Solutions of Problems of the book
Quantum Communications, Springer 2015
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Abstract This report collects the solutions to the problems proposed in each chapter of the book *Quantum Communications* by G.,ariolaro. Springer, May 2015

As a telecommunications engineer I know that in the real world the error probability is never zero. This is particularly true for the solutions of problems. I do hope that, with the help of the readers, the error probability in the solutions will be considerably reduced

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Problems of Chapter 2

Problem 2.1 $\star$ [Sect. 2.4] A basis in $\mathcal{H} = \mathbb{C}^2$ is usually denoted by $\{|0\rangle, |1\rangle\}$. Write the standard basis and a nonorthogonal basis.

Solution The standard basis is

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Another basis is

$$|0\rangle = \begin{bmatrix} 3\mathrm{i} \\ 2 \end{bmatrix}, \qquad |1\rangle = \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}. \tag{P1}$$

The two kets in (P1) are independent (check it), but not orthogonal. In fact

$$\langle 0|1\rangle = [-\mathrm{i}, 2] \begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix} = -3\mathrm{i} + 2\sqrt{3} \neq 0\,.$$

Problem 2.2 $\star\star$ [Sect. 2.4] An important basis in $\mathcal{H} = \mathbb{C}^n$ is given by the columns of the Discrete Fourier Transform (DFT) matrix of order n, given by

$$|w_i\rangle = \frac{1}{\sqrt{n}} \left[1, W_n^{-i}, W_n^{-2i}, \ldots, W_n^{-i(n-1)}\right]^{\mathrm{T}}, \qquad i = 0, 1, \ldots, n-1 \tag{E1}$$

where $W_n := \exp(\mathrm{i}2\pi/n)$ is the nth root of 1. Prove that this basis is orthonormal.

Solution We recall that the standard inner product in $\mathbb{C}^n$ is (see (2.23))

$$\langle y|x\rangle = x_0 y_0^* + \cdots x_{n-1} y_{n-1}^* = \sum_{r=0}^{n-1} x_r y_r^*\,.$$

Here, we have to evaluate the inner products $\langle w_j|w_i\rangle$. The rth entries of $|w_i\rangle$ and $\langle w_j|$ are respectively

$$x_r = \frac{1}{\sqrt{n}} W_n^{-ri}, \qquad y_r^* = \frac{1}{\sqrt{n}} W_n^{rj}\,.$$

Hence

$$\langle w_i|w_j\rangle = \frac{1}{n} \sum_{r=0}^{n-1} W_n^{r(j-i)} = \frac{1}{n} \sum_{r=0}^{n-1} \exp(\mathrm{i}2\pi r(i-j)/n)\,.$$

Next, using the property of the nth roots of 1

$$\sum_{r=0}^{n-1} \exp(\mathrm{i}2\pi rk/n) = 0 \quad \forall \text{ integers } k \neq 0,$$

one gets

$$\langle w_i | w_j \rangle = \begin{cases} 1 & i = j \\ 0 & i \neq j \, . \end{cases}$$

Problem 2.3 $\star$ [Sect. 2.4] Find the Fourier coefficients of ket

$$|x\rangle = \begin{bmatrix} 1 \\ \mathrm{i} \\ 2 \end{bmatrix} \in \mathbb{C}^3$$

with respect to the orthonormal basis (E1).

Solution The Fourier coefficients with an orthonormal basis are given by (2.24a), which now reads

$$a_i = \langle w_i | x \rangle \, .$$

Then

$$a_k = \frac{1}{\sqrt{3}} [W_3^0, W_3^k, W_3^{2k}] \begin{bmatrix} 1 \\ \mathrm{i} \\ 2 \end{bmatrix} , \qquad k = 0, 1, 2$$

where $W_3 = \exp(\mathrm{i}2\pi/3) = -\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}$. The explicit result is

$$\begin{aligned} a_0 &= \frac{3 + \mathrm{i}}{\sqrt{3}} \\ a_1 &= \frac{1 + \mathrm{i}\left[-\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}\right] + 2\left[-\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}\right]^2}{\sqrt{3}} = -\frac{1}{6}\mathrm{i}\left[(6 - 3\mathrm{i}) + \sqrt{3}\right] \\ a_2 &= \frac{1 + \mathrm{i}\left[-\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}\right]^2 + 2\left[-\frac{1}{2} + \frac{\mathrm{i}\sqrt{3}}{2}\right]^4}{\sqrt{3}} = \frac{1}{6}\left[(3 + 6\mathrm{i}) - \mathrm{i}\sqrt{3}\right] \, . \end{aligned}$$

Problem 2.4 $\star$ [Sect. 2.4] Write the Fourier expansion (2.24) and (2.25) with a general orthonormal basis $\mathcal{B} = \{|b_i\rangle | i \in I\}$.

Solution The compact form is

$$|x\rangle = \sum_{i \in I} a_i |b_i\rangle \quad \text{with} \quad a_i = \langle b_i | x \rangle \, .$$

Then it can also be written as

$$|x\rangle = \sum_{i \in I} \left(\langle b_i | x \rangle\right) |b_i\rangle \, .$$

Problem 2.5 $\star$ [Sect. 2.5] Prove that the image of an operator on $\mathcal{H}$ is a subspace of $\mathcal{H}$.

Solution We have to prove that

$$\mathrm{im}(A) := A\mathcal{H} = \{A|x\rangle \,|\, |x\rangle \in \mathcal{H}\}$$

has the properties of a vectors space as listed in Definition 2.1.

Problem 2.6 ⋆ [Sect. 2.5] Define the 2D operator that inverts the entries of a ket and write its matrix representation with respect to the standard basis.

Solution The relation $|y\rangle = A\,|x\rangle$ of the operator is explicitly

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

which implies that

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

This is also the matrix representation of the operator A with respect to the standard basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} .$$

Problem 2.7 ⋆⋆ [Sect. 2.5] Find the matrix representation of the operator of the previous problem with respect to the DFT basis.

Solution The DFT basis for $n = 2$ is

$$\mathcal{W} = \frac{1}{\sqrt{2}} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} .$$

Hence, we apply (2.32), which reads

$$a_{ij\mathcal{W}} = \langle w_i|A|w_j\rangle , \qquad i,j = 0,1 .$$

Explicitly

$$a_{00\mathcal{W}} = 1 , \qquad a_{01\mathcal{W}} = 1 , \qquad a_{10\mathcal{W}} = 0 , \qquad a_{11\mathcal{W}} = -1 \quad \to \quad A_{\mathcal{W}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Note that $\mathrm{Tr}[A_{\mathcal{W}}] = \mathrm{Tr}[A] = 0$, which confirms that the trace is independent of the matrix representation.

Problem 2.8 ⋆ [Sect. 2.8] Classify the so called *Pauli matrices*

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \; \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \; \sigma_y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} , \; \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{(E2)}$$

which have an important role in quantum computation.

Solution All the Pauli matrices are both unitary and Hermitian.

Problem 2.9 $\star$ [Sect. 2.9] Prove properties (2.65), (2.66) and (2.67) for a projector and its complement.

Solution To prove (2.65), let $|s\rangle = P|x\rangle$ and $|s^\perp\rangle = P_c|y\rangle$; bearing in mind that $\langle s^\perp| = \langle y|P_c^*$ and that P_c is Hermitian, we get

$$\langle s^\perp|s\rangle = \langle y|P_cP|x\rangle, \quad \text{where} \quad P_cP = (I-P)P = 0\,.$$

Hence (2.65) follows. For the other properties, the proof is analogous.

Problem 2.10 $\star$ [Sect. 2.9] Prove that projectors are positive semidefinite operators.

Solution See (2.63), Theorem 2.6 and Corollary 2.1.

Problem 2.11 $\star\star\star$ [Sect. 2.12] Let A be an arbitrary operator of the Hilbert space $\mathcal{H}$. Show that the operator AA^* is always positive semidefinite.

Hint: use diagonalization of A.

Solution Recalling the rules $(AB)^* = B^*A^*$ and $(A^*)^* = A$, we find

$$(AA^*)^* = (A^*)^*A^* = AA^*$$

and therefore AA^* is Hermitian. Let $U\Lambda U^*$ be the diagonalization of A. Then, the diagonalization of A^* is $U\Lambda^*U^*$ and the diagonalization of AA^* is

$$AA^* = U\Lambda U^*U\Lambda^*U^* = U\Lambda\Lambda^*U^*\,.$$

Now, if λ_i is the ith eigenvalue in Λ, the corresponding eigenvalue in $\Lambda\Lambda^*$ is $\lambda_i\lambda_i^* = |\lambda_i|^2 \geq 0$ and the conclusion follow from Theorem 2.6.

Problem 2.12 $\star$ [Sect. 2.13] Prove that if A and B are Hermitian operators, also $A\otimes B$ is a Hermitian operator.

Solution From the Hermitian conditions, $A^* = A$, $B^* = B$, and from the second of (2.103) one gets

$$(A\otimes B)^* = A^*\otimes B^* = A\otimes B\,.$$

Problem 2.13 $\star\star$ [Sect. 2.13] Establish the compatibility conditions for the dimensions of the matrices in the mixed–product law (2.104).

Solution The compatibility conditions are concerned with the ordinary product. Let $m_A \times n_A$ be the dimension of A and so for the other matrices. Then, the compatibility conditions on the right hand side are

$$m_C = n_A\,, \qquad m_D = n_B\,.$$

These two conditions ensure the compatibility on the left hand side. In fact, on the left hand side the conditions are

$$m_{C\otimes D} = m_C m_D = n_{A\otimes B} = n_A n_B \ .$$

Problem 2.14 $\star\star$ [Sect. 2.13] Prove property (2.107) of the Kronecker product and, more specifically, prove that, if λ is an eigenvalue of A with eigenvector $|\lambda\rangle$ and μ is an eigenvalue of B with eigenvector $|\mu\rangle$, then $\lambda\mu$ is an eigenvalue of $A\otimes B$ with eigenvector $|\lambda\rangle\otimes|\mu\rangle$.

Solution We have

$$A|\lambda\rangle = \lambda|\lambda\rangle \ , \qquad B|\mu\rangle = \mu|\mu\rangle \ .$$

Hence, using definition (2.99) and property (2.98a), we get

$$(A\otimes B)|(\lambda\rangle\otimes|\mu\rangle) = (A|\lambda\rangle)\otimes(B|\mu\rangle) = (\lambda|\lambda\rangle)\otimes(\mu|\mu\rangle) = \lambda\mu(|\lambda\rangle\otimes|\mu\rangle) \ .$$

Problem 2.15 $\star\star\star$ [Sect. 2.13] The mixed–product law can be extended in several ways. In particular,

$$(A_1\otimes A_2)(B_1\otimes B_2)(C_1\otimes C_2) = (A_1B_1C_1)\otimes(A_2B_2C_2) \ . \qquad \text{(E5)}$$

Prove this relation using (2.104).

Solution The mixed–product law (2.104) gives

$$(A_1\otimes A_2)(B_1\otimes B_2) = (A_1B_1)\otimes(A_2B_2) \ .$$

Then we right multiply both sides by $C_1\otimes C_2$ to get

$$(A_1\otimes A_2)(B_1\otimes B_2)(C_1\otimes C_2) = (A_1B_1)\otimes(A_2B_2)(C_1\otimes C_2)$$

and (E5) follows after a new application of (2.104) at the right hand side.

Problem 2.16 $\star\star$ [Sect. 2.13] Prove that, if the matrices A_1 and A_2 have respectively the diagonalizations (see (2.87))

$$A_1 = U_1\Lambda_1U_1^* \ , \qquad A_2 = U_2\Lambda_2U_2^*$$

then

$$A_1\otimes A_2 = (U_1\otimes U_2)(\Lambda_1\otimes\Lambda_2)(U_1^*\otimes U_2^*) \qquad \text{(E6)}$$

is a diagonalization of $A_1\otimes A_2$.

Solution We have

$$A_1\otimes A_2 = (U_1\Lambda_1U_1^*)\otimes(U_2\Lambda_2U_2^*) \ .$$

Then, using the extension of the mixed-product law (see the previous problem), (E6) follows at once.

Problems of Chapter 3

Problem 3.1 $\star$ [Sect. 3.3] Prove that the density operator ρ of a quantum system in a pure state is *idempotent*.

Solution The density operator is given by $\rho = |\psi\rangle\langle\psi|$. Then $\rho^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi|$, where $\langle\psi|\psi\rangle = 1$ and therefore $\rho^2 = \rho$.

Problem 3.2 $\star\star\nabla$ [Sect. 3.3] Prove that, if and only if $\mathrm{Tr}[\rho^2] = 1$, the density operator ρ represents a pure state.

Hint: see Proposition 3.5.

Solution Proposition 3.5 gives the expansion (reduced EID)

$$\rho = \sum_{i=1}^{r} \sigma_i^2 \, |u_i\rangle\langle u_i|$$

where r is the rank of ρ, σ_i^2 are the r positive eigenvalues of ρ, and $|u_i\rangle$ are the corresponding orthonormal eigenvectors. Then

$$\rho^2 = \sum_{i=1}^{r} \sigma_i^4 \, |u_i\rangle\langle u_i|$$

and, considering that the σ_i^2 are probabilities ($\sigma_i^2 \le 1$),

$$\mathrm{Tr}[\rho^2] = \sum_{i=1}^{r} \sigma_i^4 \le \sum_{i=1}^{r} \sigma_i^2 = \mathrm{Tr}[\rho] \, .$$

The equality holds if and only if $r = 1$ and $\sigma_1^2 = 1$, which gives

$$\rho = \sigma_1^2 |u_1\rangle\langle u_1| = |u_1\rangle\langle u_1|$$

and ρ represents a pure state.

Problem 3.3 $\star\star$ [Sect. 3.3] Prove relation (3.9), which states that a cavity at thermal equilibrium is in a mixed state.

Solution In fact,

$$\begin{aligned}\rho^2 &= (1-\varepsilon)^2 \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \varepsilon^m \varepsilon^n |m\rangle\langle m|n\rangle\langle n| \\ &= (1-\varepsilon)^2 \sum_{m=0}^{\infty} \varepsilon^{2m} |m\rangle\langle m| \, .\end{aligned}$$

Hence

$$\mathrm{Tr}(\rho^2) = (1-\varepsilon)^2 \sum_{m=0}^{\infty} \varepsilon^{2m} = \frac{(1-\varepsilon)^2}{1-\varepsilon^2} = \frac{1-\varepsilon}{1+\varepsilon} < 1.$$

Problem 3.4 ⋆⋆ [Sect. 3.3] Verify that the ensembles (3.10), (3.11) and (3.12) give the same density operator.

Solution It is sufficient to use the definition (3.7). For the ensemble (3.12), where the states are given by a linear combination of the orthonormal basis $\{|0\rangle, |1\rangle\}$, see Example 3.2.

Problem 3.5 ⋆ [Sect. 3.4] Prove that if the temporal evolution operator $U = U(t,t_0)$ is unitary, as assumed in Postulate 2, then the norm of the wave function $|\psi(t)\rangle$ at time t remains of unit length, as it must be from Postulate 1. Moreover, prove that the inner product of two wave functions $|\psi_1(t)\rangle$ and $|\psi_2(t)\rangle$ doesn't change during the evolution.

Solution It is sufficient to prove the statement concerning the wave functions, that is,

$$\langle\psi_1(t)|\psi_2(t)\rangle = \langle\psi_1(t_0)|\psi_2(t_0)\rangle \ .$$

In fact, for both wave functions the evolution is

$$|\psi_i(t)\rangle = U(t,t_0)|\psi_i(t_0)\rangle \ , \qquad i = 1,2 \ .$$

Then the inner product at time t is given by

$$\begin{aligned}\langle\psi_1(t)|\psi_2(t)\rangle =& \langle\psi_1(t_0)|U^*(t,t_0)U(t,t_0)|\psi_2(t_0)\rangle \\ =& \langle\psi_1(t_0)|I_{\mathcal{H}}|\psi_2(t_0)\rangle \\ =& \langle\psi_1(t_0)|\psi_2(t_0)\rangle \ .\end{aligned}$$

In particular, for $|\psi_1(t)\rangle = |\psi_2(t)\rangle = |\psi(t)\rangle$ we get

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(t_0)|\psi(t_0)\rangle = 1 \ .$$

Problem 3.6 ⋆ [Sect. 3.4] Suppose that A_S, B_S, and C_S are three observables in the Schrödinger picture that verify the commutation condition

$$[A_S, B_S] = \mathrm{i} C_S \ .$$

Prove that in the Heisenberg picture the commutation condition becomes

$$[A_H(t), B_H(t)] = \mathrm{i} C_H(t) \ .$$

Solution The condition is explicitly

$$A_S B_S - B_S A_S = \mathrm{i} C_S \ .$$

Multiplying both sides from the left by $U^*(t_0,t)$ and from the right by $U(t_0,t)$ one gets

$$U^*(t_0,t)\,\{A_S B_S - B_S A_S\}\,U(t_0,t) = \mathrm{i}\,U^*(t_0,t)\,C_S\,(t_0,t)\,U(t_0,t)$$

where, considering that $U^*(t_0,t)U(t_0,t) = I_{\mathcal{H}}$, we can write

$$A_S B_S = A_S U^*(t_0,t)U(t_0,t)B_S\,, \qquad B_S A_S = B_S U^*(t_0,t)U(t_0,t)A_S\,.$$

Finally, using (3.23), one obtains the result.

Problem 3.7 $\star$ [Sect. 3.5] Apply Postulate 3 to a quantum system "prepared" in a pure state $|\psi\rangle$, when the measurement is obtained by a set of orthonormal *measurement vectors* $\{|a_0\rangle, |a_1\rangle, \ldots, |a_{M-1}\rangle\}$. Find the probability distribution of the measure m *when the state of the system is one of the measurement vectors*. Which is the state of the system after the measurement?

Solution Suppose that the system is in the state $|\psi\rangle = |a_1\rangle$. Then, considering that the vectors are orthonormal, application of (3.29) gives

$$\mathrm{P}\,[m = i|a_1] = |\langle a_i|a_1\rangle|^2 = \begin{cases} 1 & i = 1 \\ 0 & i \neq 1 \end{cases}$$

which is a degenerate distribution. In other words, the measurement gives $m = 1$ with probability 1. Since the result of the measurement is $m = 1$, from (3.7) we get

$$|\psi^{(1)}_{\mathrm{post}}\rangle = |a_1\rangle\,.$$

removed problem(MQ37), identical to prolem(MQ30)

Problem 3.8 $\star$ [Sect. 3.6] Consider the Hermitian operator

$$H = \frac{1}{2}\begin{bmatrix} 3 & -\mathrm{i} \\ \mathrm{i} & 3 \end{bmatrix}$$

and use it as an observable for the measurement in a qubit system prepared in the pure state

$$|\psi\rangle = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2\mathrm{i} \end{bmatrix}\,.$$

Evaluate the probability of the measurement outcome m and the post–measurement states.

Solution The spectral decomposition theorem (Theorem 3.1) gives

$$H = a_1|a_1\rangle\langle a_1| + a_2|a_2\rangle\langle a_2|$$

where a_1 and a_2 are the eigenvalues and the corresponding eigenvectors are

$$|a_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -\mathrm{i} \\ 1 \end{bmatrix}, \qquad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathrm{i} \\ 1 \end{bmatrix}.$$

The corresponding projectors are

$$\Pi_1 = |a_1\rangle\langle a_1| = \frac{1}{2}\begin{bmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{bmatrix}, \qquad \Pi_2 = |a_1\rangle\langle a_1| = \frac{1}{2}\begin{bmatrix} 1 & -\mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}$$

that is, the ones given by (3.39). Then the probabilities of the outcome and the post–measurement states have been evaluated in Example 3.4. Note in particular that, considering that the projectors are elementary, the post–measurement states are given by the eigenvectors of the observable H, according to (?BB2?).

Problem 3.9 ⋆⋆ [Sect. 3.6] Let A be an observable with spectrum $\sigma(A)$. Show that the *moments* of a measurement m obtained with the observable when the state $|\psi\rangle$ is set to an eigenket $|a\rangle$ of A, are simply given by

$$\mathrm{E}[m^k|a] = a^k, \qquad k = 1, 2, \ldots \tag{E2}$$

where $a \in \sigma(A)$ is the eigenvalue corresponding to the eigenket $|a\rangle$. Explain why.

Solution For the proof it is sufficient to apply the *eigenvalue equation*

$$A|a\rangle = a|a\rangle, \qquad a \in \sigma(A)$$

which defines the eigenvalues and eigenvectors of A. For instance for $k = 3$ one gets

$$\mathrm{E}[m^3|a] = \langle a|A^3|a\rangle = \langle a|A^2A|a\rangle$$

where $A|a\rangle = a\,|a\rangle$. Hence,

$$\mathrm{E}[m^3|a] = \langle a|A^2|a\rangle = a\langle a|A^2|a\rangle$$

and, repeating the substitution one gets $\mathrm{E}[m^3|a] = a^3$.

To explain this result it is sufficient to evaluate the probability distribution of random variable m in the case of elementary projectors. Using (3.29) with $|\psi\rangle = |a\rangle \in \sigma(A)$, one gets

$$\mathrm{P}\,[m = i|a] = |\langle a|a_i\rangle|^2 = \begin{cases} 1 & a = a_i \\ 0 & a \neq a_i \end{cases}$$

which state that the outcome of the measurement is $m = a$ with probability 1. In practice this means that the outcome is always $m = a$ and therefore the expectation of m^k is a^k.

Problem 3.10 ⋆⋆ [Sect. 3.9] Prove that the state after the second measurement with the same projector system remains the same as the one in which the system was after the first measurement, as stated by (3.55).

Solution In fact, it results

$$|\psi^{(i)}_{\text{post,post}}\rangle = \frac{\Pi_i|\psi^{(i)}_{\text{post}}\rangle}{\sqrt{p'_i}} = \frac{\Pi^2|\psi\rangle}{\sqrt{p'_i}} = |\psi^{(i)}_{\text{post}}\rangle$$

as soon as we bear in mind that $p'_i = 1$ and $\Pi^2 = \Pi$.

Problem 3.11 ⋆⋆ [Sect. 3.10] Consider the non normalized state

$$|\psi'\rangle = 2\,|0\,0\rangle + \mathrm{i}\,|0\,1\rangle + 3\,|0\,1\rangle$$

of a two–qubit system with basis $\mathcal{B} = \{|0\,0\rangle, |0\,1\rangle, |1\,0\rangle, |1\,1\rangle\}$ (here $|0\,0\rangle$ stands for $|0\rangle \otimes |0\rangle$, etc.). Find the normalized form, $\langle\psi|\psi\rangle = 1$, and prove that the two qubits $|\psi\rangle$ are entangled.

Solution Considering that the basis is orthonormal, the square amplitude of $|\psi'\rangle$ is

$$\langle\psi'|\psi'\rangle = 2^2 + |\mathrm{i}|^2 + 3^2 = 14$$

and therefore the normalized form is

$$|\psi\rangle = \frac{2}{\sqrt{14}}|0\,0\rangle + \frac{\mathrm{i}}{\sqrt{14}}|0\,1\rangle + \frac{3}{\sqrt{14}}|0\,1\rangle\,.$$

Considering the general form (3.68) of non entangled qubits, now rewritten in the form

$$|\psi_1\rangle \otimes |\psi_2\rangle = u_0v_0|0\,0\rangle + u_0v_1|0\,1\rangle + u_1v_0|1\,0\rangle + u_1v_1|1\,1\rangle$$

we find that the non entanglement conditions should be

$$u_0v_0 = \frac{2}{\sqrt{14}}\,, \qquad u_0v_1 = \frac{\mathrm{i}}{\sqrt{14}}\,, \qquad u_1v_0\frac{3}{\sqrt{14}}\,, \qquad u_1v_1 = 0$$

where the latter implies $u_1 = 0$ or $v_1 = 0$, which are incompatible with the other conditions. Hence, the two qubits $|\psi\rangle$ are entangled.

In Chapter 10, Proposition 10.3, we shall see Schmidt's decomposition, which states in general whether a composite state is entangled or not.

Problem 3.12 ⋆⋆ [Sect. 3.11] Minimum factor from an arbitrary factor. Let $\widehat{\Psi}$ be an arbitrary k–factor of ρ. The **reduced SVD** of the $n \times k$ matrix $\widehat{\Psi}$ has the form (see Section 2.12)

$$\widehat{\Psi} = \sum_{i=1}^{r} \sigma_i\,|u_i\rangle\langle v_i| = U\,\boldsymbol{\Sigma} V^* \tag{3.1}$$

where the σ_i are the square roots of the r positive eigenvalues σ_i^2 of $\widehat{\Psi}\,\widehat{\Psi}^* = \rho$, $\boldsymbol{\Sigma} = \mathrm{diag}\,\{\sigma_1, \dots, \sigma_r\}$, $|u_i\rangle$ and U are the same as in the EID of (3.81), $|v_i\rangle$ are orthonormal vectors of length k, and $V = [|v_1\rangle, \dots, |v_r\rangle]$. Prove that a minimum orthonormal factor of ρ is given by

$$\widehat{\Psi}_0 = U\,\boldsymbol{\Sigma}\,. \tag{3.2}$$

Solution The expression of $\widehat{\Psi}_0$ is the same as in Proposition 3.5. Note that from (3.84), considering that $V^* V = I_r$ (the $r \times r$ identity matrix), we find

$$\widehat{\Psi}\,\widehat{\Psi}^* = U\,\boldsymbol{\Sigma} V^* V\,\boldsymbol{\Sigma}\,U^* = U\,\boldsymbol{\Sigma}\boldsymbol{\Sigma}\,U^* = \widehat{\Psi}_0\,\widehat{\Psi}_0^* = \rho$$

which represents the EID of ρ.

Problem 3.13 $\star\star\star$ [Sect. 3.11] Generation of all possible factors of a density operator. Let $\widehat{\Psi}$ be a k–factor of ρ, that is, $\widehat{\Psi}\,\widehat{\Psi}^* = \rho$, and let A be an arbitrary $k \times p$ complex matrix that verifies the condition $AA^* = I_k$. Prove that

$$\Phi = \widehat{\Psi} A \tag{3.3}$$

is a p–factor of ρ. This relation allows us to generate all the possible factors of a given density operator

Solution In fact, $\Phi\,\Phi^* = \widehat{\Psi} A A^* \widehat{\Psi}^* = \widehat{\Psi}\,\widehat{\Psi}^* = \rho$. Note that $k \geq r = \text{rank}(\rho) = \text{rank}(\widehat{\Psi})$. Condition $A^* A = I_k$ states that the k columns a_i of A are orthonormal vectors, that is, they verify the condition $a_i a_j^* = \delta_{ij}$. Considering that $a_i \in \mathbb{C}^p$, we have the condition $p \geq k$, because we cannot find in $\mathbb{C}^p$ an orthonormal set $\{a_1, \ldots, a_k\}$ with more than p components.

Now we relate an arbitrary factor $\widehat{\Phi}$ of ρ to a minimum orthonormal factor. A k–factor $\widehat{\Phi}$ of ρ is related to a minimum orthonormal factor $\widehat{\Psi}_0$ in the form

$$\widehat{\Phi} = \widehat{\Psi}_0 A_0 \tag{3.4}$$

where A_0 is an $r \times k$ matrix given by

$$A_0 = \boldsymbol{\Sigma}^{-2}\,\widehat{\Psi}_0^*\,\widehat{\Phi} \tag{3.4a}$$

$\boldsymbol{\Sigma}^2$ being the diagonal matrix formed by the positive eigenvalues of ρ. The matrix A_0 always verifies the condition $A_0 A_0^* = I_r$.

Considering that a minimum factor has the form (see (3.82)) $\widehat{\Psi}_0 = U\,\boldsymbol{\Sigma}$, by left–multiplying (3.4) by U^*, we get

$$U^*\widehat{\Phi} = U^*\widehat{\Psi}_0 A_0 = U^* U\,\boldsymbol{\Sigma} A_0 = \boldsymbol{\Sigma} A_0$$

and, considering that $\widehat{\Psi}_0 = U\,\boldsymbol{\Sigma}$, (3.4a) follows. Next, from (3.4a), considering that $\widehat{\Phi}\,\widehat{\Phi}^* = \rho$ and also $\widehat{\Psi}_0\,\widehat{\Psi}_0^* = \rho$, we get

$$A_0 A_0^* = \boldsymbol{\Sigma}^{-2}\,\widehat{\Psi}_0^*\,\widehat{\Phi}\,\widehat{\Phi}^*\widehat{\Psi}_0\,\boldsymbol{\Sigma}^{-2} = \boldsymbol{\Sigma}^{-2}\,\widehat{\Psi}_0^*\,\widehat{\Psi}_0\,\widehat{\Psi}_0^*\,\widehat{\Psi}_0\,\boldsymbol{\Sigma}^{-2}$$

where $\widehat{\Psi}_0^*\,\widehat{\Psi}_0 = \boldsymbol{\Sigma}^2$. Hence, $AA^* = I_r$.

Relation (3.4) states that, starting from a minimum orthonormal factor $\widehat{\Psi}_0$, of dimension $n \times r$, *one can generate all the possible factors of a given density operator* in the form $\widehat{\Psi} = \widehat{\Psi}_0 A_0$, where A_0 is an arbitrary $k \times r$ matrix with orthonormal rows, that is, with $A_0 A_0^* = I_r$. Note that $k \geq r$ may be arbitrarily large.

Problem 3.14 ⋆⋆ [Sect. 3.11] Find the reduced SVD of the factor (3.79) and show that it gives the same minimum factor $\widehat{\Psi}_0$ obtained with the EID of ρ.

Solution The factor given by (3.79) has rank $r = 2$ and its SVD is $\widehat{\Psi} = U\,\boldsymbol{\Sigma} V^*$ with

$$U = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{\mathrm{i}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \qquad V = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \boldsymbol{\Sigma} = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Hence, we find

$$\widehat{\Psi}_0 = U\,\boldsymbol{\Sigma} = \begin{bmatrix} -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ \frac{\mathrm{i}\sqrt{3}}{4} & -\frac{1}{4} \\ \frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} \end{bmatrix}$$

that is, the same minimum factor $\widehat{\Psi}_0$ obtained with the EID of ρ.

Problem 3.15 ⋆⋆ [Sect. 3.11] Consider the minimum factor given by (3.83) and find a 2×3 matrix to generate a 3–factor. Also, apply the 2×8 matrix

$$A = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} & -\mathrm{i} & \mathrm{e}^{-\frac{3\mathrm{i}\pi}{4}} & -1 & \mathrm{e}^{\frac{3\mathrm{i}\pi}{4}} & \mathrm{i} & \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \end{bmatrix}$$

to generate an 8–factor.

