A Appendix: Proofs of intermediate results of Section 3

A.1 Proof of Lemma 1

Proof. We start to show that there is a one-to-one correspondence between the positive definite solutions of the the discrete algebraic Riccati Equation

$$P = FPF^T - FPH_{\mathbf{w}}^T(H_{\mathbf{w}}PH_{\mathbf{w}}^T + R_{\mathbf{w}})^{-1}H_{\mathbf{w}}PF^T + LL^T, \tag{14}$$

and of its counterpart in information form

$$M = (FM^{-1}F^T + LL^T)^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}}. \tag{15}$$

Let M be a positive definite solution of (15), and set $P := FM^{-1}F^T + LL^T$. Note that this matrix is positive definite, because it can be written as

$$P = \begin{pmatrix} F & L \end{pmatrix} \begin{pmatrix} M^{-1} & \\ & \mathbf{I}_\ell \end{pmatrix} \begin{pmatrix} F & L \end{pmatrix}^T$$

and our controllability assumption (A1) implies that the matrix $[F,L]$ has full row rank. (Indeed, If $C = [L, FL, F^2L, \dots, F^{n-1}L]$ has rank n then CC^T has also rank n, but $CC^T = [F,L]B$, where B is some matrix. Thus, the $n \times (n+\ell)-$matrix $[F,L]$ must have full rank n.) So we can write $M = P^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}}$. By the Woodbury matrix identity we have $M^{-1} = P - PH_{\mathbf{w}}^T(H_{\mathbf{w}}PH_{\mathbf{w}}^T + R_{\mathbf{w}})^{-1}H_{\mathbf{w}}P$, and it is now clear that P solves (14).

Conversely, if P is a positive solution of Eq (14), set $M := P^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}}$, which is also a positive definite matrix. As above, we can write $M^{-1} = P - PH_{\mathbf{w}}^T(H_{\mathbf{w}}PH_{\mathbf{w}}^T + R_{\mathbf{w}})^{-1}H_{\mathbf{w}}P$, and so we have $P = FM^{-1}F^T + LL^T$. Finally, we have $M := P^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}} = (FM^{-1}F^T + LL^T)^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}}$, i.e. M solves (15).

Moreover, the two maps that we have defined above are inverses of each other, between the set of positive definite solutions of the DARE and of the I-DARE. Indeed, the map $M \mapsto FM^{-1}F^T + LL^T$ can be rewritten as $M \mapsto (M - H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}})^{-1}$ when M is a solution of the I-DARE, which is clearly the inverse mapping of $P \mapsto P^{-1} + H_{\mathbf{w}}^T R_{\mathbf{w}}^{-1} H_{\mathbf{w}}$.

Then, we can apply known results about the former Riccati equation (14): under our assumption (A1) that (F,L) is controllable, there exists a positive semidefinite solution to Eq. (14) if only if the pair $(F, H_{\mathbf{w}})$ is detectable, and moreover this solution is unique and positive definite (see e.g. Theorems 23 and 25 in [Sim06]). So the proof of the lemma is complete.

A.2 Proof of Proposition 2

Let $\mathbf{w} \in \mathscr{W}$, $X \in \mathscr{X}(\mathbf{w})$ and U be a matrix satisfying (i) and (ii) in the definition (10) of $\mathscr{X}(\mathbf{w})$. Define $M(\mathbf{w}) := \sum_{i=1}^{s} \mu_i(w_i) H_i^T H_i$. In particular,

$$\begin{pmatrix} U + M(\mathbf{w}) - F^T U F & F^T U L \\ L^T U F & \mathbf{I}_\ell - L^T U L \end{pmatrix} \succeq 0.$$

Since (F, L) is controllable, there is a gain matrix N such that $F + LN$ has all his eigenvalues outside the unit disc of $\mathbb{C}$, see [KSH00, Sections C3 and C4], and hence every eigenvalue of $\Gamma = (F + LN)^{-1}$ lies inside the open unit disc. Now, we obtain another positive semidefinite matrix by a congruence operation:

$$\begin{pmatrix} \mathbf{I}_n & -N^T \end{pmatrix} \begin{pmatrix} U + M(\mathbf{w}) - F^T U F & F^T U L \\ L^T U F & \mathbf{I}_\ell - L^T U L \end{pmatrix} \begin{pmatrix} \mathbf{I}_n \\ -N \end{pmatrix} \succeq 0.$$

This gives:

$$U + M(\mathbf{w}) + N^T N - (F + LN)^T U (F + LN) \succeq 0.$$

Pre and post-multiplying by Γ^T (resp. Γ), we obtain

$$U \preceq \Gamma^T U \Gamma + Q_{\mathbf{w}},$$

where $Q_{\mathbf{w}} := \Gamma^T (M(\mathbf{w}) + N^T N) \Gamma \succeq 0$. Denote by U_0 the unique solution of the discrete-time Lyapunov equation $U = \Gamma^T U \Gamma + Q_{\mathbf{w}}$, see [KSH00, Section D1]. We have

$$U - U_0 \preceq \Gamma^T (U - U_0) \Gamma \preceq \ldots \preceq (\Gamma^T)^k (U - U_0) \Gamma^k \preceq \ldots,$$

and so $U \preceq U_0$ because Γ^k converges to 0 as $k \to \infty$.

