X(\omega_n^2-\omega_f^2)\cos\varphi+2X\zeta\omega_n\omega_f\sin\varphi=\frac{F_0}{m} \tag{5.2.56}$$

eq:no8

$$X(\omega_n^2-\omega_f^2)\sin\varphi-2X\zeta\omega_n\omega_f\cos\varphi=0, \tag{5.2.57}$$

eq:no9

and squaring and summing the Eq. (5.2.56) and Eq. (5.2.57) we obtain

$$X^2(\omega_n^2-\omega_f^2)^2+4X^2\zeta^2\omega_n^2\omega_f^2=(\frac{F_0}{m})^2 \tag{5.2.58}$$

eq:no10

and

$$(\omega_n^2-\omega_f^2)\tan\varphi-2\zeta\omega_n\omega_f=0, \tag{5.2.59}$$

eq:no11

from which we deduce the amplitude X

inserire un esempio pratic
può avere senso?

$$X=\frac{\frac{F_0}{m}}{\sqrt{(\omega_n^2-\omega_f^2)^2+4\zeta^2\omega_n^2\omega_f^2}} \tag{5.2.60}$$

eq:no12

and the phase

$$\tan\varphi=\frac{2\zeta\omega_n\omega_f}{\omega_n^2-\omega_f^2}. \tag{5.2.61}$$

eq:no13

Eq. (5.2.60) can be simplified if we put $r=\frac{\omega}{\omega_n}$

$$X=\frac{\frac{F_0}{m}}{\sqrt{\omega_n^4\left[\left(\frac{\omega_n^2-\omega_f^2}{\omega_n^2}\right)^2+4\zeta^2\frac{\omega_n^2\omega_f^2}{\omega_n^4}\right]}}=\frac{\frac{F_0}{m}}{\omega_n^2\sqrt{(1-r^2)^2+4\zeta^2r^2}}= \tag{5.2.62}$$
$$\frac{\frac{F_0}{m}}{\frac{k}{m}\sqrt{(1-r^2)^2+4\zeta^2r^2}}=\frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2+4\zeta^2r^2}};$$

eq:no14

We observe that in the limit $r^2\ll 1$, the response is independent on the mass; for $r=1$, the amplitude of the resonance depends inversely on the damping constant ζ and for $r^2\gg 1$ the response is independent on the spring constant. Analogously, we have

$$\tan\varphi=\frac{2\zeta\frac{\omega_n}{\omega_n^2}\omega_f}{\frac{\omega_n^2-\omega_f^2}{\omega_n^2}}=\frac{2\zeta r}{1-r^2}. \tag{5.2.63}$$

eq:no15

Therefore, if we consider only the steady-state solution we can write

$$x(t)=\frac{\frac{F_0}{k}}{\sqrt{(1-r^2)^2+4\zeta^2r^2}}\sin\left(\omega_f t-\arctan\frac{2\zeta r}{1-r^2}\right). \tag{5.2.64}$$

eq:no16

The notebook `Chap5-1-1DOF-Under-Harmonic-Force.nb` illustrates the behaviour of the amplitude X and phase φ for different values of r.

5.2.3.2 Influence of ground motion

ground

MD: qui siamo verticali

In many instances, for example during the execution of very accurate measurements, it is necessary to consider the effects of the vibrations of the base on which the system rests. Let $Y_s(t) = Y \sin \omega_f t$ denote the displacement of the base and $y(t)$ the displacement of the mass from its static equilibrium position at time t. The equation of motion can be written in the form

$$m\ddot{y} + c(\dot{y} - \dot{Y}_s) + k(y - Y_s) = 0. \tag{5.2.65}$$

eq:no30

The following relations

$$\begin{aligned} Y_s &= Y \sin \omega_f t \\ \dot{Y}_s &= Y \omega_f \cos \omega_f t \end{aligned}$$

yield

$$m\ddot{y} + c\dot{y} + ky = kY \sin \omega_f t + c\omega_f Y \cos \omega_f t, \tag{5.2.66}$$

eq:no31

or, equivalently,

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 Y \sin \omega_f t + 2\zeta\omega_n\omega_f Y \cos \omega_f t. \tag{5.2.67}$$

eq:no32

This means that the excitation of the base is equivalent to applying two different harmonic driving forces to the free system[14]. Then, by splitting Eq. 5.2.67 into two equivalent equations, we can solve them separately exploiting the results obtained in Eq. (5.2.64). Therefore we have,

$$\begin{aligned} &\text{Eq. } \mathbf{A}: \quad && \ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = \omega_n^2 Y \sin \omega_f t \\ &\text{Eq. } \mathbf{B}: \quad && \ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = 2\zeta\omega_n\omega_f Y \cos \omega_f t. \end{aligned}$$

With the help of Eq. (5.2.62), since $\omega_n^2 Y$ is equivalent to $\frac{F}{m}$, in the case of Eq. **A** the amplitude of the steady-state solution is

$$\frac{\omega_n^2 Y}{\omega_n^2 \sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}} = \frac{Y}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}}. \tag{5.2.68}$$

eq:no33

and, in the case of Eq. **B**, being $2\zeta\omega_n\omega_f$ equivalent to F/m, the amplitude is[15]

$$\frac{2\zeta\omega_n\omega_f Y}{\omega_n^2 \sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}} = \frac{2\zeta r Y}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}}. \tag{5.2.69}$$

eq:no34

If we apply the principle of superposition, the solution of Eq. (5.2.67) is

$$\frac{Y}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}} \sin(\omega_f t - \varphi_1) + \frac{2\zeta r Y}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2}} \cos(\omega_f t - \varphi_1) \tag{5.2.70}$$

eq:no35

where φ_1 is obtained from Eq. (5.2.63).

[14] but the sum of two harmonic forces is still an harmonic force

[15] here we observe that if the right hand side of Eq. (5.2.51) is $F_0 \cos \omega_f t$, it is sufficient to replace $F_0 \sin \omega_f t$ in Eq. (5.2.64)

Letting

$$\bar{Y} = \frac{Y}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}} \tag{5.2.71}$$

eqnew

into the Eq. (5.2.70), we obtain

$$\bar{Y}\sin(\omega_f t - \varphi_1) + 2\zeta r \bar{Y}\cos(\omega_f t - \varphi_1) \tag{5.2.72}$$

eq:no36

that, as we are going to show, can be written also as

MD: definizione di Y^*

$$Y^* \sin(\omega_f t - \bar{\varphi}). \tag{5.2.73}$$

eq:no37

Now, to find out the relationship between the above Y^* and Y, we exploit again the properties of the trigonometric functions in equating Eq. (5.2.72) and Eq. (5.2.73)

$$\begin{aligned}
&\bar{Y}(\sin\omega_f t\cos\varphi_1 - \cos\omega_f t\sin\varphi_1) + 2\zeta r\bar{Y}(\cos\omega_f t\cos\varphi_1 + \sin\omega_f t\sin\varphi_1) = \\
&Y^*(\sin\omega_f t\cos\bar{\varphi} - \cos\omega_f t\sin\bar{\varphi})\,.
\end{aligned} \tag{5.2.74}$$

eq:no38

By grouping the terms $\sin\omega_f t$ and $\cos\omega_f t$

$$\begin{aligned}
&(\bar{Y}\cos\varphi_1 + 2\zeta r\bar{Y}\sin\varphi_1)\sin\omega_f t + (2\zeta r\bar{Y}\cos\varphi_1 - \bar{Y}\sin\varphi_1)\cos\omega_f t = \\
&(Y^*\cos\bar{\varphi})\sin\omega_f t - (Y^*\sin\bar{\varphi})\cos\omega_f t\,,
\end{aligned} \tag{5.2.75}$$

eq:no39

we obtain

$$\bar{Y}\cos\varphi_1 + 2\zeta r\bar{Y}\sin\varphi_1 = Y^*\cos\bar{\varphi} \tag{5.2.76}$$

eq:no40

$$2\zeta r\bar{Y}\cos\varphi_1 - \bar{Y}\sin\varphi_1 = -Y^*\sin\bar{\varphi}. \tag{5.2.77}$$

eq:no41

By squaring and summing the above equations, we have

$$\bar{Y}^2 + 4\zeta^2 r^2\bar{Y}^2 = Y^{*2}, \tag{5.2.78}$$

eq:no42

and

$$\bar{Y}^2(1 + 4\zeta^2 r^2) = Y^{*2} \tag{5.2.79}$$

eq:no43

Recalling the value of $\bar{Y}$ in Eq. (5.2.71) we get the amplitude of the oscillation

$$Y^{*2} = \left(\frac{Y}{\sqrt{(1-r^2)^2 + 4\zeta^2 r^2}}\right)^2 (1 + 4\zeta^2 r^2) = \frac{Y^2(1+4\zeta^2 r^2)}{(1-r^2)^2 + 4\zeta^2 r^2}. \tag{5.2.80}$$

eq:no44

Therefore, the amplitude Y^* becomes

$$Y^* = Y\sqrt{\frac{(1+4\zeta^2 r^2)}{(1-r^2)^2 + 4\zeta^2 r^2}}. \tag{5.2.81}$$

eq:no45

Furthermore, from Eq. (5.2.76) and Eq. (5.2.77), we have

$$-\tan\bar{\varphi} = \frac{2\zeta r\cos\varphi_1 - \sin\varphi_1}{\cos\varphi_1 + 2\zeta r\sin\varphi_1} = \frac{2\zeta r - \tan\varphi_1}{1 + 2\zeta r\tan\varphi_1} = \frac{2\zeta r - \frac{2\zeta r}{1-r^2}}{1 + 2\zeta r\frac{2\zeta r}{1-r^2}} = \tag{5.2.82}$$

eq:no46

$$\frac{2\zeta r - 2\zeta r^3 - 2\zeta r}{1 - r^2 + 4\zeta^2 r^2} = \frac{-2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\,,$$

from which

$$\tan\bar{\varphi} = \frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\,. \tag{5.2.83}$$ eq:no47

Finally, the solution of Eq. (eq:no30 5.2.65) takes the form

$$y(t) = Y\sqrt{\frac{(1 + 4\zeta^2 r^2)}{(1 - r^2)^2 + 4\zeta^2 r^2}}\sin\left(\omega_f t - \arctan\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right). \tag{5.2.84}$$ eq:no48

The notebook `Chap5-2-1DOF-Under-Harmonic-Motion-Base.nb` illustrates the behaviour of the amplitude X and phase φ for different values of r. It illustrates why an anti-vibration mounting must have a natural angular frequency ω_n such that the ratio $r = \omega_f/\omega_n$ is $> \sqrt{2}$ in order to reduce the amplitude of the ground motion.

5.2.3.3 Complex vector representation of harmonic motion

complex

ID: MD-GZ: **Nota:** anche qui cambierei $\ddot{x}$ con $\ddot{z}$ e di conseguenza y con Y oppure Z; da discutere; noltre, userei ω_f invece di ω.

