4.10. Exercise

P4.1 A force is gradually applied at the end of an elastoplastic bar such that it is in the plastic phase. When the total magnitude of strain is $\varepsilon = 0.003$, calculated the applied force, axial stress, elastic strain, and plastic strain. Use the following material properties: $E = 100\text{GPa}$, $H = 10\text{GPa}$, and $\sigma_Y = 100\text{MPa}$. The cross-sectional area of the bar is $A = 1.0\times10^{-4}\text{m}^2$.

Solution:

If the total strain is pure elastic, then stress will become $\sigma = E\varepsilon = 300\text{MPa}$, which is larger than the yield stress. Thus, the material is in the plastic phase. It is convenient to separate the initial elastic deformation, $\varepsilon^{(1)}$, until the yield stress from the elastoplastic deformation, $\varepsilon^{(2)}$ such that $\varepsilon = \varepsilon^{(1)} + \varepsilon^{(2)}$. The initial elastic strain can be calculated from

$$\varepsilon^{(1)} = \varepsilon_e^{(1)} = \varepsilon_Y = \frac{\sigma_Y}{E} = 0.001, \quad \sigma^{(1)} = \sigma_Y = 100\text{MPa}$$

After the initial elastic deformation, the remaining deformation is elastoplastic deformation. In this phase, the strain increment is $\Delta\varepsilon^{(2)} = 0.002$. Since the total strain increment is given, Eq. (4.9) can be used to calculate the plastic strain increment as

$$\Delta\varepsilon_p^{(2)} = \frac{\Delta\varepsilon^{(2)}}{1 + H / E} = 0.00182$$

Thus, the plastic strain is $\varepsilon_p = \Delta\varepsilon_p^{(2)} = 0.00182$, and elastic strain is $\varepsilon_e = \varepsilon - \varepsilon_p = 0.00118$. The axial stress can be calculated using the elastic strain, $\sigma = E\varepsilon_e = 118\text{MPa}$. The applied force can be calculated from the assumption that the axial stress is uniformly distributed over the cross section: $F = \sigma A = 11.8\text{kN}$. Below is the MATLAB program that solves for the problem:

```
%
% P4.1 elastoplastic bar (MPa, mm)
%
delE=0.003; A=100;
mp = [1E5, 0, 1E4, 100];
nS=0; nA=0; nep=0;
[Snew, Anew, epnew]=combHard1D(mp,delE,nS,nA,nep)
eelast=delE - epnew
Force = Snew*A
```

■

P4.2 A force 12kN is gradually applied and then removed at the end of an elastoplastic bar. When the yield stress of the material is 100MPa, calculate plastic strains and tip displacement after removing the applied force. Use the following material properties: $E = 100\text{GPa}$ and $H = 10\text{GPa}$. The cross-sectional area of the bar is $A = 1.0\times10^{-4}\text{m}^2$ and the length of the bar is $L_0 = 1\text{m}$.

Solution:

In a one-dimensional bar, it is assumed that the force is uniformly distributed over the cross section. During the loading process, since the total stress, $\sigma = F/A = 120\text{MPa}$, is larger than the yield stress, it can be concluded that the material is under plastic deformation. It is convenient to divide the entire deformation into elastic and plastic phases. The material is initially elastic until it reaches yield stress. Thus, when stress reaches yield stress $\sigma^{(1)} = \sigma_Y = 100\text{MPa}$, strain is purely elastic:

$$\Delta\varepsilon_e^{(1)} = \frac{\sigma_Y}{E} = 0.001$$

After yielding, the remaining stress increment, $\Delta\sigma = 20\text{MPa}$, is in the plastic phase. The elastic and plastic strain increments can be calculated from

$$\Delta\varepsilon_e^{(2)} = \frac{\Delta\sigma^{(2)}}{E} = 0.0002$$

$$\Delta\varepsilon_p^{(2)} = \frac{\Delta\sigma^{(2)}}{H} = 0.002$$

Thus, the total elastic and plastic strains become

$$\begin{aligned} \varepsilon_e &= \Delta\varepsilon_e^{(1)} + \Delta\varepsilon_e^{(2)} = 0.0012 \\ \varepsilon_p &= \Delta\varepsilon_p^{(2)} = 0.002 \end{aligned}$$

Now, during the unloading, elastic strain is removed, but the plastic strain remains. Thus, $\varepsilon_p = 0.002$ and $u = \varepsilon_p L = 2\text{mm}$. ■

P4.3 A uniaxial bar is under tensile force $F = 12\text{kN}$ at load step t_n. (a) When the plastic strain is $\varepsilon_p^n = 0.002$, determine the yield status of the material. (b) If the applied force is increased to $F = 15\text{kN}$ at load step t_{n+1}, calculate plastic strain and tip displacement. Assume the initial yield stress $\sigma_Y = 100\text{MPa}$, $E = 100\text{GPa}$ and $H = 10\text{GPa}$. The cross-sectional area of the bar is $A = 1.0\times10^{-4}\text{m}^2$ and the length of the bar is $L_0 = 1\text{m}$. Assume isotropic hardening model.

Solution:

(a) At load step t_n, the stress in the bar is

$$\sigma^n = \frac{F}{A} = 120\text{MPa}$$

The yield stress is

$$\sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n = 100 + 10000\times0.002 = 120\text{MPa}$$

Since $\sigma^n = \sigma_Y^n$ the material is in the loading stage.

(b) During the loading stage up to t_n, the elastic and plastic strains can be calculated by

$$\varepsilon_e^n = \frac{\sigma^n}{E} = 0.0012, \quad \varepsilon_p^n = 0.002$$

At load step t_{n+1}, the elastic and plastic strain increments can be calculated from stress increment $\Delta\sigma^{n+1} = 30\text{MPa}$ as

$$\Delta\varepsilon_e^{n+1} = \frac{\Delta\sigma^{n+1}}{E} = 0.0004, \quad \Delta\varepsilon_p^{n+1} = \frac{\Delta\sigma^{n+1}}{H} = 0.004$$

Thus, the total elastic and plastic strains become

$$\varepsilon_e^{n+1} = \varepsilon_e^n + \Delta\varepsilon_e^{n+1} = 0.0016, \quad \varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p^{n+1} = 0.006$$

The tip displacement is

$$u = (\varepsilon_e^{n+1} + \varepsilon_p^{n+1})L = 7.6\text{mm}$$

■

P4.4 An elastoplastic bar is under variable load history. At load step t_n, the stress and plastic strain are $\sigma^n = 200\text{MPa}$ and $\varepsilon_p^n = 1.0 \times 10^{-4}$, respectively. (a) Is the material in elastic or plastic state? (b) When strain increment is $\Delta\varepsilon = -0.003$, calculate stress and plastic strain. Assume isotropic hardening with $E = 200\text{GPa}$, $H = 25\text{GPa}$, and $\sigma_Y = 250\text{MPa}$.

Solution:
(a) At a given plastic strain $\varepsilon_p^n = 1.0 \times 10^{-4}$, the yield stress is

$$\sigma_Y = \sigma_Y^0 + H\varepsilon_p^n = 252.5\text{MPa}$$

Since $\sigma^n < \sigma_Y$, the material is in the elastic state.
(b) For given strain increment, the trial stress can be obtained as

$$\Delta\sigma = E\Delta\varepsilon = -600\text{MPa}, \quad \sigma^{tr} = \sigma^n + \Delta\sigma = -400\text{MPa}$$

Since $f^{tr} =| \sigma^{tr} | -\sigma_Y = 147.5 > 0$, the material is yielded in the compression side. From Eq. (4.26), the plastic strain increment becomes

$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E + H} = 6.5556 \times 10^{-4}$$

Therefore, the stress and plastic strain are updated as

$$\sigma^{n+1} = {}^{tr}\sigma - \text{sgn}({}^{tr}\sigma)E\Delta\varepsilon_p = -268.9\text{MPa}$$

$$\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p = 7.5556 \times 10^{-4}$$

Below is the MATLAB program that solves for the problem:

```
%
% P4.4 Elastoplastic bar (isotropic hardening)
%
delE=-0.003; nS=200; nA=0; nep=1E-4;
mp=[2E5, 0, 2.5E4, 250];
[Snew, Anew, epnew]=combHard1D(mp,delE,nS,nA,nep)
```

■

P4.5 Repeat Problem P4.4 using the kinematic hardening model. For back stress, use $\alpha^n = 2.5$MPa.

Solution:
(a) At a given stress $\sigma^n = 200\text{MPa}$ and back stress $\alpha^n = 2.5\text{MPa}$, the trial shifted stress is

$$\eta^n = \sigma^n - \alpha^n = 197.5\text{MPa}$$

Since $\eta^n < \sigma_Y$, the material is in the elastic state.
(b) For given strain increment, the trial stress can be obtained as

$$\Delta\sigma = E\Delta\varepsilon = -600\text{MPa}, \quad \sigma^{tr} = \sigma^n + \Delta\sigma = -400\text{MPa}$$

Since $f^{tr} = | \sigma^{tr} - \alpha^n | - \sigma_Y = 152.5 > 0$, the material is yielded in the compression side. From Eq. (4.41), the plastic strain increment becomes

$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E + H} = 6.7778 \times 10^{-4}$$

Therefore, the stress, back stress and plastic strain are updated as

$$\sigma^{n+1} = {}^{tr}\sigma - \text{sgn}({}^{tr}\eta)E\Delta\varepsilon_p = -264.4\text{MPa}$$

$$\alpha^{n+1} = {}^{n}\alpha - \text{sgn}({}^{tr}\eta)H\Delta\varepsilon_p = -14.444\text{MPa}$$

$$\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p = 7.7778 \times 10^{-4}$$

Below is the MATLAB program that solves for the problem:

```
%
% P4.5 Elastoplastic bar (kinematic hardening
%
delE=-0.003; nS=200; nA=0; nep=1E-4;
mp=[2E5, 1, 2.5E4, 250];
[Snew, Anew, epnew]=combHard1D(mp,delE,nS,nA,nep)
```

■

P4.6 Repeat Problem 4.5 using the combined hardening model with $\beta = 0.5$.

