1.6. Exercises

P1.1 Using Cartesian bases, show that $(\mathbf{u}\otimes\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{x}) = (\mathbf{v}\cdot\mathbf{w})\mathbf{u}\otimes\mathbf{x}$ where **u**, **v**, **w**, and **x** are rank 1 tensor.

Solution: Using the Cartesian basis, $(\mathbf{u}\otimes\mathbf{v})(\mathbf{w}\otimes\mathbf{x}) = (u_i\mathbf{e}_i \otimes v_j\mathbf{e}_j)\cdot(w_k\mathbf{e}_k \otimes x_l\mathbf{e}_l)$. Since the dot product occurs between adjacent bases, we have

$$\begin{aligned}
&(u_i\mathbf{e}_i \otimes v_j\mathbf{e}_j)\cdot(w_k\mathbf{e}_k \otimes x_l\mathbf{e}_l) \\
&= u_i v_j w_k x_l(\mathbf{e}_j\cdot\mathbf{e}_k)(\mathbf{e}_i\otimes\mathbf{e}_l) \\
&= u_i v_j w_k x_l \delta_{jk}(\mathbf{e}_i\otimes\mathbf{e}_l) \\
&= u_i v_j w_j x_l(\mathbf{e}_i\otimes\mathbf{e}_l) \\
&= v_j w_j(u_i\mathbf{e}_i\otimes x_l\mathbf{e}_l) \\
&= (\mathbf{v}\cdot\mathbf{w})(\mathbf{u}\otimes\mathbf{x})
\end{aligned}$$

In the above equation, we used the following properties: $\mathbf{e}_j\cdot\mathbf{e}_k = \delta_{jk}$, $w_k\delta_{jk} = w_j$, and $v_j w_j = \mathbf{v}\cdot\mathbf{w}$.

P1.2 Any rank 2 tensor **T** can be decomposed by **T** = **S** + **W**, where **S** is the symmetric part of **T** and **W** is the skew part of **T**. Let **A** be a symmetric rank 2 tensor. Show $\mathbf{A}:\mathbf{W} = 0$ and $\mathbf{A}:\mathbf{T} = \mathbf{A}:\mathbf{S}$.

Solution: Since **A** is symmetric and **W** is skew, we have

$$\mathbf{A}:\mathbf{W} = A_{ij}W_{ij} = -A_{ij}W_{ji} = -A_{ji}W_{ji}$$

Since in the above equation, the repeated indices *i* and *j* are dummy, the above equation can be rewritten as

$$A_{ij}W_{ij} = -A_{ij}W_{ij} = 0$$

In addition, from the relation **T** = **S** + **W**,

$$\mathbf{A}:\mathbf{T} = \mathbf{A}:(\mathbf{S}+\mathbf{W}) = \mathbf{A}:\mathbf{S} + \mathbf{A}:\mathbf{W} = \mathbf{A}:\mathbf{S}$$

P1.3 For a symmetric rank-two tensor $\mathbf{E}$, using the index notation, show that $\mathbf{I}:\mathbf{E} = \mathbf{E}$, where $\mathbf{I} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$ is a symmetric unit tensor of rank-4.

Solution: Using index notation, the contraction operator can be written as

$$(\mathbf{I}:\mathbf{E})_{ij} = \tfrac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]E_{kl}$$

Since the Kronecker-delta symbol replaces indices, the above equation can be written as

$$(\mathbf{I} : \mathbf{E})_{ij} = \tfrac{1}{2}[E_{ij} + E_{ji}] = E_{ij} = (\mathbf{E})_{ij}$$

The symmetric property of $\mathbf{E}$ is used.

P1.4 The deviator of a symmetric rank-2 tensor is defined as $\mathbf{A}_{dev} = \mathbf{A} - A^m \mathbf{1}$ where $A^m = \frac{1}{3}(A_{11} + A_{22} + A_{33})$. Find the rank-4 deviatoric identity tensor $\mathbf{I}_{dev}$ that satisfies $\mathbf{A}_{dev} = \mathbf{I}_{dev} : \mathbf{A}$.

Solution: From Problem P1.3, it can be shown that $\mathbf{I} : \mathbf{A} = \mathbf{A}$. In addition, A^m can be written in the tensor notation as $A^m = \frac{1}{3}\mathbf{1} : \mathbf{A}$. Therefore, $\mathbf{A}_{dev} = \mathbf{A} - A^m \mathbf{1}$ and it can be written as

$$\mathbf{A}_{dev} = \left[\mathbf{I} - \tfrac{1}{3}\mathbf{1} \otimes \mathbf{1}\right] : \mathbf{A} = \mathbf{I}_{dev} : \mathbf{A}$$

The last equality defined the rank-4 deviatoric identity tensor $\mathbf{I}_{dev}$.

P1.5 The norm of a rank-2 tensor is defined as $\left\|\mathbf{A}\right\| = \sqrt{\mathbf{A} : \mathbf{A}}$. Calculate the following derivative $\partial \left\|\mathbf{A}\right\| / \partial \mathbf{A}$. What is the rank of the derivative?

Solution: From the definition

$$\frac{\partial \left\|\mathbf{A}\right\|}{\partial \mathbf{A}} = \frac{\partial}{\partial \mathbf{A}}\left[(\mathbf{A} : \mathbf{A})^{1/2}\right] = \frac{1}{2}(\mathbf{A} : \mathbf{A})^{-1/2}(2\mathbf{A} : \mathbf{I}) = \frac{\mathbf{A}}{\left\|\mathbf{A}\right\|}$$

The result is a rank-2 tensor. Note that the property that $\partial \mathbf{A} / \partial \mathbf{A} = \mathbf{I}$ is used.

P1.6 A unit rank-2 tensor in the direction of rank-2 tensor $\mathbf{A}$ can be defined as $\mathbf{N} = \mathbf{A} / \left\|\mathbf{A}\right\|$. Show that $\partial \mathbf{N} / \partial \mathbf{A} = [\mathbf{I} - \mathbf{N} \otimes \mathbf{N}] / \left\|\mathbf{A}\right\|$.

Solution: Using chain-rule of differentiation, the unit normal tensor can be differentiated as

$$\frac{\partial \mathbf{N}}{\partial \mathbf{A}} = \frac{\partial}{\partial \mathbf{A}}\left(\frac{\mathbf{A}}{\left\|\mathbf{A}\right\|}\right) = \frac{1}{\left\|\mathbf{A}\right\|^2}\left(\frac{\partial \mathbf{A}}{\partial \mathbf{A}}\left\|\mathbf{A}\right\| - \mathbf{A} \otimes \frac{\partial \left\|\mathbf{A}\right\|}{\partial \mathbf{A}}\right)$$

It is straightforward to show that $\partial \mathbf{A} / \partial \mathbf{A} = \mathbf{I}$. From Problem 1.5, we have

$$\frac{\partial \left\|\mathbf{A}\right\|}{\partial \mathbf{A}} = \frac{\partial}{\partial \mathbf{A}}\left[(\mathbf{A} : \mathbf{A})^{1/2}\right] = \frac{1}{2}(\mathbf{A} : \mathbf{A})^{-1/2}(2\mathbf{A}) = \frac{\mathbf{A}}{\left\|\mathbf{A}\right\|}$$

Therefore, we have

$$\frac{\partial \mathbf{N}}{\partial \mathbf{A}} = \frac{1}{\|\mathbf{A}\|}\left(\mathbf{I} - \mathbf{N} \otimes \mathbf{N}\right)$$

P1.7 Through direct calculation of a rank-2 tensor, show that the following identity $e_{rst}\det[\mathbf{A}] = e_{ijk}A_{ir}A_{js}A_{kt}$ is true

Solution: In the index notation, (r, s, t) are real indices, while (i, j, k) are dummy indices. Since (r, s, t) only appears in the permutation symbol, it is enough to show the cases of even and odd permutation. Consider the following case of even permutation: (r, s, t) = (1, 2, 3). In such a case, non-zero components of the right-hand side can be written as

$$\begin{aligned} e_{ijk}A_{i1}A_{j2}A_{k3} &= e_{123}A_{11}A_{22}A_{33} + e_{132}A_{11}A_{32}A_{23} \\ &+ e_{231}A_{21}A_{32}A_{13} + e_{213}A_{21}A_{12}A_{33} \\ &+ e_{312}A_{31}A_{12}A_{23} + e_{321}A_{31}A_{22}A_{13} \end{aligned}$$

In the above equation, we have $e_{123} = e_{231} = e_{312} = 1$ and $e_{132} = e_{213} = e_{321} = -1$. Therefore, the above equation becomes

$$e_{ijk}A_{i1}A_{j2}A_{k3} = A_{11}(A_{22}A_{33} - A_{32}A_{23}) + A_{21}(A_{32}A_{13} - A_{12}A_{33}) + A_{31}(A_{12}A_{23} - A_{22}A_{13})$$

which is the definition of $\det[\mathbf{A}]$. By following a similar approach, it can be shown that the odd permutation of (r, s, t) will yield $-\det[\mathbf{A}]$.

P1.8 For a vector $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ and its norm $r = |\mathbf{r}|$, prove $\nabla \cdot (r\mathbf{r}) = 4r$.