We can obtain the response to harmonic excitation also by using complex vector representation of the excitation and of the response itself. Then, if in the case of sinusoidal excitation we introduce in the right hand side of the slightly modified Eq (eq:no32 5.2.67),

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \omega_n^2 y + 2\zeta\omega_n\dot{y}, \tag{5.2.85}$$ eq:no49

the notation

$$y = Y\sin\omega t \equiv (\Im Y e^{i\omega t}), \tag{5.2.86}$$

the response of the system can be represented by

$$x(t) = X(i\omega)e^{i\omega t} \tag{5.2.87}$$ eq:no50

$$\dot{x}(t) = i\omega X(i\omega)e^{i\omega t} \tag{5.2.88}$$ eq:no51

$$\ddot{x}(t) = -\omega^2 X(i\omega)e^{i\omega t}. \tag{5.2.89}$$ eq:no52

By substituting these expressions into Eq. (eq:no49 5.2.85), we have

$$-\omega^2 X e^{i\omega t} + 2i\zeta\omega\omega_n X e^{i\omega t} + \omega_n^2 X e^{i\omega t} = \omega_n^2 Y e^{i\omega t} + 2i\zeta\omega\omega_n Y e^{i\omega t} \tag{5.2.90}$$ eq:no53

from which we obtain the amplitude $X(i\omega)$

$$X(i\omega) = Y\frac{\omega_n^2 + 2i\zeta\omega\omega_n}{\omega_n^2 + 2i\zeta\omega\omega_n - \omega^2} = Y\frac{1 + 2i\zeta\frac{\omega}{\omega_n}}{1 + 2i\zeta\frac{\omega}{\omega_n} - \left(\frac{\omega}{\omega_n}\right)^2}, \tag{5.2.91}$$ eq:no54

that can be transformed into the form

$$X(i\omega) = a(\omega) + ib(\omega) = \sqrt{a^2 + b^2}e^{i\varphi}. \tag{5.2.92}$$ eq:no55

We can now calculate the modulus of $X(i\omega)$

$$|X(i\omega)|^2 = X(i\omega)\cdot X^*(i\omega) = Y\frac{1 + 2i\zeta r}{(1 - r^2) + 2i\zeta r}\cdot Y\frac{1 - 2i\zeta r}{(1 - r^2) - 2i\zeta r} = \tag{5.2.93}$$ eq:no56

$$Y^2 \frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}.$$

For the phase, after some manipulations in Eq. (eq:no54 5.2.91),

$$X(i\omega) = Y \frac{(1 + 2i\zeta r)(1 - 2i\zeta r - r^2)}{(1 - r^2)^2 + 4\zeta^2 r^2} = Y \frac{1 - 2i\zeta r - r^2 + 2i\zeta r + 4\zeta^2 r^2 - 2i\zeta r^3}{(1 - r^2)^2 + 4\zeta^2 r^2} = \\ Y \left[\frac{1 - r^2 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2} - i \frac{2\zeta r^3}{(1 - r^2)^2 + 4\zeta^2 r^2} \right], \tag{5.2.94}$$

eq:no57

we get

$$\tan\varphi = \frac{-2\zeta r^3}{1 + (4\zeta^2 - 1) r^2} \tag{5.2.95}$$

eq:no58

or

$$\varphi = -\arctan \frac{2\zeta r^3}{1 + (4\zeta^2 - 1) r^2}. \tag{5.2.96}$$

eq:no59

Finally, we obtain again the solution of Eq. (eq:no49 5.2.85)

$$x(t) = \Im[X(i\omega) e^{i\omega t}] = |X(i\omega)| \, \Im[e^{i\varphi} e^{i\omega t}] = |X(i\omega)| \sin(\omega t + \varphi) = \\ Y \sqrt{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}} \sin\left(\omega t - \arctan \frac{2\zeta r^3}{1 + (4\zeta^2 - 1) r^2} \right). \tag{5.2.97}$$

eq:no60

5.3 Systems with Two Degrees of Freedom

2DOF

We can now make a further step towards the system depicted in Fig. (5.1) which, we repeat, is a simplified model of a real anti-vibration mounting. For pedagogic reasons we shall consider two distinct two-degree-of-freedom systems. The first (section 5.14) is relevant to the case of two masses m_1, m_2 which can move only along the vertical axis and the governing equations will be derived through the second Newton's law and the Lagrange method; these equations are preparatory to the analysis of the natural modes (section 5.3.1.3). In the second system (section 5.3.2) we have only one mass m but a rotation θ around one axis passing through its centre of mass is allowed. Several notebooks in this Chapter examine the effects of the damping coefficient and the dependence of the response on the frequency of the disturbing ground motion $x_g(t)$. Some numerical examples simulates the response when the elements have definite values.

5.3.1 Two masses and three springs

2m3k

We start with the more symmetric system of Fig. (5.14), owing to the presence of an additional spring k_3 and damper c_3 which make slightly simpler the analysis of the eigenfrequencies and eigenmodes.

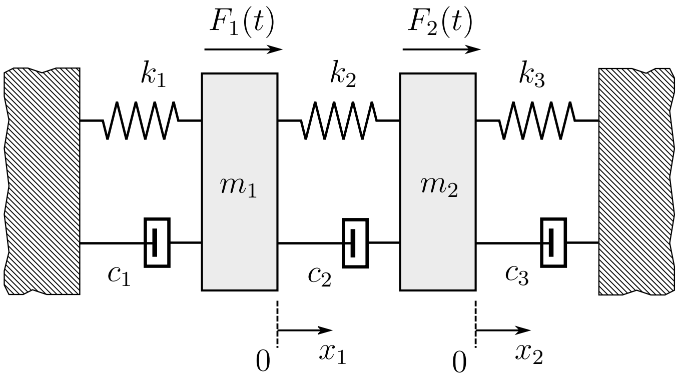


Figure 5.14: A two-degree-of-freedom system; correggere in figura due pedici di c; Chap5-Fig-5-14-2m3k-ab-Bozza.pdf

2m3k

At any instant of time, let the displacement of the masses be x_1 and x_2 from the position of equilibrium. The displacement is assumed positive when it is directed along the axis of gravity and the damping is viscous. We can follow two paths to write the equations of motion.

5.3.1.1 Newton's method

Newt

We apply Newton's second law to each of the masses. For example, the spring k_1 exerts a force $-k_1x_1$ on mass m_1 and the spring k_2, owing to the elongation $x_1 - x_2$, exerts a force $-k_2(x_1 - x_2)$ again on m_1. Analogously, the dashpot c_1 exerts a force $-c_1\dot{x}_1$ on mass m_1 and the dashpot c_2, owing to the difference of velocities $\dot{x}_1 - \dot{x}_2$, exerts a force $-c_2(\dot{x}_1 - \dot{x}_2)$ again on m_1. If we consider also the external forces

$F_1(t)$ and $F_2(t)$, the equation on motion for the masses m_1 and m_2 are

$$m_1\,\ddot{x}_1 + c_1\,\dot{x}_1 + c_2\,(\dot{x}_1 - \dot{x}_2) + k_1\,x_1 + k_2\,(x_1 - x_2) = F_1\,(t) \tag{5.3.1a}$$

$$m_2\,\ddot{x}_2 + c_3\,\dot{x}_2 + c_2\,(\dot{x}_2 - \dot{x}_1) + k_3\,x_2 + k_2\,(x_2 - x_1) = F_2\,(t) \tag{5.3.1b}$$

or, in matrix notation,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} \tag{5.3.2}$$

$$[\,M\,]\{\,\ddot{x}\,\} + [\,C\,]\{\,\dot{x}\,\} + [\,K\,]\{\,x\,\} = \{\,F(t)\,\}\,. \tag{5.3.3}$$

5.3.1.2 Lagrange's method

Lagrange's equations, for non conservative forces and for n degrees of freedom, can be stated as

$$\frac{d}{d\,t}\frac{\partial\,T}{\partial\,\dot{q}_i} - \frac{\partial\,T}{\partial\,q_i} + \frac{\partial\,V}{\partial\,q_i} = Q_i, \quad i = 1, 2, ..., n \tag{5.3.4}$$

where q_i and $\dot{q}_i$ are the generalized coordinates and velocities, respectively. Q_i represent non conservative forces and some of them deserve special consideration, namely, those those due to viscous damping. If the damping forces are proportional to the generalized velocities, the *Rayleigh's dissipation function*, in the form

$$\mathcal{R} = \frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n} c_{rs}\dot{q}_r\dot{q}_r \tag{5.3.5}$$

can be introduced. In this way, as we can derive viscous damping forces in a manner analogous to that for conservative forces, we can write

$$Q_i = -\frac{\partial\mathcal{R}}{\partial\dot{q}_i}, \quad i = 1, 2... \tag{5.3.6}$$

and Eq. (5.3.4) becomes

$$\frac{d}{d\,t}\frac{\partial T}{\partial\dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} + \frac{\partial\mathcal{R}}{\partial\dot{q}_i} = Q_i, \quad i = 1, 2, ... \tag{5.3.7}$$

where this time the terms Q_i denote only impressed forces.
In the case of Fig. (5.14) we

$$T = \frac{1}{2}\,m_1\,\dot{x}_1^2 + \frac{1}{2}\,m_2\,\dot{x}_2^2 \tag{5.3.8}$$

$$\mathcal{R} = \frac{1}{2}\,c_1\,\dot{x}_1^2 + \frac{1}{2}\,c_2\,(\dot{x}_1^2 - \dot{x}_2^2) + \frac{1}{2}\,c_3\,\dot{x}_2^2 \tag{5.3.9}$$

$$V = \frac{1}{2}\, k_1\, x_1^2 + \frac{1}{2}\, k_2\, (x_1 - x_2)^2 + \frac{1}{2}\, k_3\, x_2^2. \tag{5.3.10}$$

If we introduce these expressions into Eq. (5.3.7), and derive with respect to x_1, we obtain

$$\begin{aligned}
\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) &= m_1\, \ddot{x}_1 && (5.3.11)\\
\frac{\partial T}{\partial x_1} &= 0 && (5.3.12)\\
\frac{\partial V}{\partial x_1} &= k_1\, x_1 + k_2\, (x_1 - x_2) && (5.3.13)\\
\frac{\partial \mathcal{R}}{\partial \dot{x}_1} &= c_1\, \dot{x}_1 + c_2\, (\dot{x}_1 - \dot{x}_2). && (5.3.14)
\end{aligned}$$

which yield the equation relevant to m_1

$$m_1\, \ddot{x}_1 + (c_1 + c_2)\, \dot{x}_1 + (k_1 + k_2)\, x_1 - c_2\, \dot{x}_2 - k_2\, x_2 = F_1(t). \tag{5.3.15}$$

Similarly, if we derive with respect to x_2, we have

$$\begin{aligned}
\frac{d}{d\,t}\left(\frac{\partial T}{\partial \dot{x}_2}\right) &= m_2\, \ddot{x}_2 && (5.3.16)\\
\frac{\partial T}{\partial x_2} &= 0 && (5.3.17)\\
\frac{\partial V}{\partial x_2} &= k_3\, x_2 - k_2\, (x_1 - x_2) && (5.3.18)\\
\frac{\partial \mathcal{R}}{\partial \dot{x}_2} &= c_3\, \dot{x}_2 - c_2\, (\dot{x}_1 - \dot{x}_2). && (5.3.19)
\end{aligned}$$

which, for m_2, leads to

$$m_2\, \ddot{x}_2 + (c_2 + c_3)\, \dot{x}_2 + (k_2 + k_3)\, x_2 - c_2\, \dot{x}_1 - k_2\, x_1 = F_2(t). \tag{5.3.20}$$

We have thus reobtained the system (5.3.1).

5.3.1.3 Free vibration analysis

The search of the natural frequencies and modes of vibrations of systems with two or more degrees of freedom is - as expected - not as direct as in the case of one-degree-of-freedom. Their determination is important for two important reasons. On one hand, the general motion of the system is the superposition of the modes of vibration, each one characterized by its natural frequency. On the other end, in order to realize an effective anti-vibrating mounting, it is mandatory that its natural frequencies are distant from the frequencies of the impressed forces (directly on the masses or through the support).

The search of the natural frequencies requires that all the external and dissipative forces are set to zero, that is, $F_1 = F_2 = 0$ and $c_1 = c_2 = c_3 = 0$. Hence, the equation of motion from Eq.(5.3.1) reduces to

$$\begin{cases} m_1 \ddot{x}_1 + (k_1 + k_2)\, x_1 - k_2\, x_2 = 0 \\ m_2 \ddot{x}_2 - k_2\, x_1 + (k_2 + k_3)\, x_2 = 0, \end{cases} \tag{5.3.21}$$

or, in matrix notation

$$[\; M \;]\{\; \ddot{x} \;\} + [\; K \;]\{\; x \;\} = \{\; 0 \;\}. \tag{5.3.22}$$

We are now facing the problem of determining the nature of the solution of the system (5.3.22).

Let us assume[16] it is possible to have harmonic motion of m_1 and m_2 with the same frequency ω and with the same phase angle φ; thenwe can take the solutions of the system (5.3.21) as

$$x_1(t) = X_1 \sin(\omega\, t + \varphi) \qquad x_2(t) = X_2 \sin(\omega\, t + \varphi), \tag{5.3.23}$$

where X_1 and X_2 denote the maximum amplitude of $x_1(t)$ and $x_2(t)$.

Substituting Eqs. (5.3.23) into the system (5.3.21), and dividing out the factor $\sin(\omega\, t + \varphi)$, we obtain

$$[-m_1\,\omega^2 + (k_1 + k_2)]\, X_1 - k_2\, X_2 = 0 \tag{5.3.24}$$

$$-\, k_2\, X_1 + [-m_2\,\omega^2 + (k_2 + k_3)]\, X_2 = 0\,. \tag{5.3.25}$$

The non-trivial solutions of X_1 and X_2 can be found by imposing the determinant of the coefficients of X_1 and X_2 equal to zero

$$\begin{vmatrix} -m_1\,\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{vmatrix} = 0 \tag{5.3.26}$$

or,

$$m_1\, m_2\, \omega^4 - [m_2\,(k_1 + k_2) + m_1\,(k_2 + k_3)]\,\omega^2 + [(k_1 + k_2)(k_2 + k_3) - k_2^2] = 0 \tag{5.3.27}$$

The roots ω_1 and ω_2 of Eq. (5.3.27) are called the *angular natural frequencies* of the system and are given by

$$\begin{Bmatrix} \omega_1^2 \\ \omega_2^2 \end{Bmatrix} = \frac{1}{2}\,\frac{(k_1 + k_2)\, m_2 + (k_2 + k_3)\, m_1}{m_1\, m_2} \mp \tag{5.3.28}$$

$$\frac{1}{2}\sqrt{\left[\frac{(k_1 + k_2)\, m_2 + (k_2 + k_3)\, m_1}{m_1\, m_2}\right]^2 - 4\,\frac{(k_1 + k_2)(k_2 + k_3) - k_2^2}{m_1\, m_2}} \tag{5.3.29}$$

Under matrix form, Eq. (5.3.24) and Eq. (5.3.25) can be written as

$$([K] - \omega^2[M])\{X\} = 0 \tag{5.3.30}$$

[16] A small digression about this assumption is in Appendix ??

and, pre-multiplying by $[M^{-1}]$, we obtain

$$([M^{-1}][K] - \omega^2[I])\{X\} = 0 \tag{5.3.31}$$

or

$$([D] - \lambda[I])\{X\} = 0, \tag{5.3.32}$$

whose eigenvalues λ are obtained imposing the determinant equal to zero.

