3.9. Exercise

P3.1 Derive the expression of the Eulerian strain in Eq. (3.17).

Solution:

From the definistion of the Eulerian strain, it is necessary to define the inverse of the deformation gradient in terms of the displacement gradient. From the relation $\mathbf{X} = \mathbf{x} - \mathbf{u}$,

$$\mathbf{F}^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}} = \frac{\partial(\mathbf{x} - \mathbf{u})}{\partial \mathbf{x}} = \mathbf{1} - \nabla_n \mathbf{u}$$

Then, the Eulerian strain in Eq. (4.17) can be written as

$$\begin{aligned}
\mathbf{e} &= \tfrac{1}{2}(\mathbf{1} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}) \\
&= \tfrac{1}{2}\Big(\mathbf{1} - (\mathbf{1} - \nabla_n \mathbf{u})^{\mathrm{T}}(\mathbf{1} - \nabla_n \mathbf{u})\Big) \\
&= \tfrac{1}{2}(\nabla_n \mathbf{u} + \nabla_n \mathbf{u}^{\mathrm{T}} - \nabla_n \mathbf{u}^{\mathrm{T}} \nabla_n \mathbf{u})
\end{aligned}$$

■

P3.2 Derive the relation in volume change in Eq. (3.26) for an infinitesimal hexahedron whose edges are initially parallel to the coordinate directions.

Solution:

Let $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ be the unit basis vectors for the coordinate system, and the legths of the hexahedron are $\mathrm{d}\mathbf{X}_1$, $\mathrm{d}\mathbf{X}_2$, and $\mathrm{d}\mathbf{X}_3$. Then, the three edges at the undeformed state can be written as

$$\mathrm{d}\mathbf{X}_1 = \mathrm{d}X_1\mathbf{E}_1, \quad \mathrm{d}\mathbf{X}_2 = \mathrm{d}X_2\mathbf{E}_2, \quad \mathrm{d}\mathbf{X}_2 = \mathrm{d}X_2\mathbf{E}_2$$

With these three vectors, the volume of the infinitesimal hexahedron can be calculated by

$$\mathrm{d}V_0 = \mathrm{d}\mathbf{X}_1 \cdot (\mathrm{d}\mathbf{X}_2 \times \mathrm{d}\mathbf{X}_3) = \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3) = \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3$$

Now, after deformation, the three edges are deformed to

$$\mathrm{d}\mathbf{x}_1 = \mathbf{F}\mathrm{d}\mathbf{X}_1 = \mathrm{d}X_1\frac{\partial \mathbf{x}}{\partial X_1}, \quad \mathrm{d}\mathbf{x}_2 = \mathbf{F}\mathrm{d}\mathbf{X}_2 = \mathrm{d}X_2\frac{\partial \mathbf{x}}{\partial X_2}, \quad \mathrm{d}\mathbf{x}_3 = \mathbf{F}\mathrm{d}\mathbf{X}_3 = \mathrm{d}X_3\frac{\partial \mathbf{x}}{\partial X_3}$$

The deformed volume becomes

$$\mathrm{d}V_x = \mathrm{d}\mathbf{x}_1 \cdot (\mathrm{d}\mathbf{x}_2 \times \mathrm{d}\mathbf{x}_3) = \mathrm{d}X_1\mathrm{d}X_2\mathrm{d}X_3\frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3}\right)$$

The determinant of the deformation gradient can be written as

$$J = \det(\mathbf{F}) = \frac{\partial \mathbf{x}}{\partial X_1} \cdot \left(\frac{\partial \mathbf{x}}{\partial X_2} \times \frac{\partial \mathbf{x}}{\partial X_3} \right)$$

Thus, the deformed volume becomes

$$\mathrm{d}V_x = J\,\mathrm{d}V_0$$

■

P3.3 Consider a square block under oscillating simple shear deformation. The relation between undeformed and deformed geometry is given as

$$x_1 = X_1 + aX_2 \sin \omega t, \quad x_2 = X_2, \quad x_3 = X_3$$

Calculate the deformation gradient and the change in volume.

Solution:
The deformation gradient can be calculated from its definition as

$$\mathbf{F} = \begin{bmatrix} 1 & a \sin \omega t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of the deformation gradient is $J = \det(\mathbf{F}) = 1$. Thus, the volume of the block does not change due to the given deformation. ■

P3.4 Many materials often show very different behavior between volume-changing deformation and volume-preserving deformation. The former is called dilatation, while the latter is called distortion. In such a case, it is necessary to separate the dilatational and distortional parts from the deformation gradient. Let $\mathbf{F}$ be the deformation gradient, decompose it such that $\mathbf{F} = \mathbf{F}_v \cdot \mathbf{F}_d$, where $\mathbf{F}_v$ is the dilatational part and $\mathbf{F}_d$ is the distortional part. Calculate $\mathbf{F}_v$ and $\mathbf{F}_d$ using the third invariant of the deformation gradient.

Solution:
The determinant of the product of two tensors satisfies the following relation:

$$J = \det(\mathbf{F}) = \det(\mathbf{F}_v)\det(\mathbf{F}_d)$$

From the definition, $\det(\mathbf{F}_v) = J$ and $\det(\mathbf{F}_d) = 1$. In the view of Example 3.1, the dilatational deformation gradient must have the following form:

$$\mathbf{F}_v = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

where the principal stretch can be written in terms of J by

$$\det(\mathbf{F}_v) = \lambda^3 = J \quad \Rightarrow \quad \lambda = J^{\frac{1}{3}}$$

Thus, the following $\mathbf{F}_v$ is obtained:

$$\mathbf{F}_v = J^{\frac{1}{3}}\mathbf{1}$$

and the distortional part of the deformation gradient can be written as

$$\mathbf{F}_d = \mathbf{F}_v^{-1} \cdot \mathbf{F} = J^{-\frac{1}{3}}\mathbf{F}$$

■

P3.5 Repeat Problem P3.4 for the Cauchy-Green deformation tensor; i.e., decompose $\mathbf{C} = \mathbf{C}_v \cdot \mathbf{C}_d$.

Solution:
From the definition of the Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the determinant of $\mathbf{C}$ becomes $\det(\mathbf{C}) = \det(\mathbf{F}^{\mathrm{T}})\det(\mathbf{F}) = J^2$. The volumetric part of the Cauchy-Green tensor comes from the volumetric part of the deformation gradient: $\mathbf{C}_v = \mathbf{F}_v^{\mathrm{T}}\mathbf{F}_v$, whose determinant becomes

$$\det(\mathbf{C}_v) = \det(\mathbf{F}_v^{\mathrm{T}})\det(\mathbf{F}_v) = J^{\frac{2}{3}}$$

Thus, the following $\mathbf{C}_v$ is obtained:

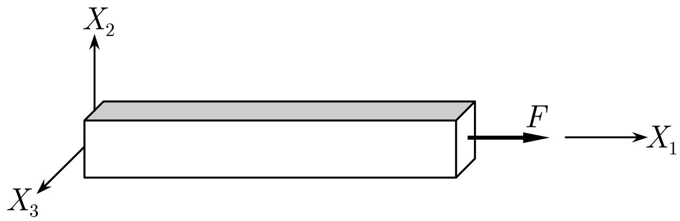
$$\mathbf{C}_v = J^{\frac{2}{3}}\mathbf{1}$$

and the distortional part of the Cauchy-Green tensor can be written as

$$\mathbf{C}_d = \mathbf{C}_v^{-1} \cdot \mathbf{C} = J^{-\frac{2}{3}}\mathbf{C}$$

■

P3.6 Consider a bar with a square cross section in the figure under uniaxial tension loading. The principal stretch in X_1 direction is given by $\lambda > 1$. When material is incompressible, compare X_1 component of normal strain using Lagrangian, Eulerian, and engineering strains.



Solution:

Since the bar will maintain rectangular shape, there is no shear deformation. In addition, since both X_2 and X_3 directions are unconstrained, and the cross-sectional geometries are identical, the principal stretches in these two directions will be the same. Thus, the relation between undeformed and deformed geometries can be written as

$$x_1 = \lambda X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_2 X_3$$

Since the material is incompressible, the volume should be preserved:

$$\lambda_1 \lambda_2^2 = 1 \quad \Rightarrow \quad \lambda_2 = \lambda^{-1/2}$$

Thus, the deformation gradient can be obtained as

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{bmatrix}$$

And the right Cauchy-Green deformation tensor can be obtained as

$$\mathbf{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix}$$

The Lagrangian strain becomes

$$\mathbf{E} = \frac{1}{2} \begin{bmatrix} \lambda^2 - 1 & 0 & 0 \\ 0 & \lambda^{-1} - 1 & 0 \\ 0 & 0 & \lambda^{-1} - 1 \end{bmatrix}$$

For the Eulerian strain, the inverse of the left Cauchy-Green deformation tensor can be calculated by

$$\mathbf{G}^{-1} = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

And the Eulerian strain becomes

$$\mathbf{e} = \frac{1}{2}\begin{bmatrix} 1-\lambda^{-2} & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

By differentiating the following displacement

$$u_1 = (\lambda - 1)X_1, \quad u_2 = (\lambda_2 - 1)X_2, \quad u_3 = (\lambda_2 - 1)X_3$$

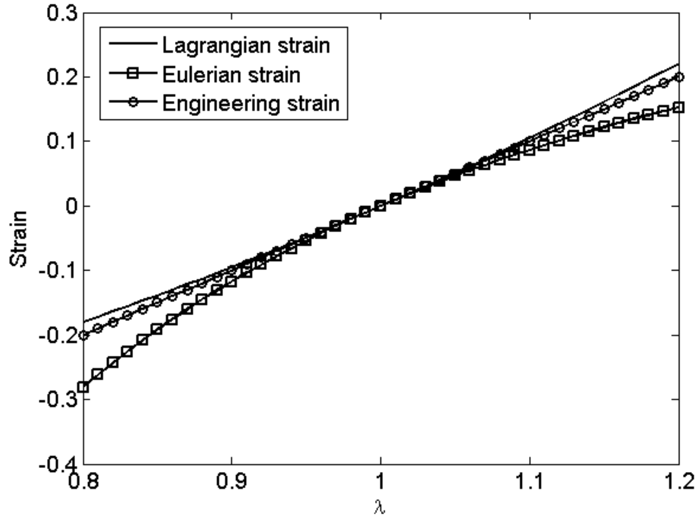
the engineering strain can be obtained as

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \lambda - 1 & 0 & 0 \\ 0 & \lambda^{-1/2} - 1 & 0 \\ 0 & 0 & \lambda^{-1/2} - 1 \end{bmatrix}$$

Below are the normal strains in the X_1 direction from the three different strains:

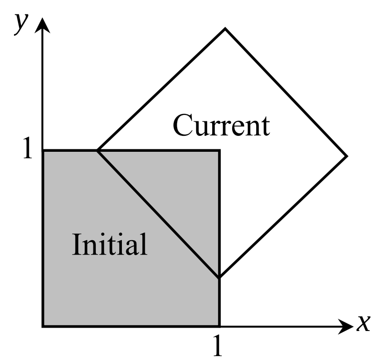
$$E_{11} = \frac{1}{2}(\lambda^2 - 1) \quad e_{11} = \frac{1}{2}(1 - \lambda^{-2}) \quad \varepsilon_{11} = \lambda - 1$$

The figure below shows the difference between these three strain components. When the strain is small; i.e., $\lambda \approx 1$, all three strains are similar. However, the difference becomes large as the strain increase.



■

P3.7 A four node square element undergoes large displacement and rotation in the XY plane, as shown in the figure. The node initially at the origin is moved to $(1,\, 1 - \sin\pi/4)$ and the element is rotated by $45°$. Calculate the deformation gradient. Compute the Lagrangian strain and demonstrate that no strain occurs during rigid body motion.



Solution:
From geometry the coordinates in the initial (X, Y) and the deformed (x,y) configurations are as follows.