Solution A 2×3 matrix that verifies the condition $A_0 A_0^* = I_2$ is

$$A_0 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

The corresponding 3–factor is

$$\widehat{\Phi} = \widehat{\Psi}_0 A = \begin{bmatrix} -\frac{1}{4\sqrt{3}} & -\frac{2+3\sqrt{2}}{8\sqrt{3}} & \frac{-2+3\sqrt{2}}{8\sqrt{3}} \\ -\frac{1}{4\sqrt{3}} & \frac{-2+3i\sqrt{2}}{8\sqrt{3}} & \frac{-2-3i\sqrt{2}}{8\sqrt{3}} \\ -\frac{1}{4\sqrt{3}} & \frac{-2+3\sqrt{2}}{8\sqrt{3}} & -\frac{2+3\sqrt{2}}{8\sqrt{3}} \\ -\frac{1}{4\sqrt{3}} & -\frac{2+3\sqrt{2}}{8\sqrt{3}} & \frac{-2+3\sqrt{2}}{8\sqrt{3}} \end{bmatrix}.$$

The application of the 2×8 matrix A, which verifies the condition $AA^* = I_2$, gives the 8-factor

$$\widehat{\Phi} = \frac{1}{8\sqrt{2}} \begin{bmatrix} -1-a & -e_{-1}-a & \mathrm{i}-a & -e_{-3}-a & 1-a & -e_3-a & -\mathrm{i}-a & -e_1-a \\ -1+\mathrm{i}a & \mathrm{i}a-e_{-1} & \mathrm{i}[1+a] & \mathrm{i}a-e_{-3} & 1+\mathrm{i}a & \mathrm{i}a-e_3 & \mathrm{i}[-1+a] & \mathrm{i}a-e_1 \\ -1+a & a-e_{-1} & \mathrm{i}+a & a-e_{-3} & 1+a & a-e_3 & -\mathrm{i}+a & a-e_1 \\ -1-a & -e_{-1}-a & \mathrm{i}-a & -e_{-3}-a & 1-a & -e_3-a & -\mathrm{i}-a & -e_1-a \end{bmatrix}$$

where $a = \sqrt{3}$ and $e_k = \exp(\mathrm{i}2\pi k/4)$.

Problem 3.16 ⋆⋆ [Sect. 3.12] Prove that the mixed state qubit expressed in the form (3.90) represents a pure state if and only if the vector **r** has unit length. Under this condition, from (3.90) find the corresponding pure state.

Solution The condition that ρ represents a pure state is (see Section 3.3) $\mathrm{Tr}(\rho^2) = 1$. Then, developing this expression, we have to find that $\|\,\mathbf{r}\| = 1$. An alternative proof can be found using the following property of Pauli's matrices

$$\mathrm{Tr}(\sigma_i \sigma_j) = 2\,\delta_{ij}$$

in (3.92). Rewriting (3.92) in the form

$$\rho = \frac{1}{2}\sum_{i=0}^{3} r_i\,\sigma_i\,, \qquad r_0 = 1$$

we find

$$\mathrm{Tr}(\rho^2) = \mathrm{Tr}\left(\frac{1}{4}\sum_{i=0}^{3}\sum_{j=0}^{3} r_i\,\sigma_i r_j\,\sigma_j\right) = \frac{1}{2}(1 + r_x^2 + r_y^2 + r_z^2) = 1\,.$$

The pure state $|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ is found imposing the condition $|\psi\rangle\langle\psi| = \rho$, where, without restriction we can assume that a is real and positive. Then

$$\begin{bmatrix} a \\ b \end{bmatrix}[a^* \quad b^*] = \begin{bmatrix} |a|^2 & ab^* \\ a*b & |b|^2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - \mathrm{i}r_y \\ r_x + \mathrm{i}r_y & 1 - r_z \end{bmatrix}\,.$$

which gives

$$|a|^2 = \tfrac{1}{2}(1 + r_z)\,, \qquad |b|^2 = \tfrac{1}{2}(1 - r_z)\,, \qquad ab^* = \tfrac{1}{2}(r_x - \mathrm{i}r_y)\,.$$

With $a := u_a \mathrm{e}^{\mathrm{i}\phi_a}$ and $b := u_b \mathrm{e}^{\mathrm{i}\phi_b}$ we get

$$u_a^2 = \tfrac{1}{2}(1 + r_z)\,, \qquad u_b^2 = \tfrac{1}{2}(1 - r_z)\,, \qquad u_a u_b \mathrm{e}^{\mathrm{i}(\phi_a - \phi_b)} = \tfrac{1}{2}(r_x - \mathrm{i}r_y)\,.$$

Hence

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} u_a \mathrm{e}^{\mathrm{i}\phi_a} \\ u_b \mathrm{e}^{\mathrm{i}\phi_b} \end{bmatrix} = \mathrm{e}^{\mathrm{i}\phi_a}\begin{bmatrix} u_a \\ u_b \mathrm{e}^{\mathrm{i}(\phi_b - \phi_a)} \end{bmatrix}$$

where the phasor $\mathrm{e}^{\mathrm{i}\phi_a}$ can be neglected. In conclusion we find

$$|\psi\rangle = \begin{bmatrix} u_a \\ \dfrac{r_x - \mathrm{i}r_y}{2\,u_a} \end{bmatrix} \quad \text{with} \quad u_a = \sqrt{\tfrac{1}{2}(1 + r_z)}\,.$$

Problem 3.17 ⋆⋆∇ [Sect. 3.12] Using Schmidt's decomposition given in Chapter 10, prove Proposition 3.6.

Solution In Section 10.3 it is proved that a pure state is separable if and only if all but one of the singular values of the coefficient matrix are zero. In the present case the singular values are given by the eigenvalues of the matrix $A = [a_{ij}]$, that is, the solution of the equation

$$\det(A - \lambda I) = (\lambda - a_{11})(\lambda - a_{22} - a_{12}a_{21}) \ .$$

The two solutions are λ_1 and 0 if and only if $a_{01}a_{10} = a_{00}a_{11}$.

Problem 3.18 [Sect. 3.12] Prove Proposition 3.6 using the considerations of Section 3.10.2, in particular relations (3.71) to (3.73).

Solution In the present case relation (3.71) is given by (3.94)

$$|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|01\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle \tag{3.5}$$

while (3.72), which expresses $|\psi\rangle$ as the tensor product of two kets, becomes

$$|\psi\rangle = (u_0|0\rangle + u_1|1\rangle) \otimes (v_0|0\rangle + v_1|1\rangle) = u_0v_0|00\rangle + u_0v_1|01\rangle + u_1v_0|10\rangle + u_1v_1|11\rangle \ .$$

Then the separability conditions are

$$a_{00} = u_0v_0 \, , \qquad a_{01} = u_0v_1 \, , \qquad a_{10} = u_1v_0 \, , \qquad a_{11} = u_1v_1 \ .$$

Combination of the first and the third and combination of the second with the fourth give

$$u_0/u_1 = a_{00}/a_{10} \, , \qquad u_0/u_1 = a_{01}/a_{11}$$

which leads to the condition claimed by Proposition 3.6

Problems of Chapter 4

Problem 4.1 ⋆ [Sect. 4.2] A still image (photo) is quantized in 800×800 pixels with 8 bit/pixel and transmitted by a digital channel with nominal rate $R_0 = 100$ kbit/s. Find 1) the signal–to–quantization error Λ_q, 2) the error probability P_e of the digital channel such that the channel error is negligible, and 3) the time needed to transmit the photo. Note that the global SNR is given by[1]

$$\Lambda = \Lambda_q/(1+P_e L^2)\,.$$

Solution We have:

$$L = 2^8 = 512\,, \qquad \Lambda_q = 512^2 = 262\,144$$

and

$$H = 800 \times 800 \times 8 = 2\,097\,152 \text{ bit} \simeq 2.1 \text{ Mbit}\,.$$

It is reasonable to assume that

$$P_e = \frac{1}{10} L^{-2} = 3.8\ 10^{-7}$$

in order that the channel error is negligible, that is, $\Lambda \simeq \Lambda_q$. Then the time needed to transmit the photo is

$$T = H/R_0 = 20.9 \text{ s} \simeq 21 \text{ s}\,.$$

Problem 4.2 ⋆ [Sect. 4.2] A video signal (produced by a TV camera) has bandwidth $B = 5$ MHz. Evaluate the A/D conversion parameters that ensure $\Lambda_q = 60$ dB, and in particular the nominal rate of the digital channel.

Solution We have from (4.10)

$$\Lambda_q = 2^m \simeq 10^6 \quad \to \quad m = 10 \text{ bit/symbol}\,.$$

Then

$$R_0 = 2B = 100 \text{ Mbit/s.}$$

The error probability of the channel can be chosen as

$$P_e = \frac{1}{10} L^{-2} \simeq\ 10^{-7}$$

in order that the channel error is negligible.

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[1] A. J. Viterbi and J. K. Omura, *Principles of digital communication and coding*. Dover Books on Electrical Engineering, 2009.

Note that with the technique of compression (Mpeg) nowadays the rate for a good TV channels is of a few Mbit/s.

Problem 4.3 $\star$ [Sect. 4.4] Write the expressions of a PAM optical power $P(t)$, with a generic fundamental pulse $g(t)$, valid for all $t \in \mathbb{R}$,

Solution Denoting by $\{A_n = A(nT)\}$ the sequence of symbols, in the PAM modulation the contribution of the nth symbol is given by

$$A_n\, g(t - nT).$$

The PAM signal is given by all the contributions and therefore the expression is given by

$$P(t) = \sum_{n=\infty}^{+\infty} A_n\, g(t - nT) \ , \qquad t \in \mathbb{R} \ .$$

Note that in the interval $(nT, nT + T]$ the power $P(t)$ is not equal to $A_n\, g(t - nT)$, because in this interval we have also the contribution of the other pulses. Only when the duration of the pulse $g(t)$ is confined to the interval $(0, T]$ we find

$$P(t) = A_n\, g(t - nT)\,, \qquad nT < t \leq nT + T \ .$$

Problem 4.4 $\star$ [Sect. 4.4] The physical parameters of the transmitter (on spacecraft board) and of the receiver (at Earth, Goldstone, California) of NASA Voyager 2 mission at Jupiter (1979) were: radio frequency $\nu = 8.9$ GHz, transmitted power $P_T = 24$ W, transmitter's antenna diameter $d_T = 3.660$ m, receiver's antenna diameter $d_R = 64$ m, noise temperature $T_r = 14$ K, accepted error probability $P_e = 10^{-3}$. Find the available rate. Repeat the evaluation at the optical frequency $\nu = 300$ THz.

Solution

Not available

Problem 4.5 $\star\star$ [Sect. 4.5] Consider a Poisson random variable n. Prove that the variance σ_n^2 is equal to the mean $m_n = \mathrm{E}[n]$ and that the characteristic function is given by (4.23).

Solution We first evaluate the characteristic function

$$\begin{aligned}\Psi_n(z) :=& \mathrm{E}[\mathrm{e}^{\mathrm{i}nz}] = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}zk}\, \mathrm{P}[n = k] \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}zk} \mathrm{e}^{-\Lambda} \frac{\Lambda^k}{k!} = \mathrm{e}^{-\Lambda} \sum_{k=0}^{\infty} \frac{\left(\mathrm{e}^{\mathrm{i}z}\Lambda\right)^k}{k!}\end{aligned}$$

where we have found the expansion of e^x with $x = \mathrm{e}^{\mathrm{i}z\Lambda}$. Then

$$\Psi_n(z) = \mathrm{e}^{-\Lambda}\,\mathrm{e}^{\mathrm{e}^{\mathrm{i}z\Lambda}} = \mathrm{e}^{\Lambda\,(\mathrm{e}^{\mathrm{i}z}-1)} \ .$$

From the characteristic function we can evaluate the moments as

$$m_n(k) = \frac{1}{\mathrm{i}^k} \frac{\mathrm{d}^k \Psi_n(z)}{\mathrm{d}z^k}\Big|_{z=0} .$$

Considering that

$$d\Psi(z)/dz = \mathrm{i}\Lambda e^{\Lambda\left[-1+e^{\mathrm{i}z}\right]+\mathrm{i}z}, \qquad d^2\Psi(z)/dz^2 = \mathrm{i}\Lambda \mathrm{e}^{\Lambda\left[-1+e^{\mathrm{i}z}\right]+\mathrm{i}z}\left[\mathrm{i}\mathrm{e}^{\mathrm{i}z}\Lambda + \mathrm{i}\right]$$

we find

$$m_n(1) = \Lambda\,, \qquad m_n(2) = \Lambda + \Lambda^2 .$$

Hence

$$\sigma_n^2 = m_n(2) - m_n^2(1) = \Lambda .$$

Problem 4.6 $\star$ [Sect. 4.5] In the the technique of **single photon** the following probabilities are of interest

$$p_0 = \mathrm{P}[n=0]\,, \qquad p_1 = \mathrm{P}[n=1]\,, \qquad p_{>1} = \mathrm{P}[N>1] .$$

Assuming that the arrival are described by a Poisson process, write and plot these probabilities. Moreover, find the average of photon arrivals such that $p_{>1} = 0.1 p_1$

Solution The probabilities are obtained from the Poisson distribution (4.21), that is,

$$p_n(k) = \mathrm{e}^{-\Lambda} \frac{\Lambda^k}{k!} .$$

We have the expressions

$$\begin{aligned} p_0 =& \mathrm{e}^{-\Lambda}\,, \qquad p_1 = \Lambda\,\mathrm{e}^{-\Lambda} \\ p_{>1} =& 1 - \mathrm{e}^{-\Lambda} - \Lambda\,\mathrm{e}^{-\Lambda} \end{aligned}$$

which are illustrated in Fig. 4.1. The relation $p_{>1} = 0.1 p_1$ is verified with $\Lambda = 0.828863$ and with such value of Λ we have

$$p_0 = 0.828863\,, \qquad p_1 = 0.155578\,, \qquad p_2 = \mathrm{P}[n=2] = 0.014601 .$$

Thus we have the condition of a single photons in the 15.5% of cases, but in the 85.8% of cases we have no arrivals.

Problem 4.7 $\star$ [Sect. 4.5] In the the technique of **single photon** the optical power is attenuated to realize the condition of the arrival of a single photon in a given symbol period $(0,T]$. Assuming that the power produced by the laser be $P_0 = 10$ mW at the frequency $\nu = 300$ THz and that the symbol period be $T = 10$ ns, find the attenuation A needed to ensure that the condition of a single photon is verified in the 15% of the symbol periods.

Solution The energy in a symbol period is $E_0 = P_0 T = 10^{-3}\,10^{-8} = 10^{-11}$ J. The quantum at the frequency $\nu = 300$ THz has energy

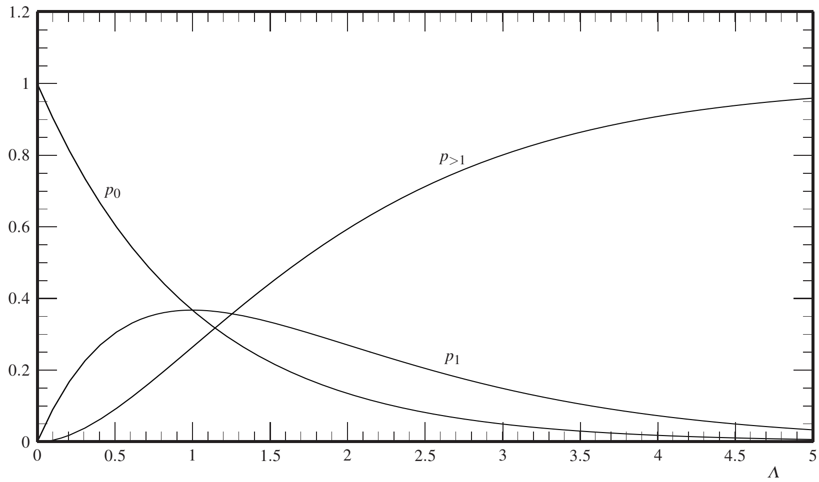

Fig. 4.1 Probabilities from a Poisson distribution as functions of the average number of arrivals Λ.

$$h\nu = 6.626\,10^{-34}\,300\,10^{12} = 1.927810^{-19}\text{ J}\,.$$

Then the average number of photons is

$$\Lambda_0 = \frac{E_0}{h\nu} = \frac{10^{-11}}{1.9278\,10^{-19}} = 5.03069\,10^9 \text{ photons per period}\,.$$

In the previous problem we have seen that to have of a single photon in the 15% of the symbol periods it is required an average number of photon $\Lambda = 0.828863$. Then the needed attenuation is $6.06939\,10^9 = 97.8314$ dB.

Problem 4.8 ⋆⋆ [Sect. 4.6] Evaluate the mean $m_y(t)$ and the variance $\sigma_y(t)^2$ of a filtered Poisson process $y(t)$, where the intensity is constant $\lambda(t) = \lambda_0$, and the fundamental pulse is rectangular of amplitude h_0 in $(0, T]$.

Solution From the first Campbell's theorem we have

$$m_y = \lambda_0 \int_{-\infty}^{+\infty} h(t)\mathrm{d}t = \lambda_0\, h_0\, T$$

and

$$k_y(\tau) = \lambda_0 \int_{-\infty}^{+\infty} h(t+\tau)h(t)\mathrm{d}t = \begin{cases} \lambda_0\, h_0^2\, T\,(1 - |\tau|/T) & |t| \le T \\ 0 & |t| > T\,. \end{cases}$$

In particular the variance is

$$\sigma_y^2 = k_y(0) = \lambda_0\, h_0^2\, T\,.$$

Problem 4.9 ⋆⋆⋆ [Sect. 4.6] Evaluate the mean $m_y(t)$ and the variance $\sigma_y(t)^2$ of a marked and filtered Poisson process $y(t)$, where the intensity is constant $\lambda(t) = \lambda_0$, the fundamental pulse is rectangular of amplitude h_0 in $(0,T]$, and the gains has the geometrical distribution. $p_g(k) = (1-a)a^k$, $k = 0,1,2,\ldots$ with a a positive constant.

Solution Since the intensity is constant, the process $y(t)$ is stationary and the mean and the variance are independent of t. From the second Campbell's theorem we find

$$m_y = G\lambda_0 \int_{-\infty}^{+\infty} h(t)\mathrm{d}t = G\lambda_0 h_0 T$$

and

$$\sigma_y^2 = \lambda_0 \int_{-\infty}^{+\infty} h^2(t)\mathrm{d}t = G_2\,\lambda_0\, h_0^2 T\ .$$

For the evaluation of the gain $G = \mathrm{E}[g_i]$ and of the quadratic gain $G_2 = \mathrm{E}[g_i^2]$ we use the moments' theorem (see solution of Problem 4.5)

$$m_n(k) = \frac{1}{\mathrm{i}^k}\,\frac{\mathrm{d}^k\Psi_n(z)}{\mathrm{d}z^k}\Big|_{z=0}\ .$$

which gives

$$G := m_g(1) = -\mathrm{i}\frac{\mathrm{d}\Psi_n(z)}{\mathrm{d}z}\Big|_{z=0}$$
$$G_2 := m_g(2) = -\frac{\mathrm{d}^2\Psi_n(z)}{\mathrm{d}z^2}\Big|_{z=0}\ .$$

The characteristic function of the gains is

$$\begin{aligned}\Psi_g(z) :=& \mathrm{E}[\mathrm{e}^{\mathrm{i}gz}] = \sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}zk}\,\mathrm{P}[g=k]\\ =&(1-a)\sum_{k=0}^{\infty} \mathrm{e}^{\mathrm{i}zk}\,a^k = (1-a)\sum_{k=0}^{\infty}\left(\mathrm{e}^{\mathrm{i}z}\,a\right)^k\\ =&(1-a)\frac{1}{1-\mathrm{e}^{\mathrm{i}z}\,a}\ .\end{aligned}$$

Hence

$$G = \frac{a}{1-a}, \qquad G_2 = \frac{a(1+a)}{(1-a)^2}\ .$$

Problem 4.10 ⋆⋆ [Sect. 4.6] To illustrate the conditioning of a double stochastic Poisson process we consider a (simplified) binary modulation, where the intensity of the optical power in $(0,T]$ has the values $\lambda_0 = 10^9$ photon/s with the symbol $A = 0$ and $\lambda_1 = 4\ 10^9$ photon/s with the symbol $A = 1$. Find the *conditioned* distribution of the number of arrivals

$$p_n(k|A=0)\,, \qquad p_n(k|A=1)$$

and the *unconditioned* distribution $p_n(k)$, assuming $\mathrm{P}[A=1]=1/4$ and $T=1$ ns.

Solution The conditioned distributions, where the intensity is *given* through the condition of the symbol, are both Poisson distributions, and therefore specified by the conditioned mean of the arrivals. These means are given by

$$\Lambda_0 = \lambda_0 T = 10^9\, 10^{-9} = 1\,, \qquad \Lambda_1 = \lambda_1 T = 10^9\, 4\, 10^{-9} = 4$$

and then

$$p_n(k|A=0) = \frac{\Lambda_0^k}{k!}\exp(-\Lambda_0)\,, \qquad p_n(k|A=1) = \frac{\Lambda_1^k}{k!}\exp(-\Lambda_1)\;.$$

The unconditional distribution is obtained by averaging the two conditioned distributions

$$p_n(k) = \tfrac{3}{4} p_n(k|A=0) + \tfrac{1}{4} p_n(k|A=1)$$

and is not a Poisson distribution.

The distributions are illustrated in Fig.4.2.

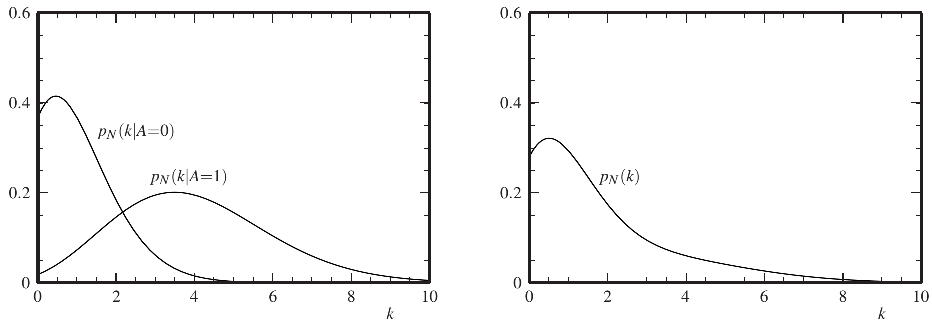

Fig. 4.2 The conditioned distributions $p_N(k|A=0)$ and $p_N(k|A=1)$ (Poissonian) and the unconditioned distribution $p_N(k)$ (non Poissonian). For semplicity the discrete plot is represented by a continuous line.

Problem 4.11 $\star$ [Sect. 4.8] Consider the counting of the random variable $u = n+\eta$, where n is Poissonian with mean N_R and η is Gaussian with zero mean and variance σ_η^2, with u and η independent. Since u is continuous, for the counting we have to introduce a rounding. Find the probability distribution of $v = \mathrm{round}(u)$.

Solution The probability density of u is given by (4.75)

$$f_u(a) = \sum_{k=0}^{\infty} p_n(k)\,\frac{1}{\sigma_\eta}\,\varphi\left(\frac{a-k}{\sigma_\eta}\right)\,, \qquad p_n(k) = e^{-N_R}\,\frac{N_R^k}{k!}\;.$$

Introducing the rounding

$$v = \mathrm{round}(u) = \begin{cases} 0 & u < \frac{1}{2} \\ s & s - \frac{1}{2} < u < s + \frac{1}{2} \end{cases}$$

we find for $s = 0$

$$p_v(0) = \mathrm{P}\left[u < \frac{1}{2}\right] = \int_{-\infty}^{\frac{1}{2}} f_u(a)\mathrm{d}a = \sum_{k=0}^{\infty} p_n(k)\,\Phi\left(\frac{\frac{1}{2} - k}{\sigma_\eta}\right)$$

and for $s > 0$

$$\begin{aligned} p_v(s) =& \mathrm{P}\left[s = \frac{1}{2} < u < s + \frac{1}{2}\right] = \int_{s-\frac{1}{2}}^{s+\frac{1}{2}} f_u(a)\mathrm{d}a \\ =& \sum_{k=0}^{\infty} p_n(k)\ \left[\Phi\left(\frac{s + \frac{1}{2} - k}{\sigma_\eta}\right) - \Phi\left(\frac{s - \frac{1}{2} - k}{\sigma_\eta}\right)\right] \end{aligned}$$

where $\Phi(x)$ is the normalized Gaussian distribution. The probability distribution depends on the same parameters, $N_\gamma = N_R$ and $\mathcal{N} = \sigma_\eta^2$, as of Laguerre distribution. The two distributions are similar but non coincident, as shown below for $N_\gamma = 0.2$ and $\mathcal{N} = 0.2$

$$p_v(k) = \begin{bmatrix} 0 & 0.367953 \\ 1 & 0.343445 \\ 2 & 0.192001 \\ 3 & 0.0714811 \\ 4 & 0.0198199 \\ 5 & 0.00436307 \\ 6 & 0.00079474 \\ 7 & 0.000123301 \\ 8 & 0.0000166456 \\ 9 & 1.987673\,10^{-6} \\ 10 & 2.126917\,10^{-7} \end{bmatrix}, \qquad p_L(k) = \begin{bmatrix} 0 & 0.362165 \\ 1 & 0.311864 \\ 2 & 0.181222 \\ 3 & 0.0865139 \\ 4 & 0.0364775 \\ 5 & 0.014087 \\ 6 & 0.00509043 \\ 7 & 0.00174521 \\ 8 & 0.000573145 \\ 9 & 0.000181567 \\ 10 & 0.0000557765 \end{bmatrix}$$

Problems of Chapter 5

Problem 5.1 ⋆⋆ [Sect. 5.2] Prove Proposition 5.1. *Hint*: see Section 3.6.4.

Solution From the orthogonality and the idempotency of the P_k, we find that the Q_i are idempotent (projectors) and mutually orthogonal

$$Q_i^2 = \Big(\sum_{k\in\mathcal{M}_i} P_k\Big)^2 = \sum_{k\in\mathcal{M}_i} P_k = Q_i$$
$$Q_iQ_j = \Big(\sum_{k\in\mathcal{M}_i} P_k\Big)\Big(\sum_{h\in\mathcal{M}_j} P_h\Big) = 0 \quad \text{if} \quad i\neq j$$

where we used the fact that $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for $i \neq j$.

Problem 5.2 ⋆⋆ [Sect. 5.2] **Optimization of decision element**. In a post–measurement decision the decision element is a mapping: $\mathcal{M} \to \mathcal{A}$, where $|\mathcal{M}| \geq |\mathcal{A}|$, in which every point $k \in \mathcal{M}$ must be associated to a symbol $a \in \mathcal{A}$, thus creating a partition of $\mathcal{M}$ into K sets $\mathcal{M}_a, a \in \mathcal{A}$. For given a priori probabilities $\{q_i\}$ and transition probabilities $\{p_c(j|i)\}$, one can optimize the decision element with the criterion to get the maximum correct decision probability. Prove the following statement: *Define the K decision functions as*

$$f_a(k) := q_a\, p_c(k|a), , \qquad a \in \mathcal{A}, k \in \mathcal{M}\ .$$

Then, for each $k \in \mathcal{M}$, find the decision function $f_a(k)$ such that

$$f_a(k) \geq f_b(k)\,, \qquad \forall b \neq a. \tag{5.1}$$

The value of a that verifies (5.1) is placed in $\mathcal{M}_a$. This defines the sets $\mathcal{M}_a$ that determine the optimum decision element.

Solution Recall that in a post–measurement decision we have two alphabets: the source alphabet $\mathcal{A} = \{0, 1, \ldots, K-1\}$ and the measurement alphabet $\mathcal{M} = \{1, \ldots, K'\}$.

The correct decision probability is given by (see (5.11) and (5.12))

$$P_c := \mathrm{P}\left[\widehat{A} = A\right] = \sum_{a\in\mathcal{A}} q_a\, p_c(a|a)$$
$$\sum_{a\in\mathcal{A}} \sum_{k\in\mathcal{M}_a} q_a\, p(k|a)$$

and introducing the decision functions we have

$$P_c = \sum_{a\in\mathcal{A}} \sum_{k\in\mathcal{M}_a} f_a(k)\ .$$

Clearly, if the partition $\mathcal{M}_a$ is chosen as in (5.17), P_c assumes its maximum value.

In the subsequent two problems we apply the above statement to two specific cases.

Problem 5.3 ⋆⋆ [Sect. 5.2] In a binary system $\{0,1\}$, where the a priori probabilities are $q(0) = 1/3$ and $q(1) = 2/3$, the quantum measurement, obtained with a photon counting, gives two Poisson variables with averages $\Lambda_0 = \mathrm{E}[m|A=0] = 5$ and $\Lambda_1 = \mathrm{E}[m|A=1] = 20$.

Apply the statement of the previous problem to find the optimum decision element.

Solution The alphabet of the measurement is $\mathcal{M} = \{0,1,2,\ldots\}$ and we have to find a partition of $\mathcal{M}$ into two subsets $\mathcal{M}_0$ and $\mathcal{M}_1$. In the binary case (5.17) can be written as

$$f_0(k) \geq f_1(k)$$

and the values of k that verify this condition give the set $\mathcal{M}_0$. The condition is explicitly

$$\frac{1}{3}\,\mathrm{e}^{-\Lambda_0}\,\frac{\Lambda_0^k}{k!} \geq \frac{2}{3}\,\mathrm{e}^{-\Lambda_1}\,\frac{\Lambda_1^k}{k!}\,.$$

With $\Lambda_0 = 5$ and $\Lambda_1 = 20$ we get the table

k	$f_0(k)$	$f_1(k)$
0	0.00224598	$1.3741024149590386\,10^{-9}$
1	0.0112299	$2.748204829918077\,10^{-8}$
2	0.0280748	$2.748204829918077\,10^{-7}$
3	0.0467913	$1.8321365532787178\,10^{-6}$
4	0.0584891	$9.16068276639359\,10^{-6}$
5	0.0584891	0.0000366427
6	0.0487409	0.000122142
7	0.034815	0.000348978
8	0.0217593	0.000872446
9	0.0120885	0.00193877
10	0.00604426	0.00387754
11	0.00274739	0.00705007
12	0.00114475	0.0117501
13	0.000440287	0.0180771
14	0.000157245	0.0258244
15	0.0000524151	0.0344326
16	0.0000163797	0.0430407
17	$4.817568453464372\,10^{-6}$	0.0506361
18	$1.338213459295659\,10^{-6}$	0.0562624
19	$3.521614366567524\,10^{-7}$	0.0592235
20	$8.80403591641881\,10^{-8}$	0.0592235

Thus we find that $f_0(k) \geq f_1(k)$ for $k = 0,1,\ldots,9,10$ and therefore

$$\mathcal{M}_0 = \{0,1,\ldots,9,10\}\,, \qquad \mathcal{M}_1 = \{11,12,13,\ldots\}\,.$$

The correct decision probability is

$$P_c = \sum_{k=0}^{10} f_0(k) + \sum_{k=11}^{\infty} f_1(k) = 0.988227\,.$$

Problem 5.4 $\star$ [Sect. 5.2] As in the previous problem but with $\Lambda_0 = 0$ and $\Lambda_1 = 20$ and equally likely symbols.

Solution In this case the conditional probability $p_c(k|0)$ degenerates as

$$p_c(k|0) = \begin{cases} 1 & k = 0 \\ 0 & k \geq 1 \end{cases}$$

and we have

$$\tfrac{1}{2} p_c(k|0) \begin{cases} > \tfrac{1}{2} p_c(k|1) & k = 0 \\ < \tfrac{1}{2} p_c(k|1) & k \geq 1\,. \end{cases}$$

Then the optimum decision criterion is obtained with

$$\mathcal{M}_0 = \{0\}\,, \qquad \mathcal{M}_1 = \{1, 2, \dots\}\,.$$

After this choice the conditional error probabilities result in

$$P_e(0) = \mathrm{P}\,[n > 0|A = 0] = 0\,, \qquad P_e(1) = P[n = 0|A = 1] = \mathrm{e}^{-\Lambda_1}$$

and the error probability is

$$P_e = \frac{1}{2} P_e(0) + \frac{1}{2} P_e(1) = \frac{1}{2} \mathrm{e}^{-\Lambda_1} \sim 10^{-9}\,.$$

Note that with the transmission of the symbol $A = 0$ there is no error. An error happens when, having transmitted $A = 1$, no photon is detected.