Now, observe that U_0 has a closed-form expression that can be obtained by vectorizing the Lyapunov equation: $\operatorname{vec}(U_0) = (\mathbf{I}_{n^2} - \Gamma^T \otimes \Gamma^T)^{-1} \operatorname{vec}(Q_{\mathbf{w}})$, where $\otimes$ denotes the Kronecker product. Using this relation, we obtain $\|U_0\|_F \leq \|(\mathbf{I}_{n^2} - \Gamma^T \otimes \Gamma^T)^{-1}\|_2 \cdot \|Q_{\mathbf{w}}\|_F$, where $\|M\|_F := \sqrt{\operatorname{trace}(MM^T)}$ denotes the Frobenius norm of M and $\|M\|_2$ is its spectral norm. By using the definition of $Q_{\mathbf{w}}$, we can thus conclude that there exists a constant $\alpha' \geq 0$ such that $\|U\|_F \leq \alpha'(1 + \sum_i \mu_i(w_i) \|H_i\|_F^2)$. Finally, we obtain the bound of the proposition, $\|X\|_2 \leq \alpha(1 + \sum_i \mu_i(w_i) \|H_i\|_2^2)$ for some $\alpha \geq 0$, by using (i) in Eq. (10) and the fact that the Frobenius and spectral norms are equivalent.

A.3 Proof of Proposition 3

Let $\mathbf{w} \in \mathscr{W}$.We introduce the function g that maps $\mathbb{S}_n^{++}$ onto $\mathbb{S}_n$, defined by $g(X) = f(X, \mathbf{w}) = (F X^{-1} F^T + L L^T)^{-1} + M(\mathbf{w}) - X$. Note that $X \in \mathscr{X}^+(\mathbf{w})$ if and only if $g(X) \succeq 0$.

The directional derivative of g in the direction of $\Delta \succeq 0$ can be found by using the formula $\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}$ and the chain rule:

$$\begin{aligned} Dg(M)[\Delta] := \lim_{\varepsilon \to 0} \quad & \frac{1}{\varepsilon}\left(g(M+\varepsilon\Delta) - g(M)\right) \\ & = (FM^{-1}F^T + LL^T)^{-1}FM^{-1}\Delta M^{-1}F^T(FM^{-1}F^T + LL^T)^{-1} - \Delta. \end{aligned}$$

Let X be a matrix in $\mathscr{X}^+(\mathbf{w})$, such that $g(X) \neq 0$. We are going to show that there exists a matrix X' in the neighborhood of X satisfying $X' \succeq X$, $X' \neq X$, $g(X') \succeq 0$. This is equivalent to the following first order property (see e.g., [HUL93, Section VI.5.1]):

$$\exists \Delta \succeq 0 : \Delta \neq 0, \qquad \forall \mathbf{u} \in \mathrm{Ker}(g(X)),\ \mathbf{u}^T Dg(X)[\Delta]\mathbf{u} \geq 0.$$

This condition is satisfied for $\Delta := g(X)$ indeed, because the first term of $Dg(X)[\Delta]$ is a positive semidefinite matrix, and the term $-\mathbf{u}^T\Delta\mathbf{u}$ vanishes for all $\mathbf{u} \in \mathrm{Ker}(g(X))$. Hence, for all $X_0 \in \mathscr{X}^+(\mathbf{w})$ we can define a sequence of matrices $X_i \in \mathscr{X}^+(\mathbf{w})$ satisfying $X_0 \preceq X_1 \preceq \ldots X_n \preceq \ldots$ as follows: $X_{n+1} = \Psi(X_n)$, where

$$\Psi(X) = X + \left(\operatorname*{argmax}_{t>0}\left\{t :\ g\big(X + tg(X)\big) \succeq 0\right\}\right) g(X).$$

The sequence (X_i) is bounded (by Propositions 1 and 2) and hence it converges. (Indeed, if it has two accumulation points X_1^* and X_2^*, then for all $\mathbf{u} \in \mathbb{R}^n$ the non-decreasing sequence $\mathbf{u}^T X_i \mathbf{u}$ converges to some value, which must be equal to both $\mathbf{u}^T X_1^* \mathbf{u}$ and $\mathbf{u}^T X_2^* \mathbf{u}$. Hence, $\mathbf{u}^T(X_1^* - X_2^*)\mathbf{u} = 0$ for all $\mathbf{u}$, which proves $X_1^* = X_2^*$.) Denote this limit by X^*. The above discussion shows that $g(X^*) = 0$, otherwise X^* cannot be a fixed point of Ψ. This means that X^* is the unique positive definite solution of the I-DARE (3). We have thus $X_0 \preceq X^* = M_\infty(\mathbf{w})$, and the proposition is proved.

References

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