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Analytical Mechanics

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To my wife,
for her patience and understanding

Preface

This is a solutions manual for the exercises in the text Analytical Mechanics.

Goshen, Indiana *Carl Helrich*

Contents

1 History

1.1. We claimed that we must begin the history of our subject in ancient Greece. Particularly we went back to the Presocratics. These people we claimed were physicists because they were more interested in how the world functioned than in how it should function. In a brief essay speak to the concepts of modern and classical physics that find roots in these Presocratics.

Solution:

Thales sought a unifying principle and Anaximander made that more explicit in calling this something the boundless. This unification we may claim to find in our concept of energy. Anaximenes made an underlying substance somewhat more tangible by identifying the substance as air. And he spoke to what we now know as phase changes. Xenophanes said that humans are limited to our direct experience, which we may, if we choose, push to the questions of modern quantum theory, as well as the basis of experimental physics. Heraclitus spoke of process and logos, rather than of matter. The universe for Heraclitus was changing, but according to law. We may think of this as a basis of theoretical physics. What is important is the laws that govern the dynamics of matter. And Pythagoras spoke about numbers and of mathematics as the basis of all. Parmenides claimed that we must look beyond the human senses if we seek truth. We know that very well as modern scientists. Empedocles warns us that the human mind is also fallible. As Newton later pointed out, Anaxagoras claimed that the same forces that govern the cosmos govern action on earth. And Democritus claimed that all that existed was atoms and the void. Although we cannot build our atomic theory on Democritus' ideas, the concept is familiar.

1.2. Cicero wrote that Socrates called down philosophy from the skies. With Socrates the truths that the philosophers, the physicists before him, had sought found their importance with people and with human structures on earth. Philosophy became also the study of ethics and of morals. But Socrates wrote nothing. What we know of Socrates comes to us from Plato, his student. What characterized Plato's thought?

Solution:

Plato's primary interest was in educating people to govern Athens. How do you create in real people the character that will place the good of society beyond personal benefit? The reality on which Plato wanted his students to focus was beyond what they saw about them. Reality is in the Forms. What we have in our existence on earth is only an image of reality. Plato wrote in dialogue form, as many great teachers after him have done. This is a venue for learning. We learn best from a story. The stories

Plato presented were not to instruct in matters of truth as absolute. That is beyond us as humans. All we can convey as humans is the likely story. The truth is present, but beyond us.

1.3. Aristotle Has had a greater influence on philosophical and scientific thought than any other individual. He was Plato's student. But he disagreed with his master about the structure of reality. How would you characterize Aristotle's approach to understanding the world?

Solution:

Aristotle accepted what we observe around us as reality. This differed from Plato's position. Aristotle accepted causality is fundamental in the world we observe, although his concept of causality was more complex than that of the modern scientist. He demanded a teleology and considered that causality was multi-layered. His logic was syllogistic, which differs from the logic pursued in modern physics. Syllogism is not, however, wholly absent from modern thought. The greatest difficulty seems to be in Aristotle's metaphysics, which resulted in his axioms. His logic, based on those axioms, was not problematic. His axioms admitted of a separation of motion in the cosmos from that on earth and found its way into aspects of motion. Notorious is his claim that a vacuum cannot exist because it would result in motion at an infinite velocity, since motion of matter was inversely proportional to the density of the fluid through which the matter was moving.

1.4. Greek was the language of the Eastern Roman Empire, which did not so quickly suffer the same deterioration that eventually destroyed the Western Roman Empire in 476 CE (Common Era). The academy in Athens was closed by the Eastern Roman emperor Justinian I in 529 CE. But this alone did not stifle communication between the intellectuals of the East and those of the West. It did, however, move the intellectual center farther East. Reflect on the growth of Muslim (Islamic) science that resulted.

Solution:

The closing of the academy, although often presented as a narrow consequence of Christian thought, seems to have been a more personal economic result of Justinian's endeavors in creating a competitive university. It did, however, result in the dispersal of intellectuals, including scientists and philosophers from Athens to other homes among the Arabs. There were some who later returned to Alexandria. But the center of gravity had moved. And this almost coincided with Muhammad's revelations in 610 CE. The result was a simultaneous rise in both scientific and religious thought among the Arabs. This was also influenced by the fact that the Caliph (685 − 705 CE) changed the language in which business was conducted from Greek or Persian to Arabic. This changed the educational system and had the result that governmental service opportunities became more based on talent than on language.

1.5. We concentrated on Muslim developments in mathematics and in astronomy because these were the areas of most interest to us. There we found distinct theological as well as philosophical and scientific interactions that determined the directions of the development of Muslim science. Outline these interactions and the influence they had on the form of Aristotle and Ptolemy that eventually appeared in western Europe.

Solution:

The difficulties that the Muslims encountered were twofold. First there was a foundational difference between the philosophy of Aristotle and the astronomy finally adopted by Ptolemy. The metaphysics Aristotle had adopted, which claimed that the motion of the cosmos was spherical, did not fit the facts Ptolemy confronted. Aristotle had accepted the proposal of Plato that cosmic motion was spherical. Indeed the spheres of Plato had taken on a reality for Aristotle that Plato may not have accepted. And they did not fit the data to which Ptolemy had access. So Ptolemy modified the picture to fit the data, a common scientific approach, which often precedes a time of change in our thinking. But this could not pass serious muster if one is trying to reconcile the picture with the philosophically accepted ideas. A further difficulty was that Aristotle's concept of God did not compare with the Islamic concept of a personal God. So there were already difficulties in trying to match Ptolemy with Aristotle. And the dilemma moved Muslim astronomers to seek new mathematical answers to questions regarding the motion. This was a notable new approach that might have opened more doors prior to the ideas of Copernicus and Kepler. But it did at least provide Copernicus with some of the mathematical steps he needed. The result was at least an Aristotle and a Ptolemy that were corrected before entering western Europe.

1.6. In 711 CE the Muslims took Spain creating *al Andalus* (Muslim Spain), which lasted until 1492. This was not a continuous period of stable borders separating al Andalus from Christian Europe. The invasion of forces from the Holy Roman Empire pushed the border of al Andalus progressively southward. But the people of al Andalus became inseparably mixed during the 700 years of Muslim rule. The Muslims brought with them a culture, an educational system, and manuscripts that had a great effect on western Europe. What was the impact of this on Europe?

Solution:

Intellectually the impact was dramatic. European intellectuals had been without direct contact with the Greek heritage. The physics and the philosophy of Aristotle worked, as did Ptolemaic (Muslim adjusted) astronomy. And the Muslim university system resulted in some of the most famous of European universities. But the manuscripts, including Aristotle and Ptolemy, were in Arabic. These needed translation into Latin for use in western Europe. That initiated a period of translation which lasted centuries and brought together Jews, Christians, and Muslims in centers such as Toledo. Muslim culture, at that time, was tolerant. And the period may almost be seen as a golden period of scientific renaissance.

1.7. As physicists and Engineers we tend to write off medieval physics as of little interest in part because of the influence of the Catholic Church. But there were serious scientists even among those taking Holy Orders. What were the issues that were considered during the late medieval period?

Solution:

We limited our considerations to only three people: Robert Grosseteste, Roger Bacon, and Jean Buridan. The issues were the role of mathematics and experimentation in physics, as well as seeking an understanding of the concepts of motion and force, or impetus. Realizing that only Euclidian geometry had been available to Aristotle it is almost logical that Aristotle would have rejected the idea that mathematics would have anything to say about physics. It may speak to the motion of the cosmos, but certainly not to motion on earth. But the question was unresolved in general. And with the advances of Muslim science and astronomy it became a serious question. Grosseteste was particularly interested in this. And Bacon was concerned about the use of experiment. We presently have no question about the importance of experiment. But Aristotle was suspicious of experiments that may affect the outcome of the physics, particularly based on his metaphysics. And Buridan introduced the idea of impetus, as a sort of force carried by a moving body. We know see this as momentum and recognize the importance of Buridan. But the concept also reveals a complete lack of understanding of what we now call a force.

1.8. The modern scientist will quickly point to the idea of Nicolaus Copernicus and his work as the beginning of what we now call classical physics. This was followed by the revolutionary steps taken by Johannes Kepler. With Kepler, as well as the ideas of others, Newton was able to synthesize the basis of mechanics. Place these ideas in context.

Solution:

No ideas are completely unique to an individual. Copernicus encountered the ideas of a sun centered system as a student. And they were discussed in ancient Greece. The obvious advantage was that it first seemed that placing the sun at the center would eliminate the plethora of epicycles in the Ptolemaic system. But the problems were not so easily solved by trying to fit circles into a system, which was not based on circles. Kepler discovered this following a tortuous intellectual journey with the best data in the world at that time, those of Tycho Brahe. Kepler came to base his thoughts on hard data rather than on philosophy. He made astrophysics out of astronomy. There are more twists to both of these stories, but we elected to economize space.

Newton was a tortured individual bordering on nervous breakdown. He was an isolated individual by choice and pursued his studies essentially on his own, which was probably the cause of his second class degree from Cambridge. He was able to synthesize the ideas of others, including René Descartes and Christiaan Huygens, into four laws of mechanics. The turning point that led to a true publication of those ideas was when Edmond Halley asked Newton for the orbit of a body moving under an inverse square centralized force. In the course of this encounter Newton became grasped by the science itself and moved with unrelenting energy to the product, which was the *Principia*. Had Newton been more ready to use the calculus this would have been the dawn of Analytical Mechanics, perhaps.

1.9. The 18$^{\text{th}}$ century saw a great advance in the methods of mechanics beyond what Newton had done. A real Analytical Mechanics emerged at the hands of Pierre Louis Maupertuis, Leonhard Euler and Joseph-Louis Lagrange. Provide a brief explanation of what these people did.

Solution:

Maupertuis began the quest for a variational principle that would produce a formulation of mechanics. He was able to produce a principle for equilibrium. This idea of a variational principle was originally Maupertuis' with a nod to the ideas of Johann Bernoulli. Maupertuis only applied this to equilibrium. But the idea was very fruitful. Euler picked this up and sought to obtain a formulation of dynamics using it. With some caveats he did so. The mathematics was not as neat as he wished and he had some end-point problems. But the very young Lagrange put the picture together developing what we now know as the calculus of variations. This included his method of undetermined multipliers, that provided for the incorporation of constraints. With this we had an Analytical Mechanics.

1.10. William Rowan Hamilton and Carl Gustav Jacobi put together the (almost) last parts of the Analytical Mechanics. We have provided an outline of their papers in the chapter. Very briefly describe what they did.

Solution:

We need not discuss Hamilton's desire to produce a theory of mathematical optics. His goal in the first essay was to produce a function (the Principal Function) that contained all that can be known about the system. He found that this function could be obtained from the solution of two (essentially identical) partial differential equations written for separate times. Once the function was found the variables could be obtained by partial differentiation of the function. In the second essay Hamilton obtained his canonical equations via Legendre Transformation and conducted the derivation without a variational principle.

Jacobi had a great admiration for the work of Hamilton. However, he wondered why Hamilton claimed he needed two partial differential equations instead of just one. And he wondered why Hamilton had limited the theory to situations in which the force function was independent of time. Jacobi also pointed out that, although the partial differential equation(s) for the Principal Function were elegant the simplicity of ordinary differential equations was lost. In many cases it was possible to reduce a partial differential equation to a set of ordinary differential equations. He developed a proof of each of his contentions in the course of his two papers. Jacobi also placed his work in his lectures as soon as practicable. This provides an accessible explanation of his ideas.

Hamilton's ideas, modified and simplified by Jacobi form the basis of what is known as the Hamilton-Jacobi approach.

2 Lagrangian Mechanics

2.1. Cylindrical coordinates are $\{r, \vartheta, z\}$. The position vector from the origin is $\boldsymbol{r} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}$. Show that the velocity vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \dot{r}\hat{e}_{\mathrm{r}} + r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

Solution:

We realize that the unit vector $\hat{e}_{\mathrm{r}}$ follows the particle and so is a function of time. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \left(\frac{\mathrm{d}}{\mathrm{d}t}r\right)\hat{e}_{\mathrm{r}} + r\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_{\mathrm{r}}\right) + \left(\frac{\mathrm{d}}{\mathrm{d}t}z\right)\hat{e}_{\mathrm{z}}.$$

With a quick drawing of the change $\delta\hat{e}_{\mathrm{r}}$ in the vector $\hat{e}_{\mathrm{r}}$ as the particle moves through an angle $\delta\vartheta$ we should be able to see that

$$\delta\hat{e}_{\mathrm{r}} = \delta\vartheta\hat{e}_{\vartheta}$$

and that, therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{e}_{\mathrm{r}} = \dot{\vartheta}\hat{e}_{\vartheta}.$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \dot{r}\hat{e}_{\mathrm{r}} + r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

2.2. Spherical coordinates are $\{\rho, \vartheta, \phi\}$. The position vector is $\overrightarrow{r} = \rho\hat{e}_{\rho}$. Show that the velocity vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \dot{\rho}\hat{e}_{\rho} + \rho\dot{\vartheta}\sin\phi\hat{e}_{\vartheta} + \rho\dot{\phi}\hat{e}_{\phi}.$$

Solution:

We know that the unit vector $\hat{e}_{\rho}$ tracks the particle. It, therefore, changes direction as the particle moves. If only the magnitude of the particle distance from the origin changes, while the direction $\hat{e}_\rho$ does not change the contribution to the velocity vector is

$$\dot{\rho}\hat{e}_\rho$$

The radial distance from the vertical axis (the $z-$axis in the preceding exercise) is $\rho \sin\phi$. So the rate of change of the vector $\overrightarrow{r} = \rho\hat{e}_\rho$ as the azimuthal angle ϑ changes at a rate $\dot{\vartheta}$ is

$$\rho\dot{\vartheta}\sin\phi\hat{e}_\vartheta.$$

The rate of change of the vector $\overrightarrow{r} = \rho\hat{e}_\rho$ as the polar angle ϕ changes at a rate $\dot{\phi}$ (a little picture will soon convince us) is

$$\rho\dot{\phi}\hat{e}_\phi.$$

The velocity of the particle is the linear sum of these three velocity components, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} = \dot{\rho}\hat{e}_\rho + \rho\dot{\vartheta}\sin\phi\hat{e}_\vartheta + \rho\dot{\phi}\hat{e}_\phi.$$

Remark 1. These exercises should serve to provide a familiarity with the coordinates that will be of interest to us. The solution will require some drawing and the consideration of small variations.

2.3. In this chapter we introduced Lagrange Undertermined Multipliers to add constraints to a variational principle. The reasoning should work just as well if we wish to find the extremum of an algebraic expression subject to a constraint. For example, if we wish to find the minimum distance from the origin to the straight line

$$y = 3x + 2$$

we seek the minimum distance from the origin to a point (x, y) in the plane and then introduce the constraint that the point lies on the line. The calculation will be easier if we minimize the square of the distance from the origin to the point (x, y), which by Pythagoras' Theorem is

$$f(x, y) = x^2 + y^2.$$

Carry out the calculation to find the point on the line.

Show that the shortest line between the origin and the straight line $y = 3x + 2$ is perpendicular to the line $y = 3x + 2$.

Solution:

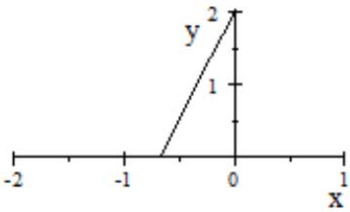
The function to minimize is the square of the distance from the origin to the point (x, y), which is

$$f(x, y) = x^2 + y^2.$$

The points (x, y) lie on the straight line. The solutions for (x, y) are then constrained by the requirement that

$$g(x, y) = y - 3x - 2 = 0,$$

be satisfied. We have graphed $g(x, y)$ in the figure below.



Plot of $y = 3x + 2$.

Using Lagrange undertermined multipliers we must find extrema of

$$h(x, y) = x^2 + y^2 + \lambda(y - 3x - 2).$$

We require then

$$\frac{\partial h}{\partial x} = 2x - 3\lambda = 0$$

and

$$\frac{\partial h}{\partial y} = 2y + \lambda = 0$$

with

$$y = 3x + 2.$$

We first eliminate λ, which is of no importance.

$$x + 3y = 0.$$

The last two equations then yield

$$y = 3(-3y) + 2.$$

Then $y = 1/5$. The value of x on the line at this value of y is $-3/5$. The vector from the origin to this point is then $(-3/5, 1/5)$. The vector along the line $y = 3x + 2$

is $\left(\frac{2}{3}, 2\right)$. Then the vectors along the line $y = 3x + 2$ and the shortest line from the origin to this line are

$$\left(\frac{2}{3}, 2\right) \text{ and } (-3/5, 1/5)$$

scalar product of these vectors is

$$\begin{aligned}&\left(\frac{2}{3}, 2\right) \cdot (-3/5, 1/5)\\ &= -\frac{2}{5} + \frac{2}{5} = 0.\end{aligned}$$

So they are perpendicular.

2.4. Show that D'Alembert's Principle results in conservation of mechanical energy $\mathrm{d}(T + V) = 0$ for impressed forces derivable from a scalar potential as $\boldsymbol{F} = -\operatorname{grad} V$.

Solution:

Motion is consistent with constraints. So, writing $\delta \boldsymbol{r}_\mathrm{i} = \mathrm{d}\boldsymbol{r}_\mathrm{i}$

$$\begin{aligned} m_\mathrm{i}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_\mathrm{i} \cdot \delta\boldsymbol{r}_\mathrm{i} &= m_\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_\mathrm{i} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{i}\\ &= m_\mathrm{i}\frac{\mathrm{d}\boldsymbol{v}_\mathrm{i}}{\mathrm{d}t} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{i}\\ &= m_\mathrm{i}\boldsymbol{v}_\mathrm{i} \cdot \mathrm{d}\boldsymbol{v}_\mathrm{i}\\ &= \mathrm{d}\left(\frac{1}{2}m_\mathrm{i}v_\mathrm{i}^2\right) = \mathrm{d}T, \end{aligned}$$

where

$$T_\mathrm{i} = \frac{1}{2}m_\mathrm{i}v_\mathrm{i}^2$$

is the kinetic energy of the i^{th} particle. And, if the forces all come from potentials,

$$\begin{aligned} \boldsymbol{F}_\mathrm{i} \cdot \mathrm{d}\boldsymbol{r}_\mathrm{i} &= -\operatorname{grad} V \cdot \mathrm{d}\boldsymbol{r}_\mathrm{i}\\ &= -\mathrm{d}V_\mathrm{i}, \end{aligned}$$

where V_i is the potential at the location of the i^{th} particle. Then D'Alembert's Principle is

$$\begin{aligned} \sum_\mathrm{i}\left[m_\mathrm{i}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{r}_\mathrm{i} - \vec{F}_\mathrm{i}\right] \cdot \delta\boldsymbol{r}_\mathrm{i} &= \sum_\mathrm{i}\mathrm{d}\left(T_\mathrm{i} + V_\mathrm{i}\right)\\ &= d\left(T + V\right), \end{aligned}$$

which is conservation of mechanical energy for the system.

2.5. Consider the parabola

$$y_1 = -2x_1^2 - 4$$

and the straight line

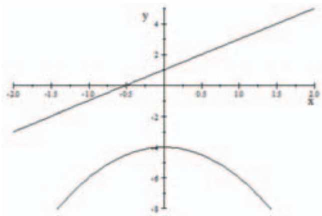
$$y_2 = 2x_2 + 1$$

First show that the graphs of these two functions never intersect. Having convinced yourself of this, then go on to find the minimum distance between these two functions.

Show also that the minimum distance is a line perpendicular to the given straight line.

Solution:

In the figure below we have plotted graphs of the straight line and the parabola.



Plot of the functions $y = -2x^2 - 4$ and $y = 2x + 1$. Note the scales on abscissa and ordinate differ.

We want to minimize the function (square of the distance between points on the curves)

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

subject to the constraints

$$g_1 = -y_1 - 2x_1^2 - 4 = 0$$

and

$$g_2 = -y_2 + 2x_2 + 1 = 0.$$

Then

$$\frac{\partial g_1}{\partial x_1} = -4x_1 \text{ and } \frac{\partial g_1}{\partial y_1} = -1$$

and

$$\frac{\partial g_2}{\partial x_2} = -4x_1 \text{ and } \frac{\partial g_2}{\partial y_2} = -1\ .$$

That is we want the extremum of

$$\ell^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + \lambda_1 \left(-y_1 - 2x_1^2 - 4\right) + \lambda_2 (-y_2 + 2x_2 + 1),$$

which is numerically equal to the first equation. At the extremum all partial derivatives vanish. The partial derivatives with respect to $\lambda_{1,2}$ vanish automatically because $g_1 = g_2 = 0$. Then

$$-2(x_2 - x_1) - 4\lambda_1 x_1 = 0,$$

$$2(x_2 - x_1) + 2\lambda_2 = 0,$$

$$-2(y_2 - y_1) - \lambda_1 = 0,$$

$$2(y_2 - y_1) - \lambda_2 = 0,$$

$$y_1 = -2x_1^2 - 4,$$

and

$$y_2 = 2x_2 + 1.$$

Because

$$\lambda_1 = -2(y_2 - y_1)$$

and

$$\lambda_2 = 2(y_2 - y_1).$$

we see

$$\lambda_1 = -\lambda_2 = \lambda.$$

We then have the equations

$$\lambda = -2(y_2 - y_1)$$

$$-(x_2 - x_1) - 2\lambda x_1 = 0$$

$$\lambda = (x_2 - x_1).$$

Combining we have

$$\lambda = (x_2 - x_1) = -2(y_2 - y_1)$$

and

$$-\lambda - 2\lambda x_1 = 0.$$

We have as well the equations for the curves

$$y_1 = -2x_1^2 - 4$$

and

$$y_2 = 2x_2 + 1.$$

We may now solve for x_1

$$x_1 = -\frac{1}{2}.$$

From the parabola we have y_1 as

$$y_1 = -2x_1^2 - 4 = -\frac{9}{2}.$$

The equation for λ then becomes

$$\begin{aligned}\lambda &= (x_2 - x_1) = -2\,(y_2 - y_1)\\ &= \left(x_2 + \frac{1}{2}\right) = -2\left(y_2 + \frac{9}{2}\right)\end{aligned}$$

With $y_2 = 2x_2 + 1$ This is

$$\left(x_2 + \frac{1}{2}\right) = -2\left(2x_2 + \frac{11}{2}\right)$$

The solution is

$$x_2 = -\frac{23}{10}.$$

Then

$$y_2 = 2\left(-\frac{23}{10}\right) + 1 = -\frac{18}{5}$$

We then have the points on the two curves as

$$(x_1, y_1) = \left(-\frac{1}{2}, -\frac{9}{2}\right)$$

and

$$(x_2, y_2) = \left(-\frac{23}{10}, -\frac{18}{5}\right).$$

The vector from the parabola to the straight line has the components

$$(x_2 - x_1, y_2 - y_1) = \left(-\frac{18}{5}, \frac{9}{10}\right).$$

The vector direction along the straight line is $\left(\frac{1}{2}, 1\right)$. The scalar product of the vectors is then

$$\left(-\frac{18}{5}, \frac{9}{10}\right) \cdot \left(\frac{1}{2}, 1\right)$$
$$= -\frac{9}{10} + \frac{9}{10} = 0.$$

The shortest line between the curves is then perpendicular to the straight line.

2.6. In statistical mechanics we find that the Gibbs expression for the entropy is

$$S = -k_{\mathrm{B}} \sum_{\mathrm{r}} P_{\mathrm{r}} \ln P_{\mathrm{r}},$$

where k_{B} is the Boltzmann constant and P_{r} is the coefficient of probability for the r^{th} state, which is a measure of the density of states in the system phase space[1]. Thermodynamics teaches us that under conditions of constant energy and volume the *entropy of a system will be a maximum*. That is, we have the constraint that

$$\mathcal{E} = \sum_{\mathrm{r}} P_{\mathrm{r}} \mathcal{E}_{\mathrm{r}},$$

which is that the average energy of the system in the ensemble is a constant. We also realize that there is another constraint in the definition of probability. That is

$$1 = \sum_{r} P_r.$$

Show that the probability that a system of atoms will be in a particular state of total energy $\mathcal{E}_{\mathrm{r}}$ is given by

$$P_{\mathrm{r}} = \exp\left[-1 - \alpha - \beta \mathcal{E}_{\mathrm{r}}\right],$$

where α and β are constants. Do this by maximizing the Gibbs entropy subject to the constraints. The α and β are the Lagrange Undetermined Multipliers.

[1]The system phase space has an axis for every canonical coordinate and every canonical momentum of every particle (atom/molecule) in the system. This space is called $\Gamma-$space.

Solution:

We seek the maximum (extremum) of

$$S = -k_{\mathrm{B}} \sum_{\mathrm{r}} P_{\mathrm{r}} \ln P_{\mathrm{r}}$$

subject to

$$g_1 = \alpha \left(\sum_{r} P_r - 1 \right) = 0$$

and

$$g_2 = \beta \left(\sum_{\mathrm{r}} P_{\mathrm{r}} \mathcal{E}_{\mathrm{r}} - \mathcal{E} \right) = 0.$$

That is, we seek an extremum of

$$h(P) = -k_{\mathrm{B}} \sum_{\mathrm{r}} P_{\mathrm{r}} \ln P_{\mathrm{r}} + \alpha \left(\sum_{r} P_r - 1 \right) + \beta \left(\sum_{\mathrm{r}} P_{\mathrm{r}} \mathcal{E}_{\mathrm{r}} - \mathcal{E} \right)$$

subject to (now) independent variations of the probabilities P_{r}. Then

$$\delta h(P) = 0 = -k_{\mathrm{B}} \sum_{\mathrm{r}} \delta P_{\mathrm{r}} \left(1 + \ln P_{\mathrm{r}} + \alpha + \beta \mathcal{E}_{\mathrm{r}}\right).$$

Because the P_{r} are now independent, this requires that

$$1 + \ln P_{\mathrm{r}} + \alpha + \beta \mathcal{E}_{\mathrm{r}} = 0$$

or

$$P_{\mathrm{r}} = \exp\left(-1 - \alpha - \beta \mathcal{E}_{\mathrm{r}}\right).$$

2.7. Study the following functional

$$J[y] = \int_0^1 \mathrm{d}x \left(y'\right).$$

Determine whether or not it has an extremum. If it does, find that extremum.

Solution:

The Euler-Lagrange equation is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}\left(y'\right) &= 0 \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(1)\,,
\end{aligned}
$$

which is satisfied uniquely for all possible functions $y = y(x)$. If we return to the original functional we see that, with $y' = \mathrm{d}y/\mathrm{d}x$,

$$J[y] = \int_0^1 \mathrm{d}x\left(y'\right) = \int_0^1 \mathrm{d}y = 1$$

for all possible $y = y(x)$. So the functional is always a constant and there is no extremum.

2.8. Consider the functional

$$J[y] = \int_0^1 \mathrm{d}x\left(yy'\right).$$

Determine whether or not this functional has an extremum. If so, find that extremum.

Solution:

The Euler-Lagrange Equation is

$$\frac{\partial}{\partial y}\left(yy'\right) - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}\left(yy'\right) = 0$$

or

$$y' - \frac{\mathrm{d}}{\mathrm{d}x}y = y' - y' = 0$$

identically. We also notice that

$$yy' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}y^2\right)$$

so the functional becomes

$$J[y] = \int_0^1 \mathrm{d}\left(\frac{1}{2}y^2\right) = \frac{1}{2}$$

regardless of the function y. There is then no extremum.

2.9. Study the functional

$$J[y] = \int_0^1 \mathrm{d}x\left(xyy'\right).$$

Determine whether or not this functional has an extremum. If it does, find that extremum.

Solution:

The Euler-Lagrange Equation is

$$\begin{aligned}\frac{\partial}{\partial y}\left(xyy'\right)-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}\left(xyy'\right)&=xy'-\frac{\mathrm{d}}{\mathrm{d}x}xy\\&=xy'-y-xy'\\&=y=0.\end{aligned}$$

The Euler-Lagrange Equation is then solved by the function $y(x)=0$. There is then a minimum for $y=0$ and this minimum is 0.

2.10. Find the differential equation for the extremum of the functional

$$J[y]=\int_0^1\mathrm{d}x\left[y^2+\left(y'\right)^2-2y\sin(x)\right]$$

[answer: $y''-y=-\sin(x)$]

Solution:

The Euler-Lagrange Equation is

$$\begin{aligned}&\frac{\partial}{\partial y}\left[y^2+\left(y'\right)^2-2y\sin(x)\right]-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial}{\partial y'}\left[y^2+\left(y'\right)^2-2y\sin(x)\right]\\&=2y-2\sin(x)-\frac{\mathrm{d}}{\mathrm{d}x}\left(2y'\right)\\&=2y-2\sin(x)-2y''=0,\end{aligned}$$

or

$$y''-y=-\sin(x).$$

2.11. Show that the functional of two functions:

$$S[x,y]=\int_{t_1}^{t_2}\mathrm{d}t\Psi[t,x,y,\dot{x},\dot{y}]$$

has an extremum when Euler-Lagrange equations for each of the functions are satisfied.. That is, show that this functional has an extremum when

$$\frac{\partial}{\partial x}\Psi-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{x}}\Psi=0$$

and

$$\frac{\partial}{\partial y}\Psi-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{y}}\Psi=0$$

provided the variations vanish at the end points.

This requires the details of what we indicated in the text.

Solution:

We assume that an extremum of the functional exists for the functions $x = \xi$ and $y = \eta$. Then we define infinitesimal functions h_{x} and h_{y}, which vanish at the end points. Then the functional has an extremum when

$$\Psi = \Psi\left(t, \xi, \eta, \xi', \eta'\right)$$

and the functions $\xi + h_{\mathrm{x}}$ and $\eta + h_{\mathrm{y}}$ provide a deviation of the functional from the extremum. That is

$$\begin{aligned}\delta S[x, y] = &\int_{\mathrm{t}_1}^{\mathrm{t}_2} \mathrm{d}t \Psi\left[t, \xi + h_{\mathrm{x}}, \eta + h_{\mathrm{y}}, \dot{\xi} + \dot{h}_{\mathrm{x}}, \dot{\eta} + \dot{h}_{\mathrm{y}}\right] \\ &- \int_{\mathrm{t}_1}^{\mathrm{t}_2} \mathrm{d}t \Psi\left[t, \xi, \eta, \dot{\xi}, \dot{\eta}\right].\end{aligned}$$

We then conduct a Taylor expansion of Ψ around the extremum condition.

$$\begin{aligned}&\Psi\left[t, \xi + h_{\mathrm{x}}, \eta + h_{\mathrm{y}}, \dot{\xi} + \dot{h}_{\mathrm{x}}, \dot{\eta} + \dot{h}_{\mathrm{y}}\right] \\ = &\Psi\left[t, \xi, \eta, \dot{\xi}, \dot{\eta}\right] + \frac{\partial \Psi}{\partial \xi} h_{\mathrm{x}} + \frac{\partial \Psi}{\partial \dot{\xi}} \dot{h}_{\mathrm{x}} \\ &+ \frac{\partial \Psi}{\partial \eta} h_{\mathrm{y}} + \frac{\partial \Psi}{\partial \dot{\eta}} \dot{h}_{\mathrm{y}}\end{aligned}$$

neglecting terms higher than first order in $h_{\mathrm{x,y}}$. Then

$$\delta S[x, y] = \int_{\mathrm{t}_1}^{\mathrm{t}_2} \mathrm{d}t \left(\frac{\partial \Psi}{\partial \xi} h_{\mathrm{x}} + \frac{\partial \Psi}{\partial \dot{\xi}} \dot{h}_{\mathrm{x}} + \frac{\partial \Psi}{\partial \eta} h_{\mathrm{y}} + \frac{\partial \Psi}{\partial \dot{\eta}} \dot{h}_{\mathrm{y}}\right).$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Psi}{\partial \dot{\xi}} h_{\mathrm{x}}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \Psi}{\partial \dot{\xi}}\right) h_{\mathrm{x}} + \frac{\partial \Psi}{\partial \dot{\xi}} \dot{h}_{\mathrm{x}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Psi}{\partial \dot{\eta}} h_{\mathrm{y}}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \Psi}{\partial \dot{\eta}}\right) h_{\mathrm{y}} + \frac{\partial \Psi}{\partial \dot{\eta}} \dot{h}_{\mathrm{y}}.$$

And, because the functions $h_{\mathrm{x,y}}$ vanish at the end points,

$$\begin{aligned}&\int_{\mathrm{t}_1}^{\mathrm{t}_2} \mathrm{d}t \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Psi}{\partial \dot{\xi}} h_{\mathrm{x}}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Psi}{\partial \dot{\eta}} h_{\mathrm{y}}\right)\right] \\ = &\left(\frac{\partial \Psi}{\partial \dot{\xi}} h_{\mathrm{x}}\right) + \left(\frac{\partial \Psi}{\partial \dot{\eta}} h_{\mathrm{y}}\right)\Bigg]_{\mathrm{t=t}_1}^{\mathrm{t=t}_2} = 0.\end{aligned}$$

Then

$$\delta S[x, y] = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial \Psi}{\partial \xi} - \frac{d}{dt} \frac{\partial \Psi}{\partial \dot{\xi}} \right) h_x + \left(\frac{\partial \Psi}{\partial \eta} - \frac{d}{dt} \frac{\partial \Psi}{\partial \dot{\eta}} \right) h_y \right].$$

because the infinitesimal functions $h_{x,y}$ are independent of one another, this integral vanishes if

$$\frac{\partial \Psi}{\partial \xi} - \frac{d}{dt} \frac{\partial \Psi}{\partial \dot{\xi}} = 0$$

and

$$\frac{\partial \Psi}{\partial \eta} - \frac{d}{dt} \frac{\partial \Psi}{\partial \dot{\eta}} = 0$$

independently of one another.

2.12. Consider that in the functional

$$S[y] = \int_{t_1}^{t_2} dt\, F(y, \dot{y})$$

the function F does not depend explicitly on the time t. Show that as a consequence

$$F - \dot{y} \frac{\partial F}{\partial \dot{y}} = \text{constant}.$$

Solution:
We begin with the Euler-Lagrange Equation.

$$\begin{aligned} &\frac{\partial F}{\partial y} - \frac{d}{dt} \frac{\partial F}{\partial \dot{y}} \\ &= \frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial \dot{y}^2} \ddot{y} - \frac{\partial^2 F}{\partial y \partial \dot{y}} \dot{y} = 0. \end{aligned}$$

If we multiply both sides through by $\dot{y}$ we have

$$\begin{aligned} &\dot{y} \left(\frac{\partial F}{\partial y} - \frac{d}{dt} \frac{\partial F}{\partial \dot{y}} \right) \\ &= \frac{\partial F}{\partial y} \dot{y} - \frac{\partial^2 F}{\partial \dot{y}^2} \ddot{y} \dot{y} - \frac{\partial^2 F}{\partial y \partial \dot{y}} \dot{y}^2 = 0. \end{aligned}$$

Now we note that

$$\frac{d}{dt} \frac{\partial F}{\partial \dot{y}} \dot{y} = \frac{\partial^2 F}{\partial \dot{y}^2} \ddot{y} \dot{y} + \frac{\partial^2 F}{\partial y \partial \dot{y}} \dot{y}^2 + \frac{\partial F}{\partial \dot{y}} \ddot{y}.$$

Then

$$\frac{d}{dt} \left(F - \dot{y} \frac{\partial F}{\partial \dot{y}} \right) = \frac{\partial F}{\partial y} \dot{y} - \frac{\partial^2 F}{\partial \dot{y}^2} \ddot{y} \dot{y} - \frac{\partial^2 F}{\partial y \partial \dot{y}} \dot{y}^2,$$

and, comparing, we see that

$$\dot{y}\left(\frac{\partial F}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{y}}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(F-\dot{y}\frac{\partial F}{\partial \dot{y}}\right)=0.$$

Therefore

$$F-\dot{y}\frac{\partial F}{\partial \dot{y}}=\text{constant}.$$

2.13. Using the results of the preceding exercise, i.e.

$$F-\dot{y}\frac{\partial F}{\partial \dot{y}}=\text{constant},$$

show that for F as the Lagrangian $F = T - V$ for a single particle of mass m that the total mechanical energy is constant and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F-\dot{y}\frac{\partial F}{\partial \dot{y}}\right)=-\frac{\mathrm{d}}{\mathrm{d}t}\left(T+V\right).$$

Solution:
If F is the Lagrangian for a single particle

$$F=\frac{1}{2}m\dot{y}^2-V.$$

Then

$$\begin{aligned}F-\dot{y}\frac{\partial F}{\partial \dot{y}}&=\frac{1}{2}m\dot{y}^2-V-\dot{y}\left(m\dot{y}\right)\\&=-\left(\frac{1}{2}m\dot{y}^2+V\right).\end{aligned}$$

Therefore

$$\left(\frac{1}{2}m\dot{y}^2+V\right)=\text{constant}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F-\dot{y}\frac{\partial F}{\partial \dot{y}}\right)=-\frac{\mathrm{d}}{\mathrm{d}t}\left(T+V\right).$$

2.14. Among all curves joining the points (x_0, y_0) and (x_1, y_1) find that which generates the minimum surface area when rotated around the $x-$axis. Begin with the Pythagorean Theorem that the differential length between two points in the $(x, y)-$plane is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1 + (y')^2}.$$

At the point x the distance to the curve $y = y(x)$ is equal to the value of y. So the differential area of the surface of rotation defined by the points x and $x+\mathrm{d}x$ on the $x-$axis is

$$\mathrm{d}x\left(2\pi y\sqrt{1 + (y')^2}\right).$$

The area of the surface of revolution between x_0 and $x_1$is

$$A_\mathrm{S} = 2\pi \int_{\mathrm{x}_{0-}}^{\mathrm{x}_1} \mathrm{d}x\left(y\sqrt{1 + (y')^2}\right).$$

[Answer: $y = K\cosh[(x + C)/K]$, where K and C are (integration) constants]
Solution:
We note that the function in the functional

$$F = 2\pi y\sqrt{1 + (y')^2}$$

is independent of the coordinate x and depends only on y and y'. From the results of a preceding exercise, then, we know that

$$F - y'\frac{\partial F}{\partial y'} = \text{constant},$$

replacing t with x. Now

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}2\pi y\sqrt{1 + (y')^2} = 2\pi y\frac{y'}{\sqrt{1 + (y')^2}}.$$

Then

$$y\sqrt{1 + (y')^2} - y\frac{(y')^2}{\sqrt{1 + (y')^2}} = K,$$

where K is a constant. Multiplying through by $\sqrt{1 + (y')^2}$ we have

$$y = K\sqrt{1 + (y')^2},$$

or (dropping $\pm$)

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{K}\sqrt{y^2 - K^2},$$

which is

$$\mathrm{d}y \frac{K}{\sqrt{y^2 - K^2}} = \mathrm{d}x.$$

This integrates to (CRC 157)

$$\begin{aligned} x + C' = K \int \frac{\mathrm{d}y}{\sqrt{y^2 - K^2}} &= K \ln\left(y + \sqrt{y^2 - K^2}\right) \\ &= K \ln\left(\bar{y} + \sqrt{\bar{y}^2 - 1}\right) + K \ln K, \end{aligned}$$

where $\bar{y} = y/K$. Defining $C = C' - K \ln K$, we have

$$\frac{x + C}{K} = \ln\left(\bar{y} + \sqrt{\bar{y}^2 - 1}\right).$$

And

$$\cosh^{-1}(\bar{y}) = \ln\left(\bar{y} + \sqrt{\bar{y}^2 - 1}\right).$$

Then

$$\frac{x + C}{K} = \cosh^{-1}(\bar{y})$$

or

$$y = K \cosh\left(\frac{x + C}{K}\right).$$

2.15. A particle is released from rest at a point (x_0, y_0) and slides (without friction) down a curve in the (x, y) plane. Since the differential distance down the plane is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1 + (y')^2}$$

the speed at which the particle slides is

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{1 + (y')^2}\,\frac{\mathrm{d}x}{\mathrm{d}t}.$$

The speed, from energy conservation (no friction) is

$$v = \sqrt{2gy}.$$

Then

$$dt = dx \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}.$$

What must the curve be, down which the particle slides, so that it reaches the vertical line at $x = b\,(> x_0)$ in the shortest time? We then wish to find the extremum of the functional for the total time

$$T\,[y] = \int_{\mathrm{x=x_0}}^{\mathrm{x=b}} dx \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}}$$

This is the brachistochrone problem, which was first posed by Johann Bernoulli in 1696.

[Answer: A cycloid]

Solution:

From a previous exercise we found that for no dependence on the independent variable in the function appearing in the functional

$$F - y'\frac{\partial F}{\partial y'} = \text{constant},$$

since here the independent variable is x (it was t in the previous exercise).

Now

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} = \frac{y'}{\sqrt{2gy}\sqrt{1+(y')^2}}$$

Then

$$\begin{aligned} &F - y'\frac{\partial F}{\partial y'} \\ &= \frac{\sqrt{1+(y')^2}}{\sqrt{2gy}} - \frac{\left(y'\right)^2}{\sqrt{2gy}\sqrt{1+(y')^2}} \\ &= K' \end{aligned}$$

or

$$1 = K'\sqrt{2gy\left[1+(y')^2\right]}.$$

With foresight we called the constant K'. Squaring both sides,

$$1 = 2g\left(K'\right)^2 y\left[1 + \left(y'\right)^2\right].$$

Now we define $K = 1/\left[2g\left(K'\right)^2\right]$. Then

$$y\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] = K$$

or

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{K - y}{y}}.$$

This can be integrated if we are a bit crafty. We are releasing the particle (mass) from the top of the curve. This we called (x_0, y_0), but we may call it the origin. And the particle slides down the curve to a point $a > x_0 = 0$. If we choose the angle ϕ to be the angle between the vertical axis and the tangent to the curve, and choose to measure y as positive downward, then $\mathrm{d}x/\mathrm{d}y = \tan\phi$. From our last equation, this means

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \tan\phi = \sqrt{\frac{y}{K - y}},$$

or

$$\frac{y}{K - y} = \frac{\sin^2\phi}{\cos^2\phi},$$

which becomes

$$\begin{aligned} &y\cos^2\phi + y\sin^2\phi \\ &= y = K\sin^2\phi. \end{aligned}$$

We then have a solution for y. To obtain a solution for x we return to the differential relationship, which we write as

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = \sqrt{\frac{K - y}{y}}\frac{\mathrm{d}x}{\mathrm{d}\phi} = \frac{1}{\tan\phi}\frac{\mathrm{d}x}{\mathrm{d}\phi}.$$

With

$$\frac{\mathrm{d}y}{\mathrm{d}\phi} = \frac{\mathrm{d}}{\mathrm{d}\phi}\left(K\sin^2\phi\right) = 2K\sin\phi\cos\phi$$

this becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = \frac{\mathrm{d}y}{\mathrm{d}\phi} = 2K \tan\phi \sin\phi \cos\phi$$

or

$$\frac{\mathrm{d}x}{\mathrm{d}\phi} = 2K \sin^2\phi.$$

But

$$\sin^2\phi = \left\{\frac{1}{2i}\left[\exp(i\phi) - \exp(-i\phi)\right]\right\}^2$$
$$= \frac{1}{2}(1 - \cos 2\phi).$$

The a simple integral with respect to $\xi = 2\phi$ results in

$$x = \frac{K}{2}(2\phi - \sin 2\phi) + C$$

If we choose the angle ϕ to be zero at the top of the path the additive constant $C = 0$. And

$$y = K\sin^2\phi = \frac{K}{2}(1 - \cos 2\phi).$$

These last equations describe the cycloid. So the curve which minimizes the sliding time is the cycloid.

2.16. A particle of mass m moves under no forces in the direction x.

Find the Lagrangian and the canonical momentum. Show that the canonical momentum is conserved. Find the energy and show that its total time derivative is zero so that the energy is a constant.

Do this using the Euler-Lagrange equations.

Solution:

The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2$$

and the potential energy in this case is zero. The Lagrangian is then

$$L(x, \dot{x}, t) = T - V = \frac{1}{2}m\dot{x}^2$$

and the canonical momentum is

$$p_{\mathrm{x}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}.$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = \frac{\mathrm{d}}{\mathrm{d}t} p_{\mathrm{x}} = 0,$$

which means that

$$p_{\mathrm{x}} = \text{constant}.$$

The energy is

$$\begin{aligned}\mathcal{E} &= m\dot{x}\dot{x} - L \\ &= m\dot{x}^2 - \frac{1}{2}m\dot{x}^2 \\ &= \frac{1}{2}m\dot{x}^2,\end{aligned}$$

Which, in terms of the canonical momentum, is

$$\mathcal{E} = \frac{p_{\mathrm{x}}^2}{2m}.$$

The time derivative of the energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{p_{\mathrm{x}}}{m}\frac{\mathrm{d}}{\mathrm{d}t} p_{\mathrm{x}} = 0,$$

since $\mathrm{d}p_{\mathrm{x}}/\mathrm{d}t = 0$ from the Euler-Lagrange equation.

2.17. Consider a particle of mass m in free fall under the influence of gravity. Find the Lagrangian, the canonical momentum, the Euler-Lagrange equation. Show that the energy is constant and integrate the Euler-Lagrange equation.

Solution:

The kinetic and potential energies are

$$\begin{aligned}T &= \frac{1}{2}m\dot{y}^2 \\ V &= mgy.\end{aligned}$$

Then the Lagrangian is

$$L = \frac{1}{2}m\dot{y}^2 - mgy.$$

The canonical momentum is

$$p_{\mathrm{y}} = \frac{\partial L}{\partial \dot{y}} = m\dot{y}.$$

The Euler-Lagrange equation is then

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} = -mg - \frac{\mathrm{d}}{\mathrm{d}t}p_y$$
$$= -mg - \frac{\mathrm{d}}{\mathrm{d}t}m\dot{y} = 0.$$

This equation is integrated as

$$\int_{\mathrm{t}_0}^{\mathrm{t}} \mathrm{d}\left(m\dot{y}\right) = -\int_{\mathrm{t}_0}^{\mathrm{t}} \mathrm{d}t\,(mg)\,,$$

which is

$$m\dot{y}\,(t) - m\dot{y}_0 = -mg\,(t - t_0)\,,$$

where

$$\dot{y}_0 = \dot{y}\,(t_0)\,.$$

Since

$$\dot{y}\,(t) = \frac{\mathrm{d}y}{\mathrm{d}t},$$

the next integration is also immediate. Integrating

$$m\frac{\mathrm{d}y}{\mathrm{d}t} = -mg\,(t - t_0) + m\dot{y}_0$$

yields

$$my\,(t) - my_0 = (m\dot{y}_0 + mgt_0)\,(t - t_0) - \frac{1}{2}mg\left(t^2 - t_0^2\right).$$

Choosing $t_0 = 0$, this is

$$y\,(t) = y_0 + \dot{y}_0 t - \frac{1}{2}mgt^2.$$

The energy is

$$\mathcal{E} = p_{\mathrm{y}}\dot{y} - L$$
$$= \frac{1}{2}m\dot{y}^2 + mgy.$$

With the canonical momentum, this is

$$\mathcal{E} = \frac{p_{\mathrm{y}}^2}{2m} + mgy.$$

The time derivative of the energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{p_\mathrm{y}}{m}\frac{\mathrm{d}}{\mathrm{d}t}p_\mathrm{y} + mg\dot{y}.$$

The Euler-Lagrange equation is

$$-mg = \frac{d}{dt}p_y$$

and the canonical momentum is

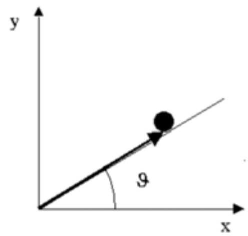
$$p_\mathrm{y} = \frac{\partial L}{\partial \dot{y}} = m\dot{y},$$

so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} &= \frac{p_\mathrm{y}}{m}\frac{\mathrm{d}}{\mathrm{d}t}p_\mathrm{y} + mg\dot{y}\\ &= \dot{y}\left(\frac{\mathrm{d}}{\mathrm{d}t}p_\mathrm{y} + mg\right)\\ &= 0,\end{aligned}$$

and the energy is constant.

2.18. Consider a particle of mass m sliding without friction down an inclined plane. We show this in the figure below.



Particle sliding on incline.

Find the Lagrangian, the Euler Lagrange equations, and the energy. Show that the energy is constant and solve the Euler-Lagrange equations. Find the reaction force with the incline using the Lagrange multipliers.

Solution:

The kinetic and potential energies are

$$\begin{aligned}T &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)\\ V &= mgy.\end{aligned}$$

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy. \end{aligned}$$

The canonical momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m\dot{x} \\ p_y &= \frac{\partial L}{\partial \dot{y}} = m\dot{y}. \end{aligned}$$

The energy is then

$$\begin{aligned} \mathcal{E} &= p_x\dot{x} + p_y\dot{y} - L \\ &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + mgy \\ &= \frac{1}{2m}\left(p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2\right) + mgy \end{aligned}$$

The constraint is the relationship between x and y, which is

$$y = \alpha x,$$

where

$$\alpha = \tan\vartheta.$$

To incorporate the constraint directly into the Lagrangian we differentiate the constraint to get

$$\dot{y} = \alpha\dot{x}.$$

Then the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2\left(1 + \alpha^2\right) - mg\alpha x.$$

For this Lagrangian the canonical momentum is

$$\frac{\partial L}{\partial \dot{x}} = m\left(1 + \alpha^2\right)\dot{x}.$$

The Euler-Lagrange equation is then

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = -mg\alpha - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\left(1 + \alpha^2\right)\dot{x}\right) = 0.$$

This equation is integrated to give

$$x = x_0 + \dot{x}_0 t - \frac{1}{2}\left(\frac{g\alpha}{1+\alpha^2}\right)t^2.$$

This is, then, the same as free fall if we replace g by $g\alpha/\left(1+\alpha^2\right)$. The inclined plane, then, models free fall, as Galileo claimed.

If we choose to use Lagrange multipliers, we first note that in standard form the constraint is

$$g = y - \alpha x.$$

Then

$$\mathrm{d}g = 0 = \mathrm{d}y - \alpha \mathrm{d}x,$$

from which

$$\frac{\partial g}{\partial y} = 1 \text{ and } \frac{\partial g}{\partial x} = -\alpha.$$

The Euler-Lagrange equations are then

$$\begin{aligned}
\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial g}{\partial x} &= 0 \\
\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial g}{\partial y} &= 0.
\end{aligned}$$

These become

$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) - \lambda\alpha &= 0 \\
-mg - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) + \lambda &= 0.
\end{aligned}$$

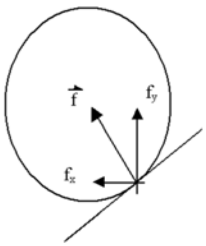
Written in the form

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) &= -\lambda\alpha \\
\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{y}) &= -mg + \lambda,
\end{aligned}$$

we easily see that the reaction forces with the incline are

$$\begin{aligned}
f_{\mathrm{x}} &= -\lambda\alpha = \lambda\frac{\partial g}{\partial x} \\
f_{\mathrm{y}} &= +\lambda = \lambda\frac{\partial g}{\partial y}.
\end{aligned}$$

The picture is



Forces for particle on incline.

We now have three equations to solve

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) &= -\lambda\alpha \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{y}\right) &= -mg + \lambda \\
y &= \alpha x,
\end{aligned}
$$

We use the constraint to eliminate y from the second in favor of x.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) = -\frac{mg}{\alpha} + \frac{\lambda}{\alpha}.$$

Combining this with the first we have an equation involving only λ, which is solved by

$$\lambda = \frac{mg}{1+\alpha^2}.$$

With this the first equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) = -\lambda\alpha = -\frac{m\alpha g}{1+\alpha^2}.$$

This is, of course, the same equation obtained above by incorporating the constraint directly into the Lagrangian, which is no mystery. We have gained in use of the Lagrange multipliers by obtaining the constraint forces. In more complicated problems the gain is more because the algebra of incorporating the constraints directly is simply too complicated to consider.

With the solution for λ the $y-$equation is

$$
\begin{aligned}
\frac{d}{dt}\dot{y} &= -g + \frac{g}{1+\alpha^2} \\
&= -g\left(\frac{\alpha^2}{1+\alpha^2}\right),
\end{aligned}
$$

which is also the same equation as obtained before.

Differentiating the energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{1}{m}\left(p_x \dot{p}_x + p_y \dot{p}_y\right) + mg\dot{y}$$
$$= \frac{p_x}{m}\dot{p}_x + \frac{p_y}{m}\left(\dot{p}_y + mg\right).$$

But, from the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_x\right) = \dot{p}_x = \lambda\alpha$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_y\right) = \dot{p}_y = -mg - \lambda$$
$$y = \alpha x.$$

Then

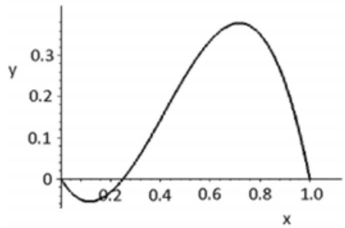
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \frac{p_x}{m}\lambda\alpha - \frac{p_y}{m}\lambda$$
$$= \dot{x}\lambda\alpha - \dot{y}\lambda$$
$$= \dot{x}\lambda\alpha - \dot{x}\lambda\alpha = 0,$$

using the constraint. That is, the energy is a constant of the motion.

2.19. Consider a mass, m, sliding without friction on the hilly terrain described by the function

$$y = -4x^3 + 5x^2 - x.$$

We have shown the hilly terrain in the figure below.



Hilly terrain described by $y = -4x^3 + 5x^2 - x$.

Obtain the equations of motion for the particle using Lagrange multipliers.
Solution:
The Lagrangian is

$$L = T - V$$
$$= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy.$$

The constraint is

$$g = y + 4x^3 - 5x^2 + x = 0.$$

The partial derivatives of the constraint are

$$\frac{\partial g}{\partial x} = 12x^2 - 10x + 1$$
$$\frac{\partial g}{\partial y} = 1.$$

The canonical momenta are

$$p_\mathrm{x} = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$
$$p_\mathrm{y} = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

and

$$\frac{\partial L}{\partial x} = 0$$
$$\frac{\partial L}{\partial y} = 0.$$

The Euler-Lagrange equations are then

$$\begin{aligned}\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) + \lambda\left(\frac{\partial g}{\partial x}\right) &= -m\ddot{x} + \lambda\left(12x^2 - 10x + 1\right)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) + \lambda\left(\frac{\partial g}{\partial y}\right) &= -m\ddot{y} + \lambda\\ &= 0.\end{aligned}$$

That is

$$\begin{aligned}-m\ddot{x} + \lambda\left(12x^2 - 10x + 1\right) &= 0\\ -m\ddot{y} + \lambda &= 0\end{aligned}$$

and the constraint is

$$y + 4x^3 - 5x^2 + x = 0.$$

We note that λ may be a function of the time.

These equations are sufficient to solve for the motion in principle. It may, however, be helpful to also use the constant energy

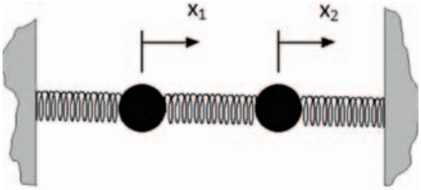
$$\begin{aligned}\mathcal{E} &= \left(\frac{\partial L}{\partial \dot{x}}\right)\dot{x} + \left(\frac{\partial L}{\partial \dot{y}}\right)\dot{y} - L\\ &= \frac{1}{2m}\left(p_\mathrm{x}^2 + p_\mathrm{y}^2\right) + mgy\end{aligned}$$

in attempting a solution. The numerical value of the energy may be found from the initial conditions (at the origin).

$$\mathcal{E} = mv_0^2.$$

However, the actual solution for the motion may not be at all easy. The Euler-Lagrange equations are nonlinear.

2.20. In the figure here we have two masses connected by identical springs to one another and to two vertical walls.



Two masses connected by identical springs to fixed vertical walls.

We neglect gravitational influences.

Study the motion of the system. Find the natural (eigen) frequencies and the corresponding eigenvectors.

Solution:

The extensions of the springs resulting from the displacements $x_{1,2}$ of the masses are

$$\begin{aligned} \text{spring 1 } &= x_1 \\ \text{spring 2 } &= (x_2 - x_1) \\ \text{spring 3 } &= x_2. \end{aligned}$$

Each spring, owing to its compression or extension, has a potential energy. The potential energies are

$$\begin{aligned} V_1 &= \frac{1}{2} k x_1^2 \\ V_2 &= \frac{1}{2} k \left(x_2 - x_1\right)^2 \\ V_3 &= \frac{1}{2} k x_2^2. \end{aligned}$$

The kinetic energies of the two masses are each of the form

$$T_\mu = \frac{1}{2} m \dot{x}_\mu^2.$$

The Lagrangian is then

$$L = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} k \left[x_1^2 + (x_2 - x_1)^2 + x_2^2\right].$$

There are two (generalized) coordinates. The canonical momenta are

$$\begin{aligned} p_{\mathrm{x}} &= \frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1 \\ p_{\mathrm{y}} &= \frac{\partial L}{\partial \dot{x}_2} = m \dot{x}_2. \end{aligned}$$

And the derivatives with respect to the generalized coordinates are

$$\frac{\partial L}{\partial x_1} = -kx_1 + k\,(x_2 - x_1)$$
$$\frac{\partial L}{\partial x_2} = -kx_2 - k\,(x_2 - x_1)\,.$$

The Euler-Lagrange equations are then

$$-m\ddot{x}_1 - kx_1 + k\,(x_2 - x_1) = 0$$
$$-m\ddot{x}_2 - kx_2 - k\,(x_2 - x_1) = 0.$$

Using the Ansatz

$$x_\mu = \widetilde{x}_\mu \exp\,(i\omega t)\,,$$

the Euler-Lagrange equations become

$$-\omega_0^2\widetilde{x}_1 + \omega_0^2\,(\widetilde{x}_2 - \widetilde{x}_1) = -\omega^2\widetilde{x}_1$$
$$-\omega_0^2\widetilde{x}_2 - \omega_0^2\,(\widetilde{x}_2 - \widetilde{x}_1) = -\omega^2\widetilde{x}_2,$$

or

$$2\omega_0^2\widetilde{x}_1 - \omega_0^2\widetilde{x}_2 = \omega^2\widetilde{x}_1$$
$$-\omega_0^2\widetilde{x}_1 + 2\omega_0^2\widetilde{x}_2 = \omega^2\widetilde{x}_2,$$

in which

$$\omega_0^2 = \frac{k}{m}.$$

Written in the form

$$\begin{bmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{bmatrix}\begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{bmatrix} = \omega^2 \begin{bmatrix} \widetilde{x}_1 \\ \widetilde{x}_2 \end{bmatrix},$$

we have an eigenvalue problem. The eigenvalues are

$$\omega^2 = \omega_0^2,\ 3\omega_0^2.$$

And the corresponding normalized eigenvectors are

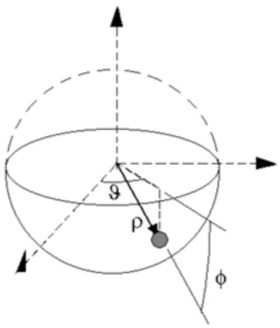
$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ for the eigenvalue } \omega_0^2$$
$$\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \text{ for the eigenvalue } 3\omega_0^2.$$

We also notice that the principal matrix is Hermitian so that the eigenvalues, which are the values of $-\omega^2$ are real. The (natural) physical motion is then clear. There is a high frequency motion at $\omega = \sqrt{3}\,\omega_0$ in which the masses move opposed to one another and a low frequency motion at $\omega = \sqrt{3}\,\omega_0$ in which the masses move together. Since this is a linear system, the general solution is a linear sum of these motions. That is, the general solution is of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{A}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \exp(i\omega_0 t) + \frac{B}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \exp\left(i\sqrt{3}\omega_0 t\right),$$

where A and B are, in general, complex numbers. That is. there are four constants in the solution. There are also four initial conditions, which are the initial positions and velocities of the masses.

2.21. Consider a mass sliding without friction inside a sphere. We have drawn the picture here.



Mass Inside Sphere. Here a mass slides without friction inside a sphere, which we choose to be of glass so the motion can be observed. The coordinates are spherical (ρ, ϑ, ϕ) with the polar angle ϕ measured from zero in the central horizontal plane.

Study the motion by incorporating all constraints directly into the Lagrangian. Find the equilibrium orbit. Study small perturbations around this orbit.