Problem 5.5 $\star\star$ [Sect. 5.3] Prove that the operators Q_1 and Q_0, defined by (5.24), form a projector system.

Solution The completeness $Q_0 + Q_1 = I$ has been imposed at the beginning of the proof. Q_1 is actually a projector, because the $|\eta_k\rangle$ are orthonormal, and therefore $Q_1^2 = Q_1$. Analogously, we prove the idempotency of Q_0, and finally from (5.24) we have $Q_1 Q_0 = 0$, that is, the orthogonality.

Problem 5.6 $\star\star$ [Sect. 5.3] Consider the following density operators

$$\rho_0 = \frac{1}{208} \begin{bmatrix} 46 & 13 - 37\mathrm{i} & -16 & 13 + 37\mathrm{i} \\ 13 + 37\mathrm{i} & 58 & 13 - 37\mathrm{i} & -32 \\ -16 & 13 + 37\mathrm{i} & 46 & 13 - 37\mathrm{i} \\ 13 - 37\mathrm{i} & -32 & 13 + 37\mathrm{i} & 58 \end{bmatrix}$$

$$\rho_1 = \frac{1}{208}\begin{bmatrix} 58 & 29-29\mathrm{i} & 8 & 21+29\mathrm{i} \\ 29+29\mathrm{i} & 58 & 29-21\mathrm{i} & -8 \\ 8 & 29+21\mathrm{i} & 46 & 21-21\mathrm{i} \\ 21-29\mathrm{i} & -8 & 21+21\mathrm{i} & 46 \end{bmatrix}.$$

First verify that they are "true" density operators. Then, assuming that they are the states in a binary transmission with a priory probabilities $q_0 = 1/5$ and $q_1 = 4/5$, find the correct decision probability P_c.

Solution By inspection ρ_0 and ρ_1 are Hermitian and have unitary trace. More complicate is to check that they are PSD. To this end we can use Theorem 2.6 which claims that a matrix is PSD if all its eigenvalues are nonnegative. The eigenvalues of ρ_1 and ρ_0 are respectively

$$\{0.546889, 0.444022, 0.00750357, 0.00158567\}$$
$$\{0.727465, 0.259985, 0.00924611, 0.00330442\}$$

and are all positive. Then we conclude that ρ_0 and ρ_1 are "true" density operators.

To find the correct decision probability we apply (5.23)

$$P_c = q_0 + \sum_{\eta_k>0} \eta_k$$

where η_k are the eigenvalues of the decision operator

$$D = q_1\rho_1 - q_0\rho_0 = \frac{1}{1040}\begin{bmatrix} 186 & 103-79\mathrm{i} & 48 & 71+79\mathrm{i} \\ 103+79\mathrm{i} & 174 & 103-47\mathrm{i} & 0 \\ 48 & 103+47\mathrm{i} & 138 & 71-47\mathrm{i} \\ 71-79\mathrm{i} & 0 & 71+47\mathrm{i} & 126 \end{bmatrix}.$$

The eigenvalues are

$$\{0.360242, 0.236339, 0.00442223, -0.00100303\}$$

and then

$$P_c = 1/5 + 0.360242 + 0.236339 + 0.00442223 = 0.801003\,.$$

Problem 5.7 $\star\star$ [Sect. 5.4] Find the coefficients a_{01} and a_{11} in the expression of the measurement vectors (5.35), assuming equally likely symbols and X real.

Solution The measurement vectors are given by

$$|\eta_0\rangle = a_{00}|\gamma_0\rangle + a_{01}|\gamma_1\rangle\,, \qquad |\eta_1\rangle = a_{10}|\gamma_0\rangle + a_{11}|\gamma_1\rangle\,.$$

For the geometrical independence of $|\gamma_0\rangle$ and $|\gamma_1\rangle$ we found (see (5.35))

$$|\eta_0\rangle = a_{00}\left(|\gamma_0\rangle + \frac{q_1X^*}{\eta_0 - q_1}|\gamma_1\rangle\right), \qquad |\eta_1\rangle = a_{11}\left(-\frac{q_0X}{\eta_1 + q_0}|\gamma_0\rangle + |\gamma_1\rangle\right)$$

Then, with equally likely symbols and X real we get

$$|\eta_0\rangle = a_{00}\left(|\gamma_0\rangle + \frac{X}{2\eta_0 - 1}|\gamma_1\rangle\right), \qquad |\eta_1\rangle = a_{11}\left(-\frac{X}{2\eta_1+1}|\gamma_0\rangle + |\gamma_1\rangle\right)$$

where

$$\eta_{0,1} = \mp\frac{1}{2}\Delta, \qquad \Delta = \sqrt{1-X^2}.$$

Hence

$$a_{01} = \frac{X}{2\eta_0 - 1}\,a_{00} = -\frac{X}{1+\sqrt{1-X^2}}\,a_{00}$$
$$a_{10} = -\frac{X}{2\eta_1 + 1}\,a_{11} = -\frac{X}{1+\sqrt{1-X^2}}\,a_{11}.$$

The nomalizations of $|\eta_0\rangle$ and $|\eta_1\rangle$ give

$$1 = ||\eta_0\rangle|^2 = \frac{2\left[X^2-1\right]}{\sqrt{1-X^2}-1}\,a_{00}^2$$

$$1 = ||\eta_1\rangle|^2 = \frac{2\left[X^2-1\right]}{\sqrt{1-X^2}-1}\,a_{11}^2.$$

Hence

$$a_{00} = a_{11} = \sqrt{\frac{\sqrt{1-X^2}-1}{2\left[X^2-1\right]}}.$$

In conclusion, the measurement vectors are obtained from the states as

$$M = \Gamma A$$

where

$$\Gamma = [|\gamma_0\rangle, |\gamma_1\rangle], \; M = [|\mu_0\rangle, |\mu_1\rangle], \; A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}.$$

The natrix A is given by

$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{\sqrt{1-X^2}-1}{2[X^2-1]}} & -\frac{X}{1+\sqrt{1-X^2}}\sqrt{\frac{\sqrt{1-X^2}-1}{2[X^2-1]}} \\ -\frac{X}{1+\sqrt{1-X^2}}\sqrt{\frac{\sqrt{1-X^2}-1}{2[X^2-1]}} & \sqrt{\frac{\sqrt{1-X^2}-1}{2[X^2-1]}} \end{bmatrix}.$$

Note that with some algebra the result can be written in the form

$$A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

with (see (5.40))

$$a = \frac{1}{2}\left[\frac{1}{\sqrt{1-|X|}} + \frac{1}{\sqrt{1+|X|}}\right], \qquad b = \frac{1}{2}\left[\frac{1}{\sqrt{1+|X|}} - \frac{1}{\sqrt{1-|X|}}\right]. \tag{5.2}$$

Problem 5.8 ⋆⋆ [Sect. 5.4] Write the fundamental relations of the geometrical approach in matrix form, using the matrices

$$\Gamma = [|\gamma_0\rangle, |\gamma_1\rangle], \qquad U = [|u_0\rangle, |u_1\rangle], \qquad M = [|\mu_0\rangle, |\mu_1\rangle] .$$

Solution We want to check Proposition 5.2 and in particular the evaluation of the optimal measurement matrix. We assume a **real** inner product

$$Y := \langle\gamma_0|\gamma_1\rangle$$

so that M_{opt} is given by (5.46).

Quantum states in terms of basis kets

The relations

$$|\gamma_0\rangle = \cos\theta|u_0\rangle + \sin\theta|u_1\rangle$$
$$|\gamma_1\rangle = \cos\theta|u_0\rangle - \sin\theta|u_1\rangle$$

become

$$\Gamma = B\,U \quad \text{with} \quad B == \begin{bmatrix} \cos(\theta) & \cos(\theta) \\ \sin(\theta) & -\sin(\theta) \end{bmatrix} .$$

The inverse of this relation is

$$U = \Gamma\, B^{-1} \quad \text{with} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} \sec(\theta) & \csc(\theta) \\ \sec(\theta) & -\csc(\theta) \end{bmatrix} . \tag{5.3}$$

We now impose the orthonormality of the vector U, that is,

$$U^*U = B^{-1^*}\Gamma^*\Gamma B^{-1} = B^{-1^*}GB^{-1} = I_2$$

where G is the Gram matrix

$$G = \Gamma^*\Gamma = \begin{bmatrix} 1 & Y \\ Y & 1 \end{bmatrix} .$$

Explicitly we have

$$U^*U = \begin{bmatrix} \frac{1}{2}(Y+1)\sec^2(\theta) & 0 \\ 0 & -\frac{1}{2}(Y-1)\csc^2(\theta) \end{bmatrix}$$

which implies

$$\cos^2\theta = \frac{1}{2}(1+Y)\ ,\ \sin^2\theta = \frac{1}{2}(1-Y)\ .$$

Then we find

$$B^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}\sqrt{Y+1}} & \frac{1}{\sqrt{2}\sqrt{1-Y}} \\ \frac{1}{\sqrt{2}\sqrt{Y+1}} & -\frac{1}{\sqrt{2}\sqrt{1-Y}} \end{bmatrix}\ .$$

Measurement vectors in terms of basis kets

The relations

$$|\mu_0\rangle = \cos\phi|u_0\rangle + \sin\phi|u_1\rangle$$
$$|\mu_1\rangle = \sin\phi|u_0\rangle - \cos\phi|u_1\rangle$$

become

$$M = U\,C \quad \text{with} \quad C := \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ \sin(\phi) & -\cos(\phi) \end{bmatrix}\ .$$

Measurement vectors in terms of quantum states

We combine the previous relations to find

$$M = \Gamma B^{-1}\,C = \Gamma\,A \tag{5.4}$$

thus we have

$$\boxed{A = B^{-1}C}$$

and explicitly

$$A = \begin{bmatrix} \frac{\cos(\phi)}{\sqrt{2}\sqrt{X+1}} + \frac{\sin(\phi)}{\sqrt{2}\sqrt{1-X}} & \frac{\sin(\phi)}{\sqrt{2}\sqrt{X+1}} - \frac{\cos(\phi)}{\sqrt{2}\sqrt{1-X}} \\ \frac{\cos(\phi)}{\sqrt{2}\sqrt{X+1}} - \frac{\sin(\phi)}{\sqrt{2}\sqrt{1-X}} & \frac{\cos(\phi)}{\sqrt{2}\sqrt{1-X}} + \frac{\sin(\phi)}{\sqrt{2}\sqrt{X+1}} \end{bmatrix}\ . \tag{5.5}$$

Check of orthonormality of measurement vectors

In the formulation we have imposed the orthonormality of the measurement vectors (see Fig. 5.6). The condition is expressed as $M^*M = I_2$. Using the previous matrix relations we have

$$M^*M = A^*GA = C^*B^{-1^*}GB^{-1}C = I_2$$

where B^{-1} is given by (5.3) and C by (5.4). The expression we obtained for $C^*B^{-1^*}GB^{-1}C$ is a very long function of Y and ϕ, but with `Mathematica` we were able to simplify it to I_2, independently of Y and ϕ.

Introduction of the optimal ϕ

We now introduce the angle ϕ giving the maximum correct decision probability, stated by (5.42), that is,

$$\tan 2\phi = \frac{1}{q_0 - q_1} \tan 2\theta = \frac{1}{q_0 - q_1} \frac{\sqrt{1 - Y^2}}{Y}$$

which gives

$$\sin 2\phi = \frac{1}{R} \sin 2\theta \quad , \qquad \cos 2\phi = \frac{q_0 - q_1}{R} \cos 2\theta$$

$$\sin 2\phi = \frac{1}{R} \sin 2\theta \quad , \qquad \cos 2\phi = \frac{q_0 - q_1}{R} \cos 2\theta$$

where $R = \sqrt{1 - 4q_0 q_1 Y^2}$. In (5.5) we have to express $\cos\phi$ and $\sin\phi$ in terms of $t := \tan 2\phi$, using the identities given in a footnote, that is,

$$\sin\phi = 2^{-1/2}\sqrt{1 - 1/\sqrt{1+t^2}}, \qquad \cos\phi = 2^{-1/2}\sqrt{1 + 1/\sqrt{1+t^2}}$$

where (see (5.42))

$$t := \tan 2\phi = \frac{1}{q_0 - q_1} \frac{\sqrt{1 - Y^2}}{Y} .$$

Hence

$$\sin\phi = 2^{-1/2}\sqrt{1 - L}, \qquad \cos\phi = 2^{-1/2}\sqrt{1 + L}$$

where

$$R = \sqrt{1 - 4q_0 q_1 Y^2}, \qquad L = |(q_0 - q_1)Y|/R .$$

Problem 5.9 $\star\star$ [Sect. 5.7] From the following normalized states of $\mathcal{H} = \mathbb{C}^4$

$$|\gamma_1\rangle = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ \frac{1}{\sqrt{13}} \end{bmatrix} \quad |\gamma_2\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \quad |\gamma_3\rangle = \begin{bmatrix} \frac{1}{2} \\ -\frac{i}{2} \\ -\frac{1}{2} \\ \frac{i}{2} \end{bmatrix} \quad |\gamma_4\rangle = \begin{bmatrix} \frac{2}{\sqrt{13}} \\ -\frac{2i}{\sqrt{13}} \\ -\frac{1}{\sqrt{13}} \\ \frac{2i}{\sqrt{13}} \end{bmatrix} \quad |\gamma_5\rangle = \begin{bmatrix} \frac{1}{\sqrt{13}} \\ -\frac{2i}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \\ \frac{2i}{\sqrt{13}} \end{bmatrix}$$

form the density operators

$$\rho_1 = \frac{3}{4}|\gamma_1\rangle\langle\gamma_1| + \frac{1}{4}|\gamma_2\rangle\langle\gamma_2|, \qquad \rho_2 = \frac{3}{4}|\gamma_3\rangle\langle\gamma_3| + \frac{1}{8}|\gamma_4\rangle\langle\gamma_4|\frac{1}{8}|\gamma_5\rangle\langle\gamma_5|$$

and find their *minimum* factors γ_1 and γ_2. Find also factorizations in which the matrices γ_1 and γ_2 have the same dimensions.

Solution The density operators are explicitly

$$\rho_1 = \frac{3}{4}|\gamma_1\rangle\langle\gamma_1| + \frac{1}{4}|\gamma_2\rangle\langle\gamma_2| = \begin{bmatrix} \frac{61}{208} & \frac{61}{208} & \frac{61}{208} & \frac{37}{208} \\ \frac{61}{208} & \frac{61}{208} & \frac{61}{208} & \frac{37}{208} \\ \frac{61}{208} & \frac{61}{208} & \frac{61}{208} & \frac{37}{208} \\ \frac{37}{208} & \frac{37}{208} & \frac{37}{208} & \frac{25}{208} \end{bmatrix}$$

$$= \begin{bmatrix} 0.29327 & 0.29327 & 0.29327 & 0.17788 \\ 0.29327 & 0.29327 & 0.29327 & 0.17788 \\ 0.29327 & 0.29327 & 0.29327 & 0.17788 \\ 0.17788 & 0.17788 & 0.17788 & 0.12019 \end{bmatrix}$$

$$\rho_2 = \frac{3}{4}|\gamma_3\rangle\langle\gamma_3| + \frac{1}{8}|\gamma_4\rangle\langle\gamma_4| + \frac{1}{8}|\gamma_5\rangle\langle\gamma_5|$$

$$= \begin{bmatrix} \frac{49}{208} & \frac{51\mathrm{i}}{208} & -\frac{47}{208} & -\frac{51\mathrm{i}}{208} \\ -\frac{51\mathrm{i}}{208} & \frac{55}{208} & \frac{51\mathrm{i}}{208} & -\frac{55}{208} \\ -\frac{47}{208} & -\frac{51\mathrm{i}}{208} & \frac{49}{208} & \frac{51\mathrm{i}}{208} \\ \frac{51\mathrm{i}}{208} & -\frac{55}{208} & -\frac{51\mathrm{i}}{208} & \frac{55}{208} \end{bmatrix} = \begin{bmatrix} 0.23558 & \mathrm{i}0.24519 & -0.22596 & -\mathrm{i}0.24519 \\ -\mathrm{i}0.24519 & 0.26442 & \mathrm{i}0.24519 & -0.26442 \\ -0.22596 & -\mathrm{i}0.24519 & 0.23558 & \mathrm{i}0.24519 \\ \mathrm{i}0.24519 & -0.26442 & -\mathrm{i}0.24519 & 0.26442 \end{bmatrix}.$$

Consider now the factorizations $\rho_1 = \gamma_1\gamma_1^*$ and $\rho_2 = \gamma_2\gamma_2^*$. The matrix ρ_1 has rank $h_1 = 2$ with positive eigenvalues $0.994517,\ 0.10458$, therefore the factor γ_1 becomes 4×2 and precisely

$$\gamma_1 = \begin{bmatrix} -0.54117 & -0.02018 \\ -0.54117 & -0.02018 \\ -0.54117 & -0.02018 \\ -0.33238 & 0.09857 \end{bmatrix}.$$

The matrix ρ_2 has rank $h_2 = 3$ with positive eigenvalues $0.993343,\ 0.0980581,$ 0.0604505 therefore the factor γ_2 becomes 4×3 and precisely

$$\gamma_2 = \begin{bmatrix} -0.47937 & -0.06934 & 0.03124 \\ \mathrm{i}0.51339 & 0 & \mathrm{i}0.02917 \\ 0.47937-\mathrm{i}0 & -0.06934+\mathrm{i}0 & -0.03124 \\ -\mathrm{i}0.51339 & 0 & -\mathrm{i}0.02917 \end{bmatrix}.$$

These are minimum orthogonal factorizations, but the matrices γ_1 and γ_2 have different dimensions, respectively 4×2 and 4×3. To get the same dimensions, e.g.,4×3, we can modify γ_1 in the form $\gamma_1 Z$, where Z is a 2×3 matrix with $ZZ^* = I_2$ (see Section 3.11). A trivial solution is

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

giving

$$\gamma_1 Z = \begin{bmatrix} -0.54117 & -0.02018 & 0 \\ -0.54117 & -0.02018 & 0 \\ -0.54117 & -0.02018 & 0 \\ -0.33238 & 0.09857 & 0 \end{bmatrix}.$$

Problem 5.10 $\star$ [Sect. 5.7] Consider the transition probabilities given by (5.72). Prove that, if γ_i is replaced by $\gamma_j Z$, with $ZZ^* = I_h$, and μ_j by $\mu_j W$, with $WW^* = I_h$, the transition probabilities do not change.

Solution The proof is immediate after use of the properties of the matrices Z and W. In fact

$$p(j|i) = \mathrm{Tr}(\mu_j \mu_j^* \gamma_i \gamma_i^*) = \mathrm{Tr}(\mu_j W W^* \mu_j^* \gamma_i Z Z^* \gamma_i^*) = \mathrm{Tr}(\mu_j \mu_j^* \gamma_i \gamma_i^*) = p(j|i)\,.$$

Problem 5.11 $\star\star$ [Sect. 5.7] Prove that the measurement matrix M defined by (5.59) and its generalization to mixed states (5.69), allows us to express the resolution of the identity in the form $MM^* = I_{\mathcal{H}}$.

Solution In a POVM system the resolution of the identity is imposed to the measurement operators Q_i as

$$\sum_{i=0}^{K-1} Q_i = I_{\mathcal{H}}\ .$$

Now, for pure states the measurement matrix (5.59) gives

$$M\,M^* = [|\mu_0\rangle, |\mu_1\rangle, \dots, |\mu_{K-1}\rangle] \begin{bmatrix} \langle\mu_0| \\ \langle\mu_1| \\ \vdots \\ \langle\mu_{K-1}| \end{bmatrix} = \sum_{i=0}^{K-1} |\mu_i\rangle\langle\mu_i| = \sum_{i=0}^{K-1} Q_i = I_{\mathcal{H}}\ . \quad (5.6)$$

For mixed states the measurement matrix (5.69) gives exactly the same results.

Problem 5.12 $\star$ [Sect. 5.12] Write the relations of Example 5.4 using the results of Helstrom's theory.

Solution We have found that with pure states the measurement vectors are given by

$$|\eta_0\rangle = a_0|\gamma_0\rangle + b_0|\gamma_1\rangle\,, \qquad |\eta_1\rangle = a_1|\gamma_0\rangle + b_1|\gamma_1\rangle\ .$$

Then we get the relation of the form (5.104a)

$$\underset{1\times 2}{M} = \underset{1\times 2}{\Gamma}\ \underset{2\times 2}{A} \quad \to \quad [|\eta_0\rangle, |\eta_1\rangle] = [|\gamma_0\rangle, |\gamma_1\rangle] \begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \end{bmatrix}$$

To get the form (5.104b) the explicit component of the kets would be requested (not necessary in Helstrom's theory).

Problem 5.13 $\star\star$ [Sect. 5.13] Prove that the quantum states of $\mathcal{H} = \mathbb{C}^4$

$$|\gamma_0\rangle = \frac{1}{2}[1, -1, 1, -1]^{\mathrm{T}}, \qquad |\gamma_1\rangle = \frac{1}{2}[1, 1, -1, 1]^{\mathrm{T}}$$

verify the GUS for a binary transmission. Find the symmetry operator S, verify that S has the properties of a symmetry operator and that $|\gamma_1\rangle$ is obtained from $|\gamma_0\rangle$ as $|\gamma_1\rangle = S|\gamma_0\rangle$.

Solution The symmetry operator of a binary system is obtained from (5.122)

$$S = I - 2\frac{|w\rangle\langle w|}{\langle w|w\rangle}$$

where

$$|w\rangle = |\gamma_1\rangle - |\gamma_0\rangle$$

if the two states have inner product $\langle\gamma_0|\gamma_1\rangle$ real. In the present case the inner product is $\langle\gamma_0|\gamma_1\rangle = -\frac{1}{2}$, and therefore we can apply the above formula, which gives

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} .$$

This matrix is unitary and gives $S^2 = I_4$ and therefore S is a correct symmetry operator. Moreover

$$S|\gamma_0\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = |\gamma_1\rangle .$$

Problem 5.14 $\star$ [Sect. 5.13] Find the EID of the symmetry operator S of the previous problem.

Solution We have

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} , \qquad S^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4 .$$

The EID oof S, say $S = U_S \Lambda_S U_S^*$, verifies the condition

$$S^2 = U_S \Lambda_S U_S^* U_S \Lambda_S U_S^* = U_S \Lambda_S^2 U_S^* \quad \rightarrow \quad \Lambda_S^2 = I_4 .$$

Hence the eigenvalues λ_i of S are $\lambda_i = \pm 1$. The EID $S = U_S \Lambda_S U_S^*$ gives explicitly

$$U_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \Lambda_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 5.15 ⋆⋆ [Sect. 5.13] Prove that the two quantum states of $\mathcal{H} = \mathbb{C}^4$

$$|\gamma_0\rangle = \frac{1}{2}[1, -1, 1, -1]^{\mathrm{T}}, \qquad |\gamma_1\rangle = \frac{1}{2}[1, 1, -\mathrm{i}, 1]^{\mathrm{T}}$$

verify the GUS for a binary transmission, and find the corresponding symmetry operator S. Note that in this case the inner product $X := \langle\gamma_0|\gamma_1\rangle$ is complex.

Solution The state matrix and the Gram matrix are respectively

$$\Gamma = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -\mathrm{i} \\ -1 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & -\frac{1}{4} - \frac{\mathrm{i}}{4} \\ -\frac{1}{4} + \frac{\mathrm{i}}{4} & 1 \end{bmatrix}$$

In particular the inner product is

$$X := \langle\gamma_0|\gamma_1\rangle = -\frac{1}{4}(1 + \mathrm{i}) = \frac{\sqrt{2}}{4}\mathrm{e}^{-\mathrm{i}(3/4)\pi}$$

Thus, we have to modify $|\gamma_1\rangle$ as

$$|\tilde{\gamma}_1\rangle = \mathrm{e}^{\mathrm{i}3\pi/4}|\gamma_1\rangle\,.$$

Then the state matrix and the Gram matrix become respectively

$$\Gamma = \frac{1}{2\sqrt{2}}\begin{bmatrix} \sqrt{2} & -1+\mathrm{i} \\ -\sqrt{2} & -1+\mathrm{i} \\ \sqrt{2} & 1+\mathrm{i} \\ -\sqrt{2} & -1+\mathrm{i} \end{bmatrix}, \qquad G = \frac{1}{4}\begin{bmatrix} 4 & \sqrt{2} \\ \sqrt{2} & 4 \end{bmatrix}$$

The vector $|w\rangle = |\gamma_1\rangle - |\gamma_0\rangle$ is

$$|w\rangle = \frac{1}{2\sqrt{2}}\left[(-1-\mathrm{i}) - \sqrt{2}, (-1-\mathrm{i}) + \sqrt{2}, (1-\mathrm{i}) - \sqrt{2}, (-1-\mathrm{i}) + \sqrt{2}\right]^{\mathrm{T}}$$

and from (5.122) we find that the symmetry operator results in

$$S = \frac{1}{-4+\sqrt{2}} \begin{bmatrix} 2\left[-1+\sqrt{2}\right] & \mathrm{i}\sqrt{2} & 1+\mathrm{i} & \mathrm{i}\sqrt{2} \\ -\mathrm{i}\sqrt{2} & -2 & (1-\mathrm{i})\left[-1+\sqrt{2}\right] & 2-\sqrt{2} \\ 1-\mathrm{i} & (1+\mathrm{i})\left[-1+\sqrt{2}\right] & -2 & (1+\mathrm{i})\left[-1+\sqrt{2}\right] \\ -\mathrm{i}\sqrt{2} & 2-\sqrt{2} & (1-\mathrm{i})\left[-1+\sqrt{2}\right] & -2 \end{bmatrix} .$$

Note that the eigenvalues of S are $\{-1,1,1,1\}$.

Problem 5.16 $\star\star$ [Sect. 5.15] Prove that the evaluation of the transition probabilities in the compressed space is based on the same formula as in the uncompressed space. that is,

$$p(j|i) = \mathrm{Tr}(\Pi_j\,\rho_i) = \mathrm{Tr}(\overline{\Pi}_j\,\overline{\rho}_i) .$$

Hint: Use orthonormality relationship $U_r^*\,U_r = I_r$, where I_r is the $r \times r$ identity matrix.

Solution In fact, using the orthogonality $U_r^* U_r = I_r$ and the cyclic property of the trace, we find

$$\begin{aligned} \mathrm{Tr}(\Pi_j\,\rho_i) &= \mathrm{Tr}(U_r\,\overline{\Pi}_j\,U_r^*\,U_r\,\overline{\rho}_i\,U_r^*) \\ &= \mathrm{Tr}(U_r\overline{\Pi}_j\,\overline{\rho}_i\,U_r^*) = \mathrm{Tr}(U_r^*\,U_r\,\overline{\Pi}_j\,\overline{\rho}_i) \\ &= \mathrm{Tr}(\overline{\Pi}_j\,\overline{\rho}_i) . \end{aligned}$$

Problem 5.17 $\star\star\star$ [Sect. 5.15] Prove Proposition 5.10, which states that the GUS is preserved after a compression. *Hint:* Use orthonormality relationship $U_r^*\,U_r = I_r$, where I_r is the $r \times r$ identity matrix.

Solution We have to prove that $\overline{S}$ is unitary and $\overline{S}^K = I_r$. In fact, considering that $S = U_r\,\overline{S}\,U_r^*$ and that, by assumption, $S^*\,S = I_{\mathcal{H}}$, we find

$$I_{\mathcal{H}} = U_r\,\overline{S}^*\,U_r^* U_r\,\overline{S}\,U_r^* = U_r\,\overline{S}^*\overline{S}\,U_r^*$$

and reversing we get

$$\overline{S}^*\overline{S} = U_r^*\,I_{\mathcal{H}}\,U_r = U_r^*\,U_r = I_r .$$

Hence, $\overline{S}$ is unitary. Analogous is the proof that $\overline{S}^K = I_r$.

Problem 5.18 $\star\star$ [Sect. 5.15] Consider the state matrix of $\mathcal{H} = \mathbb{C}^4$

$$\Gamma = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

Find the compressor U_r^* and the compressed versions of the state matrix Γ and of the Gram operator T.

Solution The reduced SVD of Γ results in

$$\Gamma = U_r \Lambda_r V^*$$

where

$$U_r = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \qquad \Lambda_r = \begin{bmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then the compressed dimension is $r = 2$, from $\mathbb{C}^4$ to $\mathbb{C}^2$, and the compressor is

$$U_r^* = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The compression of the state matrix gives

$$\overline{\Gamma} = U_r^* \Gamma = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The Gram operator results in

$$T = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and its compressed version is

$$\overline{T} = U_r^* T U_r = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} .$$

Problem 5.19 ⋆⋆ [Sect. 5.15] Consider a binary transmission where the quantum states are specified by the state matrix of the previous problem. Apply Helstrom theory with $q_0 = 1/3$ to find the probability of a correct decision P_c. Then apply the compression and evaluate P_c from the compressed states.

Solution For the evaluation of P_c with pure states it is sufficient to evaluate the quadratic inner product $\Gamma^2 = |\langle \gamma_1 | \gamma_0 \rangle|^2$. In fact from (5.33) we have

$$P_c = \frac{1}{2}\left(1 + \sqrt{1 - 4 q_0 q_1 \Gamma^2}\right) .$$

The quantum states are

$$|\gamma_0\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \qquad |\gamma_1\rangle = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

and their inner product is $\langle\gamma_1|\gamma_0\rangle = -1/2$. The probability of a correct decision P_c results in

$$P_c = \frac{1}{2}\left[1 + \frac{\sqrt{\frac{17}{2}}}{3}\right] \simeq 0.985913\,.$$

The compressed states are (see solution of Problem 5.18)

$$|\overline{\gamma}_0\rangle = \frac{1}{2}\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}, \qquad |\overline{\gamma}_1\rangle = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

and have the same inner product as the uncompressed states, in agreement with the general theory on quantum compression.

Problem 5.20 ⋆⋆ [Sect. 5.15] Consider the binary constellation of Problem 5.13, where we determined the symmetry operator S. Find the compressor U_r^* showing in particular that the compressed symmetry operator $\overline{S}$ is diagonal.

Solution The reduced EID of the state matrix

$$\Gamma = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

is $\Gamma = U_r \Lambda_r V^*$ with

$$U_r = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \qquad \Lambda_r = \begin{bmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then, the expander and the compressor are respectively

$$U_r = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \qquad U_r^* = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The uncompressed symmetry operator found in the solution of Problem 5.13 is

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

and the compressed version is

$$\overline{S} = U_r^* S U_r = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Note that $\overline{S}$ is diagonal as stated in the general theory of compression.

Problems of Chapter 6

Problem 6.1 [Sect. 6.2] Prove that $T^{-1/2}T^{1/2}$ does not yield, in general, the identity $I_{\mathcal{H}}$, but the projector $P_{\mathcal{U}} = U_r U_r^*$. Only if $r = K$ one actually produces the identity $I_{\mathcal{H}}$.

Solution From (6.13) we have

$$T^{\pm 1/2} = U_r \Sigma_r^{\pm 1} U_r^*$$

and then

$$T^{-1/2}T^{1/2} = U_r U_r^* = P_{\mathcal{U}} \,.$$

Note that U_r is not a unitary matrix for $r < K$. If $r = K$ it becomes unitary.

Problem 6.2 ⋆⋆ [Sect. 6.2] Consider the following state matrix of $\mathcal{H} = \mathbb{C}^4$

$$\Gamma = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} .$$

Find the inverse square root $G^{-1/2}$ and $T^{-1/2}$ based on the two approaches: 1) the Moore–Penrose pseudo inverse and 2) the reduced EID.