In the notebook `Chap5-3-2DOF-2m2k-IMGC-only-eigenval-eigenfun.nb` the eigenvalues and eigenmodes are first calculated symbolically and, successively, determined for a real case in which k_3 is set to zero. It is interesting to compare these results with those obtained in section 5.3.2, when the two masses are allowed to rotate around their centers of mass[17].

We have now to determine the values of X_1 and X_2 which are dependent on the natural frequencies ω_1 and ω_2. We shall denote with $X_1^{(1)}, X_2^{(1)}$ the values of X_1 and X_2 corresponding to ω_1 and with $X_1^{(2)}, X_2^{(2)}$ the values of X_1 and X_2 corresponding to ω_2. Furthermore, as above system is homogeneous, only the ratios

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} \ \text{ and } \ r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} \tag{5.3.33}$$

can be found.

The Eqs. (5.3.24,5.3.25) give, when $\omega = \omega_1^2$,

$$[-m_1\,\omega_1^2 + (k_1 + k_2)]\,X_1^{(1)} - k_2\,X_2^{(1)} = 0 \therefore \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\,\omega_1^2 + (k_1 + k_2)}{k_2} \tag{5.3.34}$$

$$-k_2\,X_1^{(1)} + [-m_2\,\omega_1^2 + (k_2 + k_3)]\,X_2^{(1)} = 0 \therefore \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\,\omega_1^2 + (k_2 + k_3)} \tag{5.3.35}$$

and, when $\omega = \omega_2^2$

$$[-m_1\,\omega_2^2 + (k_1 + k_2)]\,X_1^{(2)} - k_2\,X_2^{(2)} = 0 \therefore \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\,\omega_2^2 + (k_1 + k_2)}{k_2} \tag{5.3.36}$$

$$-k_2\,X_1^{(2)} + [-m_2\,\omega_2^2 + (k_2 + k_3)]\,X_2^{(2)} = 0 \therefore \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\,\omega_2^2 + (k_2 + k_3)} \tag{5.3.37}$$

We remark that the two expressions for r_1 are equal and similarly the two expressions for r_2. The ratios $X_2^{(1)}/X_1^{(1)}$ and $X_2^{(2)}/X_1^{(2)}$ determine the shape assumed by the system during the synchronous motion with frequencies ω_1 and ω_2, respectively. The resulting pair of values $(X_2^{(1)}, X_1^{(1)})$ and $(X_2^{(2)}, X_1^{(2)})$ are known as the *natural modes of vibration* or *modal vectors* of the system and can be represented as

$$\mathbf{X}^{(1)} = \left\{ \begin{array}{c} X_1^{(1)} \\ X_2^{(1)} \end{array} \right\} = \left\{ \begin{array}{c} X_1^{(1)} \\ r_1\,X_1^{(1)} \end{array} \right\} \tag{5.3.38}$$

[17] see the notebook `Chap5-4-4DOF-2m2k-BaseMotion-Lagr+NDSolve-Damp-Y-Symb.nb`

and

$$\mathbf{X}^{(2)} = \left\{ \begin{array}{c} X_1^{(2)} \\ X_2^{(2)} \end{array} \right\} = \left\{ \begin{array}{c} X_1^{(2)} \\ r_2\, X_1^{(2)} \end{array} \right\}. \tag{5.3.39}$$

Then, the free vibration solution for the first mode can be expressed as

$$\mathbf{x}^{(1)} = \left\{ \begin{array}{c} x_1^{(1)}\,(t) \\ x_2^{(1)}\,(t) \end{array} \right\} = \left\{ \begin{array}{c} X_1^{(1)}\, \sin\,(\omega_1\, t\, +\, \varphi_1) \\ r_1\, X_1^{(1)}\, \sin\,(\omega_1\, t\, +\, \varphi_1) \end{array} \right\} \tag{5.3.40}$$

and, for the second mode, as

$$\mathbf{x}^{(2)} = \left\{ \begin{array}{c} x_1^{(2)}\,(t) \\ x_2^{(2)}\,(t) \end{array} \right\} = \left\{ \begin{array}{c} X_1^{(2)}\, \sin\,(\omega_2\, t\, +\, \varphi_2) \\ r_2\, X_2^{(2)}\, \sin\,(\omega_2\, t\, +\, \varphi_2) \end{array} \right\} \tag{5.3.41}$$

where the constants $X_1^{(1)}$, $X_1^{(2)}$, φ_1, φ_2, have to be determined through the initial conditions.

Before considering how it is possible to let the system vibrate only on its first mode or on its second mode, it is convenient to examine more general initial conditions. In this case, both modes are excited and the resulting motion obtainable from the solution of Eq. (5.3.21) consists of the superposition of the two normal modes given by (5.3.40) and (5.3.41), that is

$$\mathbf{x}\,(t) = \mathbf{x}^{(1)}\,(t) + \mathbf{x}^{(2)}\,(t) = \tag{5.3.42}$$

$$\left\{ \begin{array}{c} x_1^{(1)}(t) \\ x_2^{(1)}(t) \end{array} \right\} + \left\{ \begin{array}{c} x_1^{(2)}(t) \\ x_2^{(2)}(t) \end{array} \right\} = \tag{5.3.43}$$

$$\left\{ \begin{array}{c} X_1^{(1)}\, \sin(\omega_1\, t\, +\, \varphi_1)\, +\, X_1^{(2)}\, \sin(\omega_2\, t\, +\, \varphi_2) \\ X_2^{(1)}\, \sin(\omega_1\, t\, +\, \varphi_1)\, +\, X_2^{(2)}\, \sin(\omega_2\, t\, +\, \varphi_2) \end{array} \right\} \tag{5.3.44}$$

where, owing to the presence of r_1 and r_2, only four unknowns ($X_1^{(1)}$, $X_1^{(2)}$, φ_1 and φ_2) have to be determined

$$\left\{ \begin{array}{c} X_1^{(1)}\, \sin(\omega_1\, t\, +\, \varphi_1)\, +\, X_1^{(2)}\, \sin(\omega_2\, t\, +\, \varphi_2) \\ r_1 X_1^{(1)}\, \sin(\omega_1\, t\, +\, \varphi_1)\, +\, r_2 X_1^{(2)}\, \sin(\omega_2\, t\, +\, \varphi_2). \end{array} \right\} \tag{5.3.45}$$

If we apply the four initial conditions, we have

$$x_1\Big|_{t=0} = x_1\,(0) = X_1^{(1)}\, \sin\, \varphi_1\, +\, X_1^{(2)}\, \sin\, \varphi_2 \tag{5.3.46}$$

$$\dot{x}_1\Big|_{t=0} = \dot{x}_1\,(0) = \omega_1\, X_1^{(1)}\, \cos\, \varphi_1\, +\, \omega_2\, X_1^{(2)}\, \sin\, \varphi_2 \tag{5.3.47}$$

$$x_2\big|_{t=0} = x_2\,(0) \;=\; r_1\,X_1^{(1)}\,\sin\,\varphi_1 \;+\; r_2\,X_1^{(2)}\,\sin\,\varphi_2 \tag{5.3.48}$$

$$\dot{x}_2\big|_{t=0} = \dot{x}_2\,(0) \;=\; \omega_1\,r_1\,X_1^{(1)}\,\cos\,\varphi_1 \;+\; \omega_2\,r_2\,X_1^{(2)}\,\cos\,\varphi_2 \tag{5.3.49}$$

¿From Eqs. (5.3.46) and 5.3.48 we have

$$X_1^{(2)}\,\sin\,\varphi_2 \;=\; \frac{x_2(0)\;-\;r_1\,x_1(0)}{r_2\;-\;r_1} \qquad X_1^{(1)}\,\sin\,\varphi_1 \;=\; \frac{r_2\,x_1(0)\;-\;x_2(0)}{r_2\;-\;r_1} \tag{5.3.50}$$

and, from Eqs. (5.3.47) and 5.3.47, we have

$$X_1^{(2)}\,\cos\,\varphi_2 \;=\; \frac{\dot{x}_2(0)\;-\;r_1\,\dot{x}_1(0)}{\omega_2\,(r_2\;-\;r_1)} \qquad X_1^{(1)}\,\cos\,\varphi_1 \;=\; \frac{r_2\,\dot{x}_1(0)\;-\;\dot{x}_2(0)}{\omega_1\,(r_2\;-\;r_1)}. \tag{5.3.51}$$

Therefore,

$$X_1^{(1)} \;=\; \frac{1}{r_2\;-\;r_1}\sqrt{(r_2\,x_1(0)\;-\;x_2(0))^2\;+\;\frac{(r_2\,\dot{x}_1(0)\;-\;\dot{x}_2(0))^2}{\omega_1^2}} \tag{5.3.52}$$

$$X_1^{(2)} \;=\; \frac{1}{r_2\;-\;r_1}\sqrt{(x_2(0)\;-\;r_1\,x_1(0))^2\;+\;\frac{(\dot{x}_2(0)\;-\;r_1\,\dot{x}_1(0))^2}{\omega_2^2}} \tag{5.3.53}$$

$$tan\,\varphi_1 \;=\; \frac{\omega_1(r_2\,x_1(0)\;-\;x_2(0))}{r_2\dot{x}_1(0)\;-\;\dot{x}_2(0)} \qquad and \qquad tan\,\varphi_2 \;=\; \frac{\omega_2(x_2(0)\;-\;r_1\,x_1(0))}{\dot{x}_2(0)\;-\;r_1\,\dot{x}_1(0)} \tag{5.3.54}$$

Finally, from these equations, we see what kind of initial condition have to be applied to let the system oscillate only on its first mode $X_1^{(1)}$, namely, we have to set to zero the terms in Eq. (5.3.53),

$$\frac{x_2(0)}{x_1(0)} = \frac{\dot{x}_2(0)}{\dot{x}_1(0)} = r_1, \tag{5.3.55}$$

Similarly, for the second natural mode $X_1^{(2)}$, we have to set equal to zero the terms in Eq. (5.3.52).

In the next subsection, we give a simple example to clarify the role of the initial conditions.