Solution:

(a) At a given stress $\sigma^n = 200\text{MPa}$, back stress $\alpha^n = 2.5\text{MPa}$, and plastic strain $\varepsilon_p^n = 1.0\times10^{-4}$, the trial shifted stress and the yield stress are

$$\eta^n = \sigma^n - \alpha^n = 197.5\text{MPa}$$

$$\sigma_Y = \sigma_Y^0 + (1-\beta)H\varepsilon_p^n = 251.25\text{MPa}$$

Since $\eta^n < \sigma_Y$, the material is in the elastic state.

(b) For given strain increment, the trial stress can be obtained as

$$\Delta\sigma = E\Delta\varepsilon = -600\text{MPa}, \quad \sigma^{tr} = \sigma^n + \Delta\sigma = -400\text{MPa}$$

Since $f^{tr} = \mid \sigma^{tr} - \alpha^n \mid - \sigma_Y = 151.25 > 0$, the material is yielded in the compression side. From Eq. (4.41), the plastic strain increment becomes

$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E+H} = 6.2222\times10^{-4}$$

Therefore, the stress, back stress and plastic strain are updated as

$$\sigma^{n+1} = {}^{tr}\sigma - \text{sgn}({}^{tr}\eta)E\Delta\varepsilon_p = -265.6\text{MPa}$$

$$\alpha^{n+1} = {}^{n}\alpha - \text{sgn}({}^{tr}\eta)\beta H\Delta\varepsilon_p = -5.9028\text{MPa}$$

$$\varepsilon_p^{n+1} = \varepsilon_p^n + \Delta\varepsilon_p = 7.2222\times10^{-4}$$

Below is the MATLAB program that solves for the problem:

```
%
% P4.6 Elastoplastic bar (combined hardening)
%
delE=-0.003; nS=200; nA=0; nep=1E-4;
mp=[2E5, 0.5, 2.5E4, 250];
[Snew, Anew, epnew]=combHard1D(mp,delE,nS,nA,nep)
```

■

P4.7 For the combined isotropic/kinematic hardening model, derive the expression of plastic strain increment from the plastic consistency condition.

Solution:

The plastic consistency condition means that the yield function at the current time step, t_{n+1}, remains zero; that is,

$${}^{n+1}f = \left|{}^{n+1}\eta\right| - {}^{n+1}\sigma_Y = 0$$

The following stress update formulas and yield stress are used for the plastic consistency condition:

$${}^{n+1}\sigma = {}^{tr}\sigma - \text{sgn}({}^{tr}\eta)E\Delta\varepsilon_p$$

$$^{n+1}\alpha = {}^{n}\alpha + \mathrm{sgn}({}^{tr}\eta)\beta H\Delta\varepsilon_p$$

$$^{n+1}\sigma_Y = {}^{n}\sigma_Y + (1-\beta)H\Delta\varepsilon_p$$

By substituting the above 3 formulas into the plastic consistency condition,
The above consistency condition can be expanded in terms of plastic strain increment as

$$\begin{aligned} &\left|{}^{tr}\sigma - \mathrm{sgn}({}^{tr}\eta)E\Delta\varepsilon_p - {}^{tr}\alpha - \mathrm{sgn}({}^{tr}\eta)\beta H\Delta\varepsilon_p\right| - ({}^{n}\sigma_Y + (1-\beta)H\Delta\varepsilon_p) = 0 \\ \Rightarrow\ &\left|{}^{tr}\sigma - {}^{tr}\alpha\right| - {}^{n}\sigma_Y - (E+H)\Delta\varepsilon_p = 0 \end{aligned}$$

Note that the above formula works for both positive and negative trial stress. Therefore, the plastic strain increment can be obtained as

$$\Delta\varepsilon_p = \frac{{}^{tr}f}{E+H}$$

Note that the formula for plastic strain increment is identical to that of isotropic hardening in Eq. (4.26) and kinematic hardening in Eq. (4.41). ■

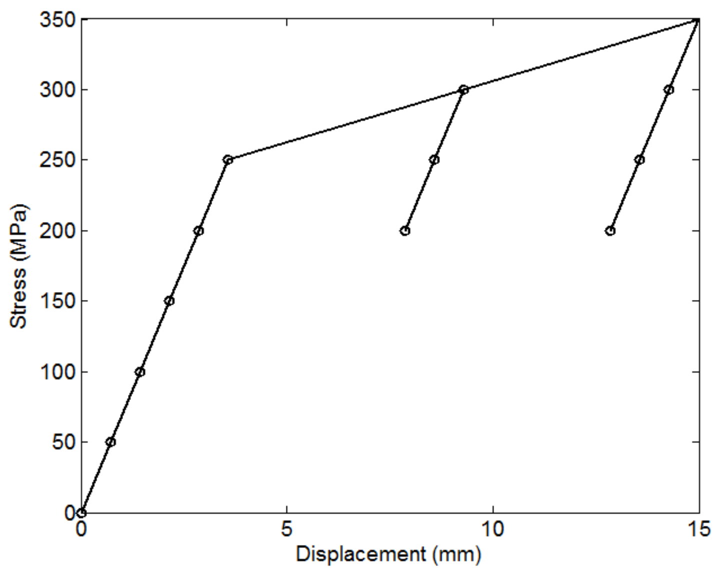
P4.8 An elastoplastic bar is clamped at the left end, and variable loads are applied at the right end, as shown in the table. Plot the stress-strain curve by changing the applied forces by $5\mathrm{kN}$ increments. Assume the following material properties with isotropic hardening: $E = 70\mathrm{GPa}$, $H = 10\mathrm{GPa}$, $\sigma_Y = 250\mathrm{MPa}$. The length of the bar is $L = 1\mathrm{m}$, and the cross-sectional area is $A = 1.0\times10^{-4}\mathrm{m}^2$.

Load step	1	2	3	4
Force (kN)	30	20	35	20

Solution:
Since the given condition is applied force at the tip, the convergence iteration must be performed to find equilibrium. Below is the MATLAB program that solves for the problem:

```
%
% P4.8 Variable loadings
%
E=70000; H=10000;Et=1000; sYield=250;
mp = [E, 0, H, sYield];
Et=E*H/(E+H);
nS=0; nep=0; nA=0;
A = 100; L = 1000;
tol = 1.0E-5; u = 0; iter=0; Res=0;
Force = 1000*[5:5:30 25 20 25 30 35 30 25 20];
N = size(Force',1);
X=zeros(N,1);Y=zeros(N,1);
fprintf('\nstep  iter        u       S       ep   Residual');
fprintf('\n %3d %3d  %7.4f %7.3f %8.6f %10.3e',i,iter,u,nS,nep,Res);
for i=1:N
 P = Force(i); iter = 0;
 Res = P - nS*A;
 conv = Res^2/(1+P^2);
```

```
 du=0;
 while conv > tol && iter < 20
   Eep = E; if epnew>nep; Eep = Et; end
   delu = Res / (Eep*A/L);
   du = du + delu;
   delE = du / L;
   [Snew, Anew, epnew]=combHard1D(mp,delE,nS,nA,nep);
   Res = P - Snew*A;
   conv = Res^2/(1+P^2);
   iter = iter + 1;
 end
 u=u+du;
 nS = Snew; nep = epnew;
 X(i) = u; Y(i) = nS;
 fprintf('\n %3d %3d  %7.4f %7.3f %8.6f %10.3e',i,iter,u,nS,nep,Res);
end
X=[0;X];Y=[0;Y];plot(X,Y);
```



■

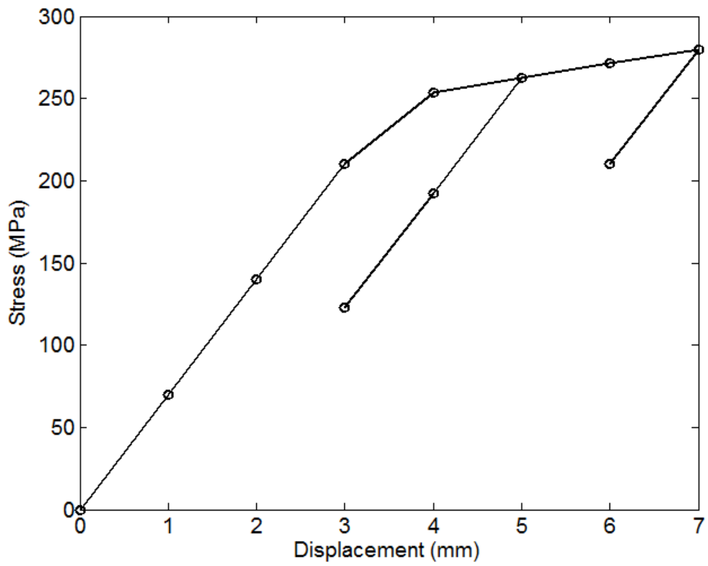
P4.9 An elastoplastic bar is clamped at the left end, and variable displacements are applied at the right end, as shown in the table. Plot the stress-strain curve by changing the tip displacement by 1mm increments. Assume the following material properties with isotropic hardening: $E = 70\text{GPa}$, $H = 10\text{GPa}$, $\sigma_Y = 250\text{MPa}$. The length of the bar is $L = 1\text{m}$, and the cross-sectional area is $A = 1.0{\times}10^{-4}\text{m}^2$.