Solution The state matrix has rank $r = 2$. The Gram matrix is

$$G = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

and has rank 2 and therefore the pseudoinverse coincides with the ordinary inverse. The EID $G = U \Lambda U^*$ is given by

$$U = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} , \qquad \Lambda = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} .$$

The inverse square root is simply obtained by evaluating the ordinary inverse square root of Λ, that is,

$$\Lambda^{-1/2} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$$

Then we get

$$G^{-1/2} = U\Lambda^{-1/2}V^* = \begin{bmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{6}} \end{bmatrix}$$

The Gram operator is given by

$$T = \Gamma\Gamma^* = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and has rank 2. Its EID $T = U\Lambda U^*$ is given by

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we have to evaluate the pseudoinverse by introducing the diagonal matrix

$$\Lambda^{-1/2} = \begin{bmatrix} \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to get

$$T^{-1/2} = U\Lambda^{-1/2}V^* = \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{\frac{2}{3}}}{3} & -\frac{\sqrt{\frac{2}{3}}}{3} & \frac{\sqrt{\frac{2}{3}}}{3} \\ 0 & -\frac{\sqrt{\frac{2}{3}}}{3} & \frac{\sqrt{\frac{2}{3}}}{3} & -\frac{\sqrt{\frac{2}{3}}}{3} \\ 0 & \frac{\sqrt{\frac{2}{3}}}{3} & -\frac{\sqrt{\frac{2}{3}}}{3} & \frac{\sqrt{\frac{2}{3}}}{3} \end{bmatrix}.$$

We arrive to the same result by using the **reduced EID** $T = U_r\Lambda_r U_r^*$, which is given by

$$U_r = \begin{bmatrix} 0 & 1 \\ \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}, \qquad \Lambda_r = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

In this case Λ is regular and the passage to the pseudoinverse is not needed.

Finally note that $T^{-1/2}T^{1/2}$ does not give the identity, but

$$T^{-1/2}T^{1/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

and we can check that $T^{-1/2}T^{1/2} = U_r^*U_r$.

Problem 6.3 ⋆⋆ [Sect. 6.3] Consider the state matrix Γ given by (6.31) of Example 6.1. Check that the methods based on the EIDs of G and T give the same transition probabilities as obtained with the SVD of Γ.

Solution The Gram matrix and the Gram operator are respectively

$$G = \Gamma^*\Gamma = \begin{bmatrix} 1 & \frac{5}{13} \\ \frac{5}{13} & 1 \end{bmatrix}$$

$$T = \Gamma\Gamma^* = \frac{1}{26}\begin{bmatrix} 13 & 9+4\mathrm{i} & 5 & 9-4\mathrm{i} \\ 9-4\mathrm{i} & 13 & 9+4\mathrm{i} & 5 \\ 5 & 9-4\mathrm{i} & 13 & 9+4\mathrm{i} \\ 9+4\mathrm{i} & 5 & 9-4\mathrm{i} & 13 \end{bmatrix} .$$

The reduced EID of G is $G = V_r\Sigma_r^2 V_r^*$ and gives

$$G^{-1/2} = G^{-1/2} = V_r\Sigma_r^{-1}V_r^* = \frac{\sqrt{\frac{13}{2}}}{12}\begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} .$$

Then

$$M_{opt} = \Gamma\, G^{-1/2} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 1-\mathrm{i} & 1+\mathrm{i} \\ 0 & 2 \\ 1+\mathrm{i} & 1-\mathrm{i} \end{bmatrix} .$$

The reduced EID of T is $T = U_r\Sigma_r^2 U_r^*$ and gives

$$T^{-1/2} = U_r\Sigma_r^{-1}U_r^* = \frac{1}{24}\sqrt{\frac{13}{2}}\begin{bmatrix} 5 & 2+3\mathrm{i} & -1 & 2-3\mathrm{i} \\ 2-3\mathrm{i} & 5 & 2+3\mathrm{i} & -1 \\ -1 & 2-3\mathrm{i} & 5 & 2+3\mathrm{i} \\ 2+3\mathrm{i} & -1 & 2-3\mathrm{i} & 5 \end{bmatrix} .$$

Then

$$M_{opt} = T^{-1/2}\Gamma = \frac{1}{2\sqrt{2}}\begin{bmatrix} 2 & 0 \\ 1-\mathrm{i} & 1+\mathrm{i} \\ 0 & 2 \\ 1+\mathrm{i} & 1-\mathrm{i} \end{bmatrix} .$$

Thus we have found that in both cases the same optimal measurement matrix evaluated in Example 6.1 and this is sufficient to conclude that the tree method gives the same transition probabilities (see (6.32)).

Problem 6.4 ⋆⋆ [Sect. 6.3] With the data of the previous problem, find the relations

$$\mu_1 = C\gamma_1 , \qquad \mu_2 = C\gamma_2 .$$

These relations are somewhat intriguing since they lead to think that μ_1 depends only on γ_1 and not on γ_2 and μ_2 only on γ_2. Explain why not.

Solution The measurement vectors has been found in Example 6.1 and are given by

$$|\mu_1\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\1-\mathrm{i}\\0\\1+\mathrm{i}\end{bmatrix}, \qquad |\mu_2\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}0\\1+\mathrm{i}\\2\\1-\mathrm{i}\end{bmatrix}.$$

From the SRM theory we know that the above relations holds with $C = T^{-1/2}$. Now we have

$$T^{-1/2}\gamma_1 = \frac{1}{2\sqrt{2}}\begin{bmatrix}2 & 0\\1-\mathrm{i} & 1+\mathrm{i}\\0 & 2\\1+\mathrm{i} & 1-\mathrm{i}\end{bmatrix}\frac{1}{2\sqrt{13}}\begin{bmatrix}5\\3-2\mathrm{i}\\15\\3+2\mathrm{i}\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\1-\mathrm{i}\\0\\1+\mathrm{i}\end{bmatrix}$$

$$T^{-1/2}\gamma_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix}2 & 0\\1-\mathrm{i} & 1+\mathrm{i}\\0 & 2\\1+\mathrm{i} & 1-\mathrm{i}\end{bmatrix}\frac{1}{2\sqrt{13}}\begin{bmatrix}5\\3+2\mathrm{i}\\15\\3-2\mathrm{i}\end{bmatrix} = \frac{1}{2\sqrt{2}}\begin{bmatrix}0\\1+\mathrm{i}\\2\\1-\mathrm{i}\end{bmatrix}.$$

Thus the relations are really verified with $C = T^{-1/2}$. But μ_1 does not depends only on γ_1 because in $T^{1/2}$ it is also encoded the information on γ_2.

Problem 6.5 ⋆⋆ [Sect. 6.5] Apply the SRM approach to find the optimal decision in a binary system with equiprobable symbols and with a real inner product X.

Solution When the inner product is real the Gram matrix becomes

$$G = \Gamma^*\Gamma = \begin{bmatrix}1 & X\\X & 1\end{bmatrix}, \qquad X := \langle\gamma_0|\gamma_1\rangle\,.$$

and it is circulant. Therefore we can apply the approach based on the DFT.

The two eigenvalues of G are $\lambda_0 := \sigma_0^2 = 1+|X|$ and $\lambda_1 := \sigma_1^2 = 1-|X|$. The corresponding eigenvectors are given by the columns of the DFT matrix

$$\Big[|w_0\rangle, |w_1\rangle\Big] = \tfrac{1}{2}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

Then the EID of G is

$$G = U\Lambda U^* \quad \text{with} \quad U = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, \Lambda = \begin{bmatrix}\sigma_0^2 & 0\\0 & \sigma_1^2\end{bmatrix}$$

The correct decision probability is

$$P_c = \left[\frac{1}{2}(\lambda_0^{1/2} + \lambda_1^{1/2})\right]^2 = \left[\frac{1}{2}(\sqrt{1+|X|} + \sqrt{1-|X|})\right]^2 = \frac{1}{2}\left(1 + \sqrt{1-|X|^2}\right)$$

that is, the Helstrom bound.

The square roots of G are given by

$$G^{1/2} = U\Lambda^{1/2}U^* = \frac{1}{2}\begin{bmatrix} \sigma_0+\sigma_1 & \sigma_0-\sigma_1 \\ \sigma_0-\sigma_1 & \sigma_0+\sigma_1 \end{bmatrix}$$
$$G^{-1/2} = U\Lambda^{-1/2}U^* = \frac{1}{2}\begin{bmatrix} \sigma_0^{-1}+\sigma_1^{-1} & \sigma_0^{-1}-\sigma_1^{-1} \\ \sigma_0^{-1}-\sigma_1^{-1} & \sigma_0^{-1}+\sigma_1^{-1} \end{bmatrix} .$$

Problem 6.6 $\star$ [Sect. 6.6] Write explicitly the block DFT matrix, defined by (6.59), for $K = 4$ and $h_0 = 2$ and prove that it is a unitary matrix.

Solution We use the symbol $W_{[K]}$ to denote the DFT matrix of order K and the symbol $W_{[K,h]}$ to denote the block DFT matrix of order K with blocks of size h.

The block DFT matrix can be written in the form

$$W_{[K,h_0]} = W_{[K]} \otimes I_{h_0}$$

where $W_{[K]}$ is the ordinary K–DFT matrix of dimension $K \times K$ and I_{h_0} is the identity matrix of dimension $h_0 \times h_0$. Then $W_{[K,h_0]}$ has dimension $Kh_0 \times Kh_0$. In the specific case $Kh_0 = 8$.

We have

$$W_{[4]} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ 1 & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ 1 & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix} \quad , \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} .$$

Hence

$$W_{[4,2]} = W_{[4]} \otimes I_2 = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} \end{bmatrix} .$$

The matrix $W_{([4,2]}$ is unitary and in fact

$$W_{[4,2]}W^*_{[4,2]} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} & 0 \\ 0 & 1 & 0 & \mathrm{i} & 0 & -1 & 0 & -\mathrm{i} \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} & 0 \\ 0 & 1 & 0 & -\mathrm{i} & 0 & -1 & 0 & \mathrm{i} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = I_8 \,.$$

Problem 6.7 ⋆⋆ [Sect. 6.6] Prove in general that the block DFT matrix, defined by (6.59), is a unitary matrix.

Solution As in the previous solution, we use the symbol $W_{[K]}$ to denote the DFT matrix of order K and the symbol $W_{[K,h]}$ to denote the block DFT matrix of order K with blocks of size h. The block DFT matrix can be written in the form

$$W_{[K,h_0]} = W_{[K]} \otimes I_{h_0}$$

where $W_{[K]}$ is the ordinary K–DFT matrix of dimension $K \times K$ and I_{h_0} is the identity matrix of dimension $h_0 \times h_0$.

For the proof we apply two rules on the Kronecker products of matrices (see Section 2.13). The first rule is

$$(A \otimes B)^* = A^* \otimes B^*$$

and the second rule is given by the mixed–product law

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \,.$$

Hence, considering that the ordinary DFT matrix is unitary, we find

$$\begin{aligned} W_{[K,h_0]}W^*_{[K,h_0]} &= (W_{[K]} \otimes I_{h_0})\,(W^*_{[K]} \otimes I_{h_0}) \\ &= (W_{[K]}\,W^*_{[K]}) \otimes (I_{h_0}\,I_{h_0}) \\ &= I_K \otimes I_{h_0} = I_{Kh_0} \,. \end{aligned}$$

Problem 6.8 $\star\star\star$ [Sect. 6.6] Extend Theorem 6.3 on circulant matrices to block circulant matrices.

Solution The proof is perfectly similar to the proof given in Appendix A of Chapter 6 for pure states, proceeding with blocks instead of scalars. Thus, we consider the matrix

$$Z := W^*_{[K,h_0]} G \, .$$

From inspection of the structure of the block Z_{ij} of Z and bearing in mind the condition (6.58), we have

$$\begin{aligned}
Z_{ij} &= \frac{1}{\sqrt{K}} \sum_{t=0}^{K-1} W_K^{it} G_{tj} = \frac{1}{\sqrt{K}} \sum_{t=0}^{K-1} W_K^{it}\, r_{j-t \,(\mathrm{mod}\, K)} \\
&= \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} W_K^{i(j-k)} r_k = \frac{1}{\sqrt{K}} W_K^{ij} \sum_{k=0}^{K-1} W_K^{-ik} r_k \\
&= \frac{1}{\sqrt{K}} W_K^{ij} D_i
\end{aligned}$$

where

$$D_i := \sum_{k=0}^{K-1} W_K^{-ik} r_k \, .$$

From the above result we infer that the matrix Z can be written in the form

$$Z = \Lambda W^*_{(h_0)}$$

and, to conclude, it is sufficient to note that also the block DFT matrix $W_{[K,h_0]}$ is unitary.

Problem 6.9 $\star\star$ [Sect. 6.6] To check the fundamental formulas of the SRM with mixed states having the GUS, consider the following degenerate case of reference state factor in a quaternary system

$$\gamma_0 = \tfrac{1}{\sqrt{3}} [|\beta_0\rangle, |\beta_0\rangle, |\beta_0\rangle]$$

where $|\beta_0\rangle$ is an arbitrary pure state, and the symmetry operator S generates the other state factor in the form $\gamma_i = S^i \gamma_0$, $i = 1,2,3$. Find the correct decision probability P_c.

Solution The reference density operator is

$$\rho_0 = \gamma_0 \gamma_0^* = |\beta_0\rangle\langle\beta_0|$$

and corresponds to a pure state. Then we can apply the theory of SRM with pure states having the GUS. The first row of the 4×4 Gram matrix G is given by

$$G_0 = [\langle\beta_0|\beta_0\rangle, \langle\beta_0|\beta_1\rangle, \langle\beta_0|\beta_2\rangle, \langle\beta_0|\beta_3\rangle]$$

where $|\beta_i\rangle = S^i|\beta_0\rangle$. The DFT of this row gives the eigenvalues of G (see (6.45a))

$$\lambda_p = \sum_{q=0}^{3} G_{0q} W_4^{-pq} = \sum_{q=0}^{3} \langle\beta_0|\beta_j\rangle \,. \tag{6.1}$$

The correct decision probability is then given by (see (6.48)

$$P_c = \left[\frac{1}{4}\sum_{p=0}^{3} \lambda_p^{\frac{1}{2}}\right]^2 \,. \tag{6.2}$$

But we want to apply the theory of SRM with mixed states having the GUS to find the same result. In this case the Gram matrix $\tilde{G}$ consists of block 3×3 and has dimension 12×12. The blocks of the first block–row are given by

$$\tilde{G}_{0j} = \gamma_0^*\gamma_j = \frac{1}{3}\langle\beta_0|\beta_j\rangle Z_3 \quad \text{with} \quad Z_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \,.$$

Then we evaluate the matrices

$$D_k = \sum_{j=0}^{3} G_{0q} W_4^{-jk} = \frac{1}{3} Z_3 \sum_{j=0}^{3} \langle\beta_0|\beta_j\rangle W_4^{-jk}$$

which, considering (6.1), are given by $D_k = \frac{1}{3}\lambda_k Z_3$. Next, we have to find the square roots D_k, which results in

$$D_k^{1/2} = (\frac{1}{3})^{\frac{1}{2}} \lambda_k^{\frac{1}{2}} Z_3^{1/2}$$

The evaluation of the square root of the matrix Z_3 gives $Z_3^{1/2} = \frac{1}{\sqrt{3}} Z_3$, but it is not needed. In fact

$$\begin{aligned} P_c = &\,\mathrm{Tr}\left(\frac{1}{4}\sum_{k=0}^{3} D_k^{1/2}\right)^2 = \mathrm{Tr}\left(\frac{1}{4}(\frac{1}{3})^{\frac{1}{2}} Z_3^{1/2} \sum_{k=0}^{3} \lambda_k^{\frac{1}{2}}\right)^2 \\ = &\left(\frac{1}{4}\sum_{k=0}^{3}\lambda_k^{\frac{1}{2}}\right)^2 \mathrm{Tr}\left((\frac{1}{3})^{\frac{1}{2}} Z_3^{1/2}\right)^2 = \left(\frac{1}{4}\sum_{k=0}^{3}\lambda_k^{\frac{1}{2}}\right)^2 \mathrm{Tr}\left(\frac{1}{3}Z_3\right)^2 \end{aligned}$$

where $\mathrm{Tr}(Z_3) = 3$. Hence

$$P_c = \left(\frac{1}{4}\sum_{k=0}^{3}\lambda_k^{\frac{1}{2}}\right)^2$$

in agreement with (6.2).

Problem 6.10 $\star\star$ [Sect. 6.7] Solve Problem 6.3 introducing compression.

Solution The state matrix in $\mathbb{C}^4$ is given by

$$\Gamma = \frac{1}{2} \begin{bmatrix} 5 & 1 \\ 3-2\mathrm{i} & 3+2\mathrm{i} \\ 1 & 5 \\ 3+2\mathrm{i} & 3-2\mathrm{i} \end{bmatrix}$$

From the reduced SVD of $\Gamma = U_r \Lambda V_r^*$ we find

$$U_r = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & \mathrm{i} \\ 1 & 1 \\ 1 & -\mathrm{i} \end{bmatrix} \quad \to \quad U_r^* = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -\mathrm{i} & 1 & \mathrm{i} \end{bmatrix}$$

where U_r^* gives the compressor. Then the compressed state matrix results in

$$\overline{\Gamma} = U_r^* \, \Gamma = \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix}$$

We know that the Gram matrix is not reduced by compression, so we consider only the Gram operator $T = \Gamma\Gamma^*$, which is 4×4. In the compressed space $\mathbb{C}^2$ it becomes

$$\overline{T} = \overline{\Gamma}\,\overline{\Gamma}^* = \begin{bmatrix} \frac{18}{13} & 0 \\ 0 & \frac{8}{13} \end{bmatrix} .$$

As expected, $\overline{T}$ is diagonal.

Then it is immediate to find the optimum measurement matrix as $M_{\text{opt}} = \overline{T}^{-1/2}\overline{\Gamma}$. We find

$$\overline{T}^{-1/2} = \begin{bmatrix} \frac{\sqrt{\frac{13}{2}}}{3} & 0 \\ 0 & \frac{\sqrt{\frac{13}{2}}}{2} \end{bmatrix}$$

and

$$M_{\text{opt}} = \begin{bmatrix} \frac{\sqrt{\frac{13}{2}}}{3} & 0 \\ 0 & \frac{\sqrt{\frac{13}{2}}}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} .$$

Problem 6.11 $\star$ [Sect. 6.8] Consider the binary system specified by the pure states

$$|\gamma_0\rangle = \frac{1}{\sqrt{13}}[5, 3-2\mathrm{i}, 1, 3+2\mathrm{i}]^{\mathrm{T}}, \qquad |\gamma_1\rangle = \frac{1}{2\sqrt{13}}[1, 3+2\mathrm{i}, 5, 3-2\mathrm{i}]^{\mathrm{T}} .$$

Check that: 1) Helstrom's theory gives $P_e = 1/26$, Chernoff bound gives $P_e = 25/338$.

Solution (file pp406.m) The inner product is $X = \frac{5}{13}$ and, from Helstrom's formula with pure states and equiprobable symbols, we

$$P_c = \frac{25}{26}\,, \qquad P_e = \frac{1}{26} = 0.038465\,.$$

The density operators are

$$\rho_0 = \frac{1}{52}\begin{bmatrix} 1 & 3+2\mathrm{i} & 5 & 3-2\mathrm{i} \\ 3-2\mathrm{i} & 13 & 15-10\mathrm{i} & 5-12\mathrm{i} \\ 5 & 15+10\mathrm{i} & 25 & 15-10\mathrm{i} \\ 3+2\mathrm{i} & 5+12\mathrm{i} & 15+10\mathrm{i} & 13 \end{bmatrix}$$

$$\rho_1 = \frac{1}{52}\begin{bmatrix} 25 & 15-10\mathrm{i} & 5 & 15+10\mathrm{i} \\ 15+10\mathrm{i} & 13 & 3+2\mathrm{i} & 5+12\mathrm{i} \\ 5 & 3-2\mathrm{i} & 1 & 3+2\mathrm{i} \\ 15-10\mathrm{i} & 5-12\mathrm{i} & 3-2\mathrm{i} & 13 \end{bmatrix}$$

and there square roots are

$$\sqrt{\rho_0} = \frac{1}{52}\begin{bmatrix} 1 & 3+2\mathrm{i} & 5 & 3-2\mathrm{i} \\ 3-2\mathrm{i} & 13 & 15-10\mathrm{i} & 5-12\mathrm{i} \\ 5 & 15+10\mathrm{i} & 25 & 15-10\mathrm{i} \\ 3+2\mathrm{i} & 5+12\mathrm{i} & 15+10\mathrm{i} & 13 \end{bmatrix}$$

$$\sqrt{\rho_1} = \frac{1}{52}\begin{bmatrix} 25 & 15-10\mathrm{i} & 5 & 15+10\mathrm{i} \\ 15+10\mathrm{i} & 13 & 3+2\mathrm{i} & 5+12\mathrm{i} \\ 5 & 3-2\mathrm{i} & 1 & 3+2\mathrm{i} \\ 15-10\mathrm{i} & 5-12\mathrm{i} & 3-2\mathrm{i} & 13 \end{bmatrix}\,.$$

Hence, from the Chernoff bound

$$P_e = \frac{25}{338} = 0.0739645\,.$$

Problems of Chapter 7

Problem 7.1 $\star$ [Sect. 7.2] Prove that the inner product $X = \langle\alpha|\beta\rangle$ of two coherent states is real if and only if $\arg\alpha - \arg\beta = 0$ or $\arg\alpha - \arg\beta = \pm\pi$.

Solution The inner product is given by (7.9)

$$\langle\alpha|\beta\rangle = \mathrm{e}^{-\frac{1}{2}(|\alpha|^2+|\beta|^2-2\alpha^*\beta)}\ .$$

Considering that

$$|\alpha|^2+|\beta|^2-2\alpha^*\beta = |\alpha-\beta|^2+\alpha^*\beta-\alpha\beta^* = |\alpha-\beta|^2+2\,\mathrm{i}\,\Im(\alpha^*\beta)$$

we can write

$$X = \mathrm{e}^{-\frac{1}{2}(|\alpha-\beta|^2}\mathrm{e}^{2\mathrm{i}\,|\alpha||\beta|\sin(\arg\beta-\arg\alpha)}\ .$$

Problem 7.2 $\star\star$ [Sect. 7.2] The map (7.2) gives for any $\alpha\in\mathbb{C}$ a coherent state $|\alpha\rangle$. Given $|\alpha\rangle$ is it possible to find the complex number α?

Solution We consider the inner product $\langle 0|\alpha\rangle$ and $\langle 1|\alpha\rangle$ to get

$$\langle 0|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle 0|n\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}$$
$$\langle 1|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\langle 1|n\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\alpha$$

Hence

$$\alpha = \frac{\langle 1|\alpha\rangle}{\langle 0|\alpha\rangle}\ .$$

Problem 7.3 $\star\star$ [Sect. 7.2] Examine the effect of the introduction of a phasor $z=\mathrm{e}^{i\varphi}$ into the complex parameter α that identifies the state $|\alpha\rangle$, that is, evaluate $|\mathrm{e}^{i\varphi}\alpha\rangle$.

Solution The effect is equivalent to the application of the operator

$$S_z := \sum_{n=0}^{\infty} z^n\,|n\rangle\langle n| = z^N\ .$$

In fact, from (7.2) we obtain

$$S_z\,|\alpha\rangle = \sum_{m=0}^{\infty} z^m|m\rangle\langle m|\;\mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle\ .$$

Hence, by the orthonormality of the number states $|n\rangle$, we get the coherent state $|z\,\alpha\rangle$.

Problem 7.4 ⋆⋆⋆ [Sect. 7.2] Let $|\alpha\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle$ be a two–mode coherent states. The number of photons m_i associated to each component state is a Poisson variable with mean $\Lambda_i = |\alpha_i|^2$. Considering that m_1 and m_2 are statistically independent (see Section 3.10), prove that the total number of photons $m = m_1 + m_2$ is a Poisson variable.

Hint: use the characteristic function given by (4.23).

Solution The characteristic functions are

$$\Psi_{m_1}(z) = \mathrm{e}^{\Lambda_1[\exp(\mathrm{i}z)-1]}\,, \qquad \Psi_{m_1}(z) = \mathrm{e}^{\Lambda_2[\exp(\mathrm{i}z)-1]}\,.$$

Recalling from Probability Theory that the characteristic function of the sum of two independent random variables is given by the product of their characteristic functions, we find

$$\Psi_m(z) = \mathrm{e}^{(\Lambda_1+\Lambda_2)[\exp(\mathrm{i}z)-1]}$$

which states that $m = m_1 + m_2$ is a Poisson variable with mean $\Lambda_1 + \Lambda_2$.

Problem 7.5 ⋆⋆ [Sect. 7.3] Show that the PPM must be considered a vector modulation. Find explicitly the waveform $\gamma_i(t)$ and the vector γ_i of the coefficients.

Solution If we assume as basis of orthonormal functions $\{p_{K-1}(t),\dots,p_1(t),p_0(t)\}$, where $p_i(t)$ is a unitary rectangle on $[iT_0,(i+1)T_0)$, and if we develop the PPM waveforms with this basis, we obtain as coefficients exactly the symbols γ_{ij} of the words γ_i.

Problem 7.6 ⋆⋆ [Sect. 7.3] The n-DFT matrix $W_{[n]}$ is unitary and has the property $W_{[n]}^n = I_n$. Then it allows the construction of n-ary constellations in $\mathcal{H} = \mathbb{C}^n$. Find a quaternary constellation using $S = W_{[4]}$ and reference state $|\gamma_0\rangle = [1,1,0,0]^{\mathrm{T}}$. Also prove also that the four states are linearly independent.

Solution The n–DFT matrix is defined by

$$W_{[n]} = \left[\frac{1}{\sqrt{n}} W_n^{-rs}\right]_{r,s=0,1,\dots,n-1} \quad \text{with} \quad W_n = \mathrm{e}^{\mathrm{i}2\pi/n}\,.$$

For $n = 4$ it results in

$$W_{[4]} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}.$$

It is unitary and its fourth power gives

$$W_{[4]}^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4\,.$$

Then $W_{[4]}$ is a symmetry operator for a quaternary system.

With the reference state $|\gamma_0\rangle = [1,1,0,0]^{\mathrm{T}}$ the following constellation is generated

$$|\gamma_0\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\2\\0\\0\end{bmatrix},\ |\gamma_1\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\1-\mathrm{i}\\0\\1+\mathrm{i}\end{bmatrix},\ |\gamma_2\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\0\\0\\2\end{bmatrix},\ |\gamma_3\rangle = \frac{1}{2\sqrt{2}}\begin{bmatrix}2\\1+\mathrm{i}\\0\\1-\mathrm{i}\end{bmatrix}.$$

These kets are linearly independent as stated by their Gram matrix having rank 4

$$G = \frac{1}{4}\begin{bmatrix}4 & 3-\mathrm{i} & 2 & 3+\mathrm{i}\\ 3+\mathrm{i} & 4 & 3-\mathrm{i} & 2\\ 2 & 3+\mathrm{i} & 4 & 3-\mathrm{i}\\ 3-\mathrm{i} & 2 & 3+\mathrm{i} & 4\end{bmatrix}.$$

Problem 7.7 $\star\star\nabla$ [Sect. 7.4] Find the shape factor μ_k of the 16–QAM constellation (see Fig.7.28).

Solution We subdivide the constellation into three parts, where the states have the same number of signal photons $N_\gamma = |\gamma|^2$:

- the 4 inner states, as $|(1+\mathrm{i})\Delta\rangle$, giving $4\times 2\,\Delta^2$,
- the 4 corner states, as $|(3+3\mathrm{i})\Delta\rangle$, giving $4\times 18\,\Delta^2$,
- the 8 lateral states, as $|(3+\mathrm{i})\Delta\rangle$, giving $8\times 10\,\Delta^2$.

The global number of signal photon in the constellation is $160\Delta^2$ and the number of signal photons per symbol is therefore

$$N_s = (160/16)\Delta^2 = 10\Delta^2 \quad \to \quad \mu_K = 10\,.$$

Problem 7.8 $\star$ [Sect. 7.5] Consider the 4–QAM (which is equivalent to 4–PSK) where the normalized constellation is $\mathcal{C}_0 = \{\gamma = \pm 1 + \pm\mathrm{i}\}$ and the constellation of received values is given by

$$\{(\pm 1 + \pm\mathrm{i})U_0 + (1+\mathrm{i})N_L\}\,.$$

Find the optimal decision regions and prove that the minimum error probability P_c is given by $P_c = 1 - \left(1 - Q(\sqrt{\Lambda})^2\right)$ with $\Lambda = 4N_R$.

Solution With he Gaussian approximation, for symmetry reasons, the optimal decision regions are given by

$$\mathcal{R}(1+\mathrm{i}) = \{x > N_L\,,\ y > N_L\}\,, \qquad \mathcal{R}(-1+\mathrm{i}) = \{x < N_L\,,\ y > N_L\}$$
$$\mathcal{R}(-1-\mathrm{i}) = \{x < N_L\,,\ < N_L\}\,, \qquad \mathcal{R}(1-\mathrm{i}) = \{x > N_L\,,\ y < N_L\}\,.$$

For the symbol $\gamma = 1+\mathrm{i}$ we have

$$
\begin{aligned}
p(\gamma|\gamma) =& \mathrm{P}\left[\widehat{C}_0 = \gamma | C_0 = \gamma\right] = \int_{N_L}^{\infty}\int_{N_L}^{\infty} f_{n_a}(a|A_0 = 1) f_{n_b}(b|B_0 = 1)\,\mathrm{d}a\mathrm{d}b \\
=& \int_{N_L}^{\infty} \frac{1}{\sigma_n}\phi\left(\frac{a-\bar{n}(1)}{\sigma_n}\right)\,\mathrm{d}a \int_{N_L}^{\infty} \frac{1}{\sigma_n}\phi\left(\frac{b-\bar{n}(1)}{\sigma_n}\right)\,\mathrm{d}b
\end{aligned}
$$

where $\bar{n}(1) = N_L + 2\sqrt{N_L N_R}$ and $\sigma_n = \sqrt{N_L}$. Then

$$
p(\gamma|\gamma) = \Phi\left(\frac{2\sqrt{N_L N_R}}{\sqrt{N_L}}\right)\Phi\left(\frac{2\sqrt{N_L N_R}}{\sqrt{N_L}}\right) = \Phi(\sqrt{4N_R})^2\ .
$$

This result holds also for the other three symbols. Hence the maximum correct decision probability is

$$
P_c = \Phi(\sqrt{4N_R})^2
$$

and

$$
P_e = 1 - P_c = 1 - \Phi(\sqrt{4N_R})^2 = 1 - (1 - Q(\sqrt{4N_R}))^2\ .
$$

Problem 7.9 ⋆⋆⋆ [Sect. 7.6] The error probability in classical homodyne BPSK has been evaluated assuming equiprobable symbols. When the symbols are not equiprobable the number of signal photons per bit N_R is still independent of the symbols and gives the SNR as $\Lambda = 4N_R$. The only change is in the evaluation is the decision element, given for equiprobable symbol by (7.78), as

$$
\widehat{A}_0 = \begin{cases} 1 & n \le S \\ 0 & n > S \end{cases}
$$

where S is the threshold to be optimized.