Solution: From the product rule,

$$\nabla \cdot (r\mathbf{r}) = \nabla r \cdot \mathbf{r} + r\nabla \cdot \mathbf{r}$$

Now consider

$$(\nabla r)_i = \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} = \frac{1}{2(x_k x_k)^{1/2}}\frac{\partial}{\partial x_i}(x_j x_j) = \frac{1}{2r}\left(\frac{\partial x_j}{\partial x_i}x_j + x_j\frac{\partial x_j}{\partial x_i}\right) = \frac{1}{r}\delta_{ij}x_j = \frac{x_i}{r}$$

Therefore,

$$\nabla \cdot (r\mathbf{r}) = \nabla r \cdot \mathbf{r} + r\nabla \cdot \mathbf{r} = \frac{x_i}{r}x_i + r\frac{\partial x_i}{\partial x_i} = \frac{r^2}{r} + 3r = 4r$$

This completes the proof.

P1.9 A velocity gradient is decomposed into symmetric and skew parts, $\nabla \mathbf{v} = \mathbf{d} + \boldsymbol{\omega}$, where

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad \omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Show that

(a) For a symmetric stress tensor, $\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \mathbf{d}$.

(b) $w_{ij} = \dfrac{1}{2} e_{ijk} e_{mnk} \dfrac{\partial v_m}{\partial x_n}$

Solution:
(a) From Prob. 1.2, Since stress tensor is symmetric, $\boldsymbol{\sigma} : \boldsymbol{\omega} = 0$. Therefore, it is obvious that $\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \boldsymbol{\omega} + \boldsymbol{\sigma} : \mathbf{d} = \boldsymbol{\sigma} : \mathbf{d}$.

(b) The direct substitution method can be used to show the identity. We will show the case when $i = 1, j = 2$. The other cases can also be shown in the same way. Knowing that the permutation symbol becomes zero when indices are repeated, in this case the only nonzero situation happens when $k = 3$. For the second permutation symbol, the only non-zero situations are $m = 1, n = 2$ and $m = 2, n = 1$, where the former is even permutation and the latter is odd permutation. Therefore,

$$w_{12} = \frac{1}{2} e_{123} e_{mn3} \frac{\partial v_m}{\partial x_n} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right)$$

Other cases can also be shown in the same way.

P1.10 A symmetric rank four tensor is defined by $\mathbf{D} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbf{I}$ where $\mathbf{1} = [\delta_{ij}]$ is a unit tensor of rank-two and $\mathbf{I} = \frac{1}{2}[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]$ is a symmetric unit tensor of rank-four. When $\mathbf{E}$ is an arbitrary symmetric rank-two tensor, calculate $\mathbf{S} = \mathbf{D} : \mathbf{E}$ in terms of $\mathbf{E}$.

Solution: Using index notation, the contraction can be written as

$$S_{ij} = D_{ijkl} E_{kl} = \left[\lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\right] E_{kl}$$

Since the Kronecker-delta symbol replaces indices, the above equation can be simplified as

$$S_{ij} = D_{ijkl} E_{kl} = \lambda E_{kk} \delta_{ij} + \mu(E_{ij} + E_{ji}) = \lambda E_{kk} \delta_{ij} + 2\mu E_{ij}$$

In the tensor notation, the above relation can be written as

$$\mathbf{S} = \mathbf{D} : \mathbf{E} = \lambda tr(\mathbf{E}) \mathbf{1} + 2\mu \mathbf{E}$$

P1.11 Using integration by parts, calculate $I = \int x\cos(x)\,\mathrm{d}x$.

Solution: Let $u = x$ and $v' = \cos(x)$. Then

$$\begin{aligned}\int x\cos(x)\,\mathrm{d}x &= \int uv'\,\mathrm{d}x \\ &= uv - \int u'v\,\mathrm{d}x \\ &= x\sin(x) - \int \sin(x)\,\mathrm{d}x \\ &= x\sin(x) + \cos(x) + C\end{aligned}$$

P1.12 Using integration by parts, calculate $I = \int e^x \cos(x)\,\mathrm{d}x$.

Solution: Let $u = \cos(x)$ and $v' = e^x$. Then

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \cos(x) + \int e^x \sin(x)\,\mathrm{d}x$$

Now, to evaluate the second terms on the right-hand side using additional integration by parts with $u = \sin(x)$ and $v' = e^x$, as

$$\int e^x \sin(x)\,\mathrm{d}x = e^x \sin(x) - \int e^x \cos(x)\,\mathrm{d}x$$

Therefore, putting these together, we have

$$\int e^x \cos(x)\,\mathrm{d}x = e^x \cos(x) + e^x \sin(x) - \int e^x \cos(x)\,\mathrm{d}x$$

After rearranging, the original integral can be obtained as

$$\int e^x \cos(x)\,\mathrm{d}x = \frac{1}{2}(e^x \cos(x) + e^x \sin(x)) + C$$

P1.13 Calculate the surface integral of the vector function $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2$ over the portion of the surface of the unit sphere, $S : x^2 + y^2 + z^2 = 1$, above the xy plane; i.e., $z \geq 0$.

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

Solution: If we close the surface of integration by adding the portion of the xy plane which spans the hemisphere, we notice that the surface integral of $\mathbf{F}$ over the added surface is zero, since

$$\mathbf{F}\cdot\mathbf{n} = \mathbf{F}\cdot(-\mathbf{e}_3) = 0$$

over this area. Thus, the divergence theorem states that we may calculate the required surface integral of $\mathbf{F}$ by evaluating

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V$$

where V is the volume interior of the hemisphere. Since $\nabla\cdot\mathbf{F} = 2$, the result is merely twice the volume of the unit hemisphere, or $4\pi/3$.

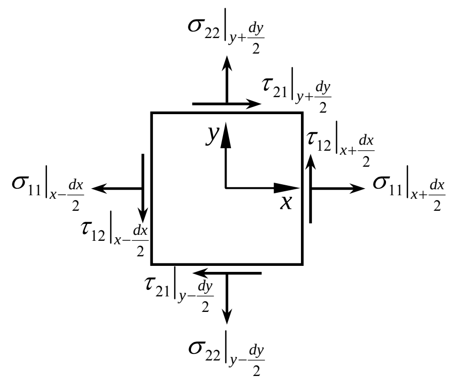
P1.14 Evaluate the surface integral of a vector, $\mathbf{F} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, over the closed surface of the cube bounded by the planes, $x = \pm 1, y = \pm 1, z = \pm 1$, using the divergence theorem.

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

Solution: Using the divergence theorem and $\nabla\cdot\mathbf{F} = 3$,

$$\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \iint_V \nabla\cdot\mathbf{F}\,\mathrm{d}V = \iint_V 3\,\mathrm{d}V = 24$$

P1.15 Consider a unit-depth (in *z*-axis) infinitesimal element as shown in the figure. Using force equilibrium, derive the governing differential equation in two-dimension (equilibrium in *x*- and *y*-directions). Assume that a uniform body force, $\mathbf{f}^B = [f_1^B, f_2^B]$, is applied to the infinitesimal element.



Solution: Equilibrium in the *x*–direction yields the following equation:

$$\left(\sigma_{11}\Big|_{x+\frac{dx}{2}}\right)dy - \left(\sigma_{11}\Big|_{x-\frac{dx}{2}}\right)dy + \left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)dx - \left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)dx + f_1^B dxdy = 0$$

If the first-order Taylor series expansion is used to represent stresses on the surfaces of the rectangle in terms of stresses at the center, the first two terms in the above equation can be approximated by

$$\left(\sigma_{11}\Big|_{x+\frac{dx}{2}}\right)dy - \left(\sigma_{11}\Big|_{x-\frac{dx}{2}}\right)dy$$
$$= \left(\sigma_{11}\Big|_{x} + \frac{\partial\sigma_{11}}{\partial x}\frac{dx}{2}\right)dy - \left(\sigma_{11}\Big|_{x} - \frac{\partial\sigma_{11}}{\partial x}\frac{dx}{2}\right)dy = \frac{\partial\sigma_{11}}{\partial x}dxdy$$

Similarly, the next two terms can be approximated by

$$\left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)dx - \left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)dx$$
$$= \left(\tau_{21}\Big|_{y} + \frac{\partial\tau_{21}}{\partial y}\frac{dy}{2}\right)dx - \left(\tau_{21}\Big|_{y} - \frac{\partial\tau_{21}}{\partial y}\frac{dy}{2}\right)dx = \frac{\partial\tau_{21}}{\partial y}dxdy$$

By substituting these two equations into the original equation, we obtain an equilibrium equation in the *x*–direction as

$$\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\tau_{21}}{\partial y} + f_1^B = 0$$

Similarly, equilibrium in the *y*-direction yields the following equation:

$$\frac{\partial\tau_{12}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + f_2^B = 0$$

P1.16 In the above unit-depth (in z-axis) infinitesimal element, show that the stress tensor is symmetric using moment equilibrium.