Node	X	Y	x	y
1	0	0	1	$1 - \sin\pi/4$
2	1	0	$1 + \sin\pi/4$	1
3	1	1	1	$1 + \sin\pi/4$
4	0	1	$1 - \sin\pi/4$	1

We first need to develop mapping of the current configuration in terms of the initial configuration. A systematic way to develop this mapping in the finite element context is to use the interpolation functions to map the given configurations into a 2×2 reference element in Chapter 1. The interpolation functions are as follows.

$$\begin{cases} N_1 = \frac{1}{4}(1-s)(1-t) \\ N_2 = \frac{1}{4}(1+s)(1-t) \\ N_3 = \frac{1}{4}(1+s)(1+t) \\ N_4 = \frac{1}{4}(1-s)(1+t) \end{cases}$$

Using these interpolation functions, the initial configuration is mapped as follows.

$$X = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 = \frac{s+1}{2}$$
$$Y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4 = \frac{t+1}{2}$$

By inverting this mapping, the s and t can be written in terms of X and Y as follows.

$$s = 2X - 1$$
$$t = 2Y - 1$$

The current configuration can also be mapped into $s,\ t$ using the interpolation functions.

$$x = \frac{1}{4}\left(\sqrt{2}s - \sqrt{2}t + 4\right)$$
$$y = \frac{1}{4}\left(\sqrt{2}s + \sqrt{2}t + 4\right)$$

Now the desired mapping between the initial and the current configurations can be written by substituting for $s,\ t$ in terms of $X,\ Y$ giving

$$x = \frac{1}{4}\left(\sqrt{2}(2X-1) - \sqrt{2}(2Y-1) + 4\right)$$
$$y = \frac{1}{4}\left(\sqrt{2}(2X-1) + \sqrt{2}(2Y-1) + 4\right)$$
$$z = Z$$

The deformation gradient can now be easily computed by direct differentiation.

$$\mathbf{F} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using the deformation gradient, the matrix of Lagrangian strains is as follows.

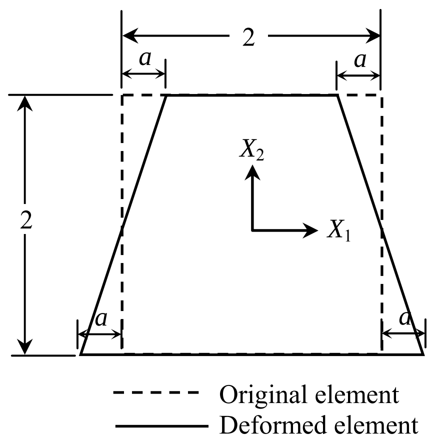
$$
\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

This matrix mathematically states the obvious fact that in this example any differential line segment in the original configuration has the same length in the current configuration. This example demonstrates that Lagrangian strains are invariant with respect to rigid body rotations and displacements. ■

P3.8 A square plane strain element is deformed as shown in the figure. The relation between deformed and undeformed coordinates is given as

$$
x_1 = X_1 - aX_1X_2, \quad x_2 = X_2, \quad x_3 = X_3
$$

Compare the engineering strain and Lagrangian strain. Show that the two strain measures become identical when 'a' approach zero.



Solution:
The displacement vector can be written as

$$
\mathbf{u} = [-aX_1X_2, 0, 0]^T
$$

For linear elastic model, the engineering strain tensor can be defined as

$$
\boldsymbol{\varepsilon} = sym(\nabla_0\mathbf{u}) = \begin{bmatrix} -aX_2 & -\frac{1}{2}aX_1 \\ -\frac{1}{2}aX_1 & 0 \end{bmatrix}
$$

Thus, the strain varies linearly.

For the geometric nonlinear model, the deformation gradient and Lagrangian strain can be calculated by

$$
\mathbf{F} = \mathbf{1} + \nabla_0\mathbf{u} = \begin{bmatrix} 1 - aX_2 & -aX_1 \\ 0 & 1 \end{bmatrix}
$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \begin{bmatrix} -aX_2(1 - \frac{1}{2}aX_2) & -\frac{1}{2}aX_1(1 - aX_2) \\ -\frac{1}{2}aX_1(1 - aX_2) & \frac{1}{2}a^2X_1^2 \end{bmatrix} = \boldsymbol{\varepsilon} + \begin{bmatrix} \frac{1}{2}a^2X_2^2 & \frac{1}{2}a^2X_1X_2 \\ \frac{1}{2}a^2X_1X_2 & \frac{1}{2}a^2X_1^2 \end{bmatrix}$$

Note that the the Lagrangian strain is the engineering strain plus nonlinear terms, which will approach zero fast when '*a*' approaches zero. ■

P3.9 The relation between deformed and undeformed coordinates for pure bending of a plane strain solid is given as

$$x_1 = X_1 - aX_1X_2, \quad x_2 = X_2 + \tfrac{1}{2}aX_1^2, \quad x_3 = X_3$$

Compare the engineering strain and Lagrangian strain. Show that the two strain measures become identical when '*a*' approach zero.

Solution:
The displacement vector can be written as

$$\mathbf{u} = [-aX_1X_2, \tfrac{1}{2}aX_1^2, 0]^T$$

For linear elastic model, the engineering strain tensor can be defined as

$$\boldsymbol{\varepsilon} = sym(\nabla_0\mathbf{u}) = \begin{bmatrix} -aX_2 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, ε_{11} is only non-zero component and varies linearly with respect to X_2.

For the geometric nonlinear model, the deformation gradient and Lagrangian strain can be calculated by

$$\mathbf{F} = \mathbf{1} + \nabla_0\mathbf{u} = \begin{bmatrix} 1 - aX_2 & -aX_1 \\ aX_1 & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{E} &= \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) \\ &= \begin{bmatrix} -aX_2 + \frac{1}{2}a^2(X_1^2 + X_2^2) & \frac{1}{2}a^2X_1X_2 \\ \frac{1}{2}a^2X_1X_2 & \frac{1}{2}a^2X_1^2 \end{bmatrix} \\ &= \boldsymbol{\varepsilon} + \tfrac{1}{2}a^2\begin{bmatrix} X_1^2 + X_2^2 & X_1X_2 \\ X_1X_2 & X_1^2 \end{bmatrix} \end{aligned}$$

Note that the Lagrangian strain is the engineering strain plus nonlinear terms, which will approach zero fast when '*a*' approaches zero. ■

P3.10 In the small deformation theory, the volumetric strain $(\mathrm{d}V_x - \mathrm{d}V_0)/\mathrm{d}V_0$ is approximated by $\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, while in the large deformation theory, it is given by $J - 1$. Show that when the deformation is small, the latter can be approximated by the former.

Solution:
The deformation gradient can be written in terms of displacement gradient as

$$\mathbf{F} = \begin{bmatrix} 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & 1+\frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

The determinant of the deformation gradient can be written as

$$\begin{aligned} \det \mathbf{F} &= \left(1+\frac{\partial u_1}{\partial X_1}\right)\left[\left(1+\frac{\partial u_2}{\partial X_2}\right)\left(1+\frac{\partial u_3}{\partial X_3}\right) - \frac{\partial u_2}{\partial X_3}\frac{\partial u_3}{\partial X_2}\right] \\ &+ \frac{\partial u_1}{\partial X_2}\left[\frac{\partial u_2}{\partial X_3}\frac{\partial u_3}{\partial X_1} - \frac{\partial u_2}{\partial X_1}\left(1+\frac{\partial u_3}{\partial X_3}\right)\right] \\ &+ \frac{\partial u_1}{\partial X_3}\left[\frac{\partial u_2}{\partial X_1}\frac{\partial u_3}{\partial X_2} - \frac{\partial u_3}{\partial X_1}\left(1+\frac{\partial u_2}{\partial X_2}\right)\right] \\ &= 1 + \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} + \text{H.O.T.} \end{aligned}$$

When the deformation is small, the higher-order terms will approach zero quickly. Thus the volumetric strain can be approximated by

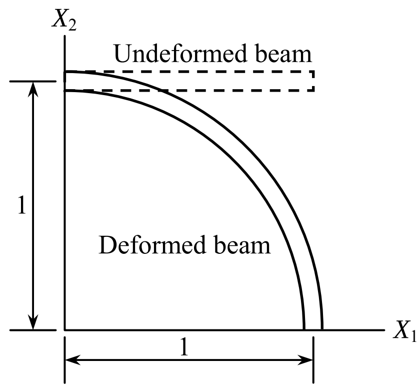
$$J - 1 = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} \approx \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

■

P3.11 An initially straight beam AB is bent into a circular arc A'B' as shown in the figure. The deformation is specified as

$$x_1 = g(X_2)\cos\frac{\pi(1-X_1)}{2}, \quad x_2 = g(X_2)\sin\frac{\pi(1-X_1)}{2}, \quad x_3 = X_3$$

where $g(X_2)$ is a simple function of X_2. (a) Find the deformation gradient in terms of $g(X_2)$. (b) If the volume of the beam does not change, find $g(X_2)$. (c) Using $g(X_2)$ in (b), find **U**, **Q**, and **V**.



Solution:

(a) For the given deformation, the deformation gradient can be obtained as

$$\mathbf{F} = \begin{bmatrix} \frac{\pi}{2} g(X_2) \sin\frac{\pi}{2}(1 - X_1) & \frac{\mathrm{d}g}{\mathrm{d}X_2} \cos\frac{\pi}{2}(1 - X_1) & 0 \\ -\frac{\pi}{2} g(X_2) \cos\frac{\pi}{2}(1 - X_1) & \frac{\mathrm{d}g}{\mathrm{d}X_2} \sin\frac{\pi}{2}(1 - X_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) In order to preserve the volume, the determinant of the deformation gradient must be one.

$$\det \mathbf{F} = \frac{\pi}{2} g(X_2) \frac{\mathrm{d}g}{\mathrm{d}X_2} = 1$$

Using the separation of variables,

$$g\mathrm{d}g = \frac{2}{\pi}\mathrm{d}X_2$$

Integrating both sides

$$\frac{1}{2}g^2 = \frac{2}{\pi}X_2$$

Thus, the expression of $g(X_2)$ is obtained as

$$g(X_2) = \sqrt{\frac{4X_2}{\pi}}$$

(c) By substituting $g(X_2)$ into the deformation gradient,

$$\mathbf{F} = \begin{bmatrix} \sqrt{\pi X_2}\sin\frac{\pi}{2}(1-X_1) & \frac{1}{\sqrt{\pi X_2}}\cos\frac{\pi}{2}(1-X_1) & 0 \\ -\sqrt{\pi X_2}\cos\frac{\pi}{2}(1-X_1) & \frac{1}{\sqrt{\pi X_2}}\sin\frac{\pi}{2}(1-X_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The left Cauchy-Greed deformation tensor becomes

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{bmatrix} \pi X_2 & 0 & 0 \\ 0 & \frac{1}{\pi X_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the left Cauchy-Green deformation tensor has only diagonal components. Thus, the matrices of eigenvectors and eigenvalues become

$$\mathbf{\Phi} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix}, \quad \mathbf{\Lambda} = \begin{bmatrix} \pi X_2 & 0 & 0 \\ 0 & \frac{1}{\pi X_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Accordingly, **U**, **Q**, and **V** can be calculated by

$$\mathbf{U} = \mathbf{\Phi}\sqrt{\mathbf{\Lambda}}\mathbf{\Phi}^{\mathrm{T}} = \begin{bmatrix} \sqrt{\pi X_2} & 0 & 0 \\ 0 & \frac{1}{\sqrt{\pi X_2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{F}\mathbf{U}^{-1} = \begin{bmatrix} \sin & \cos & 0 \\ -\cos & \sin & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

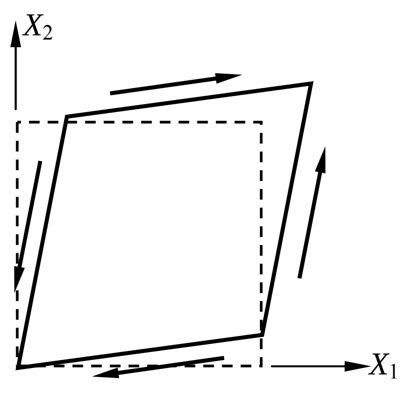
$$\mathbf{V} = \mathbf{F}\mathbf{Q}^{\mathrm{T}} = \begin{bmatrix} \sqrt{\pi X_2}\sin^2 + \dfrac{1}{\sqrt{\pi X_2}}\cos^2 & \left(-\sqrt{\pi X_2} + \dfrac{1}{\sqrt{\pi X_2}}\right)\sin\cos & 0 \\ \left(-\sqrt{\pi X_2} + \dfrac{1}{\sqrt{\pi X_2}}\right)\sin\cos & \sqrt{\pi X_2}\cos^2 + \dfrac{1}{\sqrt{\pi X_2}}\sin^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the above expressions, simplified notations are used such that $\sin = \sin\frac{\pi}{2}(1 - X_1)$ and $\cos = \cos\frac{\pi}{2}(1 - X_1)$. ■

P3.12 Consider a square element under pure shear deformation as shown in the figure. The relation between deformed and undeformed coordinates becomes

$$x_1 = X_1 + kX_2, \quad x_2 = kX_1 + X_2$$

(a) Calculate deformation gradient **F**, Lagrangian strain **E**, Eulerian strain **e**, and engineering strain ε. (b) Calculate principal stretch tensors **U** and **V**, and rotation tensor **Q**.