Find the optimal decision threshold and prove that the minimum error probability is given by

$$
P_e = q_1\,Q\left(\sqrt{\Lambda} + \frac{1}{2\sqrt{\Lambda}}\log\frac{q_1}{q_0}\right) + q_0\,Q\left(\sqrt{\Lambda} - \frac{1}{2\sqrt{\Lambda}}\log\frac{q_1}{q_0}\right)\ . \tag{7.1}
$$

Solution The correct decision probability is given by

$$
\begin{aligned}
P_c =& q_1 \mathrm{P}\left[n < S | A_0\right] = 1 + q_0 \mathrm{P}\left[n < S | A_0 = 0\right] \\
=& q_1 \int_{-\infty}^{S} f_n(a|A_0 = 1)\,\mathrm{d}a + q_0 \int_{S}^{+\infty} f_n(a|A_0 = 0)\,\mathrm{d}a
\end{aligned}
$$

where $n = \bar{n}_{A_0} + u$ with $n_{A_0} = U_0 \cos\pi A_0$ and u a Gaussian noise independent of A_0 (see (7.77)). Then

$$
\begin{aligned}
P_c =& q_1 \int_{-\infty}^{S-\bar{n}_1} f_u(b)\,\mathrm{d}b + q_0 \int_{S-\bar{n}_0}^{+\infty} f_n(b)\,\mathrm{d}b \\
=& q_1\,\Phi\left(\frac{S-\bar{n}_1}{\sigma_u}\right) + q_0\,Q\left(\frac{S-\bar{n}_1}{\sigma_u}\right)
\end{aligned}
$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy$$

is the normalized Gaussian distribution function and $Q(x) = 1 - \Phi(x)$ is its complement.

The threshold S is determined by maximizing the probability P_c. The value S_0 giving the maximum is

$$S_0 = \frac{\bar{n}_0 - \bar{n}_1}{2\sigma_u} + \frac{\sigma_u}{\bar{n}_0 + \bar{n}_1} \log \frac{q_1}{q_0} .$$

Considering that $Q(-x) = \Phi(x)$, $\bar{n}_0 = U_0 + N_0$, $\bar{n}_1 = U_0 - N_0$, and $U_0/\sigma_u = \sqrt{\Lambda}$, we find that the maximum probability of correct decision is

$$P_c = q_1\, \Phi\left(\sqrt{\Lambda} + \frac{1}{2\sqrt{\Lambda}} \log \frac{q_1}{q_0}\right) + q_0\, \Phi\left(\sqrt{\Lambda} - \frac{1}{2\sqrt{\Lambda}} \log \frac{q_1}{q_0}\right)$$

and (7.87) follows.

Problem 7.10 ⋆⋆ [Sect. 7.8] Prove that with the optimization the a posteriori probabilities $q(i|i) := \mathrm{P}\left[A_0 = i|\hat{A}_0 = i\right]$ are equal and coincides with the correct decision probability P_c.

Solution Let

$$\begin{aligned}|\gamma_0\rangle &= \cos\theta|x\rangle + \sin\theta|y\rangle \\ |\gamma_1\rangle &= \cos\theta|x\rangle - \sin\theta|y\rangle\end{aligned}$$

be the two states of the system expressed in terms of the orthonormal basis $\{|x\rangle, |y\rangle\}$, where, without restriction, we can choose $0 < \theta < \pi/4$. The superposition degree results in

$$X = |\langle\gamma_0|\gamma_1\rangle| = \cos^2\theta - \sin^2\theta = \cos 2\theta .$$

Let q_0 and q_1 be the a priori probabilities of the states $|\gamma_0\rangle$ e $|\gamma_1\rangle$.

From Kennedy's theorem we have that the optimal measurement vector are orthonormal, say

$$\begin{aligned}|\mu_0\rangle &= \cos\phi|x\rangle - \sin\phi|y\rangle \\ |\mu_1\rangle &= \sin\phi|x\rangle + \cos\phi|y\rangle .\end{aligned}$$

Hence the transition probabilities are given by

$$\begin{aligned}p(0|0) &= |\langle\gamma_0|\mu_0\rangle|^2 = (\cos\theta\cos\phi + \sin\theta\sin\phi)^2 = \cos^2(\theta - \phi) \\ p(1|1) &= |\langle\gamma_1|\mu_1\rangle|^2 = (\cos\theta\sin\phi + \sin\theta\cos\phi)^2 = \sin^2(\theta + \phi)\end{aligned}$$

and

$$p(1|0) = \sin^2(\theta - \phi) \qquad , \qquad p(0|1) = \cos^2(\theta + \phi) .$$

The output probabilities are given by

$$p(0) = q_0 p(0|0) + q_1 p(0|1) = q_0 \cos^2(\theta-\phi) + q_1 \cos^2(\theta+\phi)$$
$$p(1) = q_0 p(1|0) + q_1 p(1|1) = q_0 \sin^2(\theta-\phi) + q_1 \sin^2(\theta+\phi)$$

and the a posteriori probabilities by

$$\begin{aligned} p(0|0) &= \frac{q_0 p(0|0)}{p(0)} = \frac{q_0 \cos^2(\theta-\phi)}{q_0 \cos^2(\theta-\phi) + q_1 \cos^2(\theta+\phi)} \\ p(1|1) &= \frac{q_1 p(1|1)}{p(1)} = \frac{q_1 \sin^2(\theta+\phi)}{q_0 \sin^2(\theta-\phi) + q_1 \sin^2(\theta+\phi)} \; . \end{aligned}$$

The corresponding correct decision probability is

$$P_c = q_0 |\langle\gamma_0|\mu_0\rangle|^2 + q_1 |\langle\gamma_1|\mu_1\rangle|^2 = q_0 \cos^2(\theta-\phi) + q_1 \sin^2(\theta+\phi) \; .$$

To find the optimal decision we impose that the derivative with respect to ϕ be zero

$$\begin{aligned} 0 &= 2q_0 \cos(\theta-\phi)\sin(\theta-\phi) + 2q_1 \sin(\theta+\phi)\cos(\theta+\phi) \\ &= q_0 \sin(2\theta-2\phi) + q_1 \sin(2\theta+2\phi) \\ &= q_0(\sin 2\theta \cos 2\phi - \cos 2\theta \sin 2\phi) + q_1(\sin 2\theta \cos 2\phi + \cos 2\theta \sin 2\phi) \\ &= \sin 2\theta \cos 2\phi - (q_0 - q_1)\cos 2\theta \sin 2\phi \; . \end{aligned}$$

Hence

$$\cos 2\phi = \frac{q_0 - q_1}{R}\cos 2\theta \quad , \quad \sin 2\phi = \frac{q_0 + q_1}{R}\sin 2\theta$$

where R is determined by the condition

$$R^2 = (q_0 - q_1)^2 \cos^2 2\theta + (q_0 + q_1)^2 \sin^2 2\theta = 1 - 4q_0 q_1 \cos^2 2\theta = 1 - 4q_0 q_1 X^2$$

that is,

$$X^2 = \frac{1 - R^2}{4q_0 q_1} \; .$$

The optimal value of ϕ results in

$$\phi = \frac{1}{2}\arctan\left(\frac{1}{q_0 - q_1}\tan 2\theta\right) \; .$$

Hence the optimal transition probabilities become

$$\begin{aligned}
p\,(0|0) &= \cos^2(\theta-\phi) = \frac{1}{2}[1+\cos(2\theta-2\phi)] \\
&= \frac{1}{2}[1+\cos 2\phi\cos 2\theta+\sin 2\phi\sin 2\theta] \\
&= \frac{1}{2}\left[1+\frac{q_0-q_1}{R}\cos^2 2\theta+\frac{1}{R}\sin^2 2\theta\right] \\
&= \frac{1}{2}\left[1+\frac{q_0-q_1}{R}X^2+\frac{1}{R}(1-X^2)\right] \\
&= \frac{1}{2}\left[1+\frac{1}{R}-\frac{2q_1}{R}X^2\right] = \frac{1}{2}\left[1+\frac{1}{R}-\frac{1-R^2}{2q_0R}\right]
\end{aligned}$$

$$\begin{aligned}
p\,(1|1) &= \sin^2(\theta+\phi) = \frac{1}{2}[1-\cos(2\theta+2\phi)] \\
&= \frac{1}{2}[1-\cos 2\phi\cos 2\theta+\sin 2\phi\sin 2\theta] \\
&= \frac{1}{2}\left[1-\frac{q_0-q_1}{R}\cos^2 2\theta+\frac{1}{R}\sin^2 2\theta\right] \\
&= \frac{1}{2}\left[1-\frac{q_0-q_1}{R}X^2+\frac{1}{R}(1-X^2)\right] \\
&= \frac{1}{2}\left[1+\frac{1}{R}-\frac{2q_0}{R}X^2\right] = \frac{1}{2}\left[1+\frac{1}{R}-\frac{1-R^2}{2q_1R}\right] .
\end{aligned}$$

Finally, the optimal correct decision probability and the error probability are

$$P_c = q_0 p\,(0|0)+q_1 p\,(1|1) = \frac{1}{2}\left[1+\frac{1}{R}-\frac{4q_0q_1}{R}X^2\right] = \frac{1}{2}\,[1+R]$$

$$P_e = \frac{1}{2}[1-R]$$

which corresponds to Helstrom's bound.

The a posteriori probabilities of the state are given by

$$\begin{aligned}
p_{x|y}(0|0) &= \frac{q_0 p\,(0|0)}{p_y(0)} = q_0\frac{1+\frac{1}{R}-\frac{1-R^2}{2q_0R}}{1+\frac{q_0-q_1}{R}} = \frac{1}{2}\,\frac{2q_0(R+1)-1+R^2}{R+q_0-q_1} \\
&= \frac{1}{2}\,\frac{q_0-q_1+2q_0R+R^2}{q_0-q_1+R} = \frac{1}{2}[1+R] = P_c
\end{aligned}$$

$$\begin{aligned}
p_{x|y}(0|1) &= \frac{q_0 p\,(1|0)}{p_y(1)} = q_0\frac{1-\frac{1}{R}+\frac{1-R^2}{2q_0R}}{1-\frac{q_0-q_1}{R}} = \frac{1}{2}\,\frac{2q_0(R-1)+1-R^2}{R-(q_0-q_1)} \\
&= \frac{1}{2}\,\frac{q_0-q_1-2q_0R+R^2}{q_0-q_1-R} = \frac{1}{2}[1-R] = P_e
\end{aligned}$$

In conclusion, with the optimal decision we find that the a posteriori probabilities $p(0|0)$ and $p(1|1)$ are equal and given by the correct decision probability.

Problem 7.11 $\star$ [Sect. 7.8] Prove that in a binary system with equiprobable symbols the error probability can be expressed as function of $N_R(0)$, $N_R(1)$, and of the relative phase of the complex parameters γ_0 and γ_1. that determine the coherent states.

Solution The error probability depends only the superposition parameter $|X|^2$, which is given by (7.104a), that is,

$$|X|^2 = \mathrm{e}^{-|\gamma_0-\gamma_1|^2}$$

where

$$|\gamma_0-\gamma_1|^2 = |\gamma_0|^2\left|1-\frac{\gamma_0}{\gamma_1}\right|^2 = N_R(0)\left(1+\frac{N_R(0)}{N_R(1)}-2\sqrt{\frac{N_R(0)}{N_R(1)}}\cos\theta\right)$$

with $\theta=\arg(\gamma_0/\gamma_1)$. Then the minimal error probability (7.104) depends on $N_R(0)$, $N_R(1)$ and on the relative phase θ of the complex parameters γ_0 and γ_1.

In the cases of interest (OOK and 2–PSK) P_e depends only on $N_R(0)$ and $N_R(1)$.

Problem 7.12 $\star\star\star$ [Sect. 7.12] Prove that the operator S defined by (7.123) is the symmetry operator of the K–PSK modulation.

Solution Let us recall from Functional Analysis the meaning of the exponential of an operator (see Section 2.10.3), and observe that the eigenvalues of the number operator N are $n=0,1,2,\dots$ with the corresponding eigenvectors $|n\rangle$, and that N in (7.1) is expressed by its spectral decomposition. Therefore, the exponential of N that appears in (7.123) is defined as (see (2.90))

$$S=\sum_{n=0}^{\infty}\exp\Big(\frac{\mathrm{i}2\pi n}{K}\Big)|n\rangle\langle n| = \sum_{n=0}^{\infty} W_K^n\,|n\rangle\langle n|.$$

Hence, using the orthonormality of the number states $|n\rangle$, we get $S^m=\sum_{n=0}^{\infty}W_K^{mn}\,|n\rangle\langle n|$; in particular, for $m=K$ we obtain

$$S^K=\sum_{n=0}^{\infty}|n\rangle\langle n| = I_{\mathcal{H}}$$

because the $|n\rangle$ form an orthonormal basis (see (2.50)). Next we verify the other conditions of GUS, namely, that all the states (7.121) are obtained from the state $|\gamma_0\rangle$ as $|\gamma_m\rangle = S^m|\gamma_0\rangle$. This is easily proved by property (7.124) of the rotation operator.

Problem 7.13 $\star$ [Sect. 7.12] Find explicitly the formula for the error probability P_e of quantum 4–PSK system, with the target to show that P_e depends only on $\Delta^2 = N_s$.

Solution We apply (7.125) with $K=4$ to get the circulant vector

$$[G_{00},G_{01},G_{02},G_{03}] = [1,\mathrm{e}^{-N_s[1-\mathrm{i}]},\mathrm{e}^{-N_s[1+1]},\mathrm{e}^{-N_s[1+\mathrm{i}]}]\;.$$

Then we get the eigenvalues as

$$\lambda_i = \sum_{k=0}^{3} G_{0k} \mathrm{i}^{ik}.$$

The four real eigenvalues can be written in the forms

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 2\mathrm{e}^{-N_s}\cos N_s + \mathrm{e}^{-2N_s} + 1 \\ 2\mathrm{e}^{-N_s}\sin N_s - \mathrm{e}^{-2N_s} + 1 \\ -2\mathrm{e}^{-N_s}\cos N_s + \mathrm{e}^{-2N_s} + 1 \\ -2\mathrm{e}^{-N_s}\sin N_s - \mathrm{e}^{-2N_s} + 1 \end{bmatrix} = \begin{bmatrix} 2(\cos N_s + \cosh N_s)(\cosh N_s - \sin hN_s) \\ 2(\cosh N_s - \sin hN_s)(\sin N_s + \sin hN_s) \\ (\cosh N_s - \cos N_s)(2\cosh N_s - 2\sin hN_s) \\ 2(\cosh N_s - \sin hN_s)(\sin hN_s - \sin N_s) \end{bmatrix}$$

Finally we get the correct decision probability as

$$P_c = \frac{1}{16}\left[\sqrt{\lambda_0} + \sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3}\right]^2 .$$

Problems of Chapter 8

Problem 8.1 $\star\star\star$ [Sect. 8.2] Starting from the integral representation (8.2) of the density operator ρ_{th}, find the Fock representation (8.4). *Hint*: use polar coordinates.

Solution Use of (8.1) gives

$$|\alpha\rangle\langle\alpha| = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^m(\alpha^n)^*}{\sqrt{m!n!}} |m\rangle\langle n| \,.$$

Then

$$\rho_{\text{th}} = \frac{1}{\pi\mathcal{N}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \int_{\mathbb{C}} \exp\left[-|\alpha|^2/u\right] \frac{\alpha^m(\alpha^n)^*}{\sqrt{m!n!}} \mathrm{d}\alpha\, |m\rangle\langle n|$$

where $u = (1+\mathcal{N})/\mathcal{N}$. Introducing the polar coordinates $\alpha = r\mathrm{e}^{\mathrm{i}\phi}$ we get

$$\rho_{\text{th}} = \frac{1}{\pi\mathcal{N}} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \int_0^{\infty}\int_0^{2\pi} \exp\left[-r^2/u\right] \frac{r^{m+n}\mathrm{e}^{\mathrm{i}(m-n)\phi}}{\sqrt{m!n!}} r\,\mathrm{d}r\,\mathrm{d}\phi\, |m\rangle\langle n| \,.$$

where

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}(m-n)\phi}\mathrm{d}\phi = 2\pi\delta_{mn} \,.$$

Hence

$$\begin{aligned}\rho_{\text{th}} =& \frac{2}{\mathcal{N}} \sum_{m=0}^{\infty} \int_0^{\infty} \exp(-r^2/u)\frac{r^{2m}}{m!} r\,\mathrm{d}r\, |m\rangle\langle m| \\ =& \frac{1}{\mathcal{N}} \sum_{m=0}^{\infty} \int_0^{\infty} \exp(-x/u)\frac{x^m}{m!} \mathrm{d}x\, |m\rangle\langle m| \,.\end{aligned}$$

The integral gives[2]

$$\int_0^{\infty} \exp(-x/u) x^m \mathrm{d}x = \frac{u^{m+1}}{m!}$$

so that

$$\rho_{\text{th}} = \frac{1}{\mathcal{N}} \sum_{m=0}^{\infty} u^{m+1} |m\rangle\langle m| = \sum_{m=0}^{\infty} \frac{\mathcal{N}^m}{(\mathcal{N}+1)^m} |m\rangle\langle m| \,.$$

Problem 8.2 $\star$ [Sect. 8.2] Organize a quantum measurement with the system in the state ρ_{th} defined by (8.2). The outcome n should have the geometrical distribution $p_m(k|\rho_{\text{th}})$ given by (8.6).

Solution We recall from Postulate 3 of Quantum Mechanics (see (3.36)) that the probability of a outcome of the measurement obtained with a projector system $\{\Pi_i\}$, when the quantum system is in the state ρ, is given by

$$p(i) := \mathrm{P}[m = i|\rho] = \mathrm{Tr}[\rho\,\Pi_i] \,.$$

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series, and products*, 7th ed. Amsterdam: Elsevier, 2007, p. 340.

Now we consider the projector system obtained by the Fock basis $\{\Pi_i = |i\rangle\langle i|\}$. Since

$$\rho_{\text{th}} = \sum_{k=0}^{\infty} p_m(k)|k\rangle\langle k| \quad \text{with} \quad p_m(k) = \frac{\mathcal{N}^k}{(\mathcal{N}+1)^{k+1}}$$

we obtain

$$p(i) = \text{Tr}[\rho\,\Pi_i] = \sum_{k=0}^{\infty} p_m(k)|k\rangle\langle k|i\rangle\langle i|$$

where $\langle k|i\rangle = \delta_{ik}$. Hence

$$p(i) = \text{Tr}[\rho\,\Pi_i] = p_m(k)\,\text{Tr}(|k\rangle\langle k|) = p_m(k)\;.$$

Problem 8.3 $\star$ [Sect. 8.2] Prove (8.5), that is, $\text{E}[m|\rho_{\text{th}}] = \text{Tr}[\rho_{\text{th}}N] = \mathcal{N}$, where ρ_{th} is the density operator of thermal noise given by (8.4) and N is the number operator.

Solution In $\text{E}[m|\rho_{\text{th}}] = \text{Tr}[\rho_{\text{th}}N]$ we have

$$\rho_{\text{th}} = \sum_{k=0}^{\infty} p(k)|k\rangle\langle k| \quad \text{with} \quad p(k) = \frac{\mathcal{N}^k}{(\mathcal{N}+1)^{k+1}}$$

and

$$N = \sum_{n=0}^{\infty} n|n\rangle\langle n|\;.$$

Then

$$\text{E}[m|\rho_{\text{th}}] = \text{Tr}\left[\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} n\,p(k)|k\rangle\langle k|n\rangle\langle n|\right]$$

where $\langle k|n\rangle = \delta_{in}$ and $\text{Tr}[|n\rangle\langle n|] = 1$. Hence

$$\text{E}[m|\rho_{\text{th}}] = \sum_{k=0}^{\infty} k\,p(k)$$

which establishes that $\text{E}[m|\rho_{\text{th}}]$ gives the mean of the random variable having $p(k)$ as probability distribution. On the other hand, developing the calculation and using the identity [3]

$$\sum_{k=0}^{\infty} k\,u^k = \frac{u}{(1-u)^2}$$

one obtains the desired result.

Problem 8.4 $\star\star$ [Sect. 8.2] Representations (8.8) and (8.10) on thermal noise hold for $\gamma \neq 0$ and $\mathcal{N} > 0$. Find and discuss the representations in the degenerate cases $\gamma = 0$ (absence of signal) and $\mathcal{N} = 0$ (absence of noise).

[3] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series, and products*, 7th ed. Amsterdam: Elsevier, 2007, p. 8

Solution For $\gamma = 0$ representation (8.8) gives (8.2), while (8.10) gives (8.4). For $\mathcal{N} = 0$, both (8.8) and (8.10) degenerate. The state becomes pure and therefore we have

$$\rho(\gamma) = |\gamma\rangle\langle\gamma|$$

with $|\gamma\rangle$ given by (8.1). Hence the matrix representation (8.10) becomes

$$R_{mn}(\gamma) = \mathrm{e}^{-\frac{1}{2}|\gamma|^2} \frac{\gamma^n (\gamma^*)^n}{\sqrt{m!n!}} |m\rangle\langle n| \, .$$

Problem 8.5 ⋆⋆ [Sect. 8.10] Check that the further condition (6.66), that is, $\mu_0^* \beta_0 = \alpha I$, is not verified in 4 PSK with the data of Example 8.19.

Solution It results

$$\mu_0^* \beta_0 = \begin{bmatrix} 0.456 & 0.001 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.001 & 0.184 & 0.003 & -0.001 & -0.001 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.003 & 0.067 & 0.007 & -0.001 & -0.001 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.001 & 0.007 & 0.020 & 0.005 & 0.000 & -0.001 & 0.000 & 0.000 \\ 0.000 & -0.001 & -0.001 & 0.005 & 0.006 & 0.002 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.001 & 0.000 & 0.002 & 0.002 & 0.001 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & -0.001 & 0.000 & 0.001 & 0.001 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

that it is not proportional to the identity matrix, and therefore the SRM decision is not optimal.

Problems of Chapter 9

Problem 9.1 ⋆⋆ [Sect. 9.2] Consider the model of a photon counter where the dark current and the non unitary efficiency are taken into account. Prove that the measurement operators are given by (9.22).

Solution Provided that the input state is the Fock state $|n\rangle$ with n photons, the probability of no electron count is

$$\langle n|Q_0|n\rangle = \mathrm{e}^{-\mu}(1-\eta)^n$$

coinciding with the probability the no dark electron are emitted and each photon is missed with probability $1-\eta$. Then the first of (9.22) holds, while the second follows by the identity resolution. In the presence of input states $|0\rangle$ and $|\alpha\rangle$ the transition probabilities of the equivalent binary channel become

$$\begin{aligned}
p(0|0) =& \langle 0|Q_0|0\rangle = \mathrm{e}^{-\mu} \\
p(0|1) =& \langle \alpha|Q_0|\alpha\rangle = \mathrm{e}^{-\mu}\sum_{n=0}^{\infty}(1-\eta)^n|\langle n|n\rangle\alpha|^2 \\
=& \mathrm{e}^{-\mu}\mathrm{e}^{-|\alpha|^2}\sum_{n=0}^{\infty}(1-\eta)^n\frac{|\alpha|^{2n}}{n!} \\
=& \mathrm{e}^{-\mu-|\alpha 2|+(1-\eta)|\alpha|^2} = \mathrm{e}^{-\mu-\eta|\alpha|^2}\,.
\end{aligned}$$

Problems of Chapter 10

Problem 10.1 $\star\star\star$ [Sect. 10.3] To check Schmidt's decomposition consider a finite–dimensional bipartite system with $\mathcal{H}_A = \mathbb{C}^2$ and $\mathcal{H}_B = \mathbb{C}^4$, where the coefficient matrix C is 2×4. Suppose that the matrix has the form

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \beta \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \sqrt{\frac{5}{8} - \beta^2} \end{bmatrix}$$

where β is a parameter. Find the values of $beta$, if any, which correspond to a separable state and to a maximally entangled state.

Solution The SVD gives the two Schmidt coefficients

$$d_{\pm} = \frac{1}{4}\sqrt{8 \pm \sqrt{\Delta}} \quad \text{with} \quad \Delta = 24\sqrt{2}\beta\sqrt{5 - 8\beta^2} + 34$$

and $d_-^2 + d_+^2 = 1$. Now, we can see that in general the state is entangled. For $\beta = (\pm 1 \pm \mathrm{i}/4)/\sqrt{3}$, we have $d_{\pm} = \sqrt{2}/2$ and the state is maximally entangled. For $\beta = \sqrt{5}/4$ we have $d_- = 0$ and $d_+ = 1$, so that the state is separable. In fact, the matrix of the coefficients becomes

$$C = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{\sqrt{5}}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{\sqrt{5}}{4} \end{bmatrix}$$

and the linear combination (10.7) gives

$$|\psi\rangle\rangle = \left(\frac{1}{4}|b_1^A\rangle + \frac{1}{4}|b_2^A\rangle + \frac{1}{4}|b_3^A\rangle + \frac{\sqrt{5}}{4}|b_4^A\rangle\right) \otimes \left(|b_1^B\rangle + |b_2^B\rangle\right)$$

which clearly is the tensor product of two states.

Problems of Chapter 11

An extra problem, Problem 11.19E, not included in the book, is introduced.

Problem 11.1 ⋆ [Sect. 11.5] Prove relation (11.58), which states that all the number states $|n\rangle$ can be obtained from the ground state $|0\rangle$.

Solution Relation (11.58) is proved by induction considering that it holds for $n=0$ and in the iterations we can use (11.57).

Problem 11.2 ⋆⋆ [Sect. 11.6] Using the general definition of the exponential of a matrix, find explicitly the exponential of a 2×2 matrix.

Solution First we have to find the EID of the matrix, say

$$A = U\,\Lambda\,U^* \quad \rightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} u_{11}^* & u_{21}^* \\ u_{12}^* & u_{22}^* \end{bmatrix} .$$

Then we have

$$\exp A = U\,\exp\Lambda\,U^* \quad \text{with} \quad \exp\Lambda = \begin{bmatrix} \mathrm{e}^{\lambda_1} & 0 \\ 0 & \mathrm{e}^{\lambda_2} \end{bmatrix} .$$

The secular equation is $\lambda^2 - (a+d)\lambda + ad - bc = 0$ whose solutions are $\lambda_\mp = \frac{1}{2}(a+d\mp\sqrt{\Delta})$. Hence

$$\Lambda = \begin{bmatrix} \frac{1}{2}(a+d-\sqrt{\Delta}) & 0 \\ 0 & \frac{1}{2}(a+d+\sqrt{\Delta}) \end{bmatrix} .$$

It remains to evaluate the eigenvectors to arrive at the formula of the text.

Problem 11.3 ⋆⋆ [Sect. 11.6] Prove relation (11.81) linking the complex vectors and the real vectors defined by (11.80). Note that the entries of the matrix Ω can be written in the form

$$\Omega_{2(h-1)+r,2(k-1)+s} = \delta_{hk}(\delta_{r,s-1} - \delta_{r-1,s}) = \delta_{hk}\,\varepsilon_{rs}\,, \qquad h,k = 1,\ldots,N \quad r,s = 1,2$$

where

$$\varepsilon_{rs} = \begin{cases} 1 & r=1, s=2 \\ -1 & r=2, s=1 \\ 0 & \text{otherwise}\,. \end{cases}$$

Solution Letting $X = [X_1,\ldots,X_{2N}]^{\mathrm{T}}$ and $Y = [Y_1,\ldots,Y_{2N}]^{\mathrm{T}}$ we find

$$X^{\mathrm{T}}\,\Omega\,Y = \sum_{h=1}^{N} \left[X_{2(h-1)+1}Y_{2(h-1)+2} - X_{2(h-1)+2}Y_{2(h-1)+1}\right]$$

Next we replace X and Y with x_y and u_v, respectively, and use the fact that $x_i = \frac{1}{2}(\lambda_i + \lambda_i^*)$, etc. Thus (11.81) follows.

Problem 11.4 ⋆⋆ [Sect. 11.7] Prove the relation (11.100) between the covariance matrices R and V in the single mode.

Solution The covariances given by (11.99) are

$$V_{11} = \frac{1}{2}\langle\{|\Delta q, \Delta q\}\rangle = \langle(\Delta q)^2\rangle = R_{11}$$

$$\begin{aligned} V_{12} =& \frac{1}{2}\langle\{\Delta q, \Delta p\}\rangle = \frac{1}{2}\langle(q-\overline{q})(p-\overline{p}) + (p-\overline{p})(q-\overline{q})\rangle \\ =& \frac{1}{2}[q\,p + pq + 2\overline{q}\,\overline{p} - 2p\overline{q} - 2q\overline{p}] \end{aligned}$$

where we can express pq using the commutation relation (11.51), that is, $p\,q = q\,p - 2\mathrm{i}$. Hence

$$V_{12} = q\,p + \mathrm{i} + \overline{q}\,\overline{q} - p\overline{p} - q\overline{p} = \Delta q \Delta p - \mathrm{i} = R_{12} - \mathrm{i}$$

and

$$V_{12} = \langle q\,p + \mathrm{i} + \overline{q}\,\overline{p} - p\overline{q} - q\overline{p}\rangle = \langle q\,p\rangle + \mathrm{i} - \overline{q}\,\overline{p} = V_{qp} + \mathrm{i}\,.$$

Analogously we find $V_{21} = R_{21} - \mathrm{i}$ and $V_{22} = R_{22}$. In conclusion, we find the matrix relation

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} - \mathrm{i}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which can be written in the form

$$V = R - \mathrm{i}\,\Omega\,.$$

Problem 11.5 ⋆⋆ [Sect. 11.7] Compare conditions (11.101) and (11.102) in the single mode.

Solution Condition (11.101) gives

$$V_{11}V_{22} \geq 1$$

while condition (11.102) reads

$$V + \mathrm{i}\Omega = \begin{bmatrix} V_{11} & V_{12} + \mathrm{i} \\ V_{12} - \mathrm{i} & V_{22} \end{bmatrix} \geq 0\,.$$

We recall that a Hermitian matrix is PSD if and only if its principal minors are nonnegative. In the case of matrix $V + \mathrm{i}\Omega$ we have the conditions

$$V_{11} \geq 0\,, \qquad V_{22} \geq 0\,, \qquad \det(V + \mathrm{i}\Omega) = V_{11}V_{22} - V_{12}^2 + 1 \geq 0$$

which constrains the nondiagonal element as

$$V_{12}^2 \leq V_{11} V_{22} \leq 1 \,.$$

▽ **Problem 11.6** ⋆⋆⋆ [Sect. 11.7] Prove condition (11.113), which states that the characteristic function $\chi(\xi)$ refers to a pure state.

Hint Use the Fock expansion of the pure state and Proposition 11.9.

Solution We prove the statement in the single mode. Let

$$|\psi\rangle = \sum_n f_n \, |n\rangle$$

be the Fock expansion of the pure state $|\psi\rangle$. Then (11.112) gives

$$\chi(\xi) = \langle\psi|D(\xi)|\psi\rangle = \sum_m \sum_n f_m^* f_n \langle m|D(\xi)|n\rangle = \sum_m \sum_n f_m^* f_m \, D_{mn}(\xi)$$

and

$$|\chi(\xi)|^2 = \chi(\xi)\chi^*(\xi) = \sum_m \sum_n f_m^* f_m \, D_{mn}(\xi) \sum_r \sum_s f_r f_s^* \, D_{rs}^*(\xi) \,.$$

Now in the integral we can use Proposition 11.9 to get

$$\int_{\mathbb{C}} \mathrm{d}\xi \, |\chi(\xi)|^2 = \sum_m \sum_n f_m^* f_m \sum_r \sum_s f_r f_s^* \, \delta_{mr} \delta_{ns} = \sum_m \sum_n |f_m|^2 |f_n|^2 = 1$$

where $\sum_m |f_m|^2 = 1$ is due to the normalization of the ket $|\psi\rangle$.