Solution:

The kinetic energy is

$$T = \frac{1}{2} m \left[\dot{\rho}^2 + \rho^2 \left(\cos^2\phi\right) \dot{\vartheta}^2 + \rho^2 \dot{\phi}^2\right],$$

where ϕ is measured as positive downward from the center of the sphere. Since the radius of the sphere is constant

$$\rho = R,$$

and the kinetic energy is

$$T = \frac{mR^2}{2} \left[\left(\cos^2\phi\right) \dot{\vartheta}^2 + \dot{\phi}^2\right].$$

The potential energy is

$$V = -mgR \sin\phi,$$

since ϕ is measured from the horizontal plane at the center of the sphere. The Lagrangian is then

$$L = \frac{mR^2}{2}\left[\left(\cos^2\phi\right)\dot{\vartheta}^2 + \dot{\phi}^2\right] + mgR\sin\phi.$$

This Lagrangian specifies the complete problem. We note that this is cyclic in ϑ and the time, t. Therefore the angular momentum,

$$p_\vartheta = \frac{\partial L}{\partial\dot{\vartheta}} = mR^2\left(\cos^2\phi\right)\dot{\vartheta} = \ell$$

is a constant. That is

$$\dot{\vartheta} = \frac{\ell}{mR^2\left(\cos^2\phi\right)}.$$

We can then write the Lagrangian as

$$\begin{aligned} L &= \frac{mR^2}{2}\left(\cos^2\phi\right)\left(\frac{\ell}{mR^2\left(\cos^2\phi\right)}\right)^2 + \frac{mR^2}{2}\dot{\phi}^2 + mgR\sin\phi \\ &= \frac{\ell^2}{2mR^2\cos^2\phi} + \frac{mR^2}{2}\dot{\phi}^2 + mgR\sin\phi \end{aligned}$$

The energy,

$$\mathcal{E} = \frac{\ell^2}{2mR^2\cos^2\phi} + \frac{mR^2}{2}\dot{\phi}^2 - mgR\sin\phi$$

is also a constant. We have only a single Euler-Lagrange equation. With

$$\frac{\partial L}{\partial\phi} = \frac{\ell^2}{mR^2\cos^3\phi}\sin\phi + mgR\cos\phi$$

and

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mR^2\dot{\varphi}.$$

the Euler-Lagrange equation is

$$
\begin{aligned}
&\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) \\
&= \frac{\ell^2}{mR^2 \cos^3 \varphi} \sin\phi + mgR\cos\phi - mR^2\ddot{\phi} = 0.
\end{aligned}
$$

This is a complicated, nonlinear differential equation. We may find graphical solutions. But analytical solutions do not exist. Nevertheless we can study situations of interest. For example we expect to find an equilibrium solution with the mass moving in a circle in a plane some distance below the central plane.

The equilibrium solution is found when $\ddot{\phi} = 0$. That is,

$$
\sin\phi + \frac{m^2 g R^3}{\ell^2} \cos^4 \phi = 0.
$$

We may convert this to an expression involving only sines of ϕ using

$$
\cos^2 \phi = 1 - \sin^2 \phi.
$$

Then, for $\ell \neq 0$,

$$
\begin{aligned}
&\sin\phi + \frac{m^2 g R^3}{\ell^2}\left(1 - \sin^2\phi\right)^2 \\
&= \sin\phi + m^2 g \frac{R^3}{\ell^2} - 2m^2 g \frac{R^3}{\ell^2} \sin^2\phi + m^2 g \frac{R^3}{\ell^2} \sin^4\phi \\
&= 0.
\end{aligned}
$$

This can be solved numerically for $\sin\phi$ if values are provided for the mass, radius, and angular momentum. We do note, however, that for infinite angular momentum $\sin\phi = 0$ which means $\phi = 0$. The motion is then circular in the central horizontal plane. For zero angular momentum we must go back to the original Euler-Lagrange equation, before we divided by ℓ, to obtain

$$
mgR\cos\phi = 0,
$$

or $\phi = \pi/2$. These two limits agree with our expectations.

We can consider small oscillations about equilibrium at a particular angle ϕ_0. Let us define the angle ε such that

$$
\phi = \phi_0 + \varepsilon.
$$

Then

$$
\begin{aligned}
\cos\phi &= \cos\left(\phi_0 + \varepsilon\right) \\
&= \cos\phi_0 - \varepsilon \sin\phi_0,
\end{aligned}
$$

and

$$\sin\phi = \sin\left(\phi_0 + \varepsilon\right)$$
$$= \sin\phi_0 + \varepsilon\cos\phi_0,$$

for small ε. The Euler-Lagrange equation is then

$$\frac{1}{mR^2\left(\cos\phi_0 - \varepsilon\sin\phi_0\right)^3}\ell^2\left(\sin\phi_0 + \varepsilon\cos\phi_0\right) + mgR\left(\cos\phi_0 - \varepsilon\sin\phi_0\right) - mR^2\ddot{\varepsilon} = 0$$

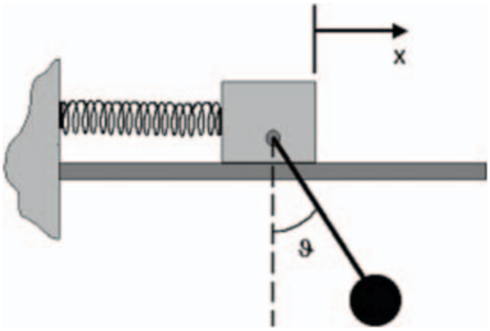
Expanding to first order in ε we have

$$\ddot{\varepsilon} = \left(\frac{g}{R}\cos\phi_0 + \frac{1}{m^2R^4\cos^3\phi_0}\ell^2\sin\phi_0\right)$$
$$-\left(\frac{g}{R}\sin\phi_0 + \left(\frac{\ell}{mR^2}\right)^2\frac{2\cos^2\phi_0 - 3}{\cos^4\phi_0}\right)\varepsilon.$$

The motion around the equilibrium point is then sinusoidal with a frequency

$$\omega = \sqrt{\left(\frac{g}{R}\sin\phi_0 + \left(\frac{\ell}{mR^2}\right)^2\frac{2\cos^2\phi_0 - 3}{\cos^4\phi_0}\right)}.$$

2.22. Consider the block, spring and pendulum system shown here.



Block, spring, and pendulum

Obtain the Euler-Lagrange equations for this system. Then simplify for equal masses ($M = m$). Consider small vibrations (small x and ϑ). Make the Ansatz that the time dependence is $\exp(i\omega t)$ and find the normal modes of motion.

Solution:

The position vectors for the center of mass of the block $\boldsymbol{r}_1$ and the pendulum bob $\boldsymbol{r}_2$ are

$$\boldsymbol{r}_1 = (a + x)\,\hat{e}_{\mathrm{x}}$$

$$\boldsymbol{r}_2 = (a + x + \ell\sin\vartheta)\,\hat{e}_{\mathrm{x}} - \ell\cos\vartheta\,\hat{e}_{\mathrm{y}}.$$

The velocities are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_1 &= (\dot{x})\,\hat{e}_\mathrm{x} \\ &= \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_2 &= \left(\dot{x} + \ell\dot{\vartheta}\cos\vartheta\right)\hat{e}_\mathrm{x} + \ell\dot{\vartheta}\sin\vartheta\,\hat{e}_\mathrm{y}. \\ &= \begin{bmatrix} \dot{x} + \ell\dot{\vartheta}\cos\vartheta \\ \ell\dot{\vartheta}\sin\vartheta \end{bmatrix},\end{aligned}$$

where we have introduced a matrix representation for the vectors. The squares of the velocities are

$$\begin{aligned}\dot{r}_1^2 &= \begin{bmatrix} \dot{x} & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \\ &= \dot{x}^2\end{aligned}$$

and

$$\begin{aligned}\dot{r}_2^2 &= \begin{bmatrix} \dot{x} + \ell\dot{\vartheta}\cos\vartheta & \ell\dot{\vartheta}\sin\vartheta \end{bmatrix}\begin{bmatrix} \dot{x} + \ell\dot{\vartheta}\cos\vartheta \\ \ell\dot{\vartheta}\sin\vartheta \end{bmatrix} \\ &= \dot{x}^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta}\cos\vartheta.\end{aligned}$$

Then the kinetic energy is

$$\begin{aligned}T &= \frac{1}{2}M\dot{r}_1^2 + \frac{1}{2}m\dot{r}_2^2 \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta}\cos\vartheta\right).\end{aligned}$$

The potential energy is

$$V = \frac{1}{2}kx^2 - mg\ell\cos\vartheta.$$

The Lagrangian is then

$$\begin{aligned}L &= T - V \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta}\cos\vartheta\right) - \frac{1}{2}kx^2 + mg\ell\cos\vartheta.\end{aligned}$$

From here the partial derivatives we need are

$$\frac{\partial L}{\partial\vartheta} = -m\ell\dot{x}\dot{\vartheta}\sin\vartheta - mg\ell\sin\vartheta$$

$$p_\vartheta = \frac{\partial L}{\partial\dot{\vartheta}} = m\left(\ell^2\dot{\vartheta} + \ell\dot{x}\cos\vartheta\right)$$

$$\frac{\partial L}{\partial x} - kx$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\,\dot{x} + m\ell\dot{\vartheta}\cos\vartheta$$

The Euler-Lagrange equations are then

$$\begin{aligned}\frac{\partial L}{\partial \vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\vartheta}}\right) &= 0 = -m\ell\dot{x}\dot{\vartheta}\sin\vartheta - mg\ell\sin\vartheta - m\frac{d}{dt}\left(\ell^2\dot{\vartheta} + \ell\dot{x}\cos\vartheta\right)\\ &= -m\ell\dot{x}\dot{\vartheta}\sin\vartheta - mg\ell\sin\vartheta - m\left(\ell^2\ddot{\vartheta} + \ell\ddot{x}\cos\vartheta - \ell\dot{x}\dot{\vartheta}\sin\vartheta\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= 0 = -kx - \frac{\mathrm{d}}{\mathrm{d}t}\left((M+m)\,\dot{x} + m\ell\dot{\vartheta}\cos\vartheta\right)\\ &= -kx - \left((M+m)\,\ddot{x} + m\ell\ddot{\vartheta}\cos\vartheta - m\ell\dot{\vartheta}^2\sin\vartheta\right).\end{aligned}$$

We can simplify these equations by choosing the masses equal and by considering small vibrations. That is, we assume that ϑ and x and all derivatives are small. Neglecting product terms in these quantities the equations are then (with $M = m$)

$$\begin{aligned}&-m\ell\dot{x}\dot{\vartheta}\vartheta - mg\ell\vartheta - m\left(\ell^2\ddot{\vartheta} + \ell\ddot{x} - \ell\dot{x}\dot{\vartheta}\vartheta\right)\\ &= -m\ell\left(\ddot{x} + \ell\ddot{\vartheta} + \ell\frac{g}{\ell}\vartheta\right) = 0\end{aligned}$$

and

$$\begin{aligned}&-kx - \left(2m\ddot{x} + m\ell\ddot{\vartheta} - m\ell\dot{\vartheta}^2\vartheta\right)\\ &= -m\left(\frac{k}{m}x + 2\ddot{x} + \ell\ddot{\vartheta}\right) = 0.\end{aligned}$$

Introducing the natural frequency for the pendulum,

$$\omega_p = \sqrt{\frac{g}{\ell}}$$

and for the spring-mass

$$\omega_s = \sqrt{\frac{k}{m}},$$

we have

$$\ddot{x} + \ell\ddot{\vartheta} + \ell\omega_p^2\vartheta = 0$$

$$2\ddot{x} + \omega_s^2 x + \ell\ddot{\vartheta} = 0.$$

Making the Ansatz that the solutions are complex exponentials

$$x = X\exp(i\omega t)$$

and

$$\vartheta = \Theta \exp(i\omega t).$$

With this Ansatz, the equations are

$$-\omega^2 X + \ell\left(-\omega^2 + \omega_\mathrm{p}^2\right)\Theta = 0$$

$$\left(-2\omega^2 + \omega_\mathrm{s}^2\right)X - \ell\omega^2\Theta = 0.$$

In matrix form,

$$\begin{bmatrix} \omega^2 & -\omega^2 + \omega_\mathrm{p}^2 \\ -2\omega^2 + \omega_\mathrm{s}^2 & \omega^2 \end{bmatrix}\begin{bmatrix} X \\ \ell\Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We have incorporated the length of the pendulum, ℓ, into the angle definition for convenience. For a non-trivial solution the determinant of the principal matrix must vanish. That is

$$\omega^4 - \left(\omega_\mathrm{p}^2 - \omega^2\right)\left(\omega_\mathrm{s}^2 - 2\omega^2\right) = 0.$$

We solve this as a quadratic equation for

$$\zeta = \omega^2.$$

That is

$$\zeta^2 - \left(\omega_\mathrm{p}^2 - \zeta\right)\left(\omega_\mathrm{s}^2 - 2\zeta\right) = 0.$$

The solutions are

$$\zeta = \omega^2 = \begin{cases} \omega_\mathrm{p}^2 + \frac{1}{2}\omega_\mathrm{s}^2 + \frac{1}{2}\sqrt{4\omega_\mathrm{p}^4 + \omega_\mathrm{s}^4} \\ \omega_\mathrm{p}^2 + \frac{1}{2}\omega_\mathrm{s}^2 - \frac{1}{2}\sqrt{4\omega_\mathrm{p}^4 + \omega_\mathrm{s}^4} \end{cases}.$$

We can find the corresponding motions by inserting these solutions back into the original problem.

For $\omega^2 = \omega_\mathrm{p}^2 + \frac{1}{2}\omega_\mathrm{s}^2 + \frac{1}{2}\sqrt{4\omega_\mathrm{p}^4 + \omega_\mathrm{s}^4}$,

$$\frac{\ell\Theta}{X} = -\frac{\left(\omega_{\mathrm{p}}^2 + \frac{1}{2}\omega_{\mathrm{s}}^2 + \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}\right)}{\left(\frac{1}{2}\omega_{\mathrm{s}}^2 + \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}\right)}$$

$$= -\left(1 + \frac{\omega_{\mathrm{p}}^2}{\frac{1}{2}\omega_{\mathrm{s}}^2 + \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}}\right).$$

For $\omega^2 = \omega_{\mathrm{p}}^2 + \frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}$,

$$\frac{\ell\Theta}{X} = -\frac{\omega_{\mathrm{p}}^2 + \frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)}}{\frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)}}$$

$$= -\left(1 + \frac{\omega_{\mathrm{p}}^2}{\frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)}}\right).$$

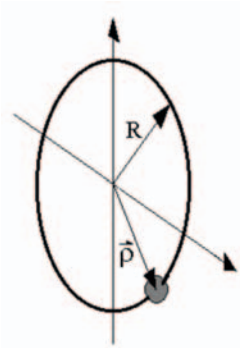
The first of these is the high frequency mode and the second is the low frequency mode. For both the motion of the block and the pendulum are opposed. That is $\frac{\ell\Theta}{X} < 0$ in both cases. However, because

$$\frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)} < 0,$$

$$\left(1 + \frac{\omega_{\mathrm{p}}^2}{\frac{1}{2}\omega_{\mathrm{s}}^2 + \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)}}\right) > \left(1 + \frac{\omega_{\mathrm{p}}^2}{\frac{1}{2}\omega_{\mathrm{s}}^2 - \frac{1}{2}\sqrt{\left(4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4\right)}}\right),$$

The high frequency mode has a higher amplitude than the low frequency mode. The general motion is, of course, a combination of these.

2.23. In the figure below we have drawn a stationary wire loop with a bead of mass m. The bead is free to move with no friction on the wire.



Bead on frictionless, stationary wire loop.

Find the Euler-Lagrange equations. Do not attempt a solution.
Solution:
In spherical coordinates the velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\boldsymbol{r}} = \dot{\rho}\hat{e}_\rho + (\rho \sin\phi)\,\dot{\vartheta}\,\hat{e}_\vartheta + \rho\dot{\phi}\hat{e}_\phi.$$

Then

$$\dot{r}^2 = \dot{\rho}^2 + \rho^2\dot{\vartheta}^2 \sin^2\phi + \rho^2\dot{\phi}^2$$

and the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2 \sin^2\phi + \rho^2\dot{\phi}^2\right).$$

For the situation with $\rho = R =$ constant, and $\vartheta = 0$ we have

$$T = \frac{1}{2}m\left(R^2\dot{\phi}^2\right).$$

The gravitational Potential energy is

$$V = mgR\cos\phi$$

So the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(R^2\dot{\phi}^2\right) - mgR\cos\phi \end{aligned}$$

There is only one coordinate and only one Euler-Lagrange equation.

$$\frac{\partial L}{\partial \phi} = mgR\sin\phi$$
$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mR^2\dot\phi.$$

Then

$$\frac{\partial L}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\phi} = mgR\sin\phi - mR^2\ddot\phi = 0$$

That is

$$\ddot\phi = \frac{g}{R}\sin\phi$$

where ϕ is measured from the top of the circular wire. We may choose to measure this from the bottom, which is the normal measurement for the simple pendulum, by adding π to ϕ. Since $\sin(\pi+\phi) = -\sin\phi$, we have then

$$\ddot\phi = -\frac{g}{R}\sin\phi,$$

which is he equation of motion for the simple pendulum.

2.24. Consider now that the loop in the preceding exercise rotates at a constant angular velocity about the vertical axis. That is $\dot\vartheta = \Omega =$ constant and $\rho = R =$ constant.

Find the Euler-Lagrange equations. What is the equilibrium location of the bead? Show that the equilibrium is stable, that is small deviations form equilibrium result in sinusoidal oscillations around equilibrium, provided

$$1 + \frac{g}{R\Omega^2} - 2\left(\frac{g}{R\Omega^2}\right)^2 > 0.$$

Solution:

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot\rho^2 + \rho^2\dot\vartheta^2\sin^2\phi + \rho^2\dot\phi^2\right).$$

With $\dot\vartheta = \Omega =$ constant and $\rho = R =$ constant,

$$T = \frac{1}{2}mR^2\left(\Omega^2\sin^2\phi + \dot\phi^2\right).$$

The potential energy is

$$V = mgR\cos\phi$$

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m R^2 \left(\Omega^2 \sin^2 \phi + \dot{\phi}^2 \right) - mgR \cos \phi \end{aligned}$$

There is again only one coordinate. So we have only one Euler-Lagrange equation. The derivatives we need are

$$\begin{aligned} \frac{\partial L}{\partial \phi} &= mR^2 \Omega^2 \sin \phi \cos \phi + mgR \sin \phi \\ p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = mR^2 \dot{\phi}. \end{aligned}$$

Then

$$\begin{aligned} &\frac{\partial L}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\phi}} \\ &= mR^2 \Omega^2 \sin \phi \cos \phi + mgR \sin \phi - mR^2 \ddot{\phi} = 0 \end{aligned}$$

or

$$mR^2 \ddot{\phi} = mR^2 \Omega^2 \sin \phi \cos \phi + mgR \sin \phi.$$

We may again choose to measure ϕ from the bottom of the circular wire by adding π to ϕ. Noting that $\sin(\pi + \phi) = -\sin \phi$ and $\cos(\pi + \phi) = -\cos \phi$, we have

$$mR^2 \ddot{\phi} = mR^2 \Omega^2 \sin \phi \cos \phi - mgR \sin \phi.$$

The bead is in equilibrium if $\ddot{\phi} = 0$. This requires that

$$R \Omega^2 \sin \phi_0 \cos \phi_0 = g \sin \phi_0,$$

where we have designated the equilibrium angle as ϕ_0. Then

$$\cos \phi_0 = \frac{g}{R \Omega^2}.$$

That is, the equilibrium position for the bead is at an angle ϕ_0 given by

$$\phi_0 = \cos^{-1} \left(\frac{g}{R \Omega^2} \right).$$

To study oscillations about equilibrium we write $\phi = \phi_0 + \varepsilon$ and linearize the Euler-Lagrange equation for small ε. Noting that

$$\begin{aligned}\sin\phi\cos\phi &= \sin\left(\phi_0+\varepsilon\right)\cos\left(\phi_0+\varepsilon\right)\\ &= \cos\phi_0\sin\phi_0+\varepsilon\left(\cos^2\phi_0-\sin^2\phi_0\right)\\ &= \cos\phi_0\sqrt{1-\cos^2\phi_0}+\varepsilon\left(2\cos^2\phi_0-1\right)\\ &= \frac{g}{R\Omega^2}\sqrt{1-\left(\frac{g}{R\Omega^2}\right)^2}+\varepsilon\left[2\left(\frac{g}{R\Omega^2}\right)^2-1\right],\end{aligned}$$

we have

$$\ddot{\phi}=\ddot{\varepsilon}=\frac{g}{R}\sqrt{1-\left(\frac{g}{R\Omega^2}\right)^2}-\Omega^2\left[1+\frac{g}{R\Omega^2}-2\left(\frac{g}{R\Omega^2}\right)^2\right]\varepsilon.$$

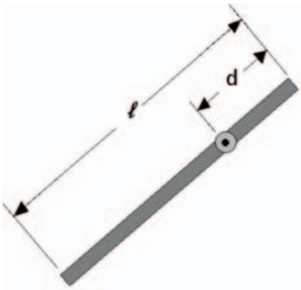
Now

$$1+\frac{g}{R\Omega^2}-2\left(\frac{g}{R\Omega^2}\right)^2>0$$

and the solution for ε is sinusoidal provided $g/R\Omega^2<1$, which is a condition on the rotational velocity. The frequency is

$$\Omega\sqrt{1+\frac{g}{R\Omega^2}-2\left(\frac{g}{R\Omega^2}\right)^2}$$

2.25. A physical pendulum is a uniform rod suspended on an axis constrained to move in one plane about a point other than the center of the rod. In the figure below we have shown a physical pendulum.



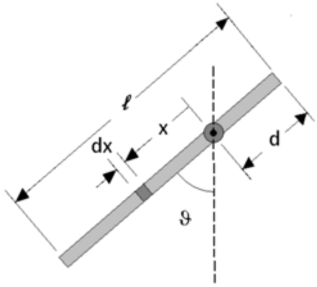
Physical pendulum.

Write the Lagrangian for this physical pendulum. Begin by writing the Lagrangian for a differential mass located at a distance x from the axis and then integrate over the rod.

Find the Euler-Lagrange equation for this physical pendulum.

Solution:

In the figure here we have indicated a segment dx of the rod at a distance x from the pivot.



Physical pendulum with differential segment dx.

The mass of the segment is d$m = \lambda$dx where λ is the mass per unit length of the rod. The square of the velocity of this segment is

$$x^2\dot{\vartheta}^2,$$

using polar (cylindrical) coordinates. The kinetic energy of the segment is

$$\mathrm{d}T = \mathrm{d}m\left(\frac{1}{2}x^2\dot{\vartheta}^2\right).$$

The potential energy of the segment is

$$\mathrm{d}V = -\mathrm{d}m\,(gx\cos\vartheta)$$

for segments on the lower side, i.e. below the pivot, and

$$\mathrm{d}V = +\mathrm{d}m\,(gx\cos\vartheta)$$

for those above. Then the contribution to the Lagrangian for segments below the pivot is

$$\begin{aligned}\mathrm{d}L_{\text{below}} &= \mathrm{d}m\left[\frac{1}{2}x^2\dot{\vartheta}^2 + (gx\cos\vartheta)\right]\\ &= \lambda\mathrm{d}x\left[\frac{1}{2}x^2\dot{\vartheta}^2 + (gx\cos\vartheta)\right],\end{aligned}$$

and for those above,

$$\begin{aligned}\mathrm{d}L_{\text{above}} &= \mathrm{d}m\left[\frac{1}{2}x^2\dot{\vartheta}^2 - (gx\cos\vartheta)\right]\\ &= \lambda\mathrm{d}x\left[\frac{1}{2}x^2\dot{\vartheta}^2 - (gx\cos\vartheta)\right].\end{aligned}$$

We obtain the total Lagrangian by integrating. We integrate over dL_{below} from $0 \to (\ell - d)$ and over dL_{above} from $0 \to (d)$. Then

$$
\begin{aligned}
L &= \lambda \int_0^{\ell\text{-d}} \mathrm{d}x \left[\frac{1}{2}x^2\dot{\vartheta}^2 + (gx\cos\vartheta)\right] + \lambda \int_0^{\mathrm{d}} \mathrm{d}x \left[\frac{1}{2}x^2\dot{\vartheta}^2 - (gx\cos\vartheta)\right] \\
&= \lambda \left\{\frac{1}{6}x^3\dot{\vartheta}^2 + \frac{1}{2}gx^2\cos\vartheta\right\}_0^{\ell\text{-d}} + \lambda \left\{\frac{1}{6}x^3\dot{\vartheta}^2 - \frac{1}{2}gx^2\cos\vartheta\right\}_0^{\mathrm{d}} \\
&= m\left\{\frac{1}{6}\left[\ell^2 - 3\ell d + 3d^2\right]\dot{\vartheta}^2 + \frac{1}{2}g\left[\ell - 2d\right]\cos\vartheta\right\}.
\end{aligned}
$$

The Euler-Lagrange equation is found from

$$
\begin{aligned}
\frac{\partial L}{\partial \vartheta} &= -\frac{m}{2}g\left[\ell - 2d\right]\sin\vartheta \\
\frac{\partial L}{\partial \dot{\vartheta}} &= \frac{m}{3}\left[\ell^2 - 3\ell d + 3d^2\right]\dot{\vartheta}
\end{aligned}
$$

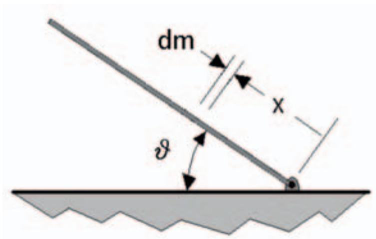
as

$$
\begin{aligned}
&\frac{\partial L}{\partial \vartheta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\vartheta}} \\
&= -\frac{m}{2}g\left[\ell - 2d\right]\sin\vartheta - \frac{m}{3}\left[\ell^2 - 3\ell d + 3d^2\right]\ddot{\vartheta} = 0
\end{aligned}
$$

or

$$
\ddot{\vartheta} = -\frac{3}{2}\frac{g\left[\ell - 2d\right]}{\left[\ell^2 - 3\ell d + 3d^2\right]}\sin\vartheta\,.
$$

2.26. Consider a uniform rod with linear mass density λ, which is fastened to the floor by a hinge. We have a drawing of the falling rod here.



Falling hinged rod.

We release the rod from the vertical with a slight nudge so that the angular momentum is initially zero. Obtain the time of fall as an integral. Do not attempt the integration.

[Hint: $\ddot{\vartheta} = (1/2)\, d\dot{\vartheta}^2/\mathrm{d}\vartheta$]

Solution:

The kinetic energy of the mass $\mathrm{d}m = \lambda \mathrm{d}x$ is

$$
\mathrm{d}T = \frac{1}{2}\mathrm{d}m\left(x^2\dot{\vartheta}^2\right)
$$

and the potential energy is

$$\mathrm{d}V = \mathrm{d}m\,(gx\sin\vartheta)\,.$$

The differential Lagrangian for dm is then

$$\mathrm{d}L = \frac{1}{2}\mathrm{d}m\left(x^2\dot{\vartheta}^2\right) - \mathrm{d}mgx\sin\vartheta\,.$$

The Lagrangian for the rod is found by integration

$$\begin{aligned}L &= \lambda\int_0^\ell \mathrm{d}x\left(\frac{1}{2}x^2\dot{\vartheta}^2 - gx\sin\vartheta\right)\\ &= \lambda\left[\frac{1}{6}x^3\dot{\vartheta}^2 - \frac{1}{2}gx^2\sin\vartheta\right]_0^\ell\\ &= m\left(\frac{1}{6}\ell^2\dot{\vartheta}^2 - \frac{1}{2}g\ell\sin\vartheta\right).\end{aligned}$$

The Euler-Lagrange equation is then found from

$$\begin{aligned}\frac{\partial L}{\partial\vartheta} &= -m\frac{1}{2}g\ell\cos\vartheta\\ p_\vartheta &= \frac{\partial L}{\partial\dot{\vartheta}} = m\frac{1}{3}\left(\ell^2\dot{\vartheta}\right),\end{aligned}$$

as

$$\begin{aligned}&\frac{\partial L}{\partial\vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\vartheta}}\\ &= -m\frac{1}{2}g\ell\cos\vartheta - m\frac{1}{3}\left(\ell^2\ddot{\vartheta}\right) = 0,\end{aligned}$$

or

$$\ddot{\vartheta} = -\frac{3}{2}\frac{g}{\ell}\cos\vartheta\,.$$

To solve for the motion we first note that

$$\begin{aligned}\ddot{\vartheta} &= \frac{\mathrm{d}}{\mathrm{d}t}\dot{\vartheta} = \frac{\mathrm{d}\vartheta}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\dot{\vartheta}\\ &= \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\dot{\vartheta} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\dot{\vartheta}^2.\end{aligned}$$

Then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\dot{\vartheta}^2 = -\frac{3}{2}\frac{g}{\ell}\cos\vartheta\,.$$

Integrating

$$\int\mathrm{d}\vartheta\frac{\mathrm{d}}{\mathrm{d}\vartheta}\dot{\vartheta}^2 = -3\frac{g}{\ell}\int\mathrm{d}\vartheta\cos\vartheta + K_1,$$

or

$$\dot{\vartheta}^2 = -3\frac{g}{\ell}\sin\vartheta + K_1.$$

At time $t = 0$ the angular velocity is zero. Then

$$K_1 = 3\frac{g}{\ell}$$

and

$$\dot{\vartheta} = \pm\sqrt{3\frac{g}{\ell}(1 - \sin\vartheta)}$$

Realizing that ϑ must decrease in time, we choose the negative sign.

$$\dot{\vartheta} = \frac{d\vartheta}{dt} = -\sqrt{3\frac{g}{\ell}(1 - \sin\vartheta)}.$$

Then

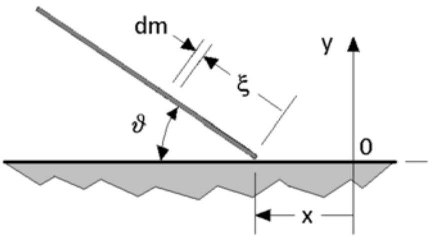
$$dt = -\frac{d\vartheta}{\sqrt{3\frac{g}{\ell}(1 - \sin\vartheta)}}$$

and

$$t = \int dt = -\int \frac{d\vartheta}{\sqrt{3\frac{g}{\ell}(1 - \sin\vartheta)}} + K_2.$$

This can be integrated. But the result involves elliptic integrals.

2.27. Now consider the falling rod as in the preceding exercise, except that instead of being hinged the end of the rod is free and the floor is frictionless. We again release the rod at $\vartheta = \pi/2$ with a slight nudge. We have drawn the rod in the figure below.



Free rod falling.

Find the Euler-Lagrange equations for the falling rod.
Solution:

First we locate the segment $\mathrm{d}m = \lambda \mathrm{d}\xi$ in the fixed coordinates (x, y). The location of the segment $\mathrm{d}m$ is

$$\boldsymbol{R}_{\mathrm{dm}} = (x + \zeta \cos\vartheta)\,\hat{e}_{\mathrm{x}} + (\zeta \sin\vartheta)\,\hat{e}_{\mathrm{y}}.$$

The velocity of the segment is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\boldsymbol{R}}_{\mathrm{dm}} = \left(\dot{x} - \zeta\dot{\vartheta}\sin\vartheta\right)\hat{e}_{\mathrm{x}} + \left(\zeta\dot{\vartheta}\cos\vartheta\right)\hat{e}_{\mathrm{y}}.$$

We notice that for the segment $\mathrm{d}m$ the locating length ζ is not a function of time. The kinetic energy of the segment $\mathrm{d}m$ is then

$$\begin{aligned}
\mathrm{d}T &= \frac{1}{2}\mathrm{d}m\left[\left(\dot{x} - \zeta\dot{\vartheta}\sin\vartheta\right)^2 + \left(\zeta\dot{\vartheta}\cos\vartheta\right)^2\right] \\
&= \frac{1}{2}\lambda d\xi\left[\dot{x}^2 + \zeta^2\dot{\vartheta}^2 - 2\zeta\dot{x}\dot{\vartheta}\sin\vartheta\right].
\end{aligned}$$

The potential energy of the segment $\mathrm{d}m$ is

$$\begin{aligned}
\mathrm{d}V &= \mathrm{d}mg\zeta\sin\vartheta \\
&= \lambda\mathrm{d}\xi g\zeta\sin\vartheta.
\end{aligned}$$

And the Lagrangian for the segment $\mathrm{d}m$ is

$$\mathrm{d}L = \lambda\mathrm{d}\xi\left[\frac{1}{2}\left(\dot{x}^2 + \zeta^2\dot{\vartheta}^2 - 2\zeta\dot{x}\dot{\vartheta}\sin\vartheta\right) - g\zeta\sin\vartheta\right].$$

We again integrate to get the total Lagrangian of the rod as

$$\begin{aligned}
L &= \lambda\int_0^{\ell}\mathrm{d}\xi\left[\frac{1}{2}\left(\dot{x}^2 + \zeta^2\dot{\vartheta}^2 - 2\zeta\dot{x}\dot{\vartheta}\sin\vartheta\right) - g\zeta\sin\vartheta\right] \\
&= \lambda\left[\frac{1}{2}\left(\dot{x}^2\xi + \frac{1}{3}\zeta^3\dot{\vartheta}^2 - \zeta^2\dot{x}\dot{\vartheta}\sin\vartheta\right) - \frac{1}{2}g\zeta^2\sin\vartheta\right]_{\zeta=0}^{\ell} \\
&= m\left[\frac{1}{2}\left(\dot{x}^2 + \frac{1}{3}\ell^2\dot{\vartheta}^2 - \ell\dot{x}\dot{\vartheta}\sin\vartheta\right) - \frac{1}{2}g\ell\sin\vartheta\right]
\end{aligned}$$

The Euler-Lagrange equations follow from

$$\begin{aligned}
\frac{\partial L}{\partial\vartheta} &= -\frac{1}{2}m\ell\dot{x}\dot{\vartheta}\cos\vartheta - \frac{1}{2}mg\ell\cos\vartheta \\
\frac{\partial L}{\partial\dot{\vartheta}} &= \frac{1}{3}m\ell^2\dot{\vartheta} - \frac{1}{2}m\ell\dot{x}\sin\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial\dot{x}} &= m\dot{x} - m\ell\dot{\vartheta}\sin\vartheta,
\end{aligned}$$

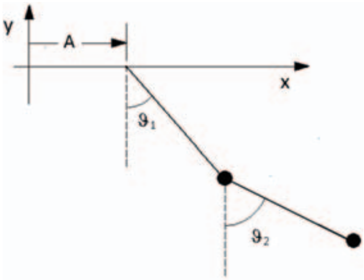
as

$$\begin{aligned}
&\frac{\partial L}{\partial \vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} \\
&= -\frac{1}{2}m\ell\dot{x}\dot{\vartheta}\cos\vartheta - \frac{1}{2}mg\ell\cos\vartheta - m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{3}\ell^2\dot{\vartheta} - \frac{1}{2}\ell\dot{x}\sin\vartheta\right) \\
&= -\frac{1}{2}mg\ell\cos\vartheta - m\left(\frac{1}{3}\ell^2\ddot{\vartheta} - \frac{1}{2}\ell\ddot{x}\sin\vartheta\right)
\end{aligned}$$

and

$$m\dot{x} - m\ell\dot{\vartheta}\sin\vartheta = \text{constant}$$

2.28. We have drawn a double pendulum in the figure here.



Double pendulum with equal lengths and bobs of equal mass.

Both pendulum lengths are ℓ and the masses of the pendulum bobs are both m. We consider the masses of the rods connecting the bobs to be zero.

Obtain the Euler-Lagrange equations, linearize these for small angles and find the normal modes of oscillation.

[Answers: for the Euler-Lagrange equations

$$\begin{aligned}
&-2m\ell^2\ddot{\vartheta}_1 - m\ell^2\ddot{\vartheta}_2(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2) \\
&-m\ell^2\dot{\vartheta}_2^2(-\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2) \\
&-2mg\ell\sin\vartheta_1 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
&-m\ell_1^2\ddot{\vartheta}(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2) - m\ell_2^2\ddot{\vartheta} \\
&-m\ell^2\dot{\vartheta}_1^2(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2) \\
&-mg\ell\sin\vartheta_2 \\
&= 0.
\end{aligned}$$

For the linearized equations

$$2\ddot{\vartheta}_1 + 2\omega_0^2\vartheta_1 + \ddot{\vartheta}_2 = 0$$

$$\ddot{\vartheta}_1 + \ddot{\vartheta}_2 + \omega_0^2\vartheta_2 = 0,$$

where $\omega_0 = \sqrt{g/\ell}$. For the normal modes

$$\omega = \pm\omega_0\sqrt{2+\sqrt{2}}$$
$$\omega = \pm\omega_0\sqrt{2-\sqrt{2}}$$

the eigenvectors are

$$\text{For } \omega = \omega_0\sqrt{\left(2+\sqrt{2}\right)}\text{:}\ \begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix} = \frac{2}{\sqrt{6}}\begin{bmatrix}-\frac{1}{2}\sqrt{2}\\ 1\end{bmatrix}$$

and

$$\text{For } \omega = \omega_0\sqrt{\left(2-\sqrt{2}\right)}\text{:}\ \begin{bmatrix}\Theta_1\\ \Theta_2\end{bmatrix} = \frac{2}{\sqrt{6}}\begin{bmatrix}\frac{1}{2}\sqrt{2}\\ 1\end{bmatrix}\Bigg].$$

Solution:

The position vectors to the masses are

$$\boldsymbol{R}_1 = [A + \ell\sin\vartheta_1]\,\hat{e}_\mathrm{x} + [-\ell\cos\vartheta_1]\,\hat{e}_\mathrm{y}$$

$$R_2 = [A + \ell\sin\vartheta_1 + \ell\sin\vartheta_2]\,\hat{e}_\mathrm{x} + [-\ell\cos\vartheta_1 - \ell\cos\vartheta_2]\,\hat{e}_\mathrm{y}$$

The velocity vectors are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}_1 = \ell\left[\dot{\vartheta}_1\cos\vartheta_1\right]\hat{e}_\mathrm{x} + \ell\left[\dot{\vartheta}_1\sin\vartheta_1\right]\hat{e}_\mathrm{y}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}R_2 = \ell\left[\dot{\vartheta}_1\cos\vartheta_1 + \dot{\vartheta}_2\cos\vartheta_2\right]\hat{e}_\mathrm{x} + \ell\left[\dot{\vartheta}_1\sin\vartheta_1 + \dot{\vartheta}_2\sin\vartheta_2\right]\hat{e}_\mathrm{y}.$$

The squares of these are

$$\dot{R}_1^2 = \ell^2\dot{\vartheta}_1^2$$

and

$$\begin{aligned}\dot{R}_2^2 &= \ell^2\left[\left(\dot{\vartheta}_1\cos\vartheta_1 + \dot{\vartheta}_2\cos\vartheta_2\right)^2 + \left(\dot{\vartheta}_1\sin\vartheta_1 + \dot{\vartheta}_2\sin\vartheta_2\right)^2\right]\\ &= \ell^2\left[\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right].\end{aligned}$$

From here we have the kinetic energy

$$T = \frac{1}{2}m\ell^2\left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right],$$

and the potential energy

$$V = -2mg\ell\cos\vartheta_1 - mg\ell\cos\vartheta_2.$$

The Lagrangian is then

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}m\ell^2\left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right] \\
&\quad +2mg\ell\cos\vartheta_1 + mg\ell\cos\vartheta_2.
\end{aligned}$$

The derivatives we need are

$$\begin{aligned}
\frac{\partial L}{\partial\vartheta_1} &= m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\left(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2\right) \\
&\quad -2mg\ell\sin\vartheta_1 \\
\frac{\partial L}{\partial\dot{\vartheta}_1} &= 2m\ell^2\dot{\vartheta}_1 + m\ell^2\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right) \\
\frac{\partial L}{\partial\vartheta_2} &= m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\left(-\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2\right) \\
&\quad -mg\ell\sin\vartheta_2 \\
\frac{\partial L}{\partial\dot{\vartheta}_2} &= m\ell^2\dot{\vartheta}_2 + m\ell^2\dot{\vartheta}_1\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)
\end{aligned}$$

Then the Euler-Lagrange equation for ϑ_1 is

$$\begin{aligned}
&m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\left(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2\right) - 2mg\ell\sin\vartheta_1 \\
&-\frac{\mathrm{d}}{\mathrm{d}t}\left[2m\ell^2\dot{\vartheta}_1 + m\ell^2\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right] \\
&= 0,
\end{aligned}$$

or

$$\begin{aligned}
&m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\left(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2\right) - 2mg\ell\sin\vartheta_1 \\
&-\left[2m\ell^2\ddot{\vartheta}_1 + m\ell^2\ddot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right] \\
&-m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\left(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2\right) \\
&-m\ell^2\dot{\vartheta}_2^2\left(-\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2\right) \\
&= 0,
\end{aligned}$$

or

$$\begin{aligned}
&-2m\ell^2\ddot{\vartheta}_1 - m\ell^2\ddot{\vartheta}_2\,(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2)\\
&-m\ell^2\dot{\vartheta}_2^2\,(-\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2)\\
&-2mg\ell\sin\vartheta_1\\
=\;&0.
\end{aligned}$$

And the ϑ_2-equation is

$$\begin{aligned}
&m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\,(-\cos\vartheta_1\sin\vartheta_2 + \sin\vartheta_1\cos\vartheta_2) - mg\ell\sin\vartheta_2\\
&-\frac{d}{dt}\left[m\ell^2\dot{\vartheta}_2 + m\ell^2\dot{\vartheta}_1\,(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2)\right]\\
=\;&0,
\end{aligned}$$

or

$$\begin{aligned}
&-m\ell_1^2\ddot{\vartheta}\,(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2) - m\ell_2^2\ddot{\vartheta}\\
&-m\ell^2\dot{\vartheta}_1^2\,(-\sin\vartheta_1\cos\vartheta_2 + \cos\vartheta_1\sin\vartheta_2)\\
&-mg\ell\sin\vartheta_2\\
=\;&0.
\end{aligned}$$

To linearize these we first set

$$\begin{aligned}
\cos\vartheta &= 1\\
\sin\vartheta &= \vartheta.
\end{aligned}$$

Then our equations are

$$\begin{aligned}
&-2m\ell^2\ddot{\vartheta}_1 - m\ell^2\ddot{\vartheta}_2\,(1 + \vartheta_1\vartheta_2)\\
&-m\ell^2\dot{\vartheta}_2^2\,(-\vartheta_2 + \vartheta_1)\\
&-2mg\ell\vartheta_1\\
=\;&0
\end{aligned}$$

and

$$\begin{aligned}
&-m\ell^2\ddot{\vartheta}_1\,(1 + \vartheta_1\vartheta_2) - m\ell^2\ddot{\vartheta}_2\\
&-m\ell^2\dot{\vartheta}_1^2\,(-\vartheta_1 + \vartheta_2)\\
&-mg\ell\vartheta_2\\
=\;&0
\end{aligned}$$

If we neglect all terms containing powers of ϑ and its derivatives these become

$$2\ddot{\vartheta}_1 + 2\omega_0^2\vartheta_1 + \ddot{\vartheta}_2 = 0$$

$$\ddot{\vartheta}_1 + \ddot{\vartheta}_2 + \omega_0^2\vartheta_2 = 0,$$

where $\omega_0 = \sqrt{g/\ell}$. With the Ansatz

$$\vartheta = \Theta\exp i\omega t$$

for both angles, we have

$$-2\omega^2\Theta_1 + 2\omega_0^2\Theta_1 - \omega^2\Theta_2 = 0$$
$$-\omega^2\Theta_1 - \omega^2\Theta_2 + \omega_0^2\Theta_2 = 0,$$

Or, in matrix form

$$\begin{bmatrix} 2\left(-\omega^2 + \omega_0^2\right) & -\omega^2 \\ -\omega^2 & -\omega^2 + \omega_0^2 \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Setting the determinant of the principal matrix to zero

$$2\left(-\omega^2 + \omega_0^2\right)^2 - \omega^4 = 0$$

or

$$\omega^4 + 2\omega_0^4 - 4\omega^2\omega_0^2 = 0.$$

Solving for ω we get the normal modes:

$$\omega = \pm\omega_0\sqrt{2 + \sqrt{2}}$$
$$\omega = \pm\omega_0\sqrt{2 - \sqrt{2}}.$$

That is we have a high and a low frequency mode. We next find the eigenvectors for these modes.

For $\omega = \omega_0\sqrt{\left(2 + \sqrt{2}\right)}$ we have, upon substitution

$$\begin{bmatrix} 2\left(1 + \sqrt{2}\right) / \left(2 + \sqrt{2}\right) & 1 \\ 1 & \left(1 + \sqrt{2}\right) / \left(2 + \sqrt{2}\right) \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \frac{2}{\sqrt{6}} \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

For $\omega = \omega_0\sqrt{\left(2 - \sqrt{2}\right)}$ we have, upon substitution

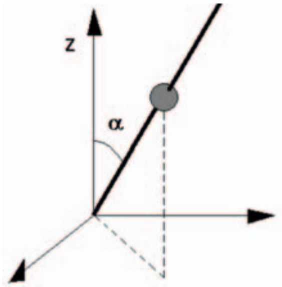
$$\begin{bmatrix} 2\left(1 - \sqrt{2}\right) / \left(2 - \sqrt{2}\right) & 1 \\ 1 & \left(1 - \sqrt{2}\right) / \left(2 - \sqrt{2}\right) \end{bmatrix} \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} = \frac{2}{\sqrt{6}} \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 1 \end{bmatrix}$$

In the first of these the masses move opposed and in the second they move together. Even when they move together there is a difference in the angles. The angle ϑ_2 is slightly larger.

2.29. In the figure below we have a bead moving without friction on a wire. The wire makes an angle α with the vertical and is free to rotate about the vertical axis, also without friction.



Bead on a frictionless wire.

We neglect the mass of the wire.

Show that motion can be described as that of a particle moving in an (effective) potential well

$$V_{\text{eff}} = -\frac{1}{2}\frac{\ell^2}{mr^2} + mg\frac{r}{\tan\alpha}$$

where ℓ is the angular momentum of the bead.

Is there a position of stable equilibrium? This requires consideration of both the radial velocity $\dot{r}$ and the radial acceleration $\ddot{r}$.

Show that the Lagrange Undetermined multiplier is

$$\lambda = -\frac{\ell^2}{mr^3\left(1+\tan^2\alpha\right)} - \frac{mg\tan\alpha}{1+\tan^2\alpha}$$

and that the forces of the wire on the bead are then

$$f_{\text{r}} = -\frac{\ell^2}{mr^3\left(1+\tan^2\alpha\right)} - \frac{mg\tan\alpha}{1+\tan^2\alpha}$$

and

$$f_{\text{z}} = \frac{\ell^2\tan\alpha}{mr^3\left(1+\tan^2\alpha\right)} + \frac{mg\tan^2\alpha}{1+\tan^2\alpha}.$$

Comment on the time dependence of the λ.

Solution:

We use cylindrical coordinates. The position vector to the bead is

$$\boldsymbol{R} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

The velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{r}\hat{e}_{\mathrm{r}} + r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right).$$

The only source of potential energy is the gravitational field. The potential energy is

$$V = mgz.$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) - mgz. \end{aligned}$$

In our first approach to this problem we incorporate the constraint directly into the Lagrangian. The constraint is

$$z = \frac{r}{\tan\alpha}$$

and the velocity relationship is

$$\dot{z} = \frac{\dot{r}}{\tan\alpha}.$$

Then the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \frac{\dot{r}^2}{\tan^2\alpha}\right) - mg\frac{r}{\tan\alpha} \\ &= \frac{1}{2}m\left(\left(\frac{1+\tan^2\alpha}{\tan^2\alpha}\right)\dot{r}^2 + r^2\dot{\vartheta}^2\right) - mg\frac{r}{\tan\alpha}. \end{aligned}$$

Now

$$\frac{1+\tan^2\alpha}{\tan^2\alpha} = \frac{1}{\sin^2\alpha}.$$

The Lagrangian is then

$$L = \frac{1}{2}m\left(\frac{\dot{r}^2}{\sin^2\alpha} + r^2\dot{\vartheta}^2\right) - mg\frac{r}{\tan\alpha}.$$

This Lagrangian is cyclic in ϑ, so the angular momentum of the bead, which we designate as ℓ, is constant.

$$\begin{aligned} p_\vartheta &= \frac{\partial L}{\partial\dot{\vartheta}} \\ &= mr^2\dot{\vartheta} = \ell. \end{aligned}$$

With this constant angular momentum the Lagrangian becomes

$$L = \frac{1}{2}m\frac{\dot{r}^2}{\sin^2\alpha} + \frac{1}{2}\frac{\ell^2}{mr^2} - mg\frac{r}{\tan\alpha}.$$

This is the Lagrangian for the radial motion of a point particle of mass m in an effective potential

$$V_{\text{eff}} = -\frac{1}{2}\frac{\ell^2}{mr^2} + mg\frac{r}{\tan\alpha}.$$

The derivative of this potential is always positive. There is then no relative minimum (or relative maximum) and, therefore, no position of stable equilibrium.

The Lagrangian does not depend explicitly on time, so the energy

$$\mathcal{E} = \frac{1}{2}m\frac{\dot{r}^2}{\sin^2\alpha} - \frac{1}{2}\frac{\ell^2}{mr^2} + mg\frac{r}{\tan\alpha}$$

is constant. We then have a single Euler-Lagrange equation.

From

$$\begin{aligned} \frac{\partial L}{\partial r} &= -\frac{\ell^2}{mr^3} - \frac{mg}{\tan\alpha} \\ \frac{\partial L}{\partial\dot{r}} &= \frac{m}{\sin^2\alpha}\dot{r}, \end{aligned}$$

the Euler-Lagrange equation in r (the $r-$equation) is

$$\begin{aligned} &\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial\dot{r}} \\ &= -\frac{\ell^2}{mr^3} - \frac{mg}{\tan\alpha} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m}{\sin^2\alpha}\dot{r}\right] \\ &= -\frac{\ell^2}{mr^3} - \frac{mg}{\tan\alpha} - \frac{m}{\sin^2\alpha}\ddot{r} \\ &= 0. \end{aligned}$$

Then we have the equation

$$-\frac{\ell^2}{mr^3}-\frac{mg}{\tan\alpha}-\frac{m}{\sin^2\alpha}\ddot{r}=0$$

and the constant energy

$$\mathcal{E}=\frac{1}{2}m\frac{\dot{r}^2}{\sin^2\alpha}-\frac{1}{2}\frac{\ell^2}{mr^2}+mg\frac{r}{\tan\alpha}$$

for our study of the motion. The constant energy is also just a first integral of the $r-$equation. So we really could study the motion with just the energy.

We release the bead at r_1 with no radial velocity, but a non-zero ℓ. Then the bead has an energy

$$\mathcal{E}=-\frac{1}{2}\frac{\ell^2}{mr_1^2}+mg\frac{r_1}{\tan\alpha}=\mathcal{E}_1,$$

which may be positive or negative. The energy is then

$$\begin{aligned}\mathcal{E}&=\frac{1}{2}m\frac{\dot{r}^2}{\sin^2\alpha}-\frac{1}{2}\frac{\ell^2}{mr^2}+mg\frac{r}{\tan\alpha}\\&=\mathcal{E}_1,\end{aligned}$$

from which we have

$$\frac{1}{2}m\frac{\dot{r}^2}{\sin^2\alpha}=\frac{1}{2}\frac{\ell^2}{m}\left(\frac{1}{r^2}-\frac{1}{r_1^2}\right)+mg\frac{(r_1-r)}{\tan\alpha}.$$

In order that the radial velocity is real, the radial position $r<r_1$. So the radial position at which we started the motion is the maximum radial distance possible. And at this radial position $\dot{r}=0$. The bead will remain at this position provided the radial acceleration is also zero at his point.

The radial Euler-Lagrange equation requires that

$$\frac{m}{\sin^2\alpha}\ddot{r}=-\frac{\ell^2}{mr^3}-\frac{mg}{\tan\alpha},$$

which is always negative. Therefore, even if we start the bead with no radial velocity at $r=r_1\neq 0$, it immediately begins to accelerate toward the $z-$axis. There is then no equilibrium position regardless of the magnitude of the angular momentum. So $\ddot{r}<0$ and the bead just falls down the wire as the wire increases angular velocity.

Using Lagrange Multipliers:

We define the constraint as

$$g=r-z\tan\alpha=0.$$

As before, the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) - mgz.$$

We still have conservation of angular momentum from the fact that the Lagrangian is cyclic in ϑ.

$$\begin{aligned} p_\vartheta &= \frac{\partial L}{\partial \dot{\vartheta}} \\ &= mr^2\dot{\vartheta} = \ell. \end{aligned}$$

So we may initially include this conservation in the Lagrangian.

$$L = \frac{1}{2}m\left(\dot{r}^2 + \frac{\ell^2}{m^2r^2} + \dot{z}^2\right) - mgz.$$

Writing the differential of the constraint

$$\begin{aligned} \mathrm{d}g &= \frac{\partial g}{\partial r}\mathrm{d}r + \frac{\partial g}{\partial z}\mathrm{d}z \\ &= \mathrm{d}r - (\tan\alpha)\,\mathrm{d}z = 0, \end{aligned}$$

we identify the partial derivatives

$$\begin{aligned} \frac{\partial g}{\partial r} &= 1 \\ \frac{\partial g}{\partial z} &= -\tan\alpha. \end{aligned}$$

We have the general form for the Euler Lagrange equations with Lagrange Multipliers as

$$\frac{\partial L}{\partial q_\mu} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_\mu} + \sum_\nu \lambda_\nu \frac{\partial g_\nu}{\partial q_\mu} = 0.$$

Noting that the $\vartheta-$equation has already produced the angular constancy of momentum we have only equations for r and z. These are

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial g}{\partial r} &= 0 \\ \frac{\partial L}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} + \lambda\frac{\partial g}{\partial z} &= 0, \end{aligned}$$

or

$$\begin{aligned} \frac{\ell^2}{mr^3} - m\ddot{r} + \lambda &= 0 \\ -mg - m\ddot{z} - \lambda\tan\alpha &= 0. \end{aligned}$$

We still have the constraint

$$r = z \tan \alpha.$$

These last three equations are three equations for the unknowns (λ, r, z).

Forces of Reaction

We identify the constraint forces as

$$f_{\mathrm{r}} = \lambda \frac{\partial g}{\partial r} = \lambda$$

and

$$f_{\mathrm{z}} = \lambda \frac{\partial g}{\partial z} = -\lambda \tan \alpha.$$

So our problem is to solve for λ. We can eliminate the $\ddot{r}$ and the $\ddot{z}$ terms using our constraint written as.

$$\ddot{r} = \ddot{z} \tan \alpha$$

The radial Euler-Lagrange equation becomes

$$m\ddot{z} \tan \alpha = \frac{\ell^2}{mr^3} + \lambda$$

and, from the $z-$equation, multiplying through by $\tan \alpha$,

$$m\ddot{z} \tan \alpha = -mg \tan \alpha - \lambda \tan^2 \alpha.$$

Equating these equations, we have

$$\frac{\ell^2}{mr^3} + \lambda = -mg \tan \alpha - \lambda \tan^2 \alpha.$$

Solving for λ

$$\lambda = -\frac{\ell^2}{mr^3 \left(1 + \tan^2 \alpha\right)} - \frac{mg \tan \alpha}{1 + \tan^2 \alpha}$$

Our forces are then

$$f_{\mathrm{r}} = -\frac{\ell^2}{mr^3 \left(1 + \tan^2 \alpha\right)} - \frac{mg \tan \alpha}{1 + \tan^2 \alpha}$$

and

$$f_z = \frac{\ell^2 \tan\alpha}{mr^3\left(1+\tan^2\alpha\right)} + \frac{mg\tan^2\alpha}{1+\tan^2\alpha}.$$

Recall that in our general development of the treatment in terms of Lagrange multipliers we realized that in general the Lagrange multiplier could be a function of the time. This was because of the fact that we introduced the constraints in terms of time integrations and generality demanded a possible time dependence. Here we see that time dependence in the fact that the λ depends on r, which depends on t.

2.30. In the preceding exercise the wire was free to rotate about the vertical axis, while the angle remained constant. We now choose to drive the wire at a constant angular velocity Ω about the vertical axis. The angle with the axis will still remain constant at α.

Incorporate the angular constraint and the constant angular velocity using Lagrange undetermined multipliers.

Solution:

The Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) - mgz. \end{aligned}$$

We now have two constraints. The first is the same as before for the wire, which we now identify with the subscript 1:

$$g_1 = r - (\tan\alpha)\, z = 0.$$

The second is the constancy of the angular velocity, which we identify with the subscript 2:

$$g_2 = \vartheta - \Omega\,(t - t_0) = 0.$$

This second constraint is actually

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t} = \dot{\vartheta} = \Omega,$$

which we have written in integrated form. Of course this constraint means that the angle ϑ is forever known. But we no longer have a conservation of angular momentum. So we should include the $\vartheta-$equation.

We have then equations for r and z as before, but now with two constraints

$$\begin{aligned} \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} + \lambda_1\frac{\partial g_1}{\partial r} + \lambda_2\frac{\partial g_2}{\partial r} &= 0 \\ \frac{\partial L}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} + \lambda_1\frac{\partial g_1}{\partial z} + \lambda_2\frac{\partial g_2}{\partial z} &= 0, \end{aligned}$$

and a $\vartheta-$equation

$$\frac{\partial L}{\partial \vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} + \lambda_1 \frac{\partial g_1}{\partial \vartheta} + \lambda_2 \frac{\partial g_2}{\partial \vartheta} = 0$$

Identifying the partial derivatives,

$$\frac{\partial g_1}{\partial r} = 1, \frac{\partial g_2}{\partial r} = 0$$
$$\frac{\partial g_1}{\partial z} = -\tan\alpha, \frac{\partial g_2}{\partial z} = 0$$
$$\frac{\partial g_1}{\partial \vartheta} = 0, \frac{\partial g_2}{\partial \vartheta} = 1$$

Our equations are then

$$\begin{aligned} mr\dot{\vartheta}^2 - m\ddot{r} + \lambda_1 &= 0 \\ -mg - m\ddot{z} - \lambda_1 \tan\alpha &= 0, \end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\vartheta}\right) + \lambda_2 = 0$$

Of course we retain our constraint equations

$$g_1 = r - (\tan\alpha)\, z = 0$$

and

$$g_2 = \vartheta - \Omega\,(t - t_0) = 0.$$

These last five equations now specify the problem in the coordinates $(r, \vartheta, z, \lambda_1, \lambda_2)$. If we incorporate the constraints into the time derivative equations, we have

$$\begin{aligned} m\Omega^2 z \tan\alpha - m\ddot{z}\tan\alpha + \lambda_1 &= 0 \\ -mg\tan\alpha - m\ddot{z}\tan\alpha - \lambda_1 \tan^2\alpha &= 0, \end{aligned}$$

and

$$2m\Omega\,(z)\,(\dot{z})\tan^2\alpha + \lambda_2 = 0.$$

A solution scheme presents itself immediately. Between the first two of these we can eliminate λ_1 obtaining an equation for $\ddot{z}$ in terms of z. The third equation allows us to use the solution for z to obtain a solution for λ_2. This can then be used to obtain r from the original form of this equation.

Eliminating $\ddot{z}$ in the first two equations,

$$m\Omega^2 z\tan\alpha + \lambda_1 = -mg\tan\alpha - \lambda_1\tan^2\alpha$$

$$m\Omega^2 z\tan\alpha + \lambda_1\left(1 + \tan^2\alpha\right) = -mg\tan\alpha$$

$$\lambda_1 = -\frac{m\Omega^2 z \tan\alpha + mg\tan\alpha}{\tan^2\alpha + 1}.$$

The first equation is then

$$m\Omega^2 z \tan\alpha - m\ddot{z}\tan\alpha - \frac{m\Omega^2 z + mg}{\tan^2\alpha + 1}\tan\alpha = 0$$

or

$$\begin{aligned}\ddot{z} &= \Omega^2\left(\frac{\tan^2\alpha}{\tan^2\alpha + 1}\right)z - \frac{g}{\tan^2\alpha + 1}\\ &= \frac{1}{\tan^2\alpha + 1}\left(\Omega^2\tan^2\alpha z - g\right).\end{aligned}$$

Then if

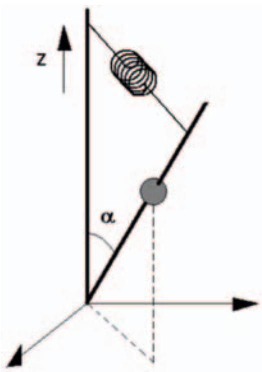
$$z < \frac{g}{\Omega^2\tan^2\alpha}$$

the bead slides down the wire to the origin. But if

$$z > \frac{g}{\Omega^2\tan^2\alpha}$$

the bead accelerates away from the central axis with an acceleration that increases with distance.