Solution: Moment equilibrium with respect to the center of the element becomes

$$\left(\tau_{12}\Big|_{x+\frac{dx}{2}}\right)\frac{dxdy}{2} + \left(\tau_{12}\Big|_{x-\frac{dx}{2}}\right)\frac{dxdy}{2} - \left(\tau_{21}\Big|_{y+\frac{dy}{2}}\right)\frac{dxdy}{2} - \left(\tau_{21}\Big|_{y-\frac{dy}{2}}\right)\frac{dxdy}{2} = 0$$

If the first-order Taylor series expansion is used to represent stresses on the surfaces of the rectangle in terms of stresses at the center,

$$\tau_{12}dxdy - \tau_{21}dxdy = 0$$

Thus, the stress tensor is symmetric. The same relation can be shown for 3-D stress tensor.

P1.17 The principal stresses at a point in a body are given by $\sigma_1 = 4, \sigma_2 = 2, \sigma_3 = 1$, and the principal directions of the first two principal stresses are given by $\mathbf{n}^{(1)} = \frac{1}{\sqrt{2}}(0,1,-1)$ and $\mathbf{n}^{(2)} = \frac{1}{\sqrt{2}}(0,1,1)$. Determine the state of stress at the point; i.e., 6 components of stress tensor.

Solution:
Since the three principal directions are mutually orthogonal, the third principal direction can be calculated by using the cross-product of the two principal directions, as

$$\mathbf{n}^{(3)} = \mathbf{n}^{(1)} \times \mathbf{n}^{(2)} = (1, 0, 0)$$

Since these three principal directions are mutually orthogonal, they can be considered as a basis of coordinate system. In this new coordinate system, the stress tensor will only have diagonal components, which is the same as the three principal stresses. Then, the transformation between the two coordinate systems for a rank-2 tensor can be written as

$$[\boldsymbol{\sigma}]_{123} = [\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\sigma}]_{xyz}[\mathbf{Q}]$$

where $[\mathbf{Q}] = [\mathbf{n}^{(1)} \quad \mathbf{n}^{(2)} \quad \mathbf{n}^{(3)}]$ is the orthogonal transformation matrix between the two coordinate systems. Using the property that the inverse of an orthogonal matrix is the same as the transpose, the reverse relationship can be obtained as

$$[\boldsymbol{\sigma}]_{xyz} = [\mathbf{Q}][\boldsymbol{\sigma}]_{123}[\mathbf{Q}]^{\mathrm{T}}$$

Or,

$$[\boldsymbol{\sigma}]_{xyz} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

The last matrix defines all 6 components of stress tensor.

P1.18 Find the principal stresses and the corresponding principal stress directions for the following cases of plane stress:

(a) $\sigma_{11} = 40$ MPa, $\sigma_{22} = 0$ MPa, $\sigma_{12} = 80$ MPa
(b) $\sigma_{11} = 140$ MPa, $\sigma_{22} = 20$ MPa, $\sigma_{12} = -60$ MPa
(c) $\sigma_{11} = -120$ MPa, $\sigma_{22} = 50$ MPa, $\sigma_{12} = 100$ MPa

Solution:
(a) The stress matrix becomes

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} 40 & 80 \\ 80 & 0 \end{bmatrix} \mathrm{MPa}$$

To find the principal stresses, the standard eigen value problem can be written as

$$\left[\boldsymbol{\sigma} - \sigma\mathbf{I}\right]\left\{\mathbf{n}\right\} = 0$$

The above problem will have non-trivial solution when the determinant of the coefficient matrix becomes zero:

$$\begin{vmatrix} \sigma_{xx} - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \sigma \end{vmatrix} = \begin{vmatrix} 40 - \sigma & 80 \\ 80 & 0 - \sigma \end{vmatrix} = 0$$

The equation of the determinant becomes:

$$\big(\big(40 - \sigma\big) \cdot -\sigma\big) - \big(80 \cdot 80\big) = \sigma^2 - 40\sigma - 6400 = 0$$

The above quadratic equation yields two principal stresses, as

$$\sigma_1 = 102.46\,\mathrm{MPa} \text{ and } \sigma_2 = -62.46\,\mathrm{MPa}\,.$$

To determine the orientation of the first principal stresses, substitute σ_1 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 - 102.46 & 80 \\ 80 & 0 - 102.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Since the determinant is zero, two equations are not independent

$$62.46 \cdot n_x = 80 \cdot n_y \text{ and } 80 \cdot n_x = -102.46 \cdot n_y\,.$$

Thus, we can only get the relation between n_x and n_y. Then using the condition $|\mathbf{n}| = 1$ we obtain

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.788 \\ 0.615 \end{Bmatrix}$$

To determine the orientation of the second principal stress, substitute σ_2 in the original eigen value problem to obtain

$$\begin{bmatrix} 40 + 62.46 & 80 \\ 80 & 0 + 62.46 \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$102.46 \cdot n_x = -80 \cdot n_y \text{ and } 80 \cdot n_x = -62.46 \cdot n_y\,.$$

Using similar procedures as above, the eigen vector of σ_2 can be obtained as

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.615 \\ -0.788 \end{Bmatrix}$$

Note that if **n** is a principal direction, −**n** is also a principal direction

(b) Repeat the procedure in (a) to obtain

$$\sigma_1 = 164.85\,\mathrm{MPa} \text{ and } \sigma_2 = -4.85\,\mathrm{MPa}\,.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} -0.924 \\ 0.383 \end{Bmatrix} \text{ and } \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0.383 \\ 0.924 \end{Bmatrix}$$

(c) Repeat the procedure in (a) to obtain

$$\sigma_1 = 96.24\,\mathrm{MPa} \quad \text{and} \quad \sigma_2 = -166.24\,\mathrm{MPa}\,.$$

$$\begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0.420 \\ 0.908 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}^{(2)} = \begin{Bmatrix} -0.908 \\ 0.420 \end{Bmatrix}$$

Note that for the case of plane stress $\sigma_3=0$ is also a principal stress and the corresponding principal stress direction is given by $n^{(3)}=(0,0,1)$

P1.19 Determine the principal stresses and their associated directions, when the stress matrix at a point is given by

$$[\boldsymbol{\sigma}] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \mathrm{MPa}$$

Solution:
Use Eq. (1.50) with the coefficients of $I_1=3$, $I_2=-3$, and $I_3=-1$,

$$\lambda^3 - 3\lambda^2 - 3\lambda + 1 = 0$$

By solving the above cubic equation,

$$\sigma_1 = 3.73\,\mathrm{MPa}, \quad \sigma_2 = 0.268\,\mathrm{MPa}, \quad \sigma_3 = -1.00\,\mathrm{MPa}$$

(a) Principal direction corresponding to σ_1:

$$\begin{aligned}
(1-3.7321)n_x^1 + n_y^1 + n_z^1 &= 0 \\
n_x^1 + (1-3.7321)n_y^1 + 2n_z^1 &= 0 \\
n_x^1 + 2n_y^1 + (1-3.7321)n_z^1 &= 0
\end{aligned}$$

Solving the above equations with $|\mathbf{n}^1| = 1$ yields

$$\mathbf{n}^1 = \{\pm 0.4597, \pm 0.6280, \pm 0.6280\}$$

(b) Principal direction corresponding to σ_2:

$$\begin{aligned}
(1-0.2679)n_x^1 + n_y^1 + n_z^1 &= 0 \\
n_x^2 + (1-0.2679)n_y^2 + 2n_z^2 &= 0 \\
n_x^2 + 2n_y^2 + (1-0.2679)n_z^2 &= 0
\end{aligned}$$

Solving the above equations with $|\mathbf{n}^2| = 1$ yields

$$\mathbf{n}^2 = \{\pm 0.8881, \mp 0.3251, \mp 0.3251\}$$

(c) Principal direction corresponding to σ_3:

$$
\begin{aligned}
(1+1)n_x^3 + n_y^3 + n_z^3 &= 0 \\
n_x^3 + (1+1)n_y^3 + 2n_z^3 &= 0 \\
n_x^3 + 2n_y^3 + (1+1)n_z^3 &= 0
\end{aligned}
$$

Solving the above equations with $|\mathbf{n}^2| = 1$ yields

$$\mathbf{n}^3 = \{0, \pm 0.7071, \mp 0.7071\}$$

P1.20 Let $x'y'z'$ coordinate system be defined using the three principal directions obtained from Problem P1.19. Determine the transformed stress matrix $[\boldsymbol{\sigma}]_{x'y'z'}$ in the new coordinates system.

Solution:
The three principal directions in Problem 6 can be used for the coordinate transformation matrix:

$$\left[\mathbf{N}\right] = \begin{bmatrix} n_x^{(1)} & n_x^{(2)} & n_x^{(3)} \\ n_y^{(1)} & n_y^{(2)} & n_y^{(3)} \\ n_z^{(1)} & n_z^{(2)} & n_z^{(3)} \end{bmatrix} = \begin{bmatrix} 0.460 & -0.888 & 0 \\ 0.628 & 0.325 & 0.707 \\ 0.628 & 0.325 & -0.707 \end{bmatrix}$$

To determine the stress components in the new coordinates we use Eq. (1.30):

$$\left[\boldsymbol{\sigma}\right]_{x'y'z'} = \left[\mathbf{N}\right]^T \left[\boldsymbol{\sigma}\right]\left[\mathbf{N}\right] = \begin{bmatrix} 3.732 & 0 & 0 \\ 0 & .268 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Note that the transformed stress matrix is a diagonal matrix with the original principal stresses on the diagonal.