Solution:
(a) From the relation between deformed and undeformed coordinates, the deformation gradient and Lagrangian strain can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}$$

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \begin{bmatrix} \frac{1}{2}k^2 & k \\ k & \frac{1}{2}k^2 \end{bmatrix}$$

The engineering strain becomes

$$\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix}$$

Note that the Lagrangian strain has normal components in higher order.

(b) The relation in the polar decomposition is given as $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$. First, the right Cauchy-Green deformation tensor becomes

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1+k^2 & 2k \\ 2k & 1+k^2 \end{bmatrix}$$

The above Cauchy-Green tensor will have two eigenvalues:

$$\begin{aligned} \left|\mathbf{C} - \lambda\mathbf{1}\right| &= (1 + k^2 - \lambda)^2 - 4k^2 \\ &= (1 + k^2 - \lambda - 2k)(1 + k^2 - \lambda + 2k) = 0 \end{aligned}$$

Thus, two eigenvalues are $\lambda_1 = (1-k)^2$ and $\lambda_2 = (1+k)^2$ and

$$\Lambda = \begin{bmatrix} (1-k)^2 & 0 \\ 0 & (1+k)^2 \end{bmatrix}$$

Two eigenvectors corresponding to the two eigenvalues can be calculated as

$$\mathbf{E}_1 = \tfrac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}, \quad \mathbf{E}_2 = \tfrac{1}{\sqrt{2}}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \Phi = \tfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then, the principal stretch tensor $\mathbf{U}$ can be calculated from the following relation:

$$\mathbf{U} = \Phi\sqrt{\Lambda}\Phi^T = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1-k & 0 \\ 0 & 1+k \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}$$

Note that the principal stretch tensor is identical to the deformation gradient, which means there is no rotation involved in the deformation. Thus, we have

$$\mathbf{Q} = \mathbf{F}\mathbf{U}^{-1} = \mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because $\mathbf{Q} = \mathbf{1}$, the following relation can also be obtained:

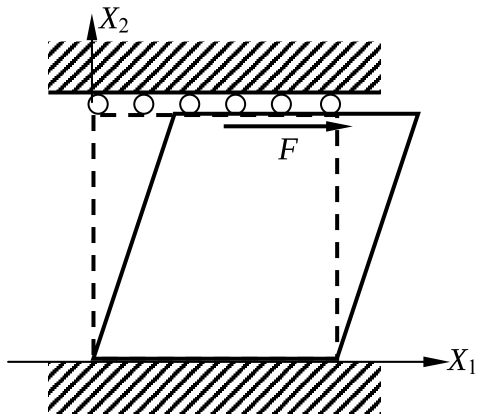
$$\mathbf{V} = \mathbf{F}\mathbf{Q}^{-1} = \mathbf{F} = \mathbf{U}$$

■

P3.13 A square block of surface area A on all sides is under pure shear deformation due to the uniformly distributed load F on the top surface, as shown in the figure. The deformation of the block is such that the deformed coordinates can be written as

$$x_1 = X_1 + aX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

Calculate Cauchy stress, 1-st and 2-nd Piola-Kirchhoff stresses.



Solution:
Since the force is uniformly distributed over the area, the Cauchy stress will be

$$\sigma = \begin{bmatrix} 0 & F/A & 0 \\ F/A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using the relation of $\mathbf{P} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$, the first Piola-Kirchhoff stress can be calculated by

$$\mathbf{P} = \begin{bmatrix} -aF/A & F/A & 0 \\ F/A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

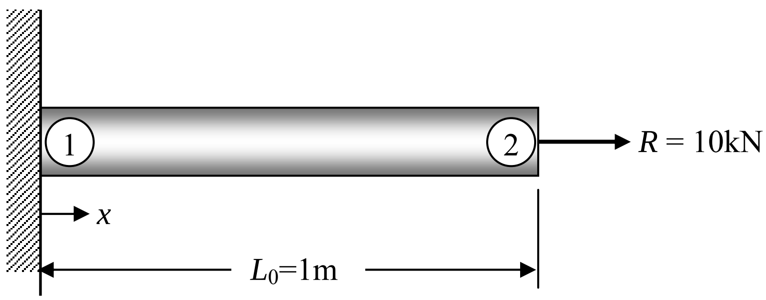
And from $\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-T}$ the second Piola-Kirchhoff stress becomes

,

$$\mathbf{S} = \begin{bmatrix} -2aF / A & F / A & 0 \\ F / A & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the shear stress components are all the same, but negative normal stress component exists in the case of **P** and **S**. However, this component becomes small as the shear deformation becomes small. ■

P3.14 A force R is applied at the tip of the uniform bar element shown in the figure. The initial length and the cross-sectional area of the bar are, respectively, A_0 and L_0. The elastic modulus of the material is E. Calculate the tip displacement by solving the total Lagrangian variational equation with St. Vernant-Kirchhoff nonlinear elastic material model. Assume the following numerical values: $E = 700\text{MPa}$, $A_0 = 1.0\times10^{-4}\text{m}^2$, $L_0 = 1.0\text{m}$, and $R = 10\text{kN}$. Compare the tip displacement with that from the linear elastic model when (a) $E = 700\text{MPa}$ and (b) $E = 70\text{GPa}$.



Solution:
If the tip displacement is u_{NL}, the displacement in the bar can be approximated by

$$u(X) = \frac{X}{L_0} u_{NL} = \lambda X \tag{1}$$

where $\lambda = u_{LN}/L_0$ is the stretch ratio. Using Eq. (1), the displacement gradient can be calculated by

$$\nabla_0 u = \frac{du}{dX} = \lambda \tag{2}$$

Since the problem is 1D, the displacement gradient becomes a scalar. Note that the above displacement gradient is in fact engineering strain. The deformation gradient becomes

$$F_{11} = 1 + \nabla_0 u = 1 + \lambda = \frac{L}{L_0} \tag{3}$$

where $L = L_0 + u_{LN}$ is the deformed length of the bar. The Lagrangian strain is given as

$$E_{11} = \tfrac{1}{2}(F_{11}^T F_{11} - 1) = \tfrac{1}{2}[(1+\lambda)^2 - 1] = \lambda + \tfrac{1}{2}\lambda^2 \tag{4}$$

Comparing with the engineering strain, the Lagrangian strain has an additional quadratic term. Assuming that the material is St. Vernant-Kirchhoff nonlinear elastic, the strain energy density becomes

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E} : \mathbf{D} : \mathbf{E} = \tfrac{1}{2}E \cdot E_{11}^2 = \tfrac{1}{2}E(\lambda + \tfrac{1}{2}\lambda^2)^2 \tag{5}$$

By differentiating the above strain energy density with respect to Lagrangian strain, the 2nd Piola-Kirchhoff stress can be calculated as

$$S_{11} = \frac{dW}{dE_{11}} = E \cdot E_{11} = E(\lambda + \tfrac{1}{2}\lambda^2) \tag{6}$$

In order to obtain the variational equation, the variation of the strain needs to be calculated. Since the displacement of the bar is expressed in terms of the tip displacement in Eq. (1), the variation of the displacement can also be represented by that of the tip displacement. Let $\overline{u}_{NL}$ be the variation of the tip displacement, and $\overline{\lambda} = \overline{u}_{NL} / L_0$ be the variation of the stretch ratio, the variation of displacement and that of the Lagrangian strain can be obtained as

$$\overline{u}(X) = \frac{X}{L_0}\overline{u}_{NL} = \overline{\lambda} X \tag{7}$$

$$\overline{E}_{11} = \tfrac{1}{2}(F_{11}^T \nabla_0 \overline{u} + \nabla_0 \overline{u}^T F_{11}) = (1+\lambda)\overline{\lambda} \tag{8}$$

Note that $\overline{E}_{11}$ is linear with respect to $\overline{\lambda}$. Using Eq. (8), the structural energy form for the total Lagrangian formulation becomes

$$a_0(u,\overline{u}) = \int_0^{L_0} S_{11}\overline{E}_{11}\, A_0 \,\mathrm{d}X = \int_0^{L_0} EA_0(\lambda + \tfrac{1}{2}\lambda^2)(1+\lambda)\overline{\lambda}\,\mathrm{d}X \tag{9}$$

Since the integrand is independent of X, the integral can be evaluated analytically, as

$$a_0(u,\overline{u}) = EA_0L_0\overline{\lambda}(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3) \tag{10}$$

Since the point load is applied at the tip of the bar, the load form can be evaluated without integration as

$$\ell_0(\overline{u}) = \overline{u}(L_0)R = \overline{\lambda} L_0 R \tag{11}$$

By equating Eqs. (10) and (11), the variational equation can be written as

$$EA_0L_0\overline{\lambda}(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3) = \overline{\lambda} L_0 R, \quad \forall \overline{\lambda} \in \mathbb{Z}_h \tag{12}$$

In the discrete domain, the space of kinematically admissible displacements will be the space of real numbers. In order to satisfy the above equation for all real number $\bar{\lambda}$, the coefficients of $\bar{\lambda}$ should be equal in the above equation, to yield

$$\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3 = \frac{R}{EA_0} \tag{13}$$

For linear elastic material, the above equation becomes

$$\lambda = \frac{R}{EA_0} \tag{14}$$

Note that both equations have the same right-hand side. Thus, the nonlinear displacement will always be smaller than that of linear displacement. In addition, when the displacement is small; i.e., $\lambda \ll 1$, the higher-order terms can be negligible, $\lambda^2 \approx \lambda^3 \approx 0$. Thus, the nonlinear displacement will approach to the linear one.

(a) When E = 700 MPa, Eq. (13) yields $\lambda = 0.12028 \quad \Rightarrow \quad u_{NL} = 120.28\text{mm}$, while Eq. (14) yields $\lambda = 0.14286 \quad \Rightarrow \quad u_L = 142.86\text{mm}$. Thus, the linear elastic model predicts about 19% larger tip displacement.

(b) When E = 70 GPa, Eq. (13) yields $\lambda = 0.001426 \quad \Rightarrow \quad u_{NL} = 1.426\text{mm}$, while Eq. (14) yields $\lambda = 0.001429 \quad \Rightarrow \quad u_L = 1.429\text{mm}$. Thus, the linear elastic model predicts about 0.2% larger tip displacement.

■

P3.15 Solve Problem P3.14 using force equilibrium; i.e., internal force caused by stress is equal to external force.

