SDP Example with Nonzero Duallity Gap

This is an example of an SDP with nonzero duallity gap (i.e. strong duallity does not hold). It demonstrates that SDP solvers may return unreliable results for problems where strong duallity does not hold.

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Requirements

This script requires the CVX toolbox: http://cvxr.com/cvx/

Primal Problem

The primal problem is

minimize z_1

subject to

$$\begin{bmatrix} z_1+1 & 0 & 0 \\ 0 & z_2 & z_1 \\ 0 & z_1 & 0 \end{bmatrix} \geq 0$$

The semidefinite constraint holds only if $z_1 = 0$ and $z_2 \geq 0$, therefore the primal optimal value is $p = 0$.

Dual Problem

The dual of the primal problem is

maximize $-y_{11}$

subject to

$y_{11} + 2y_{23} = 1$

$y_{22} = 0$

$Y \geq 0$

The constraints hold only if $y_{23} = 0$ and $y_{11} = 1$, therefore the dual optimal value is $d = -1$.

Problem Data in Standard Form

```
c = [1 0];
B = [-1 0 0; 0 0 0; 0 0 0];
A1 = [1 0 0; 0 0 1; 0 1 0];
A2 = [0 0 0; 0 1 0; 0 0 0];
```

Primal Problem in CVX

The optimal primal value is inaccurate and the optimal primal variable z(1) is slightly infeasible while z(2) is very large

```
cvx_begin sdp quiet
    variable z(2,1)
    minimize c*z
         z(1)*A1 + z(2)*A2 - B >= 0;
cvx_end

% Solver status
cvx_status

% Primal optimal value returned by CVX
cvx_optval
z
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate.
RCOND =  1.703915e-16.

cvx_status =

Solved


cvx_optval =

  -3.0095e-07


z =

  -3.0095e-07
   2.4797e+08
```

Dual Problem in CVX

The optimal dual value is incorrect and the optimal dual variable Y is very poorly conditioned

```
cvx_begin sdp quiet
    variable Y(3,3) semidefinite
    maximize trace(B*Y)
         trace(A1*Y) == c(1);
         trace(A2*Y) == c(2);
```

```
cvx_end

% Solver Status
cvx_status

% Dual optimal value returned by CVX
cvx_optval
Y
```

```
Warning: Matrix is close to singular or badly scaled. Results may be inaccurate.
RCOND =  1.703915e-16.

cvx_status =

Solved


cvx_optval =

  -4.8121e-09


Y =

   4.8121e-09            0            0
            0   6.6705e-16   5.0000e-01
            0   5.0000e-01   3.7869e+14
```

Conclusion

For this problem the solutions and optimal decision variables returned are inaccurate. Furthermore, the decision variables z and Y are very poorly conditioned. The solver tries to simultaneously solve the primal and dual problem, but encounters numerical difficulty since the primal and dual optimal values are not equal.

Attribution

This example supplements the book "Networks of Dissipative Systems: Compositional Certification of Stability, Performance, and Safety" by Murat Arcak, Chris Meissen, and Andrew Packard.

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