5.3.1.4 Natural frequencies and modes of a simple two-degree-of-freedom system

Let us now solve numerically the case relevant to the system in the following figure where $m_1 = m_2$ and the springs have the same stiffness coefficient k. To find the

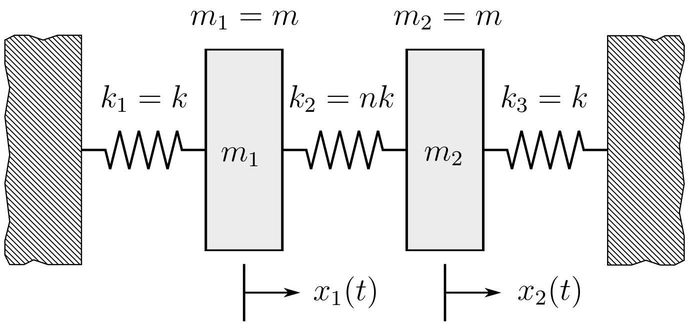


Figure 5.15: System with $m_1 = m_2$ and $k_1 = k_2 = k_3 = k$. Meiro p335 - da Chap5-Fig-5-15-mugu-ab-Bozza.pdf

natural frequencies, the modes and the displacement vector $\{\mathbf{x}(t)\}$ when the initial conditions are $\mathbf{x}(\mathbf{0}) = \{1 \;\; 0\}$ and $\dot{\mathbf{x}}(\mathbf{0}) = \{0 \;\; 0\}$, we apply the equations Eq. (5.3.21) which become

$$\begin{cases} m\,\ddot{x}_1 + 2\,k\,x_1 - k\,x_2 = 0 \\ m\,\ddot{x}_2 - k\,x_1 + 2\,k\,x_2 = 0. \end{cases} \tag{5.3.56}$$

With the following assumption,

$$x_i\,(t) = X_i \sin\left(\omega\,t + \varphi\right) \qquad i = 1,\,2\,, \tag{5.3.57}$$

from Eq. 5.3.56 we obtain

$$\begin{aligned} \{[-m\,\omega^2 + 2\,k\,]\,X_1 - k\,X_2\} \sin\left(\omega\,t + \varphi\right) = 0 \\ \{-k\,X_1 + [-m\,\omega^2 + 2\,k\,]\,X_2\} \sin\left(\omega\,t + \varphi\right) = 0. \end{aligned} \tag{5.3.58}$$

The natural frequencies are determined by imposing the relevant determinant equal to zero, that is,

$$\begin{vmatrix} (-m\,\omega^2 + 2\,k\,) & -k \\ -k & (-m\,\omega^2 + 2\,k\,) \end{vmatrix} = 0. \tag{5.3.59}$$

From the characteristic polynomial

$$m^2\,\omega^4 - 4\,k\,m\,\omega^2 + 3\,k^2 = 0 \tag{5.3.60}$$

we obtain the roots

$$\omega_1^2 = \frac{2\,k\,m - \sqrt{4\,k^2\,m^2 - 3\,m^2\,k^2}}{m^2} \Rightarrow \omega_1 = \sqrt{\frac{k}{m}} \tag{5.3.61}$$

$$\omega_2^2 = \frac{2\,k\,m + \sqrt{4\,k^2\,m^2 - 3\,m^2\,k^2}}{m^2} \Rightarrow \omega_2 = \sqrt{\frac{3\,k}{m}}. \tag{5.3.62}$$

If we substitute these results into Eqs.(5.3.34) and (5.3.36), we have

$$r_1 = \frac{-m\,\omega_1^2 + 2\,k}{k} = 1 \tag{5.3.63}$$

$$r_2 = \frac{-m\,\omega_2^2 + 2\,k}{k} = -1. \tag{5.3.64}$$

The natural modes follow from Eqs. (5.3.40) and (5.3.41)

$$\mathbf{x}^{(1)}(t) = \left\{ \begin{array}{c} X_1^{(1)} \sin\left(\sqrt{\frac{k}{m}}\,t + \varphi_1\right) \\ X_1^{(1)} \sin\left(\sqrt{\frac{k}{m}}\,t + \varphi_1\right) \end{array} \right\} \tag{5.3.65}$$

$$\mathbf{x}^{(2)}(t) = \left\{ \begin{array}{c} X_1^{(2)} \sin\left(\sqrt{\frac{3\,k}{m}}\,t + \varphi_2\right) \\ -X_1^{(2)} \sin\left(\sqrt{\frac{3\,k}{m}}\,t + \varphi_2\right) \end{array} \right\}. \tag{5.3.66}$$

We notice that, when the system vibrates in its first mode, the amplitudes of the two masses are equal; it follows that the length of the middle spring remains constant and the motions of m_1 and m_2 are in phase. When the system vibrates in its second mode, the displacements of the two masses have opposite sign and same magnitude; in this case the motions of m_1 and m_2 are 180° out of phase and the centre of the middle spring remains stationary for all time t.
The two modes are illustrated in Fig. (5.16) and (5.17)

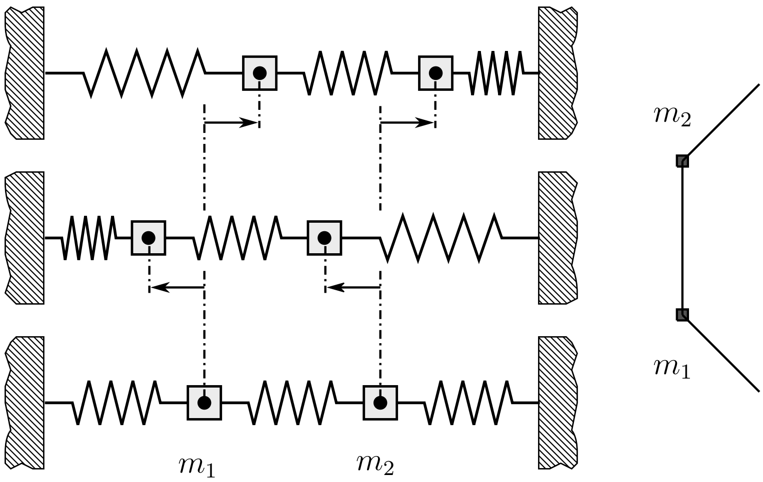


Figure 5.16: First natural mode. Tipo Meiro p336 - da Chap5-Fig-5-16-firstm-ab-Bozza.pdf

firstm

To illustrate how the solution of Eq. (5.3.56) can be represented by a superposition of its natural modes, we recall the initial conditions

$$\mathbf{x}(0) = \left\{ \begin{array}{c} x_1(0) \\ x_2(0) \end{array} \right\} = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \tag{5.3.67}$$

and

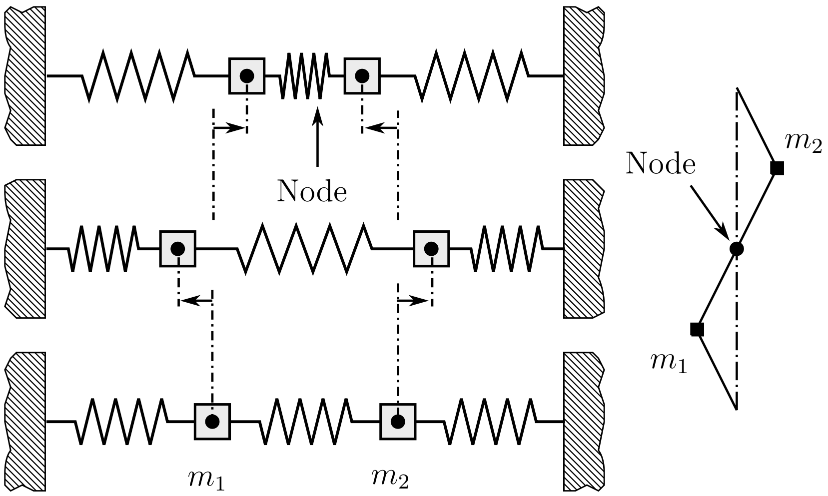


Figure 5.17: Second natural mode. Tipo Meiro p336 - da Chap5-Fig-5-17-secondm-ab-Bozza.pdf

secondm

$$\dot{\mathbf{x}}(0) = \left\{ \begin{array}{c} \dot{x}_1(0) \\ \dot{x}_2(0) \end{array} \right\} = \left\{ \begin{array}{c} 0 \\ 0 \end{array} \right\}. \tag{5.3.68}$$

¿From Eqs. (5.3.52) and (5.3.53) we have

$$X_1^{(1)} = \frac{1}{2} \tag{5.3.69}$$

$$X_1^{(2)} = \frac{1}{2} \tag{5.3.70}$$

and, from Eqs.(5.3.54),

$$tan\,\varphi_1 = \pi/2 \qquad e \qquad tan\,\varphi_2 = \pi/2. \tag{5.3.71}$$

Therefore, the solution of Eq. (5.3.56), with the relevant initial conditions, is given by

$$\mathbf{x}(t) = \left\{ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right\} = \left\{ \begin{array}{c} \frac{1}{2}\,\sin\left(\sqrt{\frac{k}{m}}\,t + \pi/2\right) + \frac{1}{2}\,\sin\left(\sqrt{\frac{3\,k}{m}}\,t + \pi/2\right) \\ \frac{1}{2}\,\sin\left(\sqrt{\frac{k}{m}}\,t + \pi/2\right) - \frac{1}{2}\,\sin\left(\sqrt{\frac{3\,k}{m}}\,t + \pi/2\right) \end{array} \right\} \tag{5.3.72}$$

The following graphs show the single natural modes and the resultant superposition when $m = 5$ kg and $k = 10000$ N/m.

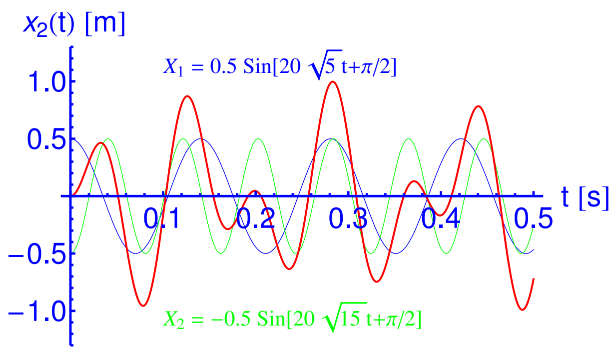


Figure 5.18: Solution $x_1(t)$; creato con Chap5-Fig-5-18-2m3k-paral-antiparal.nb

grafp

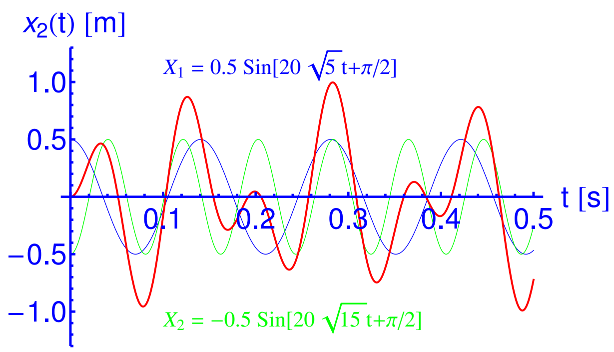


Figure 5.19: Solution $x_2(t)$; creato con Chap5-Fig-5-18-2m3k-paral-antiparal.nb

grafm

NBB *NB - Invece di radunare qui i nb, potremmo metterli nei punti dove si illustra la teoria relativa: forse però è meglio seguire il filo logico della teoria e poi alla fine avere dei riferimenti? non spezza troppo il filo del discorso?*

It is now time to illustrate the notebooks anticipated at the beginning of the section (5.3). The notebooks `Chap5-5-2DOF-2m2k-IMGC-Imped-damp-N-direct-method.nb` and `Chap5-6-2DOF-2m2k-IMGC-Imped-damp-Y-direct-method.nb` use the complex vector representation (also known under the name of so called *Impedence Method* to examine how the position and the amplitude of the response of systems with 2 DOF (undamped and damped, respectively), depend on the frequency of the harmonic motion of the base.

In `Chap5-7-2DOF-2m2k-IMGC-Base-Motion-fs-1-Decoupling-CYMC-Damp-N.nb` we show how it is possible to decouple the equations of motion by introducing the modal matrix **C**, whose columns are the eigenmodes of the undamped system.

Finally, the notebook `Chap5-8-2DOF-2m2k-BaseMotion-Lagr-Symb-Manipulate-6-parameters.nb` show how the responses $x_1(t), x_2(t)$ depend upon the several parameters previously seen m_1, m_2, k_1, k_2, c_1, c_2 and on the frequency of the base motion.

NBB Although these notebooks are only relevant to two-degree-of-freedom systems, they can supply useful hints during the design phase of anti-vibration mountings.

5.3.2 Motion of one platform with two degrees of freedom

2DOF1m

To see the effect of the rotation of a rigid slab of total mass m around an axis passing trough its centre of mass C_m, instead of considering only a translation along a vertical axis, we consider the platform in Fig. (5.20)

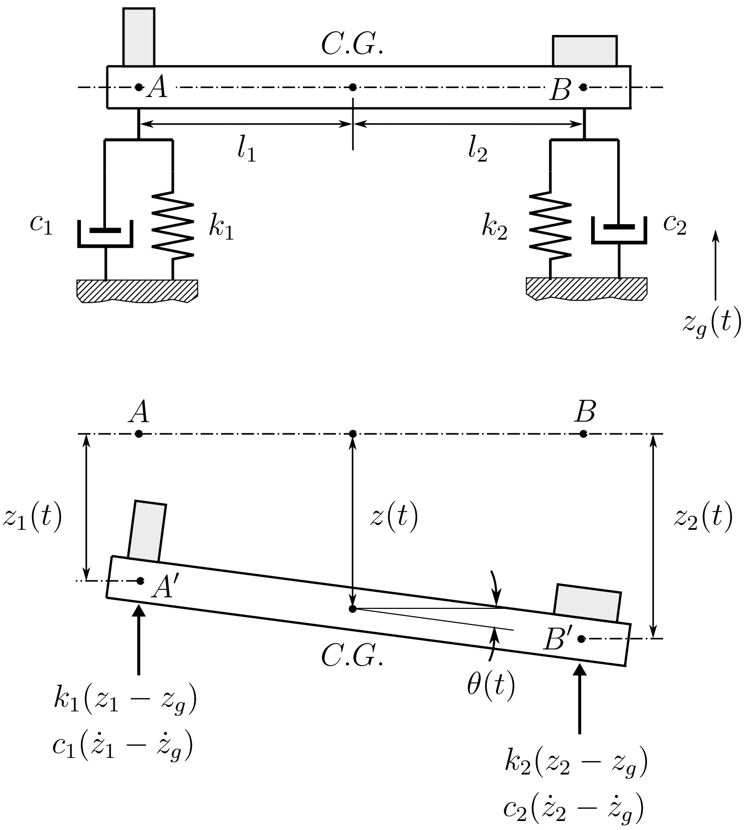


Figure 5.20: The translation $x(t)$ is positive when directed downwards and the rotation $\vartheta(t)$ is positive when clockwise. La figura è da rifare (vedi mio schizzo) coerentemente con la posizione delle molle-dash pot, e dei valori di l_1, l_2 nei nb !; da Chap5-Fig-5-20-plat2dof-ab-Bozza.pdf

plat2dof

where the springs k_1, k_2 and the dashpots c_1, c_2 the springs k_1, k_2 and the dashpots c_1, c_2, respectively.are at distances l_1 and l_2 from C_m, respectively. , È questo il NBB senso?] The platform angular mass J is about C_m; the displacement consist of the vertical translation $x(t)$ of C_m and the rotation $\vartheta(t)$ around C_m. The translation is measured from the equilibrium position and the angular displacement is supposed small; the motion of the ground is represented by $x_g(t)$.