Load step	1	2	3	4
Displacement (mm)	5.0	3.0	7.0	6.0

Solution:

Since the tip displacements are given, there is no need to perform convergence iteration. At each step, displacement increment is calculated by the difference between current and previous load increments. After that, the strain increment is calculated from the small deformation assumption. The program combHard1D will calculate stress and plastic strain for the given strain increment. Below is the MATLAB program that solves for the problem:

```
%
% P4.9 Variable displacement
%
E=70000; H=10000;Et=1000; sYield=250;
mp = [E, 0, H, sYield];
Et=E*H/(E+H);
nS=0; nep=0; nA=0;
A=100; L=1000;
tol = 1.0E-5; u=0; iter=0; Res=0;
disp=[0 1 2 3 4 5 4 3 4 5 6 7 6];
N = size(disp',1);
X=zeros(N,1);Y=zeros(N,1);
fprintf('\nstep         u       S       ep');
fprintf('\n %3d %7.4f %7.3f %8.6f',i,u,nS,nep);
for i=2:N
 delu = disp(i) - disp(i-1);
 delE = delu / L;
 [Snew, Anew, epnew]=combHard1D(mp, delE, nS, nA, nep);
 nS = Snew; nep = epnew;
 X(i) = disp(i); Y(i) = nS;
 fprintf('\n %3d %7.4f %7.3f %8.6f',i,u,nS,nep);
end
plot(X,Y);
```



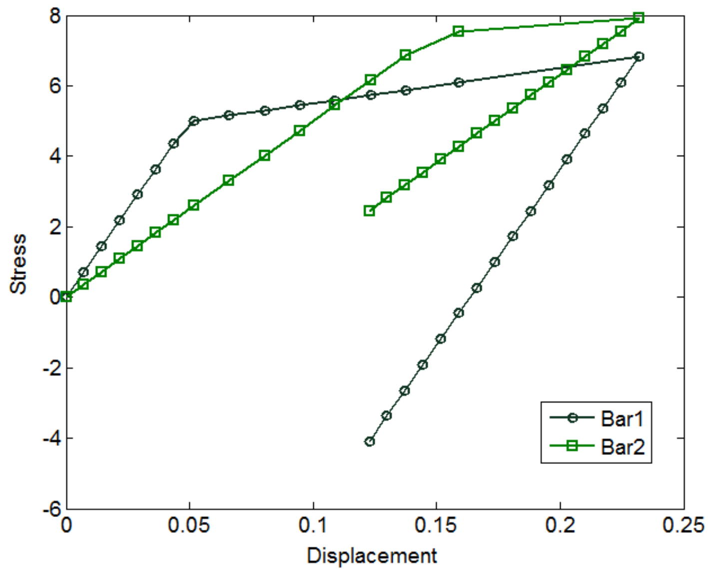
P4.10 A force of $P = 15$ is applied to the two parallel bars in Example 4.2 and then removed. Using **`combHard1D`** program, calculate tip displacement and residual stresses for the two bars after unloading. Use 15 load increments for each loading and unloading cycle. Plot stresses vs. tips displacement in the XY graph.

Solution:

Below is the MATLAB program to solve loading/unloading cycle for two parallel bars in Example 4.2. After unloading, the tip displacement is $u = 0.1227$, and stresses are $\sigma_1 =$

-4.091, $\sigma_2 = 2.455$. Since the area of bar2 is twice than that of bar1, the stress σ_2 is half of σ_1.

```
%
% P4.10 Two-bar in parallel - unloading
%
E1=10000; Et1=1000; sYield1=5;
E2=5000;  Et2=500;  sYield2=7.5;
mp1 = [E1, 0, E1*Et1/(E1-Et1), sYield1];
mp2 = [E2, 1, E2*Et2/(E2-Et2), sYield2];
nS1 = 0; nA1=0; nep1 = 0; epnew1=0;
nS2 = 0; nA2=0; nep2 = 0; epnew2=0;
A1 = 0.75; L1 = 100;
A2 = 1.25; L2 = 100;
tol = 1.0E-5; u = 0;
Force = [1:15 14:-1:0];
N = size(Force',1);
X=zeros(N,1);Y1=zeros(N,1);Y2=zeros(N,1);
fprintf('\nstep iter        u      S1      S2      ep1      ep2   Residual');
fprintf('\n %3d %3d  %7.4f %7.3f %7.3f %8.6f %8.6f %10.3e',...
      0,0,u,nS1,nS2,nep1,nep2,0);
for i=1:N
 P = Force(i); iter = 0;
 Res = P - nS1*A1 - nS2*A2;
 conv = Res^2/(1+P^2);
 while conv > tol && iter < 20
   Eep1 = E1; if epnew1>nep1; Eep1 = Et1; end
   Eep2 = E2; if epnew2>nep2; Eep2 = Et2; end
   delu = Res / (Eep1*A1/L1 + Eep2*A2/L2);
   u = u + delu;
   delE = delu / L1;
   [Snew1, Anew1, epnew1]=combHard1D(mp1, delE, nS1, nA1, nep1);
   [Snew2, Anew2, epnew2]=combHard1D(mp2, delE, nS2, nA2, nep2);
   Res = P - Snew1*A1 - Snew2*A2;
   conv = Res^2/(1+P^2);
   iter = iter + 1;
   nS1 = Snew1; nep1 = epnew1; nA1 = Anew1;
   nS2 = Snew2; nep2 = epnew2; nA2 = Anew2;
 end
 X(i) = u; Y1(i) = nS1; Y2(i) = nS2;
 fprintf('\n %3d %3d  %7.4f %7.3f %7.3f %8.6f %8.6f %10.3e',...
      i,iter,u,nS1,nS2,nep1,nep2,Res);
end
X=[0;X];Y1=[0;Y1];Y2=[0;Y2];plot(X,Y1,X,Y2);
```



P4.11 A force 12kN is gradually applied at the end of an elastoplastic bar. When the yield stress of the material is 100MPa, calculate displacement at the tip. Use the following material properties: $E = 100\text{GPa}$ and $H = 10\text{GPa}$. The cross-sectional areas of the bars are $A^{(1)} = 1.0{\times}10^{-4}\text{m}^2$ and $A^{(2)} = 0.5{\times}10^{-4}\text{m}^2$.

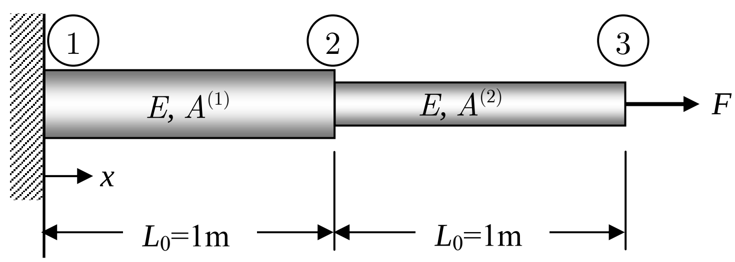


Figure P4.11

Solution:

Since the two bars are connected in parallel, the element forces of the two bars are the same as the applied force at the tip. By considering the cross-sectional areas of the bars, the stresses of the bars can be calculated by

$$\sigma^{(1)} = \frac{F}{A^{(1)}} = 120\text{MPa}, \quad \sigma^{(2)} = \frac{F}{A^{(2)}} = 240\text{MPa}$$

Since both bars are in the plastic state, it is possible to separate the entire deformation into the initial yielding stage (at $\sigma = 100\text{MPa}$) and followed by the elastoplastic state. For bar1, the elastic strain at the initial yielding stage can be calculated from

$$\varepsilon_e^{(1)} = \frac{\sigma_Y}{E} = 0.001$$

After that, the remaining $\Delta\sigma = 20\text{MPa}$ is in the elastoplastic stage:

$$\varepsilon_e^{(1)} = \frac{\Delta\sigma^{(1)}}{E} = 0.0002, \quad \varepsilon_p^{(1)} = \frac{\Delta\sigma^{(1)}}{H} = 0.002, \quad \varepsilon^{(1)} = \varepsilon_e^{(1)} + \varepsilon_p^{(1)} = 0.0022$$

Thus the total strain becomes $\varepsilon^{(1)} = 0.001 + 0.0002 + 0.002 = 0.0032$, and the displacement at Node 2 becomes

$$u_2 = \varepsilon^{(1)} L_0 = 0.32\text{mm}$$

For bar2, the strain at the initial yielding stage can be calculated from

$$\varepsilon_e^{(2)} = \frac{\sigma_Y}{E} = 0.001$$

After that, the remaining $\Delta\sigma = 140\text{MPa}$ is in the elastoplastic stage:

$$\varepsilon_e^{(2)} = \frac{\Delta\sigma^{(2)}}{E} = 0.0014, \quad \varepsilon_p^{(2)} = \frac{\Delta\sigma^{(2)}}{H} = 0.014, \quad \varepsilon^{(2)} = \varepsilon_e^{(2)} + \varepsilon_p^{(2)} = 0.0154$$

the total strain becomes $\varepsilon^{(1)} = 0.001 + 0.0014 + 0.014 = 0.0164$, and the displacement at Node 3 becomes