▽ **Problem 11.7** ⋆⋆⋆ [Sect. 11.7] Evaluate the integral (11.114) using Williamson's theorem (Theorem 11.2).

Solution From (11.105a) we have that the integrand is

$$|\chi(\xi)|^2 = |\chi(u,v)|^2 = \exp\left[-u_V{}^{\mathrm{T}} \left(\Omega \, V \, \Omega^{\mathrm{T}}\right) u_V\right] \,.$$

By Williamson's theorem the covariance matrix can be decomposed as

$$V = S_w V^{\oplus} S_w^{\mathrm{T}} \,, \qquad V^{\oplus} = \mathrm{diag}\left[\sigma_1^2, \sigma_1^2, \ldots, \sigma_N^2, \sigma_N^2\right] \,.$$

Then

$$\begin{aligned} \int_{\mathbb{C}^N} \mathrm{d}\xi \, |\chi(\xi)|^2 &= \int_{\mathbb{R}^{2N}} \mathrm{d}u \mathrm{d}v \, |\chi(u,v)|^2 \\ &= \int_{\mathbb{R}^{2N}} \mathrm{d}u \, \mathrm{d}v \, \exp\left[-u_V{}^{\mathrm{T}} \left(\Omega \, S_w V^{\oplus} S_w^{\mathrm{T}} \Omega^{\mathrm{T}}\right) u_V\right] \,. \end{aligned}$$

Now it is convenient to make the change of variables

$$X = S_w^{\mathrm{T}} \Omega^{\mathrm{T}} \, u_V \quad \rightarrow \quad \mathrm{d}X = |\det(S_w \Omega)| \mathrm{d}u \mathrm{d}v = \mathrm{d}u \mathrm{d}v$$

where $|\det(S_w \Omega)| = 1$ because S_w is a symplectic matrix (see (11.149)). Hence

$$\int_{\mathbb{C}^N} \mathrm{d}\xi\, |\chi(\xi)|^2 = \int_{\mathbb{R}^{2N}} \mathrm{d}X \exp\left[-X^{\mathrm{T}} V^{\oplus} X\right] .$$

Considering that the matrix $V^{\oplus}$ is diagonal, the biquadratic form at the exponential results in

$$X^{\mathrm{T}} V^{\oplus} X = (X_1^2 + X_2^2)\sigma_1^2 + \cdots + (X_{2N-1}^2 + X_{2N}^2)\sigma_N^2$$

so that the exponential is given by product of $2N$ exponentials and

$$\int_{\mathbb{C}^N} \mathrm{d}\xi\, |\chi(\xi)|^2 = \prod_{i=1}^{N} \int_{\mathbb{R}} \mathrm{d}X_{2i-1} \exp\left[-X_{2i-1}^2\, \sigma_i^2\right] \int_{\mathbb{R}} \mathrm{d}X_{2i} \exp\left[-X_{2i}^2 \sigma_i^2\right] .$$

Each integral gives $\sqrt{\pi}/\sigma_i$. Hence

$$\int_{\mathbb{C}^N} \mathrm{d}\xi\, |\chi(\xi)|^2 = \frac{\pi^N}{\sigma_1^2 \cdot \sigma_N^2} .$$

It remains to recall that σ_i^2 are the symplectic eigenvalues of V, which can be obtained as the ordinary eigenvalues of the matrix $\mathrm{i}\,\Omega\, V$. More precisely, the eigenvalues of $\mathrm{i}\,\Omega\, V$ are $\{\pm\sigma_1^2, \ldots, \pm\sigma_N^2\}$. On the other hand, we recall that the product of the eigenvalues of a square matrix gives the determinant of the matrix (see (2.46)). Hence $\det(\mathrm{i}\,\Omega\, V) = (-1)^N \sigma_1^4 \cdots \sigma_N^4$, where $\det(\mathrm{i}\,\Omega) = (-1)^N$. Therefore $\sigma_1^2 \cdots \sigma_N^2 = \sqrt{\det V}$. In conclusion

$$\int_{\mathbb{C}^N} \mathrm{d}\xi\, |\chi(\xi)|^2 = \frac{\pi^N}{\sqrt{\det V}} .$$

Problem 11.8 ⋆⋆⋆ [Sect. 11.8] Prove Glauber's inversion formula (11.90) in the single mode. *Hint*: use Fock representation and the orthogonality of the D_{mn} (see Proposition 11.9).

Solution The density operator ρ can be recovered from the characteristic function $\chi(\xi)$ in the single mode as

$$\rho = \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi\, \chi(\xi) D^*(\xi) .$$

We denote by $\tilde{\rho}$ the reconstructed operator, namely

$$\tilde{\rho} = \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi D^*(\xi)\chi(\xi) = \int_{\mathbb{C}} \mathrm{d}\xi D^*(\xi)\, \mathrm{Tr}\left(\rho \mathrm{e}^{\xi a^* - \xi^* a}\right) .$$

Then, we proceed with the matrix representation of the operators to get

$$\begin{aligned}\tilde{\rho}_{mn} =& \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi \, D^*_{nm}(\xi) \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \rho_{sr} D_{rs}(\xi) \\ =& \sum_{s=0}^{\infty} \sum_{r=0}^{\infty} \rho_{sr} \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi \, D^*_{nm}(\xi) D_{rs}(\xi)\end{aligned}$$

where we apply the orthogonality of the D_{mn} (see Proposition 11.9), which gives $\tilde{\rho}_{mn} = \rho_{mn}$.

Problem 11.9 ⋆ [Sect. 11.8] Using the orthogonality of the functions $D_{mn}(\xi)$ given by (11.132), prove the orthogonality of the functions $\mathcal{D}_{mn}(\lambda)$, the Fourier transform of the $D_{mn}(\xi)$.

Solution Combination of (11.131) and (11.132) gives

$$\int_{\mathbb{C}} \mathrm{d}\lambda \, \mathcal{D}_{mn}(\lambda)^* \, \mathcal{D}_{rs}(\lambda) = (-1)^{\min(m,n)+\min(r,s)} \delta_{mn}\delta_{rs} = \delta_{mn}\delta_{rs} \, .$$

▽ **Problem 11.10** ⋆⋆⋆ [Sect. 11.9] Thermal states are defined as the bosonic states that maximize the von Neumann entropy for a fixed energy. Prove this statement using Lagrange multipliers.

Solution not available

Problem 11.11 ⋆⋆ [Sect. 11.9] Prove that, if the characteristic function $\chi(\xi)$ depends only on $|\xi|^2$, say $\chi(\xi) = f(|\xi|^2)$, the reconstruction formula (11.110) of Proposition 11.7) is simplified as

$$\boxed{\rho_{nm} = \delta_{mn} \int_0^{\infty} \mathrm{d}x \, \mathrm{e}^{-\frac{1}{2}x} f(x) \, L_n(x)} \tag{11.1}$$

where $L_n(x)$ is the ordinary Laguerre polynomial.

Solution The general reconstruction formula of the matrix representation of the density operator is given by (11.110), that is,

$$\rho_{nm} = \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi \, \chi(\xi) D^*_{mn}(\xi) \tag{11.2}$$

where the matrix representation of the Weyl operator is given in Proposition 11.7 as

$$D_{mn}(\xi) = \mathrm{e}^{-\frac{1}{2}|\xi|^2} \sqrt{\frac{n!}{m!}} \, \xi^{m-n} L_n^{(m-n)}(|\xi|^2) \, .$$

Now, considering that $\chi(\xi) = f(|\xi|^2)$ we find

$$\rho_{nm} = \frac{1}{\pi} \int_{\mathbb{C}} \mathrm{d}\xi \, f(|\xi|^2) \, \mathrm{e}^{-\frac{1}{2}|\xi|^2} \sqrt{\frac{n!}{m!}} \, (\xi^*)^{m-n} L_n^{(m-n)}(|\xi|^2)$$

where we use polar coordinates with $\xi = \lambda \exp(\mathrm{i}\phi)$ to get

$$\rho_{nm} = \frac{1}{\pi}\int_0^\infty \mathrm{d}\lambda\, \lambda \int_0^{2\pi} \mathrm{d}\phi\, f(\lambda^2)\, \mathrm{e}^{-\frac{1}{2}\lambda^2} \sqrt{\frac{n!}{m!}}\, \lambda^{m-n} \mathrm{e}^{\mathrm{i}(n-m)\phi} L_n^{(m-n)}(\lambda^2)\,.$$

Considering that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}(n-m)\phi} = \delta_{mn}$$

we find

$$\rho_{nm} = \delta_{mn}\, 2\int_0^\infty \mathrm{d}\lambda\, \lambda\, f(\lambda^2)\, \mathrm{e}^{-\frac{1}{2}\lambda^2} L_n^{(0)}(\lambda^2)\,.$$

and (11.143) follows after the change of variable $x = \lambda^2$.

Problem 11.12 $\star\star$ [Sect. 11.9] Consider the alternative definition of a coherent state given by (11.138). Show that the Fock representation of $|\alpha\rangle$ is still given by (11.46).

Solution The alternative definition is

$$|\alpha\rangle = D(\alpha)|0\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\, \mathrm{e}^{a^*\alpha}\, \mathrm{e}^{-\alpha^* a}|0\rangle\,.$$

Now, from (11.126c), $\mathrm{e}^{-\alpha^* a}|0\rangle = |0\rangle$. Thus

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\, \mathrm{e}^{a^*\alpha}\, |0\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{n!} (a^*)^n |0\rangle$$

where we can use (11.124) to get $(a^*)^n\, |0\rangle = \sqrt{n!}|n\rangle$. Hence

$$|\alpha\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle$$

which is the Fock expansion obtained with the original definition of $|\alpha\rangle$.

Problem 11.13 $\star$ [Sect. 11.11] Prove that the Bogoliubov transformation generated by the N–mode displacement operator is given by

$$D_N^*(\alpha)\, a\, D_N(\alpha) = a + \alpha\,.$$

Then evaluate the corresponding symplectic matrix.

Solution The N–mode displacement operator $D_N(\alpha)$ is simply given by the product of N single–mode displacement operators . Hence, by Proposition 11.12, which holds also in the N–mode, one gets the Bogoliubov transformation. The symplectic transformation is obtained by using relation (11.151), that is,

$$S = \Pi \begin{bmatrix} \Re(E+F) & \Im(-E+F) \\ \Im(E+F) & \Re(E-F) \end{bmatrix} \Pi^{\mathrm{T}}, \qquad d = \Pi \begin{bmatrix} \Re z \\ \Im z \end{bmatrix}$$

with $u = I_N$ and $v = 0_N$ to get

$$S = \Pi \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} \Pi^{\mathrm{T}} = \Pi \, \Pi^{\mathrm{T}} = I_{2N} \, .$$

On the other hand the permutation matrix Π provides the interlace of the two N–column vectors, so that we can write

$$d = \Re\alpha \Im\alpha \, .$$

Problem 11.14 $\star\star\star$ [Sect. 11.11] Prove that the N–mode rotation operator (11.161) produces the Bogoliubov transformation

$$R_N(\phi)\, a\, R_N(\phi) = \mathrm{e}^{\mathrm{i}\,\phi}\, a \, .$$

Solution Considering that the matrix ϕ is Hermitian, the adjoint of the exponent $\mathrm{i}\, a^* \phi\, a$ is $-\mathrm{i}\, a^* \phi\, a$. Hence

$$R^*(\phi)\, a\, R(\phi) = \mathrm{e}^{-\mathrm{i}\, a^* \phi\, a}\; a\; \mathrm{e}^{\mathrm{i}\, a^* \phi\, a} \tag{11.3}$$

which corresponds to the BCH identity (11.168) with

$$x = -\mathrm{i} \, , \qquad H = a^* \phi\, a \, , \qquad K = a \, .$$

But we have to interpret this identity in the sense of (11.168), that is,

$$\mathrm{e}^{xH}\, a_k\, \mathrm{e}^{-xH} = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_n(k) \, , \qquad k = 1, \ldots, N$$

where

$$D_0(k) = a_k \, , \qquad D_n(k) = [H, D_{n-1}(k)] \quad \text{for} \quad n \geq 1 \, . \tag{11.3a}$$

Then

$$R^*(\phi)\, a_k\, R(\phi) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}\,\phi)^n}{n!}\, D_n(k) \, . \tag{11.4}$$

We now proceed for $N = 2$. We have $D_0(1) = a_1$, $D_0(2) = a_2$, and

$$a^* \phi\, a = a_1^* \phi_{11} a_1 + a_1^* \phi_{12} a_2 + a_2^* \phi_{21} a_1 + a_2^* \phi_{22} a_2 \, .$$

Hence, recalling that operators of different modes commute, we find

$$\begin{aligned} D_1(1) = [a^* \phi\, a, a_1] &= \{a_1^* \phi_{11} a_1 + a_1^* \phi_{12} a_2 + a_2^* \phi_{21} a_1 + a_2^* \phi_{22} a_2\}\, a_1 \\ &\qquad - a_1\, \{a_1^* \phi_{11} a_1 + a_1^* \phi_{12} a_2 + a_2^* \phi_{21} a_1 + a_2^* \phi_{22} a_2\} \\ &= -(\phi_{11} a_1 + \phi_{12} a_1) \end{aligned}$$

$$\begin{aligned} D_1(2) = [a^*\,\phi\,a, a_2] &= \{a_1^*\phi_{11}a_1 + a_1^*\phi_{12}a_2 + a_2^*\phi_{21}a_1 + a_2^*\phi_{22}a_2\}\,a_2 \\ &\qquad - a_1\,\{a_2^*\phi_{11}a_1 + a_1^*\phi_{12}a_2 + a_2^*\phi_{21}a_1 + a_2^*\phi_{22}a_2\} \\ &= -(\phi_{21}a_1 + \phi_{22}a_2) \end{aligned}$$

which can be written in the matrix form

$$\begin{bmatrix} D_1(1) \\ D_1(2) \end{bmatrix} = -\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad \to \quad D_1 = -\phi\,a\,.$$

Now we can organize (with some effort) an induction procedure to prove that

$$D_n = (-\phi)^n\,a$$

and the final result is

$$R^*(\phi)\,a\,R(\phi) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}\,\phi)^n}{n!} = \mathrm{e}^{\mathrm{i}\,\phi}\,a\,.$$

Problem 11.15 ⋆⋆ [Sect. 11.11] Write explicitly the symplectic matrix of a two–mode rotation operator in the two cases of matrix ϕ

$$\phi = \begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_0 \end{bmatrix}, \qquad \phi = \begin{bmatrix} 0 & \phi_0 \\ \phi_0 & 0 \end{bmatrix} \qquad\quad (\phi_0 \quad \text{real})\,.$$

Hint: use identities (11.74) for the exponential and the expressions of Π and Π^{T} given after Proposition 11.11.

Solution In the first case the matrix is diagonal and therefore

$$\mathrm{e}^{\mathrm{i}\,\phi} = \exp\mathrm{i}\begin{bmatrix} \phi_0 & 0 \\ 0 & \phi_0 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\,\phi_0} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\,\phi_0} \end{bmatrix} = \mathrm{e}^{\mathrm{i}\,\phi_0}\,I_2\,.$$

Hence

$$\Re\mathrm{e}^{\mathrm{i}\,\phi} = \cos(\phi_0)\,I_2\,, \qquad \Im\mathrm{e}^{\mathrm{i}\,\phi} = \sin(\phi_0)\,I_2\,.$$

The symplectic matrix is explicitly

$$S = \Pi \begin{bmatrix} \Re(E+F) & \Im(-E+F) \\ \Im(E+F) & \Re(E-F) \end{bmatrix} \Pi^{\mathrm{T}} = \Pi \begin{bmatrix} \cos(\phi_0)\,I_2 & -\sin(\phi_0)\,I_2 \\ \sin(\phi_0)\,I_2 & \cos(\phi_0)\,I_2 \end{bmatrix} \Pi^{\mathrm{T}}$$

and, considering the expressions of the permutation matrices

$$S = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix} \begin{bmatrix} \cos(\phi_0) & 0 & -\sin(\phi_0) & 0 \\ 0 & \cos(\phi_0) & 0 & -\sin(\phi_0) \\ \sin(\phi_0) & 0 & \cos(\phi_0) & 0 \\ 0 & \sin(\phi_0) & 0 & \cos(\phi_0) \end{bmatrix} \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{bmatrix}.$$

The final result is

$$S = \begin{bmatrix} \cos(\phi_0) & -\sin(\phi_0) & 0 & 0 \\ \sin(\phi_0) & \cos(\phi_0) & 0 & 0 \\ 0 & 0 & \cos(\phi_0) & -\sin(\phi_0) \\ 0 & 0 & \sin(\phi_0) & \cos(\phi_0) \end{bmatrix}.$$

In the second case

$$E := \exp\begin{bmatrix} 0 & \phi_0 \\ \phi_0 & 0 \end{bmatrix} = \begin{bmatrix} \cos(\phi_0) & \mathrm{i}\sin(\phi_0) \\ \mathrm{i}\sin(\phi_0) & \cos(\phi_0) \end{bmatrix}.$$

Then

$$\Re E = \cos(\phi_0)\, I_2\,, \qquad \Im E = \sin(\phi_0)\, J_2 \quad \text{with} \quad J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the conclusion is that the symplectic matrix S is the same as in the first case.

Problem 11.16 $\star\star\star$ [Sect. 11.11] Prove that the N–mode squeeze operator produces the Bogoliubov transformation

$$Z_N^*(z)\, a\, Z_N(z) = \cosh r\, a + \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\, a_*\,, \tag{11.5}$$

where the symmetric matrix z is written in the form $z = r\mathrm{e}^{\mathrm{i}\,\theta}$.

Solution In Ma and Rhode's paper[4] the authors assume that the matrix z is **symmetric** and claim that

$$Z_N^*(z)\, a\, Z_N(z) = a + z a_* + \frac{1}{2!} z z^*\, a + \frac{1}{3!}(zz^*) z\, a_* + \frac{1}{4!}(zz^*)^2\, a + \cdots \tag{11.6}$$

With the decomposition $z = r\mathrm{e}^{i\theta}$, the even and odd terms in (11.6) give respectively

$$\begin{aligned} E =& a + \frac{1}{2!} zz^*\, a + \frac{1}{4!}(zz^*)^2\, a + \cdots \\ =& (I_N + \frac{1}{2!} r^2 + \frac{1}{4!} r^4 + \cdots)\, a = \cosh r\, a \\ O =& z a_* + \frac{1}{3!}(zz^*) z\, a_* + \frac{1}{5!}(zz^*)^2 z\, a_* + \cdots \\ =& (r + \frac{1}{3!} r^3 + \frac{1}{5!} r^5 \cdots)\mathrm{e}^{\mathrm{i}\theta}\, a_* = \sinh r \mathrm{e}^{\mathrm{i}\theta}\, a_* \end{aligned}$$

and (11.179) follows.

We now sketch the proof of the first terms of (11.6) for $N = 2$, investigating in particular the role of the assumption of symmetry for z. We have

$$Z_N^*(z)\, a\, Z_N(z) = \mathrm{e}^{\frac{1}{2}(a^* z (a^*)^{\mathrm{T}} - a^{\mathrm{T}} z a)^*}\, a \mathrm{e}^{\frac{1}{2}(a^* z (a^*)^{\mathrm{T}} - a^{\mathrm{T}} z a)^*}$$

which corresponds to the BCH identity (11.168) with

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[4] X. Ma and W. Rhodes, "Multimode squeeze operators and squeezed states," *Phys. Rev. A*, vol. 41, pp. 4625–4631, May 1990.

$$x = \frac{1}{2}\,, \qquad H = (a^* z (a^*)^{\mathrm{T}} - a^{\mathrm{T}} z a)^* = a^{\mathrm{T}} z^* a - a^* z (a^*)^{\mathrm{T}}\,, \qquad K = a\,.$$

But we have to interpret this identity in the sense of (11.168), that is,

$$\mathrm{e}^{xA}\, a_k\, \mathrm{e}^{-xA} = \sum_{n=0}^{\infty} \frac{x^n}{n!} D_n(k)\,, \qquad k = 1, \ldots, N$$

where

$$D_0(k) = a_k\,, \qquad D_n(k) = [H, D_{n-1}(k)] \quad \text{for} \quad n \geq 1\,. \tag{11.6a}$$

Then

$$Z_N^*(z)\, a\, Z_N(z) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} D_n \tag{11.7}$$

where D_n is the $N \times 1$ vector collecting the $D_n(k)$.

Now

$$H = a^{\mathrm{T}} z^* a - a^* z (a^*)^{\mathrm{T}} = \sum_{i=0}^{N} \sum_{j=0}^{N} (z_{ji}^* a_i a_j - z_{ij} a_i^* a_j^*)$$

and

$$D_1(k) = [a, a_k] = \sum_{i=0}^{N} \sum_{j=0}^{N} \left(z_{ji}^* [a_i a_j, a_k] - z^* ij\, [a_i^* a_j^*, a_k] \right)\,.$$

In particular for $N = 2$

$$\begin{aligned} D_1(k) =& z_{11}^*[a_1 a_1, a_k] + z_{21}^*[a_1 a_2, a_k] + z_{12}^*[a_2 a_1, a_k] + z_{22}^*[a_2 a_2, a_k] \\ & - (z_{11}[a_1^* a_1^*, a_k] + z_{21}[a_1^* a_2^*, a_k] + z_{12}[a_2^* a_1^*, a_k] + z_{22}[a_2^* a_2^*, a_k]) \end{aligned}$$

where $[a_i a_j, a_k] = 0$ and

$$\begin{aligned} [a_1^* a_1^*, a_1] =& a_1^* a_1^* a_1 - a_1 a_1^* a_1^* = a_1^* a_1^* a_1 - a_1^* a_1 a_1^* + a_1^* a_1 a_1^* + a_1^* a_1 a_1^* + a_1 - a_1 a_1^* a_1^* \\ =& a_1^* (a_1^* a_1 - a_1^* a_1) + (a_1^* a_1 - a_1 a_1^*) a_1^* = -2a_1^* \\ [a_1^* a_2^*, a_1] =& a_1^* a_2^* a_1 - a_1 a_1^* a_2^* = -a_2^* \\ [a_2^* a_1^*, a_1] =& a_2^* a_1^* a_1 - a_1 a_2^* a_1^* = -a_2^* \\ [a_2^* a_2^*, a_1] =& 0 \end{aligned}$$

$$\begin{aligned} [a_1^* a_1^*, a_2] =& 0 \\ [a_1^* a_2^*, a_2] =& a_1^* a_2^* a_2 - a_2 a_1^* a_2^* = -a_1^* \\ [a_2^* a_1^*, a_2] =& a_2^* a_1^* a_2 - a_2 a_2^* a_1^* = -a_1^* \\ [a_2^* a_2^*, a_2] =& -2a_1^*\,. \end{aligned}$$

Hence

$$D_1 = \begin{bmatrix} D_1(1) \\ D_1(2) \end{bmatrix} = \begin{bmatrix} 2z_{11} a_1 + z_{21} a_2^* + z_{12} a_2^* \\ + z_{21} a_1^* + z_{12} a_1^* + 2z_{22} a_1^* \end{bmatrix} = \begin{bmatrix} 2z_{11} & z_{12} + z_{21} \\ z_{12} + z_{21} & 2z_{22} \end{bmatrix} \begin{bmatrix} a_1^* \\ a_2^* \end{bmatrix}$$

Now, if the matrix z is symmetric we find

$$D_1 = 2z a_* \quad \rightarrow \quad \frac{1}{2} D_1 = z a_*$$

in agreement with the first term of (11.109).

The next step would be the evaluation of

$$D_2 = [D_1, a] = [z a_*, a]$$

but the organization of an induction procedure becomes mandatory.

Problem 11.17 $\star$ [Sect. 11.14] Prove in the single bosonic mode condition (11.148), which states that the commutation relation is preserved after a Bogoliubov transformation.

Hint Use the bilinearity of the trace

$$[u_1 H_1 + u_2 H_2, v_1 K_1 + v_2 K_2] = u_1 v_1 [H_1, K_1] + u_1 v_2 [H_1, K_2] + u_2 v_1 [H_2, K_1] + u_2 v_2 [H_2, K_2] \tag{11.8}$$

where u_i, v_j are complex numbers and H_i, K_j are operators.

Solution The direct solution is

$$\begin{aligned}
[\tilde{a}, \tilde{a}^*] =& \tilde{a}\tilde{a}^* - \tilde{a}^*\tilde{a} \\
=& (E\,a + F\,a^* + z)(E^*\,a^* + F^*\,a + z^*) - (E^*\,a^* + F^*\,a + z^*)(E\,a + F\,a^* + z) \\
=& (EE^* - F^*F)aa^* + (EF^* - F^*E)aa + (FE^* - F^*E)a^*a^* + (FF^* - E^*E)a^*a \\
=& (EE^* - F^*F)(aa^* - aa^*) = (EE^* - F^*F)[a, a^*]\ .
\end{aligned}$$

Instead, using (11.199)

$$[E\,a + F\,a^*, E^*a^* + F^*a] = EE^*[a, a^*] + EF^*[a, a + FE^*F[a, a] + FF^*[a^*a]$$

where $[a, a] = [a^*, a^*] = 0$ and $[a^*, a] = -[a, a^*]$. Hence (11.148) follows at once.

Problem 11.18 $\star\star$ [Sect. 11.14] Prove that the rotation operator (11.207) produces the Bogoliubov transformation (11.210).

Solution We have

$$R^*(\phi)\,a\,R(\phi) = \mathrm{e}^{\mathrm{i}\,\phi}\,a \tag{11.9}$$

which correspond to the BCH identity (11.70) with

$$x = -\mathrm{i}\,\phi\,, \qquad K = a^*a,\,, \qquad K = a\,.$$

Then

$$R^*(\phi)\,a\,R(\phi) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}\,\phi)^n}{n!}\,D_n \tag{11.10}$$

where

$$
\begin{aligned}
&D_0 = a\\
&D_1 = [H, D_0] = [a^* a, a] = a^* aa - aa^* a = (a^* a - aa^*) a = [a^*, a] a = -a\\
&D_2 = [a, D_1] = [a^* a, -a] = a
\end{aligned}
$$

and in general $D_n = (-1)^n a$. Then from (11.10) we obtain (11.9).

Problem 11.19 ⋆⋆ [Sect. 11.14] Prove that a squeezing followed by a displacement is equivalent to a displacement followed by a squeezing with the change of the displacement amount indicated in (11.176).

Solution The cascade of $Z(z)$ with $z = r\mathrm{e}^{\mathrm{i}\theta}$ followed by $D(\alpha)$ gives the Bogoliubov transformation

$$Z^*(z)\, D^*(\alpha)\, a\, D(\alpha)\, Z(z) = \cosh r\, a + \sinh r\, \mathrm{e}^{\mathrm{i}\,\theta}\, a^* + \alpha\ .$$

The cascade of $D(\beta)$ followed by $Z(z)$ gives

$$
\begin{aligned}
D^*(\beta)\, Z^*(r)\, a\, Z^*(r)\, D(\beta) =& D^*(\beta)\, \left[\cosh r\, a + \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\, a^*\right]\, D(\beta)\\
=& \cosh r\, D^*(\beta)\, a\, D(\beta) + \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\, D^*(\beta)\, a^*\, D(\beta)\\
=& \cosh r (a + \beta) + \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\, (a^* + \beta^*)
\end{aligned}
$$

where the displacement amount is

$$\alpha = \beta\, \cosh r + \beta^*\, \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\ .$$

Solving with respect to β one gets

$$\beta = \alpha\, \cosh r - \alpha^*\, \sinh r\, \mathrm{e}^{\mathrm{i}\theta}\ .$$

Extra Problem (not introduced in the book)

Problem 11.19E ⋆⋆⋆ [Sect. 11.14] Starting from Yuen's formula (11.229), prove that the probability distribution $p_n(i)$ of squeezed–displaced states, for α real and $\theta = 0$, can be written in the form

$$p_n(i) = \frac{\operatorname{sech} r}{i!} \tanh^i r \, H_i^2 \left(\frac{\alpha(1+\tanh r)}{\sqrt{2\tanh r}} \right) \exp\left(-\frac{1}{2}\alpha^2(1+\tanh r) \right) .$$

Solution The above formula was obtained by Kim et al.[5]. They started from Yuen's formula for squeezed-displaced states, that is,

$$|-z,\alpha\rangle_n = \frac{1}{\sqrt{\mu\, n!}} \left(\frac{\nu}{2\mu} \right)^{n/2} H_n \left(\frac{\beta}{\sqrt{2\mu\nu}} \right) \exp\left(-\frac{1}{2}|\beta|^2 + \frac{\nu^*}{2\mu}\beta^2 \right) \qquad (11.11)$$

where

$$\mu = \cosh r\,, \qquad \nu = \sinh r \exp(\mathrm{i}\,\theta)\,, \qquad \beta = \cosh r\,\alpha + \sinh r\,\alpha^* \exp(\mathrm{i}\,\theta)\,.$$

We now elaborate the argument of the Hermite polynomial and of the exponent to meet Kim's formula. First we proceed in general and then we introduce the specific case: α real and $\theta = 0$.

- Elaboration of the argument of the Hermite polynomial.

The argument is

$$\begin{aligned} \arg =& \frac{\beta}{\sqrt{2\mu\nu}} = \frac{\cosh r\,\alpha + \sinh r\,\alpha^* \exp(\mathrm{i}\,\theta)}{\sqrt{2\cosh r \sinh r \exp(\mathrm{i}\,\theta)}} \\ =& \frac{\alpha + \tanh r\alpha^* \exp(\mathrm{i}\theta)}{\sqrt{2\tanh r \exp(\imath\theta)}}\,. \end{aligned}$$

If α is real and $\theta = 0$, we get

$$\arg = \frac{\alpha[1+\tanh r]}{\sqrt{2\tanh r}}\,.$$

Leaving μ and ν unspecified ($\nu > 0$) and $\beta = \mu\,\alpha + \nu\,\alpha^*$, the exponent is

[5] M. S. Kim, F. A. M. de Oliveira, and P. L. Knight, "Properties of squeezed number states and squeezed thermal states," *Phys. Rev. A*, vol. 40, pp. 2494–2503, Sep. 1989.

$$\begin{aligned}
e =&\frac{1}{2}|\beta|^2+\frac{\nu^*}{2\mu}\beta^2\\
=&\frac{1}{2}\left\{|\mu\alpha+\nu\alpha^*|^2-\frac{\nu^*}{2\mu}[\mu\alpha+\nu\alpha^*]^2\right\}\\
=&\frac{1}{2}\mu^2\left\{\left|\alpha+\frac{\nu}{\mu}\alpha^*\right|^2-\frac{\nu^*}{\mu}\left[\alpha+\frac{\nu}{\mu}\alpha^*\right]^2\right\}\\
=&\frac{1}{2}\mu^2\left\{|\alpha+T\alpha^*|^2-T^*[\alpha+T\alpha^*]^2\right\}
\end{aligned}$$

where

$$T=\frac{\nu}{\mu}=\tanh r\exp(\mathrm{i}\theta)\ .$$

This in general. When α is real and $\theta=0$

$$\begin{aligned}
e =&-\frac{1}{2}\mu^2\left\{|\alpha+T\alpha^*|^2-T^*[\alpha+T\alpha^*]^2\right\}\\
=&-\frac{1}{2}\mu^2\alpha^2\left[(1+T)^2-T(1+T)^2\right]\\
=&-\frac{1}{2}\mu^2\alpha^2\left[(1+T)^2(1-T)\right]\\
=&-\frac{1}{2}\mu^2\alpha^2(1+T)(1-T^2)\ .
\end{aligned}$$

Note that

$$\mu^2(1-T^2)=\cosh^2(r)(1-\tanh r)=\cosh^2(r)-\sinh^2(r)=1\ .$$

Hence

$$e=\frac{1}{2}\alpha^2(1+\tanh r)$$

in agreement with the result of Kim et al.