Solution:
From the previous problem, we have

$$S_{11} = E(\lambda + \tfrac{1}{2}\lambda^2)$$

However, the 2nd Piola-Kirchhoff stress cannot be used for force equilibrium. Thus, the 1st Piola-Kirchhoff stress is calculated using the following relation:

$$P_{11} = S_{11} \cdot F_{11}^T = E(\lambda + \tfrac{1}{2}\lambda^2)(1 + \lambda) = E(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3)$$

Note that the 1st Piola-Kirchhoff stress is defined with respect to the initial cross-sectional area. Assuming this stress is uniform over the cross-section, the force equilibrium can be obtained by

$$R = P_{11} \cdot A_0 = EA_0(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3)$$

Thus, we obtain the same equation with the variational approach, as

$$\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3 = \frac{R}{EA_0}$$

For linear elastic material, the above equation becomes

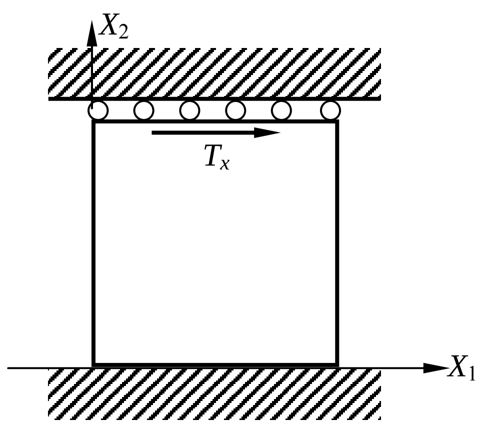
$$\lambda = \frac{R}{EA_0}$$

Note that both equations have the same right-hand side. Thus, the nonlinear displacement will always be smaller than that of linear displacement. In addition, when the displacement is small; i.e., $\lambda \ll 1$, the higher-order terms can be negligible, $\lambda^2 \approx \lambda^3 \approx 0$. Thus, the nonlinear displacement will approach to the linear one. ■

P3.16 Consider a plane strain, unit depth, square element as shown in the figure. Use St. Vernant-Kirchhoff isotropic material model with two Lame's constants λ and μ. A uniformly distributed force T_x (force per area) is horizontally applied at the top surface. Assuming it is a simple shear problem, the deformation of the element can be written as

$$\begin{cases} x_1 = X_1 + kX_2 \\ x_2 = X_2 \end{cases}$$

(a) Find the relation between k and T_x, (b) Find the reaction force in X_2 direction at the top surface, and (c) Compare the results with that of the linear elastic model.



Solution:

For the given deformation, the deformation gradient and Lagrangian strain can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & k \\ k & k^2 \end{bmatrix}$$

From the constitutive tensor of St. Vernant-Kirchhoff material $\mathbf{C} = \lambda\mathbf{1} \otimes \mathbf{1} + 2\mu\mathbf{I}$ and the Lagrangian strain, the 2nd Piola-Kirchhoff stress can be calculated by

$$\mathbf{S} = \mathbf{D} : \mathbf{E} = \lambda \mathrm{tr}(\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} = \begin{bmatrix} \frac{1}{2}\lambda k^2 & \mu k \\ \mu k & (\frac{1}{2}\lambda + \mu)k^2 \end{bmatrix}$$

Since the 2nd Piola-Kirchhoff stress does not have any physical meaning, it is converted to the 1st Piola-Kirchhoff stress as

$$\mathbf{P} = \mathbf{S} \cdot \mathbf{F}^T = \begin{bmatrix} (\frac{1}{2}\lambda + \mu)k^2 & \mu k \\ \mu k + (\frac{1}{2}\lambda + \mu)k^3 & (\frac{1}{2}\lambda + \mu)k^2 \end{bmatrix}$$

The unit normal vector of the top surface is $\mathbf{N} = [0,\ 1]^{\mathrm{T}}$. Thus, the surface traction on the top surface becomes

$$\mathbf{P}^T \cdot \mathbf{N} = \begin{Bmatrix} \mu k + (\frac{1}{2}\lambda + \mu)k^3 \\ (\frac{1}{2}\lambda + \mu)k^2 \end{Bmatrix} = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix}$$

(a) In the above equation, T_x is the horizontal surface traction on the top surface or, equivalently, uniformly distributed force: $T_x = \mu k + (\frac{1}{2}\lambda + \mu)k^3$
(b) T_y is the uniformly distributed vertical reaction: $T_y = (\frac{1}{2}\lambda + \mu)k^2$
(c) For linear elastic model, the relation between k and T_x is $T_x = \mu k$. Thus, the nonlinear model has higher-order terms. In addition, the vertical reaction in linear elastic model is zero, while the nonlinear elastic model yields non-zero vertical reaction. When $k \to 0$, the results from nonlinear model approach that of the linear elastic model. ■

P3.17 Consider a deformation of a rectangular bar whose deformed geometry is given as

$$x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3$$

When the material is incompressible and St. Venant-Kirchhoff material properties are given as $E = 600\mathrm{MPa}$ and $\nu = 0.49$, write the expression of S_{11} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{11} of the Cauchy stress as a function of α. Plot S_{11} and σ_{11} in the range of $\alpha = [0.7\ 1.5]$.

Solution:
For the given material properties, the Lame's constants can be calculated from

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 9,865.8\ \mathrm{MPa}$$
$$\mu = \frac{E}{2(1+\nu)} = 201.3\ \mathrm{MPa}$$

For given deformation, the deformation gradient and Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}$$

The relation between α and β can be obtained from incompressibility:

$$\det \mathbf{F} = \alpha\beta^2 = 1 \quad \Rightarrow \quad \beta = \alpha^{-1/2}$$

Thus, the Lagrangian strain can be calculated as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & \alpha^{-1} - 1 & 0 \\ 0 & 0 & \alpha^{-1} - 1 \end{bmatrix}$$

Since all shear components are zero, we can only consider the normal components as a vector. The second Piola-Kirchhoff stress becomes

$$\mathbf{S} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{33} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda \\ \lambda & \lambda + 2\mu & \lambda \\ \lambda & \lambda & \lambda + 2\mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ E_{33} \end{Bmatrix} = \begin{Bmatrix} \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^2 - 1) \\ \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^{-1} - 1) \\ \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^{-1} - 1) \end{Bmatrix}$$

The S_{11} component of the stress becomes

$$S_{11} = \frac{\lambda}{2}(\alpha^2 + 2\alpha^{-1} - 3) + \mu(\alpha^2 - 1)$$

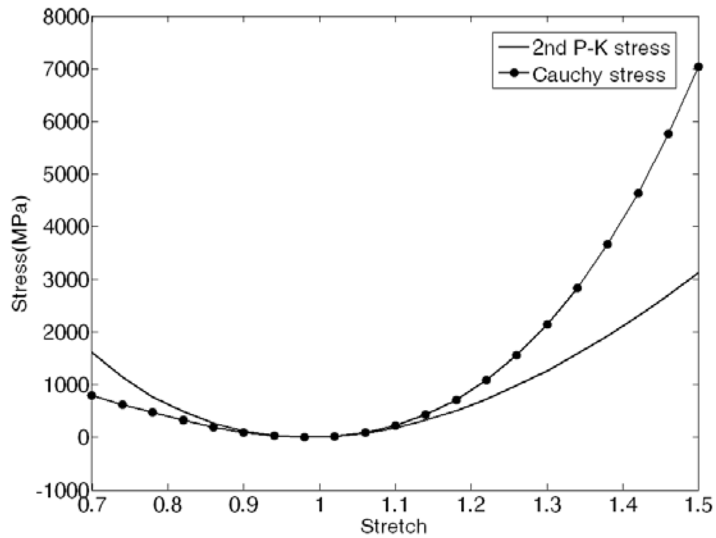
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{11} component of the stress becomes

$$\sigma_{11} = \frac{\lambda}{2}(\alpha^4 + 2\alpha - 3\alpha^2) + \mu(\alpha^4 - \alpha^2)$$

The following figure shows the two stress components as a function of the principal stretch α. Note that both stresses are highly nonlinear even if the relation between stress and strain is constant.



■

P3.18 Consider a simple shear deformation of a square whose deformed geometry is given as

$$x_1 = X_1 + \alpha X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

When the material is incompressible and St. Venant-Kirchhoff material properties are given as $E = 600\text{MPa}$ and $\nu = 0.49$, write the expression of S_{12} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{12} of the Cauchy stress as a function of α. Plot S_{12} and σ_{12} in the range of $\alpha = [0.0 \ 1.5]$.

Solution:
For the given material properties, the Lame's constants can be calculated from

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 9,865.8 \text{ MPa}$$
$$\mu = \frac{E}{2(1+\nu)} = 201.3 \text{ MPa}$$

For given deformation with incompressibility, the deformation gradient and Lagrangian strain tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & \alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In the plane strain problem, we can consider only three non-zero stress components. The second Piola-Kirchhoff stress becomes

$$\mathbf{S} = \begin{Bmatrix} S_{11} \\ S_{22} \\ S_{12} \end{Bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \begin{Bmatrix} E_{11} \\ E_{22} \\ 2E_{12} \end{Bmatrix} = \begin{Bmatrix} \frac{\lambda}{2}\alpha^2 \\ (\frac{\lambda}{2} + \mu)\alpha^2 \\ \mu\alpha \end{Bmatrix}$$

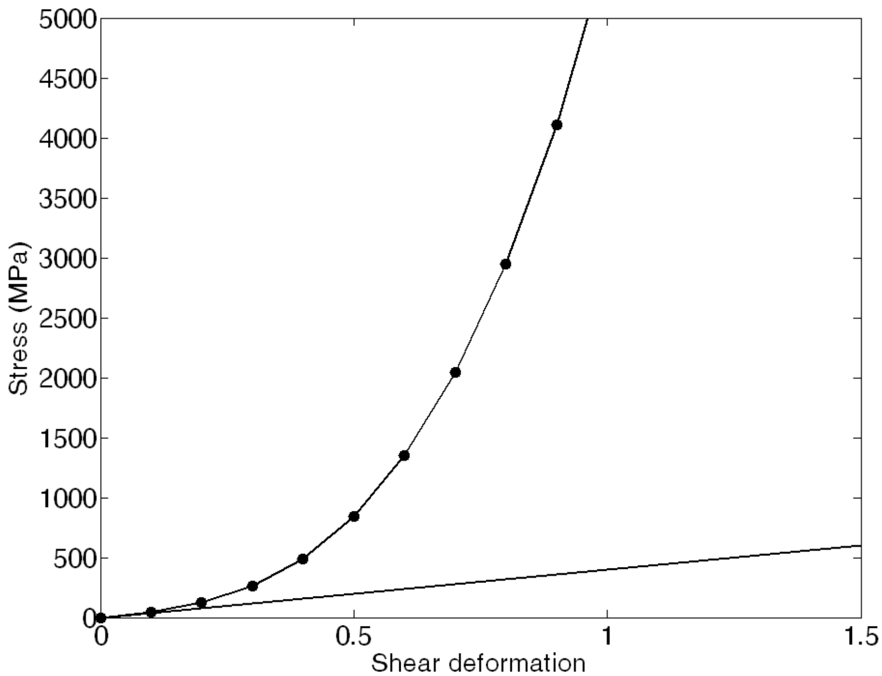
Thus, S_{12} is a linear function of α. The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{12} component of the stress becomes

$$\sigma_{12} = \left(\frac{\lambda}{2} + \mu\right)\alpha^3 + \mu\alpha$$

Different from the hyperelastic material, now σ_{12} is a cubic function, while S_{12} is a linear function for the shear deformation. The following figure compares the two stresses as a function of shear deformation.



■

P3.19 Consider the following deformation with $|\alpha| \leq 1$:

$$x_1 = X_1 + \alpha X_2, \quad x_2 = \sqrt{1-\alpha^2}X_2, \quad x_3 = X_3$$

Assume St. Venant-Kirchhoff material with two material parameters λ and μ. (a) Show that the above deformation is a pure shear deformation in terms of the Lagrangian strain, (b) Calculate the second Piola-Kirchhoff stress and Cauchy stress in terms of α, λ, and μ.