2.31. Consider the situation above with the wire mounted at the origin in a fashion that allows frictionless motion around the vertical and about the pivot point so that the angle to the vertical α varies. Let there be a vertical post erected from the origin. A spring retains the wire so that the pivot angle relative to this vertical post is limited. The spring is mounted at a distance h above the ground on the vertical post by a collar that permits rotation around the post. Consider small vibrations so that the spring remains parallel to the floor. The situation is shown below



Bead on a frictionless, massless wire with a spring tying the wire to a central pole. The height of the spring is h.

Assume that the spring is massless.
Study the motion. Is there an equilibrium orbit for the bead?
Solution:
We use cylindrical coordinates. The position vector to the bead is

$$\boldsymbol{R} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

The velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{r}\hat{e}_{\mathrm{r}} + r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right).$$

The sources of potential energy are the gravitational field and the spring. The potential energy from the gravitational field is

$$V = mgz.$$

The potential energy of the spring can be difficult. If the spring stretches too far it is no longer perpendicular to the axis. We shall then assume that the spring is always perpendicular to the axis and choose that to define how much the angle α can deviate from the value α_0 for which the spring is perpendicular to the central pole. If the spring length is x and the unextended length is x_0 then the spring extension is

$$x - x_0 = h\left(\tan\alpha - \tan\alpha_0\right).$$

Incorporating the constraint

$$\frac{r}{z} = \tan\alpha,$$

this is

$$x - x_0 = h\left(\frac{r}{z} - \tan\alpha_0\right).$$

We must have $z \neq 0$ because z appears in the denominator. The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) - mgz - \frac{1}{2}kh^2\left(\frac{r}{z} - \tan\alpha_0\right)^2. \end{aligned}$$

This Lagrangian is cyclic in ϑ and is not directly dependent on the time. Therefore the momentum

$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta}$$

is constant.

$$mr^2\dot{\vartheta} = \ell. = \text{constant.}$$

And the Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2\right) + \frac{1}{2}\frac{\ell^2}{mr^2} - mgz - \frac{1}{2}kh^2\left(\frac{r}{z} - \tan\alpha_0\right)^2.$$

Because the Lagrangian is explicitly independent of the time the energy is constant. That is

$$\mathcal{E} = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2\right) - \frac{1}{2}\frac{\ell^2}{mr^2} + mgz + \frac{1}{2}kh^2\left(\frac{r}{z} - \tan\alpha_0\right)^2,$$

There are still two other coordinates. The derivatives we need are

$$\frac{\partial L}{\partial r} = -\frac{\ell^2}{mr^3} - kh^2\frac{1}{z}\left(\frac{r}{z} - \tan\alpha_0\right),$$

$$\frac{\partial L}{\partial z} = -mg + kh^2\frac{1}{z^2}\left(\frac{r}{z} - \tan\alpha_0\right),$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r},$$

and

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}.$$

The Euler-Lagrange equations are

$$-\frac{\ell^2}{mr^3} - kh^2\frac{1}{z}\left(\frac{r}{z} - \tan\alpha_0\right) - m\ddot{r} = 0$$

$$-mg + kh^2\frac{1}{z^2}\left(\frac{r}{z} - \tan\alpha_0\right) - m\ddot{z} = 0.$$

To these we add the constancy of the energy

$$\mathcal{E} = \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2\right) - \frac{1}{2}\frac{\ell^2}{mr^2} + mgz + \frac{1}{2}kh^2\left(\frac{r}{z} - \tan\alpha_0\right)^2.$$

From the energy we see that the motion is that of a particle of mass m moving in an effective potential

$$V_{\text{eff}} = -\frac{1}{2}\frac{\ell^2}{mr^2} + mgz + \frac{1}{2}kh^2\left(\frac{r}{z} - \tan\alpha_0\right)^2.$$

If this potential has a relative minimum for values of r and z there will be an equilibrium orbit. We seek a local extremum of V_{eff} by setting the two partial derivatives equal to zero. The partial derivatives are

$$\frac{\partial V_{\text{eff}}}{\partial r} = \frac{\ell^2}{mr^3} + kh^2\frac{1}{z}\left(\frac{r}{z} - \tan\alpha_0\right)$$

and

$$\frac{\partial V_{\text{eff}}}{\partial z} = mg - \frac{1}{2}kh^2\frac{r}{z^2}\left(\frac{r}{z} - \tan\alpha_0\right).$$

If these are simultaneously zero,

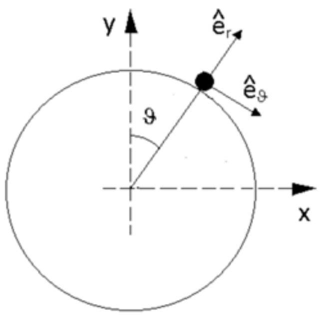
$$kh^2\left(\frac{r}{z} - \tan\alpha_0\right) = -\frac{\ell^2 z}{mr^3} = 2mg\frac{z^2}{r}$$

or

$$-\frac{\ell^2}{mr^2} = 2mgz.$$

This equation has no real solution. There is then no relative extremum in the accessible plane (r, z).

2.32. In the figure below we have drawn a cylinder of radius R lying on a laboratory table. Assume that the surface of the cylinder is frictionless. Its axis is parallel to the table top and the ground. The cylinder remains fixed. We then place a small mass m on the uppermost part of the cylinder. If we release the mass and nudge it slightly it will slide without friction on the cylinder.



Small mass on a frictionless cylinder.

At some point the small mass will fall off the cylinder. Find this point.
Solution:
The Lagrangian for the mass is

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - mgr\cos\vartheta.$$

We choose the reference point for the potential at the center of the cylinder. The energy is then

$$\mathcal{E} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + mgr\cos\vartheta.$$

Because the Lagrangian does not depend explicitly on time, the energy is constant. At the point of release the total energy is mgR. Then

$$mgR = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + mgr\cos\vartheta$$

We find the point when the mass leaves the cylinder by finding first the contact force and then equating that to zero. The contact force is found from the Lagrange multiplier for the constraint that the mass remain on the cylinder, which is

$$g = r - R = 0$$

Then

$$\mathrm{d}g = \mathrm{d}r = 0$$
$$\frac{\partial g}{\partial r} = 1.$$

The Euler-Lagrange equations are

$$mr\dot{\vartheta}^2 - mg\cos\vartheta - m\ddot{r} + \lambda = 0$$
$$mgr\sin\vartheta - m\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\vartheta}\right) = 0.$$

The contact force is $f_r = \lambda \partial g/\partial r = \lambda$. And $r = R$ as long as the mass is on the cylinder. The Euler-Lagrange equations for the motion of the mass while it remains on the cylinder are

$$mR\dot{\vartheta}^2 - mg\cos\vartheta + \lambda = 0$$
$$mgR\sin\vartheta - mR^2\ddot{\vartheta} = 0.$$

We can solve for λ from the first of these equations provided we have $\dot{\vartheta}^2$, which we can obtain from the energy with $r = R$.

$$\frac{2mg\,(1-\cos\vartheta)}{mR} = \dot{\vartheta}^2.$$

Then

$$2mg\,(1-\cos\vartheta) - mg\cos\vartheta + \lambda = 0.$$

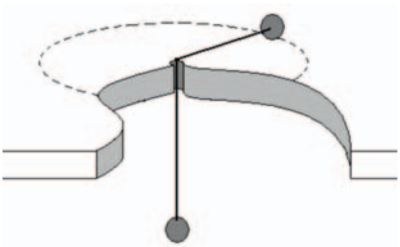
And we have

$$\lambda = -2mg + 3mg\cos\vartheta$$

The contact force vanishes and the mass leaves the cylinder when $\lambda = 0$. That is

$$\vartheta = \cos^{-1}\left(\frac{2}{3}\right).$$

2.33. Consider two balls connected by a string of length b. One is suspended through a hole in a table and the other moves on the (frictionless) top of the table. We have drawn the situation in the figure here.



Two balls connected by a string. One is suspended below a frictionless table and the other freely moves on the frictionless table.

Investigate the motion. Use Lagrange multipliers. Find an equilibrium point, if there is one. Study the general form of the motion. If you find a point of dynamic equilibrium, consider small oscillations about that point. Determine if the orbit is open or closed.

Solution:

We choose cylindrical coordinates. For the ball on the table the kinetic energy is

$$T_1 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right).$$

The suspended ball only moves in the z direction, so its kinetic energy is

$$T_2 = \frac{1}{2}m\dot{z}^2$$

The ball on the table experiences no change in potential energy. The system potential energy is solely from the suspended ball. If we choose the ground to be the table top, the potential energy is

$$V = -mgz.$$

The Lagrangian for the system is then

$$\begin{aligned} L &= T_1 + T_2 - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{1}{2}m\dot{z}^2 + mgz. \end{aligned}$$

The constraint is that the length b of the string is constant. This is

$$g = b - r - z = 0.$$

Then

$$\begin{aligned} \frac{\partial g}{\partial r} &= -1 \\ \frac{\partial g}{\partial \vartheta} &= 0 \\ \frac{\partial g}{\partial z} &= -1 \end{aligned}$$

Because the Lagrangian is cyclic in time, the energy is constant.

$$\mathcal{E} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{1}{2}m\dot{z}^2 - mgz$$

The Euler-Lagrange equations are then

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial g}{\partial r} = mr\dot{\vartheta}^2 - mg - \frac{\mathrm{d}}{\mathrm{d}t}\left(2m\dot{r}\right) - \lambda = 0.$$

$$\frac{\partial L}{\partial \vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} + \lambda\frac{\partial g}{\partial \vartheta} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\vartheta}\right) = 0.$$

$$\frac{\partial L}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} + \lambda\frac{\partial g}{\partial z} = mg - m\ddot{z} - \lambda = 0$$

The second of these is conservation of angular momentum for the ball on the table

$$mr^2\dot{\vartheta} = \ell = \text{ constant.}$$

If we include the constant angular momentum and using the constraint to eliminate z, the energy is

$$\mathcal{E} = \frac{1}{2}m\dot{r}^2 + \frac{1}{4mr^2}\ell^2 + \frac{mgr}{2}.$$

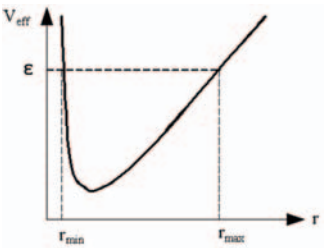
If we define an effective potential as

$$V_{\text{eff}}(r) = \frac{1}{4mr^2}\ell^2 + \frac{mgr}{2},$$

we have

$$\mathcal{E} = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r).$$

A general plot of $V_{\text{eff}}(r)$ is given below



V_{eff} for two balls connected by a string with one moving freely on a frictionless table and the other suspended through a hole in the table.

Clearly from the plot above there is an equilibrium point at the minimum of the effective potential. The radius of the orbit here is found from

$$\begin{aligned}\frac{\mathrm{d}V_{\text{eff}}(r)}{\mathrm{d}r} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{4mr^2}\ell^2 + \frac{mgr}{2}\right)\\ &= -\frac{1}{2mr^3}\ell^2 + \frac{1}{2}mg\\ &= 0.\end{aligned}$$

The real root of this is

$$r_0 = \left(\frac{\ell^2}{m^2 g}\right)^{1/3}$$

There are, certainly three roots, but the other two are complex. We ignore them as invalid physical solutions.

Let us consider now the motion near equilibrium. We define

$$\begin{aligned}\rho &= r - r_0 \\ &= r - \left(\frac{\ell^2}{m^2 g}\right)^{1/3},\end{aligned}$$

or

$$r = \rho + \left(\frac{\ell^2}{m^2 g}\right)^{1/3}.$$

Then we write V_{eff} in terms of ρ and expand around the point $\rho = 0$, which is the bottom of the well. Going to second order in ρ, we have

$$\begin{aligned}V_{\text{eff}} &= \frac{1}{4m\left[\rho + \left(\ell^2/m^2 g\right)^{1/3}\right]^2}\ell^2 + \frac{mg\left[\rho + \left(\ell^2/m^2 g\right)^{1/3}\right]}{2} \\ &= \left(\frac{1}{2}gm\left(\frac{\ell^2}{gm^2}\right)^{1/3} + \frac{1}{4m}\frac{\ell^2}{\left(\ell^2/gm^2\right)^{2/3}}\right) \\ &\quad + \frac{3}{4m}\frac{\ell^2}{\left(\ell^2/gm^2\right)^{4/3}}\rho^2 + O\left(\rho^3\right)\end{aligned}$$

There is no first order term because the first derivative vanishes at $\rho = 0$. The Euler-Lagrange equation for ρ is

$$\begin{aligned}\ddot{\rho} &= -\frac{1}{m}\frac{\mathrm{d}V_{\text{eff}}(\rho)}{\mathrm{d}\rho} \\ &= -\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\frac{3}{4m^2}\frac{\ell^2}{\left(\ell^2/gm^2\right)^{4/3}}\rho^2\right] \\ &= -\frac{3}{2m^2}\frac{\ell^2}{\left(\ell^2/gm^2\right)^{4/3}}\rho\end{aligned}$$

This is the equation for sinusoidal motion at the angular frequency

$$\omega_0 = \sqrt{\frac{3}{2m^2}\frac{\ell^2}{\left(\ell^2/gm^2\right)^{4/3}}}$$

The period for the radial motion τ_{r} is found from the requirement that in one period the angle has advanced through 2π radians. That is

$$\tau_{\text{r}} = \frac{2\pi m}{\ell\sqrt{\frac{3}{2\left(\ell^2/gm^2\right)^{4/3}}}}$$

At the equilibrium point the motion is circular. The angular velocity at equilibrium is

$$\dot{\vartheta}_0 = \frac{\ell}{mr_0^2}$$
$$= \frac{\ell}{m\left(\ell^2/m^2 g\right)^{2/3}}$$

The period for this circular motion, that is the time taken for one revolution, is

$$\tau_\vartheta = \frac{2\pi}{\dot{\vartheta}_0}$$
$$= \frac{2\pi m}{\ell}\left(\ell^2/m^2 g\right)^{2/3}$$

The orbit is called closed if the radial period τ_r is an integral number of angular periods τ_ϑ . If this is not the case the orbit will not close on itself and will be referred to as open. The orbits of the planets, with the exception, of mercury, which we shall speak of later, are closed. They are ellipses.

To see if the orbit of our ball on the table top is closed, we try to find n such that

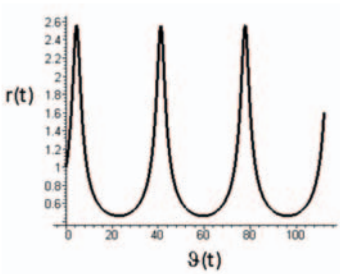
$$\tau_\mathrm{r} = n\tau_\vartheta$$

or

$$n = \sqrt{\frac{2}{3}}$$

which is not an integer. The orbit is, therefore, open.

Because we have considerable computing power at hand, we should look at this in some detail. We can solve the problem numerically for the relationship between $r(t)$ and $\vartheta(t)$. Doing so and plotting we have



Radial distance vs angle for ball on table, with second ball suspended.

In this plot we have started the ball at a radial distance of one meter from the center. We see that it can, with the angular velocity chosen, move out to about 2.6 meters and then in to about 0.5 meters. And this radial motion is also periodic in the time.

2.34. In a rocket engine the thermal energy of the burning fuel and oxidant is converted in the nozzle into kinetic energy. This high energy gas is expelled. The momentum carried away by this expelled gas results in an increase in momentum of the rocket. Consider a rocket for which

$$\begin{aligned} m_{\mathrm{r}} &= \text{ mass of the rocket excluding fuel} \\ m_{\mathrm{f}} &= \text{ mass of the fuel at any instant} \\ m_{\mathrm{e}} &= \text{ mass of exhaust gases in the nozzle at any time} \\ \dot{m} &= \text{ rate at which fuel is burned.} \end{aligned}$$

Let

$$\begin{aligned} v &= \text{ velocity of the rocket} \\ u &= \text{ velocity of exhaust gases in space} \\ \dot{v} &= \text{ acceleration of the rocket.} \end{aligned}$$

If we consider that the rocket is in a region of space in which all forces may be neglected, obtain the Euler-Lagrange equation for the rocket. This will be the standard propulsion equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p = M\dot{v} - \dot{m}U = 0,$$

where

$$M = m_{\mathrm{r}} + m_{\mathrm{f}} + m_{\mathrm{e}}.$$

and U is the velocity of the exhaust gas relative to the rocket. Note that the kinetic energy of the rocket, including unburned fuel and the exhaust gases in the nozzle is

$$T = \frac{1}{2} m_{\mathrm{r}} v^2 + \frac{1}{2} m_{\mathrm{f}} v^2 + \frac{1}{2} m_{\mathrm{e}} (v - u)^2 .$$

The exhaust gases are considered part of the rocket until they exit the nozzle.

Solution:

The kinetic energy of the rocket, including the stored fuel and the exhaust gases in the nozzle, is

$$T = \frac{1}{2} m_{\mathrm{r}} v^2 + \frac{1}{2} m_{\mathrm{f}} v^2 + \frac{1}{2} m_{\mathrm{e}} (v - u)^2 .$$

Because there is no potential this is also the Lagrangian.

$$L = \frac{1}{2} m_{\mathrm{r}} v^2 + \frac{1}{2} m_{\mathrm{f}} v^2 + \frac{1}{2} m_{\mathrm{e}} (v - u)^2 .$$

Here the velocity v is, of course, $\dot{x}$. The canonical momentum is then

$$\begin{aligned} p &= \frac{\partial L}{\partial v} \\ &= m_{\mathrm{r}} v + m_{\mathrm{f}} v + m_{\mathrm{e}} (v - u) . \end{aligned}$$

The Euler-Lagrange equation is simply

$$\frac{\mathrm{d}}{\mathrm{d}t} p = m_{\mathrm{r}} \dot{v} + m_{\mathrm{f}} \dot{v} + m_{\mathrm{e}} \dot{v} + \dot{m}_{\mathrm{e}} (v - u) = 0.$$

Defining

$$M = m_{\mathrm{r}} + m_{\mathrm{f}} + m_{\mathrm{e}},$$

the Euler-Lagrange equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} p = M \dot{v} - \dot{m}_{\mathrm{e}} U = 0,$$

in which U is the velocity of the exhaust gas relative to the rocket. This is the standard equation of rocket motion.

3 Hamiltonian Mechanics

Beginning with Exercise 3.10 we will consider electrodynamics. The correct treatment of electrodynamics is in the context of Special Relativity. In these exercises we shall then use the Hamiltonian, which we develop in the chapter on Relativity.

3.1. Written in terms of the Hamiltonian, Hamilton's Principal Function is.

$$S = \int_{t_1}^{t_2} \mathrm{d}t \left[\sum_j p_j \dot{q}_j - \mathcal{H} \right] \mathrm{d}t.$$

Hamilton's Principle requires that the $\delta-$variation (a first order variation with fixed end points) must vanish. Show that a $\delta-$variation of this form of S results in the canonical equations of Hamilton.

Solution:

Taking the $\delta-$variation of S produces

$$\delta S = 0 = \sum_j \int_{t_1}^{t_2} \mathrm{d}t \left\{ \left[\dot{q}_j - \frac{\partial \mathcal{H}}{\partial p_j} \right] \delta p_j + p_j \delta \dot{q}_j - \frac{\partial \mathcal{H}}{\partial q_j} \delta q_j \right\}.$$

Integrating the second term by parts,

$$\begin{aligned} \sum_j \int_{t_1}^{t_2} \mathrm{d}t \left(p_j \delta \dot{q}_j \right) &= \sum_j \int_{t_1}^{t_2} \mathrm{d}t \left(p_j \frac{\mathrm{d}}{\mathrm{d}t} \delta q_j \right) \\ &= \sum_j \int_{t_1}^{t_2} \mathrm{d}t \left(-\dot{p}_j \delta q_j \right) + p_j \delta q_j]_{t_1}^{t_2}. \end{aligned}$$

The second term on the right hand side vanishes, since the end points are fixed in the $\delta-$variation. That is $\delta q_j = 0$ when $t = t_1$ or t_2. Then we have

$$0 = \sum_j \int_{t_1}^{t_2} \mathrm{d}t \left\{ \left[\dot{q}_j - \frac{\partial \mathcal{H}}{\partial p_j} \right] \delta p_j + \left[-\dot{p}_j - \frac{\partial \mathcal{H}}{\partial q_j} \right] \delta q_j \right\}.$$

Because the integral is over an arbitrary time interval, the integral vanishes only if the integrand vanishes. Because q_μ and p_μ are *independent generalized coordinates*, the integrand vanishes if and only if each square bracket independently vanishes. That is

$$\dot{q}_\mu = \frac{\partial \mathcal{H}}{\partial p_\mu}$$

and

$$\dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial q_\mu}$$

as an immediate consequence of Hamilton's Principle.

3.2. In the text we worked out the Kepler Problem as an example. There we transformed the differential equation for ρ into one for $u = 1/\rho$. The solution to this equation was obviously sinusoidal. And in general it would have consisted of both a sine and a cosine. That is

$$u = \frac{1}{\rho} = A\cos\vartheta + B\sin\vartheta + \frac{mK}{\ell^2}.$$

But we claimed that we could orient the axes such that u attained a maximum, i.e. ρ attained a minimum (closest approach) at $\vartheta = 0$ and that then we needed only the cosine term. That is we chose to consider our orbits as symmetrical about the $x-$axis. This is an arbitrary choice.

Show that this choice does result in dropping the sine term in the above function.

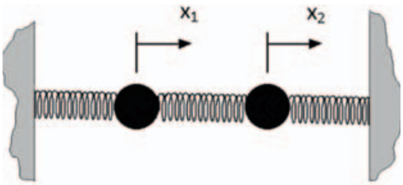
Solution:

The extrema of u (and ρ) are

$$\frac{\mathrm{d}u}{\mathrm{d}\vartheta} = 0 = -A\cos\vartheta + B\sin\vartheta.$$

If we choose this to define $\vartheta = 0$ then $B \equiv 0$.

3.3. We consider again the two masses on springs as shown in the figure here.



Two equal masses connected by identical springs between walls.

Study this system using the canonical equations of Hamilton. Obtain the normal frequencies as eigenvalues of the canonical equations and obtain the eigenvectors corresponding to those eigenvalues.

Solution:

The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2}k\left(x_1^2 + (x_2 - x_1)^2 + x_2^2\right).$$

The canonical momenta are

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$$

and

$$p_2 = \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_1^2 + p_2^2\right) + \frac{1}{2}k\left(x_1^2 + (x_2 - x_1)^2 + x_2^2\right)$$

and the canonical equations are

$$\dot{p}_1 = -2kx_1 + kx_2$$

$$\dot{p}_2 = kx_1 - 2kx_2$$

$$\dot{x}_1 = \frac{1}{m}p_1$$

$$\dot{x}_2 = \frac{1}{m}p_2.$$

We introduce the Ansatz that the time dependence is of a complex exponential form $\exp(i\omega t)$. That is $p_{1,2} = \tilde{p}_{1,2}\exp(i\omega t)$ and $x_{1,2} = \tilde{x}_{1,2}\exp(i\omega t)$. The canonical equations are then

$$i\omega\tilde{p}_1 = -2k\tilde{x}_1 + k\tilde{x}_2$$

$$i\omega\tilde{p}_2 = k\tilde{x}_1 - 2k\tilde{x}_2$$

$$i\omega\tilde{x}_1 = \frac{1}{m}\tilde{p}_1$$

$$i\omega\tilde{x}_2 = \frac{1}{m}\tilde{p}_2.$$

If we write these in matrix form

$$\begin{bmatrix} i\omega & 0 & 2k & -k \\ 0 & i\omega & -k & 2k \\ -1/m & 0 & i\omega & 0 \\ 0 & -1/m & 0 & i\omega \end{bmatrix} \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution the determinant of the square matrix on the left must vanish. That is

$$\frac{1}{m^2}\left(3k^2 - 4km\omega^2 + m^2\omega^4\right) = 0$$

The Solution is

$$\omega = \pm\sqrt{k/m}, \pm\sqrt{3k/m}.$$

These are the four natural frequencies. In magnitude, of course, there are only two. To analyze the form of the motion in each case we must realize that we are dealing presently in the complex plane. We will eventually choose the real components as the physical solutions. But for our simple analysis it is most straightforward to remain in the complex plane and look at the complex eigenvectors. These will reveal the motion.

For the eigenvalues $i\omega = \pm i\sqrt{k/m}$ the eigenvectors are

$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} -im^2\sqrt{\frac{k}{m^3}} \\ -im^2\sqrt{\frac{k}{m^3}} \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} im^2\sqrt{\frac{k}{m^3}} \\ im^2\sqrt{\frac{k}{m^3}} \\ 1 \\ 1 \end{bmatrix}$$

respectively. And for the eigenvalues $i\omega = \pm i\sqrt{3k/m}$ the eigenvectors are

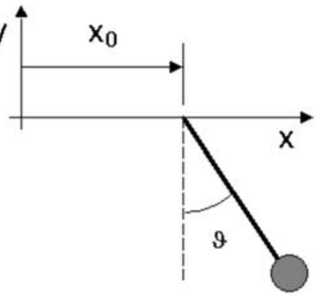
$$\begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} i\sqrt{3}m^2\sqrt{\frac{k}{m^3}} \\ -i\sqrt{3}m^2\sqrt{\frac{k}{m^3}} \\ -1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -i\sqrt{3}m^2\sqrt{\frac{k}{m^3}} \\ i\sqrt{3}m^2\sqrt{\frac{k}{m^3}} \\ -1 \\ 1 \end{bmatrix}$$

respectively. From considering either the momentum or the position components of these eigenvectors we can see that in the lower frequency modes ($\omega = \pm\sqrt{k/m}$) the masses move in concert with one another and in the high frequency modes ($\omega = \pm\sqrt{3k/m}$) they move in opposition. This we discovered in our study based on the Euler-Lagrange equations. But the first order canonical equations are mathematically easier to handle.

3.4. In the figure below we have drawn a pendulum of length ℓ and with bob of mass m located at the point x_0, which is driven horizontally as

$$x_0(t) = A + a\cos\omega t,$$

where A and a are constants.



Pendulum with driven mount.

Study the motion of this pendulum. Consider particularly small angles with corresponding small excursions of the mount. Use the canonical equations of Hamilton.

Explain why you cannot incorporate the constraint using a Lagrange multiplier.

[Answers: $\vartheta = -a\omega^2\cos\omega t/\left(\ell\omega^2 - g\right)$ and $p_\vartheta = ma\omega\ell g\sin\omega t/\left(\ell\omega^2 - g\right)$]

Solution:

We cannot approach this using a Lagrange multiplier because that approach requires that the constraint be a function of position alone and independent of time. So we incorporate the constraint in the position vector of the pendulum mass (bob).

The position vector to the bob is

$$\boldsymbol{R}(t) = [x_0 + \ell\sin\vartheta]\,\hat{e}_{\mathrm{x}} + [-\ell\cos\vartheta]\,\hat{e}_{\mathrm{y}}$$

and the velocity vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}(t) = \left[\dot{x}_0 + \ell\dot{\vartheta}\cos\vartheta\right]\hat{e}_{\mathrm{x}} + \left[\ell\dot{\vartheta}\sin\vartheta\right]\hat{e}_{\mathrm{y}}.$$

The kinetic energy is then

$$\begin{aligned} T &= \frac{1}{2}m\left[\left(\dot{x}_0^2 + \ell^2\dot{\vartheta}^2\cos^2\vartheta + 2\ell\dot{x}_0\dot{\vartheta}\cos\vartheta\right) + \ell^2\dot{\vartheta}^2\sin^2\vartheta\right] \\ &= \frac{1}{2}m\left(\dot{x}_0^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}_0\dot{\vartheta}\cos\vartheta\right) \end{aligned}$$

and the potential of the bob in the gravitational field is

$$V = -mg\cos\vartheta.$$

So the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}_0^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}_0\dot{\vartheta}\cos\vartheta\right) + mg\ell\cos\vartheta.$$

We now incorporate the constraint

$$x_0(t) = A + a\cos\omega t$$

directly in the Lagrangian with

$$\dot{x}_0 = -a\omega\sin\omega t.$$

Then

$$\begin{aligned} L &= \frac{1}{2}m\left(\dot{x}_0^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}_0\dot{\vartheta}\cos\vartheta\right) + mg\ell\cos\vartheta \\ &= \frac{1}{2}m\left(a^2\omega^2\sin^2\omega t + \ell^2\dot{\vartheta}^2 - 2a\omega\ell\dot{\vartheta}\sin\omega t\cos\vartheta\right) + mg\ell\cos\vartheta. \end{aligned}$$

The canonical momentum is

$$p_\vartheta = m\ell^2\dot{\vartheta} - ma\omega\ell\sin\omega t\cos\vartheta.$$

Then

$$\dot{\vartheta} = \frac{p_\vartheta}{m\ell^2} + \frac{ma\omega\ell\sin\omega t\cos\vartheta}{m\ell^2}$$

$$p_\vartheta\dot{\vartheta} = m\ell^2\dot{\vartheta}^2 - ma\omega\ell\dot{\vartheta}\sin\omega t\cos\vartheta$$

The Hamiltonian is

$$\begin{aligned} \mathcal{H} &= p_\vartheta\dot{\vartheta} - L \\ &= \frac{1}{2}m\ell^2\dot{\vartheta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\omega t - mg\ell\cos\vartheta \end{aligned}$$

Or, in terms of momentum,

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}m\ell^2\left(\frac{p_\vartheta + ma\omega\ell\sin\omega t\cos\vartheta}{m\ell^2}\right)^2 - \frac{1}{2}ma^2\omega^2\sin^2\omega t - mg\ell\cos\vartheta \\ &= \frac{p_\vartheta^2 + 2ma\omega\ell p_\vartheta\sin\omega t\cos\vartheta}{2m\ell^2} - \frac{m^2\ell^2a^2\omega^2\sin^2\vartheta}{2m\ell^2}\left(\sin^2\omega t\right) - mg\ell\cos\vartheta. \end{aligned}$$

The canonical equations are

$$\dot{\vartheta} = \frac{p_\vartheta + ma\omega\ell\sin\omega t\cos\vartheta}{m\ell^2}$$

$$\dot{p}_\vartheta = \frac{ma\omega\ell \sin\omega t \sin\vartheta}{m\ell^2} p_\vartheta + \frac{m^2\ell^2 a^2\omega^2 \sin\vartheta \cos\vartheta}{m\ell^2}\left(\sin^2\omega t\right) - mg\ell\sin\vartheta.$$

These equations are nonlinear. We may linearize them by dropping powers and products of ϑ and p_ϑ. The results are

$$\dot{\vartheta} = \frac{p_\vartheta}{m\ell^2} + \frac{a\omega\sin\omega t}{\ell}$$

$$\dot{p}_\vartheta = -\left(mg\ell - ma^2\omega^2\sin^2\omega t\right)\vartheta.$$

If we assume a small amplitude a on the driver, these become

$$p_\vartheta - m\ell^2\dot{\vartheta} = -ma\omega\ell\sin\omega t$$

$$\dot{p}_\vartheta + mg\ell\vartheta = 0.$$

These may be solved by variation of parameters. Choose

$$p_\vartheta = A\sin\omega t + B\cos\omega t$$

Then

$$\dot{p}_\vartheta = A\omega\cos\omega t - B\omega\sin\omega t$$

and

$$\vartheta = -A\frac{\omega}{mg\ell}\cos\omega t + B\frac{\omega}{mg\ell}\sin\omega t.$$

So

$$\dot{\vartheta} = A\frac{\omega^2}{mg\ell}\sin\omega t + B\frac{\omega^2}{mg\ell}\cos\omega t.$$

Then

$$p_\vartheta - m\ell^2\dot{\vartheta} = A\sin\omega t + B\cos\omega t - \left(A\frac{\omega^2}{g}\ell\sin\omega t + B\frac{\omega^2}{g}\ell\cos\omega t\right) = -ma\omega\ell\sin\omega t$$

or

$$A\left(1 - \frac{\omega^2}{g}\ell\right)\sin\omega t + B\left(1 - \frac{\omega^2}{g}\ell\right)\cos\omega t = -ma\omega\ell\sin\omega t.$$

Then

$$A = \frac{ma\omega\ell g}{\ell\omega^2 - g}$$

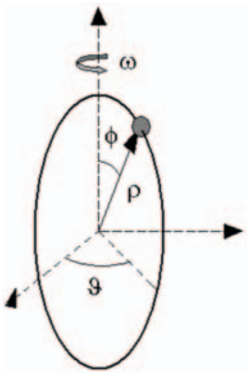
and $B = 0$. Our solutions are then

$$\vartheta = -\frac{a\omega^2}{\ell\omega^2 - g}\cos\omega t.$$

and

$$p_\vartheta = \frac{ma\omega\ell g}{\ell\omega^2 - g}\sin\omega t.$$

3.5. Consider the picture below of a bead on a circular, frictionless wire driven at a constant angular velocity ω about the vertical axis as we have shown here.



bead on a driven, circular, frictionless wire loop.

In this exercise it is advisable to use spherical coordinates. It is also convenient to take the horizontal plane through the center as the zero of potential.

Investigate the problem. There are certain things to look for. You will want to see if there are any constants of the motion, i.e. first integrals. You will also want to look for any equilibrium points and you will want to consider motion about those points to see if it is stable or not. You may want to investigate the forces of constraint as well.

Investigate the system using both the Euler-Lagrange and the canonical equations.

Solution:

The position of the bead is simply

$$\overrightarrow{R}(t) = \rho\hat{e}_\rho.$$

The velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{R}(t) = \dot{\rho}\hat{e}_\rho + (\rho\sin\phi)\,\dot{\vartheta}\hat{e}_\vartheta + \rho\dot{\phi}\hat{e}_\phi.$$

From here we have the kinetic energy as

$$T = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2\sin^2\phi + \rho^2\dot{\phi}^2\right).$$

We shall take the horizontal plane at the center of the system $\left(\phi = \frac{\pi}{2}\right)$ as ground for the potential. Then the potential energy of the system is

$$V = mg\rho\cos\phi.$$

The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2\sin^2\phi + \rho^2\dot{\phi}^2\right) - mg\rho\cos\phi.$$

There are two constraints on the system. The angular velocity about the vertical axis is constant.

$$\dot{\vartheta} = \omega = \text{ constant.}$$

The radius of the wire on which the bead moves, is constant

$$\rho = R = \text{ constant.}$$

With these the Lagrangian is

$$L = \frac{1}{2}m\left(R^2\omega^2\sin^2\phi + R^2\dot{\phi}^2\right) - mgR\cos\phi.$$

The only coordinate remaining is ϕ. We also notice that L depends on ϕ and, therefore, the canonical momentum $p_\phi = \partial L/\partial\dot{\phi}$ is not a constant.

However, we observe that L does not depend explicitly on the time. Therefore, the Hamiltonian is a constant. With

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial\dot{\phi}} \\ &= mR^2\dot{\phi} \end{aligned}$$

we have

$$\begin{aligned} \mathcal{H} &= mR^2\dot{\phi}^2 - \frac{1}{2}m\left(R^2\omega^2\sin^2\phi + R^2\dot{\phi}^2\right) + mgR\cos\phi \\ &= \frac{1}{2}mR^2\dot{\phi}^2 - \frac{1}{2}mR^2\omega^2\sin^2\phi + mgR\cos\phi, \end{aligned}$$

which is constant. It is instructive to note that this conserved Hamiltonian is not the mechanical energy, which is not conserved because of the driver maintaining the constant angular velocity. The total mechanical energy is

$$\mathcal{E} = \frac{1}{2}m\left(R^2\omega^2\sin^2\phi + R^2\dot{\phi}^2\right) + mgR\cos\phi.$$

We obtain the Euler-Lagrange equations from

$$\frac{\partial L}{\partial \phi} = mR^2\omega^2\sin\phi\cos\phi + mgR\sin\phi$$

and

$$\frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}.$$

The Euler-Lagrange equation is

$$mR^2\omega^2\sin\phi\cos\phi + mgR\sin\phi - mR^2\ddot{\phi} = 0.$$

The angular acceleration vanishes when

$$\begin{aligned} mR^2\omega^2\sin\phi\cos\phi + mgR\sin\phi &= -\frac{\mathrm{d}}{\mathrm{d}\phi}\left(-\frac{1}{2}mR^2\omega^2\sin^2\phi + mgR\cos\phi\right) \\ &= 0. \end{aligned}$$

That is,

$$V_{\mathrm{eff}} = -\frac{1}{2}mR^2\omega^2\sin^2\phi + mgR\cos\phi,$$

which we may also identify in the Hamiltonian, considered as the energy of a mass m moving in a circle of radius R. This effective potential has an extremum when

$$mR^2\omega^2\sin\phi\cos\phi + mgR\sin\phi = 0.$$

That is when $\sin\phi = 0$ or when

$$mR^2\omega^2\cos\phi + mgR = 0.$$

The sine of the angle is zero if $\phi = 0$ or π, which is at the top or bottom of the circular loop. The equilibrium at $\phi = 0$ is obviously unstable. If we consider small deviations around $\phi = \pi$,

$$R\omega^2\sin(\pi + \delta\phi)\cos(\pi + \delta\phi) + g\sin(\pi + \delta\phi) = -\delta\phi\left(g - R\omega^2\right) + O\left(\phi^3\right).$$

Then the motion of the mass is

$$mR^2\delta\ddot{\phi} = -m\left(g - R\omega^2\right)\delta\phi,$$

which is sinusoidal provided $g > R\omega^2$. The equilibrium at $\phi = \pi$ may then be stable, or not. If we consider the curvature of V_{eff} we have

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}V_{\text{eff}} &= \frac{\mathrm{d}}{\mathrm{d}\phi}\left(-mR^2\omega^2\sin\phi\cos\phi - mgR\sin\phi\right)\\ &= -mR^2\omega^2\cos^2\phi + mR^2\omega^2\sin^2\phi - mgR\cos\phi\\ &= mR^2\omega^2\left(1 - 2\cos^2\phi\right) - mgR\cos\phi\end{aligned}$$

which is positive when $\phi = \pi$ provided $g > R\omega^2$.

If we consider that $\sin\phi \neq 0$ and

$$mR^2\omega^2\cos\phi + mgR = 0.$$

we have an equilibrium point when

$$\phi = \phi_0 = -\cos^{-1}\frac{g}{R\omega^2}.$$

Here the curvature of V_{eff} is

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}V_{\text{eff}} = mR^2\omega^2 - m\frac{g^2}{\omega^2},$$

which is positive provided

$$R\omega^2 > g.$$

So, in the event that there is no stable equilibrium point at the bottom, there will be one at the angle

$$\phi_0 = -\cos^{-1}\frac{g}{R\omega^2}.$$

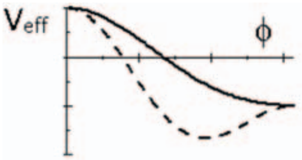
If we expand the Euler-Lagrange equation around ϕ_0 we have

$$\begin{aligned}mR^2\ddot{\phi} &= mR\omega^2\sin\left(\phi_0+\phi\right)\cos\left(\phi_0+\phi\right) + mg\sin\left(\phi_0+\phi\right)\\ &= -2mg\sqrt{1-\frac{1}{R^2}\frac{g^2}{\omega^4}} - mR\omega^2\left(1 - 3\frac{g^2}{\left(R\omega^2\right)^2}\right)\phi + O\left(\phi^2\right).\end{aligned}$$

So we have sinusoidal motion provided

$$R\omega^2 > \sqrt{3}g.$$

We have then two distinctly separate possibilities for the effective potential depending upon the angular velocity of the wire loop. These we plot in the figure here.



V_{eff} for a bead on a driven circular wire.

The canonical Equations

To obtain the canonical equations we go back to the Hamiltonian

$$\mathcal{H} = \frac{1}{2} m R^2 \dot{\phi}^2 - \frac{1}{2} m R^2 \omega^2 \sin^2 \phi + m g R \cos \phi,$$

written in terms of the canonical momentum and the coordinate ϕ. Substituting the angular velocity in terms of the canonical momentum

$$\dot{\phi} = \frac{p_\phi}{m R^2}$$

into the Hamiltonian we have

$$\mathcal{H} = \frac{1}{2 m R^2} p_\phi^2 - \frac{1}{2} m R^2 \omega^2 \sin^2 \phi + m g R \cos \phi.$$

Then

$$\frac{\partial \mathcal{H}}{\partial \phi} = -m R^2 \omega^2 \sin \phi \cos \phi - m g R \sin \phi$$
$$\frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{1}{m R^2} p_\phi.$$

And the canonical equations are

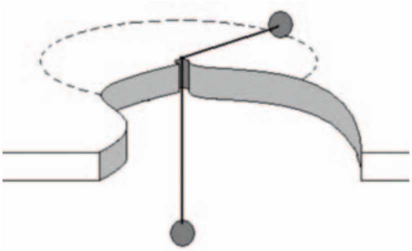
$$\begin{aligned} \dot{\phi} &= \frac{\partial \mathcal{H}}{\partial p_\phi} \\ &= \frac{1}{m R^2} p_\phi \end{aligned}$$

and

$$\begin{aligned} \dot{p}_\phi &= -\frac{\partial \mathcal{H}}{\partial \phi} \\ &= m R^2 \omega^2 \sin \phi \cos \phi + m g R \sin \phi. \end{aligned}$$

Of course the set is still nonlinear because of the trigonometric functions. But if we seek an actual solution this is a better way to go.

3.6. Consider again the two identical balls of mass m connected by a string of length b with one ball suspended through a hole in a table while the other moves on the (frictionless) top of the table. We have drawn the situation here.



Two identical balls of mass m connected by a string of length b.

Investigate the motion. Now use canonical equations and Lagrange multipliers. Find an equilibrium point, if there is one. Study the general form of the motion. Consider small oscillations about equilibrium. Determine if the orbit is open or closed.

If you have access to appropriate software, plot the effective potential, the orbit of the ball on the table for less than a complete rotation and for a long time, and the phase plot of the ball on the table (p_r versus r).

Solution:

We choose cylindrical coordinates for the ball on the table. The kinetic energy is

$$T_1 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right).$$

The suspended ball only moves in the z direction, so its kinetic energy is

$$T_2 = \frac{1}{2}m\dot{z}^2$$

The ball on the table has no change in potential energy, so the system potential energy is solely from the suspended ball. If we choose the ground of the potential to be the table top, the potential energy is

$$V = -mgz.$$

The Lagrangian is then

$$\begin{aligned} L &= T_1 + T_2 - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{1}{2}m\dot{z}^2 + mgz. \end{aligned}$$

The canonical momenta are

$$p_r = m\dot{r}$$

$$p_\vartheta = mr^2\dot{\vartheta}$$

$$p_z = m\dot{z}$$

The constraint is that the length of the string is constant. This is

$$g = b - r - z = 0.$$

Then

$$\frac{\partial g}{\partial r} = -1, \frac{\partial g}{\partial \vartheta} = 0, \frac{\partial g}{\partial z} = -1$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + \frac{1}{2}m\dot{z}^2 - mgz \\ &= \frac{p_{\mathrm{r}}^2}{2m} + \frac{p_{\vartheta}^2}{2mr^2} + \frac{p_{\mathrm{z}}^2}{2m} - mgz\end{aligned}$$

To obtain the canonical equations we need the derivatives

$$\frac{\partial \mathcal{H}}{\partial p_{\mathrm{r}}} = \frac{p_{\mathrm{r}}}{m}, \ \frac{\partial \mathcal{H}}{\partial p_{\vartheta}} = \frac{p_{\vartheta}}{mr^2}, \ \frac{\partial \mathcal{H}}{\partial p_{\mathrm{z}}} = \frac{p_{\mathrm{z}}}{m}$$

and

$$\frac{\partial \mathcal{H}}{\partial r} = -\frac{p_{\vartheta}^2}{mr^3}, \ \frac{\partial \mathcal{H}}{\partial \vartheta} = 0, \ \frac{\partial \mathcal{H}}{\partial z} = -mg.$$

The canonical equations are then

$$\dot{r} = \frac{p_{\mathrm{r}}}{m}$$

$$\dot{\vartheta} = \frac{p_{\vartheta}}{mr^2}$$

$$\dot{z} = \frac{p_{\mathrm{z}}}{m}$$

$$\dot{p}_{\mathrm{r}} = -\frac{\partial \mathcal{H}}{\partial r} + \lambda\left(\frac{\partial g}{\partial r}\right) = \frac{p_{\vartheta}^2}{mr^3} - \lambda$$

$$\dot{p}_{\vartheta} = 0$$

$$\dot{p}_{\mathrm{z}} = -\frac{\partial \mathcal{H}}{\partial z} + \lambda\left(\frac{\partial g}{\partial z}\right) = mg - \lambda(t)$$

Clearly the multiplier λ is the tension in the string, which is a function of time. We observe first that p_{ϑ} is a constant, which we call ℓ. Then

$$mr^2\dot{\vartheta} = \ell$$

Since the Lagrangian is cyclic in the time, the Hamiltonian is constant. We call this constant $\mathcal{E}$.

$$\mathcal{E} = \frac{p_{\mathrm{r}}^2}{2m} + \frac{\ell^2}{2mr^2} + \frac{p_{\mathrm{z}}^2}{2m} - mgz.$$

Our set of equations is then

$$\dot{r} = \frac{p_{\mathrm{r}}}{m}$$

$$\dot{z} = \frac{p_{\mathrm{z}}}{m}$$

$$\dot{p}_{\mathrm{r}} = \frac{\ell^2}{mr^3} - \lambda$$

$$\dot{p}_{\mathrm{z}} = mg - \lambda$$

$$b = r + z$$

and

$$\mathcal{E} = \frac{p_{\mathrm{r}}^2}{2m} + \frac{\ell^2}{2mr^2} + \frac{p_{\mathrm{z}}^2}{2m} - mgz.$$

We may eliminate λ by subtracting the canonical equations for $\dot{p}_{\mathrm{r}}$ and $\dot{p}_{\mathrm{z}}$.

$$\dot{p}_{\mathrm{r}} - \dot{p}_{\mathrm{z}} = \frac{\ell^2}{mr^3} - mg.$$

Then, since the constraint requires

$$\dot{r} = -\dot{z},$$

the first canonical equations produce

$$p_{\mathrm{r}} = -p_{\mathrm{z}}.$$

Combining this with the expression for $\dot{p}_r - \dot{p}_z$ we have

$$2\dot{p}_{\mathrm{r}} = \frac{\ell^2}{mr^3} - mg.$$

And the constancy of the Hamiltonian is

$$\mathcal{E} = \frac{p_{\mathrm{r}}^2}{m} + \frac{\ell^2}{2mr^2} - mg\,(b - r)\,.$$

So we have the differential equations

$$2\dot{p}_{\mathrm{r}} = \frac{\ell^2}{mr^3} - mg$$

and

$$\dot{r} = \frac{p_{\mathrm{r}}}{m}$$

and the algebraic equation

$$\mathcal{E} = \frac{p_{\mathrm{r}}^2}{m} + \frac{\ell^2}{2mr^2} + mgr - mgb$$

for our study. First we should find the equilibrium condition. This is the condition at which $\dot{r} = \dot{p}_r = 0$. If then $r = r_0$ and $\ell = \ell_0$,

$$\frac{\ell_0^2}{mr_0^3} = mg.$$

This is a cubic equation for r_0. There are three roots. The real root is,

$$r_0 = \left(\frac{\ell_0^2}{m^2 g}\right)^{1/3}$$

and the complex roots are

$$r_0 = -\frac{1}{2}\left(\frac{\ell_0^2}{m^2 g}\right)^{1/3} \pm \frac{1}{2} i\sqrt{3}\left(\frac{\ell_0^2}{m^2 g}\right)^{1/3}.$$

We neglect the complex roots on physical grounds.

Of course the path of the mass on the table under these conditions is circular. The system energy is

$$\begin{aligned}
\mathcal{E}_0 &= \frac{\ell_0^2}{2mr_0^2} + mgr_0 - mgb \\
&= \frac{\ell_0^2}{2m\left(\ell_0^2/m^2 g\right)^{2/3}} + mg\left(\frac{\ell_0^2}{m^2 g}\right)^{1/3} - mgb \\
&= (3/2)\left(\ell_0^2 m g^2\right)^{1/3} - mgb
\end{aligned}$$

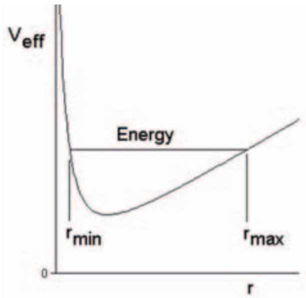
If we identify an effective potential as

$$V_{\text{eff}}(r) = \frac{1}{2mr^2}\ell^2 + mg(r-b),$$

we have

$$\mathcal{H} = \frac{p_{\text{r}}^2}{2m} + V_{\text{eff}}(r).$$

This is, of course, the Hamiltonian for a particle of mass m moving in a potential $V_{\text{eff}}(r)$. This equation provides a pictorial understanding of the motion. which we plot here.



Effective potential $V_{\text{eff}}(r) = \frac{1}{2mr^2}\ell^2 + mg(r-b)$.

To consider the motion near equilibrium $r = r_0$ we define

$$\begin{aligned}\rho &= r - r_0 \\ &= r - \left(\frac{\ell_0^2}{m^2 g}\right)^{1/3},\end{aligned}$$

Then we write V_{eff} in terms of ρ and expand around the point $\rho = 0$, which is the bottom of the potential well. Going to second order in ρ, and assuming that the angular momentum is approximately the equilibrium value, we have

$$r = \rho + \left(\frac{\ell_0^2}{m^2 g}\right)^{\frac{1}{3}}$$

$$\begin{aligned}V_{\text{eff}}(\rho) &= \frac{1}{2mr^2}\ell^2 + mg(r-b) \\ &= V_0 + \frac{3}{2m}\frac{\ell_0^2}{\left(\ell_0^2/gm^2\right)^{4/3}}\rho^2 + O\left(\rho^3\right)\end{aligned}$$

where

$$V_0 = \frac{1}{2m} \frac{\ell_0^2}{\left(\ell_0^2/gm^2\right)^{2/3}} - gm \left[b - \left(\frac{1}{gm^2} \ell_0^2 \right)^{1/3} \right].$$

V_0 is, of course, the value of the potential energy at the bottom of the well (the minimum in potential). The next term is proportional to ρ^2. There is no term proportional to ρ, because that term is zero by definition of a minimum.

The canonical equation for $\dot{p}_\mathrm{r}$ is

$$2\dot{p}_\mathrm{r} = -\frac{\mathrm{d}}{\mathrm{d}r} V_\mathrm{eff}(r).$$

Near $r = r_0$ this becomes

$$\begin{aligned} 2\dot{p}_\mathrm{r} &= -\frac{\mathrm{d}}{\mathrm{d}\rho} V_\mathrm{eff}(\rho) \\ &= -\frac{3}{m} \frac{\ell_0^2}{\left(\ell_0^2/gm^2\right)^{4/3}} \rho + O\left(\rho^2\right), \end{aligned}$$

or

$$\dot{p}_\mathrm{r} = -\frac{3}{2m} \frac{\ell_0^2}{\left(\ell_0^2/gm^2\right)^{4/3}} \rho,$$

dropping $O\left(\rho^2\right)$. If we define

$$\alpha = \frac{3}{2m} \frac{\ell_0^2}{\left(\ell_0^2/gm^2\right)^{4/3}}$$

the canonical equations are

$$\dot{p}_\mathrm{r} = -\alpha\rho$$

and

$$\dot{\rho} = \frac{p_\mathrm{r}}{m}.$$

If we make the Ansatz of the exponential solution these are

$$\begin{bmatrix} i\omega & \alpha \\ -\frac{1}{m} & i\omega \end{bmatrix} \begin{bmatrix} p_\mathrm{r} \\ \rho \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrix must be singular. That is,

$$\det\left(\begin{bmatrix} i\omega & \alpha \\ -\frac{1}{m} & i\omega \end{bmatrix} \right) = -\frac{\omega^2 m - \alpha}{m} = 0$$

and

$$\omega = \pm\sqrt{\frac{\alpha}{m}}.$$

This is the frequency of small radial oscillations about the equilibrium position.

The orbit is the radial position as a function of ϑ. to obtain differential equations in ϑ, which can be solved for the orbit, we use the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\vartheta}\frac{\mathrm{d}}{\mathrm{d}\vartheta} = \frac{\ell}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}.$$

The canonical equations are then

$$2\dot{p}_{\mathrm{r}} = 2\frac{\ell}{mr^2}\frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}\vartheta} = \frac{\ell^2}{mr^3} - mg,$$

or

$$\frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}\vartheta} = \frac{\ell}{2r} - \frac{m^2r^2}{2\ell}g.$$

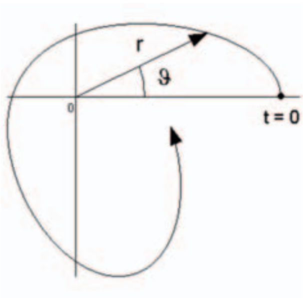
And

$$\dot{r} = \frac{\ell}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\vartheta} = \frac{p_{\mathrm{r}}}{m},$$

or

$$\frac{\mathrm{d}r}{\mathrm{d}\vartheta} = \frac{r^2}{\ell}p_{\mathrm{r}}.$$

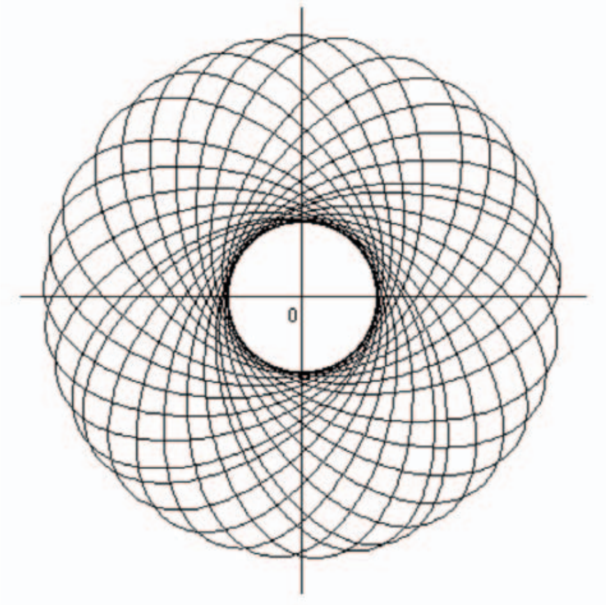
These equations can be integrated simultaneously to produce the graphical results we have here.



The orbit of the ball on the table $r = r(\vartheta)$. The initial point is the closed circle at $t = 0$. The arrow locates the ball at $\vartheta < 2\pi$.

The plot here is for a short time.

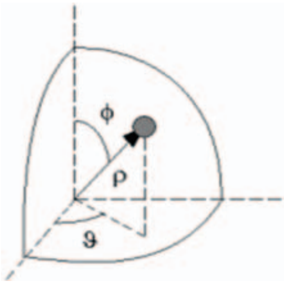
In the figure below we have a long time plot of the orbit of the ball on the table.



Long time orbit of the ball on the table.

In this figure it is particularly clear that the orbit is not closed.

3.7. In the drawing here we have a mass m sliding without friction on the surface of a sphere. We release the mass an infinitesimal distance from the top of the sphere.



Mass sliding without friction on the surface of a sphere

At some point (some value of ϕ) the mass will leave the surface of the sphere. What is this value of ϕ? Use canonical equations.

Solution:

We begin with the Lagrangian in spherical coordinates

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2 \sin^2\phi\dot{\vartheta}^2 + \rho^2\dot{\phi}^2\right) - mg\rho\cos\phi$$

The constraint is

$$\rho = R.$$

Because the Lagrangian is cyclic in ϑ

$$p_\vartheta = m\rho^2 \sin^2 \phi \dot{\vartheta} = \text{constant}.$$

Since the mass is released at the top of the sphere with $\dot{\vartheta} = 0$, this constant is zero.

$$p_\vartheta = 0.$$

The remaining canonical momenta are

$$p_\rho = m\dot{\rho}$$

and

$$p_\phi = m\rho^2\dot{\phi}.$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= p_\rho\dot{\rho} + p_\phi\dot{\phi} - \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mg\rho\cos\phi \\ &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2\right) + mg\rho\cos\phi\end{aligned}$$

or

$$\mathcal{H} = \frac{1}{2m}\left(p_\rho^2 + \frac{p_\phi^2}{\rho^2}\right) + mg\rho\cos\phi.$$

In standard form the constraint is

$$g = \rho - R$$

and

$$\frac{\partial g}{\partial \rho} = 1.$$

The canonical equations are found from

$$\frac{\partial \mathcal{H}}{\partial \rho} = mg\cos\phi - \frac{1}{\rho^3}\frac{p_\phi^2}{m}$$

$$\frac{\partial \mathcal{H}}{\partial \phi} = -mg\rho\sin\phi$$

$$\frac{\partial \mathcal{H}}{\partial p_\rho} = \frac{p_\rho}{m}$$

and

$$\frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{1}{\rho^2}\frac{p_\phi}{m}.$$

The canonical equations are

$$\dot{\rho} = \frac{p_\rho}{m}$$

$$\dot{p}_\rho = -mg\cos\phi + \frac{1}{\rho^3}\frac{p_\phi^2}{m} + \lambda$$

$$\dot{\phi} = \frac{1}{\rho^2}\frac{p_\phi}{m}$$

$$\dot{p}_\phi = mg\rho\sin\phi$$

If we introduce the constraint $\rho = R$ on the sphere, these are

$$0 = \frac{p_\rho}{m}$$

$$0 = -mg\cos\phi + \frac{1}{R^3}\frac{p_\phi^2}{m} + \lambda$$

$$\dot{\phi} = \frac{1}{R^2}\frac{p_\phi}{m}$$

$$\dot{p}_\phi = mgR\sin\phi$$

We know that the Hamiltonian is a constant. Initially, we released the mass at a very small distance from the top at rest. So the constant value is

$$\mathcal{H} = \text{constant} = mgR$$

While the mass is on the sphere, $p_\rho = 0$ and the constant Hamiltonian becomes

$$mgR = \frac{1}{R^2}\frac{p_\phi^2}{2m} + mgR\cos\phi$$

Now we have two equations with λ and $\cos\phi$. These are the constant Hamiltonian and the canonical equation for $\dot{p}_\rho$ with $p_\rho = 0$ (on the sphere).

$$mgR\cos\phi = \frac{1}{R^2}\frac{p_\phi^2}{m} + \lambda R.$$

The point at which the mass leaves the sphere is that for which the reaction force vanishes, That is, $\lambda = 0$. At this point then

$$mgR\cos\phi = \frac{1}{R^2}\frac{p_\phi^2}{m}$$

and the Hamiltonian becomes

$$mgR = \frac{3}{2}\frac{1}{R^2}\frac{p_\phi^2}{m}$$

or

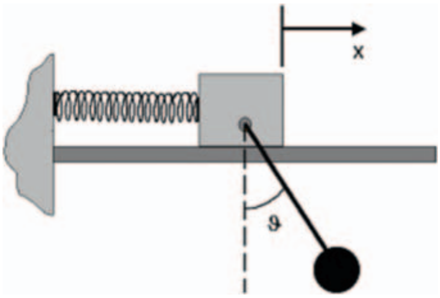
$$mgR = \frac{3}{2}mgR\cos\phi.$$

Solving for ϕ,

$$\phi = \cos^{-1}\frac{2}{3},$$

which is the value of ϕ at which the mass leaves the sphere.

3.8. Consider (again) the block, spring and pendulum we have drawn here.



A block, spring and pendulum.

Consider the motion now in terms of the canonical equations. Linearize these for small displacements and find the natural frequencies. Choose the masses of the block and the pendulum bob to be equal.