Problem 11.20 ⋆⋆ [Sect. 11.15] Prove that in a cascade of three symplectic transformations $\tilde{X}_i=S_iX_i+d_i$, $i=1,2,3$, the covariance matrix at the output is given by

$$V_{123}=S_3S_2S_1V_0S_1^{\mathrm{T}}S_2^{\mathrm{T}}S_3^{\mathrm{T}}$$

where V_0 is the covariance matrix at the input.

Solution The cascade is shown in Fig.11.1. By applying iteratively relations (11.154) of Theorem 11.1 one gets the relations for the mean vectors

$$\overline{X}_{123}=S_3S_2S_1\overline{X}_0+S_3S_2d_1+S_3d_2+d_3 \tag{11.12}$$

where $\overline{X}_0$ is the mean vector at the input. Analogously, for the covariance matrices we find

$$V_{123}=S_3S_2S_1V_0S_1^{\mathrm{T}}S_2^{\mathrm{T}}S_3^{\mathrm{T}}\ . \tag{11.13}$$

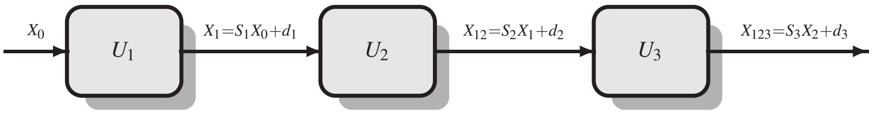

Fig. 11.1 Cascade application of Gaussian unitaries and corresponding relation of symplectic parameters.

Problem 11.21 ⋆⋆⋆ [Sect. 11.15] Prove that the covariance matrix of a squeezed–displaced–rotated state $|z,\alpha,\phi\rangle$ is given by

$$V_{\text{sq,rot}}(z,\phi) = \begin{bmatrix} \cosh(2r)+\cos(2\phi+\theta)\sinh(2r) & \sin(2\phi+\theta)\sinh(2r) \\ \sin(2\phi+\theta)\sinh(2r) & \cosh(2r)-\cos(2\phi+\theta)\sinh(2r) \end{bmatrix}. \tag{11.14}$$

Solution We apply the statement of Problem 11.20 with: 1=squeezing, 2=displacement, 3=rotation, starting form the vacuum state, which have $V_0 = I_2$. Also, $S_1 = S_{\text{sq}}(z)$, $S_2 = I_2$, $S_3 = S_{\text{rot}}(\phi)$. Then

$$V_{\text{sq,disp,rot}}(z,\phi) = S_{\text{rot}} S_{\text{sq}} S_{\text{sq}}^{\text{T}} S_{\text{rot}}^{\text{T}}$$

where (see (11.170) and (11.172))

$$S_{\text{rot}}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

and

$$S_{\text{sq}} = \begin{bmatrix} \cosh(r)+\cos(\theta)\sinh(r) & \sin(\theta)\sinh(r) \\ \sin(\theta)\sinh(r) & \cosh(r)-\cos(\theta)\sinh(r) \end{bmatrix}$$

Substitution of the expressions gives (11.233).

Problem 11.22 ⋆ [Sect. 11.16] Prove that the covariance matrix of the single–mode Gaussian state $|z,\alpha\rangle$ is given by (11.236).

Solution The covariance matrix is given by $V = S_{\text{sq}}(z) S_{\text{sq}}^{\text{T}}(z)$, where the symplectic matrix is given by (11.172)

$$S_{\text{sq}}(r\,\mathrm{e}^{\mathrm{i}\theta}) = \begin{bmatrix} \cosh r + \cos\theta \sinh r & \sin\theta \sinh r \\ \sin\theta \sinh r & \cosh r - \cos\theta \sinh r \end{bmatrix}. \tag{11.15}$$

Hence

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix} = \begin{bmatrix} \cosh 2r + \cos\theta \sinh 2r & \sin\theta \sinh 2r \\ \sin\theta \sinh 2r & \cosh 2r - \cos\theta \sinh 2r \end{bmatrix}$$

Note that this result is a special case of the covariance matrix of an N–mode Gaussian state, given by (11.185), where in the single mode r and θ are scalars and $Y_{ij} = V_{ij}$.

Problem 11.23 $\star\star$ [Sect. 11.16] Consider the Wigner function $W(x,y)$ of a general Gaussian state given by (11.237) and introduce the change of coordinate (see the left of Fig.11.13)

$$x = u\cos\tfrac{1}{2}\theta - v\sin\tfrac{1}{2}\theta\,, \qquad y = u\sin\tfrac{1}{2}\theta + v\cos\tfrac{1}{2}\theta$$

which provides a rotation of the angle $\frac{1}{2}\theta$. Prove that the new Wigner function $\tilde{W}(u,v)$ is obtained with the covariance matrix

$$V = \begin{bmatrix} \mathrm{e}^{2r} & 0 \\ 0 & \mathrm{e}^{-2r} \end{bmatrix}\,.$$

In words, the rotation of $\frac{1}{2}\theta$ removes the squeeze phase in $z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$.

Solution(CS126) It is convenient to write

$$W(x,y) = \frac{1}{2\pi}\exp\left[-\frac{1}{2}E\right]$$

where

$$\begin{aligned} E =& -2\cos\theta\cosh r\sinh(r)y^2 + \left[x^2+y^2\right]\cosh^2(r) + \left[x^2+y^2\right]\sinh^2(r) \\ &+ x(x\cos\theta + 2y\sin(\theta))\sinh(2r)\,. \end{aligned}$$

After the introduction of the rotation we find

$$\begin{aligned} E =& \left[\left[v\cos\tfrac{1}{2}\theta + u\sin\tfrac{1}{2}\theta\right]^2 + \left[u\cos\tfrac{1}{2}\theta - v\sin\tfrac{1}{2}\theta\right]^2\right]\cosh^2 r \\ &- 2\cos\theta\left[v\cos\tfrac{1}{2}\theta + u\sin\tfrac{1}{2}\theta\right]^2\sinh(r)\cosh r \\ &+ \left[\left[v\cos\tfrac{1}{2}\theta + u\sin\tfrac{1}{2}\theta\right]^2 + \left[u\cos\tfrac{1}{2}\theta - v\sin\tfrac{1}{2}\theta\right]^2\right]\sinh^2(r) \\ &+ \left[u\cos\tfrac{1}{2}\theta - v\sin\tfrac{1}{2}\theta\right]\cos\theta\left[u\cos\tfrac{1}{2}\theta - v\sin\tfrac{1}{2}\theta\right] \\ &+ 2\left[v\cos\tfrac{1}{2}\theta + u\sin\tfrac{1}{2}\theta\right]\sin(\theta)\sinh 2r \end{aligned}$$

which can be simplified as

$$E = \left[u^2+v^2\right]\cosh 2r + \left[u^2-v^2\right]\sinh 2r = \mathrm{e}^{2r}u^2 + \mathrm{e}^{-2r}v^2\,.$$

We have obtained the above simplifications with `Mathematica`. ces

Problem 11.24 $\star\star$ [Sect. 11.17] Prove that the symplectic transformation of the Gaussian unitary (11.243) for $\theta = 0$ is given by

$$S_{\mathrm{sq}}(z_0) = \begin{bmatrix} \cosh r_0\, I_2 & \sinh r_0\, Y_2 \\ \sinh r_0\, Y_2 & \cosh r_0\, I_2 \end{bmatrix}, \qquad Y_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\,. \qquad (11.16)$$

Solution For the squeeze operator (11.242) the Bogoliubov transformation is (see (11.73) and (11.74))

$$\begin{aligned}\tilde{a} = Z_2^*(z_0)\,a\,Z_2(z_0) &= \cosh\begin{bmatrix}0 & r_0\\ r_0 & 0\end{bmatrix} a + \mathrm{e}^{\mathrm{i}\theta}\sinh\begin{bmatrix}0 & r_0\\ r_0 & 0\end{bmatrix} a_*\\ &= \begin{bmatrix}\cosh r_0 & 0\\ 0 & \cosh r_0\end{bmatrix} a + \mathrm{e}^{\mathrm{i}\theta}\begin{bmatrix}0 & \sinh r_0\\ \sinh r_0 & 0\end{bmatrix} a_*\end{aligned}$$

that is,

$$\tilde{a} = \cosh r_0\, I_2\, a + \sinh r_0\, W_2\, a_*\,, \qquad W_2 := \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\,. \tag{11.17}$$

The symplectic matrix is obtained using Proposition 11.11 and reads

$$S_{\mathrm{sq}}(z_0) = \begin{bmatrix}\cosh r_0\, I_2 & \sinh r_0\, Y_2\\ \sinh r_0\, Y_2 & \cosh r_0\, I_2\end{bmatrix}, \qquad Y_2 := \begin{bmatrix}1 & 0\\ -1 & 0\end{bmatrix}\,.$$

Problem 11.25 ⋆⋆ [Sect. 11.17] Prove that the covariance matrix of the state (11.244) is given by (11.245).

Solution We use the matrices

$$W_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \qquad Y_2 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

and, to avoid confusion between scalars and matrices, we have written the squeeze matrix z as

$$z = \begin{bmatrix}0 & z_0\\ z_0 & 0\end{bmatrix} = z_0\, W_2 \quad\rightarrow\quad Z_2(z_0) = \mathrm{e}^{\frac{1}{2}\,(z_0\, a_1^* a_2^* - z_0^* a_1 a_2)}$$

with $z_0 = r_0\,\mathrm{e}^{\mathrm{i}\theta_0} \in \mathbb{C}$. Now we write z in the standard form $z = r\mathrm{e}^{\mathrm{i}\theta}$, where z is symmetric and r and θ Hermitian. We have

$$r = \begin{bmatrix}0 & r_0\\ r_0 & 0\end{bmatrix} = r_0 W_2\,, \qquad \theta = \theta_0 I_2 \rightarrow \mathrm{e}^{\mathrm{i}\theta} = \mathrm{e}^{\mathrm{i}\theta_0} I_2\,.$$

Now we can apply the general formula (11.185) giving the covariance matrix Y in the N–mode. Considering that $r^{\mathrm{T}} = r$ and

$$\cosh 2r = \cosh 2r_0\, I_2\,, \quad \sinh 2r = \sinh r_0 W_2\,, \quad \cos\theta = \cos\theta_0 I_2\,, \quad \sin\theta = \sin\theta_0 I_2$$

we find

$$Y_{qq} = \cosh 2r + \sinh 2r\ \cos\theta = \cosh 2r_0\, I_2 + \cos\theta_0 \sinh 2r_0 W_2$$
$$V_{pp} = \cosh 2r - \sinh 2r \cos\theta = \cosh 2r_0\, I_2 - \cos\theta_0 \sinh 2r_0 W_2$$
$$Y_{qp} = \sinh 2r\ \sin\theta = \sin\theta_0 \sinh 2r_0 W_2$$

and explicitly

$$Y = \begin{bmatrix} \cosh 2r_0 & \cos\theta_0 \sinh 2r_0 & 0 & \sin\theta_0 \sinh 2r_0 \\ \cos\theta_0 \sinh 2r_0 & \cosh 2r_0 & \sin\theta_0 \sinh 2r_0 & 0 \\ 0 & \sin\theta_0 \sinh 2r_0 & \cosh 2r_0 & -\cos\theta_0 \sinh 2r_0 \\ \sin\theta_0 \sinh 2r_0 & 0 & -\cos\theta_0 \sinh 2r_0 & \cosh 2r_0 \end{bmatrix}.$$

Then with

$$\Pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we get

$$V = \Pi\, Y\, \Pi^{\mathrm{T}} = \begin{bmatrix} \cosh 2r_0 & 0 & \cos\theta_0 \sinh 2r_0 & \sin\theta_0 \sinh 2r_0 \\ 0 & \cosh 2r_0 & \sin\theta_0 \sinh 2r_0 & -\cos\theta_0 \sinh 2r_0 \\ \cos\theta_0 \sinh 2r_0 & \sin\theta_0 \sinh 2r_0 & \cosh 2r_0 & 0 \\ \sin\theta_0 \sinh 2r_0 & -\cos\theta_0 \sinh 2r_0 & 0 & \cosh 2r_0 \end{bmatrix}$$

which can be written in the compact form

$$V = \begin{bmatrix} \cosh 2r_0 I_2 & \cos\theta_0 \sinh 2r_0 Y_2 + \sin\theta_0 \sinh 2r_0 W_2 \\ \cos\theta_0 \sinh 2r_0 Y_2 + \sin\theta_0 \sinh 2r_0 W_2 & \cosh 2r_0 I_2 \end{bmatrix}.$$

In particular, for $\theta_0 = 0$ we find

$$V = \begin{bmatrix} \cosh 2r_0 I_2 & \sinh 2r_0 Y_2 \\ \sinh 2r_0 Y_2 & \cosh 2r_0 I_2 \end{bmatrix}.$$

We now check the identity $Y_{qq}Y_{pp} = Y_{qp}^2 + I_2$. We find

$$Y_{qq}Y_{pp} = \cosh 2r_0^2\, I_2 - \cos^2\theta \sinh^2 2r_0 W_2^2$$

where $W_2^2 = I_2$. On the other hand

$$Y_{qp}^2 = \sin^2\theta_0 \sinh^2 2r_0 W_2^2.$$

Hence

$$Y_{qq}Y_{pp} - Y_{qp}^2 = (\cosh^2 2r_0 - \sinh^2 2r_0) I_2 = I_2\,.$$

The condition of minimum uncertainty, $Y_{qq}Y_{pp} = I_2$, is verified for $\theta_0 = 0$.

Problem 11.26 $\star\star\star$ [Sect. 11.17] Develop the Fock expansion of the general two–mode Gaussian state (11.254), considering that the exponential e^L has the structure (11.258).

Solution We begin with the non degenerate case.f

The case $u_{12} \neq 0\,,\ v_1, v_2 \neq 0$

The exponent L is explicitly

$$L = u_1 a_1 + u_2 a_2 + v_1 a_1^2 + v_2 a_2^2 + u_{12} a_1 a_2 \tag{11.18}$$

and the exponential reads

$$B(x,a) = e^{u_1 a_1 + u_2 a_2 + v_1 a_1^2 + v_2 a_2^2 + u_{12} a_1 a_2} \tag{11.19}$$

The direct expansion of the exponential gives

$$B(\alpha, a) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(u_1 a_1 + u_2 a_2 + v_1 [a_1]^2 + v_2 [a_2]^2 + + u_{12} a_1 a_2)^{\ell}$$

where

$$\begin{aligned}&\frac{1}{\ell!}(u_1 a_1 + u_2 a_2 + v_1 [a_1]^2 + v_2 [a_2]^2 + u_{12} a_1 a_2)^{\ell}\\ &= \sum_{i,j,k \in \mathcal{C}_\ell} \frac{1}{i_1! i_2! j_1! j_2! k!} u_1^{i_1} u_2^{i_2} v_1^{j_1} v_2^{j_2} u_{12}^{k} (a_1)^{i_1 + 2j_1 + k} (a_2)^{i_2 + 2j_2 + k}\end{aligned}$$

with the summation over all naturals i, j, k such that

$$\mathcal{C}_\ell \ :\ i_1 + i_2 + j_1 + j_2 + k = \ell\,. \tag{11.20}$$

To get the coefficients we let

$$i_1 + 2j_1 + k = n_1\,, \qquad i_2 + 2j_2 + k = n_2 \tag{11.21}$$

to obtain

$$\begin{aligned}b(n_1, n_2) = \sum_{j,k} &\frac{1}{(n_1 - (2j_1 + k))!(n_2 - (2j_2 + k)) j_1! j_2! k!}\\ &u_1^{n_1 - (2j_1 + k)} u_2^{n_2 - (2j_2 + k)} v_1^{j_1} v_2^{j_2} u_{12}^{k}\end{aligned} \tag{11.22}$$

Considering (11.21), we find that the range of summations is explicitly given by

$$\begin{aligned}\mathcal{C}(n_1, n_2) =& \{(k, j_1, j_2) | 0 \le k \le \max(n_1, n_2), 0 \le 2j_1 + k \le n_1, 0 \le 2j_2 + k \le n_2\}\\ =& \{(k, j_1, j_2) | 0 \le k \le \max(n_1, n_2), 0 \le j_1 \le (n_1 - k)/2, 0 \le j_2 \le (n_2 - k)/2\}\,.\end{aligned}$$

Now (11.22) can be rewritten in the form

$$b(n_1,n_2) = \sum_{k=0}^{\min(n_1,n_2)} \frac{1}{k!} u_{12}^k \sum_{j_1=0}^{\lfloor (n_1-k)/2 \rfloor} \frac{1}{(n_1-2j_1-k)j_1!} u_1^{n_1-2j_1-k} v_1^{j_1} \sum_{j_2=0}^{\lfloor (n_2-k)/2 \rfloor} \frac{1}{(n_2-2j_2-k)j_2!} u_2^{n_2-2j_2-k} v_2^{j_2}$$

Next we introduce the Hermite–like polynomial (11.231) (called Hermite 2VHKdFP in the literature), specifically

$$\mathcal{H}_m(x,y) := \sum_{j=0}^{\lfloor m/2 \rfloor} \frac{1}{(m-2j)!j!} x^{m-2j} y^j \; . \tag{11.23}$$

to get

$$\boxed{b(n_1,n_2) = \sum_{k=0}^{\min(n_1,n_2)} \frac{1}{k!} u_{12}^k \, \mathcal{H}_{n_1-k}(u_1,v_1) \, \mathcal{H}_{n_2-k}(u_2,v_2) \; .} \tag{11.24}$$

In conclusion:

Proposition 11.1. *For $N=2$ the expansion of the exponential*

$$B(x,a) = \mathrm{e}^{x^\mathrm{T} F a - \frac{1}{2} a^\mathrm{T} C a} = \mathrm{e}^{u_1 a_1 + u_2 a_2 + v_1 a_1^2 + v_2 a_2^2 + u_{12} a_1 a_2} \tag{11.25}$$

is given by

$$B(x,a) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} b(n_1,n_2) \, a_1^{n_1} a_2^{n_2} \tag{11.26}$$

where the coefficients $b(n_1,n_2)$ are given by (11.24) and the parameters u_1, u_2, v_1, v_2 and u_{12} are determined by the matrices S and T by the relations (11.259).

The case $u_{12} \neq 0 \; , \; v_1 = v_2 = 0$

This correspond to Caves–Schumaker expansion where the exponent is

$$L = u_1 a_1^* + u_2 a_2^* + u_{12} a_1^* a_2^* \; . \tag{11.27}$$

However, we can obtain the expansion form the general case of Proposition 11.1 considering that the polynomial (11.23) for $x=0$ gives

$$\mathcal{H}_m(x,0) = \frac{1}{m!} x^m$$

so that (11.24) becomes

$$\boxed{\begin{aligned} b(n_1,n_2) &= \sum_{k=0}^{\min(n_1,n_2)} \frac{1}{k!}\, u_{12}^k\, \mathcal{H}_{n_1-k}(u_1,0)\, \mathcal{H}_{n_2-k}(u_2,0) \\ &= \sum_{k=0}^{\min(n_1,n_2)} \frac{1}{k!}\, u_{12}^k\, \frac{1}{(n_1-k)!} u_1^{n_1-k}\, \frac{1}{(n_2-k)!} u_2^{n_2-k}\;. \end{aligned}} \tag{11.28}$$

The case $u_{12}=0$

The exponent (11.258) becomes

$$L = \alpha^{\mathrm{T}} S^{\mathrm{T}} a_* + \tfrac{1}{2} a^* T\, a_* = u_1 a_1^* + u_2 a_2^* + v_1 (a_1^*)^2 + v_2 (a_2^*)^2\;. \tag{11.29}$$

and leads to a separable exponential

$$\mathrm{e}^L = \mathrm{e}^{u_1 a_1^* + u_2 a_2^* + v_1 (a_1^*)^2 + v_2 (a_2^*)^2} = \mathrm{e}^{u_1 a_1^* + v_1 (a_1^*)^2}\, \mathrm{e}^{u_2 a_2^* + v_2 (a_2^*)^2}\;.$$

Then the two–mode state is factored into two single–mode squeezed displaced states, say

$$|z,\alpha\rangle = |\tilde{z}_1, \tilde{\alpha}_1\rangle \otimes |\tilde{z}_2, \tilde{\alpha}_2\rangle\;.$$

Problem 11.27 $\star$ [Sect. 11.18] Prove that the symplectic matrix of the beam splitter is given by (11.263).

Solution Using Proposition 11.11 one gets

$$S_{\mathrm{bs}} = \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0\\ -\sin\beta & \cos\beta & 0 & 0\\ 0 & 0 & \cos\beta & \sin\beta\\ 0 & 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0\\ 0 & \cos\beta & 0 & \sin\beta\\ -\sin\beta & 0 & \cos\beta & 0\\ 0 & -\sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} \cos\beta\, I_2 & \sin\beta\, I_2\\ -\sin\beta\, I_2 & \cos\beta\, I_2 \end{bmatrix}\;.$$

Problem 11.28 $\star\star\star$ Consider the beam splitter with a Caves-Schumacher state at the input. Prove that the average numbers of photons in the two modes are given by (11.265) at the input and by (11.266) at the output.

Solution We take the opportunity to develop a general theory of photon counting, not considered in the book. Finally, we give the solution to the problem.

Average photon numbers in an N–mode Gaussian state

We consider a general Bogoliubov transformation in the N–mode

$$b = E\,a + F\,a_{*} + y\,, \qquad b^{*} = a^{*}\,E^{*} + a\,F^{*} + y^{*}\,. \tag{11.30}$$

whose scalar forms are

$$b_i = \sum_r [E_{ir}a_r + F_{ir}\,a_r^{*} + y_i]\,, \qquad b_i^{*} = \sum_s [E_{is}^{*}a_s^{*} + F_{is}^{*}a_s + y_i^{*}]\,.$$

Assuming that the input is the N–mode ground state $|0_N\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_N$, the average number of photons in the i–th mode is given by

$$\begin{aligned}\overline{n}_i = {}_i\langle 0|b_i^{*}b_i|0\rangle_i = \sum_r\sum_s [&E_{is}^{*}E_{ir}\,{}_i\langle 0|a_s^{*}a_r|0\rangle_i + F_{is}^{*}F_{ir}\,{}_i\langle 0|a_s a_r^{*}|0\rangle_i \\ &+ E_{is}^{*}F_{ir}\,{}_i\langle 0|a_s^{*}a_r^{*}|0\rangle_i + F_{is}^{*}E_{ir}\,{}_i\langle 0|a_s a_r|0\rangle_i \\ &+ y_i(E_{is}^{*}\;{}_i\langle 0|a_s^{*}|0\rangle_i + F_{is}^{*}\;{}_i\langle 0|a_s|0\rangle_i) \\ &+ y_i^{*}(E_{ir}\;{}_i\langle 0|a_r|0\rangle_i + F_{ir i}\langle 0|a_r|0\rangle_i) + |y_i|^2]\,.\end{aligned}$$

Now most product are zero, except ${}_i\langle 0|a_s a_r^{*}|0\rangle_i$ for $r = s$, that is,

$${}_i\langle 0|a_r a_r^{*}|0\rangle_i = {}_i\langle 0|[a_r^{*}a_r + 1]|0\rangle_i = 1\,.$$

Thus we have

$$\boxed{\overline{n}_i = \sum_{r=1}^{N} |F_{ir}|^2 + |y_i|^2\,.} \tag{11.31}$$

In the N mode the general state Gaussian $|\alpha, z\rangle$, with $z = r\mathrm{e}^{\mathrm{i}\theta}$, is obtained from the Bogoliubov transformation, where we have

$$E = \cosh r\,, \qquad F = \sinh r\mathrm{e}^{\mathrm{i}\theta}\,, \qquad y = \alpha$$

and (11.31) requires to evaluate the entries

$$F_{ir} = \Big[\sinh r\mathrm{e}^{\mathrm{i}\theta}\Big]_{ir}\,, \qquad i, r = 1, \ldots, N\,.$$

Now, given the complex symmetric matrix z, we have to find its polar form $z = r\mathrm{e}^{\mathrm{i}\theta}$ and the EID of of r to evaluate $\sinh r$.

In particular in the two–mode the average number of photons in the two modes results in

$$\overline{n}_1 = |F_{11}|^2 + |F_{12}|^2 + |\alpha_1|^2\,, \qquad \overline{n}_2 = |F_{21}|^2 + |F_{22}|^2 + |\alpha_2|^2\,. \tag{11.32}$$

Note that (11.32) can be written in the matrix form

$$\begin{bmatrix} \overline{n}_1 \\ \overline{n}_2 \end{bmatrix} = \mathrm{vetD}[FF^* + \alpha\alpha^*] \tag{11.33}$$

where vetD$[A]$ is the column vector obtained from the diagonal elements of the matrix A.

Polar decomposition of the output squeeze matrix

We find the relation linking the polar decomposition $z' = r'\mathrm{e}^{\mathrm{i}\theta'}$ at the output to the polar decomposition $z = r\mathrm{e}^{\mathrm{i}\theta}$ at the input. For the two matrices we have found the relation

$$z' = u\,z\,u^{\mathrm{T}}$$

where u is a unitary matrix.

Proposition 11.2. *The relation $z' = u\,z\,u^{\mathrm{T}}$ for the the squeeze matrices, gives the following relations for the factors of the polar decomposition*

$$\boxed{r' = u\,r\,u^*\,, \qquad \mathrm{e}^{\mathrm{i}\theta'} = u\,\mathrm{e}^{\mathrm{i}\theta}\,u^{\mathrm{T}}\,.} \tag{11.34}$$

Proof. We follow the procedure outlined in a previous section. We evaluate the EID of $z_2' := z'\,z'^*$ using the EID $z_2 = V\,\Lambda^2\,V^*$. We find

$$z_2' = u\,z\,u^{\mathrm{T}}(u^{\mathrm{T}})^* z^*\,u^* = uz_2u^* = u V\,\Lambda^2\,V^*\,u^*$$

so that the EID of z_2' has uV as unitary matrix. Hence

$$r' = \sqrt{z_2'} == u V\,\Lambda\,V^*\,u^* = u\,r\,u^*\,. \tag{11.35}$$

Finally, if $r > 0$, we find $(r')^{-1} = u\,r^{-1}\,u^*$ and

$$\mathrm{e}^{\mathrm{i}\theta'} = (r')^{-1}\,z' = u\,r^{-1}\,u^* u\,z\,u^{\mathrm{T}} = u\,r^{-1}\,u^* u\,z\,u^{\mathrm{T}} = u\,r^{-1}\,z\,u^{\mathrm{T}} = u\,\mathrm{e}^{\mathrm{i}\theta}\,u^{\mathrm{T}}\,.$$

□

In the proof we have seen the following EIDs

$$\begin{aligned} &r = V\,\Lambda\,V^*\,, &&z_2 = V\,\Lambda^2\,V^* \\ &r' = uV\,\Lambda\,V^* u^*\,, &&z_2' = uV\,\Lambda^2\,V^* V^*\,. \end{aligned}$$

Corollary 11.1. *The matrices r and r' have the same eigenvalues. If $f(r) = V\,f(\Lambda)\,V^*$ is an arbitrary function of r, the corresponding function of r' is given by*

$$f(r') = u\,f(r)\,u^*\,. \tag{11.36}$$

In particular we have the relations

$$\begin{aligned} S &:= \operatorname{sech} r && \to && S' := \operatorname{sech} r' = u\,S\,u^* \\ T &:= \tanh r\, e^{i\theta} && \to && T' := \tanh r'\, e^{i\theta'} = u\,T\,u^T\ . \end{aligned} \tag{11.37}$$

Average number of photons in a beam splitter

We have seen that in a Gaussian state the average number of photons in the two modes are given (11.33), where the matrix F is

$$F = \sinh r e^{i\theta} := T\ .$$

Now, if $\overline{n}_1$ and $\overline{n}_2$ are the average numbers of photons at the input of the BS, to find the average numbers at the output we can evaluate the output matrix F' using the polar decomposition at the output. In alternative we can use relation(11.37), that is,

$$F' = u\,F\,u^{\mathrm{T}}$$

In conclusion, we find that the average number of photons at the output are given by

$$\begin{bmatrix} \overline{n}'_1 \\ \overline{n}'_2 \end{bmatrix} = \mathrm{vetA}[u\,F\,F^* u^* + u\,\alpha\alpha^* u^*] \tag{11.38}$$

The explicit results are (with separation of the contribution of squeezing n and of displacement m)

- Means at input

$$\overline{n}_1 = F_{11}F_{11}^* + F_{12}F_{12}^*\,, \qquad \overline{n}_2 = F_{21}F_{21}^* + F_{22}F_{22}^* \tag{11.39}$$

$$\overline{n}_1 + \overline{n}_2 = |F_{11}|^2 + |F_{12}|^2 + |F_{21}|^2 + |F_{22}|^2$$

$$\overline{m}_1 = |\alpha_1|^2\,, \qquad \overline{m}_2 = |\alpha_2|^2\,, \qquad \overline{m}_1 + \overline{m}_2 = |\alpha_1|^2 + |\alpha_2|^2$$

- Means at output

$$\begin{aligned} \overline{n}'_1 =& \sin^2(\beta)\left[|F_{21}|^2 + |F_{22}|^2\right] + \cos^2(\beta)\left[|F_{11}|^2 + |F_{12}|^2\right] \\ &+ \sin(\beta)\cos(\beta)\,[F_{11}F_{21}^* + F_{12}F_{22}^* + F_{21}F_{11}^* + F_{22}F_{12}^*] \end{aligned}$$

$$\begin{aligned} \overline{n}'_2 =& \sin^2(\beta)\left[|F_{11}|^2 + |F_{12}|^2\right] + \cos^2(\beta)\left[|F_{21}|^2 + |F_{22}|^2\right] \\ &- \sin(\beta)\cos(\beta)\,[F_{11}F_{21}^* + F_{12}F_{22}^* + F_{21}F_{11}^* + F_{22}F_{12}^*] \end{aligned}$$

$$\overline{n}'_1 + \overline{n}'_2 = |F_{11}|^2 + |F_{12}|^2 + |F_{21}|^2 + |F_{22}|^2$$

The above results can be rewritten as

$$\begin{aligned} \overline{n}'_1 &= \cos^2(\beta)\,\overline{n}_1 + \sin^2(\beta)\,\overline{n}_2 + \Delta n \sin 2\beta \\ \overline{n}'_2 &= \sin^2(\beta)\,\overline{n}_1 + \cos^2(\beta)\,\overline{n}_2 - \Delta n \sin 2\beta \end{aligned} \tag{11.40}$$

where

$$\Delta n = \Re(F_{11}F_{21}^* + F_{22}F_{12}^*) \tag{11.40a}$$

and

$$\begin{aligned} \overline{m}_1' &= \cos^2(\beta)\,\overline{m}_1 + \sin^2(\beta)\,\overline{m}_2 + \Delta m \sin 2\beta \\ \overline{m}_2' &= \sin^2(\beta)\,\overline{m}_1 + \cos^2(\beta)\,\overline{m}_2 - \Delta m \sin 2\beta \end{aligned} \tag{11.41}$$

where

$$\Delta m = \Re(\alpha_1 \alpha_2^*)\;. \tag{11.41a}$$

The global average numbers of both contributions at the output remain unchained.