P1.21 The stress-strain relationship for three-dimensional isotropic solid is given as $\sigma_{ij} = \left[K\delta_{ij}\delta_{kl} + 2\mu(\delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl})\right]\varepsilon_{kl}$ where K is the bulk modulus and μ is the shear modulus. In practice, stress and strain are written in the vector forms such that $\{\boldsymbol{\sigma}\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{12}\}^{\mathrm{T}}$ and $\{\boldsymbol{\varepsilon}\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \gamma_{12}, \gamma_{23}, \gamma_{12}\}^{\mathrm{T}}$. Then, the stress-strain can be written as $\{\boldsymbol{\sigma}\} = [\mathbf{D}]\{\boldsymbol{\varepsilon}\}$. Write the expression of 6x6 elasticity matrix $[\mathbf{D}]$ in terms of K and μ.

Solution: Based on the arrangement of stress and strain vector, the indices are arranged as

$$[\mathbf{D}] = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & \frac{1}{2}D_{1112} & \frac{1}{2}D_{1123} & \frac{1}{2}D_{1113} \\ D_{2211} & D_{2222} & D_{2233} & \frac{1}{2}D_{2212} & \frac{1}{2}D_{2223} & \frac{1}{2}D_{2213} \\ D_{3311} & D_{3322} & D_{3333} & \frac{1}{2}D_{3312} & \frac{1}{2}D_{3323} & \frac{1}{2}D_{3313} \\ \frac{1}{2}D_{1211} & \frac{1}{2}D_{1222} & \frac{1}{2}D_{1233} & \frac{1}{2}D_{1212} & \frac{1}{2}D_{1223} & \frac{1}{2}D_{1213} \\ \frac{1}{2}D_{2311} & \frac{1}{2}D_{2322} & \frac{1}{2}D_{2333} & \frac{1}{2}D_{2312} & \frac{1}{2}D_{2323} & \frac{1}{2}D_{2313} \\ \frac{1}{2}D_{1311} & \frac{1}{2}D_{1322} & \frac{1}{2}D_{1333} & \frac{1}{2}D_{1312} & \frac{1}{2}D_{1323} & \frac{1}{2}D_{1313} \end{bmatrix}$$

Note that the components corresponding to shear strains are divided by two because $\gamma_{ij} = 2\varepsilon_{ij}$. Due to Kronecker-delta symbol, many components are zero. Non-zero components are

$$D_{1111} = D_{2222} = D_{3333} = K + \tfrac{4}{3}\mu$$

$$D_{1122} = D_{1133} = D_{2211} = D_{2233} = D_{3311} = D_{3322} = K - \tfrac{2}{3}\mu$$

$$D_{1212} = D_{2323} = D_{1313} = 2\mu$$

Therefore, the elasticity matrix can be written as

$$[\mathbf{D}] = \begin{bmatrix} K + \frac{4}{3}\mu & K - \frac{2}{3}\mu & K - \frac{2}{3}\mu & 0 & 0 & 0 \\ K - \frac{2}{3}\mu & K + \frac{4}{3}\mu & K - \frac{2}{3}\mu & 0 & 0 & 0 \\ K - \frac{2}{3}\mu & K - \frac{2}{3}\mu & K + \frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

P1.22 For steel, the following material data are applicable: Young's modulus E = 207 GPa and shear modulus G = 80 GPa. For the strain matrix at a point shown below, determine the symmetric 3×3 stress matrix.

$$[\varepsilon] = \begin{bmatrix} 0.003 & 0 & -0.006 \\ 0 & -0.001 & 0.003 \\ -0.006 & 0.003 & 0.0015 \end{bmatrix}$$

Solution:

From Eq. (1.81) the elasticity matrix becomes

$$[\mathbf{D}] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}-\nu \end{bmatrix}$$

From the relation $G = E / 2(1+\nu)$, we calculate $\nu = (E / 2G) - 1 = 0.294$.

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = [\mathbf{D}] \begin{Bmatrix} 0.003 \\ -0.001 \\ 0.0015 \\ 0.006 \\ -0.012 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.879 \\ 0.239 \\ 0.639 \\ 0.480 \\ -0.960 \\ 0 \end{Bmatrix} \text{GPa}$$

In the matrix notation

$$\left[\boldsymbol{\sigma}\right] = \begin{bmatrix} 0.879 & 0 & -0.960 \\ 0 & 0.239 & 0.480 \\ -0.960 & 0.480 & 0.639 \end{bmatrix} \text{GPa}$$

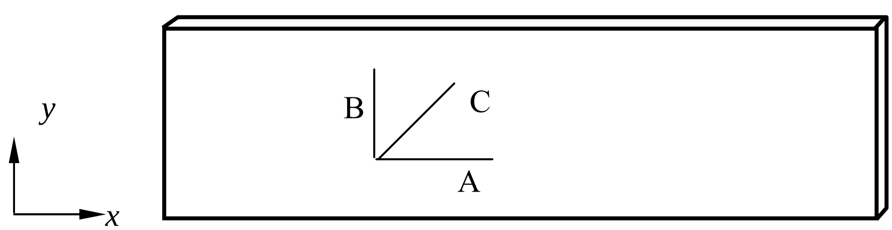
P1.23 A strain rosette consisting of three strain gages was used to measure the strains at a point in a thin-walled plate. The measured strains in the three gages are: $\varepsilon_A = 0.001$, $\varepsilon_B = -0.0006$, and $\varepsilon_C = 0.0007$. Not that Gage C is at 45° with respect to the *x*-axis.

(a) Determine the complete state of strains and stresses (all six components) at that point. Assume $E = 70$ GPa, and $\nu = 0.3$.

(b) What are the principal strains and their directions?

(c) What are the principal stresses and their directions?

(d) Show that the principal strains and stresses satisfy the stress-strain relations.



Solution:

(a) From figure it is obvious $\varepsilon_{xx} = \varepsilon_A = 0.001$ and $\varepsilon_{yy} = \varepsilon_B = -0.0006$. Shear strain can be found using the strain version of the stress transformation relation in Eq. (1.38). The 2-D version becomes

$$\varepsilon_{nn} = \varepsilon_{xx} n_x^2 + \varepsilon_{yy} n_y^2 + \gamma_{xy} n_x n_y$$

where $n_x = \cos(45°)$ and $n_y = \sin(45°)$. Thus,

$$\varepsilon_C = \varepsilon_{nn}(45°) = \varepsilon_{xx} \cos^2 45 + \varepsilon_{yy} \sin^2 45 + \gamma_{xy} \sin 45 \cos 45 = 0.0007$$

By solving the above equation, we obtain $\gamma_{xy} = 0.001$. Since the strain rosette only measures plane stress state, ε_{zz} is unknown. But, there is no shear strain in the z-direction, $\gamma_{xz} = \gamma_{yz} = 0$. In order to calculate the unknown stress ε_{zz}, we use the constitutive relation for plane stress. Since the plate is in a state of plane stress, $\sigma_{zz} = \tau_{xz}$

= τ_{yz} = 0. Other stresses can be obtained from stress-strain relations for plane stress conditions as shown below:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu \\ \nu & 1 \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \end{Bmatrix} = \begin{Bmatrix} 63.1 \\ -23.1 \end{Bmatrix} \mathrm{MPa}$$

$$\tau_{xy} = G\gamma_{xy} = 13.5\,\mathrm{MPa}$$

For plane stress condition the through-the-thickness strain is obtained, as

$$\varepsilon_{zz} = \frac{-\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) = -0.000171$$

(b) For a state of plane stress, $\varepsilon_{zz} = -0.000171$ is a principal stress and the z-axis (0,0,1) is the corresponding principal strain direction. The other two principal strains can be found from the eigen value problem in 2D strain state:

$$[\boldsymbol{\varepsilon} - \lambda\mathbf{I}]\{\mathbf{n}\} = \begin{bmatrix} \varepsilon_{xx} - \lambda & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} - \lambda \end{bmatrix}\begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Two principal strains are calculated from the condition that the determinant of the coefficient matrix is zero: $(\varepsilon_{xx} - \lambda)(\varepsilon_{yy} - \lambda) - \varepsilon_{xy}^2 = 0$. The solution of the quadratic equation becomes λ_1 = 0.0011 and λ_2 = −0.0007. Thus, the three principal strains are ε_1 = 0.0011, ε_2 = −0.000171, and ε_3 = −0.0007. Two principal directions can be obtained from the original eigen value problem. Adding z-axis, the three principal directions are

$$\mathbf{n}^1 = \begin{Bmatrix} -0.961 \\ -0.276 \\ 0 \end{Bmatrix}, \quad \mathbf{n}^2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \mathbf{n}^3 = \begin{Bmatrix} 0.276 \\ -0.961 \\ 0 \end{Bmatrix}$$

(c) Principal stresses

For plane stress condition, σ_z = 0 is a principal stress and the z-axis (0,0,1) is the corresponding principal direction. The other principal stresses and the directions can be found by solving the following eigen value problem:

$$\left[\boldsymbol{\sigma} - \lambda\mathbf{I}\right]\{\mathbf{n}\} = \begin{bmatrix} \sigma_{xx} - \lambda & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} - \lambda \end{bmatrix}\begin{Bmatrix} n_x \\ n_y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Two principal stresses are calculated from the condition that the determinant of the coefficient matrix is zero: $(\sigma_{xx} - \lambda)(\sigma_{yy} - \lambda) - \tau_{xy}^2 = 0$. The solution of the quadratic equation becomes λ_1 = 70.8 and λ_2 = −30.8. Thus, the three principal stresses are σ_1 = 70.8 MPa, σ_2 = 0.0 MPa, and σ_3 = −30.8 MPa. Two principal directions can be obtained from the original eigen value problem. Adding z-axis, the three principal directions are

$$\mathbf{n}^1 = \begin{Bmatrix} -0.961 \\ -0.276 \\ 0 \end{Bmatrix}, \quad \mathbf{n}^2 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}, \quad \mathbf{n}^3 = \begin{Bmatrix} 0.276 \\ -0.961 \\ 0 \end{Bmatrix}$$

For isotropic materials, principal stress directions and principal strain directions are the same.