$$u_3 = u_2 + \varepsilon^{(2)} L_0 = 1.96\text{mm}$$

The same problem can be solved using combHard1D program with convergence iteration. The following MATLAB program solves for the displacement and stress for the two-bar problem:

```
%
% P4.11 Two-bar in serial
%
E=100000; H=10000; sYield=100;Et=E*H/(E+H);
mp = [E, 0, H, sYield];
nS1 = 0; nA1=0; nep1=0; epnew1=0; Eep1=E;
nS2 = 0; nA2=0; nep2=0; epnew2=0; Eep2=E;
A1 = 100; L1 = 100; A2 = 50; L2 = 100;
tol = 1.0E-8; iter = 0; u = [0 0]'; F = [0 12000]';
Res = F - [nS1*A1-nS2*A2;nS2*A2];
conv = norm(Res)^2/(1+norm(F)^2);
fprintf('\n iter      u1      u2      S1      S2      ep1      ep2
Residual');
fprintf('\n %3d  %7.4f %7.4f %7.3f %7.3f %8.6f %8.6f %10.2e %10.2e',...
      iter,u(1),u(2),nS1,nS2,nep1,nep2,Res);
while conv > tol && iter < 20
  Kt = [Eep1*A1/L1+Eep2*A2/L2,-Eep2*A2/L2;-Eep2*A2/L2,Eep2*A2/L2];
  delu = Kt\Res;
  u = u + delu;
  delE1 = delu(1) / L1;
  delE2 = (delu(2)-delu(1)) / L2;
  [Snew1, Anew1, epnew1]=combHard1D(mp, delE1, nS1, nA1, nep1);
  [Snew2, Anew2, epnew2]=combHard1D(mp, delE2, nS2, nA2, nep2);
  Eep1 = E; if epnew1>nep1; Eep1 = Et; end
  Eep2 = E; if epnew2>nep2; Eep2 = Et; end
  nS1 = Snew1; nep1 = epnew1; nA1=Anew1;
  nS2 = Snew2; nep2 = epnew2; nA2=Anew2;
  Res = F - [nS1*A1-nS2*A2;nS2*A2];
  conv = norm(Res)^2/(1+norm(F)^2);
```

```
  iter = iter + 1;
  fprintf('\n %3d  %7.4f %7.4f %7.3f %7.3f %8.6f %8.6f %10.2e %10.2e',...
      iter,u(1),u(2),nS1,nS2,nep1,nep2,Res);
end
```

As shown in the output below, the Newton-Raphson iteration converges in the second iteration.

```
iter       u1      u2      S1      S2      ep1      ep2   Residual
   0   0.0000  0.0000   0.000   0.000 0.000000 0.000000  0.00e+000  1.20e+004
   1   0.1200  0.3600 101.818 112.727 0.000182 0.001273 -4.55e+003  6.36e+003
   2   0.3200  1.9600 120.000 240.000 0.002000 0.014000  0.00e+000  0.00e+000
```

■

P4.12 Two one-dimensional bars are connected serially as shown in the figure. At load step n, bar1 was plastic and bar2 was elastic. At load step n+1, the increments of nodal displacements are given as $\Delta\mathbf{u} = [\Delta u_1,\ \Delta u_2,\ \Delta u_3] = [0.0,\ -0.01,\ 0.0]$. Calculate stresses and plastic strains of both bars at load step n+1.

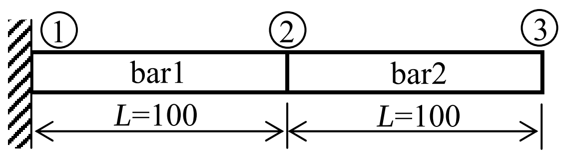


Figure P4.12

	bar1	bar2
Young modulus (E)	10,000	5,000
Tangent modulus (E_t)	1,000	500
Previous stress (σ^n)	6.0	7.4
Initial yield stress (σ_Y)	5.0	7.5
Plastic strain (ε_p)	9E-4	0.0
Yield status	Plastic	Elastic
Hardening	Isotropic	Isotropic

Solution:

Bar1: From the given nodal displacements, the strain of the element can be calculated by

$$\Delta\varepsilon = \frac{\Delta u_2 - \Delta u_1}{L_1} = -10^{-4}$$

First, assume that the strain increment is purely elastic to obtain the following trial state:

$$\sigma^{tr} = \sigma^n + E\Delta\varepsilon = 5.0, \quad \sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n = 6.0$$

Since the material is initially in the plastic state and the stress is positive at load step n, the bar yielded in tension. However, since the incremental strain is negative, the bar is under unloading. Thus, the material becomes elastic and

$$\sigma^{n+1} = \sigma^{tr} = 5.0, \quad \varepsilon_p^{n+1} = \varepsilon_p^n = 9 \times 10^{-4}$$

Bar2: From the given nodal displacements, the strain of the element can be calculated by

$$\Delta\varepsilon = \frac{\Delta u_3 - \Delta u_2}{L_1} = 10^{-4}$$

First, assume that the strain increment is purely elastic to obtain the following trial state:

$$\sigma^{tr} = \sigma^n + E\Delta\varepsilon = 7.9, \quad \sigma_Y^n = \sigma_Y^0 + H\varepsilon_p^n = 7.5$$

Since $\sigma^{tr} > \sigma_Y^n$, the material becomes plastic in this load increment. Since the element is elastic at load step n, the ratio R needs to be calculated, which can be written as

$$R = 1 - \frac{\left|\sigma^{tr}\right| - \sigma_Y^n}{\left|\Delta\sigma\right|} = 0.2$$

Thus, the updated stress and plastic strain become

$$\sigma^{n+1} = \sigma^n + R\Delta\sigma + E_t(1-R)\Delta\varepsilon = 7.54$$

$$\varepsilon_p^{n+1} = \varepsilon_p^n + \frac{1-R}{1+H\,/\,E}\left|\Delta\varepsilon\right| = 7.2 \times 10^{-5}$$

■

P4.13 Write the expression of the rank-4 unit symmetric tensor and unit deviatoric tensor in the 6×6 matrix notation.

Solution:
The index of rank-2 tensor can be converted into a vector via [11 22 33 12 23 13]. From the definition of $I_{ijkl} = (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,/\,2$, the matrix version of the fourth-order unit symmetric tensor becomes

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

In addition, from the definition of rank-4 unit deviatoric tensor $\mathbf{I}_{dev} = \mathbf{I} - \frac{1}{3}\mathbf{1} \otimes \mathbf{1}$, its matrix version becomes

$$\text{IDEV} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

■

P4.14 A solid shaft as shown in the figure is subjected to tensile force *P* and a torque *T*. The force and torque are such that the normal stress $\sigma_{xx} = \sigma$ and shear stress $\tau = \sigma$. The shear stress is along the circumference of the shaft. Using the von Mises criterion, determine the values of σ when the material yields first time. The yield stress from the uniaxial tension test is σ_Y.

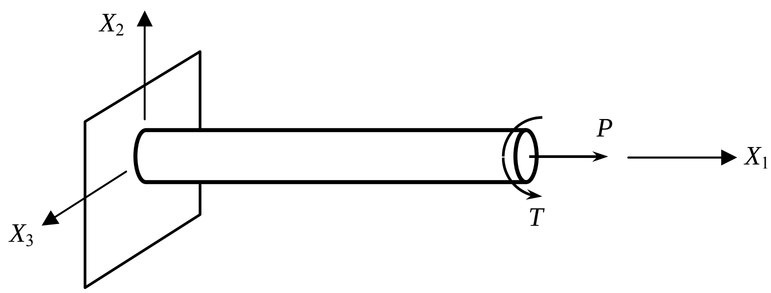


Figure P4.14

Solution:

In the case of P, the stress component is fixed to be σ_{11}. In the case of T, the shear stress component varies at different location on the boundary of the cross-section. For the simplicity, let's consider the case in which $\tau_{12} = \sigma$. Thus, the stress matrix and deviatoric stress become

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{s} = \begin{bmatrix} \frac{2}{3}\sigma & \sigma & 0 \\ \sigma & -\frac{1}{3}\sigma & 0 \\ 0 & 0 & -\frac{1}{3}\sigma \end{bmatrix}$$

The norm of deviatoric stress can be calculated by

$$\|\mathbf{s}\| = \sigma\sqrt{\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9} + 1 + 1} = \sigma\sqrt{\tfrac{8}{3}}$$

The yield criterion is

$$f = \|\mathbf{s}\| - \sqrt{\tfrac{2}{3}}\sigma_Y = \sqrt{\tfrac{8}{3}}\sigma - \sqrt{\tfrac{2}{3}}\sigma_Y = 0$$

Thus, the first yielding starts when

$$\sigma = \frac{1}{2}\sigma_Y$$

■

P4.15 A plane stress plate is under biaxial stress state in which $\sigma_{xx} = -\sigma_{yy} = \sigma$. When the applied load is proportional, determine σ when the material yields first time. The yield stress from the uniaxial tension test is σ_Y.

Solution:
In the case of biaxial loading, the stress matrix and deviatoric stress become

$$\boldsymbol{\sigma} = \mathbf{s} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The norm of deviatoric stress can be calculated by

$$\|\mathbf{s}\| = \sigma\sqrt{2}$$

The yield criterion is

$$f = \|\mathbf{s}\| - \sqrt{\tfrac{2}{3}}\sigma_Y = \sqrt{2}\sigma - \sqrt{\tfrac{2}{3}}\sigma_Y = 0$$

Thus, the first yielding starts when

$$\sigma = \sqrt{\tfrac{1}{3}}\sigma_Y$$

■

P4.16 A square is under proportional loading with shear stress $\tau_{12} = \tau_{21} = \tau$. When the effective plastic strain is $e_p = 0.1$, calculate the value of shear stress. Consider three different hardening models: (a) isotropic, (b) kinematic, and (c) combined hardening with $\beta = 0.5$. Assume that the initial yield stress is $400\mathrm{MPa}$ and the plastic modulus is $H = 200\mathrm{MPa}$.