The relations (11.40) and (11.41) give the solution to the problem.

Problem 11.29 ⋆⋆⋆ [Sect. 11.20] Consider the Fock representation of a pure state in the single mode

$$|\psi(p)\rangle = \sum_{n=0}^{\infty} f_n(p)\,|n\rangle\;.$$

Prove that the application of the rotation operator $R(\phi)$ to $|\psi(p)\rangle$ modifies the Fourier coefficients as

$$f_n(p) \quad \rightarrow \quad \mathrm{e}^{\mathrm{i}\,n\,\phi}\, f_n(p)\;. \tag{11.42}$$

Solution Consider the rotation operator expressed by the number operator $N = a^*\,a$

$$R(\phi) = \mathrm{e}^{\mathrm{i}\,\phi\,N} = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\,\phi)^k}{k!} N^k\;.$$

Starting from the expression $N = \sum_{j=0}^{\infty} j\,|j\rangle\langle j|$, we get $N^k = \sum_{j=0}^{\infty} j^k\,|j\rangle\langle j|$. Then

$$\begin{aligned} R(\phi)|\psi(p)\rangle &= \sum_{k=0}^{\infty} \frac{(\mathrm{i}\,\phi)^k}{k!} N^k \sum_{n=0}^{\infty} f_n(p)\,|n\rangle = \sum_{k=0}^{\infty} \frac{(\mathrm{i}\,\phi)^k}{k!} \sum_{j=0}^{\infty} j^k\,|j\rangle\langle j| \sum_{n=0}^{\infty} f_n(p)\,|n\rangle \\ &\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\mathrm{i}\,\phi)^k}{k!} n^k\, f_n(p)\,|n\rangle = \sum_{n=0}^{\infty} \mathrm{e}^{\mathrm{i}n\phi}\, f_n(p)\,|n\rangle\;. \end{aligned}$$

Problem 11.30 ⋆⋆⋆ [Sect. 11.20] Apply the statement of the previous problem to prove that the class of squeezed–displaced states is closed under rotations.

Solution The solution is cumbersome for the complicated expression of the Fock representation given by (11.229).

After a rotation of ϕ the n–coefficient of a squeezed state becomes

$$\mathrm{e}^{\mathrm{i}\,n\,\phi}|\alpha, z\rangle_n = \mathrm{e}^{\mathrm{i}\,n\,\phi} \frac{1}{\sqrt{\mu\,n!}} \left(\frac{\nu}{2\mu}\right)^{n/2} H_n\left(\frac{\beta}{\sqrt{2\mu\,\nu}}\right) \exp\left(-\frac{1}{2}|\beta|^2 + \frac{\nu^*}{2\mu}\beta^2\right)\;. \tag{11.43}$$

The target should be: find α_ϕ and z_ϕ such that

$$e^{in\phi}|\alpha, z\rangle_n = |\alpha_\phi, z_\phi\rangle_n$$

Now, we try to replace

$$\nu \to \nu_\phi := \nu e^{i2\phi}, \qquad \to \qquad \boxed{\nu_\phi = \nu e^{i2\phi}.}$$

so that

$$(\nu_\phi)^{n/2} = (\nu e^{i2\phi})^{n/2} = (\nu)^{n/2} e^{in\phi}$$

and

$$e^{in\phi}|\alpha, z\rangle_n = \frac{1}{\sqrt{\mu\, n!}} \left(\frac{\nu_\phi}{2\mu}\right)^{n/2} H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) \exp\left(-\frac{1}{2}|\beta|^2 + \frac{\nu^*}{2\mu}\beta^2\right). \quad (11.44)$$

But, how should the other parameters be changed? Considering that

$$\mu = \cosh r, \qquad \nu = \sinh r e^{i\theta}, \qquad \beta = \mu\alpha - \nu\alpha^*.$$

we have

$$\mu_\phi = \cosh(r_\phi), \qquad \nu_\phi = \sinh(r_\phi) e^{i\theta_\phi} = \nu e^{i2\phi}, \qquad \beta_\phi = \mu_\phi \alpha_\phi - \nu_\phi \alpha_\phi^*.$$

Next we try to set μ unchanged

$$\boxed{\mu_\phi = \mu} \qquad \to \quad \cosh(r_\phi) = \cosh r.$$

Then we can write the condition $e^{in\phi}|\alpha, z\rangle = |\alpha_\phi, z_\phi\rangle$ as

$$\begin{aligned} &\frac{1}{\sqrt{\mu\, n!}} \left(\frac{\nu_\phi}{2\mu}\right)^{n/2} H_n\left(\frac{\beta}{\sqrt{2\mu\nu}}\right) \exp\left(-\frac{1}{2}|\beta|^2 + \frac{\nu^*}{2\mu}\beta^2\right) \\ =&\frac{1}{\sqrt{\mu\, n!}} \left(\frac{\nu_\phi}{2\mu}\right)^{n/2} H_n\left(\frac{\beta_\phi}{\sqrt{2\mu\nu_\phi}}\right) \exp\left(-\frac{1}{2}|\beta_\phi|^2 + \frac{\nu_\phi^*}{2\mu}\beta_\phi^2\right) \end{aligned} \quad (11.45)$$

and we get three conditions. The first condition is

$$\frac{\beta}{\sqrt{2\mu\nu}} = \frac{\beta_\phi}{\sqrt{2\mu\nu_\phi}} = \frac{\beta_\phi}{e^{i\phi}\sqrt{2\mu\nu}}$$

that is

$$\beta_\phi = \beta e^{i\phi} \quad \to \quad \mu\alpha_\phi - \nu e^{i2\Phi}\alpha_\phi^* = \mu\alpha e^{i\phi} - \nu\alpha^* e^{i\phi}.$$

This gives

$$\alpha_\phi = \alpha e^{i\Phi}, \qquad \alpha_\phi^* e^{i\Phi} = \alpha^* \quad , \qquad \boxed{\alpha_\phi = \alpha e^{i\Phi}}$$

The second condition is

$$|\beta_\phi|^2 = |\beta|^2 \quad \to \quad |\mu\alpha_\phi - \nu_\phi \alpha_\phi^*|^2 = |\mu\alpha - \nu\alpha^*|^2$$

that is,

$$|\mu\alpha e^{i\phi} - \nu e^{i2\phi}\alpha^* e^{-i\phi}|^2 = |\mu\alpha - \nu\alpha^*|^2$$

which holds! The third condition is

$$\frac{\nu_\phi^*}{2\mu}\beta_\phi^2 = \frac{\nu^*}{2\mu}\beta^2 \quad \to \quad \nu_\phi^*\beta_\phi^2 = \nu^*\beta^2$$

that is

$$\nu^* e^{-i2\phi}(\mu\alpha_\phi - \nu_\phi\alpha_\phi^*)^2 = \nu^*(\mu\alpha - \nu\alpha^*)^2 \to e^{-i2\phi}(\mu\alpha_\phi - \nu\alpha_\phi^*)^2 = (\mu\alpha - \nu\alpha^*)^2$$

$$\nu^* e^{-i2\phi}(\mu\alpha_\phi - \nu_\phi\alpha_\phi^*)^2 = \nu^*(\mu\alpha - \nu\alpha^*)^2 \to e^{-i2\phi}(\mu\alpha e^{i\phi} - \nu e^{I2\Phi}\alpha^* e^{-i\phi})^2 = (\mu\alpha - \nu\alpha^*)^2$$

which holds!

In conclusion, the squeezed state absorbs the rotation with

$$\alpha_\phi = \alpha e^{i\phi}\,, \qquad \mu_\phi = \mu\,, \qquad \nu_\phi = \nu e^{i2\phi}$$

where

$$\begin{aligned} \mu_\phi &= \mu\,, &\to \quad \cosh(r_\phi) &= \cosh r \\ \nu_\phi &= \nu e^{i2\phi} &\to \quad \sinh(r_\phi)e^{i\theta_\phi} &= \sinh r e^{i\theta} e^{i2\phi}\,. \end{aligned} \tag{11.46}$$

Considering that $z = re^{i\theta}$, where r is not the modulus of z (it can be negative), from (11.46) we have

$$r_\phi = \pm r\,, \qquad \pm e^{i\theta_\phi} = e^{\theta + 2\phi}\,.$$

Hence

$$\boxed{\alpha_\phi = \alpha e^{i\phi}\,, \qquad z_\phi = z e^{i2\phi}\,.} \tag{11.47}$$

Problem 11.31 ⋆ [Sect. 11.20] Prove that the class of coherent states is closed with respect to rotations, using the Fock representation (11.191)

Solution The coefficient f_n of the Fock representation of the coherent state $|\alpha\rangle$ is given by

$$f_n = e^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}$$

while the coefficient $f_n(\phi)$ of the state $|\alpha e^{i\phi}\rangle$ is

$$f_n(\phi) = e^{-\frac{1}{2}|\alpha|^2}\frac{(\alpha e^{i\phi})^n}{\sqrt{n!}} = e^{in\phi}\, f_n\,.$$

and the conclusion follows from the statement of the previous problem.

Problems of Chapter 12

Problem 12.1 ⋆⋆ [Sect. 12.3] Consider a pair (A,B) of statistically independent symbols. Prove that, by imposing the condition $i(a,b) = i(a) + i(b)$, the unique function $f[\cdot]$ defining the information $i(a) = f[p(a]$ is the logarithm.

Solution The condition is

$$f[p(a,b)] = f[p(a)] + f[p(b)]$$

where $p(a,b)$ is the joint probability. Considering the statistical independence we have

$$p(a,b) = p(a)\; p(b).$$

Hence

$$f[p(a)\; p(b)] = f[p(a)] + f[p(b)] \tag{12.1}$$

which must hold for every pair of probabilities $p(a)$, $p(b)$. This leads to the functional equation

$$f(xy) = f(x) + f(y)\,, \qquad \forall x,y \in [0,\,1]\,.$$

It can be shown that the unique *continuous* solution of this equation is given by the logarithm

$$f(x) = K \log_c x$$

where both the constant K and the basis c are arbitrary $(c > 0)$. But, by imposing that (see the definition of bit) $f(\frac{1}{2}) = 1$, one gets $K \log_c \frac{1}{2} = 1$ and hence

$$f(x) = -\frac{1}{\log_c 2}\, \log_c x = -\log_2 x$$

where we have used the formula giving the change of base of the logarithmic function

$$\log_c x = (\log_c d) \log_d x.$$

Problem 12.2 ⋆⋆ [Sect. 12.4] (Thermal states) A thermal state may be defined as the bosonic state that maximizes the von Neumann entropy for a given mean number of photons $\mathcal{N}$.[6] It has the following Fock representation (see Section 11.9)

$$\rho_{\text{th}} = \sum_{n=0}^{\infty} \frac{\mathcal{N}^n}{(\mathcal{N}+1)^{n+1}}\, |n\rangle\langle n|\,. \tag{12.2}$$

Find its quantum entropy.

Solution The form (12.2) is already an EID with orthonormal states and eigenvalues

[6] C. Weedbrook, S. Pirandola, R. Garcí a Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, “Gaussian quantum information,” *Rev. Mod. Phys.*, vol. 84, pp. 621–669, May 2012.

$$\lambda_n = \frac{\mathcal{N}^n}{(\mathcal{N}+1)^{n+1}} \ .$$

Then we can apply (12.27) to get

$$S(\rho_{\rm th}) = \sum_{n=0}^{\infty} \frac{\mathcal{N}^n}{(\mathcal{N}+1)^{n+1}} \log \frac{\mathcal{N}^n}{(\mathcal{N}+1)^{n+1}}$$

The summation gives the quantum entropy

$$S(\rho_{\rm th}) = (\mathcal{N}+1)\log(\mathcal{N}+1) - \mathcal{N}\log(\mathcal{N})$$

which is illustrated in Fig. 12.1.

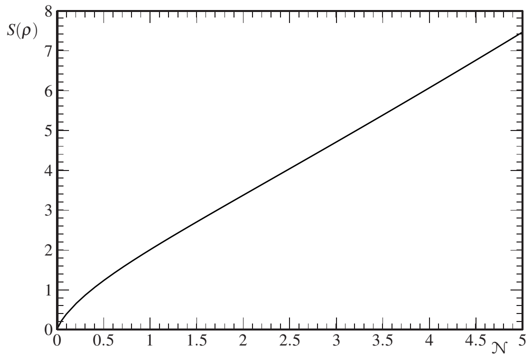

Fig. 12.1 The quantum entropy $S(\rho)$ of the thermal state as a function of the mean number of photons $\mathcal{N}$.

Problem 12.3 $\star$ [Sect. 12.7] Prove that in a binary symmetric channel with cross transition probability ε and equal a priori probabilities (see Fig. 12.15), the equivocation is given by

$$H(A|B) = -(1-\varepsilon)\log(1-\varepsilon) - \varepsilon\log\varepsilon \ .$$

Solution The general formula giving the equivocation is

$$H(A|B) = \mathrm{E}[i(A|B)] = -\sum_a \sum_b p_{AB}(a,b)\log p_{A|B}(a|b) \ .$$

In this case we have

$$p_{AB}(a,b) = \frac{1}{2} p_{A|B}(a|b) \ .$$

Then

$$
\begin{aligned}
H(A|B) &= \sum_{a,b} p_{AB}(a,b)\,\log p_{A|B}(a|b)\\
&= -\frac{1}{2}\left[p_{A|B}(0|0)\,\log p_{A|B}(0|0) + p_{A|B}(1|1)\,\log p_{A|B}(1|1) + \right.\\
&\quad \left. + p_{A|B}(0|1)\,\log p_{A|B}(0|1) + p_{A|B}(1|0)\,\log p_{A|B}(1|0)\right]\\
&= -(1-\varepsilon)\log(1-\varepsilon) - \varepsilon\log\varepsilon\ .
\end{aligned}
$$

Problem 12.4 $\star$ [Sect. 12.7] Prove formula (12.71) giving the mutual information in terms of the a priori probabilities and the transition probabilities.

Solution The general formula giving the mutual information is

$$
\begin{aligned}
I(A\,;B) =& H(A) - H(A|B) = \mathrm{E}[i(A) - i(A|B)]\\
=& \sum_a p_A(a)\log p_A(a) + \sum_a \sum_b p_{AB}(a,b)\log p_{A|B}(a|b)
\end{aligned}
$$

where the probabilities can be expressed in the form

$$
\begin{aligned}
p_{AB}(a,\,b) &= p_A(a)\,p_{B|A}(b|a) = p_A(a)\,p_c(b|a)\\
p_B(b) &= \sum_{a'} p_{AB}(a',\,b') = \sum_{a'} p_A(a')\,p_c(b|a')\ .
\end{aligned}
$$

Then

$$
I(A\,;B) = \sum_{a,b} p_A(a)\,p_c(b|a)\log\frac{p_c(b|a)}{\sum_{a'} p_A(a')\,p_c(b|a')}\ .
$$

Problem 12.5 $\star\star$ [Sect. 12.8] Find the Kraus representation of a depolarizing channel in a qubit system, using identity (12.86).

Solution We can first check that the relation

$$
I_{\mathcal{H}} = I_2 = \tfrac{1}{2}\sum_{i=0,x,y,z} \sigma_i A\,\sigma_i^*)
$$

holds for every Hermitian matrix A, as soon as we use the expression of Pauli's matrices (see (3.91)), given by

$$
\sigma_0 = I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ ,\ \sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\ ,\ \sigma_y = \begin{bmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{bmatrix}\ ,\ \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\ .
$$

Then, using the above relation with $A = \rho$ in

$$
\Phi(\rho) = (1-p)\rho + p\,(1/2)\,I_{\mathcal{H}}\,, \qquad d = 2
$$

we get

$$\begin{aligned}\Phi(\rho) =&(1-p)\rho + \tfrac{1}{4}\,p\tfrac{1}{2}\sum_{i=0,x,y,z} \sigma_i\,\rho\,\sigma_i^* \\ =&(1-p)\rho + \tfrac{1}{4}\,p\rho + \tfrac{1}{4}\,p\sum_{i=x,y,z} \sigma_i\,\rho\,\sigma_i^* \\ =&\sum_{k=0}^{3} V_k\,\rho\,V_k^*\end{aligned}$$

with

$$V_0 = \sqrt{(1-p)+\tfrac{1}{4}\,p}\,I_2\,, \qquad V_k = \tfrac{1}{2}\sqrt{p}\,\sigma_k\,,\ k=1,2,3\,.$$

Problem 12.6 $\star$ [Sect. 12.9] Consider the following ensemble in a qubit system

$$\mathcal{L}: \quad p_0 = \tfrac{1}{2}\,,\ \rho_0 = \begin{bmatrix} 0.8 & 0.25 \\ 0.25 & 0.2 \end{bmatrix}, \qquad p_1 = \tfrac{1}{2}\,,\ \rho_1 = \begin{bmatrix} 0.1 & 0.3 \\ 0.3 & 0.9 \end{bmatrix}.$$

Evaluate the Holevo χ.

Solution The eigenvalues of ρ_0 and of ρ_1 are respectively

$$\{0.890512, 0.109488\}\,, \qquad \{1., 0\}\,.$$

Note that ρ_1 has rank 1 and therefore it corresponds to a pure state. The mean density operator is given by

$$\rho = \tfrac{1}{2}\rho_0 + \tfrac{1}{2}\rho_1 = \begin{bmatrix} 0.45 & 0.275 \\ 0.275 & 0.55 \end{bmatrix}$$

and its eigenvalues are

$$\{0.779508, 0.220492\}\,.$$

From the eigenvalues we find the entropies

$$S(\rho_0) = 0.498368\,, \qquad S(\rho_1) = 0.\,, \qquad S(\rho) = 0.761064\,.$$

Hence

$$\chi = S(\rho) - \tfrac{1}{2}S(\rho_0) - \tfrac{1}{2}S(\rho_1) = 0.51188\,.$$

Problem 12.7 $\star\star$ [Sect. 12.9] With the ensemble $\mathcal{L}$ specified in the previous problem, evaluate the mutual information, assuming that Bob uses the measurement operators provided by the Helstrom theory. Then verify the Holevo bound $I(A,B) \le \chi(\mathcal{L})$.

Solution We have to calculate the mutual information $I(A;B)$,which in the binary case is given by

$$I(A;B) = H(A) - H(A|B)$$

where

$$H(A|B) = \mathrm{E}[\log p_{A|B}(a|b)] = \sum_{a,b} p_{AB}(a,b) \log p_{A|B}(a|b) \,.$$

With equal a priory probabilities the joint probabilities are evaluated from the transition probabilities as $p_{AB}(0,0) = \frac{1}{2}p_c(0|0)$, $p_{AB}(0,1) = \frac{1}{2}p_c(0|1)$, etc. where for the symmetry we have $p_c(1|1) = p_c(0,0) = P_c$ and $p_c(0|1) = p_c(1,0) = P_e$, with P_c the correct decision probability and $P_e - 1 - P_c$. Thus, we have

$$p_{AB}(0,0) = p_{AB}(1,1) = \tfrac{1}{2}P_c \,, \qquad p_{AB}(1,0) = p_{AB}(0,1) = \tfrac{1}{2}P_e \,.$$

For the symmetry we have that also the output probabilities are equal, $p_B(0) = p_B(1) = \frac{1}{2}$. Hence

$$p_{A|B}(0|0) = p_{A|B}(1|1) = P_c \,, \qquad p_{A|B}(1|0) = p_{A|B}(0|1) = P_c$$

and

$$H(A|B) = -P_c \log P_c - P_e \log P_e \,.$$

On the other hand Helstrom's theory gives

$$P_c = \tfrac{1}{2} + \sum_{\eta_k > 0} \eta_k$$

where η_k are the eigenvalues of the operator $D = \frac{1}{2}\rho_1 - \frac{1}{2}\rho_0$. In this case the eigenvalues result in $\{-0.350892, 0.350892\}$ and therefore

$$P_c = 0.850892 \,, \qquad P_e = 0.149108 \,.$$

and

$$H(A|B) = 0.607604 \,, \qquad H(A) = 1 \,, \qquad I(A;B) = 0.39239 \,.$$

On the other hand

$$S(\rho_0) = 0.498368 \,, \qquad S(\rho_1) = 0. \,, \qquad S(\rho) = 0.761064$$

and

$$\chi = S(\rho) - \tfrac{1}{2}S(\rho_0) - \tfrac{1}{2}S(\rho_1) = 0.51188 \,.$$

Thus $I(A;B) < \chi$ and the Holevo bound is verified.

Problem 12.8 $\star\star\star$ [Sect. 12.9] Prove that the Holevo bound holds with the equality sign if 1) the density operators $\{\rho_a\}$ commute, that is, they are *simultaneously diagonalizable*, and 2) the POVM measurement is performed with the common eigenbasis of the $\{\rho_a\}$.

Solution Let

$$\rho_a = U \Lambda_a U^* = \sum_k |u_k\rangle \lambda_k^a \langle u_k|$$

be the simultaneous diagonalization of the ρ_a. Then we implement the measurement with the basis $\{u_b\}, b \in \mathcal{A}$ provided by the common unitary operator U. The measurement with the elementary projectors $Q_b = |u_b\rangle\langle u_b|$ gives (see (3.36))

$$\begin{aligned} p_{B|A}(b|a) :=& \mathrm{P}[m = b|\rho_a] = \langle u_b|\rho_a|u_b\rangle = |\langle u_b|U\Lambda_a U^*|u_b\rangle \\ & \sum_k |u_b\rangle\langle u_k|\lambda_k^a|u_k\rangle\langle u_b| \\ =& \sum_k \delta_{bk}\delta_{bk}\lambda_k^a = \lambda_b^a \ . \end{aligned}$$

Comment. The probabilities $p_{B|A}(b|a) = \lambda_b^a$ are generic with no particular structure. Thus, if this is true, it does not seem possible to prove that the mutual information saturates the Holevo–χ.

On the other hand Datta[7] claims the statement. Also Holevo and Giovannetti[8] claim the statement, citing Helstrom's book.

Problem 12.9 ⋆ [Sect. 12.9] Prove that in a constellation of distinct pure states $\{\rho_a = |\psi_a\rangle\langle\psi_a|\ ,\ a \in \mathcal{A}\}$, the density operators commute if and only if the states are orthogonal.

Solution We have

$$\begin{aligned} \rho_a\rho_b =& |\psi_a\rangle\langle\psi_a|\psi_b\rangle\langle\psi_b| = X_{ab}|\psi_a\rangle\langle\psi_b| \\ \rho_b\rho_a =& |\psi_b\rangle\langle\psi_b|\psi_a\rangle\langle\psi_a| = X_{ab}|\psi_b\rangle\langle\psi_a|\ . \end{aligned}$$

If $X_{ab} = 0$ the density operators commute. Are $|\psi_a\rangle\langle\psi_b|$ and $|\psi_b\rangle\langle\psi_a|$ commutable with $b \neq a$?

Small corrections to be introduced in the Prblems

Correction N.1

Please remove Problem 3.8 at p.103, because it has the same content as Problem 3.7 at p.97., and enumerate the subsequent problems correspondingly.

The text to be removed reads

Problem 3.8 ⋆ Apply Postulate 3 to a quantum system "prepared" in a pure state $|\psi\rangle$, when the measurement is obtained by an set of orthonormal *measurement vectors* $\{|a_0\rangle, |a_1\rangle, \ldots, |a_{M-1}\rangle\}$. Find the probability distribution of the measure m *when the state of the system is one of the measurement vectors*. Which is the state of the system after the measurement?

Correction N.2

In the equation of Problem 5.8 at p. 202 there is an error. Please replace the last part of the eqaution

```
M=[ \ket{\mu_0}\ket{\mu_0}]
```

[7] N. Datta, "Quantum entropy and information," in *Quantum information, computation and cryptography*, ser. Lecture Notes in Physics, F. Benatti, M. Fannes, R. Floreanini, and D. Petritis, Eds. Springer Berlin Heidelberg, 2010, vol. 808, pp. 175–214.

[8] A. S. Holevo and V. Giovannetti, "Quantum channels and their entropic characteristics," *Reports on Progress in Physics*, vol. 75, no. 4, p. 046001, 2012.

with

```
M=[ \ket{\mu_0},\ket{\mu_1}]
```

Then the text should result in

Problem 5.8 ⋆⋆ Write the fundamental relations of the geometrical approach in matrix form, using the matrices

$$\Gamma = [|\gamma_0\rangle, |\gamma_1\rangle]\,, \qquad U = [|u_0\rangle, |u_1\rangle]\,, \qquad M = [|\mu_0\rangle, |\mu_1\rangle]\,.$$

Correction N.3

In the equation of Problem 5.10 at p.209 there is an error. Please replace the last part of the text

```
if $\gamma_1$ is replaced by
 $\gamma_1 \B(Z) $, with $\B(Z)\B(Z)^*=I_h$,
  $\BB(\mu)_1 \B(W) $,
```

with

```
if $\gamma_i$ is replaced by
 $\gamma_i \B(Z) $, with $\B(Z)\B(Z)^*=I_h$, and $\mu_j$ by
  $\mu_j \B(W) $,
```

Then the text should result in

Problem 5.10 ⋆ Consider the transition probabilities given by (5.22). Prove that, if γ_i is replaced by $\gamma_i Z$, with $ZZ^* = I_h$, and μ_j by $\mu_j W$, with $WW^* = I_h$, the transition probabilities do not change.

Corrections on page 196

Present stautus.

$$\vdots \qquad \vdots$$

$$|\eta_0\rangle = a_{00}|\gamma_{00}\rangle + a_{01}|\gamma_{10}\rangle\,, \qquad |\eta_1\rangle = a_{10}|\gamma_{00}\rangle + a_{11}|\gamma_{10}\rangle\,. \tag{5.28}$$

Now, the coefficients a_{ij} are obtained by applying the definition of eigenvector, that is,

$$D|\eta_0\rangle = \eta_0\,|\eta_0\rangle\,, \qquad D|\eta_1\rangle = \eta_1\,|\eta_1\rangle \tag{5.29}$$

where η_0 and η_1 are the eigenvalues. Substituting (?H2?) and (5.28) in (5.29), recalling that $\langle\gamma_{10}|\gamma_{10}\rangle = \langle\gamma_{00}|\gamma_{00}\rangle = 1$ and letting $X = \langle\gamma_{00}|\gamma_{10}\rangle$, we obtain

$$q_1(a_{0i}X + a_{1i})|\gamma_{10}\rangle - q_0(a_{0i} + a_{1i}X^*)|\gamma_{00}\rangle = \eta_{0i}(a_{0i}|\gamma_{00}\rangle + a_{1i}|\gamma_{10}\rangle)\,, \qquad i = 0, 1\,. \tag{5.30}$$

But, because of the assumed *independence*, in (5.30) the coefficients of $|\gamma_{10}\rangle$ and $|\gamma_{00}\rangle$ must be equal

$$q_1(a_{0i}X + a_{1i}) = \eta_i a_{1i}\,, \qquad -q_0(a_{0i} + a_{1i}X^*) = \eta_i a_{0i}\,, \qquad i = 0,1\;. \tag{5.31}$$

Solving with respect to η_i we get the equation

Please

1) **replace all γ_{00} with γ_0 and all γ_{10} with γ_1,**
2) **replace must be equal with must be equal to zeo. Hence**
3) **in Eq.(5.31) replace**

$$q_1(a_{0i}X + a_{1i}) = \eta_i a_{1i}\,, \qquad -q_0(a_{0i} + a_{1i}X^*) = \eta_i a_{0i}\,, \qquad i = 0,1\;.$$

with

$$q_1(a_{i\,0}X^* + a_{i\,1}) = \eta_i\, a_{i\,1}\,, \qquad -q_0(a_{i\,0} + a_{i\,1}X) = \eta_i\, a_{i\,0}\,, \qquad i = 0,1\;.$$

The TeX code of the last (correct) equation is

```
$$
q_1(a_{i\,0}X^*+a_{i\,1})=\eta_{i}\, a_{i\,1}
\vq -q_0(a_{i\,0}+a_{i\,1}X)=
\eta_{i}\, a_{i\,0}\vq i=0,1\;.
\e(H4B)
$$
```

Corrections on page 197

Present stautus.

represents the *(quadratic) superposition degree* between the two states. In the literature expressions (?H20?) are universally known as **Helstrom's bound**.

The optimal projectors derive from (?DD13?) and become

$$Q_0 = |\eta_0\rangle\langle\eta_0|\,, \qquad Q_1 = |\eta_1\rangle\langle\eta_1| \tag{5.34}$$

and therefore they are of the *elementary* type, with **measurement vectors** given by the eigenvectors $|\eta_0\rangle$ and $|\eta_1\rangle$ of the decision operator D.

It remains to complete the computation of these two eigenvectors, identified by the linear combinations (5.28). Considering (5.31) we find

$$|\eta_0\rangle = a_{00}\left(|\gamma_0\rangle - \frac{\eta_0 - q_0}{q_1 X^*}|\gamma_1\rangle\right)\,, \qquad |\eta_1\rangle = a_{11}\left(\frac{\eta_1 - q_1}{q_1 X}|\gamma_0\rangle + |\gamma_1\rangle\right) \tag{5.35}$$

where a_{00} and a_{11} are calculated by imposing the normalization $\langle\eta_i|\eta_i\rangle = 1$. In the general case, the calculation of the eigenvectors is very complicated

Please in Eq. (5.35) replace

$$|\eta_0\rangle = a_{00}\left(|\gamma_0\rangle - \frac{\eta_0 - q_0}{q_1 X^*}|\gamma_1\rangle\right)\,, \qquad |\eta_1\rangle = a_{11}\left(\frac{\eta_1 - q_1}{q_1 X}|\gamma_0\rangle + |\gamma_1\rangle\right)$$

with

$$|\eta_0\rangle = a_{00}\left(|\gamma_0\rangle + \frac{q_1 X^*}{\eta_0 - q_1}|\gamma_1\rangle\right)\,, \qquad |\eta_1\rangle = a_{11}\left(-\frac{q_0 X}{\eta_1 + q_0}|\gamma_0\rangle + |\gamma_1\rangle\right)$$

correzioni

- p. 112 final dot . in Eq. (3.63)
- p. 263 last but one line; remove -i0.0 and two extra plus

$$\Gamma = [\gamma_0, \gamma_1] = \begin{bmatrix} -0.54117 & -0.02018 & -0.47937 & -0.06934 & 0.03124 \\ -0.54117 & -0.02018 & \mathrm{i}0.51339 & 0.0 & \mathrm{i}0.02917 \\ -0.54117 & -0.02018 & 0.47937 - \mathrm{i}0.0 & -0.06934+ & -0.03124+ \\ -0.33238 & 0.09857 & -\mathrm{i}0.51339 & 0.0 & -\mathrm{i}0.02917 \end{bmatrix}.$$

Remove the red part

- p. 415 last symbol of last equation: replace $D\alpha$ with $\mathrm{d}\alpha$
  Tex code `\D\alpha`
- p. 445 first equation: add a final .
- p. 461 three lines before (10.12): replace "system A" with "system A"
- p. 489 last two lines of Eq.(11.80): replace $\mathbb{R}^N$ with $\mathbb{C}^N$ in both lines. Tex code `$\M(C)^N$`

- p. 659 first line after the second equation: replace $B_n \in (\mathbb{R})$ with $B_n \in \mathbb{R}$ that is, remove ( ) aroud $\mathbb{R}$.

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