Therefore, the force equilibrium equation in the vertical direction and the moment equation around the centre of gravity (C.G.) can be written as[18]

$$
\begin{aligned}
m\,\ddot{x} &= -\,k_1\,(x_1 - x_g) - k_2\,(x_2 - x_g) - c_1\,(\dot{x}_1 - \dot{x}_g) - c_2\,(\dot{x}_2 - \dot{x}_g) && (5.3.73)\\
&= -\,k_1\,(x - l_1\,\vartheta - x_g) - k_2\,(x + l_2\,\vartheta - x_g) - && (5.3.74)\\
&\quad c_1\,(\dot{x} - l_1\,\dot{\vartheta} - \dot{x}_g) - c_2\,(\dot{x} - l_2\,\dot{\vartheta} - \dot{x}_g) && (5.3.75)\\
J_0\,\ddot{\vartheta} &= k_1(x - l_1\,\vartheta - x_g)\,l_1 - k_2(x + l_2\,\vartheta - x_g)\,l_2 + && (5.3.76)\\
&\quad c_1(\dot{x} - l_1\,\dot{\vartheta} - \dot{x}_g)\,l_1 - c_2(\dot{x} + l_2\,\dot{\vartheta} - \dot{x}_g)\,l_2\,. && (5.3.77)
\end{aligned}
$$

sisx1 sisJ

NBB As we are interested in calculating the natural frequencies and the mode shapes, we have to neglect in the previous equation the dissipative forces and the external forces due to the ground motion. Therefore, we can write

$$
\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}
\begin{Bmatrix} \ddot{x} \\ \ddot{\vartheta} \end{Bmatrix}
+
\begin{bmatrix} k_1 + k_2 & -(k_1\,l_1 - k_2\,l_2) \\ -(k_1\,l_1 - k_2\,l_2) & k_1\,l_1^2 + k_2\,l_2^2 \end{bmatrix}
\begin{Bmatrix} x \\ \vartheta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.
\tag{5.3.78}
$$

V60_B

It can be seen that Eqs. (5.3.78) become uncoupled if the term $k_1 l_1 - k_2 l_2$ is equal to zero. In this system the mass matrix is diagonal.
If we assume harmonic solutions for free vibrations

$$
\begin{aligned}
x &= X\,sin\,(\omega\,t + \varphi) && (5.3.79)\\
\vartheta &= \Theta\,sin\,(\omega\,t + \varphi), && (5.3.80)
\end{aligned}
$$

we have

$$
\begin{aligned}
(-\,m\,\omega^2 + k_1 + k_2)\,X - (k_1\,l_1 - k_2\,l_2)\Theta &= 0 && (5.3.81)\\
-\,(k_1\,l_1 - k_2\,l_2)\,X + (-J_0\,\omega^2 + k_1 l_1^2 + k_2\,l_2^2)\Theta &= 0, && (5.3.82)
\end{aligned}
$$

sistemaa sistemab

and the natural frequencies are determined by imposing the relevant determinant equal to zero, that is,

$$
\begin{vmatrix} (-\,m\,\omega^2 + k_1 + k_2) & -\,(k_1 l_1 - k_2 l_2) \\ -\,(k_1 l_1 - k_2 l_2) & (-\,J_0\,\omega^2 + k_1 l_1^2 + k_2 l_2^2) \end{vmatrix} = 0. \tag{5.3.83}
$$

From the characteristic polynomial

$$
\omega^4 - \omega^2\left(\frac{k_1 + k_2}{m} + \frac{k_1 l_1^2 + k_2 l_2^2}{J_0}\right) + \frac{1}{J_0 m} k_1 k_2 (l_1 + l_2)^2 = 0. \tag{5.3.84}
$$

we obtain the roots

$$
\omega_{1,2}^2 = \frac{1}{2}\left(\frac{k_1 + k_2}{m} + \frac{k_1 l_1^2 + k_2 l_2^2}{J_0}\right) \mp \frac{1}{2}\sqrt{\left(\frac{k_1 + k_2}{m} + \frac{k_1 l_1^2 + k_2 l_2^2}{J_0}\right)^2 - \frac{1}{J_0 m} 4 k_1 k_2 (l_1 + l_2)^2}. \tag{5.3.85}
$$

[18] accenniamo al fatto che vi sono almeno due altre coppie di coordinate ? Thompson p139, Rao p345, Tse p155, mie dispense

From Eqs.(5.3.82), we have

$$r_1 = \frac{\Theta^{(1)}}{X^{(1)}} = \frac{k_1 l_1 - k_2 l_2}{-J_0\omega_1^2 + k_1 l_1^2 + k_2 l_2^2} \qquad (5.3.86)$$

$$r_2 = \frac{\Theta^{(2)}}{X^{(2)}} = \frac{k_1 l_1 - k_2 l_2}{-J_0\omega_2^2 + k_1 l_1^2 + k_2 l_2^2}. \qquad (5.3.87)$$

This means that, when $X^{(1)}(t) = 1\,\mathrm{m}$, $\Theta^{(1)} = r_1$, that is, there is a *first node*, at $1/\tan r_1$ meters from the centre of mass around which the mass oscillates; similarly, there is a *second node* at $1/\tan r_2$ meters from the centre of mass. The two modes, in the case of an undamped platform with the following parameters $(m = 4200, J = 1070, k_1 = k_2 = 20000, l_1 = 0.65, l_2 = 0.70))$, are shown in Fig. (5.21) In interpreting these results, the first mode is mostly vertical with rather small

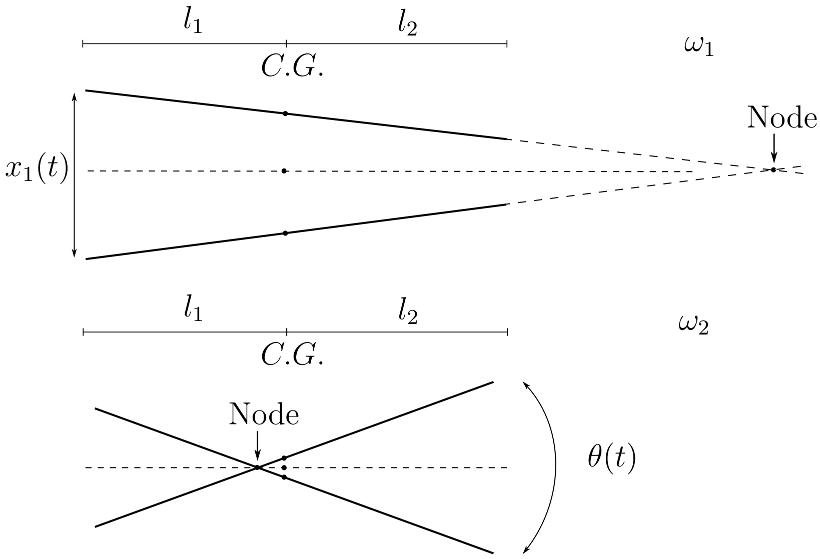


Figure 5.21: da rifare secondo mio schizzo; bisogna mantenere la scala ed essere coerenti con i nb che verranno citati; i conti nello schizzo vengono da Chap5-Fig5-21- verifica-V901.nb

nodi

rotations, whereas in the second mode rotation is more evident. The values of ω_1 and ω_2 are derived in the third of the following books.

The next step requires to calculate the natural frequencis of the platform to verify that they do not overlap with the frequency content of the ground motion[19]. To this end, we write the general solution of (5.3.78) as

$$x(t) = X^{(1)}\, sin(\omega_1 t + \varphi_1) + X^{(2)}\, sin(\omega_2 t + \varphi_2) \qquad (5.3.88)$$

sinx

$$\vartheta(t) = r_1 X^{(1)}\, sin(\omega_1 t + \varphi_1) + r_2 X^{(2)}\, sin(\omega_2 t + \varphi_2) \qquad (5.3.89)$$

sinth

and, in the first of the following notebooks, `Chap5-9-2DOF-1m1k-BaseMotion-Lagr-IMGC4000-0p85-` we use the Lagrangian formalism to determine the resonances of an undamped

[19]The main components of the frequency spectrum of the ground motion are centered near 3 and 7 Hz, as measured by means of suitable seismometers

platform. The search of the possible resonances is mandatory during the design phase of the anti-vibration mounting system.

In the second `Chap5-10-2DOF-1m1k-BaseMotion-Lagr+NDSolve-Symb-Manip-9-parameters.nb`, we examine the influence on $x(t)$ and $\vartheta(t)$ of the parameters in the equations (5.3.75) and (5.3.77) when the platform is damped.

In the third `Chap5-11-2DOF-1m1k-Eigenval-Eigenfun.nb` we re-consider the case of an undamped platform with the usual parameters and with the following initial conditions

$$x(0) \;= 0.1 \;= X^{(1)} \sin \varphi_1 + X^{(2)} \sin \varphi_2 \qquad (5.3.90)$$

$$\theta(0) \;= 0 \;= r_1 X^{(1)} \sin \varphi_1 + r_2 X^{(2)} \sin \varphi_2 \qquad (5.3.91)$$

$$\dot{x}(0) \;= 0 \;= \omega_1 X^{(1)} \cos \varphi_1 + \omega_2 X^{(2)} \cos \varphi_2 \qquad (5.3.92)$$

$$\dot{\theta}(0) \;= 0 \;= r_1 \omega_1 X^{(1)} \cos \varphi_1 + r_2 \omega_2 X^{(2)} cos\, \varphi_2. \qquad (5.3.93)$$

Here, we have done step by step the calculations whose main objective is to find out the modulations of $x(t)$ and $\vartheta(t)$. By direct inspection from the third and fourth equation we obtain $\cos \varphi_1 = 0 = \cos \varphi_2$ from which $\varphi_1 = \varphi_2 = \frac{\pi}{2}$. Then, from the first and second equation, we have

$$X^{(1)} = 0.09961 \qquad (5.3.94)$$

$$X^{(2)} = 0.0003845. \qquad (5.3.95)$$

Therefore, the Eqs. (5.3.88, 5.3.89) take the form

$$x(t) \;=\; 0.09961\, cos\, \omega_1 t + 0.0003845\, cos\, \omega_2 t \qquad (5.3.96)$$

$$\vartheta(t) \;=\; -0.01228\, cos\, \omega_1 t + 0.01228\, cos\, \omega_2 t. \qquad (5.3.97)$$

Let us now consider the motion of the centre of mass $x(t)$; as we have $\omega_1 = 3.08$ and $\omega_2 = 4.13$ we have[20] $\omega_2 - \omega_1 = \Delta\omega = 1.05$, then

$$x(t) = 0.09961 \cos \omega_1 t + 0.0003845 \cos (\omega_1 + \Delta\omega) t = A \cos (\omega_1 t + \psi) \qquad (5.3.98)$$

where, after some manipulations,

$$A = \sqrt{0.009923 + 0.0000766 \cos \Delta\omega\, t} \qquad (5.3.99)$$

and

$$tan\, \psi = \frac{0.0003845 \sin \Delta\omega\, t}{0.09961 + 0.0003845 \cos \Delta\omega\, t}. \qquad (5.3.100)$$

The rotation of the platform around the center of gravity is given by

$$\vartheta(t) = -0.01228 (\cos \omega_1 t - \cos \omega_2 t) = -0.02456 \sin \frac{\Delta\omega}{2} t \sin \frac{\omega_1 + \omega_2}{2} t. \qquad (5.3.101)$$

[20]rifatte!, ma check ancora una volta ...