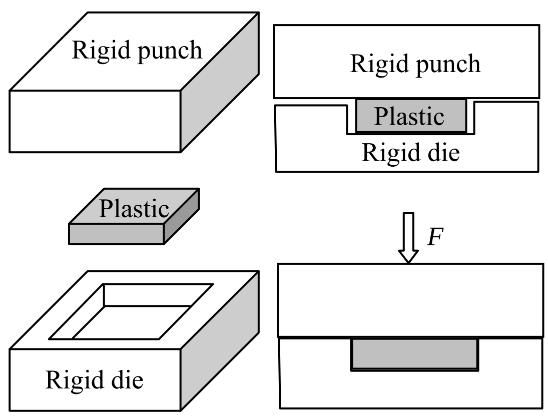
(d) Principal Stress-strain relations

From Eq. (1.55), the stress-strain relation can be written as

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{Bmatrix} 0.0011 \\ -0.0002 \\ -0.0007 \end{Bmatrix}$$

Also, all shear strains and stresses are zero because they are in the principal directions. Thus, the stress-strain relation satisfies in the principal stresses and strains.

P1.24 A rectangular plastic specimen of size 100×100×10 mm^3 is placed in a rectangular metal cavity. The dimensions of the cavity are 101×101×9 mm^3. The plastic is compressed by a rigid punch until it is completely inside the cavity. Due to Poisson effect, the plastic also expands in the *x* and *y* directions and fills the cavity. Calculate all stress and strain components and the force exerted by the punch. Assume there is no friction between all contacting surfaces. The metal cavity is rigid. Elastic constants of the plastic are E = 10 GPa, ν = 0.3.



Solution:

The strains in the specimen are calculated as the ratio of change in length to original length.

$$\varepsilon_{zz} = \frac{(9-10)}{10} = -0.1,\ \varepsilon_{xx} = \varepsilon_{yy} = \frac{(101-100)}{100} = +0.01$$

We have assumed that the plastic expands laterally and fill the cavity completely. If it does not, then we will get positive values for σ_{xx} and/or σ_{yy}, which will indicate that our assumption is wrong. Then we can assume σ_{xx} and/or σ_{yy} = 0, and redo the problem and obtain corresponding strains ε_{xx} and/or ε_{yy} which will be less than that calculated above.

Since there is no friction between contacting surfaces, all shear stresses and hence all shear strains will be identically equal to zero.

The normal stresses can be obtained from three-dimensional stress strain relations:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & \nu\\ \nu & (1-\nu) & \nu\\ \nu & \nu & (1-\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{Bmatrix}$$

Substituting for the strains and elastic constants E and ν we obtain the stresses as

$$\{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xx}\} = \{-385 \quad -385 \quad -1{,}231\}\ \mathrm{MPa}$$

Since σ_{xx} and σ_{yy} are negative (compressive), our initial assumption about the strains is correct. The punch force is obtained from σ_z and the area of cross section:

$$F = A\sigma_z = 0.1 \times 0.1 \times 1{,}231 = \boxed{12.31\,\mathrm{MN}}$$

P1.25 Repeat Problem P1.24 with elastic constants of the plastic as E = 10 GPa and ν = 0.485.

Solution:

The strains in the plastic specimen are calculated as the ratio of change in length to original length.

$$\varepsilon_z = \frac{(9-10)}{10} = -0.1,\ \varepsilon_x = \varepsilon_y = \frac{(101-100)}{100} = +0.01$$

We have assumed that the plastic expands laterally and fill the cavity completely. If it does not, then we will get positive values for σ_{xx} and/or σ_{yy}, which will indicate that our assumption is wrong. Then we can assume σ_{xx} and/or σ_{yy} = 0, and reiterate the problem and obtain corresponding strains ε_{xx} and/or ε_{yy} which will be less than that calculated above.

Since there is no friction between contacting surfaces, all shear stresses and hence all shear strains will be identically equal to zero.

The normal stresses can be obtained from three-dimensional stress strain relations:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{Bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}(1-\nu) & \nu & \nu\\ \nu & (1-\nu) & \nu\\ \nu & \nu & (1-\nu)\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\end{Bmatrix}$$

Substituting for the strains and elastic constants E and ν we obtain the stresses as

$$\{\sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xx}\} = \{-8{,}642 \quad -8{,}642 \quad -9{,}383\}\ \mathrm{MPa}$$

Since σ_{xx} and σ_{yy} are negative (compressive), our initial assumption about the strains is correct. The punch force is obtained from σ_{zz} and the area of cross section:

$$F = A\sigma_z = 0.1 \times 0.1 \times 9{,}383 = \boxed{93.83\,\mathrm{MN}}$$

Note: Punch force for this problem is almost 8 times that for Problem 24. The increase is due to Poisson's ratio. As the material compressibility decreases, Poisson's ratio increases. For example, as $\nu \to 0.5$ the material becomes incompressible, i.e., its volume cannot be changed, and the stresses become unbounded. Note the term $(1-2\nu)$ in the denominator of the above constitutive relation.

P1.26 The strain energy and work done by applied load are given in the following equations. When the solution is expressed by $u(x) = c_1 x + c_2 x^2$, calculate the solution using the principle of minimum potential energy.

$$U = \frac{1}{2}\int_0^1 (u')^2 \,\mathrm{d}x, \quad W = \int_0^1 u \,\mathrm{d}x + u(1)$$

Solution: From the given form of displacement, the virtual displacement can be expressed as $\overline{u}(x) = \overline{c}_1 x + \overline{c}_2 x^2$. The variation of the potential energy can be written as

$$\begin{aligned} \delta\Pi &= \int_0^1 u'\overline{u}' \,\mathrm{d}x - \int_0^1 \overline{u} \,\mathrm{d}x - u(1) \\ &= \int_0^1 (c_1 + 2c_2 x)(\overline{c}_1 + 2\overline{c}_2 x) \,\mathrm{d}x - \int_0^1 (\overline{c}_1 x + \overline{c}_2 x^2) \,\mathrm{d}x - \overline{c}_1 - \overline{c}_2 \\ &= 0 \end{aligned}$$

The above variational equation must satisfy for all $\overline{u}(x) \in \mathbb{Z}$. Since the virtual displacement is expressed by $\overline{u}(x) = \overline{c}_1 x + \overline{c}_2 x^2$, it is possible that the above variational equation must satisfy for arbitrary coefficients $\overline{c}_1$ and $\overline{c}_2$. Since $\overline{c}_1$ and $\overline{c}_2$ are independent, those terms that contain them must vanish independently; that is,

$$\begin{aligned} \overline{c}_1(c_1 + c_2 - \tfrac{3}{2}) &= 0 \\ \overline{c}_2(c_1 + \tfrac{4}{3}c_2 - \tfrac{4}{3}) &= 0 \end{aligned}$$

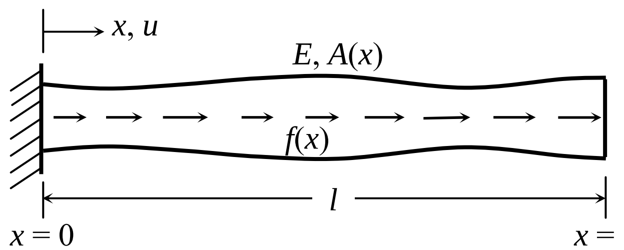
By solving the above two equations, we obtain $c_1 = 2$ and $c_2 = -\frac{1}{2}$. Thus, the solution becomes

$$u(x) = 2x - \frac{x^2}{2}$$

P1.27 The governing differential equation for the bar component in the figure is given as

$$\begin{aligned} -(EA(x)u_{,1})_{,1} &= f(x), \qquad x \in (0, l) \\ u(0) &= 0 \\ u_{,1}(l) &= 0, \end{aligned}$$

where the subscribed comma denotes differentiation with respect to the spatial coordinate, i.e., $u_{,1} = \mathrm{d}u/\mathrm{d}x$. Derive the weak form using the principle of virtual work.