Solution:

The position vectors for the center of mass of the block $\boldsymbol{r}_1$ and the pendulum bob $\boldsymbol{r}_2$ are

$$\boldsymbol{r}_1 = (a + x)\,\hat{e}_{\mathrm{x}},$$

where a is the equilibrium position of the block, and

$$\boldsymbol{r}_2 = (a + x + \ell \sin \vartheta)\, \hat{e}_\text{x} - \ell \cos \vartheta\, \hat{e}_\text{y}.$$

The velocities are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_1 = (\dot{x})\, \hat{e}_\text{x}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r}_2 = \left(\dot{x} + \ell\dot{\vartheta} \cos \vartheta\right) \hat{e}_\text{x} + \ell\dot{\vartheta} \sin \vartheta\, \hat{e}_\text{y}.$$

The squares of the velocities are

$$\dot{r}_1^2 = \dot{x}^2$$

and

$$\dot{r}_2^2 = \dot{x}^2 + \ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta} \cos \vartheta.$$

Then the kinetic energy is

$$T = \frac{1}{2}(M + m)\, \dot{x}^2 + \frac{1}{2}m\left(\ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta} \cos \vartheta\right).$$

The potential energy is

$$V = \frac{1}{2}kx^2 - mg\ell \cos \vartheta.$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}(M + m)\, \dot{x}^2 + \frac{1}{2}m\left(\ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta} \cos \vartheta\right) - \frac{1}{2}kx^2 + mg\ell \cos \vartheta. \end{aligned}$$

The canonical momenta are

$$p_\text{x} = \frac{\partial L}{\partial \dot{x}} = (M + m)\, \dot{x} + m\ell\dot{\vartheta} \cos \vartheta$$

and

$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = m\left(\ell^2\dot{\vartheta} + \ell\dot{x} \cos \vartheta\right).$$

In matrix form these are

$$\begin{bmatrix} (M+m) & m\ell\cos\vartheta \\ m\ell\cos\vartheta & m\ell^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\vartheta} \end{bmatrix} = \begin{bmatrix} p_{\mathrm{x}} \\ p_\vartheta \end{bmatrix}.$$

The inverse of the principal matrix is

$$\frac{1}{D} \begin{bmatrix} \ell & -\cos\vartheta \\ -\cos\vartheta & \frac{1}{m\ell}(M+m) \end{bmatrix},$$

where

$$\begin{aligned} D &= (M+m)\,\ell - m\ell\cos^2\vartheta \\ &= M\ell + m\ell\sin^2\vartheta. \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\vartheta} \end{bmatrix} &= \frac{1}{D} \begin{bmatrix} \ell & -\cos\vartheta \\ -\cos\vartheta & \frac{1}{m\ell}(M+m) \end{bmatrix} \begin{bmatrix} p_{\mathrm{x}} \\ p_\vartheta \end{bmatrix} \\ &= \frac{1}{D} \begin{bmatrix} \ell p_{\mathrm{x}} - p_\vartheta\cos\vartheta \\ \frac{1}{m\ell} p_\vartheta\,(M+m) - p_{\mathrm{x}}\cos\vartheta \end{bmatrix}. \end{aligned}$$

If we choose $M = m$, we have

$$\begin{aligned} L &= T - V \\ &= m\dot{x}^2 + \frac{1}{2}m\left(\ell^2\dot{\vartheta}^2 + 2\ell\dot{x}\dot{\vartheta}\cos\vartheta\right) - \frac{1}{2}kx^2 + mg\ell\cos\vartheta. \end{aligned}$$

$$\begin{aligned} D &= (M+m)\,\ell - m\ell\cos^2\vartheta \\ &= m\ell\left(1+\sin^2\vartheta\right). \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\vartheta} \end{bmatrix} &= \frac{1}{D} \begin{bmatrix} \ell & -\cos\vartheta \\ -\cos\vartheta & \frac{2}{\ell} \end{bmatrix} \begin{bmatrix} p_{\mathrm{x}} \\ p_\vartheta \end{bmatrix} \\ &= \frac{1}{D} \begin{bmatrix} \ell p_x - p_\vartheta\cos\vartheta \\ \frac{2}{\ell} p_\vartheta - p_x\cos\vartheta \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} \dot{x}^2 &= \frac{1}{D^2}\left(\ell p_{\mathrm{x}} - p_\vartheta\cos\vartheta\right)^2 \\ &= \frac{1}{D^2}\left(\ell^2 p_{\mathrm{x}}^2 + p_\vartheta^2\cos^2\vartheta - 2\ell p_{\mathrm{x}} p_\vartheta\cos\vartheta\right), \end{aligned}$$

$$\begin{aligned} \dot{\vartheta}^2 &= \frac{1}{D^2}\left(\frac{2}{\ell}p_\vartheta - p_{\mathrm{x}}\cos\vartheta\right)^2 \\ &= \frac{1}{D^2}\left(\frac{4}{\ell^2}p_\vartheta^2 + p_{\mathrm{x}}^2\cos^2\vartheta - \frac{4}{\ell}p_{\mathrm{x}}p_\vartheta\cos\vartheta\right), \end{aligned}$$

and

$$\begin{aligned}\dot{x}\dot{\vartheta} &= \frac{1}{D^2}\left(\ell p_{\mathrm{x}} - p_\vartheta \cos\vartheta\right)\left(\frac{2}{\ell}p_\vartheta - p_{\mathrm{x}}\cos\vartheta\right)\\ &= \frac{1}{D^2}\left(2p_{\mathrm{x}}p_\vartheta - \ell p_{\mathrm{x}}^2\cos\vartheta - \frac{2}{\ell}p_\vartheta^2\cos\vartheta + p_{\mathrm{x}}p_\vartheta\cos^2\vartheta\right).\end{aligned}$$

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= 2m\dot{x}^2 + m\ell^2\dot{\vartheta}^2 + 2m\ell\dot{x}\dot{\vartheta}\cos\vartheta - L\\ &= m\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\vartheta}^2 + m\ell\dot{x}\dot{\vartheta}\cos\vartheta + \frac{1}{2}kx^2 - mg\ell\cos\vartheta.\end{aligned}$$

In terms of canonical momenta

$$\begin{aligned}\mathcal{H} = {}& m\frac{1}{D^2}p_{\mathrm{x}}^2\left(\ell^2 - \frac{1}{2}\ell^2\cos^2\vartheta\right) + m\frac{1}{D^2}p_\vartheta^2\left(2 - \cos^2\vartheta\right)\\ &+ m\frac{1}{D^2}p_{\mathrm{x}}p_\vartheta\left(-2\ell + \ell\cos^3\vartheta\right) + \frac{1}{2}kx^2 - mg\ell\cos\vartheta.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial x} &= kx\\ \frac{\partial\mathcal{H}}{\partial\vartheta} &= m\frac{1}{D^2}p_{\mathrm{x}}^2\left(\ell^2\sin\vartheta\cos\vartheta\right) + m\frac{1}{D^2}p_\vartheta^2\left(2\sin\vartheta\cos\vartheta\right)\\ &\quad + m\frac{1}{D^2}p_{\mathrm{x}}p_\vartheta\left(-3\ell\sin\vartheta\cos^2\vartheta\right) + mg\ell\sin\vartheta\\ \frac{\partial\mathcal{H}}{\partial p_{\mathrm{x}}} &= 2m\frac{1}{D^2}p_{\mathrm{x}}\left(\ell^2 - \frac{1}{2}\ell^2\cos^2\vartheta\right) + m\frac{1}{D^2}p_\vartheta\left(-2\ell + \ell\cos^3\vartheta\right)\\ \frac{\partial\mathcal{H}}{\partial p_\vartheta} &= m\frac{1}{D^2}p_{\mathrm{x}}\left(-2\ell + \ell\cos^3\vartheta\right) + 2m\frac{1}{D^2}p_\vartheta\left(2 - \cos^2\vartheta\right).\end{aligned}$$

And the canonical equations are

$$\dot{p}_{\mathrm{x}} = -kx,$$

$$\begin{aligned}\dot{p}_\vartheta &= -m\frac{1}{D^2}p_{\mathrm{x}}^2\left(\ell^2\sin\vartheta\cos\vartheta\right) - m\frac{1}{D^2}p_\vartheta^2\left(2\sin\vartheta\cos\vartheta\right)\\ &\quad - m\frac{1}{D^2}p_{\mathrm{x}}p_\vartheta\left(-3\ell\sin\vartheta\cos^2\vartheta\right) - mg\ell\sin\vartheta,\\ \dot{x} &= 2m\frac{1}{D^2}p_{\mathrm{x}}\left(\ell^2 - \frac{1}{2}\ell^2\cos^2\vartheta\right) + m\frac{1}{D^2}p_\vartheta\left(-2\ell + \ell\cos^3\vartheta\right),\end{aligned}$$

and

$$\dot{\vartheta} = m\frac{1}{D^2}p_{\mathrm{x}}\left(-2\ell + \ell\cos^3\vartheta\right) + 2m\frac{1}{D^2}p_\vartheta\left(2 - \cos^2\vartheta\right).$$

If we linearize these we get

$$\dot{p}_{\mathrm{x}} = -m\omega_{\mathrm{s}}^2 x,$$

$$\dot{p}_{\vartheta} = -m\omega_{\mathrm{p}}^2 \ell^2 \vartheta,$$

$$\dot{x} = \frac{1}{m} p_{\mathrm{x}} - \frac{1}{m\ell} p_{\vartheta},$$

and

$$\dot{\vartheta} = -\frac{1}{m\ell} p_{\mathrm{x}} + 2\frac{1}{m\ell^2} p_{\vartheta}.$$

These linear, first order differential equations have exponential solutions. We write the solutions as

$$\begin{bmatrix} p_{\mathrm{x}} \\ p_{\vartheta} \\ x \\ \vartheta \end{bmatrix} = \begin{bmatrix} \tilde{p}_{\mathrm{x}} \\ \tilde{p}_{\vartheta} \\ \tilde{x} \\ \tilde{\vartheta} \end{bmatrix} \exp(i\omega t)$$

In matrix form the canonical equations are then

$$\begin{bmatrix} 0 & 0 & -m\omega_{\mathrm{s}}^2 & 0 \\ 0 & 0 & 0 & -m\ell^2\omega_{\mathrm{p}}^2 \\ \frac{1}{m} & -\frac{1}{m\ell} & 0 & 0 \\ -\frac{1}{m\ell} & 2\frac{1}{m\ell^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_{\mathrm{x}} \\ \tilde{p}_{\vartheta} \\ \tilde{x} \\ \tilde{\vartheta} \end{bmatrix} = i\omega \begin{bmatrix} \tilde{p}_{\mathrm{x}} \\ \tilde{p}_{\vartheta} \\ \tilde{x} \\ \tilde{\vartheta} \end{bmatrix}$$

or

$$\begin{bmatrix} -i\omega & 0 & -m\omega_{\mathrm{s}}^2 & 0 \\ 0 & -i\omega & 0 & -m\ell^2\omega_{\mathrm{p}}^2 \\ \frac{1}{m} & -\frac{1}{m\ell} & -i\omega & 0 \\ -\frac{1}{m\ell} & 2\frac{1}{m\ell^2} & 0 & -i\omega \end{bmatrix} \begin{bmatrix} \tilde{p}_{\mathrm{x}} \\ \tilde{p}_{\vartheta} \\ \tilde{x} \\ \tilde{\vartheta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of this last principal matrix is

$$\omega^4 - 2\omega^2\omega_{\mathrm{p}}^2 - \omega^2\omega_{\mathrm{s}}^2 + \omega_{\mathrm{p}}^2\omega_{\mathrm{s}}^2.$$

If we solve this for ω^2 we have

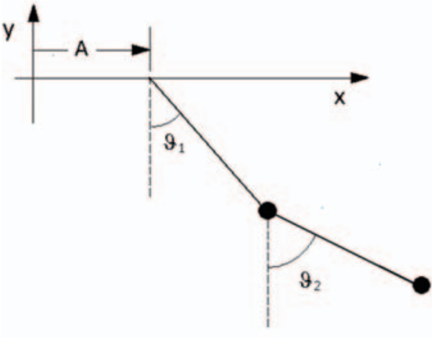
$$\omega^2 = \omega_{\mathrm{p}}^2 + \frac{1}{2}\omega_{\mathrm{s}}^2 \pm \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}.$$

Then

$$\omega = \pm\sqrt{\omega_{\mathrm{p}}^2 + \frac{1}{2}\omega_{\mathrm{s}}^2 \pm \frac{1}{2}\sqrt{4\omega_{\mathrm{p}}^4 + \omega_{\mathrm{s}}^4}}.$$

There are then actually two natural frequencies, since the $\pm$ outside of the final radical does not indicate different natural frequencies.

3.9. In the figure here we have a schematic picture of the double pendulum.



Double pendulum.

Both pendulum lengths are ℓ and both masses are m. Study the motion using the canonical equations. Obtain the modes of natural oscillation for small angles.

Solution:

The position vectors to the masses are

$$\boldsymbol{R}_1 = [A + \ell \sin\vartheta_1]\,\hat{e}_{\mathrm{x}} + [-\ell\cos\vartheta_1]\,\hat{e}_{\mathrm{y}}$$

$$R_2 = [A + \ell\sin\vartheta_1 + \ell\sin\vartheta_2]\,\hat{e}_{\mathrm{x}} + [-\ell\cos\vartheta_1 - \ell\cos\vartheta_2]\,\hat{e}_{\mathrm{y}}.$$

The velocity vectors are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}_1 = \ell\left[\dot{\vartheta}_1\cos\vartheta_1\right]\hat{e}_{\mathrm{x}} + \ell\left[\dot{\vartheta}_1\sin\vartheta_1\right]\hat{e}_{\mathrm{y}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}R_2 = \ell\left[\dot{\vartheta}_1\cos\vartheta_1 + \dot{\vartheta}_2\cos\vartheta_2\right]\hat{e}_{\mathrm{x}} + \ell\left[\dot{\vartheta}_1\sin\vartheta_1 + \dot{\vartheta}_2\sin\vartheta_2\right]\hat{e}_{\mathrm{y}}.$$

The squares of these are

$$\dot{R}_1^2 = \ell^2\dot{\vartheta}_1^2$$

and

$$\begin{aligned}\dot{R}_2^2 &= \ell^2\left[\left(\dot{\vartheta}_1\cos\vartheta_1 + \dot{\vartheta}_2\cos\vartheta_2\right)^2 + \left(\dot{\vartheta}_1\sin\vartheta_1 + \dot{\vartheta}_2\sin\vartheta_2\right)^2\right]\\ &= \ell^2\left[\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right].\end{aligned}$$

From here we have the kinetic energy

$$T = \frac{1}{2}m\ell^2\left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\left(\cos\vartheta_1\cos\vartheta_2 + \sin\vartheta_1\sin\vartheta_2\right)\right]$$
$$= \frac{1}{2}m\ell^2\left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\cos\left(\vartheta_1 - \vartheta_2\right)\right],$$

and the potential energy

$$V = -2mg\ell\cos\vartheta_1 - mg\ell\cos\vartheta_2.$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\ell^2\left[2\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + 2\dot{\vartheta}_1\dot{\vartheta}_2\cos\left(\vartheta_1 - \vartheta_2\right)\right] \\ &\quad +2mg\ell\cos\vartheta_1 + mg\ell\cos\vartheta_2. \end{aligned}$$

The canonical momenta are

$$p_{\vartheta_1} = \frac{\partial L}{\partial\dot{\vartheta}_1} = 2m\ell^2\dot{\vartheta}_1 + m\ell^2\dot{\vartheta}_2\cos\left(\vartheta_1 - \vartheta_2\right)$$

$$p_{\vartheta_2} = \frac{\partial L}{\partial\dot{\vartheta}_2} = m\ell^2\dot{\vartheta}_2 + m\ell^2\dot{\vartheta}_1\cos\left(\vartheta_1 - \vartheta_2\right)$$

The Hamiltonian is then

$$\begin{aligned} \mathcal{H} &= m\ell^2\dot{\vartheta}_1^2 + \frac{1}{2}m\ell^2\dot{\vartheta}_2^2 + m\ell^2\dot{\vartheta}_1\dot{\vartheta}_2\cos\left(\vartheta_1 - \vartheta_2\right) \\ &\quad -2mg\ell\cos\vartheta_1 - mg\ell\cos\vartheta_2 \end{aligned}$$

To obtain the Hamiltonian as a function of the momenta and coordinates we must first obtain $\dot{\vartheta}_1$ and $\dot{\vartheta}_2$ as functions of the momenta p_1 and p_2 and the coordinates ϑ_1 and ϑ_2. That is, we must invert the two equations defining the momenta. Writing the equations in matrix form,

$$m\ell^2\begin{bmatrix} 2 & \cos\left(\vartheta_1 - \vartheta_2\right) \\ \cos\left(\vartheta_1 - \vartheta_2\right) & 1 \end{bmatrix}\begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = \begin{bmatrix} p_{\vartheta_1} \\ p_{\vartheta_2} \end{bmatrix}.$$

Inverting the matrix

$$\begin{aligned} & m\ell^2\begin{bmatrix} 2 & \cos\left(\vartheta_1 - \vartheta_2\right) \\ \cos\left(\vartheta_1 - \vartheta_2\right) & 1 \end{bmatrix}^{-1} \\ &= \frac{1}{m\ell^2}\frac{1}{D}\begin{bmatrix} -1 & \cos\left(\vartheta_1 - \vartheta_2\right) \\ \cos\left(\vartheta_1 - \vartheta_2\right) & -2 \end{bmatrix} \end{aligned}$$

where

$$D = \cos^2\left(\vartheta_1 - \vartheta_2\right) - 2.$$

Solving the set of linear equations, we then have

$$\begin{bmatrix} \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = \frac{1}{m\ell^2}\frac{1}{D}\begin{bmatrix} -p_{\vartheta_1} + p_{\vartheta_2}\cos(\vartheta_1 - \vartheta_2) \\ p_{\vartheta_1}\cos(\vartheta_1 - \vartheta_2) - 2p_{\vartheta_2} \end{bmatrix}.$$

The Hamiltonian is then

$$\begin{aligned}\mathcal{H} = \frac{1}{2m\ell^2}\Bigg\{ & p_{\vartheta_1}^2 \frac{1}{2-\cos^2(\vartheta_1-\vartheta_2)} + p_{\vartheta_2}^2 \frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)} \\ & + p_{\vartheta_1}p_{\vartheta_2} \frac{2\cos^3(\vartheta_1-\vartheta_2) - 4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\Bigg\} - 2mg\ell\cos\vartheta_1 - mg\ell\cos\vartheta_2\end{aligned}$$

The derivatives we need are

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\vartheta_1} = \frac{1}{2m\ell^2}\Bigg\{ & -8p_{\vartheta_1}^2\cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & -16p_{\vartheta_2}^2(\cos(\vartheta_1-\vartheta_2))\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & +p_{\vartheta_1}p_{\vartheta_2}\frac{4}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\left(\frac{1}{2}\sin(3\vartheta_1-3\vartheta_2)+\frac{9}{2}\sin(\vartheta_1-\vartheta_2)\right)\Bigg\} \\ & +2mg\ell\sin\vartheta_1\end{aligned}$$

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial\vartheta_2} = \frac{1}{2m\ell^2}\Bigg\{ & 8p_{\vartheta_1}^2(\cos(\vartheta_1-\vartheta_2))\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & +16p_{\vartheta_2}^2(\cos(\vartheta_1-\vartheta_2))\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & -p_{\vartheta_1}p_{\vartheta_2}\frac{4}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\left(\frac{1}{2}\sin(3\vartheta_1-3\vartheta_2)+\frac{9}{2}\sin(\vartheta_1-\vartheta_2)\right)\Bigg\} \\ & +mg\ell\sin\vartheta_2\end{aligned}$$

$$\frac{\partial\mathcal{H}}{\partial p_{\vartheta_1}} = \frac{1}{2m\ell^2}\left\{p_{\vartheta_1}\frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)} + p_{\vartheta_2}\frac{2\cos^3(\vartheta_1-\vartheta_2)-4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\right\}$$

$$\frac{\partial\mathcal{H}}{\partial p_{\vartheta_2}} = \frac{1}{2m\ell^2}\left\{p_{\vartheta_1}\frac{2\cos^3(\vartheta_1-\vartheta_2)-4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2} + 2p_{\vartheta_2}\frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)}\right\}.$$

And the canonical equations are then

$$\begin{aligned}\dot{p}_{\vartheta_1} = \frac{1}{2m\ell^2}\Bigg\{ & 8p_{\vartheta_1}^2\cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & +16p_{\vartheta_2}^2\cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} \\ & -p_{\vartheta_1}p_{\vartheta_2}\frac{4}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\left(\frac{1}{2}\sin(3\vartheta_1-3\vartheta_2)+\frac{9}{2}\sin(\vartheta_1-\vartheta_2)\right)\Bigg\} \\ & -2mg\ell\sin\vartheta_1\end{aligned}$$

$$\dot{p}_{\vartheta_2} = \frac{1}{2m\ell^2}\left\{-8p_{\vartheta_1}^2 \cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2}\right.$$
$$-16p_{\vartheta_2}^2 \cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2}$$
$$\left.+p_{\vartheta_1}p_{\vartheta_2}\frac{4}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\left(\frac{1}{2}\sin(3\vartheta_1-3\vartheta_2)+\frac{9}{2}\sin(\vartheta_1-\vartheta_2)\right)\right\}$$
$$-mg\ell\sin\vartheta_2$$
$$\dot{\vartheta}_1 = \frac{1}{2m\ell^2}\left\{p_{\vartheta_1}\frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)}+p_{\vartheta_2}\frac{2\cos^3(\vartheta_1-\vartheta_2)-4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\right\}$$
$$\dot{\vartheta}_2 = \frac{1}{2m\ell^2}\left\{p_{\vartheta_1}\frac{2\cos^3(\vartheta_1-\vartheta_2)-4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}+2p_{\vartheta_2}\frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)}\right\}.$$

With

$$\cos(\vartheta_1-\vartheta_2)\frac{\sin(\vartheta_1-\vartheta_2)}{(\cos(2\vartheta_1-2\vartheta_2)-3)^2} = (\vartheta_1-\vartheta_2)+O\left(\vartheta_1^2,\vartheta_2^2\right),$$

$$\frac{4}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2}\left(\frac{1}{2}\sin(3\vartheta_1-3\vartheta_2)+\frac{9}{2}\sin(\vartheta_1-\vartheta_2)\right)$$
$$=24(\vartheta_1-\vartheta_2)+O\left(\vartheta_1^2,\vartheta_2^2\right),\ \frac{2}{2-\cos^2(\vartheta_1-\vartheta_2)} = 2+O\left(\vartheta_1^2,\vartheta_2^2\right),$$

and

$$\frac{2\cos^3(\vartheta_1-\vartheta_2)-4\cos(\vartheta_1-\vartheta_2)}{\left(\cos^2(\vartheta_1-\vartheta_2)-2\right)^2} = -2+O\left(\vartheta_1^2,\vartheta_2^2\right),$$

we linearize the canonical equations as

$$\dot{p}_{\vartheta_1} = -2mg\ell\vartheta_1$$

$$\dot{p}_{\vartheta_2} = -mg\ell\vartheta_2$$

$$\dot{\vartheta}_1 = \frac{1}{m\ell^2}\left(p_{\vartheta_1}-p_{\vartheta_2}\right)$$

$$\dot{\vartheta}_2 = \frac{1}{m\ell^2}\left(-p_{\vartheta_1}+2p_{\vartheta_2}\right).$$

If we make the Ansatz that the solutions are complex exponential (sinusoidal), we obtain the matrix formulation

$$\begin{bmatrix} 0 & 0 & -2mg\ell & 0 \\ 0 & 0 & 0 & -mg\ell \\ \frac{1}{m\ell^2} & -\frac{1}{m\ell^2} & 0 & 0 \\ -\frac{1}{m\ell^2} & 2\frac{1}{m\ell^2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{p}_{\vartheta_1} \\ \tilde{p}_{\vartheta_2} \\ \tilde{\vartheta}_1 \\ \tilde{\vartheta}_2 \end{bmatrix} = i\omega \begin{bmatrix} \tilde{p}_{\vartheta_1} \\ \tilde{p}_{\vartheta_2} \\ \tilde{\vartheta}_1 \\ \tilde{\vartheta}_2 \end{bmatrix}$$

The eigenvalues are the values of $i\omega$, which are

$$i\omega = \pm i\omega_0\sqrt{\left(2-\sqrt{2}\right)}, \pm i\omega_0\sqrt{\left(2+\sqrt{2}\right)},$$

where

$$\omega_0 = \sqrt{\frac{g}{\ell}}.$$

These are the normal modes of vibration.

3.10. In the static case the magnetic field induction vector $\boldsymbol{B}$ is obtained from the vector potential as

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A},$$

provided A satisfies the Coulomb gauge

$$\operatorname{div} \boldsymbol{A} = 0.$$

Show that the magnetic component of the Lorentz electromagnetic force (per unit charge) may be written in terms of the magnetic vector potential $\boldsymbol{A}$ as

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu - \frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu$$

where the velocity of the point charge is $v = \dot{q}_\nu \hat{e}_\nu$. We use the Einstein sum convention in which the repeated Greek indices are summed from 1 to 3 over the indices for the three Cartesian coordinates.

This will require some steps in vector algebra, which are always easier if we use subscript notation for cross products. We use the Levi-Civita density $\varepsilon_{\mu\nu\sigma}$ defined by

$$\varepsilon_{\mu\nu\sigma} = \begin{cases} +1 \text{ if } \mu\nu\sigma \text{ is an even permutation of } 1, 2, 3 \\ -1 \text{ if } \mu\nu\sigma \text{ is an odd permutation of } 1, 2, 3 \\ \ \ 0 \text{ if any 2 of the indices } \mu, \nu, \sigma \text{ are alike} \end{cases}$$

Then the μ^{th} component of $\boldsymbol{v} \times \boldsymbol{B}$ is

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \varepsilon_{\mu\nu\gamma} \dot{q}_\nu B_\gamma$$

and the γ^{th} component of $\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}$ is

$$B_\gamma = \varepsilon_{\gamma\alpha\beta} \frac{\partial A_\beta}{\partial q_\alpha}.$$

Put these together to obtain $(\boldsymbol{v} \times \boldsymbol{B})_\mu$.

Solution:

We begin with

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \varepsilon_{\mu\nu\gamma} \dot{q}_\nu B_\gamma$$

and insert

$$B_\gamma = \varepsilon_{\gamma\alpha\beta} \frac{\partial A_\beta}{\partial q_\alpha},$$

being careful about the subscripts. Then

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \varepsilon_{\mu\nu\gamma} \varepsilon_{\gamma\alpha\beta} \frac{\partial A_\beta}{\partial q_\alpha} \dot{q}_\nu.$$

Moving the γ two places to the right $\varepsilon_{\gamma\alpha\beta} = \varepsilon_{\alpha\beta\gamma}$. Then

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \varepsilon_{\mu\nu\gamma} \varepsilon_{\alpha\beta\gamma} \frac{\partial A_\beta}{\partial q_\alpha} \dot{q}_\nu.$$

We do not sum on μ, which always has a fixed value. But we must decide on the values of the other subscripts. We may have $\alpha = \mu$ and $\nu = \beta$, which has a positive sign, or we may have $\beta = \mu$ and $\nu = \alpha$, which has a negative sign. That is

$$(\boldsymbol{v} \times \boldsymbol{B})_\mu = \frac{\partial A_\nu}{\partial q_\mu} \dot{q}_\nu - \frac{\partial A_\mu}{\partial q_\nu} \dot{q}_\nu.$$

3.11. Show that the canonical equations for a charged particle with charge Q

$$\begin{aligned} \dot{q}_\mu &= \frac{\partial \mathcal{H}}{\partial p_\mu} \\ &= \frac{1}{m} \left(p_\mu - Q A_\mu \right), \end{aligned}$$

and

$$\dot{p}_\mu = -\frac{\partial \mathcal{H}}{\partial q_\mu}$$
$$= \frac{Q}{m}\left(p_\nu - QA_\nu\right)\frac{\partial A_\nu}{\partial q_\mu} - Q\frac{\partial \phi}{\partial q_\mu}$$

result in the standard form of Newton's Second Law

$$m\ddot{q}_\mu = Q\left(\boldsymbol{v} \times \boldsymbol{B}\right)_\mu - QE_\mu.$$

Solution:
From the canonical equation for $\dot{q}$ we have

$$m\dot{q}_\mu = p_\mu - QA_\mu.$$

Differentiating with respect to t,

$$m\ddot{q}_\mu = \dot{p}_\mu - Q\frac{\mathrm{d}A_\mu}{\mathrm{d}t}$$

where

$$\frac{\mathrm{d}A_\mu}{\mathrm{d}t} = \frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu + \frac{\partial A_\mu}{\partial t}.$$

Then, using the canonical equation for $\dot{p}$ we have

$$m\ddot{q}_\mu = \frac{Q}{m}\left(p_\nu - QA_\nu\right)\frac{\partial A_\nu}{\partial q_\mu} - Q\frac{\partial \phi}{\partial q_\mu} - Q\left(\frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu + \frac{\partial A_\mu}{\partial t}\right)$$
$$= Q\left(\frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu - \frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu\right) - Q\frac{\partial \phi}{\partial q_\mu} - Q\left(\frac{\partial A_\mu}{\partial t}\right).$$

We can show, for the time dependent case (see previous exercise), that

$$\left(\boldsymbol{v} \times \boldsymbol{B}\right)_\mu = \frac{\partial A_\nu}{\partial q_\mu}\dot{q}_\nu - \frac{\partial A_\mu}{\partial q_\nu}\dot{q}_\nu,$$

where $\boldsymbol{v} = \dot{q}_\nu \hat{e}_\nu$. Then

$$m\ddot{q}_\mu = Q\left(\boldsymbol{v} \times \boldsymbol{B}\right)_\mu - Q\frac{\partial \phi}{\partial q_\mu} - Q\left(\frac{\partial A_\mu}{\partial t}\right).$$

And for the time dependent case

$$\boldsymbol{E} = -\operatorname{grad}\phi - \frac{\partial}{\partial t}\boldsymbol{A}.$$

Then

$$m\ddot{q}_\mu = Q(\boldsymbol{v} \times \boldsymbol{B})_\mu - QE_\mu.$$

3.12. In the text we show that the low energy (nonrelativistic) form the relativistic Hamiltonian for a classical charged point particle with mass m and charge Q is

$$\mathcal{H} = \frac{1}{2m}\sum_\mu (p_\mu - QA_\mu)^2 + Q\varphi,$$

where φ is that scalar potential from which we find the electric field, in the static case, as $\boldsymbol{E} = -\operatorname{grad}\varphi$. We have used Q to designate the charge because q is the designation for generalized coordinate. The vector potential $\boldsymbol{A}$ has components A_μ ($\mu = 1, 2, 3$). The magnetic field induction $\boldsymbol{B}$ is obtained from the curl of $\boldsymbol{A}$ as

$$\boldsymbol{B} = \operatorname{curl}\boldsymbol{A},$$

and $\boldsymbol{A}$ is limited by the requirement that

$$\operatorname{div}\boldsymbol{A} = 0,$$

which is the Coulomb gauge for time independent fields.

Consider the motion of a charge Q in a region containing only a magnetic field with induction $\boldsymbol{B} = \hat{e}_z B$. Show that this induction results from

$$\begin{aligned}\boldsymbol{A} &= \frac{B}{2}\left(-y\hat{e}_x + x\hat{e}_y\right),\\ &= -By\hat{e}_x,\\ &= Bx\hat{e}_y,\end{aligned}$$

or

$$\boldsymbol{A} = \frac{1}{2}Br\hat{e}_\vartheta$$

in cylindrical coordinates.

Let the charge be released with non-vanishing velocities in the x and y directions. Show that the orbit of the charge is a circle.

a) Using $\boldsymbol{A} = -By\hat{e}_x$

b) Using $\boldsymbol{A} = -\frac{B}{2}y\hat{e}_x + \frac{B}{2}x\hat{e}_y$.

Because we already know that a positive charge moves clockwise in a field with induction $\hat{e}_z B$, we choose initial conditions ($t = 0$) as $x(0) = -R$, $\dot{x}(0) = 0$, $y(0) = 0$, and $\dot{y}(0) = v$.

Solution:

First we must show that the divergence vanishes for each $\boldsymbol{A}$. In rectangular coordinates

$$\begin{aligned}\operatorname{div} \boldsymbol{A} &= \frac{\partial}{\partial x}\left(-\frac{B}{2}y\right) + \frac{\partial}{\partial y}\left(\frac{B}{2}x\right)\\ &= 0 + 0.\end{aligned}$$

And in cylindrical coordinates

$$\begin{aligned}\operatorname{div} \boldsymbol{A} &= \frac{1}{r}\frac{\partial}{\partial \vartheta}\left(\frac{1}{2}Br\hat{e}_\vartheta\right)\\ &= 0.\end{aligned}$$

Then we must show that we get the correct induction.

$$\begin{aligned}\operatorname{curl} \boldsymbol{A} &= \hat{e}_\mathrm{x}\left(-\frac{\partial}{\partial z}\left(\frac{B}{2}x\right)\right) + \hat{e}_\mathrm{y}\left(\frac{\partial}{\partial z}\left(-\frac{B}{2}y\right)\right)\\ &\quad + \hat{e}_\mathrm{z}\left(\frac{\partial}{\partial x}\left(\frac{B}{2}x\right) - \frac{\partial}{\partial y}\left(-\frac{B}{2}y\right)\right)\\ &= \hat{e}_\mathrm{z}B\end{aligned}$$

and

$$\begin{aligned}\operatorname{curl} \boldsymbol{A} &= \hat{e}_\mathrm{y}\left(\frac{\partial}{\partial z}(-By)\right)\\ &\quad + \hat{e}_\mathrm{z}\left(-\frac{\partial}{\partial y}(-By)\right)\\ &= \hat{e}_\mathrm{z}B.\end{aligned}$$

Then

$$\begin{aligned}\operatorname{curl} \boldsymbol{A} &= \hat{e}_\mathrm{r}\left[-\frac{\partial}{\partial z}\left(\frac{1}{2}Br\right)\right]\\ &\quad + \hat{e}_\mathrm{z}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r^2\frac{1}{2}B\right)\right]\\ &= \hat{e}_\mathrm{z}B.\end{aligned}$$

Any of the proposed expressions for the vector potential is then acceptable.

a) Using $\boldsymbol{A} = -By\hat{e}_\mathrm{x}$. With no electric field $\varphi = 0$ and the Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m}\sum_\mu \left(p_\mu - QA_\mu\right)^2\\ &= \frac{1}{2m}\left[(p_\mathrm{x} + QBy)^2 + \left(p_\mathrm{y}\right)^2 + (p_\mathrm{z})^2\right].\end{aligned}$$

The canonical equations are

$$\dot{x} = \frac{\partial}{\partial p_\mathrm{x}}\mathcal{H} = \frac{1}{m}\left(p_\mathrm{x} + m\Omega y\right)$$

$$\dot{y} = \frac{\partial}{\partial p_\mathrm{y}}\mathcal{H} = \frac{1}{m}p_\mathrm{y}$$

$$\dot{z} = \frac{\partial}{\partial p_{\mathrm{z}}}\mathcal{H} = \frac{1}{m}p_{\mathrm{z}}.$$

$$\dot{p}_{\mathrm{x}} = -\frac{\partial}{\partial x}\mathcal{H} = 0$$

$$\dot{p}_{\mathrm{y}} = -\frac{\partial}{\partial y}\mathcal{H} = -\Omega\,(p_{\mathrm{x}} + m\Omega y)$$

$$\dot{p}_{\mathrm{z}} = -\frac{\partial}{\partial z}\mathcal{H} = 0$$

$\Omega = QB/m$ is the cyclotron frequency for the charge. The canonical momenta p_{x} and p_{z} are constants. We are only interested in motion in the (x, y) −plane. We may then either assume $p_{\mathrm{z}} = 0$ or transfer to a plane moving uniformly in the z−direction. We define $p_{\mathrm{x}} = \beta_{\mathrm{x}}$ then

$$\dot{x} = \frac{\beta_{\mathrm{x}}}{m} + \Omega y$$

$$\dot{y} = \frac{1}{m}p_{\mathrm{y}}$$

$$\dot{p}_{\mathrm{y}} = -\Omega\beta_{\mathrm{x}} - m\Omega^2 y.$$

Let us call

$$\Omega Y = \frac{\beta_{\mathrm{x}}}{m} + \Omega y$$

or

$$y = Y - \frac{\beta_{\mathrm{x}}}{m\Omega}.$$

Then

$$\dot{x} = \Omega Y,$$

$$\dot{Y} = \frac{1}{m}p_{\mathrm{y}},$$

and

$$\begin{aligned}\dot{p}_{\mathrm{y}} &= -\Omega\beta_{\mathrm{x}} - m\Omega^2\left(Y - \frac{\beta_{\mathrm{x}}}{m\Omega}\right)\\ &= -m\Omega^2 Y.\end{aligned}$$

These are solved most easily in the complex plane. We write then the quantities p_y, x, and Y as the real parts of complex quantities

$$\begin{aligned} p_y &= \operatorname{Re} \tilde{p} \exp(i\omega t), \\ x &= \operatorname{Re} \tilde{x} \exp(i\omega t), \end{aligned}$$

and

$$Y = \operatorname{Re} \tilde{Y} \exp(i\omega t)$$

We have then equations in the complex plane

$$\begin{aligned} i\omega\tilde{x} - \Omega\tilde{Y} &= 0, \\ i\omega\tilde{Y} - \frac{1}{m}\tilde{p} &= 0, \end{aligned}$$

and

$$i\omega\tilde{p} + m\Omega^2\tilde{Y} = 0.$$

If we write the resulting equations in matrix form we have

$$\begin{bmatrix} 0 & i\omega & -\Omega \\ -\frac{1}{m} & 0 & i\omega \\ i\omega & 0 & m\Omega^2 \end{bmatrix} \begin{bmatrix} \tilde{p} \\ \tilde{x} \\ \tilde{Y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, setting the determinant equal to zero,

$$\omega = \Omega.$$

Then our complex plane solutions are

$$\begin{aligned} p_y &= \tilde{p}_r \cos\Omega t - \tilde{p}_i \sin\Omega t \\ x &= \tilde{x}_r \cos\Omega t - \tilde{x}_i \sin\Omega t \\ Y &= \tilde{Y}_r \cos\Omega t - \tilde{Y}_i \sin\Omega t. \end{aligned}$$

From the initial conditions, and considering the values of $y(t)$ when $t = \tau/2$ and $\tau/4$, we have $\beta_x = 0$, $v = R\Omega$ and

$$\begin{aligned} x &= -R\cos\Omega t \\ y &= R\sin\Omega t, \end{aligned}$$

which is a circle.

b) Here we encounter a more complicated solution, which is interesting mathematically. For practical purposes, however, if a simple solution is our goal, we do not go here.

The Hamiltonian is then

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\sum_{\mu}\left(p_{\mu} - QA_{\mu}\right)^2 \\
&= \frac{1}{2m}\left(p_{\mathrm{x}} - QA_{\mathrm{x}}\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - QA_{\mathrm{y}}\right)^2 \\
&= \frac{1}{2m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right)^2 .
\end{aligned}
$$

The canonical equations are

$$
\begin{aligned}
\dot{p}_{\mathrm{x}} &= -\frac{\partial\mathcal{H}}{\partial x} = \frac{QB}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right) \\
&= \frac{1}{2}\Omega p_{\mathrm{y}} - \frac{1}{4}m\Omega^2 x
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{p}_{\mathrm{y}} &= -\frac{\partial\mathcal{H}}{\partial y} = -\frac{QB}{2m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right) \\
&= -\frac{1}{2}\Omega p_{\mathrm{x}} - \frac{1}{4}m\Omega^2 y,
\end{aligned}
$$

where $\Omega = QB/m$ is the cyclotron frequency.

And

$$
\begin{aligned}
\dot{x} &= \frac{\partial\mathcal{H}}{\partial p_{\mathrm{x}}} = \frac{1}{m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right) \\
&= \frac{1}{m}p_{\mathrm{x}} + \frac{1}{2}\Omega y
\end{aligned}
$$

$$
\begin{aligned}
\dot{y} &= \frac{\partial\mathcal{H}}{\partial p_{\mathrm{y}}} = \frac{1}{m}\left(p_{\mathrm{y}} - \frac{QB}{2c}x\right) \\
&= \frac{1}{m}p_{\mathrm{y}} - \frac{1}{2}\Omega x.
\end{aligned}
$$

The motion in the (x, y) plane is then described by the equations

$$\dot{p}_{\mathrm{x}} = \frac{1}{2}\Omega p_{\mathrm{y}} - \frac{1}{4}m\Omega^2 x,$$

$$\dot{p}_{\mathrm{y}} = -\frac{1}{2}\Omega p_{\mathrm{x}} - \frac{1}{4}m\Omega^2 y,$$

and

$$\dot{x} = \frac{1}{m}p_{\mathrm{x}} + \frac{1}{2}\Omega y,$$

$$\dot{y} = \frac{1}{m}p_{\mathrm{y}} - \frac{1}{2}\Omega x.$$

To solve these equations it is easiest to work in the complex plane. We define the complex quantities

$$Z = x + iy$$

and

$$P_{\mathrm{Z}} = p_{\mathrm{x}} + ip_{\mathrm{y}}.$$

Then the canonical equations combine to give

$$\begin{aligned}\dot{P}_{\mathrm{Z}} &= \dot{p}_{\mathrm{x}} + i\dot{p}_{\mathrm{y}} \\ &= \frac{1}{2}\Omega\left(p_{\mathrm{y}} - ip_{\mathrm{x}}\right) - \frac{1}{4}m\Omega^2\left(x + iy\right) \\ &= -\frac{i}{2}\Omega\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - \frac{1}{4}m\Omega^2\left(x + iy\right) \\ &= -\frac{i}{2}\Omega P_{\mathrm{Z}} - \frac{1}{4}m\Omega^2 Z\end{aligned}$$

and

$$\begin{aligned}\dot{Z} &= \dot{x} + i\dot{y} \\ &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) + \frac{1}{2}\Omega\left(y - ix\right) \\ &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - \frac{i}{2}\Omega\left(x + iy\right) \\ &= \frac{P_{\mathrm{Z}}}{m} - i\frac{\Omega}{2}Z.\end{aligned}$$

So we have the equations

$$\dot{P}_{\mathrm{Z}} = -\frac{i}{2}m\Omega P_{\mathrm{Z}} - \frac{1}{4}m\Omega^2 Z,$$

and

$$\dot{Z} = \frac{P_{\mathrm{Z}}}{m} - i\frac{\Omega}{2}Z.$$

These equations are obviously solved by the Ansatz

$$P_{\mathrm{Z}} = \tilde{P}_{\mathrm{Z}} \exp\left(i\omega t\right)$$

and

$$Z = \tilde{Z} \exp\left(i\omega t\right).$$

The Ω can be found by inserting the Ansatz into the equations. That is

$$i\omega\tilde{P}_Z = -\frac{i}{2}\Omega\tilde{P}_Z - \frac{1}{4}m\Omega^2\tilde{Z}$$

and

$$i\omega\tilde{Z} = \frac{\tilde{P}_Z}{m} - i\frac{\Omega}{2}\tilde{Z}.$$

In matrix form,

$$\begin{bmatrix} -i\frac{1}{2}\Omega & -\frac{1}{4}m\Omega^2 \\ \frac{1}{m} & -i\frac{1}{2}\Omega \end{bmatrix}\begin{bmatrix} \tilde{P}_Z \\ \tilde{Z} \end{bmatrix} = i\omega\begin{bmatrix} \tilde{P}_Z \\ \tilde{Z} \end{bmatrix}$$

This has a nontrivial solution only when the determinant of the matrix

$$\begin{bmatrix} -i\frac{1}{2}\Omega & -\frac{1}{4}m\Omega^2 \\ \frac{1}{m} & -i\frac{1}{2}\Omega \end{bmatrix} - i\omega\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}i\Omega - i\omega & -\frac{1}{4}m\Omega^2 \\ \frac{1}{m} & -\frac{1}{2}i\Omega - i\omega \end{bmatrix}$$

vanishes. That is when

$$\Omega\omega + \omega^2 = 0$$

which is to say

$$\omega = 0, -\Omega.$$

When $\omega = -\Omega$,

$$\begin{aligned} &\begin{bmatrix} -\frac{1}{2}i\Omega + i\Omega & -\frac{1}{4}m\Omega^2 \\ \frac{1}{m} & -\frac{1}{2}i\Omega + i\Omega \end{bmatrix}\begin{bmatrix} \tilde{P}_Z \\ \tilde{Z} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}i\Omega & -\frac{1}{4}m\Omega^2 \\ \frac{1}{m} & \frac{1}{2}i\Omega \end{bmatrix}\begin{bmatrix} \tilde{P}_Z \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

or

$$\frac{1}{m}\tilde{P}_Z + \frac{1}{2}i\Omega\tilde{Z} = 0,$$

which is

$$\tilde{P}_Z = -\frac{m}{2}i\Omega\tilde{Z},$$

The solution vector is then

$$\begin{bmatrix} \tilde{P}_Z \\ \tilde{Z} \end{bmatrix} = \tilde{Z}\begin{bmatrix} -\frac{m}{2}i\Omega \\ 1 \end{bmatrix}$$

Recalling that

$$\begin{aligned} Z(t) &= \exp(-i\Omega t)\,\tilde{Z} \\ &= x(t) + iy(t), \end{aligned}$$

we have

$$\begin{aligned} x(t) &= \operatorname{Re}(Z(t)) = \left|\tilde{Z}\right| \cos(\Omega t) \\ y(t) &= \operatorname{Im}(Z(t)) = -\left|\tilde{Z}\right| \sin(\Omega t), \end{aligned}$$

noticing the negative sign in the complex exponential. The trajectory is a plot of $y(t)$ versus $x(t)$. The geometrical form of this figure is established by the trigonometric identity $\sin^2\alpha + \cos^2\alpha = 1$. that is

$$x(t)^2 + y(t)^2 = \left|\tilde{Z}\right|^2,$$

which is a circle of radius $\left|\tilde{Z}\right|$.

At time $t = 0$, the location of the charge is $\left(\left|\tilde{Z}\right|, 0\right)$, that is on the $x-$ axis at the point $\left|\tilde{Z}\right|$ from the origin. The momentum at that point is $\left(0, -\frac{m}{2}\Omega\left|\tilde{Z}\right|\right)$. Recalling that

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} = m\dot{q}_\mu + \frac{Q}{c}A_\mu,$$

we have

$$\begin{aligned} m\dot{x} &= p_x - \frac{Q}{c}\left(-\frac{B}{2}y\right) \\ &= p_x + \frac{1}{2}m\Omega y \end{aligned}$$

and

$$\begin{aligned} m\dot{y} &= p_y - \frac{Q}{c}\left(\frac{B}{2}x\right) \\ &= p_y - \frac{1}{2}m\Omega x. \end{aligned}$$

That is, the velocity vector is

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{p_x}{m} + \frac{1}{2}\Omega y \\ \frac{p_y}{m} - \frac{1}{2}\Omega x \end{bmatrix}$$

at the time $t = 0$ we have the velocity vector $(v_x, v_y) = \left(0, -\Omega\left|\tilde{Z}\right|\right)$. In other words, looking vertically down on the plane, the charge is moving clockwise in a circle centered on the origin. A quick “right hand rule” check on the direction of the magnetic force will show this to be what is expected. We notice that the kinetic energy does not change. That is, the magnitude of the velocity is constant.

3.13. Consider the motion of a charge Q in a region containing only a magnetic field. Let the charge be released with non-vanishing velocities in the x, y, and z directions. Show that the charge will ”spiral” along the magnetic field lines. Assume that the magnetic field changes slowly in space so that it may always be considered approximately constant. This phenomenon is important in plasma physics and forms the core of some of the ideas proposed for “trapping” charges in a fusion reactor. The radiation from such charge motion also forms the “northern lights”.

Solution:

In the text we have shown that the Hamiltonian for the charged particle in the electromagnetic field is generally

$$\mathcal{H} = \frac{1}{2m} \sum_{\mu} \left(p_\mu - QA_\mu\right)^2 + Q\varphi.$$

For a static magnetic field along the direction $\hat{e}_z$ the vector potential is

$$\boldsymbol{A} = \frac{B}{2}\left(-y\hat{e}_{\mathrm{x}} + x\hat{e}_{\mathrm{y}}\right)$$

and for no electric field a the electrostatic potential is

$$\varphi = 0.$$

The Hamiltonian is then

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2m} \sum_{\mu} \left(p_\mu - QA_\mu\right)^2 \\
&= \frac{1}{2m}\left(p_{\mathrm{x}} - QA_{\mathrm{x}}\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - QA_{\mathrm{y}}\right)^2 + \frac{1}{2m}p_{\mathrm{z}}^2 \\
&= \frac{1}{2m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right)^2 + \frac{1}{2m}p_{\mathrm{z}}^2.
\end{aligned}$$

The canonical equations are then

$$\dot{p}_{\mathrm{x}} = -\frac{\partial \mathcal{H}}{\partial x} = \frac{QB}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right)$$

$$\dot{p}_{\mathrm{y}} = -\frac{\partial \mathcal{H}}{\partial y} = -\frac{QB}{2m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right)$$

$$\dot{p}_{\mathrm{z}} = 0$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_{\mathrm{x}}} = \frac{1}{m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right)$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_{\mathrm{y}}} = \frac{1}{m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right).$$

$$\dot{z} = \frac{\partial \mathcal{H}}{\partial p_{\mathrm{z}}} = \frac{1}{m}p_{\mathrm{z}}.$$

The momentum p_{z} is constant and the coordinate z varies linearly with time. The motion in the $z-$ direction can then be accounted for by transforming to a coordinate frame which is moving with the $z-$velocity initially imparted to the particle. We then have a problem in the (x, y) plane described by the equations

$$\dot{p}_{\mathrm{x}} = -\frac{\partial \mathcal{H}}{\partial x} = \frac{QB}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right)$$

$$\dot{p}_{\mathrm{y}} = -\frac{\partial \mathcal{H}}{\partial y} = -\frac{QB}{2m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right),$$

and

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_{\mathrm{x}}} = \frac{1}{m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right)$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_{\mathrm{y}}} = \frac{1}{m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right).$$

To solve these equations it is easiest to work in the complex plane. We define the complex quantities

$$Z = x + iy$$

and

$$P_{\mathrm{Z}} = p_{\mathrm{x}} + ip_{\mathrm{y}}.$$

Then the canonical equations combine to give

$$\begin{aligned}
\dot{P}_{\mathrm{Z}} &= \dot{p}_{\mathrm{x}} + i\,\dot{p}_{\mathrm{y}} \\
&= \frac{QB}{2m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right) - \frac{QB}{2m}i\left(p_{\mathrm{x}} + \frac{QB}{2}y\right) \\
&= -\frac{Q^2B^2}{4m}(x + iy) - \frac{QB}{2m}i\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) \\
&= -\frac{Q^2B^2}{4m}Z - \frac{QB}{2m}iP_{\mathrm{Z}},
\end{aligned}$$

and

$$\begin{aligned}\dot{Z} &= \dot{x} + i\dot{y} \\ &= \frac{1}{m}\left(p_{\mathrm{x}} + \frac{QB}{2}y\right) + i\frac{1}{m}\left(p_{\mathrm{y}} - \frac{QB}{2}x\right) \\ &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - i\frac{QB}{2m}(x + iy) \\ &= \frac{P_{\mathrm{Z}}}{m} - i\frac{QB}{2m}Z.\end{aligned}$$

So we have the equations

$$\dot{P}_{\mathrm{Z}} = -\frac{Q^2B^2}{4m}Z - \frac{QB}{2m}iP_{\mathrm{Z}},$$

and

$$\dot{Z} = \frac{P_{\mathrm{Z}}}{m} - i\frac{QB}{2m}Z.$$

These equations are obviously solved by the Ansatz

$$P_{\mathrm{Z}} = \tilde{P}_{\mathrm{Z}} \exp(i\Omega t)$$

and

$$Z = \tilde{Z} \exp(i\Omega t).$$

The Ω can be found by inserting the Ansatz into the equations. That is

$$i\Omega\tilde{P}_{\mathrm{Z}} = -\frac{Q^2B^2}{4m}\tilde{Z} - \frac{QB}{2m}i\tilde{P}_{\mathrm{Z}}$$

and

$$i\Omega\tilde{Z} = \frac{\tilde{P}_{\mathrm{Z}}}{m} - i\frac{QB}{2m}\tilde{Z}.$$

In matrix form,

$$\begin{bmatrix} i\left(\frac{QB}{2m} + \Omega\right) & \frac{Q^2B^2}{4m} \\ \frac{1}{m} & -i\left(\frac{QB}{2m} + \Omega\right) \end{bmatrix} \begin{bmatrix} \tilde{P}_{\mathrm{Z}} \\ \tilde{Z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This has a nontrivial solution only when the determinant vanishes. That is when

$$\frac{\Omega}{m}(QB + \Omega m) = 0,$$

which is to say

$$\Omega = -\frac{QB}{m}.$$

The orbit is then easily obtained from the $Z(t)$. Talking real and imaginary parts,

$$x(t) = \mathrm{Re}(Z(t)) = x_0 \cos(\Omega t)$$
$$y(t) = \mathrm{Im}(Z(t)) = -y_0 \sin(\Omega t).$$

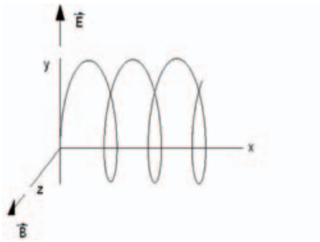
This orbit is a circle. Looking down on the $(x, y)-$plane, the direction of motion is clockwise.

We may now couple this motion with uniform motion along the (original) $z-$axis. The result is a spiral motion along the $z-$axis. Small variations in the magnetic field along the $z-$axis will not disturb this general spiral motion. We may then consider the charges as "trapped" in the spiral paths along the magnetic field lines.

3.14. Consider the motion of a charged particle in a region of space in which there is a uniform magnetic field with induction $\boldsymbol{B} = \hat{e}_z B$, with the vector potential

$$\boldsymbol{A} = -\hat{e}_x \frac{B}{2} y + \hat{e}_y \frac{B}{2} x$$

and a uniform electric field $\boldsymbol{E} = \hat{e}_y E$. Show that the motion is cycloidal as we have shown here.



The trajectory of a charged particle moving in a region containing electric and magnetic fields perpendicular to one another.

Solution:

For a static magnetic field with induction $\boldsymbol{B} = \hat{e}_z B$ the vector potential is

$$\boldsymbol{A} = -\hat{e}_x \frac{B}{2} y + \hat{e}_y \frac{B}{2} x.$$

And for a static electric field $\boldsymbol{E} = \hat{e}_y E$ the electrostatic potential is

$$\varphi = -Ey.$$

The Hamiltonian, in rectangular Cartesian coordinates, is then

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m}\left(p_{\mathrm{x}} - QA_{\mathrm{x}}\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - QA_{\mathrm{y}}\right)^2 - QEy \\ &= \frac{1}{2m}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right)^2 + \frac{1}{2m}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right)^2 - QEy.\end{aligned}$$

where $\Omega = QB/m$. Motion is then entirely in the (x, y) plane.

The canonical equations are

$$\begin{aligned}\dot{x} &= \frac{1}{m}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right) \\ \dot{y} &= \frac{1}{m}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right) \\ \dot{p}_{\mathrm{x}} &= \frac{\Omega}{2}\left(p_{\mathrm{y}} - m\frac{\Omega}{2}x\right) \\ \dot{p}_{\mathrm{y}} &= -\frac{\Omega}{2}\left(p_{\mathrm{x}} + m\frac{\Omega}{2}y\right) + QE\end{aligned}$$

These equations are nonhomogeneous because of the presence of the term QE.

We (as before) simplify the problem if we introduce the complex variables $Z = x + iy$ and $P_{\mathrm{z}} = p_{\mathrm{x}} + ip_{\mathrm{y}}$. We can then combine the canonical equations to give

$$\begin{aligned}\dot{Z} &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - i\frac{1}{2}\Omega\left(x + iy\right) \\ &= \frac{1}{m}P_{\mathrm{Z}} - i\frac{1}{2}\Omega Z.\end{aligned}$$

and

$$\begin{aligned}\dot{P}_{\mathrm{Z}} &= -\frac{1}{4}m\Omega^2\left(x + iy\right) - i\frac{1}{2}\Omega\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) + iQE \\ &= -\frac{1}{4}m\Omega^2 Z - i\frac{1}{2}\Omega P_{\mathrm{Z}} + iQE,\end{aligned}$$

Introduce

$$Z = Z' + \beta t$$

$$\dot{Z} = \dot{Z}' + \beta$$

Then

$$\begin{aligned}\dot{Z}' + \beta &= \frac{1}{m}\left(p_{\mathrm{x}} + ip_{\mathrm{y}}\right) - i\frac{1}{2}\Omega\left(x + iy\right) \\ &= \frac{1}{m}P_{\mathrm{Z}} - i\frac{1}{2}\Omega Z' - i\frac{1}{2}\Omega\beta t.\end{aligned}$$

or

$$\dot{Z}' = \frac{1}{m} P_Z - i\frac{1}{2}\Omega Z' - i\frac{1}{2}\Omega\beta t - \beta.$$

and

$$\begin{aligned}\dot{P}_Z &= -\frac{1}{4}m\Omega^2 Z - i\frac{1}{2}\Omega P_Z + iQE \\ &= -\frac{1}{4}m\Omega^2 Z' - i\frac{1}{2}\Omega P_Z - \frac{1}{4}m\Omega^2\beta t + iQE\end{aligned}$$

From the first two of the canonical equations we see that

$$P_Z = m\left(\dot{Z}\right) + im\left(\frac{\Omega}{2}\right)(Z)$$

whether or not an electric field is present.
With Z' this is

$$\begin{aligned}P_Z &= m\left(\dot{Z}\right) + im\left(\frac{\Omega}{2}\right)(Z) \\ &= m\dot{Z}' + im\left(\frac{\Omega}{2}\right)Z' + m\beta + im\left(\frac{\Omega}{2}\right)\beta t.\end{aligned}$$

If we require that

$$P_Z' = m\left(\dot{Z}'\right) + im\left(\frac{\Omega}{2}\right)\left(Z'\right)$$

We must define

$$P_Z = P_Z' + m\beta + im\left(\frac{\Omega}{2}\right)\beta t$$

Our equation for $\dot{P}_z$ then becomes

$$\begin{aligned}\dot{P}_Z &= \dot{P}_Z' + im\left(\frac{\Omega}{2}\right)\beta \\ &= -\frac{1}{4}m\Omega^2 Z' - i\frac{1}{2}\Omega P_Z' - im\left(\Omega\right)\beta + iQE,\end{aligned}$$

We recall that the equations for no electric field are

$$\begin{aligned}\dot{Z} &= \frac{1}{m}\left(p_x + ip_y\right) - \frac{1}{2}\Omega\left(ix - y\right) \\ &= \frac{1}{m}P_Z - \frac{1}{2}\Omega i Z\end{aligned}$$

and for P_Z

$$\begin{aligned}\dot{P}_Z &= -\frac{1}{2}\Omega\left(ip_x - p_y\right) - m\left(\frac{1}{2}\Omega\right)^2(x+iy)\\ &= -\frac{1}{2}\Omega i P_Z - m\left(\frac{1}{2}\Omega\right)^2 Z.\end{aligned}$$

Therefore, if we choose

$$-im\,(\Omega)\,\beta + iQE = 0$$

our solution is that for a charged particle moving in only a static magnetic field. That is, if

$$\beta = \frac{Q}{m\Omega}E = \frac{E}{B}$$

then

$$\begin{aligned}x(t) &= \mathrm{Re}\,(Z) = R\cos(\Omega t) + \frac{E}{B}t\\ y(t) &= \mathrm{Im}\,(Z) = -R\sin(\Omega t),\end{aligned}$$

which is a cycloid.

3.15. If we treat the motion of a charged point particle of mass m and charge Q moving in a constant magnetic field of induction $\boldsymbol{B} = \hat{e}_z B$ using cylindrical coordinates the vector potential is

$$\boldsymbol{A} = \frac{1}{2}Br\hat{e}_\vartheta.$$

We note that

$$\mathrm{div}\,\boldsymbol{A} = \frac{1}{r}\frac{\partial}{\partial\vartheta}\left(\frac{1}{2}Br\right) = 0$$

and with

$$\begin{aligned}\mathrm{curl}\,\boldsymbol{F} = \hat{e}_r\left[\frac{1}{r}\frac{\partial F_z}{\partial\vartheta} - \frac{\partial F_\vartheta}{\partial z}\right] + \hat{e}_\vartheta\left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right]\\ + \hat{e}_z\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\vartheta) - \frac{\partial F_r}{\partial\vartheta}\right]\end{aligned}$$

that

$$\mathrm{curl}\,\boldsymbol{A} = \hat{e}_z\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{2}Br^2\right) = \hat{e}_z\,(B).$$

So the vector potential above satisfies the Coulomb gauge and produces the magnetic field induction we desire. Obtain the canonical equations and the (constant) Hamiltonian for this situation.