Solution:
Since the applied stress is proportional loading, it is expected that the material is in the plastic phase, and all three models provide the same stress value. The difference occurs only when the direction of loading changes. In the case of pure shear loading, the stress and deviatoric stress become

$$\boldsymbol{\sigma} = \mathbf{s} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, the norm of the deviatoric stress becomes

$$\|\mathbf{s}\| = 2\tau$$

(a) Isotropic hardening: from the definition of yield function,

$$\|\mathbf{s}\| - \sqrt{\tfrac{2}{3}}(\sigma_Y^0 + He_p) = 2\tau - \sqrt{\tfrac{2}{3}}(400 + 200 \times 0.1) = 0$$

$$\sigma = 171.5\text{MPa}$$

(b) Kinematic hardening: from the definition of yield function,

$$\|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 0$$

Note that $\Delta\boldsymbol{\alpha}$ is parallel to $\boldsymbol{\eta}$ and the loading direction remains fixed, and thus, $\boldsymbol{\alpha}$ is parallel to $\mathbf{s}$. In that case, the norm of the shifted stress can be written as $\|\mathbf{s} - \boldsymbol{\alpha}\| = \|\mathbf{s}\| - \|\boldsymbol{\alpha}\|$. Thus, the yield function can be rewritten as

$$\|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = \|\mathbf{s}\| - \|\boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 2\tau - \sqrt{\tfrac{2}{3}}He_p - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 0$$

$$\tau = \tfrac{1}{2}\sqrt{\tfrac{2}{3}}(\sigma_Y^0 + He_p) = 171.5\text{MPa}$$

(c) Combined hardening: Similar to the kinematic hardening model, $\boldsymbol{\alpha}$ is parallel to $\mathbf{s}$. Thus, the yield function can be written as

$$\begin{aligned}
&\|\mathbf{s} - \boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\left[\sigma_Y^0 + (1-\beta)He_p\right] \\
&= \|\mathbf{s}\| - \|\boldsymbol{\alpha}\| - \sqrt{\tfrac{2}{3}}\left[\sigma_Y^0 + (1-\beta)He_p\right] \\
&= 2\tau - \sqrt{\tfrac{2}{3}}\beta He_p - \sqrt{\tfrac{2}{3}}\sigma_Y^0 - \sqrt{\tfrac{2}{3}}(1-\beta)He_p \\
&= 0
\end{aligned}$$

Thus, the applied stress can be solved for

$$\tau = \tfrac{1}{2}\sqrt{\tfrac{2}{3}}(\sigma_Y^0 + He_p) = 171.5\text{MPa}$$

Note that all three models provide the same stress value. ■

P4.17 A pure shear deformation is applied to the square element as shown in the figure such that $\sigma_{12} = \sigma_{21}$ is only non-zero stress component. At load step n, the stress value was $\sigma_{12} = 50$, and there was no plastic deformation. At load step n+1, incremental strain $\Delta\varepsilon_{12} = \Delta\varepsilon_{21} = 0.005$ is applied. Calculate stress components and effective plastic strain at load step n+1. Use the following material properties: shear modulus $\mu = 1{,}000$, plastic modulus $H = 100$, initial yield stress $\sigma_Y = 100$.

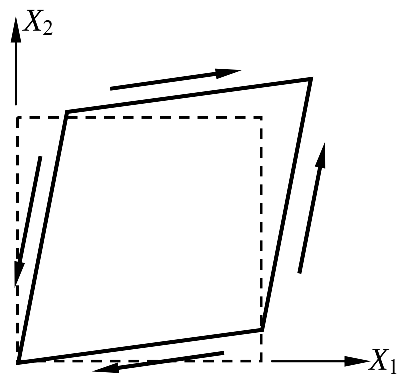


Figure P4.17

Solution:

Since σ_{12} is only non-zero component, it is convenient to work with it as a scalar rather than stress matrix. With given information of $\sigma_{12}^{n} = 50$ and $\Delta\varepsilon_{12} = 0.005$, the trial state can be calculated as

$$\sigma_{12}^{tr} = \sigma_{12}^{n} + 2\mu\Delta\varepsilon_{12} = 60$$

The yield state can be tested using the yield function as

$$f = \left\|\boldsymbol{\eta}^{tr}\right\| - \sqrt{\tfrac{2}{3}}\sigma_Y^n = \sqrt{2\sigma_{12}^{tr}} - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 3.2032 > 0$$

Thus, the material becomes plastic in this load increment. The plastic consistency parameter can be calculated by

$$\Delta\gamma = \frac{f}{2\mu + \frac{2}{3}H} = 0.00155$$

In addition, the deviatoric unit tensor for the trial state becomes

$$\mathbf{N} = \frac{\boldsymbol{\eta}^{tr}}{\left\|\boldsymbol{\eta}^{tr}\right\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, only σ_{12} will be updated due to plastic deformation. The updated stress and plastic strain become

$$\sigma_{12}^{n+1} = \sigma_{12}^{tr} - \frac{2\mu\Delta\gamma}{\sqrt{2}} = 57.808$$

$$e_p^{n+1} = e_p^{n} + \sqrt{\tfrac{2}{3}}\Delta\gamma = 0.00127$$

Below is the MATLAB program that solves the above problem:

```
%
% P4.17 shear deformation of a square
%
lambda=1000; mu=1000;
mp = [lambda, mu, 0, 100, 100];
```

```
Iden=[1 1 1 0 0 0]';
deps=[0 0 0 0.01 0 0]';
stressN=[0 0 0 50 0 0]';
alphaN=[0 0 0 0 0 0]';
epN=0;
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
[stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN)
```

■

P4.18 Displacements of a simple shear deformation in the figure can be expressed by $u_1 = kx_2$, $u_2 = 0$. At load step n, $k = 0.016$ and the material is elastic. At load step $n+1$, $\Delta k = 0.008$. Calculate stress and plastic strain. Check if the updated state is on the yield function; i.e., $f(\sigma^{n+1}, e_p^{n+1}) = 0$. Use the following material properties: shear modulus $\mu = 100$, plastic modulus $H = 10$, initial yield stress $\sigma_Y = \sqrt{12}$.

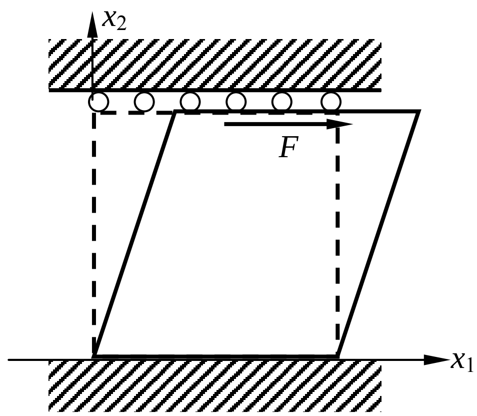


Figure P4.18

Solution:

From the infinitesimal deformation assumption, the strain at load step n can be calculated by

$$\boldsymbol{\varepsilon}^n = \begin{bmatrix} 0 & \frac{1}{2}k \\ \frac{1}{2}k & 0 \end{bmatrix} \quad \Rightarrow \quad \varepsilon_{12}^n = 0.008, \quad \Delta\varepsilon_{12} = 0.004$$

Since ε_{12} is only non-zero component, strain will be considered as a scalar in the following calculations. For the given strain at load step n, the stress becomes

$$\boldsymbol{\sigma}^n = \mathbf{s}^n = 2\mu\mathbf{e} = 2\mu\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & \mu k \\ \mu k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1.6 \\ 1.6 & 0 \end{bmatrix}, \quad \Rightarrow \quad \sigma_{12}^n = 1.6$$

Again, since σ_{12} is only non-zero component, stress will be considered as a scalar. For a given increment $\Delta\varepsilon_{12}$, the trial stress becomes

$$s_{12}^{tr} = s_{12}^n + 2\mu\Delta e = 2.4$$

The yield function can be used to check the status of the material, as

$$f = \left\| \mathbf{s}_{12}^{tr} \right\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = \sqrt{2(S_{12}^{tr})^2} - \sqrt{8} = 0.566 > 0$$

Thus, the material becomes plastic in this increment, $\text{yieldStatus} = 1$. The unit deviatoric tensor and the plastic consistency parameter can be calculated by

$$\mathbf{N} = \frac{\mathbf{s}^{tr}}{\left\| \mathbf{s}^{tr} \right\|} = \frac{1}{3.394}\begin{bmatrix} 0 & 2.4 \\ 2.4 & 0 \end{bmatrix}$$

$$\Delta\gamma = \frac{f}{2\mu + \frac{2}{3}H} = \frac{0.5657}{200 + 6.667} = 0.002737$$

Using the unit deviatoric tensor and the plastic consistency parameter, the stress and the effective plastic strain can be updated by

$$\mathbf{s}^{n+1} = \mathbf{s}^{tr} - 2\mu\Delta\gamma\mathbf{N} = \begin{bmatrix} 0 & 2.4 \\ 2.4 & 0 \end{bmatrix} - \frac{0.5473}{3.394}\begin{bmatrix} 0 & 2.4 \\ 2.4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2.013 \\ 2.013 & 0 \end{bmatrix}$$

$$e_p^{n+1} = e_p^n + \sqrt{\tfrac{2}{3}}\Delta\gamma = 0.002234$$

After the plastic deformation, the yield surface is expanded by

$$\kappa(e_p^{n+1}) = \sigma_Y^0 + He_p^{n+1} = 3.4864$$

Thus, the yield function after update becomes

$$f(\mathbf{s}^{n+1}, e_p^{n+1}) = \left\| \mathbf{s}^{n+1} \right\| - \sqrt{\tfrac{2}{3}}\kappa(e_p^{n+1}) = 0$$

Thus, the updated state of the material is on the surface of the yield function.
Below is the MATLAB program that solves the above problem:

```
%
% P4.18 shear deformation of a square
%
lambda=100; mu=100;
mp = [lambda, mu, 0, 10, sqrt(12)];
Iden=[1 1 1 0 0 0]';
stressN=[0 0 0 0 0 0]';
alphaN=[0 0 0 0 0 0]';
epN=0;
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
deps=[0 0 0 0.016 0 0]';
[stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN);
stressN=stress; alphaN=alpha;epN=ep;
deps=[0 0 0 0.008 0 0]';
[stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN)
```

■

P4.19 At load step t_n, a unit cube is under unaxial stress state with $\sigma_{11} = 100\text{MPa}$, and all other stress components and plastic variables are zero. At load step t_{n+1}, additional shear stress is applied such that $\Delta\gamma_{12} = 0.002$. Determine stress, back stress, and effective plastic strain. Assume the following material properties: $\lambda = \mu = 100\text{GPa}$, $H = 10\text{GPa}$, $\sigma_Y = 100\text{MPa}$, combined isotropic/kinematic hardening with $\beta = 0.5$.