Solution: The principle of virtual work can be obtained by multiplying the governing differential equation with an arbitrary function $\overline{u}$ (called virtual displacement) and then integrating over the domain as

$$\int_0^l u_{,1}\overline{u}_{,1}\,\mathrm{d}x - \int_0^l f\,\overline{u}\,\mathrm{d}x = \left[EAu_{,1}\overline{u}\right]\Big|_0^l,$$

where integration by parts is used once. The above equation is called the variational identity. Among arbitrary $\overline{u}$, let us choose those that satisfy the homogeneous essential boundary condition, that is, $\overline{u}(0) = 0$. Thus, the space of kinematically admissible displacements is defined as

$$\mathbb{Z} = \left\{ u \in H^1(0,l) \mid \; u(0) = 0 \right\},$$

where H^1 is the Sobolev space of the first order. Note that $\mathbb{Z}$ contains the homogeneous essential boundary condition but not the natural boundary condition. Since the derivative of the solution vanishes at $x = l$, the following variational equation can be obtained

$$\int_0^l EAu_{,1}\overline{u}_{,1}\,\mathrm{d}x = \int_0^l f\,\overline{u}\,\mathrm{d}x,$$

for all $\overline{u}$ in $\mathbb{Z}$. Note that the above variational problem is well defined for the integrable cross-sectional area $A(x)$ as well as for the continuous displacement function $u(x)$ whose first derivative is in $L_2(\Omega)$. Therefore, smoothness requirements for this variational problem are much less than for the classical differential equation.

For the homogeneous boundary condition, the solution space is the same as $\mathbb{Z}$. Therefore, the structural energy bilinear and load linear forms are defined as

$$a(u,\overline{u}) = \int_0^l EAu_{,1}\overline{u}_{,1}\,\mathrm{d}x$$

and

$$\ell(\overline{u}) = \int_0^l f\overline{u}\,\mathrm{d}x.$$

Then, the variational equation of the bar component can be represented using the energy bilinear and load linear forms as

$$a(u,\overline{u}) = \ell(\overline{u}), \quad \forall \overline{u} \in \mathbb{Z}.$$

Note that $a(\bullet,\bullet)$ is symmetrical with respect to its arguments.

P1.28 Derive the weak form of two-dimensional, steady-state heat transfer problem.

Solution: The governing differential equation of a steady-state heat transfer problem in two-dimension is

$$\frac{\partial}{\partial x_1}\left(k_1\frac{\partial T}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(k_2\frac{\partial T}{\partial x_2}\right)+Q=0 \quad \Leftrightarrow (k_1T_{,1})_{,1}+(k_2T_{,2})_{,2}+Q=0$$

For given boundary conditions, the space of kinematically admissible temperatures becomes

$$Z=\left\{\overline{T}\in[H^1(\Omega)]^2\,\middle|\,\overline{T}(x)=0 \text{ on } S_T\right\}$$

Multiplying the governing equation with a virtual temperature and integrating over the domain, we obtain the following equation:

$$\iint_\Omega \overline{T}[(k_1T_{,1})_{,1}+(k_2T_{,2})_{,2}+Q]\,\mathrm{d}\Omega=0 \quad \forall\overline{T}\in Z$$

After integrating by parts, we have

$$\begin{aligned}
&\iint_\Omega[(k_1T_{,1}\overline{T})_{,1}+(k_2T_{,2}\overline{T})_{,2}]\,\mathrm{d}\Omega-\iint_\Omega(k_1\overline{T}_{,1}T_{,1}+k_2\overline{T}_{,2}T_{,2})\,\mathrm{d}\Omega+\iint_\Omega\overline{T}Q\,\mathrm{d}\Omega=0\\
\Rightarrow&\int_S(k_1T_{,1}n_1\overline{T}+k_2T_{,2}n_2\overline{T})\,\mathrm{d}\Gamma-\iint_\Omega(k_1\overline{T}_{,1}T_{,1}+k_2\overline{T}_{,2}T_{,2})\,\mathrm{d}\Omega+\iint_\Omega\overline{T}Q\,\mathrm{d}\Omega=0\\
\Rightarrow&\int_S(-f_1n_1-f_2n_2)\overline{T}\,\mathrm{d}\Gamma-\iint_\Omega(k_1\overline{T}_{,1}T_{,1}+k_2\overline{T}_{,2}T_{,2})\,\mathrm{d}\Omega+\iint_\Omega\overline{T}Q\,\mathrm{d}\Omega=0\\
\Rightarrow&\int_{S_q}q_n\overline{T}\,\mathrm{d}\Gamma-\iint_\Omega(k_1\overline{T}_{,1}T_{,1}+k_2\overline{T}_{,2}T_{,2})\,\mathrm{d}\Omega+\iint_\Omega\overline{T}Q\,\mathrm{d}\Omega=0
\end{aligned}$$

In the above equation, we used the Green-Gauss theorem and the following properties:

$$\begin{aligned}f_1&=-k_1T_{,1}\\ f_2&=-k_2T_{,2}\end{aligned}$$

Thus, the weak form becomes

$$\iint_\Omega(k_1\overline{T}_{,1}T_{,1}+k_2\overline{T}_{,2}T_{,2})\,\mathrm{d}\Omega=\int_{S_q}q_n\overline{T}\,\mathrm{d}\Gamma+\iint_\Omega\overline{T}Q\,\mathrm{d}\Omega,\quad\forall\overline{T}\in Z$$

P1.29 Derive the weak form of simply-supported beam problem.

Solution: The governing differential equation of a beam becomes

$$EIv^{(4)}=f(x)$$

where $v^{(4)}$ is the forth-order derivative of deflection, and $f(x)$ is the distributed load. Boundary conditions are given as

$$v(0) = v(L) = 0, \quad EIv''(0) = EIv''(L) = 0$$

By including homogeneous essential boundary conditions, the space of kinematically admissible displacements becomes

$$Z = \left\{ \overline{v} \in H^2(0,L) \middle| \overline{v}(0) = \overline{v}(L) = 0 \right\}$$

The weak form can be obtained by multiplying the governing equation by a virtual deflection and integrating over the domain, as

$$\int_0^L [EIv^{(4)} - f]\overline{v}\,\mathrm{d}x = 0$$

After performing integration-by-parts twice for the first term, we have

$$\int_0^L EIv''\overline{v}''\,\mathrm{d}x - \int_0^L f\overline{v}\,\mathrm{d}x + [EIv'''\overline{v}]\Big|_0^L - [EIv''\overline{v}']\Big|_0^L = 0$$

Since the virtual displacements are zero at the both boundaries and the bending moments are zero at the both boundaries, the boundary terms in the above equation vanish. Thus, the weak form becomes

$$\int_0^L EIv''\overline{v}''\,\mathrm{d}x = \int_0^L f\overline{v}\,\mathrm{d}x, \quad \forall \overline{v} \in Z$$

P1.30 When the potential energy of P1.29 is given, derive the variational equation using the principle of minimum potential energy.

$$\Pi = \int_0^L \left(\frac{1}{2} EI(v_{,11})^2 - fv \right) \mathrm{d}x$$

Solution: It is clear that the potential energy is well defined as long as $u_{,11} \in L_2(0,l)$ and it does not require u to be $C^4(0,l)$, as in the original differential equation. Equating the first variation of Π to zero, in which the variation $\overline{u}(x)$ has the second-order derivative, $\overline{u}_{,11} \in L^2(0,l)$, and assuming that $\overline{u}$ satisfies the essential boundary conditions, we obtain

$$\begin{aligned} \delta\Pi &= \frac{d}{d\tau} \int_0^l \left[\frac{1}{2} EI(u_{,11} + \tau\overline{u}_{,11})^2 - f(u + \tau\overline{u}) \right] \mathrm{d}x \Big|_{\tau=0} \\ &= \int_0^l \left[EIu_{,11}\overline{u}_{,11} - f\overline{u} \right] \mathrm{d}x = 0. \end{aligned}$$

In order to make a consistent notation, the following energy bilinear and load linear forms are defined:

$$a(u,\overline{u}) = \int_0^l EIu_{,11}\overline{u}_{,11}\ \mathrm{d}x$$

and

$$\ell(\overline{u}) = \int_0^l f\overline{u}\ \mathrm{d}x.$$

Then, the variational equation can be written as

$$a(u,\overline{u}) = \ell(\overline{u}), \quad \forall \overline{u} \in \mathbb{Z}. \tag{0.1}$$

which is identical to the weak form obtained using the principle of virtual work.