In the cylindrical case it will be easiest to simply begin with the low energy (nonrelativistic) approximation to the electromagnetic Lagrangian we developed in our chapter on special relativity. The Lagrangian is the function appearing in the Hamilton's Principal Function. So we return to the Lagrangian for anything other than rectangular Cartesian coordinates. This is

$$L = \frac{1}{2} m \dot{q}_\mu \dot{q}_\mu - Q\varphi + Q A_\mu \dot{q}_\mu$$

where the summation is over the three spatial components.

Solution:

To identify the canonical momenta we begin with the Lagrangian above in cylindrical coordinates

$$\begin{aligned} L &= \frac{1}{2} m \dot{q}_\mu \dot{q}_\mu - Q\varphi + Q A_\mu \dot{q}_\mu \\ &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + \dot{z}^2 \right) + \frac{1}{2} Q B r^2 \dot{\vartheta}. \end{aligned}$$

The (spatial components of the) canonical momenta are then

$$\begin{aligned} p_{\mathrm{r}} &= m\dot{r} \\ p_\vartheta &= m r^2 \dot{\vartheta} + \frac{1}{2} Q B r^2 \\ p_{\mathrm{z}} &= m\dot{z}. \end{aligned}$$

The Legendre transformation producing the Hamiltonian (for cylindrical coordinates) is then

$$\begin{aligned} \mathcal{H} &= \frac{1}{2} m \left(\dot{r}^2 + \dot{z}^2 \right) + \frac{1}{2} m r^2 \left(\frac{p_\vartheta}{m r^2} - \frac{\Omega}{2} \right)^2 \\ &= \frac{1}{2m} \left(p_{\mathrm{r}}^2 + p_{\mathrm{z}}^2 \right) + \frac{1}{2} m r^2 \left(\frac{p_\vartheta}{m r^2} - \frac{\Omega}{2} \right)^2, \end{aligned}$$

where $\Omega = QB/m$ is the cyclotron frequency.

The only coordinate in the Hamiltonian is r. Then

$$\begin{aligned} \dot{p}_{\mathrm{r}} = -\frac{\partial \mathcal{H}}{\partial r} &= -mr \left(\frac{p_\vartheta}{m r^2} - \frac{\Omega}{2} \right)^2 + 2\frac{p_\vartheta}{r} \left(\frac{p_\vartheta}{m r^2} - \frac{\Omega}{2} \right) \\ &= mr \left(\frac{p_\vartheta}{m r^2} + \frac{\Omega}{2} \right) \left(\frac{p_\vartheta}{m r^2} - \frac{\Omega}{2} \right) \end{aligned}$$

and

$$\dot{p}_\vartheta = \dot{p}_{\mathrm{z}} = 0.$$

We gain no insight by choosing p_{z} to be anything other than zero. And we shall call $p_\vartheta = \ell$. Then

$$\ell = mr^2\left(\dot{\vartheta} + \frac{1}{2}\Omega\right)$$

and

$$\dot{p}_{\mathrm{r}} = mr\left(\frac{\ell}{mr^2} + \frac{\Omega}{2}\right)\left(\frac{\ell}{mr^2} - \frac{\Omega}{2}\right).$$

With

$$\frac{\ell}{mr^2} = \dot{\vartheta} + \frac{\Omega}{2}$$

we have

$$\dot{p}_{\mathrm{r}} = mr\left(\dot{\vartheta} + \Omega\right)\left(\dot{\vartheta}\right).$$

and

$$\dot{r} = \frac{1}{m}p_{\mathrm{r}}$$

as the canonical equations.

The Lagrangian does not depend explicitly on the time. Therefore the Hamiltonian is a constant of the motion. We call this constant $\mathcal{E}$. That is

$$\begin{aligned}\mathcal{H} = \mathcal{E} &= \frac{1}{2m}p_{\mathrm{r}}^2 + \frac{1}{2}mr^2\left(\frac{\ell}{mr^2} - \frac{\Omega}{2}\right)^2\\ &= \frac{1}{2m}p_{\mathrm{r}}^2 + \frac{1}{2}mr^2\dot{\vartheta}^2,\end{aligned}$$

and release the charge with $p_{\mathrm{r}} = 0$. Then initially $\dot{\vartheta}_0 = -\Omega$ and

$$\mathcal{E} = \frac{1}{2}mr_0^2\Omega^2.$$

We note that the canonical equations and the Hamiltonian are consistent with the circular orbit with $\dot{\vartheta} - \Omega$. But the canonical equations are nonlinear. So we cannot guarantee that this is the only solution.

3.16. In the early work on magnetic confinement of fusion plasmas we considered magnetic bottles to trap the electric charges. Magnetic fields of (almost) any geometry can be produced by arrangements of external electric currents. Magnetic bottles are based on the universal principle that charged particles move on circles with radii that decrease with increasing magnetic induction.

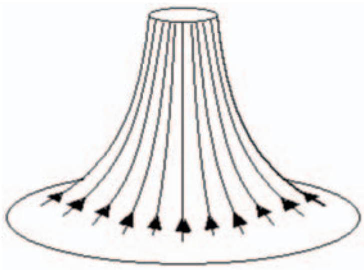
The vector potential

$$\boldsymbol{A} = -\hat{e}_{\mathrm{x}}y\frac{B}{2}\exp(az) + \hat{e}_{\mathrm{y}}x\frac{B}{2}\exp(az),$$

for example, produces the magnetic induction

$$B_{\mathrm{x}} = -x\left(a\frac{B}{2}\right)\exp(az)$$
$$B_{\mathrm{y}} = -y\left(a\frac{B}{2}\right)\exp(az)$$
$$B_{\mathrm{z}} = B\exp(az)\,,$$

which, with $z-$axis vertical, has the form shown here.

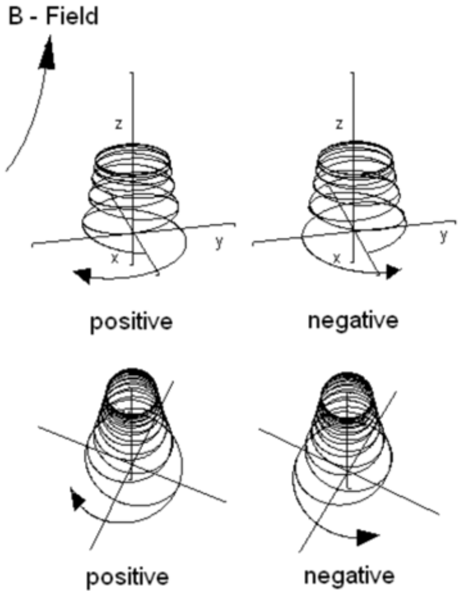


Magnetic field induction $\boldsymbol{B}$ from the vector potential $\boldsymbol{A} = -\hat{e}_{\mathrm{x}}y\frac{B}{2}\exp(az) + \hat{e}_{\mathrm{y}}x\frac{B}{2}\exp(az)$.

Charged particles in this region follow the lines of induction in a corkscrew motion of decreasing radius.

We obtained the particle trajectory from a numerical integration of the canonical equations using a Runge-Kutta algorithm[1]. In the numerical solution we released the charged particle on the $x-$axis at $x = 1$ with a momentum in the $y-$and $z-$directions. The result was the trajectory shown here.

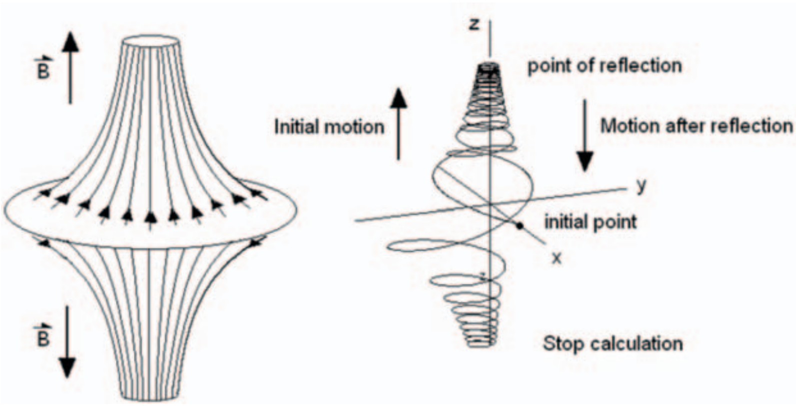
[1] These very important numerical techniques for the solution of first order differential equations were developed around 1900 by the German mathematicians C. Runge and M.W. Kutta.



Spiral motion of charges in spatially varying magnetic field.

In this figure we have plotted results for both positive and for negative charges. The charges spiral along the magnetic field lines moving in the positive $z-$direction until they are deflected and then they spiral out with growing radius along the negative $z-$direction. The top images are for a small initial momentum and the bottom for a larger initial momentum. The larger momentum makes the spiral of the charge more evident.

The results from a region containing oppositely converging magnetic fields demonstrates the magnetic bottle effect. We show this in the figure.



Spiral motion of charges in spatially varying magnetic field.

Show that the vector potential actually results in the magnetic field induction above and that the Coulomb gauge div $\boldsymbol{A} = 0$ is satisfied by $\boldsymbol{A}$. Then obtain the

Hamiltonian and the canonical equations for motion of a charge in this field. Consider planes of constant z to show the decrease in radius with increasing z.

Solution:

With

$$\boldsymbol{A} = -\hat{e}_{\mathrm{x}} y \frac{B}{2} \exp(az) + \hat{e}_{\mathrm{y}} x \frac{B}{2} \exp(az),$$

$$\begin{aligned}
\operatorname{curl} \boldsymbol{A} &= \hat{e}_{\mathrm{x}} \left(\frac{\partial A_{\mathrm{z}}}{\partial y} - \frac{\partial A_{\mathrm{y}}}{\partial z} \right) + \hat{e}_{\mathrm{y}} \left(\frac{\partial A_{\mathrm{x}}}{\partial z} - \frac{\partial A_{\mathrm{z}}}{\partial x} \right) \\
&\quad + \hat{e}_{\mathrm{z}} \left(\frac{\partial A_{\mathrm{y}}}{\partial x} - \frac{\partial A_{\mathrm{x}}}{\partial y} \right) \\
&= -\hat{e}_{\mathrm{x}} x \left(a \frac{B}{2} \right) \exp(az) - \hat{e}_{\mathrm{y}} y \left(a \frac{B}{2} \right) \exp(az) + \hat{e}_{\mathrm{z}} B \exp(az)
\end{aligned}$$

and

$$\operatorname{div} \boldsymbol{A} = \frac{\partial A_{\mathrm{x}}}{\partial x} + \frac{\partial A_{\mathrm{y}}}{\partial y} + \frac{\partial A_{\mathrm{z}}}{\partial z} = 0.$$

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left[\left(p_{\mathrm{x}} + \frac{m\Omega}{2} y \exp(az) \right)^2 + \left(p_{\mathrm{y}} - \frac{m\Omega}{2} x \exp(az) \right)^2 \right] + \frac{1}{2m} p_{\mathrm{z}}^2.$$

With $\Omega = QB/m$. The canonical equations are

$$\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p_{\mathrm{x}}} = \frac{1}{m} p_{\mathrm{x}} + \frac{1}{2} \Omega y \exp(az) \\
\dot{y} &= \frac{\partial \mathcal{H}}{\partial p_{\mathrm{y}}} = \frac{1}{m} p_{\mathrm{y}} - \frac{1}{2} \Omega x \exp(az) \\
\dot{z} &= \frac{\partial \mathcal{H}}{\partial p_{\mathrm{z}}} = \frac{1}{m} p_{\mathrm{z}} \\
\dot{p}_{\mathrm{x}} &= -\frac{\partial \mathcal{H}}{\partial x} = \frac{1}{2} \Omega \left(p_{\mathrm{y}} - \frac{1}{2} m\Omega x \exp(az) \right) \exp(az) \\
\dot{p}_{\mathrm{y}} &= -\frac{\partial \mathcal{H}}{\partial y} = -\frac{1}{2} \Omega \left(p_{\mathrm{x}} + \frac{1}{2} m\Omega y \exp(az) \right) \exp(az) \\
\dot{p}_{\mathrm{z}} &= -\frac{\partial \mathcal{H}}{\partial z} = -\frac{1}{2} \Omega a \left[\left(p_{\mathrm{x}} + \frac{1}{2} m\Omega y \exp(az) \right) y \right. \\
&\quad \left. - \left(p_{\mathrm{y}} - \frac{1}{2} m\Omega x \exp(az) \right) x \right] \exp(az).
\end{aligned}$$

We notice that the $\dot{p}_{\mathrm{z}}$ equation is nonlinear because of the terms in x^2 and y^2. We may ignore this difficulty if we confine our interest to planes of constant z assuming that the variation of induction with z is weak, i.e. that a is small. To keep the equations simple we shall define $\Omega_\mathrm{z} = \Omega \exp(az)$. The canonical equations in a plane of constant z are then

$$\begin{aligned}
\dot{x} &= \frac{\partial \mathcal{H}}{\partial p_\mathrm{x}} = \frac{1}{m} p_\mathrm{x} + \frac{1}{2}\Omega_\mathrm{z} y \\
\dot{y} &= \frac{\partial \mathcal{H}}{\partial p_\mathrm{y}} = \frac{1}{m} p_\mathrm{y} - \frac{1}{2}\Omega_\mathrm{z} x \\
\dot{p}_\mathrm{x} &= -\frac{\partial \mathcal{H}}{\partial x} = \frac{1}{2}\Omega_\mathrm{z}\left(p_\mathrm{y} - \frac{1}{2} m \Omega_\mathrm{z} x\right) \\
\dot{p}_\mathrm{y} &= -\frac{\partial \mathcal{H}}{\partial y} = -\frac{1}{2}\Omega_\mathrm{z}\left(p_\mathrm{x} + \frac{1}{2} m \Omega_\mathrm{z} y\right)
\end{aligned}$$

These are linear equations with solutions of the form $x(t) = \tilde{x} \exp(i\omega t)$. Then

$$\begin{aligned}
i\omega\tilde{x} &= \frac{1}{m}\tilde{p}_\mathrm{x} + \frac{1}{2}\Omega_\mathrm{z}\tilde{y} \\
i\omega\tilde{y} &= \frac{1}{m}\tilde{p}_\mathrm{y} - \frac{1}{2}\Omega_\mathrm{z}\tilde{x} \\
i\omega\tilde{p}_x &= \frac{1}{2}\Omega_\mathrm{z}\left(\tilde{p}_\mathrm{y} - \frac{1}{2} m \Omega_\mathrm{z}\tilde{x}\right) \\
i\omega\tilde{p}_y &= -\frac{1}{2}\Omega_\mathrm{z}\left(\tilde{p}_\mathrm{x} + \frac{1}{2} m \Omega_\mathrm{z}\tilde{y}\right)
\end{aligned}$$

or

$$\begin{aligned}
i\omega\tilde{x} - \frac{1}{2}\Omega_\mathrm{z}\tilde{y} - \frac{1}{m}\tilde{p}_\mathrm{x} &= 0 \\
\frac{1}{2}\Omega_\mathrm{z}\tilde{x} + i\omega\tilde{y} - \frac{1}{m}\tilde{p}_\mathrm{y} &= 0 \\
\frac{1}{4} m \Omega_\mathrm{z}^2\tilde{x} + i\omega\tilde{p}_\mathrm{x} - \frac{1}{2}\Omega_\mathrm{z}\tilde{p}_\mathrm{y} &= 0 \\
\frac{1}{4} m \Omega_\mathrm{z}^2\tilde{y} + \frac{1}{2}\Omega_\mathrm{z}\tilde{p}_\mathrm{x} + i\omega\tilde{p}_\mathrm{y} &= 0.
\end{aligned}$$

This set of homogeneous linear equations has a non-trivial solution if, and only if the determinant of the coefficients vanishes. That is

$$\det\begin{bmatrix} i\omega & -\frac{1}{2}\Omega_\mathrm{z} & -\frac{1}{m} & 0 \\ \frac{1}{2}\Omega_\mathrm{z} & i\omega & 0 & -\frac{1}{m} \\ \frac{1}{4} m \Omega_\mathrm{z}^2 & 0 & i\omega & -\frac{1}{2}\Omega_\mathrm{z} \\ 0 & \frac{1}{4} m \Omega_\mathrm{z}^2 & \frac{1}{2}\Omega_\mathrm{z} & i\omega \end{bmatrix} = 0,$$

which results in

$$\omega^4 - \omega^2\Omega_\mathrm{z}^2 = 0.$$

That is

$$\omega = \pm\Omega_z,$$

which establishes that a charged particle moves in a circular orbit in each plane of constant z. The frequency of the motion increases with z. But the angular momentum is constant. So an increase in angular velocity implies a decrease in orbital radius.

3.17. In the text we discussed the problem of Rutherford scattering. In his analysis Rutherford assumed that only the Coulomb force acted on the $\alpha-$particle scattered by the nucleus. The potential was then

$$\varphi = \frac{Q_N Q}{4\pi\varepsilon_0}\frac{1}{\rho}$$

We may, depending on the nucleus, also have a nuclear magnetic moment. This will have an affect as well on the moving $\alpha-$particle. The vector potential at a distance $\boldsymbol{r}$ (using spherical coordinates $|\boldsymbol{r}| = \rho$) from a nucleus with magnetic moment $\boldsymbol{M}$ is.

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi\rho^2}\boldsymbol{M}\times\boldsymbol{r},$$

which, carrying out the cross product, becomes

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 M}{4\pi}\frac{1}{\rho^3}\sin\phi\hat{e}_\vartheta.$$

We shall simplify our problem by confining motion to the horizontal plane. In spherical coordinates the polar angle is then $\phi = \pi/2$.

Find the Hamiltonian for Rutherford scattering when the nucleus has a magnetic moment. Linearize this for small values of M. Comment on the effect of the nuclear magnetic moment.

Solution:

The Hamiltonian is, of course,

$$\mathcal{H} = \frac{1}{2m}\left(p_\mu - QA_\mu\right)^2 + Q\varphi.$$

With $\sin\phi = 1$. Then the vector potential is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 M}{4\pi}\frac{1}{\rho^3}\hat{e}_\vartheta$$

The Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}m\dot{q}_\mu\dot{q}_\mu - Q\varphi + QA_\mu\dot{q}_\mu \\ &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2\right) + Q\frac{\mu_0 M}{4\pi}\frac{1}{\rho^2}\dot{\vartheta} - Q\varphi, \end{aligned}$$

where φ is, of course, the scalar potential. The canonical momenta are

$$p_{\mathrm{r}} = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$$
$$p_{\vartheta} = \frac{\partial L}{\partial \dot{\vartheta}} = m\rho^2\dot{\vartheta} + Q\frac{\mu_0 M}{4\pi}\frac{1}{\rho^2}.$$

Then

$$\dot{\rho} = \frac{1}{m}p_{\rho}$$
$$\dot{\vartheta} = \frac{1}{m\rho^2}p_{\vartheta} - Q\frac{\mu_0 M}{4\pi}\frac{1}{m\rho^4}.$$

From the Legendre transformation,

$$\begin{aligned}\mathcal{H} &= m\dot{\rho}^2 + m\rho^2\dot{\vartheta}^2 + Q\frac{\mu_0 M}{4\pi}\frac{1}{\rho^2}\dot{\vartheta} - \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2\right) - Q\frac{\mu_0 M}{4\pi}\frac{1}{\rho^2}\dot{\vartheta} + Q\varphi \\ &= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\vartheta}^2\right) + Q\varphi,\end{aligned}$$

which becomes

$$\mathcal{H} = \frac{1}{2m}p_{\rho}^2 + \frac{1}{2m\rho^2}\left(p_{\vartheta} - Q\frac{\mu_0 M}{4\pi}\frac{1}{\rho^2}\right)^2 + \frac{Q_{\mathrm{N}}Q}{4\pi\varepsilon_0}\frac{1}{\rho}.$$

Because the Lagrangian is cyclic in ϑ we have p_{ϑ} =constant. If we expand this Hamiltonian for small values of M we have

$$\mathcal{H} = \frac{1}{2m}p_{\rho}^2 + \frac{1}{2m\rho^2}p_{\vartheta}^2 - \frac{MQ\mu_0}{4\pi m}\frac{1}{\rho^4}p_{\vartheta} + \frac{Q_{\mathrm{N}}Q}{4\pi\varepsilon_0}\frac{1}{\rho}.$$

The effective potential is then decreased in the neighborhood of the nucleus if the nucleus possessed a magnetic moment. The $1/\rho^4$ dependence of this potential, however, makes it a weak potential at large distances.

4 Solid Bodies

4.1. In the text we obtained the matrices (operators) for infinitesimal rotations about three basis vectors as

$$\mathbf{R}_1\left(\delta\vartheta_1'\right) \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\delta\vartheta_1' \\ 0 & \delta\vartheta_1' & 1 \end{bmatrix},$$

$$\mathbf{R}_2\left(\delta\vartheta_2'\right) \Longrightarrow \begin{bmatrix} 1 & 0 & \delta\vartheta_2' \\ 0 & 1 & 0 \\ -\delta\vartheta_2' & 0 & 1 \end{bmatrix},$$

and

$$\mathbf{R}_3\left(\delta\vartheta_3'\right) \Longrightarrow \begin{bmatrix} 1 & -\delta\vartheta_3' & 0 \\ \delta\vartheta_3' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Sow that these commute by carrying out the calculation. Pick any two matrices you wish for this demonstration.

Solution:

We compute the two products

$$\begin{aligned} &\mathbf{R}_2\left(\delta\vartheta_2'\right)\mathbf{R}_3\left(\delta\vartheta_3'\right) \\ &= \begin{bmatrix} 1 & 0 & \delta\vartheta_2' \\ 0 & 1 & 0 \\ -\delta\vartheta_2' & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\delta\vartheta_3' & 0 \\ \delta\vartheta_3' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -\delta\vartheta_3' & \delta\vartheta_2' \\ \delta\vartheta_3' & 1 & 0 \\ -\delta\vartheta_2' & \delta^2\vartheta_2'\vartheta_3' & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}
&\mathbf{R}_3\left(\delta\vartheta_3'\right)\mathbf{R}_2\left(\delta\vartheta_2'\right)\\
&=\begin{bmatrix} 1 & -\delta\vartheta_3' & 0 \\ \delta\vartheta_3' & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & \delta\vartheta_2' \\ 0 & 1 & 0 \\ -\delta\vartheta_2' & 0 & 1 \end{bmatrix}\\
&=\begin{bmatrix} 1 & -\delta\vartheta_3' & \delta\vartheta_2' \\ \delta\vartheta_3' & 1 & \delta^2\vartheta_2'\vartheta_3' \\ -\delta\vartheta_2' & 0 & 1 \end{bmatrix}.
\end{aligned}$$

They are equal.

4.2. The kinetic energy of rotation of a rigid body has the abstract form

$$T_{\mathrm{rot}} = \frac{1}{2}\left\langle\omega\right| I \left|\omega\right\rangle$$

If we project this onto the basis fixed in the body using

$$P_{\mathrm{x}} = \left|x_\mu\right\rangle\left\langle x_\mu\right| = \mathbf{1}$$

we have

$$T_{\mathrm{rot}} = \frac{1}{2}\left\langle\omega\right.\left|x_\mu\right\rangle\left\langle x_\mu\right| I \left|x_\nu\right\rangle\left\langle x_\nu\right|\left.\omega\right\rangle .$$

Show that if the rigid body is a sphere this kinetic energy has the same form represented in the fixed system with basis $\left\{\left|X_\mu\right\rangle\right\}$.

Solution:

For a sphere

$$I = I_0\,\mathbf{1} = I_0\left|x_\rho\right\rangle\left\langle x_\rho\right| .$$

So

$$\begin{aligned}
T_{\mathrm{rot}} &= \frac{1}{2}\boldsymbol{I}_0\left\langle\omega\right.\left|x_\mu\right\rangle\left\langle x_\mu\right|\left.x_\rho\right\rangle\left\langle x_\rho\right.\left|x_\nu\right\rangle\left\langle x_\nu\right|\left.\omega\right\rangle\\
&= \frac{1}{2}\boldsymbol{I}_0\left\langle\omega\right.\left|x_\rho\right\rangle\left\langle x_\rho\right|\left.\omega\right\rangle
\end{aligned}$$

We transform to the fixed basis by introducing the projector

$$P_{\mathrm{X}} = \left|X_\mu\right\rangle\left\langle X_\mu\right| = \mathbf{1},$$

which we may introduce at any point in our expression for the kinetic energy. Then

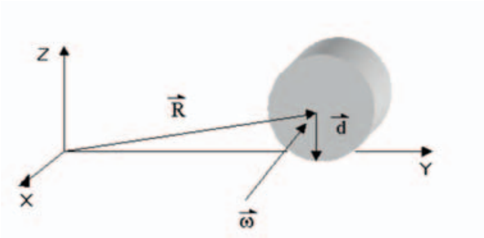
$$\begin{aligned}
T_{\mathrm{rot}} &= \frac{1}{2}I_0\left\langle\omega\right.\left|X_\lambda\right\rangle\left\langle X_\lambda\right|\left.x_\rho\right\rangle\left\langle x_\rho\right.\left|X_\sigma\right\rangle\left\langle X_\sigma\right|\left.\omega\right\rangle\\
&= \frac{1}{2}I_0\left\langle\omega\right.\left|X_\lambda\right\rangle\delta_{\lambda\sigma}\left\langle X_\sigma\right|\left.\omega\right\rangle\\
&= \frac{1}{2}I_0\left\langle\omega\right.\left|X_\lambda\right\rangle\left\langle X_\lambda\right|\left.\omega\right\rangle ,
\end{aligned}$$

which is the same form as

$$T_{\mathrm{rot}} = \frac{1}{2}\boldsymbol{I}_0 \left\langle \omega \left| x_\rho \right\rangle \left\langle x_\rho \right| \omega \right\rangle ,$$

except the representation is in a different basis.

4.3. A spinning disk of radius a is lowered onto a table. The disk is rotating about the axis parallel to the table at an initial angular velocity ω_0. The disk begins slipping and eventually starts to roll. Study the motion and determine the point at which rolling begins.



Disk released onto table with initial angular velocity $\omega = \omega_0$.

Recall that the horizontal kinetic frictional force has a magnitude

$$f_{\mathrm{friction}} = \mu_{\mathrm{k}} m g.$$

This is the horizontal force while the disk is slipping. In the Hamiltonian formulation this appears as a Lagrange multiplier. That is

$$\dot{p}_\ell = -\frac{\partial \mathcal{H}}{\partial q_\ell} + \sum_{\mathrm{k}=1}^{\mathrm{n}} \lambda_{\mathrm{k}}(t)\frac{\partial g_{\mathrm{k}}}{\partial q_\ell}.$$

Solution:
The rolling constraint is

$$\mathbf{0} = \boldsymbol{V} + \boldsymbol{\omega} \times \boldsymbol{d},$$

with

$$\boldsymbol{V} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{X}\,\hat{e}_{\mathrm{X}} + \dot{Y}\,\hat{e}_{\mathrm{Y}},$$

and

$$\boldsymbol{d} = -a\,\hat{e}_{\mathrm{Z}}$$

$$\boldsymbol{\omega} = -\dot{\vartheta}\,\hat{e}_{\mathrm{X}}.$$

Then

$$\boldsymbol{\omega} \times \boldsymbol{d} = -a\dot{\vartheta}\ \hat{e}_{\mathrm{Y}}$$

So the rolling constraint is

$$\mathbf{0} = \left(\dot{X}\right)\ \hat{e}_{\mathrm{X}} + \left(\dot{Y} - a\dot{\vartheta}\right)\ \hat{e}_{\mathrm{Y}},$$

or

$$\dot{X} = 0,\ \dot{Y} - a\dot{\vartheta} = 0.$$

This condition is not reached until slipping stops and rolling begins.

While the disk is slipping the force in the (negative) $Y-$direction retarding the motion is

$$f_{\mathrm{friction}} = \mu_{\mathrm{k}} mg$$

and this force provides a (negative) torque

$$\tau_{\mathrm{friction}} = a f_{\mathrm{friction}}$$

slowing the rotational angular momentum. Because this frictional force does not depend functionally on any of the coordinates describing the system, we cannot include it in our usual fashion as a constraint. It simply is not a constraint, which would confine the motion. We shall, therefore, simply insert the frictional force in the canonical equations where appropriate. We can easily do this because we understand the meaning of forces in the canonical equations.

There is no potential energy since the only conservative force is gravity and there is no motion in the vertical direction. The Lagrangian is then only the kinetic energy

$$L = \frac{1}{2} m\ \dot{Y}^2 + \frac{1}{2} I \dot{\vartheta}^2,$$

where I is the moment of inertia of the disk about its axis. The momenta are

$$P_{\mathrm{Y}} = \frac{\partial L}{\partial \dot{Y}} = m\dot{Y}$$
$$P_{\vartheta} = \frac{\partial L}{\partial \dot{\vartheta}} = I\dot{\vartheta}.$$

The Hamiltonian is

$$\mathcal{H} = \frac{P_{\mathrm{Y}}^2}{2m} + \frac{P_{\vartheta}^2}{2I}.$$

Inserting the frictional force and the frictional torque, the canonical equations are

$$\dot{P}_Y = \mu_k mg \text{ and } \dot{Y} = \frac{P_Y}{m}$$

and

$$\dot{P}_\vartheta = -a\mu_k mg \text{ and } \dot{\vartheta} = \frac{P_\vartheta}{I}.$$

These equations hold until $\dot{Y} = a\dot{\vartheta}$, which is when rolling begins.

It is easiest to integrate the two canonical equations for the momenta directly. That is

$$\int_0^t \mathrm{d}t\, \dot{P}_Y = P_Y = \mu_k mgt$$
$$\int_0^t \mathrm{d}t\, \dot{P}_\vartheta = P_\vartheta - I\omega_0 = -a\mu_k mgt.$$

When rolling begins,

$$\dot{Y} = a\dot{\vartheta}$$

or

$$P_{\mathrm{Y}} = m\dot{Y} = ma\dot{\vartheta} = ma\frac{P_\vartheta}{I}.$$

That is,

$$\mu_k mgt_{\mathrm{roll}} = ma\frac{(I\omega_0 - \mu_k magt_{\mathrm{roll}})}{I}$$

when rolling begins at the time t_{roll}. Then

$$\frac{I}{ma^2}\mu_k magt_{\mathrm{roll}} = I\omega_0 - \mu_k magt_{\mathrm{roll}}$$

or

$$t_{\mathrm{roll}} = \frac{I\omega_0/\mu_k mag}{\left(1 + I/ma^2\right)}.$$

For the disk of radius a, density ρ and thickness w, the moment of inertia is

$$
\begin{aligned}
I &= \int r^2 \mathrm{d}m \\
&= \int_{\mathrm{r}=0}^{\mathrm{a}} \int_{\vartheta=0}^{2\pi} (\rho w r \mathrm{d}\vartheta \mathrm{d}r)\, r^2 \\
&= \left(\pi a^2 w \rho\right) \frac{a^2}{2} \\
&= \frac{1}{2} m a^2,
\end{aligned}
$$

so

$$t_{\mathrm{roll}} = \frac{a\omega_0}{3\mu_{\mathrm{k}} g}.$$

From the canonical equation

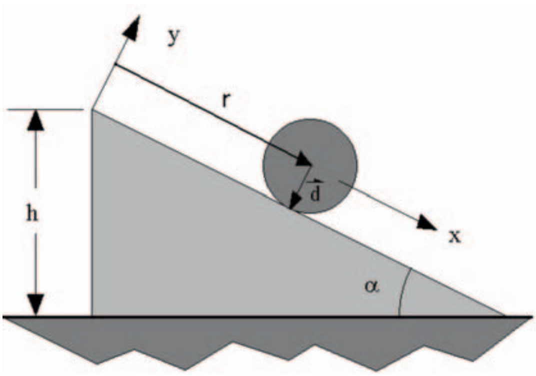
$$\dot{Y} = \frac{P_Y}{m}$$

we have the distance at which rolling begins as

$$
\begin{aligned}
Y_{\mathrm{roll}} &= \int_0^{\mathrm{t_{roll}}} \mathrm{d}t\, (\mu_{\mathrm{k}} g t) \\
&= \frac{1}{2} \mu_{\mathrm{k}} g t_{\mathrm{roll}}^2 \\
&= \frac{a^2 \omega_0^2}{18 \mu_{\mathrm{k}} g}.
\end{aligned}
$$

We see that Y_{roll} increases with increasing $\omega_0$and decreases with increasing μ_k.

4.4. Consider the disk of mass M and radius a rolling down a hill, as shown below.



Disk rolling down hill.

Obtain the description of the motion by solving the canonical equations for $r = x\,(t)$.

Solution:

The position vector, $\boldsymbol{r}$, to the CM of the disk and the vector $\boldsymbol{d}$ from the CM to the contact point with the incline, are

$$\boldsymbol{r} = x\ \hat{e}_{\mathrm{x}}$$

and

$$\boldsymbol{d} = -a\ \hat{e}_{\mathrm{y}}.$$

The angular velocity of the disk is

$$\boldsymbol{\omega} = -\dot{\vartheta}\ \hat{e}_{\mathrm{z}}.$$

The rolling constraint is then

$$\begin{aligned}\mathbf{0} &= \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{r} + \boldsymbol{\omega} \times \boldsymbol{d}\\ &= \dot{x}\ \hat{e}_{\mathrm{x}} - a\dot{\vartheta}\ \hat{e}_{\mathrm{x}}.\end{aligned}$$

We note that the constraint

$$0 = \dot{x}\ - a\dot{\vartheta}$$

can be written as

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \dot{x}\ - a\dot{\vartheta} = 0$$

with

$$g = \text{constant}.$$

Then

$$\mathrm{d}g = \mathrm{d}x - a\mathrm{d}\vartheta$$

and we have

$$\begin{aligned}\frac{\partial g}{\partial x} &= 1\\ \frac{\partial g}{\partial \vartheta} &= -a.\end{aligned}$$

The Lagrangian is

$$L = \frac{M}{2}\dot{x}^2 + \frac{I}{2}\dot{\vartheta}^2 + Mgx\sin\alpha.$$

The canonical momenta are

$$p_{\mathrm{x}} = M\dot{x}$$
$$p_{\vartheta} = I\dot{\vartheta}$$

and the Hamiltonian is

$$\mathcal{H} = \frac{p_{\mathrm{x}}^2}{2M} + \frac{p_{\vartheta}^2}{2I} - Mgx\sin\alpha.$$

The canonical equations are then

$$\dot{p}_{\mathrm{x}} = Mg\sin\alpha + \lambda$$

$$\dot{p}_{\vartheta} = -a\lambda$$

and

$$\dot{x} = \frac{p_{\mathrm{x}}}{M}$$
$$\dot{\vartheta} = \frac{p_{\vartheta}}{I}.$$

Combining the canonical momenta equations

$$\dot{p}_{\mathrm{x}} = Mg\sin\alpha - \frac{\dot{p}_{\vartheta}}{a}.$$

The rolling constraint may be written in terms of the canonical momenta as

$$p_{\mathrm{x}} = \frac{Ma}{I}p_{\vartheta},$$

or

$$\dot{p}_{\mathrm{x}} = \frac{Ma}{I}\dot{p}_{\vartheta},$$

Then we have the set of equations

$$\begin{bmatrix} 1 & -\frac{1}{a} \\ 1 & -\frac{Ma}{I} \end{bmatrix} \begin{bmatrix} \dot{p}_{\mathrm{x}} \\ \dot{p}_{\vartheta} \end{bmatrix} = \begin{bmatrix} Mg\sin\alpha \\ 0 \end{bmatrix}$$

The solution is

$$\begin{bmatrix} \dot{p}_{\mathrm{x}} \\ \dot{p}_{\vartheta} \end{bmatrix} = \frac{1}{Ma^2 - I} \begin{bmatrix} M^2a^2g\sin\alpha \\ aIMg\sin\alpha \end{bmatrix}.$$

This is actually sufficient for the solution, since the integration of these equations is straightforward. Carrying out the integration

$$\begin{bmatrix} p_{\mathrm{x}} \\ p_{\vartheta} \end{bmatrix} = \frac{1}{Ma^2 - I} \begin{bmatrix} M^2 a^2 gt \sin\alpha \\ aIMgt \sin\alpha \end{bmatrix},$$

since the disk is released from rest. If we want solutions for x and ϑ, we write these as

$$\begin{bmatrix} \dot{x} \\ \dot{\vartheta} \end{bmatrix} = \frac{1}{Ma^2 - I} \begin{bmatrix} Ma^2 gt \sin\alpha \\ aMgt \sin\alpha \end{bmatrix},$$

and integrate again.

$$\begin{bmatrix} x \\ \vartheta \end{bmatrix} = \frac{1}{2\left(Ma^2 - I\right)} \begin{bmatrix} Ma^2 gt^2 \sin\alpha \\ aMgt^2 \sin\alpha \end{bmatrix},$$

noting again the release from rest.

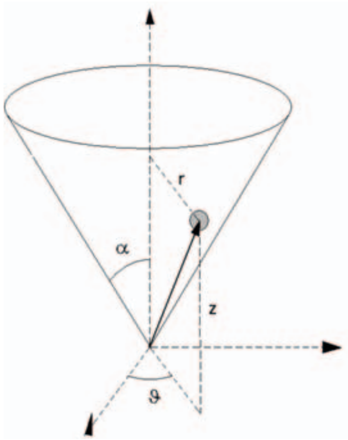
For the disk of radius a, density ρ and thickness w, the moment of inertia is

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_{r=0}^{a} \int_{\vartheta=0}^{2\pi} (\rho w r d\vartheta dr)\, r^2 \\ &= \left(\pi a^2 w \rho\right) \frac{a^2}{2} \\ &= \frac{1}{2} Ma^2. \end{aligned}$$

Then

$$\begin{aligned} \begin{bmatrix} x \\ \vartheta \end{bmatrix} &= \frac{1}{Ma^2} \begin{bmatrix} Ma^2 gt^2 \sin\alpha \\ aMgt^2 \sin\alpha \end{bmatrix} \\ &= \begin{bmatrix} gt^2 \sin\alpha \\ \left(gt^2/a\right) \sin\alpha \end{bmatrix}. \end{aligned}$$

4.5. Consider a small ball of mass m is rolling inside of a cone with axis along the $z-$axis of cylindrical coordinates and defined by the angle α from the central axis. The situation is shown here



A small ball rolling without slipping inside a cone.

Obtain the canonical equations for the motion of the ball. The fact that the ball moves on the inner surface of the cone introduces a constraint. And there will be three rolling constraints corresponding to the three cylindrical coordinates.

Solution:

The inner surface of the cone is defined by

$$\tan\alpha = \frac{r}{z}.$$

This is one constraint, which we write as

$$g_1 = r - z\tan\alpha.$$

Then

$$\mathrm{d}g_1 = \mathrm{d}r - (\tan\alpha)\,\mathrm{d}z,$$

so that

$$\frac{\partial g_1}{\partial r} = 1$$
$$\frac{\partial g_1}{\partial z} = -\tan\alpha$$

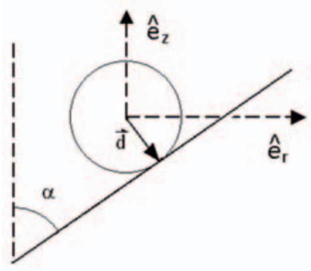
The other constraint is the rolling constraint. To find the rolling constraint we begin with the vector positioning the CM of the ball.

$$\boldsymbol{R} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

The CM velocity is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{r}\hat{e}_{\mathrm{r}} + r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

Here is a drawing of the ball rolling inside the cone.



Constraint for ball rolling inside cone.

So

$$\boldsymbol{\omega} = \omega_{\mathrm{r}}\hat{e}_{\mathrm{r}} + \omega_{\vartheta}\hat{e}_{\vartheta} + \omega_{\mathrm{z}}\hat{e}_{\mathrm{z}}.$$

The basis $\left(\hat{e}_{\mathrm{r}}, \hat{e}_{\vartheta}, \hat{e}_{\mathrm{z}}\right)$ forms a right-handed triad with

$$\hat{e}_{\mathrm{r}} \times \hat{e}_{\vartheta} = \hat{e}_{\mathrm{z}}.$$

Then

$$\begin{aligned}
\boldsymbol{\omega} \times \boldsymbol{d} = \det &\begin{bmatrix} \hat{e}_{\mathrm{r}} & \hat{e}_{\vartheta} & \hat{e}_{\mathrm{z}} \\ \omega_{\mathrm{r}} & \omega_{\vartheta} & \omega_{\mathrm{z}} \\ a\cos\alpha & 0 & -a\sin\alpha \end{bmatrix} \\
&= \hat{e}_{\mathrm{r}}\left(-a\omega_{\vartheta}\sin\alpha\right) \\
&\quad + \hat{e}_{\vartheta}\left(a\omega_{\mathrm{r}}\sin\alpha + a\omega_{\mathrm{z}}\cos\alpha\right) \\
&\quad + \hat{e}_{\mathrm{z}}\left(-a\omega_{\vartheta}\cos\alpha\right).
\end{aligned}$$

The rolling constraint is then

$$\begin{aligned}
\boldsymbol{0} &= \dot{\boldsymbol{R}} + \boldsymbol{\omega} \times \boldsymbol{d} \\
&= \left(\dot{r} - a\omega_{\vartheta}\sin\alpha\right)\hat{e}_{\mathrm{r}} + \left(r\dot{\vartheta} + a\omega_{\mathrm{r}}\sin\alpha + a\omega_{\mathrm{z}}\cos\alpha\right)\hat{e}_{\vartheta} + \left(\dot{z} - a\omega_{\vartheta}\cos\alpha\right)\hat{e}_{\mathrm{z}} \\
&= \left(\frac{\mathrm{d}g_2}{\mathrm{d}t}\right)\hat{e}_{\mathrm{r}} + \left(\frac{\mathrm{d}g_3}{\mathrm{d}t}\right)\hat{e}_{\vartheta} + \left(\frac{\mathrm{d}g_4}{\mathrm{d}t}\right)\hat{e}_{\mathrm{z}}.
\end{aligned}$$

In the last line here we have introduced the three functions g_2, g_3, g_4 whose derivatives are the components of the rolling constraint. For the rotation of the ball we define the angles $(\vartheta_1, \vartheta_2, \vartheta_3)$ such that

$$\omega_{\mathrm{r}} = \frac{\mathrm{d}}{\mathrm{d}t}\vartheta_1$$
$$\omega_{\vartheta} = \frac{\mathrm{d}}{\mathrm{d}t}\vartheta_2$$
$$\omega_{\mathrm{z}} = \frac{\mathrm{d}}{\mathrm{d}t}\vartheta_3.$$

The (r, ϑ, z) notation can then be used for the motion of the CM. We then have the constraints from the three components of the rolling constraint above as

$$\frac{\mathrm{d}g_2}{\mathrm{d}t} = 0 = \frac{\mathrm{d}r}{\mathrm{d}t} - a\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t}\sin\alpha$$

$$\frac{\mathrm{d}g_3}{\mathrm{d}t} = 0 = r\frac{\mathrm{d}\vartheta}{\mathrm{d}t} + a\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t}\sin\alpha + a\frac{\mathrm{d}\vartheta_3}{\mathrm{d}t}\cos\alpha$$

$$\frac{\mathrm{d}g_4}{\mathrm{d}t} = 0 = \frac{\mathrm{d}z}{\mathrm{d}t} - a\frac{\mathrm{d}\vartheta_2}{\mathrm{d}t}\cos\alpha.$$

From these we have

$$\mathrm{d}g_2 = 0 = \mathrm{d}r - (a\sin\alpha)\,\mathrm{d}\vartheta_1$$

$$\mathrm{d}g_3 = 0 = r\mathrm{d}\vartheta + (a\sin\alpha)\,\mathrm{d}\vartheta_1 + (a\cos\alpha)\,\mathrm{d}\vartheta_3$$

$$\mathrm{d}g_4 = 0 = \mathrm{d}z - (a\cos\alpha)\,\mathrm{d}\vartheta_2.$$

We can then identify the partial derivatives for the canonical equations as

$$\frac{\partial g_2}{\partial r} = 1$$
$$\frac{\partial g_2}{\partial \vartheta_1} = -(a\sin\alpha)$$
$$\frac{\partial g_3}{\partial \vartheta} = r$$
$$\frac{\partial g_3}{\partial \vartheta_1} = (a\sin\alpha)$$
$$\frac{\partial g_3}{\partial \vartheta_3} = (a\cos\alpha)$$
$$\frac{\partial g_4}{\partial z} = 1$$
$$\frac{\partial g_4}{\partial \vartheta_2} = -(a\cos\alpha)$$

The Lagrangian is then

$$L = \frac{1}{2}m\left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}\right|^2 + \frac{1}{2}I_0\left(\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + \dot{\vartheta}_3^2\right) - mgz$$
$$= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) + \frac{1}{2}I_0\left(\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2 + \dot{\vartheta}_3^2\right) - mgz.$$

The canonical momenta are

$$p_\mathrm{r} = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta}$$

$$p_\mathrm{z} = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$p_1 = \frac{\partial L}{\partial \dot{\vartheta}_1} = I_0\dot{\vartheta}_1$$

$$p_2 = \frac{\partial L}{\partial \dot{\vartheta}_2} = I_0\dot{\vartheta}_2$$

$$p_3 = \frac{\partial L}{\partial \dot{\vartheta}_3} = I_0\dot{\vartheta}_3.$$

The Hamiltonian is then

$$\mathcal{H} = \frac{p_\mathrm{r}^2 + p_\mathrm{z}^2}{2m} + \frac{p_\vartheta^2}{2mr^2} + \frac{p_1^2 + p_2^2 + p_3^2}{2I_0} + mgz.$$

The canonical equations with multipliers are

$$\dot{p}_\mu + \frac{\partial \mathcal{H}}{\partial q_\mu} + \sum_k \lambda_k \frac{\partial g_k}{\partial q_\mu} = 0$$

and

$$\dot{q}_\mu - \frac{\partial \mathcal{H}}{\partial p_\mu} = 0.$$

These are obtained as

$$\dot{p}_\mathrm{r} - \frac{p_\vartheta^2}{mr^3} + \lambda_1 + \lambda_2 = 0,$$

$$\dot{p}_\vartheta + \lambda_3 r = 0,$$

$$\dot{p}_{\mathrm{z}} + mg - \lambda_1 \tan\alpha + \lambda_4 = 0,$$

$$\dot{p}_1 - \lambda_2 (a \sin\alpha) + \lambda_3 a \sin\alpha = 0,$$

$$\dot{p}_2 - \lambda_4 (a \cos\alpha) = 0,$$

$$\dot{p}_3 + \lambda_3 (a \cos\alpha) = 0,$$

and

$$\dot{r} = \frac{p_{\mathrm{r}}}{m},$$

$$\dot{\vartheta} = \frac{p_{\vartheta}}{mr^2},$$

$$\dot{z} = \frac{p_{\mathrm{z}}}{m},$$

$$\dot{\vartheta}_1 = \frac{p_1}{I_0},$$

$$\dot{\vartheta}_2 = \frac{p_2}{I_0},$$

and

$$\dot{\vartheta}_3 = \frac{p_3}{I_0}.$$

The constraints add the equations

$$\begin{aligned}\frac{\mathrm{d}g_1}{\mathrm{d}t} &= 0 = \dot{r} - (\tan\alpha)\,\dot{z}\\ &= \frac{p_{\mathrm{r}}}{m} - (\tan\alpha)\,\frac{p_{\mathrm{z}}}{m},\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}g_2}{\mathrm{d}t} &= 0 = \frac{\mathrm{d}r}{\mathrm{d}t} - a\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t}\sin\alpha\\ &= \frac{p_{\mathrm{r}}}{m} - \frac{p_1}{I_0}a\sin\alpha\end{aligned}$$

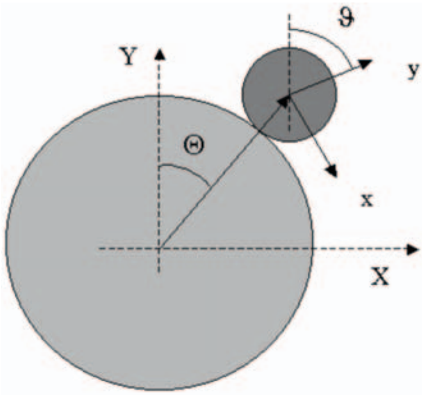
$$\begin{aligned}\frac{\mathrm{d}g_3}{\mathrm{d}t} &= 0 = r\frac{\mathrm{d}\vartheta}{\mathrm{d}t} + a\frac{\mathrm{d}\vartheta_1}{\mathrm{d}t}\sin\alpha + a\frac{\mathrm{d}\vartheta_3}{\mathrm{d}t}\cos\alpha\\ &= \frac{p_{\vartheta}}{mr} + \frac{p_1}{I_0}a\sin\alpha + \frac{p_3}{I_0}a\cos\alpha\end{aligned}$$

$$\frac{\mathrm{d}g_4}{\mathrm{d}t} = 0 = \frac{\mathrm{d}z}{\mathrm{d}t} - a\frac{\mathrm{d}\vartheta_2}{\mathrm{d}t}\cos\alpha$$
$$= \frac{p_\mathrm{z}}{m} - \frac{p_2}{I_0}a\cos\alpha.$$

There are then 16 equations for the 12 canonical variables (coordinates and momenta) and the 4 λs. That is, the mathematical representation of the system is complete.

The set of equations appears tantalizingly simple at first glance. Were it not for the presence of the r in the equations for $\dot{p}_\mathrm{r}$, $\dot{p}_\vartheta$ and $\mathrm{d}g_3/\mathrm{d}t$ we would have a linear set of equations in the momenta alone. But the presence of the r makes the equations nonlinear. The solution is not easy.

4.6. In the figure below we have drawn a small right circular cylinder of radius a and length ℓ rolling without slipping on a larger right circular cylinder with radius $A > a$ and length $\geq \ell$. The larger cylinder is fastened to the laboratory bench and does not move.



Small solid right circular cylinder of radius a and length ℓ rolling on large right circular cylinder of radius $A > a$ and length $> \ell$.

The fixed coordinate system is (X, Y, Z). The unit vector $\hat{e}_\mathrm{Z}$ is, according to the right hand system, along the axis of the larger (fixed) cylinder and oriented out of the figure. The system (x, y, z) is fixed in the smaller cylinder with the unit vector $\hat{e}_\mathrm{z}$ along the axis of the small cylinder and out of the figure. The unit vectors $\hat{e}_\Theta$ and $\hat{e}_\vartheta$ are then parallel and positive in the direction of increasing Θ and ϑ.

We carefully balance the smaller cylinder along the top of the larger cylinder at $X = 0$ and then set it in motion with a very small nudge. At what point (value of Θ) does the smaller cylinder lose contact with the larger?

Solution:

For convenience we choose

$$R = A + a,$$

which holds as long as the small cylinder remains in contact with the larger cylinder. The vector $\boldsymbol{R} = R\,\hat{e}_\mathrm{R}$ then locates the central axis of the smaller cylinder.

Because the large cylinder is fixed in space the velocity of the contact point is zero and the rolling constraint is

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{R} + \overrightarrow{\omega} \times \overrightarrow{d},$$

with

$$\overrightarrow{d} = -a\hat{e}_\mathrm{R},$$

$$\overrightarrow{\omega} = -\dot{\vartheta}\hat{e}_\mathrm{Z},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{R} = \dot{R}\,\hat{e}_\mathrm{R} + R\dot{\Theta}\hat{e}_\Theta,$$

Then

$$\begin{aligned}\overrightarrow{\omega} \times \overrightarrow{d} &= -\dot{\vartheta}\hat{e}_\mathrm{Z} \times (-a)\,\hat{e}_\mathrm{R}\\ &= -a\dot{\vartheta}\hat{e}_\Theta,\end{aligned}$$

since $\hat{e}_\mathrm{Z} \times \hat{e}_\mathrm{R} = -\hat{e}_\Theta$. Then the rolling constraint is

$$\begin{aligned}0 &= \dot{R}\,\hat{e}_\mathrm{R} + R\dot{\Theta}\hat{e}_\Theta - a\dot{\vartheta}\hat{e}_\Theta\\ &= \dot{R}\,\hat{e}_\mathrm{R} + \left(R\dot{\Theta} - a\dot{\vartheta}\right)\hat{e}_\Theta.\end{aligned}$$

As long as there is contact

$$\dot{R} = 0.$$

and

$$R\dot{\Theta} = a\dot{\vartheta}$$

is the rolling constraint. We notice that $R\mathrm{d}\Theta$ is the distance moved by the CM in a time $\mathrm{d}t$, which is equal to $a\mathrm{d}\vartheta$ if the cylinder is rolling.

Writing these constraints in differential form,

$$\mathrm{d}g_1 = 0 = \mathrm{d}R$$

and

$$\mathrm{d}g_2 = 0 = R\mathrm{d}\Theta - a\mathrm{d}\vartheta.$$

Then

$$\frac{\partial g_1}{\partial R} = 1$$

and

$$\frac{\partial g_2}{\partial \Theta} = R \; ; \; \frac{\partial g_2}{\partial \vartheta} = -a$$

If we pick the reference level for the potential energy to be the plane $Y = 0$ the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}M\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{R}\right)^2 + \frac{1}{2}I_0\dot{\vartheta}^2 - MgR\cos\Theta \\ &= \frac{1}{2}M\left(\dot{R}^2 + R^2\dot{\Theta}^2\right) + \frac{1}{2}I_0\dot{\vartheta}^2 - MgR\cos\Theta \end{aligned}$$

The canonical momenta are

$$P_R = \frac{\partial L}{\partial \dot{R}} = M\dot{R},$$

$$P_\Theta = \frac{\partial L}{\partial \dot{\Theta}} = MR^2\dot{\Theta},$$

and

$$P_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = I_0\dot{\vartheta}.$$

The Lagrangian is cyclic in the time so the Hamiltonian is a constant of the motion.

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}M\left(\dot{R}^2 + R^2\dot{\Theta}^2\right) + \frac{1}{2}I_0\dot{\vartheta}^2 + MgR\cos\Theta \\ &= \frac{P_\mathrm{R}^2}{2M} + \frac{P_\Theta^2}{2MR^2} + \frac{P_\vartheta^2}{2I_0} + MgR\cos\Theta. \end{aligned}$$

The canonical equations are then

$$\dot{P}_\mathrm{R} = \frac{P_\Theta^2}{MR^3} - Mg\cos\Theta + \lambda_1$$

$$\dot{P}_\Theta = MgR\sin\Theta + \lambda_2 R$$

$$\dot{P}_\vartheta = -\lambda_2 a$$

and

$$\dot{R} = \frac{P_\mathrm{R}}{M}$$

$$\dot{\Theta} = \frac{P_\Theta}{MR^2}$$

$$\dot{\vartheta} = \frac{P_\vartheta}{I_0}$$

Before proceeding we see that λ_1 is the contact force in the $\hat{e}_\mathrm{R}$ direction.

We have

$$\lambda_1 = \dot{P}_\mathrm{R} - \frac{P_\Theta^2}{MR^3} + Mg\cos\Theta$$

While there is rolling the constraint $R =$ constant implies that $\dot{P}_\mathrm{R} = 0$, so

$$\lambda_1 = -\frac{P_\Theta^2}{MR^3} + Mg\cos\Theta$$

is the contact force in the R-direction. This vanishes when the cylinders lose contact. We cannot obtain the solution for the angle at which contact is lost, however, until we have P_Θ.

We can find P_Θ from the Hamiltonian. Since the small cylinder is released from essential rest at the top, we have

$$\begin{aligned}\mathcal{H} &= MgR \\ &= \frac{P_\Theta^2}{2MR^2} + \frac{P_\vartheta^2}{2I_0} + MgR\cos\Theta,\end{aligned}$$

while contact is maintained. Using the rolling constraint

$$R\dot{\Theta} = a\dot{\vartheta}$$

we have a relation between P_Θ and P_ϑ.

$$\begin{aligned}P_\vartheta &= I_0\dot{\vartheta} = I_0\left(\frac{R}{a}\right)\dot{\Theta} \\ &= I_0\left(\frac{R}{a}\right)\frac{P_\Theta}{MR^2} \\ &= \frac{I_0}{Ra}\frac{P_\Theta}{M}.\end{aligned}$$

Then the Hamiltonian is then

$$\begin{aligned}MgR &= \frac{P_\Theta^2}{2MR^2} + \frac{(I_0P_\Theta/MRa)^2}{2I_0} + MgR\cos\Theta \\ &= \frac{1}{2}P_\Theta^2\frac{Ma^2 + I_0}{R^2a^2M^2} + MgR\cos\Theta.\end{aligned}$$

From above

$$P_\Theta^2 = (Mg\cos\Theta - \lambda_1)\,MR^3.$$

Then the Hamiltonian becomes

$$MgR = \frac{1}{2}\left(Mg\cos\Theta - \lambda_1\right) MR^3 \frac{Ma^2 + I_0}{R^2a^2M^2} + MgR\cos\Theta$$
$$= \frac{R}{2}\left(Mg\cos\Theta - \lambda_1\right) \frac{Ma^2 + I_0}{Ma^2} + MgR\cos\Theta.$$

or

$$Mg = \frac{1}{2}\left(Mg\cos\Theta - \lambda_1\right) \frac{Ma^2 + I_0}{Ma^2} + Mg\cos\Theta$$

Contact is lost when $\lambda_1 = 0$.

$$\cos\Theta = \frac{2}{3 + I_0/Ma^2}$$

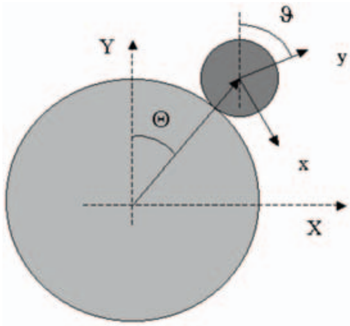
or

$$\Theta = \cos^{-1}\left(\frac{2}{3 + I_0/Ma^2}\right)$$

Including the moment of inertia for the solid cylinder rotated about the axis as $I_0 = Ma^2/2$ this is

$$\Theta = \cos^{-1}\left(\frac{2}{3 + 1/2}\right) = \cos^{-1}\frac{4}{7}.$$

4.7. In the figure below we have drawn a small solid sphere of radius a rolling without slipping on a larger sphere of radius $A > a$. The larger sphere is fastened to the laboratory bench and is stationary.



Solid sphere of radius a rolling on a fixed sphere of radius $A > a$.

We carefully balance the smaller sphere at the top of the larger sphere and then set it in motion with a very small nudge. The smaller sphere then rolls down a great circle of the larger sphere, which is in the plane of the figure. We choose spherical coordinate systems for both spheres. The angles Θ and ϑ are the azimuthal angles. The polar angle $\phi = \pi/2$, which is the plane of motion. The unit vectors $\hat{e}_\Theta$ and $\hat{e}_\vartheta$ are then parallel and positive in the direction of increasing Θ and ϑ.

At what point (value of Θ) does the smaller sphere lose contact with the larger?

Solution:

We shall designate the vector from the center of the Cartesian fixed system to the center of the rolling sphere as $\boldsymbol{R}$. The magnitude of this vector is

$$R = A + a.$$

Because the larger sphere is fixed in space the velocity of the contact point is zero and the rolling constraint is

$$\mathbf{0} = \frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{R} + \overrightarrow{\omega} \times \overrightarrow{d}.$$

The vector from the center of the smaller sphere to the contact point between the spheres we designate as

$$\overrightarrow{d} = -a\hat{e}_{\mathrm{R}},$$

where $\hat{e}_{\mathrm{R}}$ is the unit vector along the radial vector between the centers of the two spheres and directed outward from the origin of the system (X, Y, Z). The angular velocity of the smaller sphere $\boldsymbol{\omega}$ is oriented in the negative direction along the direction of the $Z-$axis. Specifically

$$\overrightarrow{\omega} = -\dot{\vartheta}\hat{e}_{\mathrm{Z}}.$$

And the velocity of the center of the smaller sphere is

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{R} = \dot{R}\,\hat{e}_{\mathrm{R}} + R\dot{\Theta}\hat{e}_{\Theta},$$

where we continue to use $\hat{e}_{\mathrm{R}}$ to designate the radial vector. Then

$$\begin{aligned}\overrightarrow{\omega} \times \overrightarrow{d} &= -\dot{\vartheta}\hat{e}_{\mathrm{Z}} \times (-a)\,\hat{e}_{\mathrm{R}} \\ &= -a\dot{\vartheta}\hat{e}_{\Theta},\end{aligned}$$

since $\hat{e}_{\mathrm{Z}} \times \hat{e}_{\mathrm{R}} = -\hat{e}_{\Theta}$. The rolling constraint is then

$$\begin{aligned}0 &= \dot{R}\,\hat{e}_{\mathrm{R}} + R\dot{\Theta}\hat{e}_{\Theta} - a\dot{\vartheta}\hat{e}_{\Theta} \\ &= \dot{R}\,\hat{e}_{\mathrm{R}} + \left(R\dot{\Theta} - a\dot{\vartheta}\right)\hat{e}_{\Theta}.\end{aligned}$$

As long as there is contact we have the constraint

$$\dot{R} = 0.$$

and

$$R\dot{\Theta} = a\dot{\vartheta}$$

is the rolling constraint. We notice that $R\mathrm{d}\Theta$ is the distance moved by the center of the small sphere in a time $\mathrm{d}t$, which is equal to $a\mathrm{d}\vartheta$ if the sphere is rolling.