Solution:
(a) For the given deformation, the deformation gradient and the Lagrangian strain become

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & \sqrt{1-\alpha^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}) = \frac{1}{2}\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

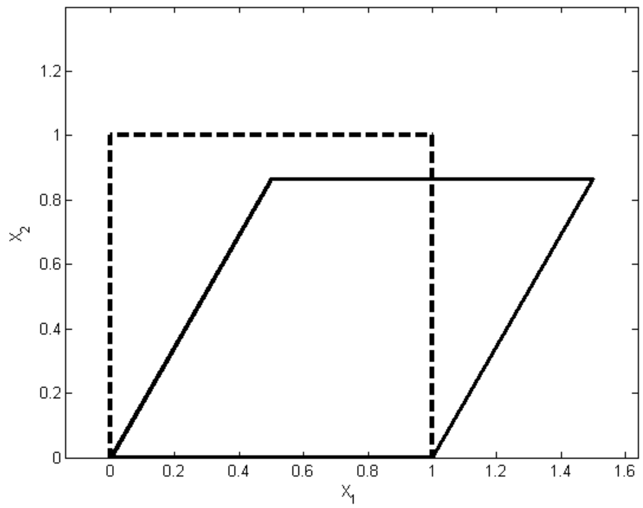
Thus, the given deformation is a pure shear deformation in terms of the Lgrangian strain.
(b) From the St. Venant-Kichhoff material, the stress-strain relation becomes

$$\mathbf{S} = \lambda(tr\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} = \mu\begin{bmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, only shear stress component S_{12} exists. The Cauchy stress becomes

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{FSF}^{\mathrm{T}} = \mu\begin{bmatrix} \frac{\alpha^2}{\sqrt{1-\alpha^2}} & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the Cauchy stress has a non-zero normal component σ_{11}. The following figure shows the deformation of a square block with $\alpha = 0.5$.



■

P3.20 A force F is applied at the tip of the uniform bar shown in the figure. The displacement of the bar is given as $u(X) = \lambda X$ where λ is the principal stretch. The initial length and the cross-sectional area of the bar are, respectively, A_0 and L_0. The elastic modulus of the material is E. Calculate the tip displacement by solving the principal stretch using the total Lagrangian formulation with the St. Venant-Kirchhoff material model. Assume the following numerical values: E = 700 MPa, $A_0 = 1.0\times10^{-4}$ m^2, L_0 = 1.0 m, and F = 10 kN. Compare the tip displacement with that from the linear elastic model when (a) E = 700 MPa and (b) E = 70 GPa.

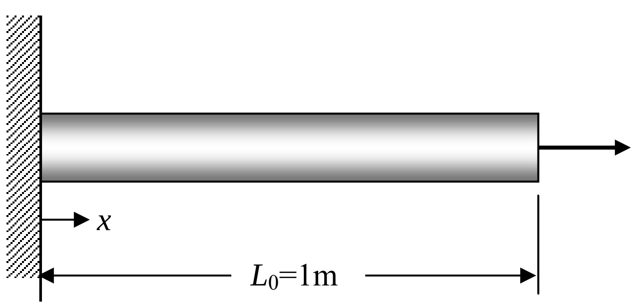


Figure P3.20

Solution:
If the tip displacement is u_{NL}, the displacement in the bar can be approximated by

$$u(X) = \frac{X}{L_0} u_{NL} = \lambda X \tag{1}$$

where $\lambda = u_{NL} / L_0$ is the stretch ratio. Using Eq. (1), the displacement gradient can be calculated by

$$\nabla_0 u = \frac{du}{dX} = \lambda \tag{2}$$

Since the problem is 1D, the displacement gradient becomes a scalar. Note that the above displacement gradient is in fact engineering strain. The deformation gradient becomes

$$F_{11} = 1 + \nabla_0 u = 1 + \lambda = \frac{L}{L_0} \tag{3}$$

where $L = L_0 + u_{NL}$ is the deformed length of the bar. The Lagrangian strain is given as

$$E_{11} = \tfrac{1}{2}(F_{11}^T F_{11} - 1) = \tfrac{1}{2}[(1+\lambda)^2 - 1] = \lambda + \tfrac{1}{2}\lambda^2 \tag{4}$$

Comparing with the engineering strain, the Lagrangian strain has an additional quadratic term. Assuming that the material is St. Vernant-Kirchhoff nonlinear elastic, the strain energy density becomes

$$W(\mathbf{E}) = \tfrac{1}{2}\mathbf{E} : \mathbf{C} : \mathbf{E} = \tfrac{1}{2}E \cdot E_{11}^2 = \tfrac{1}{2}E(\lambda + \tfrac{1}{2}\lambda^2)^2 \tag{5}$$

By differentiating the above strain energy density with respect to Lagrangian strain, the 2nd Piola-Kirchhoff stress can be calculated as

$$S_{11} = \frac{dW}{dE_{11}} = E \cdot E_{11} = E(\lambda + \tfrac{1}{2}\lambda^2) \tag{6}$$

In order to obtain the variational equation, the variation of the strain needs to be calculated. Since the displacement of the bar is expressed in terms of the tip displacement in Eq. (1), the variation of the displacement can also be represented by that of the tip displacement. Let $\overline{u}_{NL}$ be the variation of the tip displacement, and $\overline{\lambda} = \overline{u}_{NL} / L_0$ be the variation of the stretch ratio, the variation of displacement and that of the Lagrangian strain can be obtained as

$$\overline{u}(X) = \frac{X}{L_0}\overline{u}_{NL} = \overline{\lambda}X \tag{7}$$

$$\overline{E}_{11} = \tfrac{1}{2}(F_{11}^T \nabla_0 \overline{u} + \nabla_0 \overline{u}^T F_{11}) = (1+\lambda)\overline{\lambda} \tag{8}$$

Note that $\overline{E}_{11}$ is linear with respect to $\overline{\lambda}$. Using Eq. (8), the structural energy form for the total Lagrangian formulation becomes

$$a_0(u, \overline{u}) = \int_0^{L_0} S_{11}\overline{E}_{11} A_0 \,\mathrm{d}X = \int_0^{L_0} EA_0(\lambda + \tfrac{1}{2}\lambda^2)(1+\lambda)\overline{\lambda}\,\mathrm{d}X \tag{9}$$

Since the integrand is independent of X, the integral can be evaluated analytically, as

$$a_0(u,\overline{u}) = EA_0L_0\overline{\lambda}(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3) \tag{10}$$

Since the point load is applied at the tip of the bar, the load form can be evaluated without integration as

$$\ell_0(\overline{u}) = \overline{u}(L_0)F = \overline{\lambda}L_0F \tag{11}$$

By equating Eqs. (10) and (11), the variational equation can be written as

$$EA_0L_0\overline{\lambda}(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3) = \overline{\lambda}L_0F, \quad \forall\overline{\lambda} \in \mathbb{Z}_h \tag{12}$$

In the discrete domain, the space of kinematically admissible displacements will be the space of real numbers. In order to satisfy the above equation for all real number $\overline{\lambda}$, the coefficients of $\overline{\lambda}$ should be equal in the above equation, to yield

$$\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3 = \frac{F}{EA_0} \tag{13}$$

For linear elastic material, the above equation becomes

$$\lambda = \frac{F}{EA_0} \tag{14}$$

Note that both equations have the same right-hand side. Thus, the nonlinear displacement will always be smaller than that of linear displacement. In addition, when the displacement is small; i.e., $\lambda \ll 1$, the higher-order terms can be negligible, $\lambda^2 \approx \lambda^3 \approx 0$. Thus, the nonlinear displacement will approach to the linear one.

(a) When *E* = 700 MPa, Eq. (13) yields $\lambda = 0.12028 \quad \Rightarrow \quad u_{NL} = 120.28\text{mm}$, while Eq. (14) yields $\lambda = 0.14286 \quad \Rightarrow \quad u_L = 142.86\text{mm}$. Thus, the linear elastic model predicts about 19% larger tip displacement.

(b) When *E* = 70 GPa, Eq. (13) yields $\lambda = 0.001426 \quad \Rightarrow \quad u_{NL} = 1.426\text{mm}$, while Eq. (14) yields $\lambda = 0.001429 \quad \Rightarrow \quad u_L = 1.429\text{mm}$. Thus, the linear elastic model predicts about 0.2% larger tip displacement. ■

P3.21 Solve Problem P4.20 using force equilibrium; i.e., internal force caused by stress is equal to external force.

Solution:
From the previous problem, we have

$$S_{11} = E(\lambda + \tfrac{1}{2}\lambda^2)$$

However, the 2nd Piola-Kirchhoff stress cannot be used for force equilibrium. Thus, the 1st Piola-Kirchhoff stress is calculated using the following relation:

$$P_{11} = S_{11} \cdot F_{11}^T = E(\lambda + \tfrac{1}{2}\lambda^2)(1 + \lambda) = E(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3)$$

Note that the 1st Piola-Kirchhoff stress is defined with respect to the initial cross-sectional area. Assuming this stress is uniform over the cross-section, the force equilibrium can be obtained by

$$F = P_{11} \cdot A_0 = EA_0(\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3)$$

Thus, we obtain the same equation with the variational approach, as

$$\lambda + \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda^3 = \frac{F}{EA_0}$$

For linear elastic material, the above equation becomes

$$\lambda = \frac{F}{EA_0}$$

Note that both equations have the same right-hand side. Thus, the nonlinear displacement will always be smaller than that of linear displacement. In addition, when the displacement is small; i.e., $\lambda \ll 1$, the higher-order terms can be negligible, $\lambda^2 \approx \lambda^3 \approx 0$. Thus, the nonlinear displacement will approach to the linear one.
(a) When E = 700 MPa, Eq. (13) yields $\lambda = 0.12028 \quad \Rightarrow \quad u_{NL} = 120.28\text{mm}$, while Eq. (14) yields $\lambda = 0.14286 \quad \Rightarrow \quad u_L = 142.86\text{mm}$. Thus, the linear elastic model predicts about 19% larger tip displacement.

(b) When E = 70 GPa, Eq. (13) yields $\lambda = 0.001426 \quad \Rightarrow \quad u_{NL} = 1.426\text{mm}$, while Eq. (14) yields $\lambda = 0.001429 \quad \Rightarrow \quad u_L = 1.429\text{mm}$. Thus, the linear elastic model predicts about 0.2% larger tip displacement. ■

P3.22 Consider two bar elements under a force at the tip. Using the displacement-controlled method, plot the load-displacement curve (F vs. u_2 and u_3). Increase the tip displacement u_3 up to 1.0m with ten equal increments. Assume St. Venant-Kirchhoff material with $E = 100\text{MPa}$, and cross-sectional areas of $A^{(1)} = 1.0\times10^{-4}$ m^2 and $A^{(2)} = 0.5\times10^{-4}$ m^2.

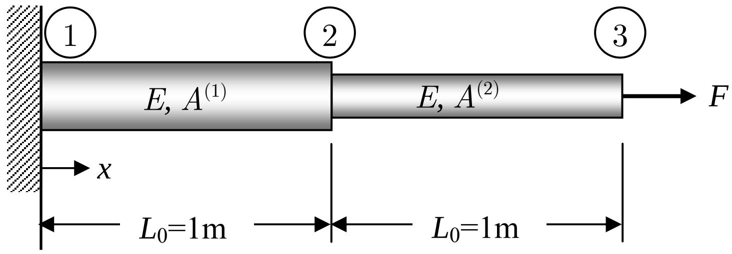


Figure P3.22

Solution:
Since this is a one-dimensional problem, only X_1-directional component will be used in the following calculations. For Elements 1 and 2, the Lagrangian strains are defined as

$$E_{11}^{(1)} = u_2 + \tfrac{1}{2}u_2^2, \quad E_{11}^{(2)} = u_3 - u_2 + \tfrac{1}{2}(u_3 - u_2)^2$$

In the above equation $u_1 = 0$ is used. For the St. Venant-Kirchhoff material, the second Piola-Kirchhoff strsses for the two elements can be calculated by

$$S_{11}^{(1)} = E(u_2 + \tfrac{1}{2}u_2^2), \quad S_{11}^{(2)} = E\left(u_3 - u_2 + \tfrac{1}{2}(u_3 - u_2)^2\right)$$

Since u_1 is fixed, its variation is also equal to zero. The variation of the Lagrangian strans become

$$\overline{E}_{11}^{(1)} = (1 + u_2)\overline{u}_2, \quad \overline{E}_{11}^{(2)} = (u_2 - u_3 - 1)(\overline{u}_2 - \overline{u}_3)$$

The energy form can be obtained by adding the contributions from two elements as

$$\begin{aligned} a(u,\overline{u}) &= \int_0^{L_0} S_{11}^{(1)}(1 + u_2)\overline{u}_2 A^{(1)}\,\mathrm{d}X + \int_{L_0}^{2L_0} S_{11}^{(2)}(u_2 - u_3 - 1)(\overline{u}_2 - \overline{u}_3)A^{(2)}\,\mathrm{d}X \\ &= \overline{u}_2\left[S_{11}^{(1)}(1 + u_2)A^{(1)}L_0 + S_{11}^{(2)}(u_2 - u_3 - 1)A^{(2)}L_0\right] - \overline{u}_3\left[S_{11}^{(2)}(u_2 - u_3 - 1)A^{(2)}L_0\right] \end{aligned}$$

The load form is simply

$$\ell(\overline{u}) = \overline{u}_3 F_3$$

Since the nonlinear variational equation must satisfy for arbitraty $\overline{u}_2$ and $\overline{u}_3$, two nonlinear equation can be obtained:

$$\begin{aligned} &S_{11}^{(1)}(1 + u_2)A^{(1)}L_0 + S_{11}^{(2)}(u_2 - u_3 - 1)A^{(2)}L_0 = 0 \\ &S_{11}^{(2)}(u_2 - u_3 - 1)A^{(2)}L_0 + F = 0 \end{aligned}$$

Note that the first equation is the coefficient of $\overline{u}_2$ and the second is that of $\overline{u}_3$. Since u_3 is prescribed for the displacement-controll method, its variation is zero, and the applied force is in fact the reaction force required prescribing the displacement. Thus, for a given u_3, the unknown u_2 is solved from the first equation. After that, the second equation is used to solve for the reaction force F.