The displacement $x(t)$ and the rotation $\theta(t)$, together with the modulation of their amplitudes, are shown in Figs. (5.22) and (5.23).

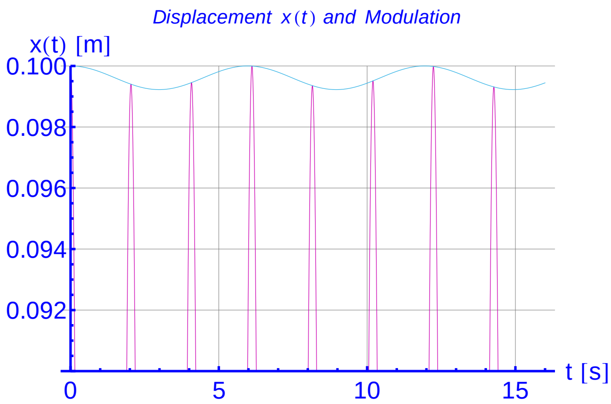


Figure 5.22: As the amplitude of the modulation of $x(t)$ is small, we show only the significant behaviour; creato da Chap5-11-2DOF-1m1k-Eigenval-Eigenfun-WithOutput-V804.nb; attenzione l'asse è ancora X invece di Z

sposCG

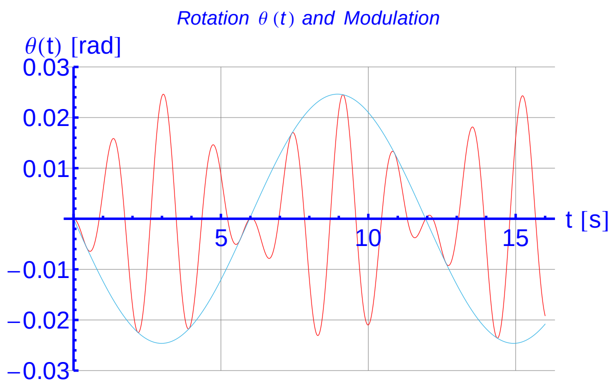


Figure 5.23: Rotation $\theta(t)$ and modulation of the relevant amplitude; creato da Chap5-11-2DOF-1m1k-Eigenval-Eigenfun-WithOutput-V804.nb

rotCG

A more complex system formed by two platforms, although still a simplified version being characterized by only four degrees of freedom, is described in `Chap5-4-4DOF-2m2k-BaseMotion-Lagr+NDSolve-Damp-Y-Symb-Manip-ultimo2.nb`. Notwithstanding this limitation, the possibility of considering simultaneously seventeen parameters $(m_1, m_2, J_1, J_2, ..., c_3, c_4)$ supplies useful hints about the design of the vibration isolation mounting.

5.4 Dynamic response of an anti-vibration mounting

6DOF

In this section we examine the behaviour of a rather realistic anti-vibrating mounting characterized by six degrees of freedom (three translational and three rotational).

B continua nel II formato del file, ma non è molto chiaro... lo riscriviamo?

In the following figure we choose O as the origin of a set of fixed coordinate axes[21]

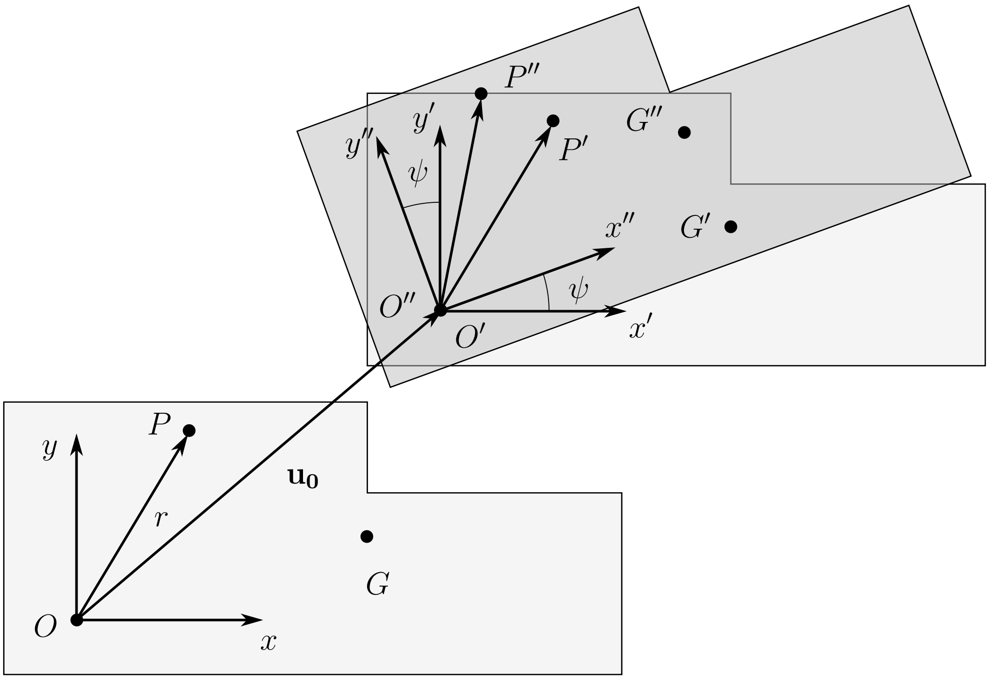


Figure 5.24: la figura è da rivedere potremmo colorarla? così non è chiara A section of a 4200-kg concrete block supported by four springs and four dash-pots (not shown). By the translation vector $\mathbf{u_0} = \overrightarrow{OO'}$, the fixed set $Oxyz$ is transformed into $O'x'y'z'$, where O' is fixed to the block and, by a rotation around the z'-axis through an angle ψ, the set $O'x'y'z'$ is transformed into $O''x''y''z''$. The z, z', z''-axes are orthogonal to the plane of the paper. The generic point P of the block has initial position $\mathbf{r} = \overrightarrow{OP}$; successively, after a displacement by $\mathbf{u}$, the new position of the point is $\mathbf{r} + \mathbf{u} = \overrightarrow{OP''}$. The vector $\overrightarrow{O'P''}$ is obtained by rotating the vector $\overrightarrow{O'P'}$ around the z', z''-axes through the angle ψ. The initial position of the centre of mass is indicated by G, and G' and G'' are the new positions after a translation by $\mathbf{u_0}$ and a rotation by ψ around the z''-axis; creato da Chap5-Fig-5-24-vibrazioninew.pdf

vibrazioni

$Oxyz$; another set of coordinate axes $O'x'y'z'$ parallel to the first is fixed to the

NBB vibrating block. At the beginning the two sets of axes coincide. We denote by

[21] that is, the perspective from which observations are made, assumed to have no translation or rotation in space

$\mathbf{i}, \mathbf{j}, \mathbf{k}$ the unit vectors in the directions of the x-, y-, z-axes, respectively. As the block vibrates, the set $O'x'y'z'$ is obtained from $Oxyz$ by the translation vector $\mathbf{u_0} = \overrightarrow{OO'} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$. At a fixed instant, a generic point P'' in the moving block is obtained by means of a rotation around a generic axis of the point P', whose position vector $\overrightarrow{O'P'}$ coincides (has the same components, to be more precise) with the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \overrightarrow{OP}$ of the point at $t = 0$ (in Fig. 5.24 we consider the case of a rotation around a axis orthogonal to the plane of the paper). The rotation $\mathbf{R}$ can be viewed, for instance, as the ordered composition of three different rotations, firstly $\mathbf{R}_\theta$, then $\mathbf{R}_\phi$ and finally $\mathbf{R}_\psi$ about the x'-, y'-, z'-axes through angles θ, ϕ, ψ, respectively. Identifying each rotation with its matrix with respect to the non-rotating reference frame $O'x'y'z'$, one has $R = R_\psi R_\phi R_\theta$. Note that in the reference frame with origin at O the position vector of P'' is given by

notazione coerente per R

$$\overrightarrow{OP''} = \mathbf{u_0} + \overrightarrow{O''P''} = \mathbf{u_0} + \mathbf{R}(\mathbf{r})\,. \tag{5.4.1}$$

eq:traslOP

Let us define the displacement vector

$$\mathbf{u} = \overrightarrow{PP''} = \overrightarrow{OP''} - \overrightarrow{OP} = \mathbf{u_0} + \mathbf{R}(\mathbf{r}) - \mathbf{r}. \tag{5.4.2}$$

eq:U=

Assuming that θ, ϕ, ψ are "small", the displacement of any point, with respect to set $Oxyz$, is given by the formula

non usiamo × ?

$$\mathbf{u} = \mathbf{u_0} + \boldsymbol{\alpha} \wedge \mathbf{r}, \tag{5.4.3}$$

eq:displ

where $\boldsymbol{\alpha} = \theta\mathbf{i} + \phi\mathbf{j} + \psi\mathbf{k}$ indicates the direction of the rotation axis.
Eq. (5.4.3) in scalar form becomes

$$u = \xi + z\phi - y\psi \tag{5.4.4}$$
$$v = \eta - z\theta + x\psi \tag{5.4.5}$$
$$w = \zeta + y\theta - x\phi\,. \tag{5.4.6}$$

As Eq. (5.4.3) is not immediate, to deduce it we first derive the second order tensor representing the rotation of a rigid body through an angle α about a rotation axis containing the origin O' and whose direction is given by a unit vector $\hat{\mathbf{n}}$. According to Fig. (5.25), the original position vector is $\mathbf{r} = \overrightarrow{O'P}$ and the position vector after rotation is $\mathbf{r}' = \overrightarrow{OP'}$. Also, A is the intersection point between the plane passing through P and orthogonal to the rotation axis and the rotation axis itself; B is the point on AP such that $\overrightarrow{P'B}$ and $\overrightarrow{AP}$ are orthogonal. Thus we have

$$\mathbf{r}' = \mathbf{r} + \overrightarrow{PP'} = \mathbf{r} + \overrightarrow{PB} + \overrightarrow{BP'}. \tag{5.4.7}$$

r'=

To obtain $\overrightarrow{PB}$, we observe that

$$\overrightarrow{O'A} = |\overrightarrow{O'A}|\hat{\mathbf{n}} = (\mathbf{r}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}}, \tag{5.4.8}$$

which yields

$$\overrightarrow{PA} = \overrightarrow{O'A} - \mathbf{r} = (\mathbf{r}\cdot\hat{\mathbf{n}})\,\hat{\mathbf{n}} - \mathbf{r} \tag{5.4.9}$$

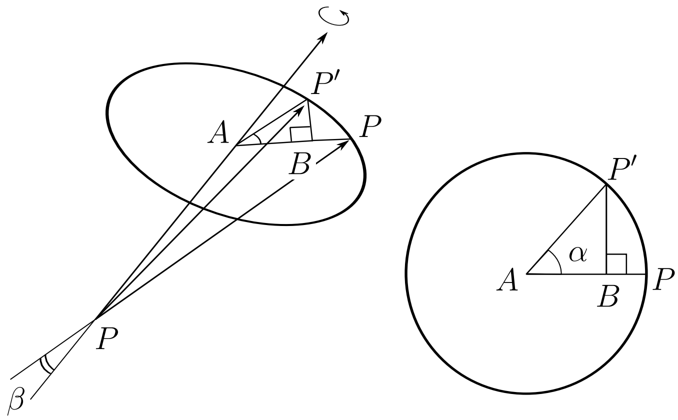


Figure 5.25: la figura è proprio brutta colorare o rifare? serve poi un commento ma dopo]; Descrizione 1....;da Chap5-Fig-5-25-rotazione-ab-Bozza.pdf

rotazione

and, as $|\overrightarrow{P'A}| = |\overrightarrow{PA}|$ (because they lie on a circle), we get

$$|\overrightarrow{PB}| = |\overrightarrow{PA}| - |\overrightarrow{PA}|\cos\alpha. \tag{5.4.10}$$

Hence, we find

$$\overrightarrow{PB} = |\overrightarrow{PB}|\frac{\overrightarrow{PA}}{|\overrightarrow{PA}|} = |\overrightarrow{PA}|\left(1-\cos\alpha\right)\frac{\overrightarrow{PA}}{|\overrightarrow{PA}|} = \left(1-\cos\alpha\right)\overrightarrow{PA} = \tag{5.4.11}$$

$$\left(\left(\mathbf{r}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\mathbf{r}\right)\left(1-\cos\alpha\right). \tag{5.4.12}$$