Solution:

At load step t_n, the material reaches the initial yield stress. At load step t_{n+1}, the trial stress becomes

$$^{tr}\boldsymbol{\sigma} = {}^{n}\boldsymbol{\sigma} + \mathbf{D} : \Delta\boldsymbol{\varepsilon} = [100 \quad 0 \quad 0 \quad 200 \quad 0 \quad 0]^{\mathrm{T}}$$

Since back stress is zero at load step t_n, the deviatoric stress is the same with the shifted stress as

$$^{tr}\boldsymbol{\eta} = {}^{tr}\mathbf{s} = {}^{tr}\boldsymbol{\sigma} - tr({}^{tr}\boldsymbol{\sigma})\mathbf{1} = [66.67 \quad -33.33 \quad -33.33 \quad 200 \quad 0 \quad 0]^{\mathrm{T}}$$

From the shifted stress, the unit deviatoric tensor that is normal to the yield surface can be obtained as

$$\mathbf{N} = \frac{^{tr}\boldsymbol{\eta}}{\left\| {}^{tr}\boldsymbol{\eta} \right\|} = [.2265 \quad -.1132 \quad -.1132 \quad .6794 \quad 0 \quad 0]^{\mathrm{T}}$$

Using the norm of the shifted stress, the yield function can be evaluated as

$$f = \left\| {}^{tr}\boldsymbol{\eta} \right\| - \sqrt{\tfrac{2}{3}}\sigma_Y^0 = 294.4 - 100\sqrt{\tfrac{2}{3}} = 212.7 > 0$$

It is clear that the material is in the plastic state. The plastic consistency parameter can be calculated from

$$\gamma = \frac{f}{2\mu + \frac{2}{3}H} = 0.001$$

Using γ and $\mathbf{N}$, the stress, back stress, and effective plastic strain can be updated as

$$^{n+1}\boldsymbol{\sigma} = {}^{tr}\boldsymbol{\sigma} - 2\mu\gamma\mathbf{N} = [53.4 \quad 23.3 \quad 23.3 \quad 60.1 \quad 0 \quad 0]^{\mathrm{T}}$$

$$^{n+1}\boldsymbol{\alpha} = {}^{n}\boldsymbol{\alpha} + \tfrac{2}{3}\beta H\gamma\mathbf{N} = [.777 \quad -.389 \quad -.389 \quad 2.331 \quad 0 \quad 0]^{\mathrm{T}}$$

$$^{n+1}e_p = {}^{n}e_p + \sqrt{\tfrac{2}{3}}\gamma = 8.405 \times 10^{-4}$$

Below is the MATLAB program that solves for the above problem:

```
lambda=100000; mu=100000;
mp = [lambda, mu, 0.5, 10000, 100];
Iden=[1 1 1 0 0 0]';
stressN=[100 0 0 0 0 0]';
alphaN=[0 0 0 0 0 0]';
epN=0;
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
deps=[0 0 0 0.002 0 0]';
[stress, alpha, ep]=combHard(mp,D,deps,stressN,alphaN,epN)
```

■

P4.20 Using Abaqus perform a uniaxial tension test of a unit cube (C3D8) in x_3-direction. Assume elastoplastic material with linear isotropic hardening (E = 2.0E5, ν = 0.3, σ_Y = 200, H = 2.0E4). Displace at x_3 = 1 surface is controlled as shown in the figure with three steps. Use 10 increments in each step. Plot stress-strain curve for all 30 increments.

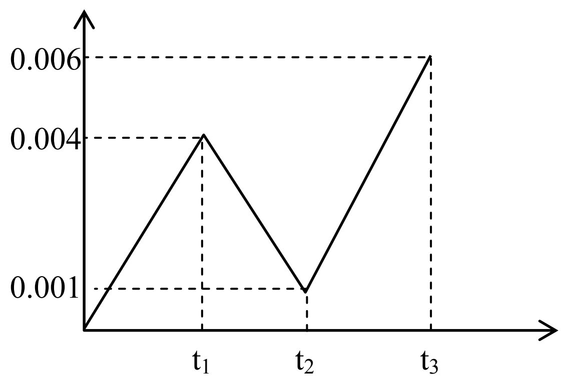
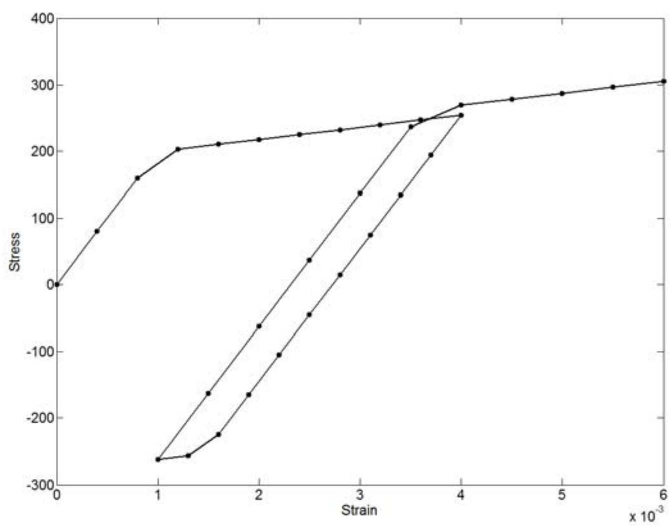


Figure P4.20

Solution:
The following program list shows the ABAQUS input file for the elastoplastic loading:

```
*HEADING
MISES PLASTICITY/LINEAR ELASTICITY,
UNIAXIAL TENSION TEST, C3D8
*NODE,NSET=ALLN
1,0.,0.,0.
2,1.,0.,0.
3,1.,1.,0.
4,0.,1.,0.
5,0.,0.,1.
6,1.,0.,1.
7,1.,1.,1.
8,0.,1.,1.
*ELEMENT,TYPE=C3D8,ELSET=ALLE
1,1,2,3,4,5,6,7,8
*SOLID
SECTION,ELSET=ALLE,MATERIAL=ALLE
*MATERIAL,NAME=ALLE
*ELASTIC
200.E3,.3
*PLASTIC
200.,0.
400.,.01
*BOUNDARY
1,PINNED
2,2
5,2
6,2
4,1
5,1
8,1
2,3
3,3
4,3
*STEP,INC=10
*STATIC,DIRECT
*BOUNDARY
7,3,,.001
5,3,,.001
6,3,,.001
8,3,,.001
*EL PRINT,FREQ=1
S,
E,
EP,
*NODE PRINT
U,RF
*EL FILE,FREQ=1
S,
E,EP
*END STEP
*STEP,INC=10
*STATIC,DIRECT
1.,10.
*BOUNDARY
7,3,,.006
5,3,,.006
6,3,,.006
8,3,,.006
*EL PRINT,FREQ=1
S,
E,
EP,
*NODE PRINT
U,RF
*EL FILE,FREQ=1
S,
E,EP
*END STEP
```

The analysis results are shown in the following figure:



■

P4.21 Calculate D_{ep} and D^{alg} for one-dimensional elastoplasticity problem using the von Mises yield criterion and linear combined isotropic/kinematic hardening. Assume material properties: $(E, H, \sigma_Y^0, \beta)$.

Solution:

(a) In one-dimensional problems, the lateral strains are ignored, and stress σ and strain ε are scalar. In addition, there is no need to calculate deviatoric stress and strain. The yield function becomes

$$f(\eta, e_p) = \left|\sigma - \alpha\right| - (\sigma_Y^0 + (1-\beta)He_p) \le 0$$

Note that the above yield function is slightly different from the multi-dimensional yield function in Eq. (4.84) because equivalent stress and effective strain are not used. The constitutive relation is $\dot{\sigma} = E(\dot{\varepsilon} - \dot{\varepsilon}^p)$ with $\dot{\varepsilon}^p = \dot{e}_p = \gamma$, and the hardening models are

$$\dot{\alpha} = \beta H \gamma$$

There is no difference between the plastic strain and effective plastic strain. From the consistency condition:

$$\dot{f} = E(\dot{\varepsilon} - \dot{\varepsilon}^p) - \dot{\alpha} - (1-\beta)H\dot{e}_p = 0$$

The plastic consistency parameter can be calculated as

$$\gamma = \frac{E\dot{\varepsilon}}{E+H}$$

Thus,

$$\dot{\sigma} = E\dot{\varepsilon} - E\dot{\varepsilon}^p = E\left(1 - \frac{E}{E+H}\right)\dot{\varepsilon} = \frac{EH}{E+H}\dot{\varepsilon}$$

Thus, the elastoplastic tangent modulus can be obtained as

$$D_{ep} = \frac{EH}{E+H}$$

Note that the above D_{ep} is the same as the tangent modulus in Eq. (4.8).