In order to solve the above nonlinear equation using the Newton-Raphson method, the increment is stresses are required:

$$\Delta S_{11}^{(1)} = E(1+u_2)\Delta u_2, \quad \Delta S_{11}^{(2)} = E(u_2 - u_3 - 1)\Delta u_2$$

Note that only the increment Δu_2 is considered because the motion of u_3 is prescribed; i.e., the convergence iteration is performed to find Δu_2 after increasing u_3 according to the displacement controlled method. Thus,

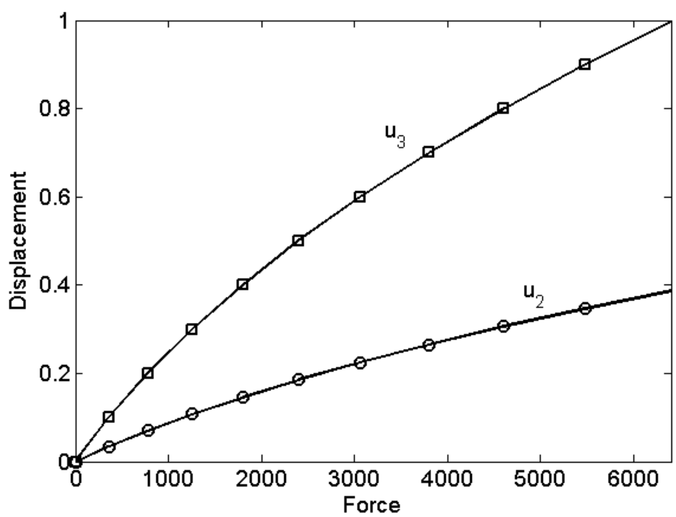
$$\begin{aligned} a^*(u;\Delta u,\overline{u}) &= \int_0^{L_0} A^{(1)}[E(1+u_2)^2 + S_{11}^{(1)}]\Delta u_2 \overline{u}_2 \, \mathrm{d}X \\ &+ \int_{L_0}^{2L_0} A^{(2)}[E(u_2 - u_3 - 1)^2 + S_{11}^{(2)}]\Delta u_2 \overline{u}_2 \, \mathrm{d}X \\ &= \overline{u}_2 L_0 \left[A^{(1)}\left(E(1+u_2)^2 + S_{11}^{(1)}\right) + A^{(2)}\left(E(u_2 - u_3 - 1)^2 + S_{11}^{(2)}\right)\right]\Delta u_2 \end{aligned}$$

Below is the list of MATLAB programs that solves for the nonlinear variational equation. The following table and figure show the converged solutions at each increment.

```
%
% P3.22 Two bar elements--displacement controlled procedure
%
tol = 1.0e-5;  conv = 0;  u2 = 0;  u2old = u2;
E = 1E8; A1 = 1E-4; A2 = .5E-4;
fprintf('\n step      u2      u3        F');
% Displacement increment loop
for i=1:10
 u3 = 0.1*i;
 S1 = E*(u2+.5*u2^2);
 S2 = E*(u3-u2+.5*(u3-u2)^2);
 P = S1*A1*(1+u2)+S2*A2*(u2-u3-1);
 F = S2*A2*(1+u3-u2);
 R = -P;
 conv = R^2;
 % Convergence loop
 iter = 0;
 while conv > tol && iter < 50
  Kt = A1*(E*(1+u2)^2+S1) + A2*(E*(u2-u3-1)^2+S2);
  delu2 = R/Kt;
  u2 = u2old + delu2;
  S1 = E*(u2+.5*u2^2);
  S2 = E*(u3-u2+.5*(u3-u2)^2);
  P = S1*A1*(1+u2)+S2*A2*(u2-u3-1);
  R = -P;
  conv= R^2;
  u2old = u2;
  iter = iter + 1;
 end
 F = S2*A2*(1+u3-u2);
 fprintf('\n %3d  %7.5f %7.5f %8.3f',i,u2,u3,F);
end
```

Increment	u_2	u_3	F

1	0.0343	0.1000	361.3
2	0.0704	0.2000	779.7
3	0.1077	0.3000	1256.9
4	0.1460	0.4000	1795.0
5	0.1851	0.5000	2396.4
6	0.2248	0.6000	3063.4
7	0.2651	0.7000	3798.5
8	0.3058	0.8000	4604.2
9	0.3469	0.9000	5482.9
10	0.3883	1.0000	6437.2



■

P3.23 Consider a nonlinear elastic uniaxial bar element under tip force $F = 100\text{N}$ shown in Figure 3.11. The stress strain relation is given in terms of Cauchy stress and engineering strain in the deformed geometry: $\sigma_{11} = E\varepsilon_{11}$. Using the updated Lagrangian formulation, solve for displacement at the tip, stress and strain of the uniaxial bar. Assume $E = 200\text{Pa}$ and the cross-sectional area $A = 1.0\text{m}^2$.

Solution:
It is easy to estimate the stress and strain in order to make equilibrium with the applied load. Since $F = 100\text{N}$ and $A = 1.0\text{m}^2$, the required Cauchy stress should be $\sigma_{11} = 100\text{Pa}$. From the stress-strain relation, the required strain should be $\varepsilon_{11} = 0.5$. Since the strain is defined using the deformed geometry, the deformed length of the bar should be

2.0m, which yields the tip displacement of $u_2 = 1.0\text{m}$. Now in the following, the updated Lagrangian method is used to solve the problem.

Since the strain is defined in the deformed geometry,

$$\varepsilon_{11}(u) = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}x} = \frac{u_2}{1+u_2}, \quad \varepsilon_{11}(\overline{u}) = \frac{\overline{u}_2}{1+u_2}, \quad J = 1+u_2$$

Thus, since a concentrated force is applied at the tip, the load form is $\ell(\overline{u}) = F\overline{u}_2$. The energy form can be wriiten as

$$\begin{aligned} a(u,\overline{u}) &= \int_0^L \sigma_{11}\varepsilon_{11}(\overline{u})A\,\mathrm{d}x \\ &= \int_0^{L_0} \sigma_{11}\varepsilon_{11}(\overline{u})AJ\,\mathrm{d}X \\ &= EAL_0\frac{u_2\overline{u}_2}{1+u_2} \end{aligned}$$

Since the nonlinear variational equation satisfies for arbitrary $\overline{u}_2$, the residual can be defined as

$$R = EAL_0\frac{u_2}{1+u_2} - F = 0$$

Since the constitutive relation is given in terms of Cauchy stress and engineering strain, it is unnecessary to transform the material description to the spatial descrition. It is more convenient to directly linearlize the energy form in the spatial form. After linearizing the residual, the incremental equation for the Newton-Raphson method becomes

$$\frac{EAL_0}{(1+u_2)^2}\Delta u_2 = -R$$

Below is the list of MATLAB programs that solve the the nonlinear variational equation. Also the following table shows the convergence interation of the Newton-Raphson method. The solution converges in the fourth iteration. As expected, the tip displacement, strain, and strss converge to $u_2 = 1.0\text{m.}$, $\varepsilon_{11} = 0.5$, and $\sigma_{11} = 100\text{Pa}$.

```
%
% P3.23 Uniaxial bar--updated Lagrangian formulation
%
tol = 1.0e-5;  iter = 0;  E = 200;
u = 0;         uold = u;  f = 100;
strain = u/(1+u);
stress = E*strain;
P = stress;
R = f - P;
conv= R^2/(1+f^2);
fprintf('\n iter      u1    Strain   Stress         conv');
fprintf('\n %3d  %7.5f  %7.5f %8.3f %12.3e %7.5f',iter,u,strain,stress,conv);
while conv > tol && iter < 20
 Kt = E/(1+u)^2;
 delu = R/Kt;
 u = uold + delu;
 strain = u/(1+u);
```

```
 stress = E*strain;
 P = stress;
 R = f - P;
 conv= R^2/(1+f^2);
 uold = u;
 iter = iter + 1;
 fprintf('\n %3d  %7.5f  %7.5f %8.3f %12.3e %7.5f',iter,u,strain,stress,conv);
end
```

Iteration	u	Strain	Stress	conv
0	0.0000	0.0000	0.000	9.999E−01
1	0.5000	0.3333	66.667	1.111E−01
2	0.8750	0.4667	93.333	4.444E−03
3	0.9922	0.4980	99.608	1.538E−05
4	1.0000	0.5000	99.998	2.328E−10

■

P3.24 Consider a deformation of a rectangular bar whose deformed geometry is given as

$$x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3$$

When the material is incompressible, Mooney-Rivlin hyperelastic material with $A_{10} = 80\text{MPa}$ and $A_{01} = 20\text{MPa}$, write the expression of S_{11} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{11} of the Cauchy stress as a function of α. Plot S_{11} and σ_{11} in the range of $\alpha = [0.7\ 1.5]$.

Solution:
For given deformation, the deformation gradient and Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{bmatrix}$$

The relation between α and β can be obtained from incompressibility:

$$\det \mathbf{F} = \alpha\beta^2 = 1 \quad \Rightarrow \quad \beta = \alpha^{-1/2}$$

The three invariants of the deformation tensor can be obtained as

$$
\begin{aligned}
I_1 &= \alpha^2 - 2\alpha^{-1} \\
I_2 &= 2\alpha + \alpha^{-2} \\
I_3 &= 1
\end{aligned}
$$

The reduced invariants become

$$
\begin{aligned}
J_1 &= I_1 I_3^{-1/3} = \alpha^2 - 2\alpha^{-1} \\
J_2 &= I_2 I_3^{-2/3} = 2\alpha + \alpha^{-2} \\
J_3 &= I_3^{-1/2} = 1
\end{aligned}
$$

In order to calculate stress, we need to differentiate the reduced invariants with respect to strain

$$
I_{1,\mathbf{E}} = 2\mathbf{1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = 2\begin{bmatrix} 2\alpha^{-1} & 0 & 0 \\ 0 & \alpha^2 + \alpha^{-1} & 0 \\ 0 & 0 & \alpha^2 + \alpha^{-1} \end{bmatrix}
$$

$$
I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = 2\begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}
$$

The derivatives of the reduced invariants become

$$
J_{1,\mathbf{E}} = I_3^{-1/3} I_{1,\mathbf{E}} - \frac{1}{3} I_1 I_3^{-4/3} I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} 2(1-\alpha^{-3}) & 0 & 0 \\ 0 & 1-\alpha^3 & 0 \\ 0 & 0 & 1-\alpha^3 \end{bmatrix}
$$

$$
J_{2,\mathbf{E}} = I_3^{-2/3} I_{2,\mathbf{E}} - \frac{2}{3} I_2 I_3^{-5/3} I_{3,\mathbf{E}} = \frac{2}{3}\begin{bmatrix} 2(\alpha^{-1}-\alpha^{-4}) & 0 & 0 \\ 0 & -\alpha^2 + \alpha^{-1} & 0 \\ 0 & 0 & -\alpha^2 + \alpha^{-1} \end{bmatrix}
$$

$$
J_{3,\mathbf{E}} = \frac{1}{2} I_3^{-1/2} I_{3,\mathbf{E}} = \begin{bmatrix} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}
$$