To obtain $\overrightarrow{BP'}$, we observe that its direction is the same as $\hat{\mathbf{n}}\wedge\mathbf{r}$ and $|\overrightarrow{BP'}| = |\overrightarrow{PA}|\sin\alpha$. So we have

$$\overrightarrow{BP'} = |\overrightarrow{BP'}|\frac{\hat{\mathbf{n}}\wedge\mathbf{r}}{|\hat{\mathbf{n}}\wedge\mathbf{r}|} = |\overrightarrow{PA}|\sin\alpha\frac{\hat{\mathbf{n}}\wedge\mathbf{r}}{|\mathbf{r}|\sin\beta} = \left(\hat{\mathbf{n}}\wedge\mathbf{r}\right)\sin\alpha, \tag{5.4.13}$$

where we have used the fact that $|\overrightarrow{PA}| = |\mathbf{r}|\sin\beta$, as one can deduce from the triangle $O'PA$. Substituting the expressions for $\overrightarrow{PB}$ and $\overrightarrow{BP'}$ into Eq. (5.4.7),we conclude that

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + \left(\left(\mathbf{r}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}-\mathbf{r}\right)\left(1-\cos\alpha\right) + \left(\hat{\mathbf{n}}\wedge\mathbf{r}\right)\sin\alpha & (5.4.14)\\ &= \mathbf{r}\cos\alpha + \left(\mathbf{r}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}\left(1-\cos\alpha\right) + \left(\hat{\mathbf{n}}\wedge\mathbf{r}\right)\sin\alpha. & (5.4.15)\end{aligned}$$

rotation0
rotation

More explicitly, having denoted by (x_1, x_2, x_3), (x_1', x_2', x_3'), (n_1, n_2, n_3) the components (x, y, z) of $\mathbf{r}$, (x', y', z') of $\mathbf{r}'$ and the components of $\hat{\mathbf{n}}$, respectively, we can write

$$x_1'\hat{\mathbf{i}} + x_2'\hat{\mathbf{j}} + x_3'\hat{\mathbf{k}} = \left(x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}}\right)\cos\alpha + \tag{5.4.16}$$

$$\left[\left(x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}}\right)\cdot\left(n_1\hat{\mathbf{i}} + n_2\hat{\mathbf{j}} + n_3\hat{\mathbf{k}}\right)\left(n_1\hat{\mathbf{i}} + n_2\hat{\mathbf{j}} + n_3\hat{\mathbf{k}}\right)\right]\left(1-\cos\alpha\right) \tag{5.4.17}$$

$$+\left[\left(n_2x_3 - n_3x_2\right)\hat{\mathbf{i}} + \left(n_3x_1 - n_1x_3\right)\hat{\mathbf{j}} + \left(n_1x_2 - n_2x_1\right)\hat{\mathbf{k}}\right]\sin\alpha, \tag{5.4.18}$$

or

$$x'_1\hat{\mathbf{i}} + x'_2\hat{\mathbf{j}} + x'_3\hat{\mathbf{k}} = \left(x_1\hat{\mathbf{i}} + x_2\hat{\mathbf{j}} + x_3\hat{\mathbf{k}}\right)\cos\alpha + \tag{5.4.19}$$

$$\left[(x_1n_1n_1 + x_2n_2n_1 + x_3n_3n_1)\,\hat{\mathbf{i}} + (x_1n_1n_2 + x_2n_2n_2 + x_3n_3n_2)\,\hat{\mathbf{j}} + (x_1n_1n_3 + x_2n_2n_3 + x_3n_3n_3)\,\hat{\mathbf{k}}\right] \tag{5.4.20}$$

$$(1-\cos\alpha) + \left[(n_2x_3 - n_3x_2)\,\hat{\mathbf{i}} + (n_3x_1 - n_1x_3)\,\hat{\mathbf{j}} + (n_1x_2 - n_2x_1)\,\hat{\mathbf{k}}\right]\sin\alpha. \tag{5.4.21}$$

We now observe that the i-th component of the vector product $\hat{\mathbf{n}} \wedge \mathbf{r}$, by exploiting the indexed notation, can be written as

$$(\hat{\mathbf{n}} \wedge \mathbf{r})_i = \eta_{ijk}n_jx_k.\text{[22]} \tag{5.4.22}$$

Therefore, Eq. (5.4.15) takes the more concise form

$$x'_i = R_{ik}x_k, \tag{5.4.23}$$

where

$$R_{ik} = \cos\alpha\,\delta_{ik} + (1-\cos\alpha)\,n_in_k + \eta_{ijk}n_j\sin\alpha. \tag{5.4.24}$$

Eq. (5.4.23) expresses the coordinates of the end point of a vector after it has been rotated through an angle α about an axis in the direction of $\hat{\mathbf{n}}$.
From expression (5.4.24), one can observe that R_{ik} is the sum of second order tensors, and thus it is a second order tensor itself.
It is interesting to examine the rotation R_ψ corresponding to the case when we rotate the block around the Z-axis through an angle ψ; in this case we have $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, $\alpha = \psi$ and $n_i = (0,0,1) = \delta_{i3}$. Hence the rotation tensor R_{ik} can be simplified as NBB

$$R_{ik} = \cos\psi\,\delta_{ik} + (1-\cos\psi)\,\delta_{i3}\delta_{k3} + \eta_{ijk}\delta_{j3}\sin\psi \tag{5.4.25}$$

and Eq. (5.4.23) becomes

$$x'_i = \cos\psi\,\delta_{ik}x_k + (1-\cos\psi)\,\delta_{i3}\delta_{k3}x_k + \eta_{i3k}x_k\sin\psi \tag{5.4.26}$$

$$= \cos\psi\,x_i + (1-\cos\psi)\,\delta_{i3}x_3 + (\eta_{i31}x_1 + \eta_{i32}x_2)\sin\psi. \tag{5.4.27}$$

---

[22] In order to check this identity, we recall that

$$\eta_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ is a cyclic permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ is an anti-cyclic permutation of } 1,2,3 \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\eta_{ijk}n_jx_k = \eta_{i1k}n_1x_k + \eta_{i2k}n_2x_k + \eta_{i3k}n_3x_k = \eta_{i12}n_1x_2 + \eta_{i13}n_1x_3 + \eta_{i21}n_2x_1 + \eta_{i23}n_2x_3 + \eta_{i31}n_3x_1 + \eta_{i32}n_3x_2,$$

which yields

$$\eta_{1jk}n_jx_k = \eta_{112}n_1x_2 + \eta_{113}n_1x_3 + \eta_{121}n_2x_1 + \eta_{123}n_2x_3 + \eta_{131}n_3x_1 + \eta_{132}n_3x_2 = n_2x_3 - n_3x_2$$

and, similarly, $\eta_{2jk}n_jx_k = n_3x_1 - n_1x_3$ and $\eta_{3jk}n_jx_k = a_1x_2 - n_2x_1$.

As a conclusion, we get

$$x' = \cos\psi\, x - \sin\psi\, y \tag{5.4.28}$$
$$y' = \sin\psi\, x + \cos\psi\, y \tag{5.4.29}$$
$$z' = z \tag{5.4.30}$$

NBB In the same way, for $\hat{\mathbf{n}} = \hat{\mathbf{i}}$, $\alpha = \theta$ and for $\hat{\mathbf{n}} = \hat{\mathbf{j}}$, $\alpha = \phi$ Eq. (5.4.7), respectively, becomes

$$x' = x \tag{5.4.31}$$
$$y' = \cos\theta\, y - \sin\theta\, z \tag{5.4.32}$$
$$z' = \sin\theta\, y + \cos\theta\, z \tag{5.4.33}$$

and

$$x' = \cos\phi\, x + \sin\phi\, z \tag{5.4.34}$$
$$y' = y \tag{5.4.35}$$
$$z' = -\sin\phi\, x + \cos\phi\, z \tag{5.4.36}$$

In the notebook `Chap5-12-6DOF-Pitch-Yaw-Roll.nb` we compute the matrix $R = R_\psi R_\phi R_\theta$ representing the composition of the three rotations and we get

$$R = \begin{bmatrix} \cos\phi\cos\psi & \cos\psi\sin\theta\sin\phi - \cos\theta\sin\psi & \cos\theta\cos\psi\sin\phi + \sin\theta\sin\psi \\ \cos\phi\sin\psi & \cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi & -\cos\psi\sin\theta + \cos\theta\sin\phi\sin\psi \\ -\sin\phi & \cos\phi\sin\theta & \cos\theta\cos\phi \end{bmatrix} \tag{5.4.37}$$

At this point, since the rotation angles are infinitesimal, we can use the following approximations:

$$\cos\theta, \cos\phi, \cos\psi \approx 1,\ \sin\theta \approx \theta,\ \sin\phi \approx \phi,\ \sin\psi \approx \psi,\ \theta\phi, \theta\psi, \phi\psi \approx 0, \tag{5.4.38}$$

obtaining

$$R = \begin{bmatrix} 1 & -\psi & \phi \\ \psi & 1 & -\theta \\ -\phi & \theta & 1 \end{bmatrix} \tag{5.4.39}$$

and thus

$$R\mathbf{r} = \begin{bmatrix} 1 & -\psi & \phi \\ \psi & 1 & -\theta \\ -\phi & \theta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - y\psi + z\phi \\ y + x\psi - z\theta \\ z - x\phi + y\theta \end{bmatrix}. \tag{5.4.40}$$

Hence, recalling Eq. (5.4.2), we get in conclusion

$$\mathbf{u} = \mathbf{u_0} + \mathbf{R}(\mathbf{r}) - \mathbf{r} = \mathbf{u_0} + [\mathbf{i}, \mathbf{j}, \mathbf{k}] \begin{bmatrix} -y\psi + z\phi \\ x\psi - z\theta \\ x\phi + y\theta \end{bmatrix} = \mathbf{u_0} + \boldsymbol{\alpha} \wedge \mathbf{r}, \tag{5.4.41}$$

which is exactly Eq. (5.4.3).

The kinetic energy of the system is

$$T = \frac{1}{2}\int_V \dot{\mathbf{u}}^2 \rho \,\mathrm{d}V \tag{5.4.42}$$ eq:ekin

where V is the total volume of the block.

We denote by $\mathbf{r}_G$ the position vector of the centre of gravity G of the block with respect to the origin O and we write $\mathbf{r} = \mathbf{r}_G + \tilde{\mathbf{r}}$ (see Fig. vibrazioni 5.24). From Eq. (eq:displ 5.4.3) it follows that

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}_\mathbf{0} + \dot{\boldsymbol{\alpha}} \wedge \mathbf{r}, \tag{5.4.43}$$ wedge1

as $\mathbf{r}$ does not change with time. Let us call theta grande the... io spiegherei questo passaggio... da vettori a tensori, sto passando ad un estensione per le varie direzioni?. This yields NBB

$$\dot{\mathbf{U}}^2 = \dot{\mathbf{U}}_\mathbf{0}^2 + \left(\dot{\boldsymbol{\Theta}} \wedge \mathbf{r}\right)^2 + 2\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\boldsymbol{\Theta}} \wedge \mathbf{r}\right) \tag{5.4.44}$$

$$= \dot{\mathbf{U}}_\mathbf{0}^2 + \left(\dot{\boldsymbol{\Theta}} \wedge \mathbf{r}\right)^2 + 2\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\boldsymbol{\Theta}} \wedge \mathbf{r}_\mathbf{0}\right) + 2\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\boldsymbol{\Theta}} \wedge \tilde{\mathbf{r}}\right). \tag{5.4.45}$$ U'^2

Setting

$$A = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}, \tag{5.4.46}$$

Eq. (wedge1 5.4.43) can be written as follows in matrix form:

$$\dot{\boldsymbol{\Theta}} \wedge \mathbf{r} = A\dot{\boldsymbol{\Theta}}. \tag{5.4.47}$$

Indeed, we have

$$\dot{\boldsymbol{\Theta}} \wedge \mathbf{r} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dot{\theta} & \dot{\phi} & \dot{\psi} \\ x & y & z \end{vmatrix} \tag{5.4.48}$$

$$= \left(\dot{\phi} z - \dot{\psi} y\right)\hat{\mathbf{i}} + \left(\dot{\psi} x - \dot{\theta} z\right)\hat{\mathbf{j}} + \left(\dot{\theta} y - \dot{\phi} x\right)\hat{\mathbf{k}} = \begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ \dot{\phi} \\ \dot{\psi} \end{bmatrix} = A\dot{\boldsymbol{\Theta}}. \tag{5.4.49}$$

Using the fact that A is antisymmetric, we get

$$\left(\dot{\boldsymbol{\Theta}} \wedge \mathbf{r}\right)^2 = A\dot{\boldsymbol{\Theta}} \cdot A\dot{\boldsymbol{\Theta}} = \left(A\dot{\boldsymbol{\Theta}}\right)^T A\dot{\boldsymbol{\Theta}} = \dot{\boldsymbol{\Theta}}^T A^T A\,\dot{\boldsymbol{\Theta}} \tag{5.4.50}$$ wedge^2

where

$$A^T A = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}\begin{bmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{bmatrix} = \begin{bmatrix} z^2+y^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & y^2+x^2 \end{bmatrix}. \tag{5.4.51}$$