(b) In numerical integration, the trial stress is obtained from

$${}^{tr}\sigma = {}^{n}\sigma + E\Delta\varepsilon, \quad {}^{tr}\alpha = {}^{n}\alpha, \quad {}^{tr}e_p = {}^{n}e_p$$

Using the property of $\Delta\varepsilon^p = \Delta e_p = \Delta\gamma$, the plastic return mapping becomes

$${}^{n+1}\sigma = {}^{tr}\sigma - E\Delta\gamma, \quad {}^{n+1}\alpha = {}^{tr}\alpha + \beta H\Delta\gamma, \quad {}^{n+1}e_p = {}^{tr}e_p + \Delta\gamma$$

The yield function becomes

$$f({}^{n+1}\eta, {}^{n+1}e_p) = \left|{}^{tr}\eta\right| - (E + \beta H)\Delta\gamma - (\sigma_Y^0 + (1-\beta)H({}^{tr}e_p + \Delta\gamma)) = 0$$

from which the plastic consistency parameter can be calculated by

$$\Delta\gamma = \frac{\left|{}^{tr}\eta\right| - (\sigma_Y^0 + (1-\beta)H\,{}^{tr}e_p)}{(E+H)}$$

For the consistent tangent stiffness, it is clear that **N** is fixed in one-dimensional problems. The derivative of the plastic consistency parameter with respect to strain increment becomes

$$\frac{\partial \Delta\gamma}{\partial \Delta\varepsilon} = \frac{E}{E+H}$$

Thus, the consistent tangent stiffness becomes

$$D^{\mathrm{alg}} = E - \frac{E^2}{E+H} = \frac{EH}{E+H}$$

Note that $D^{\mathrm{alg}} = D_{ep}$ because in the case of one-dimensional case, the direction **N** is fixed. ■

P4.22 In the saturated isotropic hardening model, the yield stress starts from initial value of σ_Y^0 and approaches σ_Y^∞ as the plastic strain increases.

$$\kappa(e_p) = \sigma_Y^0 + (\sigma_Y^\infty - \sigma_Y^0)\left[1 - \exp\left(-\frac{e_p}{e_p^\infty}\right)\right]$$

Since the hardening model is nonlinear, it is required to have a local Newton-Raphson method to find the plastic consistency parameter. Modify MATLAB program combHard so that it can solve for the above saturated isotropic hardening model. Test the program by solving the pure shear problem in P4.15. Assume the following material properties: shear modulus $\mu = 1{,}000$, initial yield stress $\sigma_Y = 100$, asymptotic yield stress $\sigma_Y^\infty = 200$, and asymptotic effective plastic strain $e_p^\infty = 0.05$.

Solution:

The return-mapping algorithm will be similar to the linear isotropic hardening model except for the yield function and the local Newton-Raphson method to find the plastic consistency parameter.

1. Yield function

$$f^k = \left\|{}^{tr}\boldsymbol{\eta}\right\| - \sqrt{\tfrac{2}{3}}\left(\sigma_Y^0 + (\sigma_Y^\infty - \sigma_Y^0)\left[1 - \exp\left(-\frac{e_p^k}{e_p^\infty}\right)\right]\right)$$

2. Jacobian relation

$$\frac{\partial f}{\partial \gamma} = 2\mu + \frac{2}{3}\frac{(\sigma_Y^\infty - \sigma_Y^0)}{e_p^\infty}\left[1 - \exp\left(-\frac{e_p^k}{e_p^\infty}\right)\right]$$

3. Update the plastic consistency parameter and effective plastic strain

$$\gamma^{k+1} = \gamma^k + \frac{f^k}{\partial f / \partial \gamma}, \quad e_p^{k+1} = {}^n e_p + \sqrt{\tfrac{2}{3}}\gamma^{k+1}$$

4. Check convergence

$$\text{If } (|f^k| > f_{\text{TOL}})\ k = k + 1 \text{ and go to Step 1}$$

$$\text{If } (k > k_{\text{MAX}}) \text{ stop with error message}$$

Below is MATLAB program, combHardSat, which can solve for stress and effective plastic strain from the saturated hardening model:

```
%
% Saturated isotropic hardening model
%
function [stress, ep]=combHardSat(mp,D,deps,stressN,epN)
% Inputs:
% mp = [lambda, mu, epinf, Y0, Yinf];
% D = elastic stiffness matrix
% stressN = [s11, s22, s33, t12, t23, t13];
%
Iden = [1 1 1 0 0 0]';
two3 = 2/3; stwo3=sqrt(two3);                      %constants
mu=mp(2);epinf=mp(3);Y0=mp(4);Yinf=mp(5);          %material properties
ftol = Y0*1E-6;                                    %tolerance for yield
stresstr = stressN + D*deps;                       %trial stress
I1 = sum(stresstr(1:3));                           %trace(sigmatr)
eta = stresstr - I1*Iden/3;                        %deviatoric stress
etat = sqrt(eta(1)^2 + eta(2)^2 + eta(3)^2 ...
          + 2*(eta(4)^2 + eta(5)^2 + eta(6)^2));%norm of eta
fyld = etat - stwo3*(Y0+(Yinf-Y0)*(1-exp(-epN/epinf)))
if fyld < ftol                                     %yield test
    stress = stresstr; ep = epN;                   %trial states are final
    return;
else
  iter =0; gamma = 0; ep = epN;
  while fyld > ftol                                %local N-R iteration
     iter=iter+1;
     if iter > 40; error('Fail to compute gamma'); end;
     dfdg = 2*mu+two3*((Yinf-Y0)*exp(-ep/epinf)/epinf)
     gamma = gamma + fyld/dfdg                     %update plast cons param
     ep = epN + stwo3*gamma;
     fyld = etat -2*mu*gamma - stwo3*(Y0+(Yinf-Y0)*(1-exp(-ep/epinf)))
  end
end
N = eta/etat;                                      %unit vector normal to f
stress = stresstr - 2*mu*gamma*N;                  %updated stress
```

Below is the MATLAB program that can solve for P4.15 with the saturated isotropic hardening model:

```
%
% P4.22 shear deformation of a square (saturated isotropic hardening)
%
lambda=1000; mu=1000;epinf=0.05;Y0=100;Yinf=200;
mp = [lambda, mu, epinf, Y0, Yinf];
Iden=[1 1 1 0 0 0]';
deps=[0 0 0 0.01 0 0]';
stressN=[0 0 0 50 0 0]';
alphaN=[0 0 0 0 0 0]';
epN=0;
D=2*mu*eye(6) + lambda*Iden*Iden';
D(4,4) = mu; D(5,5) = mu; D(6,6) = mu;
[stress, ep]=combHardSat(mp,D,deps,stressN,epN)
```

The local Newton-Raphson converged after the second iteration.
Iteration 1: $f = 3.2032$, $\mathrm{d}f/\mathrm{d}\gamma = 3.33\mathrm{E}3$, $\gamma = 9.61\mathrm{E}\text{-}4$
Iteration 2: $f = 0.01$, $\mathrm{d}f/\mathrm{d}\gamma = 3.31\mathrm{E}3$, $\gamma = 9.64\mathrm{E}\text{-}4$
Iteration 3: $f = 9.77\mathrm{E}\text{-}8$

After convergence, the stress and effective plastic strain are updated to

$$ {}^{n+1}\sigma_{12} = 58.63, \quad {}^{n+1}e_p = 7.87 \times 10^{-4} $$

■

P4.23 An plane strain square undergoes the following elastic deformation:

$$ x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 $$

Using the linear relationship between principal Kirchhoff stress and logarithmic stretch, find the Kirchhoff stress tensor when $k = 0.02$. Use the following material properties: $\lambda = \mu = 100\text{GPa}$.

Solution:
Since the cube is elastic, there is no need to separate elastic and plastic part of deformation. Thus, the superscript 'e' will be omitted in the following derivation. For given deformation, the deformation gradient and left Cauchy-Green deformation tensor becomes

$$ \mathbf{F} = \begin{bmatrix} 1 & 0.02 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}} = \begin{bmatrix} 1.0004 & 0.02 & 0 \\ 0.02 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

The three eigenvalues and eigenvectors of **b** become

$$ \begin{aligned} \lambda_1 &= 1.02, & \mathbf{n}^1 &= [-0.711 \quad -0.704 \quad 0]^{\mathrm{T}} \\ \lambda_2 &= 0.98, & \mathbf{n}^2 &= [0.704 \quad -0.711 \quad 0]^{\mathrm{T}} \\ \lambda_3 &= 1, & \mathbf{n}^3 &= [0 \quad 0 \quad 1]^{\mathrm{T}} \end{aligned} $$

Then, the logarithmic stretch can be obtained by

$$\mathbf{e} = \begin{Bmatrix} 0.02 & -0.02 & 0 \end{Bmatrix}^{\mathrm{T}}$$

The stress-strain relation in the principal space, $\boldsymbol{\tau}^p = \mathbf{c}\cdot\mathbf{e}$, can be written as

$$\boldsymbol{\tau}^p = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{Bmatrix} 0.02 \\ -0.02 \\ 0 \end{Bmatrix} \times 10^{11} = \begin{Bmatrix} 4 \\ -4 \\ 0 \end{Bmatrix} \times 10^{9}\,\mathrm{Pa}$$

Then, the Kirchhoff stress can be obtained using

$$\boldsymbol{\tau} = \sum_{i=1}^{3} \tau_i^p \mathbf{n}^i \otimes \mathbf{n}^i = \begin{bmatrix} 0.04 & 4 & 0 \\ 4 & -0.04 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathrm{GPa}$$

■

P4.24 A history of biaxial loadings is applied to a $1\mathrm{mm}\times 1\mathrm{mm}$ square, as shown in the figure. The square is constrained in the Y-direction along the bottom edge and in the X-direction along the left edge. The model is displaced in the X and Y directions at the right and top edges by $R = 2.5\times 10^{-5}\mathrm{mm}$, respectively. Calculate σ_{xx}, σ_{yy}, σ_{zz}, and vonMises stress at each load step. Use the following material properties: $E = 250\mathrm{GPa}$, $\nu = 0.25$, $\sigma_Y = 5\mathrm{MPa}$, and $E_T = 50\mathrm{GPa}$.