P1.31 Derive the principle of virtual work for the simply-supported Kirchhoff plate element from the governing equation:

$$[D(u_{,11} + \nu u_{,22})]_{,11} + [D(u_{,22} + \nu u_{,11})]_{,22} + 2(1-\nu)[Du_{,12}]_{,12} = f$$

Solution: For a simply supported plate, the space of kinematically admissible displacements is

$$\mathbb{Z} = \left\{ u \in [H^2(\Omega)]^2 \middle| u = 0 \text{ on } \Gamma \right\}$$

The principle of virtual work can be obtained by multiplying the governing differential equation with a virtual displacements $\overline{u}$ and integrate over the domain as

$$\iint_\Omega \left\{ [D(u_{,11} + \nu u_{,22})]_{,11} + [D(u_{,22} + \nu u_{,11})]_{,22} + 2(1-\nu)[Du_{,12}]_{,12} \right\} \overline{u}\, \mathrm{d}\Omega = \iint_\Omega f\overline{u}\, \mathrm{d}\Omega$$

Applying the integration by parts once and using Green-Gauss theorem, we have

$$\begin{aligned} &\int_\Gamma \left\{ [D(u_{,11} + \nu u_{,22})]_{,1}\overline{u}n_1 + [D(u_{,22} + \nu u_{,11})]_{,2}\overline{u}n_2 + 2(1-\nu)[Du_{,12}]_{,1}\overline{u}n_2 \right\} \mathrm{d}\Gamma \\ &-\iint_\Omega \left\{ [D(u_{,11} + \nu u_{,22})]_{,1}\overline{u}_{,1} + [D(u_{,22} + \nu u_{,11})]_{,2}\overline{u}_{,2} + 2(1-\nu)[Du_{,12}]_{,1}\overline{u}_{,2} \right\} \mathrm{d}\Omega = \iint_\Omega f\overline{u}\, \mathrm{d}\Omega \end{aligned}$$

In the above equation, the first boundary integral term becomes

$$\int_\Gamma \left\{ [D(u_{,11} + \nu u_{,22})]_{,1}\overline{u}n_1 + [D(u_{,22} + \nu u_{,11})]_{,2}\overline{u}n_2 + 2(1-\nu)[Du_{,12}]_{,1}\overline{u}n_2 \right\} \mathrm{d}\Gamma = -\int_\Gamma \overline{u}Nu\, \mathrm{d}\Gamma$$

where Nu is the transverse shear force on the boundary. Applying the integration by parts again and using Green-Gauss theorem, we have

$$\begin{aligned} &-\int_\Gamma \overline{u}Nu\, \mathrm{d}\Gamma - \int_\Gamma \left\{ D(u_{,11} + \nu u_{,22})\overline{u}_{,1}n_1 + D(u_{,22} + \nu u_{,11})\overline{u}_{,2}n_2 + 2(1-\nu)Du_{,12}\overline{u}_{,2}n_1 \right\} \mathrm{d}\Gamma \\ &+\iint_\Omega \left\{ D(u_{,11} + \nu u_{,22})\overline{u}_{,11} + D(u_{,22} + \nu u_{,11})\overline{u}_{,22} + 2(1-\nu)Du_{,12}\overline{u}_{,12} \right\} \mathrm{d}\Omega = \iint_\Omega f\overline{u}\, \mathrm{d}\Omega \end{aligned}$$

where the second boundary integral becomes

$$-\int_{\Gamma}\left\{D(u_{,11}+\nu u_{,22})\overline{u}_{,1}n_1 + D(u_{,22}+\nu u_{,11})\overline{u}_{,2}n_2 + 2(1-\nu)Du_{,12}\overline{u}_{,2}n_1\right\}\mathrm{d}\Gamma = -\int_{\Gamma}\frac{\partial \overline{u}}{\partial n}Mu\,\mathrm{d}\Gamma$$

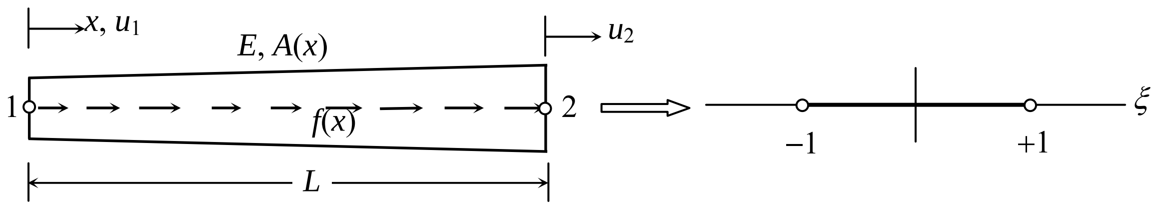
In addition, the domain integrals in the above equation can be converted to

$$\iint_{\Omega}\boldsymbol{\kappa}(\overline{u})^T\mathbf{C}^b\boldsymbol{\kappa}(u)\,\mathrm{d}\Omega = \iint_{\Omega}\overline{u}f\,\mathrm{d}\Omega + \int_{\Gamma}\overline{u}Nu\,\mathrm{d}\Gamma + \int_{\Gamma}\frac{\partial \overline{u}}{\partial n}Mu\,\mathrm{d}\Gamma$$

For the simply supported plate, $\overline{u} = Mu = 0$ on the boundary. Thus, the variational equation becomes

$$\iint_{\Omega}\boldsymbol{\kappa}(\overline{u})^T\mathbf{C}^b\boldsymbol{\kappa}(u)\,\mathrm{d}\Omega = \iint_{\Omega}\overline{u}f\,\mathrm{d}\Omega$$

P1.32 Consider a bar element as shown in the figure. The cross sectional areas are A_1 and A_2 at Nodes 1 and 2, respectively, and vary linearly. In addition, the gravitational acceleration is applied along the axial direction of the bar, such that the distributed load per unit length is $f(x) = \rho g A(x)$, where ρ is the density and g is gravitational acceleration. Construct the discrete variational equation for the element.



(a) Finite Element (b) Reference Element

Solution: The discrete variational equation of a bar element is

$$\overline{\mathbf{d}}^T\left[\int_0^L EA(x)\mathbf{B}^T\mathbf{B}\,\mathrm{d}x\right]\mathbf{d} = \overline{\mathbf{d}}^T\int_0^L \mathbf{N}^T f(x)\,\mathrm{d}x$$

In the problem statement, both cross-sectional area $A(x)$ and distributed load $f(x)$ are not constant. Since $f(x)$ is also a function of $A(x)$, it is necessary to integrate the area over the length of the element. Note that the area varies linearly between A_1 at Node 1 and A_2 at Node 2. Or, it can be considered that the cross-sectional area is interpolated using shape function, as

$$A(\xi) = N_1(\xi)A_1 + N_2(\xi)A_2$$

Then, the integral of the cross-sectional area will be

$$\int_0^L A(x)\,\mathrm{d}x = \int_{-1}^{1}(N_1(\xi)A_1 + N_2(\xi)A_2)\frac{L}{2}\mathrm{d}\xi = \frac{(A_1+A_2)L}{2}$$

Also, for the distributed load term, we need to calculate the following integrals:

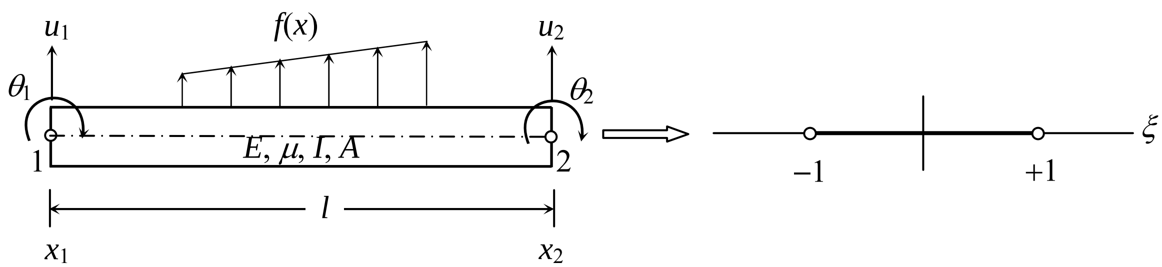
$$\int_0^L N_1(\xi)A(x)\,\mathrm{d}x = \int_{-1}^{1} N_1(\xi)(N_1(\xi)A_1 + N_2(\xi)A_2)\frac{L}{2}\mathrm{d}\xi = (\frac{2}{3}A_1 + \frac{1}{3}A_2)\frac{L}{2}$$
$$\int_0^L N_2(\xi)A(x)\,\mathrm{d}x = \int_{-1}^{1} N_2(\xi)(N_1(\xi)A_1 + N_2(\xi)A_2)\frac{L}{2}\mathrm{d}\xi = (\frac{1}{3}A_1 + \frac{2}{3}A_2)\frac{L}{2}$$

Then, the above discrete variational equation becomes $\overline{\mathbf{d}}^T\mathbf{kd} = \overline{\mathbf{d}}^T\mathbf{f}$, where the element stiffness matrix and nodal force vector are defined as

$$\mathbf{k} = \frac{E(A_1 + A_2)}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{f} = \frac{\rho g L}{6}\begin{Bmatrix} 2A_1 + A_2 \\ A_1 + 2A_2 \end{Bmatrix}$$

P1.33 For the Euler beam element shown in the figure, derive interpolation functions $N_I(\xi)$, stiffness matrix $\mathbf{k}$, nodal force vector $\mathbf{f}$. Assume uniformly distributed load $f(x) = f$. Note that the reference element is defined in the domain $\xi = [-1, 1]$.