Thus, the second Piola-Kirchhoff stress becomes

$$
\mathbf{S} = A_{10} J_{1,\mathbf{E}} + A_{01} J_{2,\mathbf{E}} + K(J_3 - 1) J_{3,\mathbf{E}}
$$

The S_{11} component of the stress becomes

$$\begin{aligned} S_{11} &= \frac{4}{3}\left[A_{10}(1-\alpha^{-3}) + A_{01}(\alpha^{-1}-\alpha^{-4})\right] \\ &= \frac{4}{3}(80+20\alpha^{-1})(1-\alpha^{-3}) \end{aligned}$$

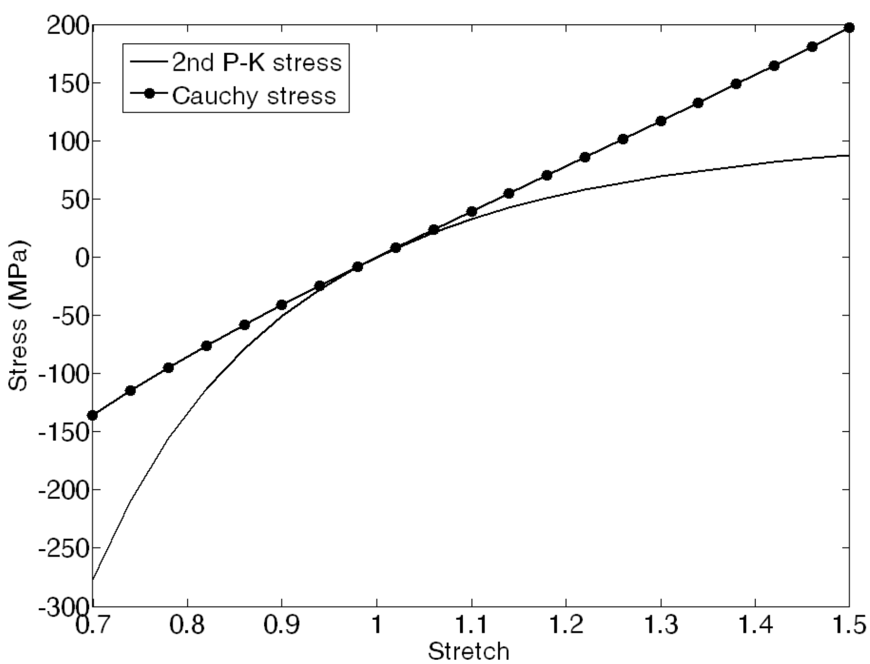
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\mathbf{S}\cdot\mathbf{F}^T$$

The σ_{11} component of the stress becomes

$$\sigma_{11} = \frac{4}{3}(80\alpha^2 + 20\alpha)(1-\alpha^{-3})$$

The following figure shows the two stress components as a function of the principal stretch α. Note that the second Piola-Kirchhoff stress is highly nonlinear, but the Cauchy stress is reasonably linear with respect to the principal stretch. Also note that the two stresses are similar when the deformation is small. However, as deformation increases, the difference also increases.



■

P3.25 Consider a simple shear deformation of a square whose deformed geometry is given as

$$x_1 = X_1 + \alpha X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

When the material is incompressible, Mooney-Rivlin hyperelastic material with $A_{10} = 80\text{MPa}$ and $A_{01} = 20\text{MPa}$, write the expression of S_{12} component of the second Piola-Kirchhoff stress as a function of α. In addition, write the expression of σ_{12} of the Cauchy stress as a function of α. Plot S_{12} and σ_{12} in the range of $\alpha = [0.0,\ 1.5]$.

Solution:
For given deformation, the deformation gradient and Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \begin{bmatrix} 1 & \alpha & 0 \\ \alpha & \alpha^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The three invariants of the deformation tensor can be obtained as

$$I_1 = \alpha^2 + 3, \quad I_2 = \alpha^2 + 3, \quad I_3 = 1$$

The reduced invariants become

$$\begin{aligned} J_1 &= I_1 I_3^{-1/3} = \alpha^2 + 3 \\ J_2 &= I_2 I_3^{-2/3} = \alpha^2 + 3 \\ J_3 &= I_3^{-1/2} = 1 \end{aligned}$$

In order to calculate stress, we need to differentiate the reduced invariants with respect to strain

$$I_{1,\mathbf{E}} = 2\mathbf{1} = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_{2,\mathbf{E}} = 2(I_1\mathbf{1} - \mathbf{C}) = \begin{bmatrix} 2\alpha^2 + 4 & -2\alpha & 0 \\ -2\alpha & 4 & 0 \\ 0 & 0 & 2\alpha^2 + 4 \end{bmatrix}$$

$$I_{3,\mathbf{E}} = 2I_3\mathbf{C}^{-1} = \begin{bmatrix} 2\alpha^2 + 2 & -2\alpha & 0 \\ -2\alpha & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The derivatives of the reduced invariants become

$$J_{1,\mathbf{E}} = I_3^{-1/3} I_{1,\mathbf{E}} - \frac{1}{3} I_1 I_3^{-4/3} I_{3,\mathbf{E}} = \frac{2}{3} \begin{bmatrix} -\alpha^4 - 4\alpha^2 & \alpha^3 + 3\alpha & 0 \\ \alpha^3 + 3\alpha & -\alpha^2 & 0 \\ 0 & 0 & -\alpha^2 \end{bmatrix}$$

$$J_{2,\mathbf{E}} = I_3^{-2/3} I_{2,\mathbf{E}} - \frac{2}{3} I_2 I_3^{-5/3} I_{3,\mathbf{E}} = \frac{2}{3} \begin{bmatrix} -2\alpha^4 - 5\alpha^2 & 2\alpha^3 + 3\alpha & 0 \\ 2\alpha^3 + 3\alpha & -2\alpha^2 & 0 \\ 0 & 0 & -2\alpha^2 \end{bmatrix}$$

$$J_{3,\mathbf{E}} = \frac{1}{2} I_3^{-1/2} I_{3,\mathbf{E}} = \begin{bmatrix} \alpha^2 + 1 & -\alpha & 0 \\ -\alpha & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the second Piola-Kirchhoff stress becomes

$$\mathbf{S} = A_{10} J_{1,\mathbf{E}} + A_{01} J_{2,\mathbf{E}} + K(J_3 - 1) J_{3,\mathbf{E}}$$

The S_{12} component of the stress becomes

$$\begin{aligned} S_{12} &= \frac{2}{3}[A_{10}(\alpha^3 + 3\alpha) + A_{01}(2\alpha^3 + 3\alpha)] \\ &= 80\alpha^3 + 200\alpha \end{aligned}$$

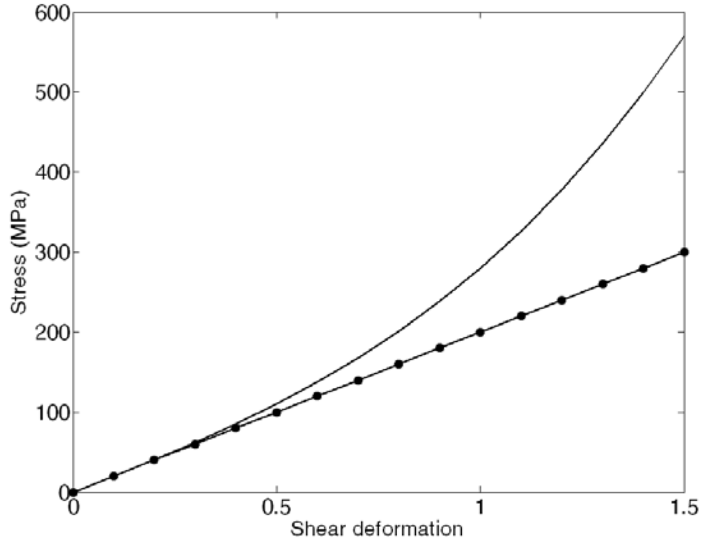
The Cauchy stress can be obtained from the relation

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \mathbf{S} \cdot \mathbf{F}^T$$

The σ_{12} component of the stress becomes

$$\sigma_{12} = 2(A_{10} + A_{01})\alpha = 200\alpha$$

Note that the shear stress S_{12} is a cubic function of α, but σ_{12} is a linear function.



■

P3.26 Derive the energy form and its linearization of a Mooney-Rivlin hyperelastic material using the perturbed Lagrangian method. Use a mixed variable $\mathbf{r} = [\mathbf{u}^{\mathrm{T}},\ p]^{\mathrm{T}}$.

Solution:
Using the distortional strain energy density in Eq. (4.116) and dilatational strain energy density in Eq. (4.123) for the Mooney-Rivlin material, the second Piola-Kirchhoff stress can be obtained by

$$W(J_1, J_2, J_3, p) = A_{10}(J_1 - 1) + A_{01}(J_2 - 3) + p(J_3 - 1) - \frac{1}{2K}p^2 \tag{1}$$

$$\mathbf{S} = W_{,\mathbf{E}} = A_{10}J_{1,\mathbf{E}} + A_{01}J_{2,\mathbf{E}} + pJ_{3,\mathbf{E}}. \tag{2}$$

Note that the independent pressure p is used. In order to derive the energy form, the first variation of the strain energy density can be written as

$$\overline{\mathrm{W}} = \mathrm{W}_{,\mathbf{E}} : \overline{\mathbf{E}} + W_{,p}\overline{p} = \mathbf{S} : \overline{\mathbf{E}} + \left(J_3 - 1 - \frac{p}{K}\right)\overline{p}. \tag{3}$$

Since both displacement and pressure are independent variables, a new combined variable is introduced as $\mathbf{r} = [\mathbf{u}^{\mathrm{T}},\ p]^{\mathrm{T}}$. Then, the energy form can be obtained by integrating Eq. (3) as

$$a(\mathbf{r}, \overline{\mathbf{r}}) \equiv \iint_{{}^0\Omega} \mathbf{S} : \overline{\mathbf{E}}\, \mathrm{d}\Omega + \iint_{{}^0\Omega} \overline{p}H\, \mathrm{d}\Omega \tag{4}$$

where $H = J_3 - 1 - p/K$ corresponds to the volumetric strain.