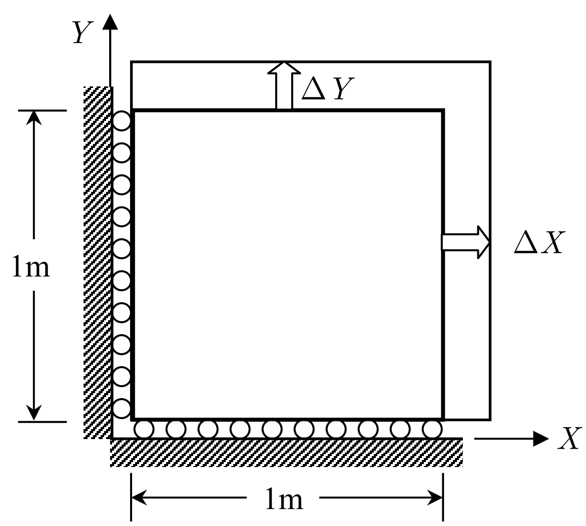


Figure P4.24

Load step	ΔX	ΔY	Description
1	R	0	First yield
2	R	0	Plastic flow
3	0	R	Elastic unloading
4	0	R	Plastic reloading

5	$-R$	0	Plastic flow
6	$-R$	0	Plastic flow
7	0	$-R$	Elastic unloading
8	0	$-R$	Plastic flow

Solution:
The problem is modeled using one square element in ANSYS. Since the strain in Z-direction is constrained, stress in Z-direction. Below is the table of stresses at each load step.

Load step	σ_{xx}	σ_{yy}	σ_{zz}	von Mises stress
1	7.50000	2.5	2.5	5.0
2	11.7	6.7	6.7	5.0
3	14.2	14.2	9.2	5.0
4	16.4	19.7	13.9	5.0
5	9.9	15.6	12.0	5.0
6	5.1	10.7	9.2	5.0
7	2.6	3.2	6.7	3.8
8	0.2	-3.0	2.8	5.0

Below is ANSYS script to model and solve the problem:

```
/COM,ANSYS MEDIA REL. 11.0 (10/27/2006) REF. VERIF. MANUAL: REL. 11.0
/VERIFY,VMR049-PL1A-182
/TITLE, VMR049-PL1A-182, 2D PLANE STRAIN PLASTICITY BENCHMARK
/COM, REFERENCE: NAFEMS REPORT-R0049

/PREP7
R = 2.5E-5
ET,1,182,,,
KEYOPT,1,3,2
N,1,,,,
N,2,0,1,,
N,3,1,0,,
N,4,1,1,,
E, 1,3,4,2
MP,EX,1,250E3,
MP,NUXY,1,0.25,
TB,BISO,1,1, , ,
TBMODIF,2,1,5
TBMODIF,3,1,0.0
TB,HILL,1
TBDATA,1,1.0,1.0,1.0,1.0,1.0,1.0
NSEL,S,LOC,X
D,ALL,UX
NSEL,S,LOC,Y
D,ALL,UY
NSEL,S,LOC,Y,1.0
D,ALL,UY
FINISH

/SOLU
NLGEOM,ON
NSEL,S,LOC,X,1.0
D,ALL,UX,R
```

```
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,X,1.0
D,ALL,UX,2*R
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,Y,1.0
D,ALL,UY,R
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,Y,1.0
D,ALL,UY,2*R
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,X,1.0
D,ALL,UX,R
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,X,1.0
D,ALL,UX,0.0
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,Y,1.0
D,ALL,UY,R
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
NSEL,S,LOC,Y,1.0
D,ALL,UY,0.0
NSEL,ALL
NSUBST,10,10,10,
OUTRES,ALL,5
SOLVE
FINISH

/POST26
/GROPT,VIEW,0
/GTHK,CURVE,1
/GROPT,FILL,OFF
/GRID,1
/GTHK,GRID,1
/GROPT,CGRID,1
/AXLAB,X,STEP
/AXLAB,Y,STRESS
/GTHK,AXIS,1
/GRTYP,0
/XRANGE,0,8
/YRANGE,-10,25
ESOL,2,1,4,S,X,
ESOL,3,1,4,S,Y,
```

```
ESOL,4,1,4,S,Z,
ESOL,5,1,4,S,EQV,
PLVAR,2,3,4,5
PRVAR,2,3,4,5

*DIM,VALUEX,ARRAY,8,1
*DO,JJ,1,8,1
*GET,VALUEX(JJ,1),VARI,2,RTIME,JJ
*ENDDO
R1=VALUEX(1,1)/7.500
R2=VALUEX(2,1)/11.666
R3=VALUEX(3,1)/14.166
R4=VALUEX(4,1)/16.418
R5=VALUEX(5,1)/9.927
R6=VALUEX(6,1)/5.134
R7=VALUEX(7,1)/2.635
R8=VALUEX(8,1)/1.218

*DIM,VALUEY,ARRAY,8,1
*DO,JJ,1,8,1
*GET,VALUEY(JJ,1),VARI,3,RTIME,JJ
*ENDDO
R9=VALUEY(1,1)/2.500
R10=VALUEY(2,1)/6.666
R11=VALUEY(3,1)/14.166
R12=VALUEY(4,1)/19.669
R13=VALUEY(5,1)/15.622
R14=VALUEY(6,1)/10.745
R15=VALUEY(7,1)/3.245
R16=VALUEY(8,1)/(-3.715)

*DIM,VALUEZ,ARRAY,8,1
*DO,JJ,1,8,1
*GET,VALUEZ(JJ,1),VARI,4,RTIME,JJ
*ENDDO
R17=VALUEZ(1,1)/2.500
R18=VALUEZ(2,1)/6.666
R19=VALUEZ(3,1)/9.166
R20=VALUEZ(4,1)/13.912
R21=VALUEZ(5,1)/11.951
R22=VALUEZ(6,1)/9.120
R23=VALUEZ(7,1)/6.620
R24=VALUEZ(8,1)/3.521

*DIM,VALUEEF,ARRAY,8,1
*DO,JJ,1,8,1
*GET,VALUEEF(JJ,1),VARI,5,RTIME,JJ
*ENDDO
R25=VALUEEF(1,1)/5.000
R26=VALUEEF(2,1)/5.000
R27=VALUEEF(3,1)/5.000
R28=VALUEEF(4,1)/5.000
R29=VALUEEF(5,1)/5.000
R30=VALUEEF(6,1)/5.000
R31=VALUEEF(7,1)/3.719
R32=VALUEEF(8,1)/5.000

*DIM,STEP,CHAR,10
*DIM,TARGETX,CHAR,10
*DIM,TARGETY,CHAR,10
*DIM,TARGETZ,CHAR,10
*DIM,TARGETEF,CHAR,10
*DIM,RATIOX,,8,1
```

```
*DIM,RATIOY,,8,1
*DIM,RATIOZ,,8,1
*DIM,RATIOEF,,8,1
STEP(1)='1.0','2.0','3.0','4.0','5.0','6.0','7.0','8.0'
TARGETX(1)='7.500','11.666','14.166','16.418','9.927','5.134','2.635','1.218'
TARGETY(1)='2.500','6.666','14.166','19.669','15.622','10.745','3.245','-3.715'
TARGETZ(1)='2.500','6.666','9.166','13.914','11.951','9.120','6.620','3.521'
TARGETEF(1)='5.000','5.000','5.000','5.000','5.000','5.000','3.719','5.000'
*VFILL,RATIOX,DATA,R1,R2,R3,R4,R5,R6,R7,R8
*VFILL,RATIOY,DATA,R9,R10,R11,R12,R13,R14,R15,R16
*VFILL,RATIOZ,DATA,R17,R18,R19,R20,R21,R22,R23,R24
*VFILL,RATIOEF,DATA,R25,R26,R27,R28,R29,R30,R31,R32
/COM,
/COM, ----------- VMR049-PL1A-182 RESULTS COMPARISON --------------
/COM,
/COM, vmr049-pl1a-182.jpeg RESULTS SHOULD MATCH R0049 NAFEMS MANUAL
/COM, GRAPH RESULTS ON PAGE 49, FIGURE 2.14(A).  THE RESULTS
/COM, DISPLAYED ARE INCREMENTED FOR THIS PURPOSE.
/COM,
/COM,
/COM, ------ VMR049-PL1A-182 STRESS RESULTS IN X DIRECTION -------
/COM,
/COM, |  STEP  |  TARGET  |  ANSYS  |  RATIO
/COM,
*VWRITE,STEP(1),TARGETX(1),VALUEX(1,1),RATIOX(1,1)
(1X,A8,'   ',1x,A8,'   ',F8.3,'   ',F8.3,'   ')
/COM,
/COM, ------ VMR049-PL1A-182 STRESS RESULTS IN Y DIRECTION -------
/COM,
/COM, |  STEP  |  TARGET  |  ANSYS  |  RATIO
/COM,
*VWRITE,STEP(1),TARGETY(1),VALUEY(1,1),RATIOY(1,1)
(1X,A8,'   ',1x,A8,'   ',F8.3,'   ',F8.3,'   ')
/COM,
/COM,
/COM, ------ VMR049-PL1A-182 STRESS RESULTS IN Z DIRECTION -------
/COM,
/COM, |  STEP  |  TARGET  |  ANSYS  |  RATIO
/COM,
*VWRITE,STEP(1),TARGETZ(1),VALUEZ(1,1),RATIOZ(1,1)
(1X,A8,'   ',1x,A8,'   ',F8.3,'   ',F8.3,'   ')
/COM,
/COM,
/COM, ------ VMR049-PL1A-182 EFFECTIVE STRESS RESULTS -------
/COM,
/COM, |  STEP  |  TARGET  |  ANSYS  |  RATIO
/COM,
*VWRITE,STEP(1),TARGETEF(1),VALUEEF(1,1),RATIOEF(1,1)
(1X,A8,'   ',1x,A8,'   ',F8.3,'   ',F8.3,'   ')
/COM,
/COM,
FINISH
```