Writing these constraints in differential form,

$$\mathrm{d}g_1 = 0 = \mathrm{d}R$$

and

$$\mathrm{d}g_2 = 0 = R\mathrm{d}\Theta - a\mathrm{d}\vartheta.$$

Then

$$\frac{\partial g_1}{\partial R} = 1$$

and

$$\frac{\partial g_2}{\partial \Theta} = R\ ;\ \frac{\partial g_2}{\partial \vartheta} = -a$$

If we pick the reference level for the potential energy to be the plane $Y = 0$ the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2}M\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{R}\right)^2 + \frac{1}{2}I_0\dot{\vartheta}^2 - MgR\cos\Theta \\ &= \frac{1}{2}M\left(\dot{R}^2 + R^2\dot{\Theta}^2\right) + \frac{1}{2}I_0\dot{\vartheta}^2 - MgR\cos\Theta \end{aligned}$$

The canonical momenta are

$$P_R = \frac{\partial L}{\partial \dot{R}} = M\dot{R},$$

$$P_\Theta = \frac{\partial L}{\partial \dot{\Theta}} = MR^2\dot{\Theta},$$

and

$$P_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = I_0\dot{\vartheta}.$$

The Lagrangian is cyclic in the time so the Hamiltonian is a constant of the motion.

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}M\left(\dot{R}^2 + R^2\dot{\Theta}^2\right) + \frac{1}{2}I_0\dot{\vartheta}^2 + MgR\cos\Theta \\ &= \frac{P_{\mathrm{R}}^2}{2M} + \frac{P_\Theta^2}{2MR^2} + \frac{P_\vartheta^2}{2I_0} + MgR\cos\Theta. \end{aligned}$$

The canonical equations are then

$$\dot{P}_{\rm R} = \frac{P_\Theta^2}{MR^3} - Mg\cos\Theta + \lambda_1$$

$$\dot{P}_\Theta = MgR\sin\Theta + \lambda_2 R$$

$$\dot{P}_\vartheta = -\lambda_2 a$$

and

$$\dot{R} = \frac{P_{\rm R}}{M}$$

$$\dot{\Theta} = \frac{P_\Theta}{MR^2}$$

$$\dot{\vartheta} = \frac{P_\vartheta}{I_0}$$

Before proceeding we see that λ_1 is the contact force in the $\hat{e}_{\rm R}$ direction.

We have

$$\lambda_1 = \dot{P}_{\rm R} - \frac{P_\Theta^2}{MR^3} + Mg\cos\Theta$$

While there is rolling the constraint $R =$ constant implies that $\dot{P}_{\rm R} = 0$, so

$$\lambda_1 = -\frac{P_\Theta^2}{MR^3} + Mg\cos\Theta$$

is the contact force in the R-direction. This vanishes when the cylinders lose contact. We cannot obtain the solution for the angle at which contact is lost, however, until we have P_Θ.

We can find P_Θ from the Hamiltonian. Since the small cylinder is released from essential rest at the top, we have

$$\begin{aligned}\mathcal{H} &= MgR \\ &= \frac{P_\Theta^2}{2MR^2} + \frac{P_\vartheta^2}{2I_0} + MgR\cos\Theta,\end{aligned}$$

while contact is maintained. Using the rolling constraint

$$R\dot{\Theta} = a\dot{\vartheta}$$

we have a relation between P_Θ and P_ϑ.

$$P_\vartheta = I_0 \dot{\vartheta} = I_0 \left(\frac{R}{a}\right) \dot{\Theta}$$
$$= I_0 \left(\frac{R}{a}\right) \frac{P_\Theta}{MR^2}$$
$$= \frac{I_0}{Ra} \frac{P_\Theta}{M}.$$

Then the Hamiltonian is then

$$MgR = \frac{P_\Theta^2}{2MR^2} + \frac{(I_0 P_\Theta / MRa)^2}{2I_0} + MgR\cos\Theta$$
$$= \frac{1}{2} P_\Theta^2 \frac{Ma^2 + I_0}{R^2 a^2 M^2} + MgR\cos\Theta.$$

From above

$$P_\Theta^2 = (Mg\cos\Theta - \lambda_1)\, MR^3.$$

Then the Hamiltonian becomes

$$Mg = \frac{1}{2}(Mg\cos\Theta - \lambda_1)\,\frac{Ma^2 + I_0}{Ma^2} + Mg\cos\Theta$$

Contact is lost when $\lambda_1 = 0$. Including the moment of inertia as $I_0 = 2Ma^2/5$, this is

$$1 = \cos\Theta\left(\frac{3Ma^2 + I_0}{2Ma^2}\right)$$

or

$$\Theta = \cos^{-1}\left(\frac{2}{3 + I_0/Ma^2}\right)$$

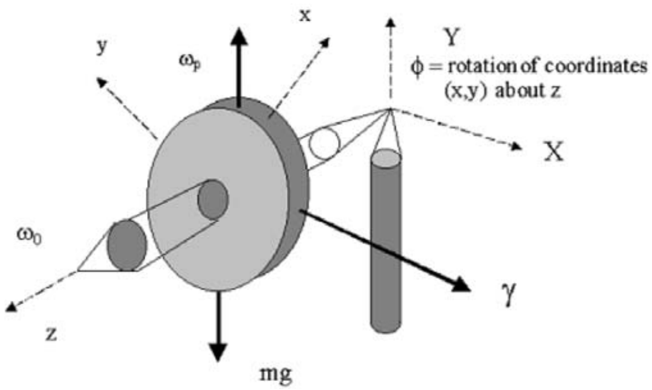
With $I_0 = 2Ma^2/5$ for the solid sphere,

$$\Theta = \cos^{-1}\left(\frac{2}{3 + 2/5}\right) = \cos^{-1}\left(\frac{10}{17}\right).$$

4.8. Here we shall seek an understanding of the rather mysterious motion of the toy gyroscope. This toy is not really a gyroscope. A real gyroscope pivots about a fixed point at the CM of the gyroscope. The toy gyroscope is actually a top, because it pivots about a point which is not the CM. In seeking an understanding we shall approach the problem by inserting the motion we have observed and asking whether or not this is consistent with Euler's Equations for rotational motion.

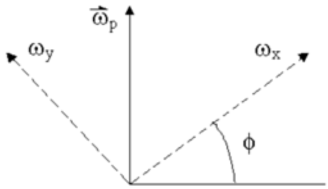
We have drawn a toy gyroscope in the figure below. The mass of the gyroscope flywheel is m. The torque about the pivot point is γ and is equal to the product of mg

and the moment arm from the center of mass of the flywheel to the the pivot point. The flywheel of the toy gyroscope rotates around the z axis with an angular velocity ω_0 and the precession velocity is ω_p. The fixed coordinates are (X, Y, Z). The body coordinates are (x, y, z). Initially the axes Z and z are aligned.



A toy gyroscope.

The two angular velocities: ω_0 and ω_p in this figure are measurable. The angular velocity of the flywheel has components ω_x, ω_y, and ω_z, which we have identified as ω_0. The angular velocity ω_0 may be measured stroboscopically before the experiment. The angular velocity of precession ω_p is the projection of ω_x and ω_y on the fixed axis Y as we have shown in the drawing here.



Combination of angular velocities of the flywheel into ω_p.

In the experiment we observe that the toy gyroscope precesses around the vertical axis Y. This is counter-intuitive. It seems to float as it precesses rather than falling over as we may expect.

Obtain the relationship between the angular velocity of precession ω_p and the angular velocity of the flywheel about the $Z-$axis ω_0.

Solution:

From the drawing the components of ω_p in the body system are

$$\omega_x = \omega_p \sin\phi$$
$$\omega_y = \omega_p \cos\phi.$$

The torque arises from the force, mg, of the support on the gyroscope. The gravitational force acts through the center of mass and has no torque about the CM. In the body coordinates, the torque is

$$\overrightarrow{\gamma} = \gamma \cos\phi \hat{e}_{\mathrm{x}} - \gamma \sin\phi \hat{e}_{\mathrm{y}}.$$

The flywheel is a body of revolution. Two of the moments of inertia are the same. We call these

$$I_{\mathrm{x}} = I_{\mathrm{y}} = I'$$
$$I_{\mathrm{z}} = I$$

Euler's Equations, which we shall take as the fundamental equations in this treatment, are

$$\begin{aligned}\gamma_{\mathrm{x}} &= \gamma \cos\phi = I'\dot{\omega}_{\mathrm{x}} + \left(I - I'\right)\omega_{\mathrm{z}}\omega_{\mathrm{y}}\\ \gamma_{\mathrm{y}} &= -\gamma \sin\phi = I'\dot{\omega}_{\mathrm{y}} - \left(I - I'\right)\omega_{\mathrm{z}}\omega_{\mathrm{x}}\\ 0 &= I\dot{\omega}_{\mathrm{z}}.\end{aligned}$$

The last of these equations immediately yields

$$\omega_{\mathrm{z}} = \text{constant} = \omega_0.$$

Then

$$\phi = \omega_0 t.$$

If we define

$$\alpha \equiv \frac{\gamma}{I'}$$

and

$$\lambda \equiv \frac{I - I'}{I'}\omega_0$$

the Euler equations are

$$\begin{aligned}\alpha \cos\phi &= \dot{\omega}_{\mathrm{x}} + \lambda\omega_{\mathrm{y}}\\ -\alpha \sin\phi &= \dot{\omega}_{\mathrm{y}} - \lambda\omega_{\mathrm{x}}.\end{aligned}$$

These equations have the asymmetry of the charge in the magnetic field. We can solve them in a straightforward fashion in the complex plane. We write

$$\Omega_{\mathrm{Z}} = \omega_{\mathrm{x}} + i\omega_{\mathrm{y}}.$$

We then write the Euler equations as

$$\begin{aligned}\alpha \cos\phi &= \dot{\omega}_{\mathrm{x}} - i\lambda\left(i\omega_{\mathrm{y}}\right)\\ -i\alpha \sin\phi &= i\dot{\omega}_{\mathrm{y}} - i\lambda\left(\omega_{\mathrm{x}}\right)\end{aligned}$$

and add these together to get

$$\alpha \exp(-i\phi) = \dot{\Omega}_Z - i\lambda\Omega_Z.$$

We shall solve this equation using the standard Green's Function approach [see e.g. [1] Chapt. 8]. We first find the homogeneous solution and then the Green's Function (GF) from that homogenous solution.

The homogeneous equation is

$$0 = \dot{\Omega}_Z^{(h)} - i\lambda\Omega_Z^{(h)},$$

which has the solution

$$\Omega_Z^{(h)} = \exp(i\lambda t).$$

The GF is a linear combination of the homogeneous solutions. Of course, since there is only one homogeneous solution, the GF here must be simply proportional to this solution. This is also an initial value problem so the GF vanishes for $t < \tau$. That is

$$G(t,\tau) = \begin{cases} 0 \text{ for } t < \tau \\ A(\tau)\exp(i\lambda t) \text{ for } t > \tau \end{cases}$$

We only have a single condition to employ because this is a first order equation. This is the jump condition:

$$\frac{\mathrm{d}^{\text{n-1}}}{\mathrm{d}t^{\text{n-1}}} G(t,\tau) \text{ is discontinuous at } t = \tau \text{ by an amount } \frac{1}{a_0(\tau)} = 1.$$

But $n = 1$ for this first order problem. So we have a discontinuity in the zeroth order derivative, which is the GF itself. Then

$$A(\tau)\exp(i\lambda\tau) = 1$$

or

$$A(\tau) = \exp(-i\lambda\tau).$$

The GF is then

$$G(t,\tau) = \begin{cases} 0 \text{ for } t < \tau \\ \exp(i\lambda(t-\tau)) \text{ for } t > \tau \end{cases}$$

And the particular solution is

$$\Omega_Z^{(p)} = \int_0^t \mathrm{d}\tau \exp(i\lambda(t-\tau))\left[\alpha \exp(-i\phi(\tau))\right].$$

With $\phi = \omega_0 t$ this becomes

$$\Omega_{\mathrm{Z}}^{(\mathrm{p})} = \alpha \exp(i\lambda t) \int_0^{\mathrm{t}} \mathrm{d}\tau \exp(-i(\lambda + \omega_0)\tau) .$$

Integrating

$$\begin{aligned}
\Omega_{\mathrm{Z}}^{(\mathrm{p})} &= i\alpha \exp(i\lambda t) \frac{1}{\lambda + \omega_0} \exp(-i(\lambda + \omega_0)\tau)\Big]_0^{\mathrm{t}} \\
&= \frac{\alpha}{\lambda + \omega_0} [(\sin\lambda t + \sin\omega_0 t) + i(\cos\omega_0 t - \cos\lambda t)] .
\end{aligned}$$

The general solution is the sum of this and the homogeneous solution.

$$K \exp(i\lambda t)$$

where K is an arbitrary (complex) constant, which is evaluated from initial conditions. Expanding this, the homogeneous solution is

$$K \exp(i\lambda t) = (K_{\mathrm{r}} \cos\lambda t - K_{\mathrm{i}} \sin\lambda t) + i(K_{\mathrm{i}} \cos\lambda t + K_{\mathrm{r}} \sin\lambda t) .$$

The complex solution, Ω_{Z} is then

$$\begin{aligned}
\Omega_{\mathrm{Z}} &= \frac{K\alpha}{\lambda + \omega_0} \exp(i\lambda t) + \Omega_{\mathrm{Z}}^{(\mathrm{p})} \\
&= \frac{\alpha}{\lambda + \omega_0} \{[K_{\mathrm{r}} \cos\lambda t - K_{\mathrm{i}} \sin\lambda t + (\sin\lambda t + \sin\omega_0 t)] \\
&\quad + i[K_{\mathrm{r}} \sin\lambda t + K_{\mathrm{i}} \cos\lambda t + (\cos\lambda t - \cos\omega_0 t)]\} ,
\end{aligned}$$

where K_{r} and K_{i} are dependent on initial conditions. To obtain values for these constants these we must find the explicit forms for the angular momentum components ω_{x} and ω_{y} and evaluate them at $t = 0$. The expressions for ω_{x} and ω_{y} are obtained from the complex angular velocity Ω_{Z}.

Since $\Omega_{\mathrm{Z}} = \omega_{\mathrm{x}} + i\omega_{\mathrm{y}}$, we have

$$\begin{aligned}
\omega_{\mathrm{x}} &= \frac{\alpha}{\lambda + \omega_0} [K_{\mathrm{r}} \cos\lambda t - K_{\mathrm{i}} \sin\lambda t + (\sin\lambda t + \sin\omega_0 t)] \\
\omega_{\mathrm{y}} &= \frac{\alpha}{\lambda + \omega_0} [K_{\mathrm{r}} \sin\lambda t + K_{\mathrm{i}} \cos\lambda t + (\cos\lambda t - \cos\omega_0 t)] .
\end{aligned}$$

At $t = 0$, ω_{x} is zero and ω_{y} is equal to the precession angular velocity ω_{p}. Then

$$\omega_{\mathrm{x}} = 0 = \frac{K_{\mathrm{r}}\alpha}{\lambda + \omega_0}$$

$$\omega_{\mathrm{y}} = \omega_{\mathrm{p}} = \frac{K_{\mathrm{i}}\alpha}{\lambda + \omega_0}.$$

So

$$K_{\mathrm{r}} = 0$$

and

$$K_{\mathrm{i}} = \frac{\omega_{\mathrm{p}}(\lambda + \omega_0)}{\alpha}.$$

With these values for K_{r} and K_{i} the expressions for ω_{x} and ω_{y} become

$$\begin{aligned}
\omega_{\mathrm{x}} &= \frac{\alpha}{\lambda + \omega_0}\left[\left(1 - \frac{\omega_{\mathrm{p}}(\lambda + \omega_0)}{\alpha}\right)\sin\lambda t + \sin\omega_0 t\right] \\
\omega_{\mathrm{y}} &= \frac{\alpha}{\lambda + \omega_0}\left[\left(\frac{\omega_{\mathrm{p}}(\lambda + \omega_0)}{\alpha} - 1\right)\cos\lambda t + \cos\omega_0 t\right].
\end{aligned}$$

The relationship among the components $\left(\omega_{\mathrm{x}}, \omega_{\mathrm{y}}\right)$ and the precession angular velocity, ω_{p} is

$$\begin{aligned}
\omega_{\mathrm{x}} &= \omega_{\mathrm{p}}\sin\omega_0 t \\
\omega_{\mathrm{y}} &= \omega_{\mathrm{p}}\cos\omega_0 t.
\end{aligned}$$

Using these we obtain

$$\begin{aligned}
\omega_{\mathrm{p}}\sin\omega_0 t &= \frac{\alpha}{\lambda + \omega_0}\left[\left(1 - \frac{\omega_{\mathrm{p}}(\lambda + \omega_0)}{\alpha}\right)\sin\lambda t + \sin\omega_0 t\right] \\
\omega_{\mathrm{p}}\cos\omega_0 t &= \frac{\alpha}{\lambda + \omega_0}\left[\left(\frac{\omega_{\mathrm{p}}(\lambda + \omega_0)}{\alpha} - 1\right)\cos\lambda t + \cos\omega_0 t\right].
\end{aligned}$$

Collecting terms,

$$\begin{aligned}
\left(\omega_{\mathrm{p}} - \frac{\alpha}{\lambda + \omega_0}\right)\sin\omega_0 t &= -\left(\omega_{\mathrm{p}} - \frac{\alpha}{\lambda + \omega_0}\right)\sin\lambda t \\
\left(\omega_{\mathrm{p}} - \frac{\alpha}{\lambda + \omega_0}\right)\cos\omega_0 t &= \left(\omega_{\mathrm{p}} - \frac{\alpha}{\lambda + \omega_0}\right)\cos\lambda t.
\end{aligned}$$

Because of the definition

$$\lambda \equiv \frac{I - I'}{I'}\omega_0,$$

we cannot have $\omega_0 = \lambda$. So the above equations are only satisfied if

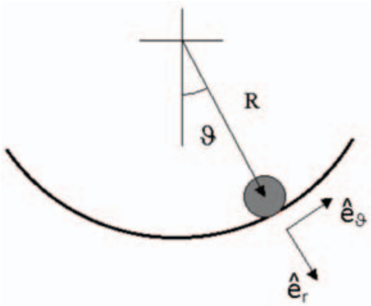
$$\omega_{\mathrm{p}} = \frac{\alpha}{\lambda + \omega_0}.$$

This is then the requirement for the precession angular velocity in order that the strange motion is actually that resulting. Putting in the terms

$$\omega_{\mathrm{p}} = mg\frac{\ell}{\omega_0 I},$$

where ℓ is the distance between the CM and the support. So we do find an expression for the angular velocity of precession. This has been checked in an undergraduate laboratory. Agreement is good.

4.9. The rolling ball pendulum is a pendulum in which a bowling ball (of mass M and radius a) may be either suspended by a wire of length ℓ and allowed to swing through an arc, or removed from the suspending wire and allowed to roll on a circular track of radius $R = \ell$ constructed to follow the same path as that of the swinging pendulum. We have drawn the rolling ball pendulum in the figure here.



The rolling ball pendulum. We have shown here only the track with the ball rolling on it. The ball (a bowling ball) may also be suspended to result in a simple pendulum following the same path.

Experimentally we can measure the period in each case and compare them. The fact of the matter is that we find two periods. The rolling period is slightly longer than that of the simple pendulum. Almost all students initially claim that the difference is the result of friction. But this is not the case if the ball rolls on the track, because rolling on a smooth surface is frictionless.

Analyze the two situations and find the difference in the two periods. show that there is no friction from rolling.

Solution:

For the suspended mass (simple pendulum) the angular frequency is known to be $\omega = \sqrt{\ell/g}$. We need then only consider the motion of the bowling ball rolling back and forth on the circular track. We locate the ball with the coordinates (r, ϑ) and define the angle of rotation of the ball about its axis as φ. Then the kinetic energy of the ball is

$$T = \frac{1}{2}M\left(\ell^2\dot{\vartheta}^2\right) + \frac{1}{2}I\dot{\phi}^2.$$

The potential energy, choosing the reference to be the center of the coordinate system, is

$$V = -Mg\ell\cos\vartheta.$$

So the Lagrangian is

$$L = \frac{1}{2}M\left(\ell^2\dot{\vartheta}^2\right) + \frac{1}{2}I\dot{\phi}^2 + Mg\ell\cos\vartheta.$$

And the Hamiltonian is

$$\mathcal{H} = \frac{1}{2}M\left(\ell^2\dot{\vartheta}^2\right) + \frac{1}{2}I\dot{\phi}^2 - Mg\ell\cos\vartheta.$$

The momenta are

$$p_\vartheta = M\ell^2\dot{\vartheta}$$

and

$$p_\phi = I\dot{\varphi}.$$

So the Hamiltonian in natural coordinates is

$$\mathcal{H} = \frac{p_\vartheta^2}{2M\ell^2} + \frac{p_\phi^2}{2I} - Mg\ell\cos\vartheta.$$

The rolling constraint is found from the velocity of the CM of the bowling ball

$$\overrightarrow{V}_{\mathrm{CM}} = \ell\dot{\vartheta}\hat{e}_\vartheta$$

and the angular momentum of the ball on its axis measured in the fixed system

$$\overrightarrow{\omega}_{\mathrm{B}} = -\dot{\phi}\hat{e}_{\mathrm{z}}.$$

The vector from the center of the ball to the track, also in the fixed system, is

$$\overrightarrow{d} = -a\hat{e}_{\mathrm{r}}.$$

The rolling constraint is then

$$\begin{aligned} 0 &= \overrightarrow{V}_{\mathrm{CM}} + \overrightarrow{\omega}_{\mathrm{B}} \times \overrightarrow{d} \\ &= \ell\dot{\vartheta}\hat{e}_\vartheta - a\dot{\phi}\hat{e}_\vartheta \end{aligned}$$

or

$$\ell\dot{\vartheta} = a\dot{\phi}.$$

This constraint is written in normal form as

$$\dot{g} = 0 = \ell\dot{\vartheta} - a\dot{\phi}$$

or

$$dg = 0 = \ell d\vartheta - a d\phi.$$

That is

$$\frac{\partial g}{\partial \vartheta} = \ell$$
$$\frac{\partial g}{\partial \phi} = -a.$$

The canonical equations are then

$$\dot{p}_\vartheta = -Mg\ell \sin\vartheta + \lambda\ell,$$

$$p_\vartheta = M\ell^2\dot{\vartheta},$$

$$\dot{p}_\phi = -\lambda a,$$

$$p_\varphi = I\dot{\phi}.$$

First we eliminate the λ.

$$\lambda = -\frac{\dot{p}_\phi}{a}$$

and

$$\dot{p}_\vartheta = -Mg\ell \sin\vartheta - \frac{\dot{p}_\phi}{a}\ell.$$

Then we use the rolling constraint to find the relationship between p_ϕ and p_ϑ.

$$\begin{aligned} p_\phi &= I\dot{\phi} \\ &= I\frac{\ell}{a}\dot{\vartheta} \\ &= I\frac{\ell}{a}\left(\frac{p_\vartheta}{M\ell^2}\right) \\ &= \frac{I}{\ell a}\frac{p_\vartheta}{M}. \end{aligned}$$

Then

$$\dot{p}_\vartheta = -\frac{Mg\ell}{1 + I/\left(Ma^2\right)} \sin\vartheta.$$

To find the period we consider small displacements. For small angles we have

$$\dot{p}_\vartheta = -\frac{Mg\ell}{1 + I/\left(Ma^2\right)}\vartheta,$$

or

$$M\ell^2 = -\frac{Mg\ell}{1 + I/\left(Ma^2\right)}\vartheta,$$

which is

$$\ddot{\vartheta} = -\frac{g/\ell}{1 + I/\left(Ma^2\right)}\vartheta.$$

Then

$$\omega_{\text{rolling}} = \frac{\sqrt{g/\ell}}{\sqrt{1 + I/\left(Ma^2\right)}}$$

The moment of inertia of a sphere about any axis is

$$I = \frac{1}{2}Ma^2.$$

Therefore

$$\begin{aligned} \omega_{\text{rolling}} &= \sqrt{\frac{2}{3}}\omega_0 \\ &= 0.8165\omega_0 \end{aligned}$$

independently of the size of the ball. The period is

$$\tau = \frac{2\pi}{\omega},$$

so the relationship of the periods is

$$\begin{aligned} \tau_{\text{rolling}} &= \sqrt{\frac{3}{2}}\tau_0 \\ &= 1.2247\tau_0 \end{aligned}$$

This result is born out very well experimentally.

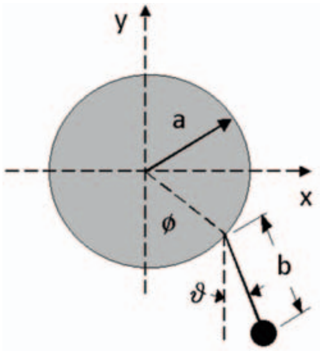
Perhaps the simplest way to show that the rolling constraint causes no loss in energy, i.e. that there is no friction due to rolling, is to show that the Lagrangian remains explicitly independent of time if the rolling constraint is incorporated into the Lagrangian. If this is the case, then $\partial L/\partial t = 0$ and $\mathrm{d}\mathcal{H}/\mathrm{d}t = 0$.

With the rolling constraint incorporated into the Lagrangian,

$$L = \frac{1}{2}\left(M\ell^2 + I\frac{\ell}{a}\right)\dot{\vartheta}^2 + Mg\ell\cos\vartheta,$$

which remains explicitly independent of the time. Therefore the Hamiltonian, which is the total energy in this case, is constant.

4.10. In the figure below we have a disk, which is free to rotate without friction about the central axis, and has a rigid-rod pendulum affixed to the rim of the disk.



Disk with a pendulum.

Obtain the angular velocities in terms of the canonical momenta. Then turn to the Euler-Lagrange Equations for the study of small vibrations. In the final analysis simplify to the case in which $a = b$.

Solution:

The position vector to the pendulum bob is

$$\boldsymbol{R} = (a \sin\phi + b \sin\vartheta)\, \hat{e}_{\mathrm{x}} - (a \cos\phi + b \cos\vartheta)\, \hat{e}_{\mathrm{y}}$$

The velocity of this mass is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \left(a\dot{\phi}\cos\phi + b\dot{\vartheta}\cos\vartheta\right)\hat{e}_x + \left(a\dot{\phi}\sin\phi + b\dot{\vartheta}\sin\vartheta\right)\hat{e}_y,$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}\right|^2 = a^2\dot{\phi}^2 + b^2\dot{\vartheta}^2 + 2ab\left(\dot{\phi}\dot{\vartheta}\right)\cos(\phi - \vartheta).$$

The kinetic energy of the bob plus disk is then

$$T = \frac{1}{2}\left[a^2\dot{\phi}^2 + b^2\dot{\vartheta}^2 + 2ab\left(\dot{\phi}\dot{\vartheta}\right)\cos(\phi - \vartheta)\right] + \frac{1}{2}I\dot{\phi}^2.$$

The potential energy is wholly contained in the bob and is

$$V = -mg\,(a\cos\phi + b\cos\vartheta)$$

where the ground is the plane through the center of the disk. The Lagrangian is then

$$L = \frac{1}{2} m \left[a^2 \dot{\phi}^2 + b^2 \dot{\vartheta}^2 + 2ab \left(\dot{\phi} \dot{\vartheta} \right) \cos (\phi - \vartheta) \right] + \frac{1}{2} I \dot{\phi}^2$$
$$+ mg \left(a \cos \phi + b \cos \vartheta \right).$$

From here we obtain the canonical momenta as

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ma^2 \dot{\phi} + mab \dot{\vartheta} \cos (\phi - \vartheta) + I \dot{\phi}$$

$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = mb^2 \dot{\vartheta} + mab \dot{\phi} \cos (\phi - \vartheta).$$

In matrix form

$$\begin{bmatrix} p_\vartheta \\ p_\phi \end{bmatrix} = \begin{bmatrix} mb^2 & mab \cos (\phi - \vartheta) \\ mab \cos (\phi - \vartheta) & \left(ma^2 + I \right) \end{bmatrix} \begin{bmatrix} \dot{\vartheta} \\ \dot{\phi} \end{bmatrix}.$$

If we choose $a = b$ and $\bar{I} = I/ma^2$ this is

$$\begin{bmatrix} p_\vartheta \\ p_\phi \end{bmatrix} = ma^2 \begin{bmatrix} 1 & \cos (\phi - \vartheta) \\ \cos (\phi - \vartheta) & \left(1 + \bar{I} \right) \end{bmatrix} \begin{bmatrix} \dot{\vartheta} \\ \dot{\phi} \end{bmatrix}.$$

Then the solution for $\begin{bmatrix} \dot{\vartheta} \\ \dot{\phi} \end{bmatrix}$ is

$$\begin{bmatrix} \dot{\vartheta} \\ \dot{\phi} \end{bmatrix} = \frac{1}{ma^2 \left[1 - \cos^2 (\phi - \vartheta) + \bar{I} \right]} \begin{bmatrix} p_\vartheta \left(\bar{I} + 1 \right) - p_\phi \cos (\phi - \vartheta) \\ p_\phi - p_\vartheta \cos (\phi - \vartheta) \end{bmatrix}.$$

We may use these solutions in the Hamiltonian. But that will introduce rather extensive trigonometric terms. We, therefore, choose to study small vibrations via the Euler-Lagrange equations.

We already have $\partial L / \partial \dot{\phi}$ and $\partial L / \partial \dot{\vartheta}$ above. The derivatives of the Lagrangian with respect to the coordinates are

$$\frac{\partial L}{\partial \phi} = -mab \left(\dot{\phi} \dot{\vartheta} \right) \sin (\phi - \vartheta) - mga \sin \phi$$

$$\frac{\partial L}{\partial \vartheta} = mab \left(\dot{\phi} \dot{\vartheta} \right) \sin (\phi - \vartheta) - mgb \sin \vartheta.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = ma^2\ddot{\phi} + mab\ddot{\vartheta}\cos(\phi - \vartheta)$$
$$-mab\dot{\vartheta}\dot{\phi}\sin(\phi - \vartheta) + mab\dot{\vartheta}^2\sin(\phi - \vartheta) + I\ddot{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} = mb^2\ddot{\vartheta} + mab\ddot{\phi}\cos(\phi - \vartheta)$$
$$-mab\dot{\phi}^2\sin(\phi - \vartheta) + mab\dot{\phi}\dot{\vartheta}\sin(\phi - \vartheta)\,.$$

The Euler-Lagrange equations are then

$$\frac{\partial L}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = -\left(ma^2 + I\right)\ddot{\phi} - (mab\cos(\phi - \vartheta))\,\ddot{\vartheta}$$
$$-mab\dot{\vartheta}^2\sin(\phi - \vartheta) - mga\sin\phi$$
$$= 0$$

$$\frac{\partial L}{\partial \vartheta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} = -mb^2\ddot{\vartheta} - (mab\cos(\phi - \vartheta))\,\ddot{\phi}$$
$$+mab\dot{\phi}^2\sin(\phi - \vartheta) - mgb\sin\vartheta$$
$$= 0$$

We linearize these equations to consider small vibrations. Doing so, noting that the sine of a difference in small angles is proportional to that difference and that the cosine of the difference is unity and choosing $a = b$, we have

$$\left(ma^2 + I\right)\ddot{\phi} + \left(ma^2\right)\ddot{\vartheta} + (mga)\,\phi = 0$$

$$\left(ma^2\right)\ddot{\vartheta} + \left(ma^2\right)\ddot{\phi} + (mga)\,\vartheta = 0$$

If we invoke the standard Ansatz of the complex exponential, these become

$$-\omega^2\left(ma^2 + I\right)\tilde{\phi} - \omega^2\left(ma^2\right)\tilde{\vartheta} + (mga)\,\tilde{\phi} = 0$$

$$-\omega^2\left(ma^2\right)\tilde{\vartheta} - \omega^2\left(ma^2\right)\tilde{\phi} + (mga)\,\tilde{\vartheta} = 0$$

In matrix form, with $I = ma^2/2$, we have

$$\begin{bmatrix} (g/a) - \omega^2\,(3/2) & -\omega^2 \\ -\omega^2 & (g/a) - \omega^2 \end{bmatrix}\begin{bmatrix} \tilde{\phi} \\ \tilde{\vartheta} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For a non-trivial solution the determinant of the principal matrix must vanish

$$a^2\omega^4 - 5ag\omega^2 + 2g^2 = 0.$$

Solving this for ω^2,

$$\omega^2 = \frac{g}{a}\left(\frac{5}{2}+\frac{1}{2}\sqrt{17}\right)$$
$$\omega^2 = \frac{g}{a}\left(\frac{5}{2}-\frac{1}{2}\sqrt{17}\right)$$

Both of these are real. So the square roots may be taken to find the actual frequencies. There are two possible frequencies. The motion may be found for each by inserting the above ω^2 values into the matrix equation and Solving in general for the ratio of $\tilde{\vartheta}$ to $\tilde{\phi}$ we have

$$\frac{\tilde{\vartheta}}{\tilde{\phi}} = \frac{\left(mga - \frac{3}{2}ma^2\omega^2\right)}{\omega^2\left(ma^2\right)}.$$

Then for $\omega^2 = (g/a)\left(5/2 + \sqrt{17}/2\right)$

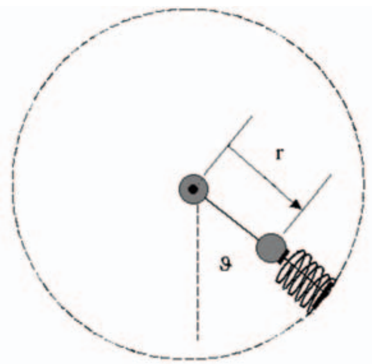
$$\frac{\tilde{\vartheta}}{\tilde{\phi}} = -\frac{1}{2}\frac{11+3\sqrt{17}}{5+\sqrt{17}} = -1.2808$$

and for $\omega^2 = (g/a)\left(5/2 - \sqrt{17}/2\right)$

$$\frac{\tilde{\vartheta}}{\tilde{\phi}} = \frac{1}{2}\frac{11-3\sqrt{17}}{5-\sqrt{17}} = -.78078.$$

So we have two modes at two frequencies. The higher frequency mode has a larger amplitude of the pendulum swing than the lower frequency mode. But in both modes the pendulum and the disk move in opposing directions.

4.11. We have devised a sort of toy we designate as a spring pendulum, which we have drawn in the figure here.

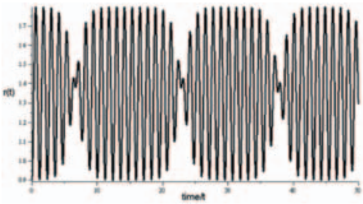


The spring pendulum.

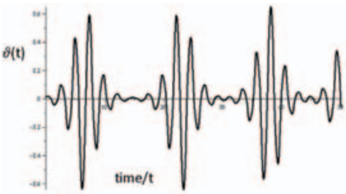
In the spring pendulum we have a rigid rod of negligible mass and of length R mounted on a bearing about which it rotates without friction. A metal ball of mass m with a hole drilled through it slides also without friction on the rod. To the ball we have affixed a spring, which is solidly mounted to the end of the rod.

Study the motion of this toy. Obtain the canonical and the Euler-Lagrange equations. Seek a numerical solution if software is available.

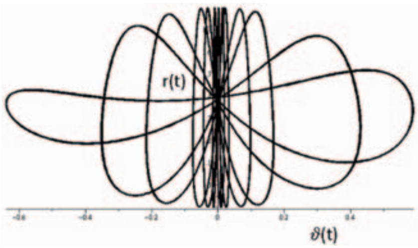
In the figures below we have plotted results from the numerical solution of the nonlinear canonical equations describing the spring pendulum.



Radial motion $r\,(t)$ vs t for the spring pendulum.



Angular motion $\vartheta\,(t)$ vs t for the spring pendulum.



Plot of $r\,(t)$ vs $\vartheta\,(t)$ for the spring pendulum.

Solution:

The ball is easily located by a vector in cylindrical coordinates with origin at the central bearing. The coordinates are then r and ϑ as shown. The square of the velocity is

$$\dot{r}^2 + r^2\dot{\vartheta}^2$$

and the kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right).$$

The potential energy arises from the spring and from the gravitational field. The gravitational potential energy is

$$-mgr\cos\vartheta,$$

with the reference plane containing the central axis, and the spring potential energy is

$$\frac{1}{2}k\left(r-r_0\right)^2,$$

where $r_0 =$ the equilibrium position of r. The Lagrangian is then

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) + mgr\cos\vartheta - \frac{1}{2}k\left(r-r_0\right)^2.$$

The canonical momenta are

$$p_{\mathrm{r}} = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = mr^2\dot{\vartheta}.$$

The Lagrangian is cyclic in neither r nor ϑ. So neither of these momenta is constant.

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2\right) - mgr\cos\vartheta + \frac{1}{2}k\left(r-r_0\right)^2.$$

With

$$\dot{r} = \frac{p_r}{m}$$
$$\dot{\vartheta} = \frac{p_\vartheta}{mr^2}$$

the Hamiltonian is

$$\mathcal{H} = \frac{p_{\mathrm{r}}^2}{2m} + \frac{p_\vartheta^2}{2mr^2} - mgr\cos\vartheta + \frac{1}{2}k\left(r-r_0\right)^2.$$

The canonical equations are the above equations for $\dot{r}$ and $\dot{\vartheta}$ and

$$\dot{p}_{\mathrm{r}} = -\frac{\partial\mathcal{H}}{\partial r} = \frac{p_\vartheta^2}{mr^3} + mg\cos\vartheta - k\left(r-r_0\right)$$
$$\dot{p}_\vartheta = -\frac{\partial\mathcal{H}}{\partial\vartheta} = -mgr\sin\vartheta$$

We have, then, a coupled set of four first order equations.

The Euler-Lagrange equations are

$$\frac{\partial L}{\partial r}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}}=mr\dot{\vartheta}^2+mg\cos\vartheta-k\left(r-r_0\right)-m\ddot{r}$$
$$=0$$

and

$$\frac{\partial L}{\partial \vartheta}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}}=-mgr\sin\vartheta-\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\vartheta}\right)$$
$$=-mgr\sin\vartheta-2mr\dot{\vartheta}\dot{r}-mr^2\ddot{\vartheta}$$
$$=0.$$

The Lagrangian does not depend explicitly on the time. So there is a conservation of energy

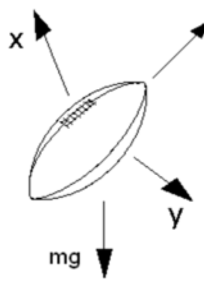
$$\mathcal{E}=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\vartheta}^2\right)-mgr\cos\vartheta+\frac{1}{2}k\left(r-r_0\right)^2.$$

If we linearize the Euler-Lagrange or the canonical equations we can seek the natural modes of vibration. But those hardly seem of any interest. This splendid toy has the possibility of very interesting motion only if we exceed the limits on linearity. So we turn, instead, to numerical solutions. We cross checked the solutions from both the canonical and the Euler-Lagrange equations, but we present only the results for the canonical equations, which, as first order equations, were more easily treated by the Runge-Kutta method.

In the plots we have above we applied initial displacements to both the radial and angular positions and initial radial and angular momenta.

4.12. In American football the forward pass is a critical part of the game. And many children, playing touch football in the backyard or on school playgrounds, dream of learning how to throw that ideal spiral pass, in which the football does not tumble awkwardly but seems to move like a bullet in slow motion to the receiver. Here we shall analyze the motion of the passed football. This is an interesting problem not only for those who have tried to throw that splendid spiral pass. It is also interesting for the spectator in the stands, who believes a certain motion was observed, which is not physically possible.

Below is a picture of an American football and the body coordinates.



American football after passing.

The moments of inertia are chosen as

$$I_{\mathrm{x}} = I_{\mathrm{y}} = I'$$
$$I_{\mathrm{z}} = I$$

Study the motion of the spiral pass, and the possible wobble that destroys the beautiful spiral.

Solution:

There are no torques acting on the football, since the gravitational force acts through the CM. Then Euler's Equations are

$$0 = I'\dot{\omega}_{\mathrm{x}} + \left(I - I'\right)\omega_{\mathrm{y}}\omega_{\mathrm{z}}$$
$$0 = I'\dot{\omega}_{\mathrm{y}} - \left(I - I'\right)\omega_{\mathrm{x}}\omega_{\mathrm{z}}$$
$$0 = I\dot{\omega}_{\mathrm{z}}.$$

The last of these demands

$$\omega_{\mathrm{z}} = \omega_0 = \text{ constant.}$$

Defining

$$\lambda = \frac{I - I'}{I'}\omega_0$$

we have as our Euler's Equations

$$0 = \dot{\omega}_{\mathrm{x}} + \lambda\omega_{\mathrm{y}}$$
$$0 = \dot{\omega}_{\mathrm{y}} - \lambda\omega_{\mathrm{x}}.$$

These equations we solve in the complex plane by defining the complex angular frequency Ω_{Z} as

$$\Omega_{\mathrm{Z}} = \omega_{\mathrm{x}} + i\omega_{\mathrm{y}}.$$

We then write our Euler's Equations as

$$0 = \dot{\omega}_{\mathrm{x}} - i\lambda\left(i\omega_{\mathrm{y}}\right)$$
$$0 = i\dot{\omega}_{\mathrm{y}} - i\lambda\omega_{\mathrm{x}}.$$

Adding these

$$0 = \Omega_{\mathrm{Z}} - i\lambda\Omega_{\mathrm{Z}}$$

The solution is

$$\Omega_{\mathrm{Z}} = \tilde{\Omega}\exp\left(i\lambda t\right),$$

where $\tilde{\Omega}$ is a complex constant. This constant may be evaluated from the initial conditions. We shall assume that ω_{x} and ω_{y} are non-zero.

$$\omega_x\,(t=0)=\omega_{x0}$$
$$\omega_y\,(t=0)=\omega_{y0}$$

Then

$$\begin{aligned}\Omega_Z\,(t=0) &= \omega_{x0}+i\omega_{y0}\\ &= \tilde{\Omega}_r\cos\lambda t-\tilde{\Omega}_i\sin\lambda t+i\left(\tilde{\Omega}_r\sin\lambda t+\tilde{\Omega}_i\cos\lambda t\right)\Big|_{t=0}\\ &= \tilde{\Omega}_r+i\tilde{\Omega}_i.\end{aligned}$$

Equating real and imaginary parts

$$\tilde{\Omega}_r=\omega_{x0}$$
$$\tilde{\Omega}_i=\omega_{y0}.$$

Then we have

$$\omega_x=\omega_{x0}\cos\lambda t-\omega_{y0}\sin\lambda t$$
$$\omega_y=\omega_{x0}\sin\lambda t+\omega_{y0}\cos\lambda t.$$

The situation is easier to visualize if we combine the sine and cosine terms and introduce a phase angle. In general

$$\begin{aligned}A\cos\alpha t+B\sin\alpha t &= \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\alpha t+\frac{A}{\sqrt{A^2+B^2}}\sin\alpha t\right)\\ &= \sqrt{A^2+B^2}\,(\sin\phi\cos\alpha t+\cos\phi\sin\alpha t)\\ &= \sqrt{A^2+B^2}\sin(\alpha t+\phi)\,,\end{aligned}$$

where

$$\phi=\tan^{-1}\frac{A}{B}.$$

For our problem we define

$$\sin\phi=\frac{\omega_{y0}}{\sqrt{\omega_{x0}^2+\omega_{y0}^2}}$$
$$\cos\phi=\frac{\omega_{x0}}{\sqrt{\omega_{x0}^2+\omega_{y0}^2}}$$

Then

$$\begin{aligned}\omega_x &= \sqrt{\omega_{x0}^2+\omega_{y0}^2}\,(\cos\phi\cos\lambda t-\sin\phi\sin\lambda t)\\ &= \sqrt{\omega_{x0}^2+\omega_{y0}^2}\cos(\lambda t+\phi)\end{aligned}$$

and

$$\begin{aligned}\omega_y &= \sqrt{\omega_{x0}^2+\omega_{y0}^2}\,(\cos\phi\sin\lambda t+\sin\phi\cos\lambda t)\\ &= \sqrt{\omega_{x0}^2+\omega_{y0}^2}\sin(\lambda t+\phi)\,.\end{aligned}$$

For shorthand we call

$$A = \sqrt{\omega_{x0}^2 + \omega_{y0}^2}.$$

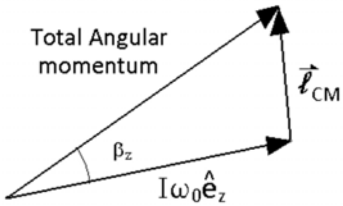
Then the vector $\vec{\omega}$ in the body coordinates is

$$\vec{\omega} = A\cos(\lambda t + \phi)\,\hat{e}_x + A\sin(\lambda t + \phi)\,\hat{e}_y + \omega_0 \hat{e}_z.$$

Because there is no torque about the CM, the angular momentum in the body coordinates is constant. This is

$$\begin{aligned}\vec{\ell}_{CM} &= \begin{bmatrix} I' & 0 & 0 \\ 0 & I' & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A\cos(\lambda t + \phi) \\ A\sin(\lambda t + \phi) \\ \omega_0 \end{bmatrix} \\ &= \begin{bmatrix} I'A\cos(\lambda t + \phi) \\ I'A\sin(\lambda t + \phi) \\ I\omega_0 \end{bmatrix} \\ &= I'A\cos(\lambda t + \phi)\,\hat{e}_X + I'A\sin(\lambda t + \phi)\,\hat{e}_Y + I\omega_0 \hat{e}_Z.\end{aligned}$$

Now we can draw a vector picture.



Football vector diagram.

From here we discover the requirement for the spiral pass.

We observe that in general the angular momentum and the angular velocity are not parallel, even for bodies of revolution. That is

$$\vec{\omega} \cdot \vec{\ell}_{CM} = |\omega|\,|\ell_{CM}|\cos\alpha_s$$

where α_s is the angle between the angular momentum and the angular velocity vectors. From our expressions for angular momentum and angular velocity we have

$$\begin{aligned}\vec{\omega} \cdot \vec{\ell}_{CM} &= \begin{bmatrix} A\cos(\lambda t + \phi) & A\sin(\lambda t + \phi) & \omega_0 \end{bmatrix} \begin{bmatrix} I'A\cos(\lambda t + \phi) \\ I'A\sin(\lambda t + \phi) \\ I\omega_0 \end{bmatrix} \\ &= A^2\left(\cos^2(\lambda t + \phi)\right) I' + A^2\left(\sin^2(\lambda t + \phi)\right) I' + \omega_0^2 I \\ &= A^2 I' + \omega_0^2 I,\end{aligned}$$

and

$$|\omega|\,|\ell_{\rm CM}| = \sqrt{\left(A^2+\omega_0^2\right)\left(I'^2A^2+I^2\omega_0^2\right)}.$$

Then the cosine of the angle between the angular velocity vector and the angular momentum is

$$\cos\alpha_s = \frac{A^2I'+\omega_0^2 I}{\sqrt{\left(A^2+\omega_0^2\right)\left(I'^2A^2+I^2\omega_0^2\right)}}.$$

In a like fashion the cosine of the angle in the drawing above between the $z-$component of the angular velocity vector and the angular momentum is

$$\cos\beta_{\rm z} = \frac{\omega_0 I}{\sqrt{I'^2A^2+I^2\omega_0^2}}.$$

Now we see what is going on. The spectator and the player cannot see the angular momentum vector. What each of them sees is the axis of the football. If the $z-$ axis is aligned with the angular momentum vector the football will not appear to wobble because the angular momentum vector has a fixed direction in space. This is the spiral pass. If $\beta_{\rm z}$ is not zero, then the angular velocity vector rotates about the angular momentum vector and the pass appears wobbly.

It is interesting to observe that in the spiral pass the axis of the football does not follow along the trajectory. the axis is along the angular momentum vector, which is fixed in space. That is, if a nice spiral pass is thrown at an angle of, say, 30^0 to the ground, the axis of the football will remain at 30^0 to the horizontal and will gently come down into the receiver's hands at this convenient angle to be plucked from the air and cradled rapidly under the arm.

5 Hamilton-Jacobi Approach

5.1. Apply the Legendre transformation to obtain generating functions of $F_2(p, P, t)$ and $F_3(p, Q, t)$.

Solution:

We transform out the q from $F_1(q, P, t)$ to obtain $F_2(p, P, t)$. The transform is

$$
\begin{aligned}
F_2(p, P, t) &= F_1(q, P, t) - \sum_i q_i \frac{\partial F_1}{\partial q_i} \\
&= F_1 - \sum_i q_i p_i .
\end{aligned}
$$

Then

$$
\begin{aligned}
dF_2(p, P, t) &= dF_1 - \sum_i q_i dp_i + p_i dq_i \\
&= \sum_i \frac{\partial F_1}{\partial q_i} dq_i + \frac{\partial F_1}{\partial P_i} dP_i - q_i dp_i - p_i dq_i \\
&= \sum_i p_i dq_i + Q_i dP_i - q_i dp_i - p_i dq_i .
\end{aligned}
$$

and

$$
dF_2(p, P, t) = \sum_i Q_i dP_i - q_i dp_i + \frac{\partial F}{\partial t} dt .
$$

Therefore we have

$$
\frac{\partial F_2}{\partial p_i} = -q_i \text{ and } \frac{\partial F_2}{\partial P_i} = Q_i .
$$

From $F(q, Q, t)$ we may transform out q in favor of p with

$$
p_i = \frac{\partial F(q, Q)}{\partial q_i} .
$$

The transform is

$$F_3(q, P, t) = F(q, Q, t) - \sum_i q_i \frac{\partial F}{\partial q_i}$$
$$= F(q, Q, t) - \sum_i q_i p_i$$

Then

$$\mathrm{d}F_3 = \mathrm{d}F - \sum_i q_i dp_i + p_i dq_i$$
$$= \mathrm{d}F - \sum_i q_i \mathrm{d}p_i + p_i \mathrm{d}q_i$$
$$= \sum_i \frac{\partial F}{\partial q_i}\mathrm{d}q_i + \frac{\partial F}{\partial Q_i}\mathrm{d}Q_i - \sum_i q_i \mathrm{d}p_i - p_i \mathrm{d}q_i$$
$$= \sum_i p_i \mathrm{d}q_i - P_i \mathrm{d}Q_i - q_i \mathrm{d}p_i - p_i \mathrm{d}q_i$$

and

$$\mathrm{d}F_3(p, Q, t) = \sum_i -P_i \mathrm{d}Q_i - q_i \mathrm{d}p_i + \frac{\partial F}{\partial t}\mathrm{d}t$$

with

$$\frac{\partial F_3}{\partial p_i} = -q_i \text{ and } \frac{\partial F_2}{\partial Q_i} = -P_i.$$

5.2. Consider a conservative system and suppose that you have solved the partial differential equation

$$\mathcal{H}\left(q, \frac{\partial F}{\partial q}\right) = \mathcal{E}$$

for the function $F(q, a, \mathcal{E})$. Now suppose that you choose to form the link to the final configuration of the system through the final canonical momenta P. That is, you choose $P_1 = \mathcal{E}$. Follow the procedure we used in the chapter for $Q_1 = \mathcal{E}$ to discuss the steps toward the final solution for the generator.

Solution:

If we choose $P_1 = \mathcal{E}$ the end point Hamiltonian is cyclic in all momenta except P_1. Since $\partial P_1 / \partial P_i = 0$ for all $i \neq 1$, the canonical equations for the Q_i are

$$\dot{Q}_i = \frac{\partial \mathcal{H}(a, \mathcal{E}, P_1)}{\partial P_i} = 0 \text{ for } i > 1,$$

and, with $P_1 = \mathcal{E}$

$$\dot{Q}_1 = \frac{\partial \mathcal{H}(a, \mathcal{E}, P_1)}{\partial P_1} = 1.$$

The end point coordinates Q_{i} for $i > 1$ are then the $n-1$ constants. That is

$$Q_{\mathrm{i}} = \text{constant for } i > 1,$$

and for $i = 1$

$$Q_1 = t - \tau,$$

where τ is an integration constant.

There is then no dependence on the canonical variables Q_{i} for $i > 1$. Then

$$\dot{P}_{\mathrm{i}} = -\frac{\partial \mathcal{H}(a, \mathcal{E}, P_1)}{\partial Q_{\mathrm{i}}} = 0 \text{ for } i > 1,$$

and

$$P_{\mathrm{i}} = \text{constant for } i > 1$$

with $P_1 = \mathcal{E}$.

5.3. In the chapter we considered the simple harmonic oscillator as an example for which we could find a generator and a final solution. There we found equations relating the initial coordinates and momenta (q, p) to the final coordinates $(Q = \mathcal{E})$ and momenta P based on a generator constructed based on a simple choice of the final coordinate. These equations we found to be

$$q = \mp\sqrt{2\mathcal{E}/k}\,\sin\sqrt{\frac{k}{m}}P$$

and

$$p = \pm\sqrt{2m\mathcal{E}}\,\cos\sqrt{\frac{k}{m}}P.$$

The physics requires that the final momentum is related to the final coordinate by the canonical equations. Use these to obtain the final momentum as a function of time and, then, the initial coordinate and momentum as functions of the time.

Solution:

We begin with

$$q = \mp\sqrt{2\mathcal{E}/k}\,\sin\sqrt{\frac{k}{m}}P$$

and

$$p = \pm\sqrt{2m\mathcal{E}}\,\cos\sqrt{\frac{k}{m}}P.$$

Now from

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = -\frac{\partial \mathcal{H}}{\partial \mathcal{E}} = -1$$

we have

$$P = \tau - t.$$

Then, recalling that the natural frequency of the harmonic oscillator is $\omega_0 = \sqrt{k/m}$, we may also write our solution as

$$q = \mp\sqrt{2\mathcal{E}/k}\,\sin\left[\omega_0\left(\tau - t\right)\right]$$

$$p = \pm\sqrt{2m\mathcal{E}}\,\cos\left[\omega_0\left(\tau - t\right)\right].$$

5.4. From the action-angle solution we obtained for the harmonic oscillator in the text

$$J_{\mathrm{q}}\left(\mathcal{E}\right) = \frac{2\pi\mathcal{E}}{\omega_0}$$

and

$$\omega_{\mathrm{q}} = \frac{\omega_0}{2\pi}t + \delta_{\mathrm{q}},$$

or

$$2\pi\omega_{\mathrm{q}} = \omega_0\left(t - \tau\right),$$

obtain the standard (q, p) description of the motion of the harmonic oscillator.

Solution:

The orbit of the representative point in the phase space (q, p) is, from

$$\mathcal{H}\left(q, p\right) = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 q^2 = \mathcal{E}$$

the ellipse

$$\frac{1}{2m\mathcal{E}}p^2 + \frac{m\omega_0^2}{2\mathcal{E}}q^2 = 1.$$

Using

$$J_{\mathrm{q}}\left(\mathcal{E}\right) = \frac{2\pi\mathcal{E}}{\omega_0}$$

this equation becomes

$$\left(\frac{\pi}{m\omega_0 J_\mathrm{q}}\right)p^2 + \left(\frac{m\pi\omega_0}{J_\mathrm{q}}\right)q^2 = 1.$$

The angle $\phi = \omega_0(t-\tau)$, which is equal to $2\pi\omega_\mathrm{q}$, locates the representative point on the orbit in the phase space (q, p). That is $q \propto \cos\phi$ and $p \propto \sin\phi$. From

$$\left(\frac{\pi}{m\omega_0 J_\mathrm{q}}\right)p^2 + \left(\frac{m\pi\omega_0}{J_\mathrm{q}}\right)q^2 = 1$$

and the requirement that $\sin^2\phi + \cos^2\phi = 1$, we have

$$q = \sqrt{\frac{J_q}{m\omega_0\pi}}\cos\phi = \sqrt{\frac{J_q}{m\omega_0\pi}}\cos\omega_0(t-\tau).$$

and

$$p = \sqrt{\frac{m\omega_0 J_q}{\pi}}\sin\phi = \sqrt{\frac{m\omega_0 J_q}{\pi}}\sin\omega_0(t-\tau).$$

We may also use

$$\begin{aligned}
J_\mathrm{q}(\mathcal{E}) = 2m\omega_0\Bigg[&\frac{q}{2}\sqrt{2\mathcal{E}/m\omega_0^2 - q^2}\\
&+ \frac{\mathcal{E}}{m\omega_0^2}\sin^{-1}\left(\frac{q}{\sqrt{2\mathcal{E}/m\omega_0^2}}\right)\Bigg]_{-\sqrt{2\mathcal{E}/\mathrm{m}\omega_0^2}}^{\sqrt{2\mathcal{E}/\mathrm{m}\omega_0^2}}\\
= \frac{2\pi\mathcal{E}}{\omega_0}.&
\end{aligned}$$

to obtain the magnitudes of q and p in terms of $\mathcal{E}$ resulting in the elementary forms of the position and momentum of the harmonic oscillator.

5.5. Show that the Poisson Bracket is unchanged by canonical transformation. That is, show that

$$\begin{aligned}
(f, g)_\mathrm{q,p} &= \frac{\partial f}{\partial q_\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q_\mu}\\
&= (f, g)_\mathrm{Q,P} = \frac{\partial f}{\partial Q_\mu}\frac{\partial g}{\partial P_\mu} - \frac{\partial f}{\partial P_\mu}\frac{\partial g}{\partial Q_\mu}.
\end{aligned}$$

Choose the generator of the canonical transformation to be dependent on (q, Q), as in the text.