Setting io inserirei una nota sui tensori o lo facciamo in appendice? con un NBB richiamo? io chiarirei nei conti quando passiamo da vettori a tensori per compattare la notazione.

$$I = \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{xy} & I_y & -I_{yz} \\ -I_{xz} & -I_{yz} & I_z \end{bmatrix}, \tag{5.4.52}$$

where

$$I_x = \int_V \rho \left(y^2 + z^2\right) dV, \quad I_{xy} = \int_V \rho xy \, dV, \quad \dots, \tag{5.4.53}$$

U'^2 wedge^2

and taking into account Eq. (5.4.45) and Eq. (5.4.50), we obtain the following expression for the kinetic energy:

$$T = \frac{1}{2}\int_V \dot{\mathbf{U}}^2 \rho dV = \frac{1}{2}\int_V \rho \left[\dot{\mathbf{U}}_\mathbf{0}^2 + \dot{\mathbf{\Theta}}^T I \, \dot{\mathbf{\Theta}} + 2\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \mathbf{r_0}\right) + 2\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \tilde{\mathbf{r}}\right)\right] dV \tag{5.4.54}$$

$$= \frac{1}{2}\left[M\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + \dot{\mathbf{\Theta}}^T I \, \dot{\mathbf{\Theta}} + 2\int_V \rho \dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \mathbf{r_0}\right) dV + 2\int_V \rho \dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \tilde{\mathbf{r}}\right) dV\right], \tag{5.4.55}$$

where M is the total mass of the foundation. Observing that $\int_V \rho \tilde{\mathbf{r}} dV = \mathbf{0}$, since in the reference frame $\mathcal{R}(O_G; \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ the centre of gravity of the block is the origin, we get

$$\int_V \rho \dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \tilde{\mathbf{r}}\right) dV = \int_V \rho \tilde{\mathbf{r}} \cdot \left(\dot{\mathbf{U}}_\mathbf{0} \wedge \dot{\mathbf{\Theta}}\right) dV = \left(\dot{\mathbf{U}}_\mathbf{0} \wedge \dot{\mathbf{\Theta}}\right) \cdot \int_V \rho \tilde{\mathbf{r}} dV = \mathbf{0}, \tag{5.4.56}$$

and thus we conclude that

$$T = \frac{1}{2}\left[M\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + \dot{\mathbf{\Theta}}^T I \, \dot{\mathbf{\Theta}} + 2M\dot{\mathbf{U}}_\mathbf{0} \cdot \left(\dot{\mathbf{\Theta}} \wedge \mathbf{r_0}\right)\right] \tag{5.4.57}$$

$$= \frac{1}{2}\Big[M\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + I_x\dot{\theta}^2 + I_y\dot{\phi}^2 + I_z\dot{\psi}^2 - 2\left(I_{xy}\dot{\theta}\dot{\phi} + I_{yz}\dot{\phi}\dot{\psi} + I_{xz}\dot{\psi}\dot{\theta}\right)$$

$$+ 2M\dot{\xi}\left(z_0\dot{\phi} - y_0\dot{\psi}\right) + 2M\dot{\eta}\left(-z_0\dot{\theta} + x_0\dot{\psi}\right) + 2M\dot{\zeta}\left(y_0\dot{\theta} - x_0\dot{\phi}\right)\Big]. \tag{5.4.58}$$

T=

Let us suppose that the origin O' coincides with the centre of gravity of the NBB foundation; then the expression found for T can be simplified as

$$T = \frac{1}{2}\left[M\left(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right) + I_x\dot{\theta}^2 + I_y\dot{\phi}^2 + I_z\dot{\psi}^2 - 2\left(I_{xy}\dot{\theta}\dot{\phi} + I_{yz}\dot{\phi}\dot{\psi} + I_{xz}\dot{\psi}\dot{\theta}\right)\right]. \tag{5.4.59}$$

Assuming that each supporting spring has spring constants k_{ix}, k_{iy}, k_{iz} in the X-,Y-,Z-directions, respectively, the potential energy of the system is given by

$$V = \frac{1}{2}\sum_{i=1}^{N_s}\left(k_{ix}u^2 + k_{iy}v^2 + k_{iz}w^2\right) \tag{5.4.60}$$

$$= \frac{1}{2}\sum_{i=1}^{N_s}\Big[k_{ix}\left(\xi + z_i\phi - y_i\psi\right)^2 + k_{iy}\left(\eta - z_i\theta + x_i\psi\right)^2 \tag{5.4.61}$$

$$+ k_{iz}\left(\zeta + y_i\theta - x_i\phi\right)^2\Big], \tag{5.4.62}$$

where N_s is the total number of supporting springs.

The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i, \tag{5.4.63}$$

where $q_1 = \xi, q_2 = \eta, q_3 = \zeta, q_4 = \theta, q_5 = \phi, q_6 = \psi$. We have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}}\right) = M\ddot{\xi}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\eta}}\right) = M\ddot{\eta}, \quad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\zeta}}\right) = M\ddot{\zeta}, \tag{5.4.64}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = I_x\ddot{\theta} - I_{xy}\ddot{\phi} - I_{xz}\ddot{\psi}, \tag{5.4.65}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = I_y\ddot{\phi} - I_{yz}\ddot{\psi} - I_{yx}\ddot{\theta}, \tag{5.4.66}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\psi}}\right) = I_z\ddot{\psi} - I_{zx}\ddot{\theta} - I_{zy}\ddot{\phi}. \tag{5.4.67}$$

In matrix form we can write

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \begin{bmatrix} M & 0 & 0 & 0 & 0 & 0 \\ 0 & M & 0 & 0 & 0 & 0 \\ 0 & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & I_x & -I_{xy} & -I_{xz} \\ 0 & 0 & 0 & -I_{xy} & I_y & -I_{yz} \\ 0 & 0 & 0 & -I_{xz} & -I_{yz} & I_z \end{bmatrix} \begin{bmatrix} \ddot{\xi} \\ \ddot{\eta} \\ \ddot{\zeta} \\ \ddot{\theta} \\ \ddot{\phi} \\ \ddot{\psi} \end{bmatrix} = M\ddot{\mathbf{q}}. \tag{5.4.68}$$

The matrix M is real and symmetric, and it turns out to be diagonal if the products of inertia vanish.

Let us now calculate $\frac{\partial V}{\partial q_i}$. We get

$$
\begin{aligned}
\frac{\partial V}{\partial \xi} &= \sum_i k_{ix}\left(\xi + z_i\phi - y_i\psi\right) = \xi\sum_i k_{ix} + \phi\sum_i k_{ix}z_i - \psi\sum_i k_{ix}y_i, \\
\frac{\partial V}{\partial \eta} &= \sum_i k_{iy}\left(\eta - z_i\theta + x_i\psi\right) = \eta\sum_i k_{iy} - \theta\sum_i k_{iy}z_i + \psi\sum_i k_{iy}x_i, \\
\frac{\partial V}{\partial \zeta} &= \sum_i k_{iz}\left(\zeta + y_i\theta - x_i\phi\right) = \zeta\sum_i k_{iz} + \theta\sum_i k_{iz}y_i - \phi\sum_i k_{iz}x_i, \\
\frac{\partial V}{\partial \theta} &= \sum_i \left(-k_{iy}z_i\left(\eta - z_i\theta + x_i\psi\right) + k_{iz}y_i\left(\zeta + y_i\theta - x_i\phi\right)\right) \\
&= -\eta\sum_i k_{iy}z_i + \zeta\sum_i k_{iz}y_i + \theta\left(\sum_i k_{iy}z_i^2 + \sum_i k_{iz}y_i^2\right) - \phi\sum_i k_{iz}y_i x_i + \\
&\quad -\psi\sum_i k_{iy}z_i x_i, \\
\frac{\partial V}{\partial \phi} &= \sum_i \left(k_{ix}z_i\left(\xi + z_i\phi - y_i\psi\right) - k_{iz}x_i\left(\zeta + y_i\theta - x_i\phi\right)\right) \\
&= \xi\sum_i k_{ix}z_i - \zeta\sum_i k_{iz}x_i - \theta\sum_i k_{iz}x_i y_i + \phi\left(\sum_i k_{ix}z_i^2 + \sum_i k_{iz}x_i^2\right) + \\
&\quad -\psi\sum_i k_{ix}z_i y_i, \\
\frac{\partial V}{\partial \psi} &= \sum_i \left(-k_{ix}y_i\left(\xi + z_i\phi - y_i\psi\right) + k_{iy}x_i\left(\eta - z_i\theta + x_i\psi\right)\right) \\
&= -\xi\sum_i k_{ix}y_i + \eta\sum_i k_{iy}x_i - \theta\sum_i k_{iy}x_i z_i - \phi\sum_i k_{ix}y_i z_i + \\
&\quad +\psi\left(\sum_i k_{iy}x_i^2 + \sum_i k_{ix}y_i^2\right).
\end{aligned}
$$

In matrix form, we can write

$$
\left[\frac{\partial V}{\partial q_i}\right] =
$$

$$
= \begin{bmatrix}
\sum_i k_{ix} & 0 & 0 & 0 & \sum_i k_{ix}z_i & -\sum_i k_{ix}y_i \\
0 & \sum_i k_{iy} & 0 & -\sum_i k_{iy}z_i & 0 & \sum_i k_{iy}x_i \\
0 & 0 & \sum_i k_{iz} & \sum_i k_{iz}y_i & -\sum_i k_{iz}x_i & 0 \\
0 & -\sum_i k_{iy}z_i & \sum_i k_{iz}y_i & \begin{matrix}\sum_i k_{iy}z_i^2 + \\ +\sum_i k_{iz}y_i^2)\end{matrix} & -\sum_i k_{iz}y_i x_i & -\sum_i k_{iy}z_i x_i \\
\sum_i k_{ix}z_i & 0 & -\sum_i k_{iz}x_i & -\sum_i k_{iz}x_i y_i & \begin{matrix}\sum_i k_{ix}z_i^2 + \\ +\sum_i k_{iz}x_i^2\end{matrix} & -\sum_i k_{ix}z_i y_i \\
-\sum_i k_{ix}y_i & \sum_i k_{iy}x_i & 0 & -\sum_i k_{iy}x_i z_i & -\sum_i k_{ix}y_i z_i & \begin{matrix}\sum_i k_{iy}x_i^2 + \\ +\sum_i k_{ix}y_i^2\end{matrix}
\end{bmatrix}.
$$

$$\cdot \begin{bmatrix} \xi \\ \eta \\ \zeta \\ \theta \\ \phi \\ \psi \end{bmatrix} = K\mathbf{q}. \tag{5.4.69}$$

def K

The Lagrange equations can be written in matrix form as

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{Q}$$

where

$$\mathbf{Q} = \begin{bmatrix} F_x \\ F_y \\ F_z \\ M_x \\ M_y \\ M_z \end{bmatrix} \sin(\omega_f t)\,.$$

We now consider the case in which there is a sussultatory action on the base of the system along the z-direction, and no driving force $\mathbf{Q}$ is applied directly to the mass; we denote by $\mathbf{q}_s = \begin{bmatrix} 0 \\ 0 \\ A_z \sin(\omega_s t) \\ 0 \\ 0 \\ 0 \end{bmatrix}$ the displacement of the base. The equation of motion becomes

$$M\ddot{\mathbf{q}} + K(\mathbf{q} - \mathbf{q}_s) = 0 \tag{5.4.70}$$

or, equivalently,

$$M\ddot{\mathbf{q}} + K\mathbf{q} = K\mathbf{q}_s. \tag{5.4.71}$$

eq sussul

If we compute $K\mathbf{q}_s$ by using the definition of K given in Eq. (5.4.69), we can rewrite the above equation in the following form

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \begin{bmatrix} 0 \\ 0 \\ \sum_i k_{iz} \\ \sum_i k_{iz} y_i \\ -\sum_i k_{iz} x_i \\ 0 \end{bmatrix} A_z \sin(\omega_s t)\,. \tag{5.4.72}$$

The notebook Chap5-13-6DOF-BaseMotion-Fs-4-Hz-Kz-0.nb shows how the displacement and the rotations around the mass centre of the above system can be determined. The solutions of the Lagrangian equations are found when the various parameters are varied and the base is subjected to an harmonic motion. The results are coherent and in excellent agreement with those obtained with simpler 1-DOF systems (e.g., Chap5-2-1DOF-Under-Harmonic-Motion-Base.nb. Furthermore, by exploiting a Finite Element code we have examined the behaviour of the system in Fig. (5.1) when all the twelve degrees of freedom (six for mass m_1 and six for mass m_2) are taken into account. Again, the simulation indicates that the results obtained with Mathematica are coherent and worth of consideration.