The energy form $a(\mathbf{r}, \overline{\mathbf{r}})$ is nonlinear through the constitutive relation and strain-displacement relation. Linearization of stress can be expressed in terms of displacement and pressure increments as

$$\Delta\mathbf{S} = W_{,\mathbf{E}\mathbf{E}} : \Delta\mathbf{E} + W_{,\mathbf{E}p}\Delta p = \mathbf{D} : \Delta\mathbf{E} + J_{3,\mathbf{E}}\Delta p \tag{4}$$

where $\mathbf{D}$ is the fourth-order constitutive tensor defined as

$$\mathbf{D} = \frac{\partial\mathbf{S}}{\partial\mathbf{E}} = A_{10}J_{1,\mathbf{EE}} + A_{01}J_{2,\mathbf{EE}} + pJ_{3,\mathbf{EE}} \tag{5}$$

and $\Delta\mathbf{E}$ and Δp are the incremental strain and pressure. Linearization of energy form can be obtained as

$$\begin{aligned} a^*(\mathbf{r};\Delta\mathbf{r},\overline{\mathbf{r}}) \equiv & \iint_{^0\Omega}\left[\overline{\mathbf{E}} : (\mathbf{D} : \Delta\mathbf{E} + J_{3,\mathbf{E}}\Delta p) + \mathbf{S} : \Delta\overline{\mathbf{E}}\right] \mathrm{d}\Omega \\ & + \iint_{^0\Omega} \overline{p}\left(J_{3,\mathbf{E}} : \Delta\mathbf{E} - \frac{\Delta p}{K}\right) \mathrm{d}\Omega \end{aligned} \tag{6}$$

The pressure term can be condensed on the finite element level by directly solving the terms that contain the pressure variation. This can easily be done if constant pressure approximation is used, which can be done within a finite element. ■

P3.27 Derive the 6×6 $[\mathbf{D}]$ matrix in Eq. (3.147) for two-dimensional Mooney-Rivlin material with three material parameters (A_{10}, A_{01}, and K). Use the penalty method for near-incompressibility

Solution:
The expression for the constitutive tensor is given in Eq. (3.128). In order to make a matrix expression, it is necessary to define the second-order derivatives of the three invariants in the matrix notation as

$$[I_{1,\mathbf{EE}}] = [\mathbf{0}]_{6\times 6}$$

$$[I_{2,\mathbf{EE}}] = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$[I_{3,\mathbf{EE}}]=\begin{bmatrix} & 4C_3 & 4C_2 & & -4C_5 & \\ 4C_3 & & 4C_1 & & & -4C_6 \\ 4C_2 & 4C_1 & & -4C_4 & & \\ & & -4C_4 & -2C_3 & 2C_6 & 2C_5 \\ -4C_5 & & & 2C_6 & -2C_1 & 2C_4 \\ & -4C_6 & & 2C_5 & 2C_4 & -2C_2 \end{bmatrix}$$

From Eq. (3.129), the second-order derivatives of the three reduced invariants can be written in the matrix notation as

$$[J_{1,\mathbf{EE}}]=-\frac{1}{3}I_3^{-\frac{4}{3}}(\{I_{1,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}}+\{I_{3,\mathbf{E}}\}\{I_{1,\mathbf{E}}\}^{\mathrm{T}})+\frac{4}{9}I_1I_3^{-\frac{7}{3}}\{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}}-\frac{1}{3}I_1I_3^{-\frac{4}{3}}[I_{3,\mathbf{EE}}]$$
$$[J_{2,\mathbf{EE}}]=I_3^{-\frac{2}{3}}[I_{2,\mathbf{EE}}]-\frac{2}{3}I_3^{-\frac{5}{3}}(\{I_{2,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}}+\{I_{3,\mathbf{E}}\}\{I_{2,\mathbf{E}}\}^{\mathrm{T}})+\frac{10}{9}I_2I_3^{-\frac{8}{3}}\{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}}-\frac{2}{3}I_2I_3^{-\frac{5}{3}}[I_{3,\mathbf{EE}}]$$
$$[J_{3,\mathbf{EE}}]=-\frac{1}{4}I_3^{-\frac{3}{2}}\{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}}+\frac{1}{2}I_3^{-\frac{1}{2}}[I_{3,\mathbf{EE}}]$$

Note that the expressions of $\{I_{1,\mathbf{E}}\}$, $\{I_{2,\mathbf{E}}\}$, and $\{I_{3,\mathbf{E}}\}$ are available in Section 3.5.2. From Eq. (3.128), the constitutive matrix can be obtained as

$$[\mathbf{D}]=A_{10}[J_{1,\mathbf{EE}}]+A_{01}[J_{2,\mathbf{EE}}]+K(J_3-1)[J_{3,\mathbf{EE}}]+K\{J_{3,\mathbf{E}}\}\{J_{3,\mathbf{E}}\}^{\mathrm{T}}$$

■

P3.28 Derive the 6×6 $[\mathbf{D}]$ matrix in Eq. (3.147) for two-dimensional Mooney-Rivlin material with three material parameters (A_{10}, A_{01}, and K). Use the perturbed Lagrangian method for near-incompressibility

Solution:
For the perturbed Lagrangian formulation, the expression of stress is given in Example 3.15, and the constitutive tensor can be written as

$$\mathbf{D}=\frac{\partial\mathbf{S}}{\partial\mathbf{E}}=A_{10}J_{1,\mathbf{EE}}+A_{01}J_{2,\mathbf{EE}}+pJ_{3,\mathbf{EE}}$$

In order to make a matrix expression, it is necessary to define the second-order derivatives of the three invariants in the matrix notation as

$$[I_{1,\mathbf{EE}}]=[\mathbf{0}]_{6\times6}$$

$$[I_{2,\mathbf{EE}}] = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 \\ 4 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{bmatrix}$$

$$[I_{3,\mathbf{EE}}] = \begin{bmatrix} & 4C_3 & 4C_2 & & -4C_5 & \\ 4C_3 & & 4C_1 & & & -4C_6 \\ 4C_2 & 4C_1 & & -4C_4 & & \\ & & -4C_4 & -2C_3 & 2C_6 & 2C_5 \\ -4C_5 & & & 2C_6 & -2C_1 & 2C_4 \\ & -4C_6 & & 2C_5 & 2C_4 & -2C_2 \end{bmatrix}$$

From Eq. (3.129), the second-order derivatives of the three reduced invariants can be written in the matrix notation as

$$[J_{1,\mathbf{EE}}] = -\frac{1}{3} I_3^{-\frac{4}{3}} (\{I_{1,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}} + \{I_{3,\mathbf{E}}\}\{I_{1,\mathbf{E}}\}^{\mathrm{T}}) + \frac{4}{9} I_1 I_3^{-\frac{7}{3}} \{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}} - \frac{1}{3} I_1 I_3^{-\frac{4}{3}} [I_{3,\mathbf{EE}}]$$

$$[J_{2,\mathbf{EE}}] = I_3^{-\frac{2}{3}} [I_{2,\mathbf{EE}}] - \frac{2}{3} I_3^{-\frac{5}{3}} (\{I_{2,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}} + \{I_{3,\mathbf{E}}\}\{I_{2,\mathbf{E}}\}^{\mathrm{T}}) + \frac{10}{9} I_2 I_3^{-\frac{8}{3}} \{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}} - \frac{2}{3} I_2 I_3^{-\frac{5}{3}} [I_{3,\mathbf{EE}}]$$

$$[J_{3,\mathbf{EE}}] = -\frac{1}{4} I_3^{-\frac{3}{2}} \{I_{3,\mathbf{E}}\}\{I_{3,\mathbf{E}}\}^{\mathrm{T}} + \frac{1}{2} I_3^{-\frac{1}{2}} [I_{3,\mathbf{EE}}]$$

Note that the expressions of $\{I_{1,\mathbf{E}}\}$, $\{I_{2,\mathbf{E}}\}$, and $\{I_{3,\mathbf{E}}\}$ are available in Section 3.5.2. Thus, the constitutive matrix can be obtained as

$$[\mathbf{D}] = A_{10}[J_{1,\mathbf{EE}}] + A_{01}[J_{2,\mathbf{EE}}] + p[J_{3,\mathbf{EE}}]$$

■

P3.29 A nearly incompressible rubber block is confined between two frictionless rigid walls as shown in the figure. When uniform pressure *P* is applied on the right end, the length of the block is changed by $x_1 = (1 - \alpha)X_1$. When $\alpha = 0.1$, (a) calculate the value of strain energy density and (b) the magnitude of applied pressure *P* on the right end. Assume plane strain problem and use Mooney-Rivlin material with $A_{10} = 80\text{MPa}$, $A_{01} = 20\text{MPa}$, and $K = 1{,}000\text{MPa}$.

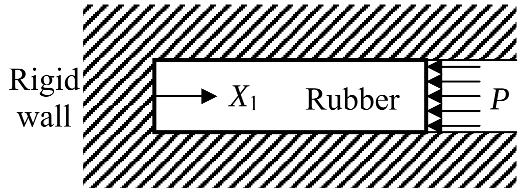


Figure P3.29

Solution:

Since both X_2 and X_3 directions are fixed, the deformation of the rubber block can be written as

$$x_1 = 0.9X_1, \quad x_2 = X_2, \quad x_3 = X_3$$

from which the deformation gradient and the right Cauchy-Green deformation tensor can be calculated as

$$\mathbf{F} = \begin{bmatrix} .9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} .81 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The three eigenvalues of the right Cauchy-Green deformation tensor can be obtained as

$$\lambda_1^2 = \lambda_2^2 = 1, \quad \lambda_3^2 = 0.81$$

Using the three eigenvalues, the three invariants can be obtained as

$$I_1 = 2.81, \quad I_2 = 2.62, \quad I_3 = 0.81$$

In addition, the reduced invariants are

$$J_1 = 3.0145, \quad J_2 = 3.0151, \quad J_3 = 0.9$$

(a) Strain energy density

$$\begin{aligned} W &= A_{10}(J_1 - 3) + A_{01}(J_2 - 3) + \frac{K}{2}(J_3 - 1)^2 \\ &= 80 \times 0.0145 + 20 \times 0.0151 + 500 \times (-0.1)^2 \\ &= 6.462 \end{aligned}$$

(b) Hydrostatic pressure

$$p = K(J_3 - 1) = -100\text{MPa}$$

■

P3.30 Consider a unit cube shown in Figure 3.15. Using an eight-node solid element, perform biaxial extension analysis using ABAQUS. Apply uniform extensions in both X_1

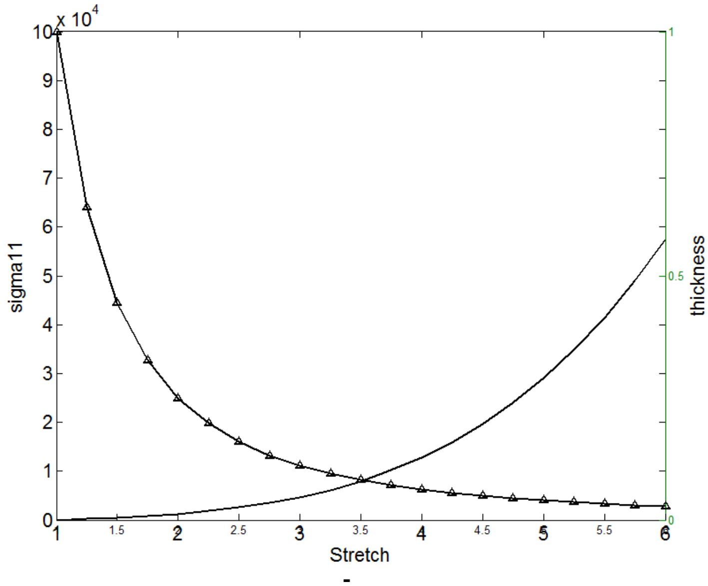
and X_2 directions so that deformed shape will be $5 \times 5 \times t_3$. Plot stress σ_{11} and thickness t_3 as a function of stretch.

Solution:

The following program list shows the ABAQUS input file for the biaxial loading:

```
*HEADING
  - INCOMPRESSIBLE HYPERELASTICITY
(MOONEY-RIVLIN), BIAXIAL TENSION
*NODE,NSET=ALL
1,
2,1.
3,1.,1.,
4,0.,1.,
5,0.,0.,1.
6,1.,0.,1.
7,1.,1.,1.
8,0.,1.,1.
*NSET,NSET=FACE1
1,2,3,4
*NSET,NSET=FACE2
5,6,7,8
*NSET,NSET=FACE3
1,2,5,6
*NSET,NSET=FACE4
2,3,6,7
*NSET,NSET=FACE5
3,4,7,8
*NSET,NSET=FACE6
4,1,8,5
*ELEMENT,TYPE=C3D8RH,ELSET=ONE
1,1,2,3,4,5,6,7,8
*SOLID SECTION,ELSET=ONE,MATERIAL=MNEY
*MATERIAL,NAME=MNEY
*HYPERELASTIC, MOONEY-RIVLIN
80., 20.,
*STEP,NLGEOM,INC=20
BIAXIAL TENSION
*STATIC,DIRECT
1.,20.
*BOUNDARY,OP=NEW
FACE1,3
FACE3,2
FACE6,1
FACE4,1,1,5.
FACE5,2,2,5.
*EL PRINT,F=1
S,
E,
*NODE PRINT,F=1
U,RF
*OUTPUT,FIELD,FREQ=1
*ELEMENT OUTPUT
S,E
*OUTPUT,FIELD,FREQ=1
*NODE OUTPUT
U,RF
*END STEP
```

The analysis results are shown in the following figure:


x 10^4
sigma11
thickness
Stretch
10
9
8
7
6
5
4
3
2
1
0
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
0.5

[redacted]