Solution:

For the generator we have chosen q and Q are independent coordinates. Then, for $f = f(Q, P)$

$$\frac{\partial f}{\partial q_\mu} = \frac{\partial f}{\partial Q_\nu}\frac{\partial Q_\nu}{\partial q_\mu} + \frac{\partial f}{\partial P_\nu}\frac{\partial P_\nu}{\partial q_\mu} = -\frac{\partial f}{\partial P_\nu}\frac{\partial}{\partial q_\mu}\left(\frac{\partial F}{\partial Q_\nu}\right)$$

$$\frac{\partial f}{\partial p_\mu} = \frac{\partial f}{\partial P_\sigma}\frac{\partial P_\sigma}{\partial p_\mu} + \frac{\partial f}{\partial Q_\sigma}\frac{\partial Q_\sigma}{\partial p_\mu} = -\frac{\partial f}{\partial P_\sigma}\frac{\partial}{\partial p_\mu}\left(\frac{\partial F}{\partial Q_\sigma}\right) + \frac{\partial f}{\partial Q_\sigma}\frac{\partial Q_\sigma}{\partial p_\mu}$$

and for $g = g(Q.P)$

$$\frac{\partial g}{\partial q_\mu} = \frac{\partial g}{\partial Q_\rho}\frac{\partial Q_\rho}{\partial q_\mu} + \frac{\partial g}{\partial P_\rho}\frac{\partial P_\rho}{\partial q_\mu} = -\frac{\partial g}{\partial P_\rho}\frac{\partial}{\partial q_\mu}\left(\frac{\partial F}{\partial Q_\rho}\right)$$

$$\frac{\partial g}{\partial p_\mu} = \frac{\partial g}{\partial P_\lambda}\frac{\partial P_\lambda}{\partial p_\mu} + \frac{\partial g}{\partial Q_\lambda}\frac{\partial Q_\lambda}{\partial p_\mu} = -\frac{\partial g}{\partial P_\lambda}\frac{\partial}{\partial p_\mu}\left(\frac{\partial F}{\partial Q_\lambda}\right) + \frac{\partial g}{\partial Q_\lambda}\frac{\partial Q_\lambda}{\partial p_\mu}$$

Then

$$\begin{aligned}(f, g)_{\mathrm{q,p}} &= \frac{\partial f}{\partial q_\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial f}{\partial p_\mu}\frac{\partial g}{\partial q_\mu} \\ &= \frac{\partial f}{\partial P_\nu}\frac{\partial g}{\partial Q_\lambda}\left(\frac{\partial P_\nu}{\partial q_\mu}\frac{\partial Q_\lambda}{\partial p_\mu}\right) - \frac{\partial f}{\partial Q_\nu}\frac{\partial g}{\partial P_\lambda}\left(\frac{\partial Q_\nu}{\partial p_\mu}\frac{\partial P_\lambda}{\partial q_\mu}\right)\end{aligned}$$

But, from the partial derivative relations obtained in the text

$$\frac{\partial P_\nu}{\partial q_\mu}\frac{\partial Q_\lambda}{\partial p_\mu} = -\frac{\partial p_\mu}{\partial Q_\lambda}\delta_{\lambda\nu}\frac{\partial Q_\lambda}{\partial p_\mu} = -\delta_{\lambda\nu} = \frac{\partial Q_\nu}{\partial p_\mu}\frac{\partial P_\lambda}{\partial q_\mu}$$

Then

$$\begin{aligned}(f, g)_{\mathrm{q,p}} &= \frac{\partial f}{\partial P_\nu}\frac{\partial g}{\partial Q_\lambda}\left(\frac{\partial P_\nu}{\partial q_\mu}\frac{\partial Q_\lambda}{\partial p_\mu}\right) - \frac{\partial f}{\partial Q_\nu}\frac{\partial g}{\partial P_\lambda}\left(\frac{\partial Q_\nu}{\partial p_\mu}\frac{\partial P_\lambda}{\partial q_\mu}\right) \\ &= \left(\frac{\partial f}{\partial Q_\nu}\frac{\partial g}{\partial P_\lambda} - \frac{\partial f}{\partial P_\nu}\frac{\partial g}{\partial Q_\lambda}\right)\delta_{\lambda\nu} \\ &= \frac{\partial f}{\partial Q_\nu}\frac{\partial g}{\partial P_\nu} - \frac{\partial f}{\partial P_\nu}\frac{\partial g}{\partial Q_\nu}\end{aligned}$$

5.6. Consider the situation we considered for the time dependent Hamiltonian in which we defined new coordinates

$$\begin{aligned}q_{\mathrm{n+1}} &= t \\ p_{\mathrm{n+1}} &= -\mathcal{H}(q, p, t)\,.\end{aligned}$$

Show that classically

$$(t, \mathcal{H}) = \mathbf{1}.$$

Note that it then follows that quantum mechanically

$$(\mathcal{H}t - t\mathcal{H}) = -i\hbar\mathbf{1}$$

and that, therefore, there is an indeterminacy relation

$$\Delta\mathcal{E}\Delta t \geq \frac{1}{2}\hbar.$$

Quantum mechanically energy levels in an atom are measured by the light emitted by the atom in transitions between states. The Planck-Einstein relation $\mathcal{E} = h\nu$, where ν is the frequency of the light quantum (photon), relates the characteristic spectrum of an element to the energies of the atom. We may consider that the indeterminacy of the time of transition from the state is the lifetime of the state. That is the atom may decay from a state at any time up to approximately the lifetime. What is then $\Delta\mathcal{E}$?

Solution:

For

$$\begin{aligned} q_{\mathrm{n+1}} &= t \\ p_{\mathrm{n+1}} &= -\mathcal{H}(q, p, t) \end{aligned}$$

and the new set of $2n + 2$ independent coordinates we have

$$\begin{aligned} (q_{\mathrm{n+1}}, p_{\mathrm{n+1}}) &= \frac{\partial q_{\mathrm{n+1}}}{\partial q_\alpha}\frac{\partial p_{\mathrm{n+1}}}{\partial p_\beta} - \frac{\partial p_{\mathrm{n+1}}}{\partial q_\alpha}\frac{\partial q_{\mathrm{n+1}}}{\partial p_\beta} \\ &= \delta_{\mathrm{n+1},\alpha}\delta_{\mathrm{n+1},\beta} \end{aligned}$$

Then the Dirac analog yields

$$\begin{aligned} (q_{\mathrm{n+1}}p_{\mathrm{n+1}} - p_{\mathrm{n+1}}q_{\mathrm{n+1}}) &= (-t\mathcal{H} + \mathcal{H}t) \\ &= i\hbar\mathbf{1} \end{aligned}$$

or

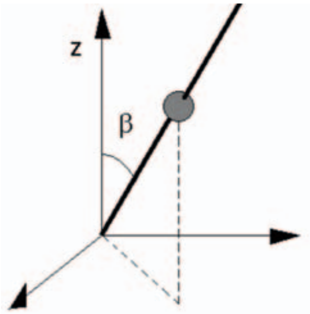
$$(\mathcal{H}t - t\mathcal{H}) = -i\hbar\mathbf{1}.$$

Since the magnitude of $\mathcal{H}$ is the energy $\mathcal{E}$, we have

$$\Delta\mathcal{E}\Delta t \geq \frac{1}{2}\hbar.$$

The quantity $\Delta\mathcal{E}$. is the spread in the energy of the state, or combined states, and from the Planck-Einstein relation this results in a spread in the frequency of the spectral line. This spread is affected by collisions with other atoms and is, therefore a measure of the density and temperature of the environment. Based on this the temperature of a star can be determined from the spread of the lines in particularly the hydrogen spectrum.

5.7. Consider a mass m moving without friction on a wire. The wire makes an angle β with the vertical and is free to rotate about the vertical axis, also without friction. The situation is shown here.



Bead on frictionless wire held at an angle β with the vertical. The wire rotates without friction about the vertical axis.

Consider that at the initial time the mass is located at a distance z_0 above the reference plane and has an angular momentum of $\ell = mr_0^2\dot{\vartheta}_0$ where r_0 and $\dot{\vartheta}_0$ are the initial radial distance from the axis and the initial angular velocity. There is no initial radial velocity.

Formulate and study the problem in the Hamilton-Jacobi formulation. Obtain the phase plot $p(r)$ vs r for the motion.

Solution:

The constraint is the relationship between r and z

$$\frac{r}{z} = \tan\beta.$$

We use cylindrical coordinates.. The position vector to the mass is

$$\boldsymbol{R} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

The velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{r}\hat{e}_{\mathrm{r}} + +r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right).$$

The only source of potential energy is the gravitational field. The potential energy of the mass is

$$V = mgz.$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + \dot{z}^2 \right) - mgz. \end{aligned}$$

The constraint is

$$z = \frac{r}{\tan\beta}$$

and the velocity relationship is

$$\dot{z} = \frac{\dot{r}}{\tan\beta}.$$

Then the Lagrangian is

$$\begin{aligned} L &= \frac{1}{2} m \left(\left(\frac{1 + \tan^2\beta}{\tan^2\beta} \right) \dot{r}^2 + r^2 \dot{\vartheta}^2 \right) - mg \frac{r}{\tan\beta} \\ &= \frac{1}{2} m \left(\frac{\dot{r}^2}{\sin^2\beta} + r^2 \dot{\vartheta}^2 \right) - mg \frac{r}{\tan\beta}. \end{aligned}$$

The canonical momenta are

$$\begin{aligned} p_\vartheta &= \frac{\partial L}{\partial \dot{\vartheta}} \\ &= m r^2 \dot{\vartheta} \end{aligned}$$

$$\begin{aligned} p_{\mathrm{r}} &= \frac{\partial L}{\partial \dot{r}} \\ &= \frac{m}{\sin^2\beta} \dot{r} \end{aligned}$$

And the Hamiltonian is then

$$\mathcal{H} = \frac{1}{2} m \left(\frac{\dot{r}^2}{\sin^2\beta} + r^2 \dot{\vartheta}^2 \right) + mg \frac{r}{\tan\beta}$$

Written in terms of momenta and coordinates, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left(\sin^2\beta \, p_{\mathrm{r}}^2 + \frac{1}{r^2} p_\vartheta^2 \right) + mg \frac{r}{\tan\beta}.$$

We use, of course, the extended method with the time as a coordinate in the generator. We identify the generator as either F_1 or F_2. And we assume a separation of the generator as

$$F_{1,2}(q, \alpha, t) = F_{\mathrm{t}}(\alpha, t) + F_{\mathrm{r}}(r, \alpha) + F_\vartheta(\vartheta, \alpha).$$

Then, with

$$p_{\mathrm{r}} = \frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha) \text{ and } p_{\vartheta} = \frac{\mathrm{d}}{\mathrm{d}\vartheta} F_{\vartheta}(\vartheta, \alpha),$$

we have the equation for the generator as

$$\frac{\mathrm{d}F_{\mathrm{t}}(\alpha, t)}{\mathrm{d}t} = -\frac{1}{2m}\left[\sin^2\beta\left(\frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha)\right)^2 + \frac{1}{r^2}\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} F_{\vartheta}(\vartheta, \alpha)\right)^2\right] - mg\frac{r}{\tan\beta}.$$

The first separation is

$$\frac{\mathrm{d}F_{\mathrm{t}}(\alpha, t)}{\mathrm{d}t} = -\alpha_1$$

and

$$\alpha_1 = \frac{1}{2m}\left[\sin^2\beta\left(\frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha)\right)^2 + \frac{1}{r^2}\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} F_{\vartheta}(\vartheta, \alpha)\right)^2\right] + mg\frac{r}{\tan\beta}.$$

The constant α_1 is, of course, the energy.

The second separation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta} F_{\vartheta}(\vartheta, \alpha)\right)^2 = \alpha_2^2 = 2m\alpha_1 r^2 - 2m^2 g\frac{r^3}{\tan\beta} - r^2\sin^2\beta\left(\frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha)\right)^2.$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta} F_{\vartheta}(\vartheta, \alpha) = \alpha_2,$$

which is the angular momentum. Now

$$\alpha_1 = \frac{1}{2m}\left[\sin^2\beta\left(\frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha)\right)^2 + \frac{1}{r^2}\alpha_2^2\right] + mg\frac{r}{\tan\beta}.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha) = \pm\frac{1}{\sin\beta}\sqrt{2m\alpha_1 - \frac{1}{r^2}\alpha_2^2 - 2m^2 g\frac{r}{\tan\beta}}.$$

We may now choose to obtain the generator. To do that requires that we integrate

$$F_{\mathrm{r}}(r, \alpha) = \pm\frac{1}{\sin\beta}\int \mathrm{d}r\sqrt{2m\alpha_1 - \frac{1}{r^2}\alpha_2^2 - 2m^2 g\frac{r}{\tan\beta}},$$

neglecting the additive constant, which is of no importance. This integral is, however, unappetizing. And we may obtain all the interesting information without carrying out the integration.

The constants we identify as

$$\alpha_1 = \mathcal{E} = \text{energy}$$

and

$$\alpha_2 = \ell = \text{angular momentum } p_\vartheta .$$

We may obtain both of these from the initial conditions on the motion. With no initial radial velocity

$$\alpha_1 = \frac{\ell^2}{2mr_0^2} + mg\frac{r_0}{\tan\beta}$$

and

$$\begin{aligned}\alpha_2 &= mr_0^2\dot{\vartheta}_0 \\ &= m\left(z_0\tan\beta\right)^2\dot{\vartheta}_0.\end{aligned}$$

Then

$$F_\mathrm{t} = -\mathcal{E}t$$

$$F_\vartheta = \ell\vartheta$$

and

$$F_\mathrm{r} = \pm\frac{1}{\sin\beta}\int \mathrm{d}r\sqrt{2m\mathcal{E} - \frac{1}{r^2}\ell^2 - 2m^2g\frac{r}{\tan\beta}}.$$

If we choose to identify the generator as F_1 then α_1 and α_2 are the final constant coordinates Q_1 and Q_2. The final constant canonical momenta are found from

$$\frac{\partial F_1}{\partial Q_\mathrm{i}} = -P_\mathrm{i}.$$

With

$$F_1 = -\mathcal{E}t + \ell\vartheta \pm \frac{1}{\sin\beta}\int \mathrm{d}r\sqrt{2m\mathcal{E} - \frac{1}{r^2}\ell^2 - 2m^2g\frac{r}{\tan\beta}}$$

we have

$$P_1 = \frac{\partial F_1}{\partial \mathcal{E}} = -t \pm \frac{m}{\sin\beta} \int \mathrm{d}r \frac{1}{\sqrt{2m\mathcal{E} - \frac{1}{r^2}\ell^2 - 2m^2 g \frac{r}{\tan\beta}}} = \beta_1$$

and

$$P_2 = \frac{\partial F_1}{\partial \ell} = \vartheta \mp \frac{\ell}{\sin\beta} \int \frac{\mathrm{d}r}{r^2} \frac{1}{\sqrt{2m\mathcal{E} - \frac{1}{r^2}\ell^2 - 2m^2 g \frac{r}{\tan\beta}}} = \beta_2.$$

The first of these is the radial position as a function of the time. The second is the radial position as a function of the angle ϑ. This detailed information about the motion of the mass requires that we either perform the first radial integration, which we found unappetizing, or the second two, equally unappetizing radial integrals. The trajectory $r = r(\vartheta)$ may be interesting, but the plot would finally be three dimensional after we re-incorporate z.

We have more insight into the system from the phase plot of p_{r} vs r. The canonical momentum $p_{\mathrm{r}} = \mathrm{d}F_{\mathrm{r}}(r, \alpha)/\mathrm{d}r$ is

$$p_{\mathrm{r}} = \pm\frac{1}{\sin\beta}\sqrt{2m\mathcal{E} - \frac{1}{r^2}\ell^2 - 2m^2 g \frac{r}{\tan\beta}}$$

with

$$\mathcal{E} = \frac{\ell^2}{2mr_0^2} + mg\frac{r_0}{\tan\beta}$$

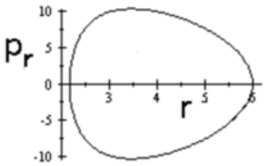
and

$$\ell = m(z_0 \tan\beta)^2 \dot{\vartheta}_0.$$

As an example we choose

$$\begin{aligned} m &= 1\,\mathrm{kg} \\ \beta &= \frac{\pi}{5}\,\mathrm{rad} \\ \dot{\vartheta}_0 &= 5\,\mathrm{rad\,s^{-1}} \\ g &= 9.8\,\mathrm{m\,s^{-2}} \\ z_0 &= 3\,\mathrm{m}. \end{aligned}$$

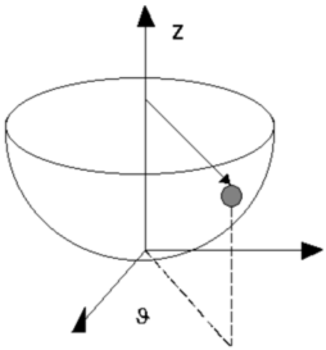
Then the phase plot is



Phase plot for radial momentum vs radial position.

We see that the mass oscillates between a minimum and a maximum distance from the vertical $z-$axis, which we designate here as the radial distance. The radial momentum vanishes at each extremum and changes sign as it must in oscillation.

5.8. Consider a marble in a fishbowl. We shall assume that the marble slides without friction so we can neglect rotation. We have illustrated the situation in the figure below. We shall use cylindrical coordinates because then it is easiest to specify the surface of the fishbowl.



Marble in a fishbowl.

Pursue the problem employing the Hamilton-Jacobi approach. Find the phase plot(s).

Solution:

The position vector to the marble is

$$\boldsymbol{R} = r\hat{e}_{\mathrm{r}} + z\hat{e}_{\mathrm{z}}.$$

The velocity is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R} = \dot{r}\hat{e}_{\mathrm{r}} + +r\dot{\vartheta}\hat{e}_{\vartheta} + \dot{z}\hat{e}_{\mathrm{z}}.$$

The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right).$$

The only source of potential energy is the gravitational field. The potential energy of the mass is

$$V = mgz.$$

The Lagrangian is then

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) - mgz. \end{aligned}$$

We will introduce the constraint directly at this point to eliminate the $z-$ dependence. That is, we use directly

$$z = \eta r^{\mathrm{n}}$$

and

$$\dot{z} = n\eta r^{\mathrm{n}\text{-}1}\dot{r}.$$

(We will later re-introduce z to obtain a second phase plot.) Then the Lagrangian is

$$L = \frac{1}{2}m\left[\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)\dot{r}^2 + r^2\dot{\vartheta}^2\right] - mg\eta r^{\mathrm{n}}.$$

There are two variables: r, ϑ. So we have two canonical momenta

$$\begin{aligned} p_\vartheta &= \frac{\partial L}{\partial \dot{\vartheta}} \\ &= mr^2\dot{\vartheta} \end{aligned}$$

and

$$\begin{aligned} p_{\mathrm{r}} &= \frac{\partial L}{\partial \dot{r}} \\ &= m\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)\dot{r}. \end{aligned}$$

Because the Lagrangian is cyclic in the angle, the angular momentum is constant. But we shall let this fact emerge later from our work.

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}m\left[\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)\dot{r}^2 + r^2\dot{\vartheta}^2\right] + mg\eta r^{\mathrm{n}}.$$

The velocities in terms of momenta are

$$\dot{r} = \frac{p_{\mathrm{r}}}{m\left(n^2\alpha^2 r^{2\mathrm{n}\text{-}2} + 1\right)}$$

and

$$\dot{\vartheta} = \frac{p_\vartheta}{mr^2}.$$

In natural coordinates, then, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left[\frac{1}{\left(n^2\eta^2 r^{2\text{n-}2}+1\right)}\left(p_\text{r}\right)^2 + \frac{1}{r^2}\left(p_\vartheta\right)^2\right] + mg\eta r^\text{n}.$$

We use, of course, the extended method with the time as a coordinate in the generator. We identify the generator as either F_1 or F_2. And we assume a separation of the generator as

$$F_{1,2}\left(q,\alpha,t\right) = F_\text{t}\left(\alpha,t\right) + F_\text{r}\left(r,\alpha\right) + F_\vartheta\left(\vartheta,\alpha\right).$$

Then, with

$$p_\text{r} = \frac{\text{d}}{\text{d}r}F_\text{r}\left(r,\alpha\right) \text{ and } p_\vartheta = \frac{\text{d}}{\text{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right),$$

we have the equation for the generator as

$$\frac{\text{d}F_\text{t}\left(\alpha,t\right)}{\text{d}t} = -\frac{1}{2m}\left[\frac{1}{\left(n^2\eta^2 r^{2\text{n-}2}+1\right)}\left(\frac{\text{d}}{\text{d}r}F_\text{r}\left(r,\alpha\right)\right)^2 + \frac{1}{r^2}\left(\frac{\text{d}}{\text{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right)\right)^2\right] - mg\eta r^\text{n}.$$

The first separation is

$$\frac{\text{d}F_\text{t}\left(\alpha,t\right)}{\text{d}t} = -\alpha_1$$

and

$$\alpha_1 = \frac{1}{2m}\left[\frac{1}{\left(n^2\eta^2 r^{2\text{n-}2}+1\right)}\left(\frac{\text{d}}{\text{d}r}F_\text{r}\left(r,\alpha\right)\right)^2 + \frac{1}{r^2}\left(\frac{\text{d}}{\text{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right)\right)^2\right] + mg\eta r^\text{n}.$$

The constant α_1 is, of course, the energy.

The second separation is

$$\left(\frac{\text{d}}{\text{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right)\right)^2 = \alpha_2^2 = 2m\alpha_1 r^2 - 2m^2 g\eta r^{\text{n}+2} - \frac{r^2}{\left(n^2\eta^2 r^{2\text{n-}2}+1\right)}\left(\frac{\text{d}}{\text{d}r}F_\text{r}\left(r,\alpha\right)\right)^2.$$

That is

$$\frac{\text{d}}{\text{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right) = \alpha_2,$$

which is the angular momentum. The radial momentum is

$$p_{\mathrm{r}} = \frac{\mathrm{d}}{\mathrm{d}r} F_{\mathrm{r}}(r, \alpha) = \pm \sqrt{\left(2m\alpha_1 - 2m^2 g\eta r^{\mathrm{n}} - \frac{1}{r^2}\alpha_2^2\right)\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)}.$$

We may now choose to obtain the generator. To do that requires that we integrate

$$F_{\mathrm{r}}(r, \alpha) = \pm \int \mathrm{d}r \sqrt{\left(2m\alpha_1 - 2m^2 g\eta r^{\mathrm{n}} - \frac{1}{r^2}\alpha_2^2\right)\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)},$$

neglecting the additive constant, which is of no importance. This integral is, however, unappetizing. And we may obtain all the interesting information without carrying out the integration.

The constants we identify as

$$\alpha_1 = \mathcal{E} = \text{energy}$$

and

$$\alpha_2 = \ell = \text{angular momentum } p_{\vartheta}.$$

We may obtain both of these from the initial conditions on the motion. With no initial radial velocity

$$\ell = m r_0^2 \dot{\vartheta}_0.$$

and

$$\mathcal{E} = \frac{\ell^2}{2m r_0^2} + m g \eta r_0^{\mathrm{n}}.$$

Then

$$F_{\mathrm{t}} = -\mathcal{E} t$$

$$F_{\vartheta} = \ell \vartheta$$

and

$$F_{\mathrm{r}}(r, \alpha) = \pm \int \mathrm{d}r \sqrt{\left(2m\mathcal{E} - 2m^2 g\eta r^{\mathrm{n}} - \frac{1}{r^2}\ell^2\right)\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)},$$

If we choose to identify the generator as F_1 then α_1 and α_2 are the final constant coordinates Q_1 and Q_2. The final constant canonical momenta are found from

$$\frac{\partial F_1}{\partial Q_\mathrm{i}} = -P_\mathrm{i}.$$

With

$$F_1 = -\mathcal{E}t + \ell\vartheta \pm \int \mathrm{d}r \sqrt{\left(2m\mathcal{E} - 2m^2 g\eta r^\mathrm{n} - \frac{1}{r^2}\ell^2\right)\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)}$$

we have

$$P_1 = \frac{\partial F_1}{\partial \mathcal{E}} = -t \pm m \int \mathrm{d}r \sqrt{\frac{\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)}{\left(2m\mathcal{E} - 2m^2 g\eta r^\mathrm{n} - \frac{1}{r^2}\ell^2\right)}} = \beta_1$$

and

$$P_2 = \frac{\partial F_1}{\partial \ell} = \vartheta \mp \ell \int \frac{\mathrm{d}r}{r^2} \sqrt{\frac{\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)}{\left(2m\mathcal{E} - 2m^2 g\eta r^\mathrm{n} - \frac{1}{r^2}\ell^2\right)}} = \beta_2.$$

The first of these is the radial position as a function of the time. The second is the radial position as a function of the angle ϑ. This detailed information about the motion of the marble requires that we either perform the first radial integration, which we found unappetizing, or the second two, equally unappetizing radial integrals.

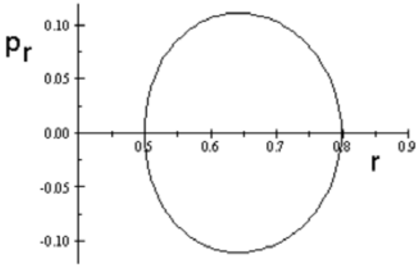
We have more insight into the system from the phase plot of p_r vs r. The canonical momentum $p_\mathrm{r} = dF_\mathrm{r}(r, \alpha)/\mathrm{d}r$ is

$$p_\mathrm{r} = \pm\sqrt{\left(2m\mathcal{E} - 2m^2 g\eta r^\mathrm{n} - \frac{1}{r^2}\ell^2\right)\left(n^2\eta^2 r^{2\mathrm{n}\text{-}2} + 1\right)}$$

If we choose values

$$r_0 = 0.5\,\mathrm{m}$$
$$m = 0.1\,\mathrm{kg}$$
$$\dot{\vartheta}_0 = 5\,\mathrm{rad\,s^{-1}}$$

and a parabolic fishbowl with $n = 2$ and $\eta = 0.5$, then the phase plot of p_r vs r is in the figure here.

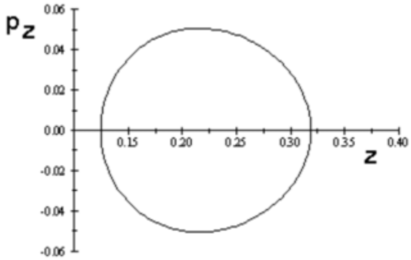


Phase plot for marble in fishbowl; radial momentum vs radial position.

We may also obtain the phase plot of the $z-$ component of the canonical momentum against the coordinate z. In doing this we must be careful with the algebra, noting the definition of p_z in terms of $\dot{r}$. The result is

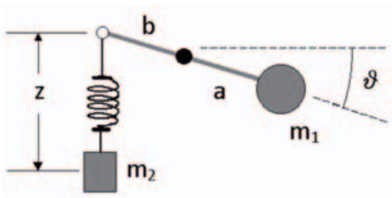
$$p_z = \pm\left(n\eta\left(\frac{z}{\eta}\right)^{1\text{-}1/\text{n}}\right)\sqrt{\frac{\left(2m\left(\mathcal{E}\right) - 2m^2 g\eta\left(\frac{z}{\eta}\right) - \left(\frac{z}{\eta}\right)^{\text{-}2/\text{n}}(\ell)^2\right)}{\left(n^2\eta^2\left(\frac{z}{\eta}\right)^{2\text{-}2/\text{n}} + 1\right)}}$$

and the phase plot is here.



Phase plot for marble in fishbowl; vertical momentum vs vertical position.

5.9. Consider the two masses connected as shown here.



Rotating and suspended masses.

Obtain the Lagrangian for unequal masses and then simplify for equal masses and $a = b$.

Show that the motion cannot be easily studied using the Hamilton-Jacobi approach. So you should turn to the canonical equations.

Linearize the canonical equations for small vibrations and find the (eigen) frequencies of vibration.

[Answers:

$$\omega = \frac{1}{a\sqrt{m}}\sqrt{K_1 + \sqrt{K_2^2}}, \frac{1}{a\sqrt{m}}\sqrt{K_1 - \sqrt{K_2^2}}$$

with

$$K_1 = k' + a^2 k$$

and

$$K_2^2 = a^4 k^2 + k'^2 - a^2 k k' \Big].$$

Solution:

We choose the origin of coordinates at the fulcrum. The vector locating the mass m_1 is then

$$\boldsymbol{R}_1 = (a \cos\vartheta)\, \hat{e}_{\mathrm{x}} + (-a \sin\vartheta)\, \hat{e}_{\mathrm{y}}.$$

The vector locating mass m_2 is

$$\boldsymbol{R}_2 = (-b \cos\vartheta)\, \widehat{e_x} + (b \sin\vartheta - z)\, \widehat{e_y}.$$

The velocities are

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}_1 = \left(-a\dot{\vartheta} \sin\vartheta\right) \widehat{e_x} + \left(-a\dot{\vartheta} \cos\vartheta\right) \widehat{e_y}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{R}_2 = \left(b\dot{\vartheta} \sin\vartheta\right) \widehat{e_x} + \left(b\dot{\vartheta} \cos\vartheta - \dot{z}\right) \widehat{e_y}.$$

The kinetic energy is then

$$T = \frac{1}{2} m_1 \left(a^2 \dot{\vartheta}^2\right) + \frac{1}{2} m_2 \left(b^2 \dot{\vartheta}^2 - 2b\dot{\vartheta}\dot{z} \cos\vartheta + \dot{z}^2\right).$$

The potential energy is

$$V = -m_1 g a \sin\vartheta + m_2 g\, (b \sin\vartheta - z) + \frac{1}{2} k \left(z^2 - \ell^2\right) + \frac{1}{2} k' \vartheta^2,$$

where ℓ is the unextended length of the spring.

The Lagrangian is then

$$L = \frac{1}{2}m_1\left(a^2\dot{\vartheta}^2\right) + \frac{1}{2}m_2\left(b^2\dot{\vartheta}^2 - 2b\dot{\vartheta}\dot{z}\cos\vartheta + \dot{z}^2\right)$$
$$+m_1ga\sin\vartheta - m_2g\left(b\sin\vartheta - z\right) - \frac{1}{2}k\left(z^2 - \ell^2\right) - \frac{1}{2}k'\vartheta^2.$$

Choose

$$m_1 = m_2$$
$$a = b$$

$$L = m\left(a^2\dot{\vartheta}^2\right) + \frac{1}{2}m\left(\dot{z}^2 - 2a\dot{\vartheta}\dot{z}\cos\vartheta\right)$$
$$+mgz - \frac{1}{2}k\left(z^2 - \ell^2\right) - \frac{1}{2}k'\vartheta^2.$$

The canonical momenta are

$$p_\vartheta = 2ma^2\dot{\vartheta} - ma\dot{z}\cos\vartheta$$

and

$$p_{\mathrm{z}} = m\dot{z} - ma\dot{\vartheta}\cos\vartheta.$$

This is a set of two equations for $\dot{\vartheta}$ and $\dot{z}$ in terms of p_ϑ and p_z. These must be solved, because we shall need the solution to formulate the Hamiltonian. The equations in matrix form are

$$\begin{bmatrix} 2ma^2 & -ma\cos\vartheta \\ -ma\cos\vartheta & m \end{bmatrix}\begin{bmatrix} \dot{\vartheta} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} p_\vartheta \\ p_{\mathrm{z}} \end{bmatrix}$$

Inverting the principal matrix, the solution to this set is

$$\begin{bmatrix} \dot{\vartheta} \\ \dot{z} \end{bmatrix} = \frac{1}{am\left(2 - \cos^2\vartheta\right)}\begin{bmatrix} 1/a & \cos\vartheta \\ \cos\vartheta & 2a \end{bmatrix}\begin{bmatrix} p_\vartheta \\ p_{\mathrm{z}} \end{bmatrix}$$
$$= \frac{1}{D}\begin{bmatrix} p_z\cos\vartheta + \frac{1}{a}p_\vartheta \\ 2ap_z + p_\vartheta\cos\vartheta \end{bmatrix},$$

where

$$D = am\left(2 - \cos^2\vartheta\right).$$

The Hamiltonian is

$$\mathcal{H} = ma^2\dot{\vartheta}^2 + \frac{1}{2}m\dot{z}^2 - ma\dot{\vartheta}\dot{z}\cos\vartheta$$
$$-mgz + \frac{1}{2}k\left(z^2 - \ell^2\right) + \frac{1}{2}k'\vartheta^2,$$

In terms of the momenta,

$$\begin{aligned}\mathcal{H} = {} & ma^2 \frac{1}{D^2}\left(\frac{1}{a}p_\vartheta + p_\mathrm{z}\cos\vartheta\right)^2 + \frac{1}{2}m\frac{1}{D^2}\left(2ap_\mathrm{z} + p_\vartheta\cos\vartheta\right)^2 \\ & -ma\frac{1}{D^2}\left(\frac{1}{a}p_\vartheta + p_\mathrm{z}\cos\vartheta\right)\left(2ap_\mathrm{z} + p_\vartheta\cos\vartheta\right)\cos\vartheta \\ & -mgz + \frac{1}{2}k\left(z^2 - \ell^2\right) + \frac{1}{2}k'\vartheta^2\end{aligned}$$

That is

$$\begin{aligned}\mathcal{H} = {} & \frac{1}{\left(2-\cos^2\vartheta\right)}\left(\frac{1}{2ma^2}p_\vartheta^2 + \frac{1}{m}p_\mathrm{z}^2 + \frac{1}{ma}p_\vartheta p_\mathrm{z}\cos\vartheta\right) \\ & -mgz + \frac{1}{2}k\left(z^2 - \ell^2\right) + \frac{1}{2}k\vartheta^2.\end{aligned}$$

We now attempt a solution using the Hamilton-Jacobi approach using, of course, the extended method with the time as a coordinate in the generator. We identify the generator as either F_1 or F_2 and assume a separation of the generator as

$$F_{1,2}\left(q,\alpha,t\right) = F_\mathrm{t}\left(\alpha,t\right) + F_\mathrm{z}\left(z,\alpha\right) + F_\vartheta\left(\vartheta,\alpha\right).$$

Then, with

$$p_\mathrm{z} = \frac{\mathrm{d}}{\mathrm{d}z}F_\mathrm{z}\left(z,\alpha\right) \text{ and } p_\vartheta = \frac{\mathrm{d}}{\mathrm{d}\vartheta}F_\vartheta\left(\vartheta,\alpha\right),$$

we have the equation for the generator as

$$\begin{aligned}\frac{\mathrm{d}F_\mathrm{t}\left(\alpha,t\right)}{\mathrm{d}t} = {} & -\frac{1}{\left(2-\cos^2\vartheta\right)}\left[\frac{1}{2ma^2}\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_\vartheta\right)^2 + \frac{1}{m}\left(\frac{\mathrm{d}}{\mathrm{d}z}F_\mathrm{z}\right)^2\right. \\ & \left. +\frac{1}{ma}\left(\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_\vartheta\right)\left(\frac{\mathrm{d}}{\mathrm{d}z}F_\mathrm{z}\right)\cos\vartheta\right] + mgz - \frac{1}{2}k\left(z^2-\ell^2\right) - \frac{1}{2}k'\vartheta^2.\end{aligned}$$

The time function can be separated. But beyond that we cannot separate the equation. That is the Hamiltonian is not separable. We are then stuck with a partial differential equation for the generator $F = F_{1,2}\left(z,\vartheta,t,\alpha\right)$.

$$\begin{aligned}\frac{\partial F_\mathrm{t}\left(\alpha,t\right)}{\partial t} = {} & -\frac{1}{\left(2-\cos^2\vartheta\right)}\left[\frac{1}{2ma^2}\left(\frac{\partial}{\partial\vartheta}F\right)^2 + \frac{1}{m}\left(\frac{\partial}{\partial z}F\right)^2\right. \\ & \left. +\frac{1}{ma}\left(\frac{\partial}{\partial\vartheta}F\right)\left(\frac{\partial}{\partial z}F\right)\cos\vartheta\right] + mgz - \frac{1}{2}k\left(z^2-\ell^2\right) - \frac{1}{2}k'\vartheta^2.\end{aligned}$$

But we may return to the approach using the canonical equations.

$$\frac{\partial}{\partial\vartheta}\mathcal{H} = -\frac{2\sin\vartheta\cos\vartheta}{\left(2-\cos^2\vartheta\right)^2}\left(\frac{1}{2ma^2}p_\vartheta^2 + \frac{1}{m}p_\mathrm{z}^2\right) - \frac{2+\cos^2\vartheta}{\left(2-\cos^2\vartheta\right)^2}\sin\vartheta\left(\frac{1}{ma}p_\vartheta p_\mathrm{z}\right) + k'\vartheta$$

$$\frac{\partial \mathcal{H}}{\partial z} = -mg + kz$$

$$\frac{\partial \mathcal{H}}{\partial p_\vartheta} = \frac{1}{\left(2 - \cos^2\vartheta\right)} \frac{p_\vartheta}{ma^2} + \frac{\cos\vartheta}{\left(2 - \cos^2\vartheta\right)} \frac{p_z}{ma}$$

$$\frac{\partial \mathcal{H}}{\partial p_z} = \frac{2}{\left(2 - \cos^2\vartheta\right)} \frac{p_z}{m} + \frac{\cos\vartheta}{\left(2 - \cos^2\vartheta\right)} \frac{p_\vartheta}{ma}$$

Then

$$\dot{\vartheta} = \frac{1}{\left(2 - \cos^2\vartheta\right)} \frac{p_\vartheta}{ma^2} + \frac{\cos\vartheta}{\left(2 - \cos^2\vartheta\right)} \frac{p_z}{ma},$$

$$\dot{z} = \frac{2}{\left(2 - \cos^2\vartheta\right)} \frac{p_z}{m} + \frac{\cos\vartheta}{\left(2 - \cos^2\vartheta\right)} \frac{p_\vartheta}{ma},$$

$$\dot{p}_\vartheta = \frac{2\sin\vartheta\cos\vartheta}{\left(2 - \cos^2\vartheta\right)^2} \left(\frac{1}{2ma^2} p_\vartheta^2 + \frac{1}{m} p_z^2\right) + \frac{2 + \cos^2\vartheta}{\left(2 - \cos^2\vartheta\right)^2} \sin\vartheta \left(\frac{1}{ma} p_\vartheta\, p_z\right) - k'\vartheta,$$

and

$$\dot{p}_z = mg - kz.$$

These canonical equations are, of course, nonlinear. But we have already limited both ϑ and z to small values. So we may linearize these and consider them to be still general.

$$\dot{\vartheta} = \frac{p_\vartheta}{ma^2} + \frac{p_z}{ma},$$

$$\dot{z} = 2\frac{p_z}{m} + \frac{p_\vartheta}{ma},$$

$$\begin{aligned}
\dot{p}_\vartheta &= \frac{2\sin\vartheta\cos\vartheta}{\left(2 - \cos^2\vartheta\right)^2} \left(\frac{1}{2ma^2} p_\vartheta^2 + \frac{1}{m} p_z^2\right) + \frac{2 + \cos^2\vartheta}{\left(2 - \cos^2\vartheta\right)^2} \sin\vartheta \left(\frac{1}{ma} p_\vartheta\, p_z\right) - k'\vartheta \\
&= 2\vartheta \left(\frac{1}{2ma^2} p_\vartheta^2 + \frac{1}{m} p_z^2\right) + 3\vartheta \left(\frac{1}{ma} p_\vartheta\, p_z\right) - k'\vartheta \\
&= -k'\vartheta,
\end{aligned}$$

and

$$\dot{p}_z = mg - kz.$$

This would be a linear homogeneous set of equations were it not for the term mg. We can easily handle a linear homogeneous set of equations. If we redefine the coordinate z as z' such that

$$mg - kz = -kz'$$

or

$$z' = \frac{kz - mg}{k}$$

Then our set of linearized equations is

$$\dot{\vartheta} = \frac{p_\vartheta}{ma^2} + \frac{p_\mathrm{z}}{ma},$$

$$\dot{z} = 2\frac{p_\mathrm{z}}{m} + \frac{p_\vartheta}{ma},$$

$$\dot{p}_\vartheta = -k'\vartheta,$$

and

$$\dot{p}_\mathrm{z} = -kz'.$$

To carry this any farther, we shall seek a solution in the complex plane. We make the Ansatz

$$\begin{aligned} p_\mu &= P_\mu \exp(i\omega t) \\ q_\mu &= Q_\mu \exp(i\omega t) \end{aligned}$$

Then our linearized equations are

$$i\omega\Theta = 2\frac{P_\vartheta}{ma^2} + \frac{P_\mathrm{z}}{ma},$$

$$i\omega Z' = \frac{P_\vartheta}{ma} + 2\frac{P_\mathrm{z}}{m},$$

$$i\omega P_\vartheta = -k'\Theta,$$

$$i\omega P_\mathrm{z} = -kZ',$$

or

$$\begin{bmatrix} 0 & 0 & 2/\left(ma^2\right) & 1/(ma) \\ 0 & 0 & 1/(ma) & 2/m \\ -k' & 0 & 0 & 0 \\ 0 & -k & 0 & 0 \end{bmatrix} \begin{bmatrix} \Theta \\ Z' \\ P_\vartheta \\ P_z \end{bmatrix} = i\omega \begin{bmatrix} \Theta \\ Z' \\ P_\vartheta \\ P_z \end{bmatrix}.$$

If we define

$$K_1 = k' + a^2 k$$

and

$$K_2^2 = a^4 k^2 + k'^2 - a^2 k k',$$

the (eigen) values of ω are

$$\omega = \pm \frac{1}{a\sqrt{m}} \sqrt{K_1 + \sqrt{K_2^2}}, \pm \frac{1}{a\sqrt{m}} \sqrt{K_1 - \sqrt{K_2^2}}.$$

Now

$$\frac{K_1^2}{K_2^2} = \frac{\left(k' + a^2 k\right)^2}{a^4 k^2 + k'^2 - a^2 k k'} = \frac{a^4 k^2 + k'^2 + 2a^2 k k'}{a^4 k^2 + k'^2 - a^2 k k'} > 0.$$

So either of the angular (eigen) frequencies

$$\omega = \frac{1}{a\sqrt{m}} \sqrt{K_1 + \sqrt{K_2^2}}, \frac{1}{a\sqrt{m}} \sqrt{K_1 - \sqrt{K_2^2}}$$

is possible. There is then a high and a low frequency mode, as we expected.

5.10. Consider a charged point particle of mass m and charge Q moving in a magnetic field of induction $\boldsymbol{B} = \hat{e}_z B$. Assume a motion in the direction of $\hat{e}_z$ as well as in the (x, y) −plane. In the text we have shown that the Hamiltonian for the charged particle in the electromagnetic field is generally

$$\mathcal{H} = (1/2m) \left(p_\mu - Q A_\mu\right)^2 + Q\varphi,$$

summing on Greek indices. In our situation there is no electric field. Therefore $\varphi = 0$.

Show that the vector potential $\boldsymbol{A} = -\hat{e}_x y B$ produces the required magnetic field induction. Obtain the trajectory of the charged particle using the Hamilton- Jacobi approach.

Solution:

The Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \left(p_\mu - Q A_\mu\right)^2.$$

We consider now a vector potential

$$\boldsymbol{A} = -\hat{e}_{\mathrm{x}} y B$$

This vector potential is obviously divergenceless. The magnetic field corresponding to this potential is

$$\begin{aligned}\overrightarrow{B} &= \operatorname{curl} \boldsymbol{A} = -\hat{e}_{\mathrm{z}} \left(\frac{\partial}{\partial y} A_{\mathrm{x}} \right) \\ &= B \hat{e}_{\mathrm{z}}\end{aligned}$$

Therefore, the Hamiltonian for a charged particle moving in a constant magnetic field is

$$\mathcal{H} = \frac{1}{2m} \left[(p_{\mathrm{x}} + QBy)^2 + \left(p_{\mathrm{y}}\right)^2 + (p_{\mathrm{z}})^2 \right]$$

We identify the generator as either F_1 or F_2 and assume a separation of the generator as

$$F_{1,2}(q, \alpha, t) = F_{\mathrm{t}}(\alpha, t) + F_{\mathrm{x}}(x, \alpha) + F_{\mathrm{y}}(y, \alpha) + F_{\mathrm{z}}(z, \alpha) \,.$$

Then, with

$$p_{\mathrm{x}} = \frac{\mathrm{d}}{\mathrm{d}x} F_{\mathrm{x}}(x, \alpha) \,, \;\; p_{\mathrm{y}} = \frac{\mathrm{d}}{\mathrm{d}y} F_{\mathrm{y}}(y, \alpha) \;\text{ and }\; p_{\mathrm{z}} = \frac{\mathrm{d}}{\mathrm{d}z} F_{\mathrm{z}}(z, \alpha) \,,$$

the equation for the generator is

$$\frac{\mathrm{d}F_{\mathrm{t}}}{\mathrm{d}t} = -\frac{1}{2m} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} F_{\mathrm{x}} + QBy \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}y} F_{\mathrm{y}} \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}z} F_{\mathrm{z}} \right)^2 \right].$$

The first separation constant α_1 is the energy

$$\alpha_1 = \frac{1}{2m} \left[\left(\frac{\mathrm{d}}{\mathrm{d}x} F_{\mathrm{x}} + QBy \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}y} F_{\mathrm{y}} \right)^2 + \left(\frac{\mathrm{d}}{\mathrm{d}z} F_{\mathrm{z}} \right)^2 \right]$$

and

$$F_{\mathrm{t}} = -\alpha_1 t$$

dropping the additive constant. For the second separation we have

$$2m\alpha_1 - \left(\frac{\mathrm{d}}{\mathrm{d}x} F_{\mathrm{x}} + QBy \right)^2 - \left(\frac{\mathrm{d}}{\mathrm{d}y} F_{\mathrm{y}} \right)^2 = \left(\frac{\mathrm{d}}{\mathrm{d}z} F_{\mathrm{z}} \right)^2 = \alpha_2^2$$

and

$$F_z = \alpha_2 z$$

dropping the additive constant.

Then

$$\pm\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{d}{dy}F_y\right)^2} - m\Omega y = \frac{d}{dx}F_x = \alpha_3$$

where we introduced the cyclotron frequency $\Omega = QB/m$. We have then completely separated the Hamiltonian.

The equation for F_y is

$$\frac{d}{dy}F_y = \pm\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}.$$

Then

$$F_y = \pm\int dy\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}.$$

Before integration of dF_y/dy the generator is

$$F_1(x, y, z, t, \alpha_1, \alpha_2, \alpha_3) = -\alpha_1 t + \alpha_3 x \pm \int dy\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2} + \alpha_2 z$$

The separation constants $\alpha_1, ... \alpha_3$ are the energy $\mathcal{E}$ the final constant coordinates Q_2, and Q_3. The final constant momenta are designated as β_1, ... β_3. These are found from $\beta_j = -\partial F_1/\partial\alpha_j$. Then

$$\beta_1 = -\frac{\partial F_1}{\partial\alpha_1} = t \mp m\int dy\frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}},$$

$$\beta_2 = -\frac{\partial F_1}{\partial\alpha_2} = -z \pm \alpha_2\int dy\frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}},$$

and

$$\beta_3 = -\frac{\partial F_1}{\partial\alpha_3} = -x \pm \int dy\frac{(\alpha_3 + m\Omega y)}{\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}}.$$

We then have two integrals to perform. We substitute

$$\xi = \alpha_3 + m\Omega y \Longrightarrow \mathrm{d}y = \mathrm{d}\xi / m\Omega.$$

$$\begin{aligned}
&\int \mathrm{d}y \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}} \\
&= \frac{1}{m\Omega} \int \mathrm{d}\xi \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - \xi^2}} \\
&= \frac{1}{m\Omega} \sin^{-1} \frac{\alpha_3 + m\Omega y}{\sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}}
\end{aligned}$$

and

$$\begin{aligned}
&\int \mathrm{d}y \frac{(\alpha_3 + m\Omega y)}{\sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}} \\
&= \frac{1}{m\Omega} \int \mathrm{d}\xi \frac{\xi}{\sqrt{2m\alpha_1 - \alpha_2^2 - \xi^2}} \\
&= -\frac{1}{m\Omega} \sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}.
\end{aligned}$$

We then have algebraic equations for the constants $\beta_1, ...\beta_3$

$$\beta_1 = -\frac{\partial F_1}{\partial \alpha_1} = t \mp m \frac{1}{m\Omega} \sin^{-1} \frac{\alpha_3 + m\Omega y}{\sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}},$$

$$\beta_2 = -\frac{\partial F_1}{\partial \alpha_2} = -z \pm \alpha_2 \frac{1}{m\Omega} \sin^{-1} \frac{\alpha_3 + m\Omega y}{\sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}},$$

and

$$\beta_3 = -\frac{\partial F_1}{\partial \alpha_3} = -x \mp \frac{1}{m\Omega} \sqrt{2m\alpha_1 - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}.$$

From the equation for β_1 we have an equation for y

$$y = \pm \frac{\sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}}{m\Omega} \sin \Omega \left(t - \beta_1\right) - \frac{\alpha_3}{m\Omega}$$

which we may write as

$$(\alpha_3 + m\Omega y) = \pm \sqrt{\left|2m\alpha_1 - \alpha_2^2\right|} \sin \Omega \left(t - \beta_1\right),$$

The equation for β_2 is then

$$\begin{aligned}\beta_2 &= -z \pm \alpha_2 \frac{1}{m\Omega} \sin^{-1}\left(\sin \Omega \left(t - \beta_1\right)\right) \\ &= -z \pm \alpha_2 \frac{1}{m}\left(t - \beta_1\right),\end{aligned}$$

or

$$z = \pm\alpha_2 \frac{1}{m} t - \left(\beta_2 \mp \alpha_2 \frac{1}{m} \beta_1\right),$$

The equation for β_3 gives us

$$\left(x + \beta_3\right)^2 = \frac{1}{m^2 \Omega^2}\left[2m\alpha_1 - \alpha_2^2 - \left(\alpha_3 + m\Omega y\right)^2\right],$$

which, with the equation for $(\alpha_3 + m\Omega y)$, becomes

$$\left(x + \beta_3\right)^2 + \left(y + \frac{\alpha_3}{m\Omega}\right)^2 = \frac{2m\alpha_1 - \alpha_2^2}{m^2 \Omega^2}.$$

In the equations

$$y = \pm \frac{\sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}}{m\Omega} \sin \Omega \left(t - \beta_1\right) - \frac{\alpha_3}{m\Omega},$$

$$z = \pm\alpha_2 \frac{1}{m} t - \left(\beta_2 \mp \alpha_2 \frac{1}{m} \beta_1\right),$$

and

$$\left(x + \beta_3\right)^2 + \left(y + \frac{\alpha_3}{m\Omega}\right)^2 = \frac{2m\alpha_1 - \alpha_2^2}{m^2 \Omega^2}$$

we have a complete description of the motion. The charge moves uniformly along the $z-$axis at a rate $\mathrm{d}z/\mathrm{d}t = \alpha_2/m$. The momentum along the $z-$axis is then α_2. This is the constant we have for the final coordinate Q_2. The motion in the (x, y) $-$plane is a circle centered at $\left(-\beta_3, -\alpha_3/m\Omega\right)$ with a radius $R = \sqrt{\left|2m\alpha_1 - \alpha_2^2\right|}/m\Omega$. And the (x, y) $-$motion in the circle is uniform at a frequency Ω as we see from the sinusoidal solution for y. We may also note that

$$\alpha_1 = \mathcal{E} = \frac{1}{2m}\left(p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2 + p_{\mathrm{z}}^2\right).$$

Therefore

$$\begin{aligned}m\alpha_1 - \alpha_2^2 &= \left(p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2 + p_{\mathrm{z}}^2\right) - p_{\mathrm{z}}^2 \\ &= p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2 > 0.\end{aligned}$$

Then

$$R = \frac{1}{m\Omega}\sqrt{p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2},$$

which is independent of motion in the z−direction.

5.11. As an example in the text we considered the motion of a charged point particle of mass m and charge Q moving in a constant magnetic field of induction $\boldsymbol{B} = \hat{e}_{\mathrm{z}}B$ using cylindrical coordinates. The vector potential is then

$$\boldsymbol{A} = \frac{1}{2}Br\hat{e}_{\vartheta}.$$

Treat this problem using the Hamilton-Jacobi approach.

[Note that a positive charge has a negative angular momentum, i.e. rotates clockwise.]

Solution:

To identify the canonical momenta in cylindrical coordinates we begin with the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m\dot{q}_{\mu}\dot{q}_{\mu} - Q\varphi + QA_{\mu}\dot{q}_{\mu} \\ &= \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\vartheta}^2 + \dot{z}^2\right) + \frac{1}{2}QBr^2\dot{\vartheta}, \end{aligned}$$

as we did in the text. The canonical momenta are then

$$\begin{aligned} p_{\mathrm{r}} &= m\dot{r} \\ p_{\vartheta} &= mr^2\dot{\vartheta} + \frac{1}{2}QBr^2 \\ p_{\mathrm{z}} &= m\dot{z}. \end{aligned}$$

The Legendre transformation producing the Hamiltonian is then

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}m\left(\dot{r}^2 + \dot{z}^2\right) + \frac{1}{2}mr^2\left(\frac{p_{\vartheta}}{mr^2} - \frac{\Omega}{2}\right)^2 \\ &= \frac{1}{2m}\left(p_{\mathrm{r}}^2 + p_{\mathrm{z}}^2\right) + \frac{1}{2}mr^2\left(\frac{p_{\vartheta}}{mr^2} - \frac{\Omega}{2}\right)^2. \end{aligned}$$

We identify the generator as either F_1 or F_2 and assume a separation of the generator as

$$F_{1,2}\left(q, \alpha, t\right) = F_{\mathrm{t}}\left(\alpha, t\right) + F_{\mathrm{r}}\left(r, \alpha\right) + F_{\vartheta}\left(\vartheta, \alpha\right) + F_{\mathrm{z}}\left(z, \alpha\right).$$

Then, with

$$p_{\mathrm{r}} = \frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\left(r, \alpha\right), \;\; p_{\vartheta} = \frac{\mathrm{d}}{\mathrm{d}\vartheta}F_{\vartheta}\left(\vartheta, \alpha\right) \text{ and } p_{\mathrm{z}} = \frac{\mathrm{d}}{\mathrm{d}z}F_{\mathrm{z}}\left(z, \alpha\right),$$

we have the equation for the generator as

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\mathrm{t}}(t,\alpha)=-\frac{1}{2m}\left[\left(\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}z}F_{\mathrm{z}}\right)^2\right]-\frac{1}{2}mr^2\left(\frac{1}{mr^2}\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_{\vartheta}-\frac{\Omega}{2}\right)^2.$$

The first separation is

$$F_{\mathrm{t}}(t,\alpha)=-\alpha_1 t$$

and

$$2m\alpha_1=\left(\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\right)^2+\left(\frac{\mathrm{d}}{\mathrm{d}z}F_{\mathrm{z}}\right)^2+\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_{\vartheta}-r\frac{m\Omega}{2}\right)^2.$$

The second separation is

$$F_{\mathrm{z}}=\alpha_2 z$$

and

$$2m\alpha_1-\left(\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\right)^2-\left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_{\vartheta}-r\frac{m\Omega}{2}\right)^2=\alpha_2^2.$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}F_{\vartheta}=r^2\frac{m\Omega}{2}\pm r\sqrt{2m\alpha_1-\alpha_2^2-\left(\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\right)^2}.$$

The third separation is

$$F_{\vartheta}=\alpha_3\vartheta$$

and

$$\alpha_3=r^2\frac{m\Omega}{2}\pm r\sqrt{2m\alpha_1-\alpha_2^2-\left(\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}\right)^2},$$

or

$$\frac{\mathrm{d}}{\mathrm{d}r}F_{\mathrm{r}}=\pm\sqrt{2m\alpha_1-\alpha_2^2-\left(\frac{1}{r}\alpha_3-r\frac{m\Omega}{2}\right)^2}.$$

Then

$$F_{\mathrm{r}} = \pm \int \mathrm{d}r \sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}.$$

The generator is then

$$\begin{aligned} &F_1\left(r, \vartheta, z, t, \alpha_1, \ldots, \alpha_3\right) \\ &= -\alpha_1 t \pm \int \mathrm{d}r \sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2} + \alpha_3\vartheta + \alpha_2 z. \end{aligned}$$

The α_1, ..., α_3 are the constant final coordinates and the energy (α_1). The final momenta will be the constants β_1, ..., β_3. We could integrate what we have here and obtain the generator directly. But the differentiation of the result looks formidable. So we calculate first the β's.

$$\beta_1 = -\frac{\partial F_1}{\partial \alpha_1} = t \mp m \int \mathrm{d}r \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}},$$

$$\beta_2 = -\frac{\partial F_1}{\partial \alpha_2} = -z \pm \alpha_2 \int \mathrm{d}r \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}},$$

and

$$\beta_3 = -\frac{\partial F_1}{\partial \alpha_3} = -\vartheta \pm \int \frac{\mathrm{d}r}{r} \frac{\left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}}.$$

The integrals in these results require some care because of the form of the expression $(\alpha_3/r - rm\Omega/2)$, which causes the difficulty.

We write the integral in the equation for β_1 as

$$\begin{aligned} &\int \mathrm{d}r \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}} \\ &= \int \mathrm{d}r \frac{r}{\sqrt{\left(2m\alpha_1 - \alpha_2^2\right) r^2 - \left(\alpha_3 - r^2\frac{m\Omega}{2}\right)^2}}. \end{aligned}$$

If we now write

$$\left(2m\alpha_1 - \alpha_2^2\right) r^2 - \left(\alpha_3 - r^2\frac{m\Omega}{2}\right)^2 = B^2 - \left(A - r^2\frac{m\Omega}{2}\right)^2$$

we find that

$$A = \frac{2m\alpha_1 - \alpha_2^2}{m\Omega} + \alpha_3.$$

and

$$B^2 = A^2 - \alpha_3^2,$$

or

$$B^2 = \left(\frac{2m\alpha_1 - \alpha_2^2}{m\Omega} \right)^2 + 2\alpha_3 \frac{2m\alpha_1 - \alpha_2^2}{m\Omega}.$$

Then our integral is

$$\int \mathrm{d}r \frac{r}{\sqrt{\left(2m\alpha_1 - \alpha_2^2\right) r^2 - \left(\alpha_3 - r^2 \frac{m\Omega}{2}\right)^2}} = \int \mathrm{d}r \frac{r}{\sqrt{B^2 - \left(A - r^2 \frac{m\Omega}{2}\right)^2}}$$

If we now make the substitution

$$\eta = A - r^2 \frac{m\Omega}{2}$$

we have

$$\frac{1}{m\Omega} \mathrm{d}\eta = r \mathrm{d}r$$

and our integral is

$$\begin{aligned} &\int \mathrm{d}r \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r \frac{m\Omega}{2}\right)^2}} \\ &= \frac{1}{m\Omega} \int \mathrm{d}\eta \frac{1}{\sqrt{B^2 - \eta^2}} \\ &= -\frac{1}{m\Omega} \cos^{-1} \frac{A - r^2 \frac{m\Omega}{2}}{|B|}. \end{aligned}$$

Then β_1 is

$$\beta_1 = t \pm \frac{1}{\Omega} \cos^{-1} \frac{A - r^2 \frac{m\Omega}{2}}{|B|},$$

or

$$\mp |B| \cos \Omega \left(t - \beta_1\right) = \left(A - r^2 \frac{m\Omega}{2} \right).$$

Solving for r^2 we have

$$r^2 = \frac{2}{m\Omega}A \pm \frac{2}{m\Omega}|B|\cos\Omega\left(t - \beta_1\right).$$

So the radial distance from the origin to the charge oscillates sinusoidally during the motion.

The same integral appears in the expression for β_2. Then

$$z + \beta_2 = \mp\alpha_2 \frac{1}{m\Omega}\cos^{-1}\frac{A - r^2\frac{m\Omega}{2}}{|B|},$$

or we could also write

$$z + \beta_2 = \pm\alpha_2 \frac{1}{m\Omega}\sin^{-1}\frac{A - r^2\frac{m\Omega}{2}}{|B|}.$$

Then either

$$\pm|B|\cos\frac{m\Omega}{\alpha_2}\left(z + \beta_2\right) = \left(A - r^2\frac{m\Omega}{2}\right).$$

or

$$\mp|B|\sin\frac{m\Omega}{\alpha_2}\left(z + \beta_2\right) = \left(A - r^2\frac{m\Omega}{2}\right).$$

Using the β_1 result we have then

$$\mp|B|\cos\Omega\left(t + \beta_1\right) = \pm|B|\cos\frac{m\Omega}{\alpha_2}\left(z + \beta_2\right)$$

or

$$\begin{aligned}|B|\cos\Omega\left(t + \beta_1\right) &= |B|\sin\frac{m\Omega}{\alpha_2}\left(z + \beta_2\right)\\ &= |B|\cos\frac{m\Omega}{\alpha_2}\left(z + \beta_2 - \frac{\pi}{2}\right).\end{aligned}$$

That is

$$\left(t + \beta_1\right) = \frac{m}{\alpha_2}\left(z + \beta_2 - \frac{\pi}{2}\right)$$

or

$$t = \frac{mz}{\alpha_2} \text{ and } \beta_1 = \frac{m\beta_2}{\alpha_2} - \frac{m\pi}{\alpha_2}.$$

The motion in the $z-$direction is then uniform. and

$$|B|^2 \sin^2 \frac{m\Omega}{\alpha_2}\left(z+\beta_2\right) = \left(A - r^2\frac{m\Omega}{2}\right)^2 .$$

To perform the integral in β_3 we need a new substitution. The difficulty is in replacing the term $(\alpha_3/r - rm\Omega/2)^2$ by a single quadratic term and still having the differential $\mathrm{d}r$ in simple form. Defining $\xi = (\alpha_3/r - rm\Omega/2)$ will not do. However, if we define

$$\xi = \left(\frac{1}{r}\alpha_3 + r\frac{m\Omega}{2}\right)$$

we see that

$$\mathrm{d}\xi = -\frac{1}{r}\left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)\mathrm{d}r.$$

So we obtain the differential in the β_3 integration. Also

$$\xi^2 = \frac{1}{r^2}\alpha_3^2 + r^2\frac{m^2\Omega^2}{4} + \alpha_3 m\Omega$$

and

$$\left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2 = \frac{1}{r^2}\alpha_3^2 + r^2\frac{m^2\Omega^2}{4} - \alpha_3 m\Omega$$

so that

$$\left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2 = \xi^2 - 2\alpha_3 m\Omega,$$

which differs from the term in the square root only by a constant. Then

$$\begin{aligned}\int \frac{\mathrm{d}r}{r}\frac{\left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)}{\sqrt{2m\alpha_1 - \alpha_2^2 - \left(\frac{1}{r}\alpha_3 - r\frac{m\Omega}{2}\right)^2}} &= -\int \mathrm{d}\xi \frac{1}{\sqrt{2m\alpha_1 - \alpha_2^2 + 2\alpha_3 m\Omega - \xi^2}} \\ &= \cos^{-1}\frac{\xi}{\left|2m\alpha_1 - \alpha_2^2 + 2\alpha_3 m\Omega\right|} \\ &= \cos^{-1}\frac{\left(\frac{1}{r}\alpha_3 + r\frac{m\Omega}{2}\right)}{\left|2m\alpha_1 - \alpha_2^2 + 2\alpha_3 m\Omega\right|}.\end{aligned}$$

This is the relation between r and the angle ϑ, which is the orbit.

$$\left|2m\alpha_1 - \alpha_2^2 + 2\alpha_3 m\Omega\right| \cos\left(\vartheta + \beta_3\right) = \pm\left(\frac{1}{r}\alpha_3 + r\frac{m\Omega}{2}\right).$$

We write this as

$$\mp 2r\frac{\left|2m\alpha_1 - \alpha_2^2 + 2\alpha_3 m\Omega\right|}{m\Omega}\cos\left(\vartheta + \beta_3\right) = 2\frac{\alpha_3}{m\Omega} + r^2,$$

or

$$2\frac{|\alpha_3|}{m\Omega} = r^2 \pm 2r\frac{\left|2m\alpha_1 - \alpha_2^2 - 2\,|\alpha_3|\,m\Omega\right|}{m\Omega}\cos\left(\vartheta + \beta_3\right),$$

noting that the angular momentum $\alpha_3 < 0$. If we identify

$$r_0 = \frac{\left|2m\alpha_1 - \alpha_2^2 - 2\,|\alpha_3|\,m\Omega\right|}{m\Omega}$$

and

$$R^2 = 2\frac{|\alpha_3|}{m\Omega} + \frac{\left(2m\alpha_1 - \alpha_2^2 - 2\,|\alpha_3|\,m\Omega\right)^2}{m^2\Omega^2}.$$

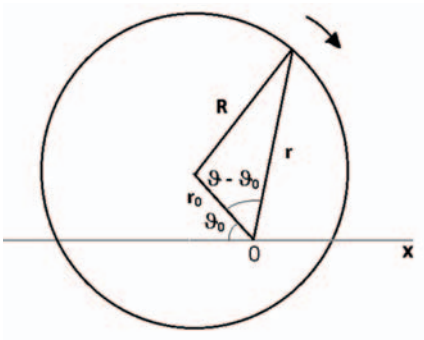
Then $R > r_0$ and the orbit is

$$R^2 = r^2 + r_0^2 - 2rr_0\cos\left(\vartheta + \beta_3\right),$$

where we have chosen the negative sign so that $\beta_3 < 0$. If we call $\beta_3 = -\vartheta_0$ the orbit has the form

$$R^2 = r^2 + r_0^2 - 2rr_0\cos\left(\vartheta - \vartheta_0\right),$$

which is the general form of the cosine law. In the figure below we have drawn the orbit, projected onto the (x, y) plane, as depicted by this result.