(a) Finite Element (b) Reference Element

Solution: The mapping relation between the physical and reference elements is

$$x = \tfrac{1}{2}(1-\xi)x_1 + \tfrac{1}{2}(1+\xi)x_2$$

Thus, the Jacobian becomes

$$J = \frac{\mathrm{d}x}{\mathrm{d}\xi} = \tfrac{1}{2}(x_2 - x_1) = \frac{L}{2}$$

Since the Euler beam element has four DOFs, the transverse deflection can be assumed as

$$u(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

Accordingly, the rotational angle can be assumed as

$$\theta(\xi) = \frac{\mathrm{d}u(\xi)}{\mathrm{d}x} = \frac{\mathrm{d}u(\xi)}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x} = \frac{2}{L}(a_1 + 2a_2\xi + 3a_3\xi^2)$$

Now, we want to express the four coefficients in terms of four DOFs, by

$$
\begin{aligned}
u_1 &= u(-1) = a_0 - a_1 + a_2 - a_3 \\
u_2 &= u(1) = a_0 + a_1 + a_2 + a_3 \\
\theta_1 &= \frac{\mathrm{d}u(-1)}{\mathrm{d}x} = \frac{2}{L}(a_1 - 2a_2 + 3a_3) \\
\theta_2 &= \frac{\mathrm{d}u(1)}{\mathrm{d}x} = \frac{2}{L}(a_1 + 2a_2 + 3a_3)
\end{aligned}
$$

By solving the above equation for four coefficients, we have

$$
\begin{aligned}
u(\xi) = \tfrac{1}{4}(2 - 3\xi + \xi^3)u_1 + \tfrac{L}{8}(1 - \xi - \xi^2 + \xi^3)\theta_1 \\
+ \tfrac{1}{4}(2 + 3\xi - \xi^3)u_2 + \tfrac{L}{8}(-1 - \xi + \xi^2 + \xi^3)\theta_2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
N_1(\xi) &= \tfrac{1}{4}(2 - 3\xi + \xi^3) \\
N_2(\xi) &= \tfrac{L}{8}(1 - \xi - \xi^2 + \xi^3) \\
N_3(\xi) &= \tfrac{1}{4}(2 + 3\xi - \xi^3) \\
N_4(\xi) &= \tfrac{L}{8}(-1 - \xi + \xi^2 + \xi^3)
\end{aligned}
$$

The displacement-strain relation for the Euler beam element becomes

$$
\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}\left(\frac{2}{L}\right)^2 = \frac{4}{L}\left[\tfrac{3}{2}\xi \quad \tfrac{L}{8}(-2+6\xi) \quad -\tfrac{3}{2}\xi \quad \tfrac{L}{8}(2+6\xi)\right]\mathbf{d} = \mathbf{Bd}
$$

Then, the element stiffness matrix becomes

$$
\mathbf{k} = \int_{-1}^{1} EI\mathbf{B}^T\mathbf{B}J\,\mathrm{d}\xi = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}
$$

For the distributed load, the nodal force vector becomes

$$
\mathbf{f} = \int_{-1}^{1} \mathbf{N}^T fJ\,\mathrm{d}\xi = \begin{Bmatrix} fL/2 \\ fL^2/12 \\ fL/2 \\ -fL^2/12 \end{Bmatrix}
$$

Note that the element stiffness matrix and the force vector are the same with the case when the reference element domain $\xi = [0, 1]$ is used.

P1.34 Below is the governing differential equation of one-dimensional bar under uniformly distributed load. Using one bar element, calculate displacement at $x = L$ and

$x = \frac{1}{2}L$. Compare these displacements with that of exact calculation. (Note: exact solution can be calculated by integrating the differential equation twice).

$$\begin{aligned} -EAu_{,11} &= f, \qquad x \in (0, L) \\ u(0) &= 0 \\ u_{,1}(L) &= 0 \end{aligned}$$

Solution: From the textbook, the finite element equation becomes

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} fL/2 \\ fL/2 \end{Bmatrix}$$

Since Node 1 is fixed, the first row and the first column can be deleted, yielding the following displacement at Node 2

$$u_2 = u(L) = \frac{fL^2}{2EA}$$

Since $u_1 = 0$, the displacement at $x = \frac{1}{2}L$ can be calculated by

$$u(\tfrac{1}{2}) = N_1(\tfrac{1}{2})u_1 + N_2(\tfrac{1}{2})u_2 = \frac{fL^2}{4EA}$$

The exact displacement can be obtained by integrating the differential equation by twice and applying two boundary conditions as

$$u(x) = \frac{f}{EA}x(L - \tfrac{1}{2}x)$$

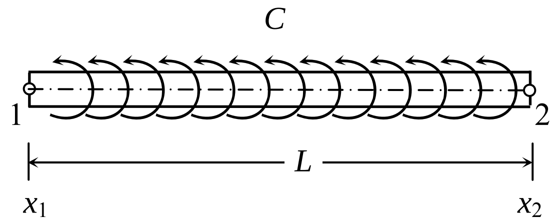
Therefore, the displacements at the end and at the center are

$$u(L) = \frac{fL^2}{2EA}$$

$$u(\tfrac{L}{2}) = \frac{3fL^2}{8EA}$$

Note that the displacement at the tip is exact, but that at the center is different. This happens because the finite element method uses linear interpolation, while the exact displacement is a quadratic function.

P1.35 An Euler beam element shown in the figure is under uniformly distributed couple C. Calculate equivalent nodal forces. Using a simply-supported beam under uniform couple, show that the reaction forces are equal and opposite directions with the equivalent nodal forces.



Solution: The loading form becomes

$$\ell(\overline{u}) = \int_0^L \overline{\theta} C \, \mathrm{d}x$$

By using the mapping relation and interpolation, the load form becomes

$$\ell(\overline{u}) = \overline{\mathbf{d}}^{\mathrm{T}} \int_0^1 \left(\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} \right)^T CJ \, \mathrm{d}\xi = \overline{\mathbf{d}}^{\mathrm{T}} \mathbf{f}$$

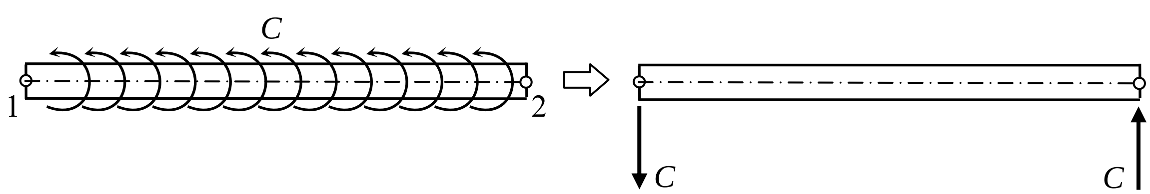
where $\mathbf{f}$ is the equivalent nodal force. The derivatives of interpolation function becomes

$$\left(\frac{\mathrm{d}\mathbf{N}}{\mathrm{d}x} \right)^{\mathrm{T}} = \frac{1}{L} \begin{Bmatrix} -6\xi + 6\xi^2 \\ L(1 - 4\xi + 3\xi^2) \\ 6\xi - 6\xi^2 \\ L(-2\xi + 3\xi^2) \end{Bmatrix}$$

By integrating the derivatives of interpolation function, the equivalent nodal forces can be obtained as

$$\mathbf{f}^T = \begin{Bmatrix} -C & 0 & C & 0 \end{Bmatrix}$$

The following figure illustrates the equivalent nodal forces.



If simply-supported boundary conditions are given at Nodes 1 and 2, the reaction forces can be calculated using the equilibrium of moments at Nodes 1 and 2, respectively:

$$\sum M \Big|_{\text{Node 1}} = CL + R_2 L = 0 \quad \Rightarrow \quad R_2 = -C$$
$$\sum M \Big|_{\text{Node 2}} = CL - R_1 L = 0 \quad \Rightarrow \quad R_1 = C$$

Therefore, the reactions are equal and opposite directions with the equivalent nodal forces.

P1.36 Integrate the following function using one–point and two–point numerical integration (Gauss quadrature). Explain how to integrate it. The exact integral is equal to 2. Compare the accuracy of the numerical integration with the exact one.

$$I = \int_0^{\pi} \sin(x)\, dx$$

Solution:

NG	Integration Points (s_i)	Weights (w_i)	Exact for polynomial of degree
1	0.0	2.0	1
2	±.57735	1.0	3

Since the numerical integration must be between the bounds [-1,1], a change of variable is needed.

$$x = as + b$$

$$\pi = a + b \qquad 0 = -a + b$$

$$b = \frac{\pi}{2} \qquad a = \frac{\pi}{2}$$

$$x = \frac{\pi}{2}(s+1)$$

$$dx = \frac{\pi}{2} ds$$

$$I = \int_0^{\pi} \sin(x) dx = \int_{-1}^{1} \frac{\pi}{2} \sin\left(\frac{\pi}{2}(s+1)\right) ds$$

$$f(s) = \frac{\pi}{2} \sin\left(\frac{\pi}{2}(s+1)\right)$$

One Point Integration: $s = 0$, $w = 2$

$$I \approx 2f(0) = 2\frac{\pi}{2} \sin\left(\frac{\pi}{2}\right) = \pi = 3.1415$$

Error = π-2 = 1.1415

Two Point Integration: s = $\pm$ .57735, w = 1

$$I \approx 1f(-.57735) + 1f(.57735) = 1\frac{\pi}{2}(.6162) + 1\frac{\pi}{2}(.6162) = 1.9358$$

Error = 2 – 1.9358 = .0642