Orbit of charge in uniform magnetic field. Origin is indicated by 0. The line of constant length r_0 is from the origin to the center of the orbit. R is the radius of the orbit. The arrow outside the circle indicates the direction of motion of the charge.

The orbit is a circle of radius R with center located by r_0, and is distinct from the origin of coordinates, which is designated by 0. The charge rotates in a clockwise fashion around the circle, which results in a negative angular momentum along the z−axis, which is out of the plot. The motion of the charge in the z−direction is uniform, and may also be stationary.

5.12. Consider the motion of a charged particle in a region of space in which there is a uniform magnetic field with induction $\boldsymbol{B} = \hat{e}_z B$ and a uniform electric field $\boldsymbol{E} = \hat{e}_y E$.

For a static magnetic field with induction $\boldsymbol{B} = \hat{e}_z B$ the vector potential is $\boldsymbol{A} = -\hat{e}_x y B$. And for an electric field $\boldsymbol{E} = \hat{e}_y E$ the electrostatic potential is

$$\varphi = -Ey.$$

Find the orbit of the charged particle using the Hamilton-Jacobi approach.
Solution:

The Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \frac{1}{2m}\left(p_x - QA_x\right)^2 + \frac{1}{2m}\left(p_y^2 + p_z^2\right) - QEy \\ &= \frac{1}{2m}\left(p_x + m\Omega y\right)^2 + \frac{1}{2m}\left(p_y^2 + p_z^2\right) - QEy.\end{aligned}$$

where $\Omega = QB/m$.
We identify the generator as either F_1 or F_2 and assume a separation of the generator as

$$F_{1,2}\left(q, \alpha, t\right) = F_t\left(\alpha, t\right) + F_x\left(x, \alpha\right) + F_y\left(y, \alpha\right) + F_z\left(z, \alpha\right).$$

Then, with

$$p_x = \frac{d}{dx} F_x\left(x, \alpha\right), \; p_y = \frac{d}{dy} F_y\left(y, \alpha\right) \text{ and } p_z = \frac{d}{dz} F_z\left(z, \alpha\right),$$

we have the equation for the generator as

$$\frac{d}{dt} F_t\left(t, \alpha\right) = -\frac{1}{2m}\left(\frac{d}{dx} F_x + m\Omega y\right)^2 - \frac{1}{2m}\left[\left(\frac{d}{dy} F_y\right)^2 + \left(\frac{d}{dz} F_z\right)^2\right] + QEy$$

The first separation is

$$F_t\left(t, \alpha\right) = -\alpha_1 t$$

and

$$2m\alpha_1 = \left(\frac{d}{dx} F_x + m\Omega y\right)^2 + \left(\frac{d}{dy} F_y\right)^2 + \left(\frac{d}{dz} F_z\right)^2 - 2mQEy.$$

The second separation is

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}F_{\mathrm{z}}\right)=\alpha_2$$

with

$$2m\alpha_1+2mQEy-\left(\frac{\mathrm{d}}{\mathrm{d}x}F_{\mathrm{x}}+m\Omega y\right)^2-\left(\frac{\mathrm{d}}{\mathrm{d}y}F_{\mathrm{y}}\right)^2=\alpha_2^2.$$

To obtain the third separation we first write

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{\mathrm{x}}=-m\Omega y\pm\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-\left(\frac{\mathrm{d}}{\mathrm{d}y}F_{\mathrm{y}}\right)^2}$$

for which all functions of y are on the right hand side. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}F_{\mathrm{x}}=\alpha_3$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}F_{\mathrm{y}}=\pm\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-(\alpha_3+m\Omega y)^2}.$$

We then have the generator as

$$F_{1,2}\left(q,\alpha,t\right)=-\alpha_1 t+\alpha_3 x\pm\int\mathrm{d}y\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-(\alpha_3+m\Omega y)^2}+\alpha_2 z.$$

Again, because the partial derivatives become complicated, we shall proceed first to the momenta $\beta_{\mathrm{j}}=-\partial F_1\left(q,t,\alpha\right)/\partial\alpha_{\mathrm{j}}$.

$$\beta_1=-\frac{\partial F_1}{\partial\alpha_1}=t\mp m\int\mathrm{d}y\frac{1}{\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-(\alpha_3+m\Omega y)^2}},$$

$$\beta_2=-\frac{\partial F_1}{\partial\alpha_2}=-z\pm\alpha_2\int\mathrm{d}y\frac{1}{\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-(\alpha_3+m\Omega y)^2}},$$

and

$$\beta_3=-\frac{\partial F_1}{\partial\alpha_3}=-x\pm\int\mathrm{d}y\frac{(\alpha_3+m\Omega y)}{\sqrt{2m\alpha_1+2mQEy-\alpha_2^2-(\alpha_3+m\Omega y)^2}}.$$

We define

$$\xi = (\alpha_3 + m\Omega y)$$

so that

$$\mathrm{d}\xi = m\Omega \mathrm{d}y$$

and

$$y = \frac{\xi - \alpha_3}{m\Omega}.$$

Then

$$\begin{aligned} &2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2 \\ &= m^2\Omega^2 \left[\frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2\right) + 2\frac{QE}{m\Omega^2}y - \left(\frac{\alpha_3}{m\Omega} + y\right)^2 \right]. \end{aligned}$$

And, after some algebra

$$\begin{aligned} &\frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2\right) + 2\frac{QE}{m\Omega^2}y - \left(\frac{\alpha_3}{m\Omega} + y\right)^2 \\ &= \frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2 - \alpha_3^2 + \left(\frac{QE}{\Omega} - \alpha_3\right)^2\right) - \left(y - \frac{1}{m\Omega}\left(\frac{QE}{\Omega} - \alpha_3\right)\right)^2. \end{aligned}$$

We then define

$$A^2 = \frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2 - \alpha_3^2 + \left(\frac{QE}{\Omega} - \alpha_3\right)^2\right)$$

and

$$\xi = y - \frac{1}{m\Omega}\left(\frac{QE}{\Omega} - \alpha_3\right)$$

so that

$$\sqrt{2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2} = m\Omega\sqrt{A^2 - \xi^2}$$

and

$$\mathrm{d}\xi = \mathrm{d}y.$$

The integral in β_1 and β_2 is then

$$\begin{aligned}
&\int \mathrm{d}y \frac{1}{\sqrt{2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}} \\
&= \frac{1}{m\Omega} \int \mathrm{d}\xi \frac{1}{\sqrt{A^2 - \xi^2}} \\
&= -\frac{1}{m\Omega} \cos^{-1} \frac{y - \left(QE/m\Omega^2 - \alpha_3/m\Omega\right)}{A}.
\end{aligned}$$

With this result

$$y = \left(\frac{QE}{m\Omega^2} - \frac{\alpha_3}{m\Omega}\right) \pm A \cos \Omega \left(t - \beta_1\right)$$

and

$$y = \left(\frac{QE}{m\Omega^2} - \frac{\alpha_3}{m\Omega}\right) \pm A \cos \frac{m\Omega}{\alpha_2} \left(z + \beta_2\right).$$

The charged particle then moves along the $z-$axis according to

$$z = \frac{\alpha_2}{m} t.$$

This may also be zero if $\alpha_2 = 0$.

We may handle the integral in β_3

$$\int \mathrm{d}y \frac{(\alpha_3 + m\Omega y)}{\sqrt{2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}}$$

in the same basic fashion. But the interpretation of the result is difficult. So we use a different approach. We first reform the term in the radical as we did above. That is

$$\begin{aligned}
&2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2 \\
&= m^2\Omega^2 \left[\frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2\right) + 2\frac{QE}{m\Omega^2} y - \left(\frac{\alpha_3}{m\Omega} + y\right)^2\right].
\end{aligned}$$

Then we reform the term in the bracket [ ] as before, with some algebra, as

$$\begin{aligned}
&\frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2\right) + 2\frac{QE}{m\Omega^2} y - \left(\frac{\alpha_3}{m\Omega} + y\right)^2 \\
&= \frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2 - \alpha_3^2 + \left(\frac{QE}{\Omega} - \alpha_3\right)^2\right) - \left(y - \frac{1}{m\Omega}\left(\frac{QE}{\Omega} - \alpha_3\right)\right)^2
\end{aligned}$$

If we now recall our definition we introduced above

$$A^2 = \frac{1}{m^2\Omega^2}\left(2m\alpha_1 - \alpha_2^2 - \alpha_3^2 + \left(\frac{QE}{\Omega} - \alpha_3\right)^2\right)$$

and define

$$y - \frac{1}{m\Omega}\left(\frac{QE}{\Omega} - \alpha_3\right) = -A\cos\lambda,$$

which is essentially our solution for y

$$y = \left(\frac{QE}{m\Omega^2} - \frac{\alpha_3}{m\Omega}\right) \pm A\cos\Omega\left(t - \beta_1\right),$$

the term in the integral within the radical becomes

$$\begin{aligned}
&2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2 \\
&= m^2\Omega^2\left\{\frac{1}{m^2\Omega^2}\left[2m\alpha_1 - \alpha_2^2 - \alpha_3 + \left(\frac{QE}{\Omega} - \alpha_3\right)^2\right] - \left[y - \left(\frac{QE}{m\Omega^2} - \frac{\alpha_3}{m\Omega}\right)\right]^2\right\} \\
&= m^2\Omega^2A^2\left(1 - \cos^2\lambda\right).
\end{aligned}$$

The differential of y is

$$\mathrm{d}y = A\sin\lambda \mathrm{d}\lambda$$

and the numerator and differential in the integrand result in

$$\begin{aligned}
(\alpha_3 + m\Omega y)\,\mathrm{d}y &= \left(\frac{QE}{\Omega} - m\Omega A\cos\lambda\right)A\sin\lambda\mathrm{d}\lambda \\
&= \frac{QE}{\Omega}A\sin\lambda\mathrm{d}\lambda - m\Omega A^2\cos\lambda\sin\lambda\mathrm{d}\lambda.
\end{aligned}$$

The integral then becomes

$$\begin{aligned}
&\int \mathrm{d}y\frac{(\alpha_3 + m\Omega y)}{\sqrt{2m\alpha_1 + 2mQEy - \alpha_2^2 - (\alpha_3 + m\Omega y)^2}} \\
&= \int \mathrm{d}\lambda\frac{(QE/\Omega)\sin\lambda - m\Omega A\cos\lambda\sin\lambda}{m\Omega\sqrt{\left(1 - \cos^2\lambda\right)}} \\
&= \int \mathrm{d}\lambda\left(\frac{QE}{m\Omega^2} - A\cos\lambda\right) \\
&= \frac{QE}{m\Omega^2}\lambda - A\sin\lambda.
\end{aligned}$$

Then, choosing the positive sign,

$$x + \beta_3 = \frac{QE}{m\Omega^2}\lambda - A\sin\lambda,$$

or

$$x + \beta_3 = \frac{QE}{m\Omega^2}\Omega\left(t - \beta_1\right) - A\sin\Omega\left(t - \beta_1\right),$$

which with

$$y + \frac{\alpha_3}{m\Omega} = \left(\frac{QE}{m\Omega^2}\right) - A\cos\Omega\left(t - \beta_1\right),$$

is the general form of the equation for a cycloid. The constants β_3 and $\alpha_3/m\Omega$ locate the cycloid relatively to the origin in the (x, y) −plane. The form of the cycloid is determined by the relative sizes of $QE/m\Omega^2$ and A. There are three general forms of the cycloid depending upon whether $QE/m\Omega^2$ is greater than, less than, or equal to A.

6 Chaos in Dynamical Systems

7.1. The equations for the Rössler system are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y - z,$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + ay,$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = bx - cz + xz.$$

Following the argument in the chapter show that the rate of change of the Rössler system volume in phase space may be either positive or negative and that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega = (a - c)\,\Omega + F_{\Omega}\,(x, y, z)\,,$$

where

$$F_{\Omega}\,(x, y, z) = \int_{\Omega} x \mathrm{d}\Omega.$$

Solution: As we showed in the chapter the rate of change of the system volume in phase for the Hamiltonian system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mathrm{d}\Omega = \int_{\Omega} \sum_{\mathrm{i}} \left(\frac{\partial}{\partial q_{\mathrm{i}}} \dot{q}_{\mathrm{i}} + \frac{\partial}{\partial p_{\mathrm{i}}} \dot{p}_{\mathrm{i}} \right) \mathrm{d}\Omega,$$

where Ω is the phase space volume. The Rössler system has three coordinates, none of which are actually identified as momenta. So the rate of change of the phase space volume for the Rössler system is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \mathrm{d}\Omega = \int_{\Omega} \sum_{\mathrm{i}} \left(\frac{\partial}{\partial q_{\mathrm{i}}} \dot{q}_{\mathrm{i}} \right) \mathrm{d}\Omega,$$

where the q_{i} are (x, y, z) and

$$\frac{\partial}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = 0$$
$$\frac{\partial}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = a$$
$$\frac{\partial}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = -c + x.$$

Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\mathrm{d}\Omega &= \int_{\Omega}\sum_{\mathrm{i}}\left(\frac{\partial}{\partial q_{\mathrm{i}}}\dot{q}_{\mathrm{i}}\right)\mathrm{d}\Omega \\ &= \int_{\Omega}(a - c + x)\,\mathrm{d}\Omega \\ &= (a - c)\,\Omega + \int_{\Omega} x\mathrm{d}\Omega\end{aligned}$$

The remaining integral cannot be easily evaluated because the phase space volume Ω is a function of time. The facial area perpendicular to the coordinate x involves the values of x as well as y and z. And Ω may not be small. We then have the result that $\mathrm{d}\Omega/\mathrm{d}t$ depends on each consecutive point in time. All we can say definitely is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Omega = (a - c)\,\Omega + F_{\Omega}\,(x, y, z)\,,$$

where

$$F_{\Omega}\,(x, y, z) = \int_{\Omega} x\mathrm{d}\Omega.$$

7.2. We consider a simplified model of the water molecule with the oxygen atom at the coordinate x_2 and the two hydrogen atoms at coordinates x_1 and x_3 along the same axis. We consider that the vibration of the hydrogen atoms is small enough that the potential energy of the bound hydrogen atoms can be approximated by a quadratic with a spring constant k. The mass of the hydrogen atom is m and the mass of the oxygen is μm. a) Obtain the Hamiltonian of this model water molecule. b) Is the Hamiltonian separable? c) What are the frequencies of vibration of this model water molecule and the corresponding eigenvectors?

Solution:

a) The Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \mu\dot{x}_2^2 + \dot{x}_3^2\right) - \frac{1}{2}k\left((x_1 - x_2)^2 + (x_2 - x_3)^2\right).$$

With momenta

$$p_{\mathrm{i}} = m\dot{x}_{\mathrm{i}}$$

the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m}\left(p_1^2 + p_3^2\right) + \frac{1}{2\mu m}p_2^2 + \frac{1}{2}k\left((x_1 - x_2)^2 + (x_2 - x_3)^2\right).$$

b) Because of the product terms x_1x_2 and x_2x_3 this Hamiltonian is not separable.
c) The canonical equations are

$$\dot{x}_1 = \frac{1}{m}p_1,$$

$$\dot{x}_2 = \frac{1}{\mu m}p_2,$$

$$\dot{x}_3 = \frac{1}{m}p_3,$$

$$\dot{p}_1 = k\left(x_2 - x_1\right),$$

$$\dot{p}_2 = k\left(x_1 - x_2\right) - k\left(x_2 - x_3\right) = kx_1 - 2kx_2 + kx_3,$$

and

$$\dot{p}_3 = k\left(x_2 - x_3\right).$$

This is a set of linear coupled ordinary differential equations. The solution is complex exponential with

$$x_{\mathrm{i}} = \tilde{x}_{\mathrm{i}} \exp\left(i\omega t\right)$$

and

$$p_{\mathrm{i}} = \tilde{p}_{\mathrm{i}} \exp\left(i\omega t\right).$$

The canonical equations are then

$$
\begin{aligned}
\omega\tilde{x}_1 &= \frac{1}{m}\tilde{p}_1,\\
\omega\tilde{x}_2 &= \frac{1}{\mu m}\tilde{p}_2,\\
\omega\tilde{x}_3 &= \frac{1}{m}\tilde{p}_3,\\
\omega\tilde{p}_1 &= k\left(\tilde{x}_2 - \tilde{x}_1\right),\\
\omega\tilde{p}_2 &= k\tilde{x}_1 - 2k\tilde{x}_2 + k\tilde{x}_3,\\
\omega\tilde{p}_3 &= k\left(\tilde{x}_2 - \tilde{x}_3\right).
\end{aligned}
$$

In matrix form these are

$$
\omega \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}
\begin{bmatrix} \tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{p}_1\\ \tilde{p}_2\\ \tilde{p}_3 \end{bmatrix}
=
\begin{bmatrix} 0&0&0&\frac{1}{m}&0&0\\ 0&0&0&0&\frac{1}{\mu m}&0\\ 0&0&0&0&0&\frac{1}{m}\\ -k&k&0&0&0&0\\ k&-2k&k&0&0&0\\ 0&k&-k&0&0&0 \end{bmatrix}
\begin{bmatrix} \tilde{x}_1\\ \tilde{x}_2\\ \tilde{x}_3\\ \tilde{p}_1\\ \tilde{p}_2\\ \tilde{p}_3 \end{bmatrix}
$$

The eigenvalues (frequencies) are:

$$
\pm i\omega_0, 0, \text{ and } \pm i\omega_0\sqrt{\left(1+\frac{2}{\mu}\right)}.
$$

And the canonical equations can be written as

$$
\begin{aligned}
\left(\omega^2+\omega_0^2\right)\tilde{x}_1 &= \omega_0^2\tilde{x}_2\\
\omega^2\tilde{x}_2 &= \frac{1}{\mu}\omega_0^2\left(\tilde{x}_1 - 2\tilde{x}_2 + \tilde{x}_3\right)\\
\left(\omega^2+\omega_0^2\right)\tilde{x}_3 &= \omega_0^2\tilde{x}_2\\
\omega\tilde{p}_1 &= k\left(\tilde{x}_2 - \tilde{x}_1\right)\\
\omega\tilde{p}_2 &= k\tilde{x}_1 - 2k\tilde{x}_2 + k\tilde{x}_3\\
\omega\tilde{p}_3 &= k\left(\tilde{x}_2 - \tilde{x}_3\right)
\end{aligned}
$$

where

$$
\omega_0^2 = \frac{k}{m}.
$$

So

$$
x_1 = x_3 = 1
$$

$$\tilde{x}_2 = \left(\frac{\omega^2}{\omega_0^2} + 1 \right).$$

Then for $\omega = i\omega_0$

$$\tilde{x}_2 = 0,$$

for $\omega = 0$

$$\tilde{x}_2 = 1,$$

and for $\omega = \pm i\omega_0 \sqrt{\left(1 + \frac{2}{\mu}\right)}$

$$\tilde{x}_2 = 1 - \left(1 + \frac{2}{\mu}\right) = -\frac{2}{\mu}.$$

7.3. In the text we encountered the algebraic system of equations

$$\begin{aligned}
0 &= -y - z \\
0 &= x + ay \\
0 &= bx - cz + xz.
\end{aligned}$$

for the condition that the velocity components for the Rössler system vanished, i.e. $\mathrm{d}x/\mathrm{d}t = \mathrm{d}y/\mathrm{d}t = \mathrm{d}z/\mathrm{d}t = 0$. Obtain the two solutions to this set of equations.

Solution:

It is obvious that the first solution is at the origin of coordinates $x = y = z = 0$. We must be cautious in seeking any other solutions to this set of equations, since we cannot divide by x, y, or z. We can, however, write the third equation in terms of each of the variables. That is, with

$$z = -y = \frac{1}{a}x,$$

we have

$$\begin{aligned}
0 &= bx - \frac{c}{a}x + \frac{1}{a}x^2 \\
&= \left[b - \frac{c}{a} + \frac{1}{a}x \right] x,
\end{aligned}$$

$$\begin{aligned}
0 &= -aby + cy + ay^2 \\
&= [-ab + c + ay]\, y,
\end{aligned}$$

or

$$\begin{aligned} 0 &= abz - cz + az^2 \\ &= [ab - c + az]\, z \end{aligned}$$

These equations are solved if (x, y, z) vanish, or if

$$b - \frac{c}{a} + \frac{1}{a}x = 0,$$

$$-ab + c + ay = 0,$$

and

$$ab - c + az = 0.$$

That is, if

$$\begin{aligned} x &= c - ab \\ y &= b - \frac{c}{a} \\ z &= \frac{c}{a} - b. \end{aligned}$$

7 Special Relativity

8.1. Using the Lorentz Transformation show that

$$\pm\left[\left(dx^0\right)^2-\left(dx^1\right)^2-\left(dx^2\right)^2-\left(dx^3\right)^2\right]$$
$$=\pm\left[\left(dx'^0\right)^2-\left(dx'^1\right)^2-\left(dx'^2\right)^2-\left(dx'^3\right)^2\right]$$

Solution:
From the Lorentz Transformation

$$\begin{aligned}x'^0 &= \gamma\left(x^0-\beta x^1\right)\\ x'^1 &= \gamma\left(x^1-\beta x^0\right)\\ x'^2 &= x^2\\ x'^3 &= x^3.\end{aligned}$$

we have

$$\begin{aligned}dx'^0 &= \gamma dx^0-\gamma\beta dx^1\\ dx'^1 &= \gamma dx^1-\gamma\beta dx^0\\ dx'^2 &= dx^2\\ dx'^3 &= dx^3\end{aligned}$$

Then

$$\begin{aligned}&\left(dx'^0\right)^2-\left(dx'^1\right)^2-\left(dx'^2\right)^2-\left(dx'^3\right)^2\\ &=\left(\gamma dx^0-\gamma\beta dx^1\right)^2-\left(\gamma dx^1-\gamma\beta dx^0\right)^2-\left(dx^2\right)^2-\left(dx^3\right)^2\\ &=\gamma^2\left(1-\beta^2\right)\left(dx^0\right)^2-\gamma^2\left(1-\beta^2\right)\left(dx^1\right)^2-2\gamma^2\beta dx^0dx^1\\ &\quad+2\gamma^2\beta dx^0dx^1-\left(dx^2\right)^2-\left(dx^3\right)^2\end{aligned}$$

Using the definition

$$\gamma^2=\frac{1}{1-\beta^2},$$

This becomes

$$\begin{aligned}&\left(\mathrm{d}x^0\right)^2-\left(\mathrm{d}x^1\right)^2-\left(\mathrm{d}x^2\right)^2-\left(\mathrm{d}x^3\right)^2\\=&\left(\mathrm{d}x'^0\right)^2-\left(\mathrm{d}x'^1\right)^2-\left(\mathrm{d}x'^2\right)^2-\left(\mathrm{d}x'^3\right)^2\end{aligned}$$

or

$$\begin{aligned}&\pm\left[\left(\mathrm{d}x^0\right)^2-\left(\mathrm{d}x^1\right)^2-\left(\mathrm{d}x^2\right)^2-\left(\mathrm{d}x^3\right)^2\right]\\=&\pm\left[\left(\mathrm{d}x'^0\right)^2-\left(\mathrm{d}x'^1\right)^2-\left(\mathrm{d}x'^2\right)^2-\left(\mathrm{d}x'^3\right)^2\right].\end{aligned}$$

8.2. Use the Lorentz Transformation matrix **A** to obtain time dilation. Consider that frame k' moves at a velocity v in the direction of the $x-$axis of k. Then $\mathrm{d}y = \mathrm{d}z = 0$ and $\mathrm{d}x = v\mathrm{d}t$, which is the distance that the origin of k' moves in the time $\mathrm{d}t$. The differential world line in k is then

$$\mathrm{d}\boldsymbol{s}=\begin{bmatrix}c\mathrm{d}t\\v\mathrm{d}t\\0\\0\end{bmatrix}.$$

This differential world line is transformed into the differential world line $\mathrm{d}\mathbf{s}'$ in k' by

$$\mathrm{d}\boldsymbol{s}'=\mathbf{A}\cdot\mathrm{d}\boldsymbol{s}.$$

Find the differential world line $\mathrm{d}\boldsymbol{s}'$ and from the result show that $\mathrm{d}t'=\sqrt{1-\beta^2}\mathrm{d}t$.

Solution:

The differential world line $\mathrm{d}\boldsymbol{s}'$ is

$$\begin{aligned}\mathrm{d}\boldsymbol{s}'&=\begin{bmatrix}\gamma&-\gamma\beta&0&0\\-\gamma\beta&\gamma&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}c\mathrm{d}t\\v\mathrm{d}t\\0\\0\end{bmatrix}\\&=\begin{bmatrix}\gamma c\mathrm{d}t-\gamma\beta v\mathrm{d}t\\\gamma v\mathrm{d}t-\gamma\beta c\mathrm{d}t\\0\\0\end{bmatrix}.\end{aligned}$$

That is

$$\begin{aligned}\mathrm{d}t'&=\gamma\left(1-\beta^2\right)\mathrm{d}t\\&=\sqrt{1-\beta^2}\mathrm{d}t.\end{aligned}$$

8.3. The Minkowski Axiom requires that the velocities of material bodies are always less than the speed of light. Using $u_{\mathrm{x}}=\left(u'_{\mathrm{x}}+c\beta\right)/\left(1+\beta u'_{\mathrm{x}}/c\right)$ show that if the inertial frame k' has a velocity $v<c$ and if the particle moving in k' also has a

velocity $u_x < c$, as it must, the velocity of the particle as measured in k is also $< c$ regardless of how close v and u_x are to c.

[Hint: Choose $\beta'_x = 1 - \kappa$ and $\beta = 1 - \lambda$]

Solution:

Choosing $\beta'_x = 1-\kappa$ and $\beta = 1-\lambda$ the equation $u_x = \left(u'_x + c\beta\right) / \left(1 + \beta u'_x/c\right)$ results in

$$\beta_x = \frac{\beta'_x + \beta}{1 + \beta\beta'_x} = \frac{2 - \kappa - \lambda}{2 - \kappa - \lambda + \kappa\lambda} < 1.$$

This is the argument that Einstein presented.

If we choose $\beta'_x = 1$, so that we are considering a light pulse in frame k' rather than a particle. Then $\kappa = 0$ and this results in $\beta_x = 1$, which is Einstein's second postulate.

8.4. Consider a collision between a high energy proton and a photon. The proton is moving at a velocity $\boldsymbol{u}$ along the positive $x-$axis and the photon of frequency ν is initially moving toward it in the direction of the negative $x-$axis. After the collision a photon of frequency ν' leaves the site of collision moving along the positive $x-$axis. Designate the 4$-$momentum of the proton before and after collision as $\boldsymbol{P}_p$ and $\boldsymbol{P}'_p$ and that of the photons before and after collision as $\boldsymbol{P}_\gamma$ and $\boldsymbol{P}'_\gamma$. What is the energy of the final photon?

Solution:

Our law of mechanics is

$$\boldsymbol{P}_p + \boldsymbol{P}_\gamma = \boldsymbol{P}'_p + \boldsymbol{P}'_\gamma.$$

Before the collision the proton 4$-$momentum is

$$\boldsymbol{P}_p = m_0\gamma_u \left(c, u\hat{e}_x\right)$$

and the photon 4$-$momentum is

$$\boldsymbol{P}_\gamma = \frac{h\nu}{c}\left(1, -\hat{e}_x\right).$$

After the collision the 4$-$momentum of the photon leaving the site is

$$\boldsymbol{P}'_\gamma = \frac{h\nu'}{c}\left(1, \hat{e}_x\right),$$

We know nothing about the scattered proton. So we use the same trick as in our example of Compton scattering. We isolate $\boldsymbol{P}'_p$ and write

$$\left(\boldsymbol{P}_p + \boldsymbol{P}_\gamma - \boldsymbol{P}'_\gamma\right)^2 = \left(\boldsymbol{P}'_p\right)^2$$

or

$$\boldsymbol{P}_{\mathrm{p}}^2 + \boldsymbol{P}_{\gamma}^2 - \boldsymbol{P}_{\gamma}'^2 + 2\boldsymbol{P}_{\mathrm{p}} \cdot \left(\boldsymbol{P}_{\gamma} - \boldsymbol{P}_{\gamma}'\right) + 2\boldsymbol{P}_{\gamma} \cdot \boldsymbol{P}_{\gamma}' = \left(\boldsymbol{P}_{\mathrm{p}}'\right)^2.$$

Realizing that $P_{\mathrm{p}}^2 = P_{\mathrm{p}}'^2 = m_0^2 c^2$ we see that $\boldsymbol{P}_{\mathrm{p}}^2$ and $\left(\boldsymbol{P}_{\mathrm{p}}'\right)^2$ cancel on each side. And $P_{\gamma}^2 = P_{\gamma}'^2 = 0$. We are left then with

$$\boldsymbol{P}_{\mathrm{p}} \cdot \left(\boldsymbol{P}_{\gamma} - \boldsymbol{P}_{\gamma}'\right) = \boldsymbol{P}_{\gamma} \cdot \boldsymbol{P}_{\gamma}'.$$

Now

$$\begin{aligned}
\boldsymbol{P}_{\mathrm{p}} \cdot \left(\boldsymbol{P}_{\gamma} - \boldsymbol{P}_{\gamma}'\right) &= P_{\mathrm{p}}^{\alpha} g_{\alpha\beta} \left(P_{\gamma}^{\beta} - P_{\gamma}'^{\beta}\right) \\
&= m_0 \gamma_{\mathrm{u}} \left(c, u\hat{e}_{\mathrm{x}}\right) \cdot \left[\frac{h\nu}{c}\left(1, \hat{e}_{\mathrm{x}}\right) - \frac{h\nu'}{c}\left(1, -\hat{e}_{\mathrm{x}}\right)\right] \\
&= m_0 \gamma_{\mathrm{u}} \left[\frac{h\nu}{c}(c+u) - \frac{h\nu'}{c}(c-u)\right],
\end{aligned}$$

noting that the scalar product involves $g_{\alpha\beta}$, which places a negative sign on the product of the spatial components. And

$$\begin{aligned}
\boldsymbol{P}_{\gamma} \cdot \boldsymbol{P}_{\gamma}' &= P_{\gamma}^{\alpha} g_{\alpha\beta} P_{\gamma}'^{\beta} \\
&= \frac{h\nu}{c}\left(1, -\hat{e}_{\mathrm{x}}\right) \cdot \frac{h\nu'}{c}\left(1, -\hat{e}_{\mathrm{x}}\right) \\
&= \left(\frac{h}{c}\right)^2 \nu\nu' (1+1) \\
&= 2\left(\frac{h}{c}\right)^2 \nu\nu'
\end{aligned}$$

Then we have

$$m_0 \gamma_{\mathrm{u}} h \left[\nu\left(1+\beta_{\mathrm{u}}\right) - \nu'\left(1-\beta_{\mathrm{u}}\right)\right] = 2\left(\frac{h}{c}\right)^2 \nu\nu'.$$

solving for ν',

$$\nu' = \frac{m_0 \gamma_{\mathrm{u}} \nu \left(1+\beta_{\mathrm{u}}\right)}{2\left(h\nu/c^2\right) + m_0 \gamma_{\mathrm{u}} \left(1-\beta_{\mathrm{u}}\right)}.$$

Or, in energy terms

$$h\nu' = \frac{\left(m_0 \gamma_{\mathrm{u}} c^2\right)(h\nu)\left(1+\beta_{\mathrm{u}}\right)}{2(h\nu) + \left(m_0 \gamma_{\mathrm{u}} c^2\right)\left(1-\beta_{\mathrm{u}}\right)}.$$

Since $m_\gamma = h\nu/c^2$ is the relativistic mass of the incoming photon, we may also write this as

$$h\nu' = \frac{m_0\gamma_{\mathrm{u}}\,(h\nu)\left(1+\beta_{\mathrm{u}}\right)}{2m_\gamma + m_0\gamma_{\mathrm{u}}\left(1-\beta_{\mathrm{u}}\right)},$$

As an example we consider that a photon from the cosmic background radiation, about 2.73 K with energy

$$h\nu = k_{\mathrm{B}}T = 2.3525 \times 10^{-4}\,\mathrm{eV}$$

collides with a cosmic ray proton with a velocity of ˜0.998c. The proton rest energy is $m_0\gamma_{\mathrm{u}}c^2 = 938.272\,\mathrm{MeV}$. We find a final energy for the photon as

$$\begin{aligned} h\nu' &= \frac{\left(938.272\times 10^6\,\mathrm{eV}\right)\left(10^{-4}\,\mathrm{eV}\right)(1+0.998)}{2\left(10^{-4}\,\mathrm{eV}\right)+\left(938.272\times 10^6\,\mathrm{eV}\right)(1-0.998)} \\ &= 9.9900\times 10^{-2}\,\mathrm{eV}. \end{aligned}$$

There has then been essentially a 400 fold increase in photon energy.

8.5. Using the concept of relativistic mass of the photon, find the relativistic mass of two photons moving in opposite directions with frequencies ν_1 and ν_2.

Solution:

The 4−momenta of the photons are

$$\boldsymbol{P}_1 = \frac{h\nu_1}{c}\left(1, \hat{n}_1\right)$$

and

$$\boldsymbol{P}_2 = \frac{h\nu_2}{c}\left(1, \hat{n}_2\right).$$

The total momentum is

$$\bar{\mathbf{P}} = \frac{h}{c}\left(\nu_1+\nu_2, (\nu_1-\nu_2)\,\hat{n}_1\right).$$

Then

$$\begin{aligned} \bar{\mathbf{P}}^2 &= \bar{P}_\alpha \bar{P}^\alpha = \bar{P}^\alpha g_{\alpha\beta}\bar{P}^\beta \\ &= \left(\frac{h}{c}\right)^2\left[(\nu_1+\nu_2)^2-(\nu_1-\nu_2)^2\right] \\ &= 4\left(\frac{h}{c}\right)^2\nu_1\nu_2 \\ &= 4\mathcal{E}_1\mathcal{E}_2/c^2 \end{aligned}$$

We then have a relativistic mass as

$$mc^2 = 2\sqrt{\mathcal{E}_1\mathcal{E}_2}$$

8.6. A photon with sufficient energy can produce an electron and a positron. The positron is the antiparticle of an electron. This was first predicted by Paul Dirac and first identified in a cloud chamber track by Carl Anderson. For the photon to produce an electron and a positron there must only be another particle present for momentum conservation. We consider a particle of mass m_{01} at rest at the origin of a frame k. A photon with frequency ν approaches this particle. At the point of "collision" the photon disappears and an electron e and a positron e^{+} appear. The electron and the positron both have rest mass $m_{0\mathrm{e}}$. What is the minimum energy of the photon required for pair production?

[After the collision refer all particles to a CM frame k_{CM}.]

Solution:

Our law of mechanics (conservation of momentum) is

$$\boldsymbol{P}_{\gamma} + \boldsymbol{P}_1 = \bar{\mathbf{P}}.$$

Squaring this

$$\mathbf{P}_{\gamma}^2 + \mathbf{P}_1^2 + 2\boldsymbol{P}_{\gamma} \cdot \boldsymbol{P}_1 = \bar{\mathbf{P}}^2.$$

Now

$$\mathbf{P}_{\gamma}^2 = 0$$

$$\mathbf{P}_1^2 = m_{01}^2 c^2$$

As in the example in the text, we do not know the velocities of the particles in k_{CM} after the collision. The minimum energy is found if we consider that the masses all have no kinetic energy after the collision. Then

$$\bar{\mathbf{P}}^2 = (2m_{0\mathrm{e}} + m_{01})^2 c^2$$

and

$$\begin{aligned}\boldsymbol{P}_{\gamma} \cdot \boldsymbol{P}_1 &= \frac{h\nu}{c} m_{01} \left(1, \hat{n}\right) \cdot (c, \mathbf{0}) \\ &= h\nu m_{01}.\end{aligned}$$

Then our equation of mechanics is

$$m_{01}^2 c^2 + 2h\nu m_{01} = \left(4m_{0\mathrm{e}}^2 + m_{01}^2 + 4m_{01}m_{0\mathrm{e}}\right) c^2$$

or

$$h\nu = 2\left(1 + \frac{m_{0\mathrm{e}}}{m_{01}}\right) m_{0\mathrm{e}}c^2.$$

If the mass m_{01} is a nucleon (proton or neutron) then $m_{0\mathrm{e}}/m_{01} \approx 0$ and

$$h\nu \approx 2m_{0\mathrm{e}}c^2.$$

That is, the absolute minimum energy of the incoming photon is the sum of the rest energies of the electron and positron.

8.7. Using the fundamental tensor, show that the covariant form of the Field Strength tensor is

$$\mathcal{F}_{\alpha\beta} = \partial_\alpha \mathcal{A}_\beta - \partial_\beta \mathcal{A}_\alpha,$$

which in matrix form, is

$$\mathcal{F}_{\alpha\beta} = \begin{bmatrix} 0 & E_\mathrm{x}/c & E_\mathrm{y}/c & E_\mathrm{z}/c \\ -E_\mathrm{x}/c & 0 & -B_\mathrm{z} & B_\mathrm{y} \\ -E_\mathrm{y}/c & B_\mathrm{z} & 0 & -B_\mathrm{x} \\ -E_\mathrm{z}/c & -B_\mathrm{y} & B_\mathrm{x} & 0 \end{bmatrix}$$

Solution:
We begin with the contravariant form

$$\mathcal{F}^{\alpha\beta} = \partial^\alpha \mathcal{A}^\beta - \partial^\beta \mathcal{A}^\alpha$$

and intersperse the fundamental (metric) tensor, which has only unity for elements,

$$g_{\mu\alpha}\mathcal{F}^{\alpha\beta}g_{\beta\nu} = g_{\mu\alpha}\partial^\alpha \mathcal{A}^\beta g_{\beta\nu} - g_{\nu\beta}\partial^\beta \mathcal{A}^\alpha g_{\alpha\mu}$$

The terms $g_{\mu\alpha}$ and $g_{\beta\nu}$ here are numbers (± 1) and, therefore commute. And $g_{\mu\alpha} = g_{\alpha\mu}$. Using the lowering property of the fundamental tensor.

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu.$$

To establish the covariant form of the Field Strength tensor, we first note that the spatial terms in the covariant and contravariant forms are the same. That is

$$\partial^\mu \mathcal{A}^\nu - \partial^\nu \mathcal{A}^\mu = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu.$$

This is because both the grad terms and the vector potential terms change sign. That is $\partial^\mu = -\partial_\mu$ and $\mathcal{A}^\mu = -\mathcal{A}_\mu$ for $\mu = 1, 2, 3$. However, neither the zeroth components of the differential nor the 4$-$potential change sign, i.e. $\partial^0 = \partial_0$ and $\mathcal{A}^0 = \mathcal{A}_0$. Therefore

$$\mathcal{F}_{0\nu} = \partial_0 \mathcal{A}_\nu - \partial_\nu \mathcal{A}_0 = -\left(\partial^0 \mathcal{A}^\nu - \partial^\nu \mathcal{A}^0\right) = -\mathcal{F}^{0\nu}$$

and likewise

$$\mathcal{F}_{\mu 0} = -\mathcal{F}^{\mu 0}.$$

There are then changes in the signs of the zeroth row and the zeroth column.

8.8. Using the differential operators show that

$$\partial_\sigma \partial^\tau \mathcal{A}^\sigma = \partial^\tau \left(\frac{1}{c^2}\frac{\partial}{\partial t}\varphi + \operatorname{div} \boldsymbol{A}\right).$$

Solution:
We exchange the order of the (partial) differential operators to obtain

$$\partial_\sigma \partial^\tau \mathcal{A}^\sigma = \partial^\tau \partial_\sigma \mathcal{A}^\sigma.$$

Then we use

$$\partial_\sigma = \left(\frac{\partial}{c\partial t}, \operatorname{grad}\right)$$

and the contravariant form of $\boldsymbol{A}$

$$\boldsymbol{A} = (\varphi/c, \boldsymbol{A})$$

to obtain

$$\partial_\sigma \partial^\tau \mathcal{A}^\sigma = \partial^\tau \left(\frac{1}{c^2}\frac{\partial}{\partial t}\varphi + \operatorname{div} \boldsymbol{A}\right).$$

8.9 Show that the elements of the 4−vector

$$\partial_\alpha \mathcal{F}_{\alpha\beta} = 0$$

are Gauss's Law

$$\operatorname{div} \boldsymbol{E} = 0$$

and Ampère's Law

$$\operatorname{curl} \boldsymbol{B} = -\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}.$$

of Maxwell's equations for empty space.
Solution:

For the component $\beta = 0$,

$$\begin{aligned}\partial_\alpha \mathcal{F}_{\alpha 0} &= \partial_0 \mathcal{F}_{00} + \partial_1 \mathcal{F}_{10} + \partial_2 \mathcal{F}_{20} + \partial_3 \mathcal{F}_{30} \\ &= \frac{1}{c}\frac{\partial}{\partial t}0 + \frac{1}{c}\frac{\partial}{\partial x}E_{\mathrm{x}} + \frac{1}{c}\frac{\partial}{\partial y}E_{\mathrm{y}} + \frac{1}{c}\frac{\partial}{\partial z}E_{\mathrm{z}} \\ &= \frac{1}{c}\,\mathrm{div}\,\boldsymbol{E}\end{aligned}$$

For the component $\beta = 1$,

$$\begin{aligned}\partial_\alpha \mathcal{F}_{\alpha 1} &= \partial_0 \mathcal{F}_{01} + \partial_1 \mathcal{F}_{11} + \partial_2 \mathcal{F}_{21} + \partial_3 \mathcal{F}_{31} \\ &= -\frac{1}{c^2}\frac{\partial}{\partial t}E_{\mathrm{x}} + \frac{\partial}{\partial x}0 + \frac{\partial}{\partial y}B_{\mathrm{z}} - \frac{\partial}{\partial z}B_{\mathrm{y}} \\ &= -\frac{1}{c^2}\frac{\partial}{\partial t}E_{\mathrm{x}} + (\mathrm{curl}\,\boldsymbol{B})_{\mathrm{x}}\end{aligned}$$

For the remaining components $\beta = 2, 3$ the y and z components are obtained. So we have Gauss's Law

$$\mathrm{div}\,\boldsymbol{E} = 0$$

and Ampère's Law

$$\mathrm{curl}\,\boldsymbol{B} = -\frac{1}{c^2}\frac{\partial}{\partial t}\boldsymbol{E}.$$

8.10. Show that the even permutations of

$$\partial_\rho \mathcal{F}_{\sigma\tau} = 0$$

or

$$\partial^\rho \mathcal{F}^{\sigma\tau} = 0$$

yield Oersted's result

$$\mathrm{div}\,B = 0$$

and Faraday's Law

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial}{\partial t}\boldsymbol{B}$$

of Maxwell's Equations in empty space.

Solution:

For $\rho\sigma\tau = 123$

$$\begin{aligned}\partial^1\mathcal{F}^{23}+\partial^3\mathcal{F}^{12}+\partial^2\mathcal{F}^{31} &= \frac{\partial}{\partial x}B_{\mathrm{x}}+\frac{\partial}{\partial z}B_{\mathrm{z}}+\frac{\partial}{\partial y}B_{\mathrm{y}} \\ &= \operatorname{div}\boldsymbol{B}=0\end{aligned}$$

For $\rho\sigma\tau = 013$

$$\begin{aligned}\partial^0\mathcal{F}^{13}+\partial^3\mathcal{F}^{01}+\partial^1\mathcal{F}^{30} &= \frac{1}{c}\frac{\partial}{\partial t}B_{\mathrm{y}}+\frac{1}{c}\frac{\partial}{\partial z}E_{\mathrm{x}}-\frac{1}{c}\frac{\partial}{\partial x}E_{\mathrm{z}} \\ &= \frac{1}{c}\left[\frac{\partial}{\partial t}B_{\mathrm{y}}+(\operatorname{curl}\boldsymbol{E})_{\mathrm{y}}\right]=0\end{aligned}$$

For $\rho\sigma\tau = 012, 023$ we obtain the remaining two components of $(1/c)\,[\partial\boldsymbol{B}/\partial t+\operatorname{curl}\boldsymbol{E}] = 0$.

8.11. Although we do have the spatial force law in the form of Newton's Second Law

$$\boldsymbol{f}=\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{p},$$

if we attempt to formulate this in terms of an acceleration, for the sake of our intuition, we encounter difficulties. For example we may attempt to consider only pure forces in the 4−vector form of Newton's Second Law

$$\begin{aligned}\boldsymbol{F} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{P} \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(mc,\boldsymbol{p}\right),\end{aligned}$$

and use the relation

$$\boldsymbol{P}=m_0\boldsymbol{U},$$

we can write

$$\boldsymbol{F}=m_0\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{U}.$$

If we then attempt to define a 4−acceleration as

$$\boldsymbol{A}=\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{U}$$

and write $\boldsymbol{F}=m_0\boldsymbol{A}$ for the pure force we find difficulty. Find the difficulty.

Solution:

The difficulty appears if we evaluate $\boldsymbol{A}$ explicitly. Differentiating the 4−velocity

$$\boldsymbol{U}=\gamma_{\mathrm{u}}\begin{bmatrix}c\\u_{\mathrm{x}}\\u_{\mathrm{y}}\\u_{\mathrm{z}}\end{bmatrix},$$

with respect to τ using

$$d\tau = dt\sqrt{1-\beta^2}$$

we have

$$\begin{aligned} \boldsymbol{A} &= \gamma_{\mathrm{u}}^2 \frac{d}{dt}(c, \boldsymbol{u}) \\ &= \gamma_{\mathrm{u}}^2 (0, d\boldsymbol{u}/dt)\,. \end{aligned}$$

But differentiating the 4−momentum and using

$$\boldsymbol{f} = \frac{d}{dt}\boldsymbol{p},$$

the 4−vector force is

$$\frac{d}{d\tau}\boldsymbol{P} = \gamma_{\mathrm{u}}\left(\frac{1}{c}\frac{d\mathcal{E}}{dt}, \boldsymbol{f}\right).$$

For a pure force, using

$$\frac{d\mathcal{E}}{dt} = \boldsymbol{f}\cdot\boldsymbol{u},$$

we have

$$\boldsymbol{F} = \frac{d}{d\tau}\boldsymbol{P} = \gamma_{\mathrm{u}}\left(\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, \boldsymbol{f}\right).$$

Equating $m_0\boldsymbol{A}$ to $d\boldsymbol{P}/d\tau$ we find that we require

$$\begin{aligned} m_0\boldsymbol{A} &= m_0\gamma_{\mathrm{u}}^2 (0, d\boldsymbol{u}/dt) \\ &= \gamma_{\mathrm{u}}\left(\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, \boldsymbol{f}\right), \end{aligned}$$

or

$$m_0\gamma_{\mathrm{u}}(0, d\boldsymbol{u}/dt) = \left(\frac{1}{c}\boldsymbol{f}\cdot\boldsymbol{u}, \boldsymbol{f}\right).$$

That is, we require

$$\boldsymbol{f}\cdot\boldsymbol{u} = 0,$$

as well as

$$\boldsymbol{f} = m_0\gamma_{\mathrm{u}}d\boldsymbol{u}/dt.$$

The latter of these is not outside of our expectations, but the former is. This (also) violates the expectation that a force, while it may not change the rest mass, in the case of a pure force, it must not be limited in such a way that it cannot change the total relativistic energy.

8.12. In our discussion of energy we followed Wolfgang Pauli to obtain the Einstein mass-energy relation. This required a proposal for the equation of motion of a material body, in addition to conservation of momentum. We chose this to be the relativistic form of Newton's Second Law written, using the spatial components of momenta, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0\gamma_{\mathrm{u}}\boldsymbol{u}\right) = \boldsymbol{F}.$$

And we later verified that this is the correct covariant form of the force law.

Begin by accepting this force law and obtain Pauli's result that

$$\begin{aligned}\mathcal{E}_{\mathrm{kin}} &= m_0\gamma_{\mathrm{u}}c^2 + \text{ constant}\\ &\approx m_0c^2 + \frac{1}{2}m_0u^2 + \text{ constant.}\end{aligned}$$

Then note that we retrieve the known classical result for the kinetic energy if we choose the constant to be $-m_0c^2$. That is

$$\mathcal{E}_{\mathrm{kin}} = m_0\gamma_{\mathrm{u}}c^2 - m_0c^2\ .$$

We may then identify identify

$$\mathcal{E} = m_0\gamma_{\mathrm{u}}c^2$$

as the total energy and

$$\mathcal{E}_{\mathrm{kin}} = \mathcal{E} - m_0c^2\ .$$

You will first need to evaluate $\mathrm{d}\left(m_0\gamma_{\mathrm{u}}\boldsymbol{u}\right)/\mathrm{d}t$. Then you will need to use your result to obtain an expression for the work done on a material body by the force $\boldsymbol{F}$. This will lead you to the result $\mathrm{d}\left(m_0\gamma_{\mathrm{u}}c^2\right)/\mathrm{d}t = \boldsymbol{F}\cdot\boldsymbol{u}$, which you integrate to obtain the result above for the kinetic energy. The next steps require Einstein's aesthetic insight. They should now follow.

Solution:

Performing the time derivative of $m_0\gamma_{\mathrm{u}}\boldsymbol{u}$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0\gamma_{\mathrm{u}}\boldsymbol{u}\right) &= m_0\gamma_{\mathrm{u}}c\left(\gamma_{\mathrm{u}}^2\boldsymbol{\beta}_{\mathrm{u}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\beta}_{\mathrm{u}}\right)\boldsymbol{\beta}_{\mathrm{u}} + m_0\gamma_{\mathrm{u}}c\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\beta}_{\mathrm{u}}\\ &= \boldsymbol{F}\end{aligned}$$

The rate at which work is done on a material body of (relativistic) mass $m\left(u\right) = m_0\gamma_{\mathrm{u}}$ moving with velocity $\boldsymbol{u}$ is $\boldsymbol{F}\cdot\boldsymbol{u}$. So we take the scalar product of this equation with

$\boldsymbol{u}$ obtaining

$$\begin{aligned}\boldsymbol{u} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(m_0 \gamma_{\mathrm{u}} \boldsymbol{u}\right) &= m_0 \gamma_{\mathrm{u}}^3 c^2 \beta_{\mathrm{u}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\beta}_{\mathrm{u}} \\ &= \boldsymbol{F} \cdot \boldsymbol{u}.\end{aligned}$$

From straightforward differentiation we also find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0 \gamma_{\mathrm{u}} c^2\right) = m_0 \gamma_{\mathrm{u}}^3 c^2 \beta_{\mathrm{u}} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\beta}_{\mathrm{u}}.$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m_0 \gamma_{\mathrm{u}} c^2\right) = \boldsymbol{F} \cdot \boldsymbol{u}.$$

The rate at which work is done on a body is the rate of change in kinetic energy. The kinetic energy of the material body is then

$$\mathcal{E}_{\mathrm{kin}} = m_0 \gamma_{\mathrm{u}} c^2 + \text{ constant.}$$

To identify the constant we expand γ_{u} in powers of β_{u}. Carrying the expansion to second order in β_{u} we have

$$\mathcal{E}_{\mathrm{kin}} \approx m_0 c^2 + \frac{1}{2} m_0 u^2 + \text{ constant.}$$

We then retrieve the known classical result for the kinetic energy if we choose the constant to be $-m_0 c^2$. This we term the *rest energy* of the material body. This is the energy present in the body when at rest.

The kinetic energy then becomes

$$\mathcal{E}_{\mathrm{kin}} = m_0 \gamma_{\mathrm{u}} c^2 - m_0 c^2 \,.$$

If we identify

$$\mathcal{E} = m_0 \gamma_{\mathrm{u}} c^2$$

as the *total energy* of the material body then what we have indicates that the kinetic energy is the difference between the total energy and the rest energy.

8.13. By direct differentiation show that

$$P\mathrm{can}_{\alpha}^{(\mathrm{t})} \equiv \frac{\partial \Lambda^{(\mathrm{t})}}{\partial \dot{x}^{\alpha}}$$

and

$$P\text{can}_{\alpha}^{(\tau)} \equiv \frac{\partial \Lambda}{\partial x^{\alpha\prime}}$$

are identical, thus verifying

$$\frac{\partial \Lambda^{(\tau)}}{\partial x^{\mu\prime}} = \frac{\partial \Lambda^{(\mathrm{t})}}{\partial \dot{x}^{\mu}}.$$

Solution:
We begin with

$$\begin{aligned} P\text{can}_{\alpha}^{(\mathrm{t})} &\equiv \frac{\partial \Lambda^{(\mathrm{t})}}{\partial \dot{x}^{\alpha}} = \frac{\partial}{\partial \dot{x}^{\alpha}} \left(-m_0 c \sqrt{g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} - Q\mathcal{A}^{\mu} g_{\mu\nu}\dot{x}^{\nu}\right) \\ &= -m_0 c \left(g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}\right)^{-1/2} g_{\alpha\nu}\dot{x}^{\nu} - Q\mathcal{A}^{\mu} g_{\mu\alpha} \\ &= -\frac{m_0 \dot{x}_{\alpha}}{\sqrt{1-u^2/c^2}} - Q\mathcal{A}_{\alpha} \end{aligned}$$

With $U_{\alpha} = \gamma_{\mathrm{u}}\dot{x}_{\alpha}$, this becomes

$$P\text{can}_{\alpha}^{(\mathrm{t})} = -P_{\alpha} - Q\mathcal{A}_{\alpha}$$

or

$$\boldsymbol{P}\text{can}^{(\mathrm{t})} = \left(-mc - Q\frac{\varphi}{c}, \boldsymbol{p} + Q\boldsymbol{A}\right),$$

which is $P\text{can}_{\alpha}^{(\tau)}$.

8.14. Show that the Canonical Equations from the relativistic Hamiltonian for the Lorentz force

$$\mathcal{H}^{(\mathrm{t})} = m_0 c^2 \left[1 + \frac{\left(\left(p_{\text{can}}^{(\mathrm{t})}\right)_{\alpha} - QA_{\alpha}\right)^2}{m_0^2 c^2}\right]^{1/2} + Q\varphi$$

are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_{\text{can}}^{(\mathrm{t})}\right)_{\mu} = Qu^{\beta}\left(\partial A_{\beta}/\partial x^{\mu}\right) - Q\frac{\partial \varphi}{\partial x^{\mu}}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}x^{\mu} = u^{\beta} = \frac{1}{m_0}\left[1 + \frac{\left(\left(p_{\text{can}}^{(\mathrm{t})}\right)_{\alpha} - QA_{\alpha}\right)^2}{m_0^2 c^2}\right]^{-1/2} \left(\left(p_{\text{can}}^{(\mathrm{t})}\right)_{\mu} - QA_{\mu}\right)$$

where $\left(p_{\text{can}}^{(\text{t})}\right)_\mu$ are the spatial components of the 4−momentum

$$\boldsymbol{P}\text{can}^{(\text{t})} = \left(-mc - Q\frac{\varphi}{c}, \boldsymbol{p} + Q\boldsymbol{A}\right).$$

Solution:
Taking the partial derivatives of the Hamiltonian

$$\mathcal{H}^{(\text{t})} = m_0 c^2 \left[1 + \frac{\left(\left(p_{\text{can}}^{(\text{t})}\right)_\alpha - QA_\alpha\right)^2}{m_0^2 c^2}\right]^{1/2} + Q\varphi$$

with respect to the spatial components of the covariant differential operator

$$\frac{\partial}{\partial x^\alpha} = \left(\frac{\partial}{c\partial t}, \text{grad}\right),$$

we have

$$\begin{aligned}\frac{\partial \mathcal{H}^{(\text{t})}}{\partial x^\mu} &= Q\frac{\partial \varphi}{\partial x^\mu} - m_0 Q c^2 \left[1 + \frac{\left(\left(p_{\text{can}}^{(\text{t})}\right)_\alpha - QA_\alpha\right)^2}{m_0^2 c^2}\right]^{-1/2} \frac{\left(\left(p_{\text{can}}^{(\text{t})}\right)_\beta - QA_\beta\right)\left(\partial A_\beta / \partial x^\mu\right)}{m_0^2 c^2} \\ &= Q\frac{\partial \varphi}{\partial x^\mu} - Qu^\beta \left(\partial A_\beta / \partial x^\mu\right).\end{aligned}$$

The canonical equations for the spatial momenta are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p_{\text{can}}^{(\text{t})}\right)_\mu = -\frac{\partial \mathcal{H}^{(\text{t})}}{\partial x^\mu} = Qu^\beta \left(\partial A_\beta / \partial x^\mu\right) - Q\frac{\partial \varphi}{\partial x^\mu}.$$

Taking the partial derivatives of the Hamiltonian with respect to the spatial components of the 4−momenta,

$$\begin{aligned}\frac{\partial \mathcal{H}^{(\text{t})}}{\partial \left(p_{\text{can}}^{(\text{t})}\right)_\mu} &= \frac{1}{2} m_0 c^2 \left[1 + \frac{\left(\left(p_{\text{can}}^{(\text{t})}\right)_\alpha - QA_\alpha\right)^2}{m_0^2 c^2}\right]^{-1/2} \frac{2\left(\left(p_{\text{can}}^{(\text{t})}\right)_\beta - QA_\beta\right)\left(\delta_\mu^\beta\right)}{m_0^2 c^2} \\ &= \frac{1}{m_0}\left[1 + \frac{\left(\left(p_{\text{can}}^{(\text{t})}\right)_\alpha - QA_\alpha\right)^2}{m_0^2 c^2}\right]^{-1/2} \left(\left(p_{\text{can}}^{(\text{t})}\right)_\mu - QA_\mu\right).\end{aligned}$$

The canonical equations for the spatial coordinates are then

$$\frac{\mathrm{d}}{\mathrm{d}t}x^{\mu} = \frac{1}{m_0}\left[1 + \frac{\left(\left(p^{(\mathrm{t})}_{\mathrm{can}}\right)_{\alpha} - QA_{\alpha}\right)^2}{m_0^2c^2}\right]^{-1/2}\left(\left(p^{(\mathrm{t})}_{\mathrm{can}}\right)_{\mu} - QA_{\mu}\right).$$

References

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