Chapter 9
Miscellany; Some applications to low-dimensional topology

Abstract In this chapter, we introduce some applications to low-dimensional topology. To be precise, in §9.1, we see the Blanchfield pairing of knots, and consider some properties. In §9.2, we give a method to produce invariants of the Hurewicz equivalence classes, including Lefschetz fibrations over S^2. In §9.3, we discuss 3-manifold invariants from 4-fold branched covering, in terms of quandles. In §9.4, we work with the Milnor invariant of links. In §9.5, we introduce bilinear forms on the twisted Alexander modules of links.

However, we should remark that discussions in this chapter are quite rough since this section is a survey of applications of quandles. The reader interested in the details should follow the references therein.

9.1 The Blanchfield pairings of knots

First, we briefly review tne Blanchfield pairing [Bla] of a knot K with Alexander polynomial Δ_K. The abelianization $\pi_1(S^3 \setminus K) \to \mathbb{Z} = \langle t \rangle$ yields the local coefficients $\mathbb{Z}[t^{\pm 1}]$ of $S^3 \setminus K$. Since $H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])$ is known to be torsion over $\mathbb{Z}[t^{\pm 1}]$, let Δ_K be the minimal polynomial that annihilates the homology. Then, from the view of the intersection form of the infinite cyclic cover $\tilde{Y}_K$, the Blanchfield pairing is defined as a sesquilinear form[1]

$$\mathrm{Bl}_K : H_1(S^3 \setminus K; \mathbb{Z}[t^{\pm 1}])^{\otimes 2} \longrightarrow \mathbb{Z}[t^{\pm 1}]/(\Delta_K). \tag{9.1}$$

The definition is roughly as follows. For simplicity, this section abbreviates $\mathbb{Z}[t^{\pm 1}]$ to Λ, and does $S^3 \setminus K$ to Y_K, as in [Hil, KawBook]. Consider the sequence

$$0 \longrightarrow \Lambda \xrightarrow{\times \Delta_K} \Lambda \longrightarrow \Lambda/(\Delta_K) \longrightarrow 0 \qquad (\text{exact}) \tag{9.2}$$

[1] In many cases (see [KawBook, Tro, Hil]), the target is described as the module $\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$. However, the pairing factors through $\mathbb{Z}[t^{\pm 1}]/(\Delta_K) \hookrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ sending $[f]$ to $[f/\Delta_K]$.

with local coefficients. Since $H_1(Y_K;\Lambda)$ is annihilated by the polynomial Δ_K from the definition, the δ-functor (i.e., Bockstein map) from (9.2) yields an isomorphism $\beta : H^1(Y_K;\Lambda/(\Delta_K)) \cong H^2(Y_K;\mathbb{Z}[t^{\pm 1}])$. To summarize, we have isomorphisms

$$H_1(Y_K;\Lambda) \xrightarrow{\sim} H_1(Y_K,\partial Y_K;\Lambda) \xrightarrow{\text{Poincaré duality}} H^2(Y_K;\Lambda) \xrightarrow{\beta^{-1}}$$
$$\longrightarrow H^1(Y_K;\Lambda/(\Delta_K)) \cong \mathrm{Hom}_\Lambda(H_1(Y_K;\Lambda),\Lambda/\Delta). \qquad (9.3)$$

Here, the first one is immediately obtained by the injection $\partial Y_K \to Y_K$ and $H_*(\partial Y_K;\Lambda) = 0$, and the last is done from the universal coefficient theorem. Then, Bl_K is defined to be the adjoint map of the composite (9.3). In particular, Bl_K is non-singular and isometry (i.e., $t^{\pm 1}$-invariant). Further, as is known, Bl_K is hermitian, and can be interpreted from the intersection form of the infinite cyclic cover of Y_K; see [KawBook, Appendix] or [Hil, Ka] for the detail.

We will give a diagrammatic computation of Bl_K. The point is to reduce Bl_K to the cohomology pairing $\mathcal{Q}_\psi$ (see §4.4 for the definition). Whereas the intersection in ordinary homology is interpreted as a cohomology pairing via the Poincaré duality, that of local coefficients (or, of abstract coverings) is little studied. As a typical difficulty, Bl_K is hermitian, in contrast to the anti-hermiticity of $\mathcal{Q}_\psi$.

However, the author showed that the Blanchfield pairing of knots can be recovered from some cohomology pairing $\mathcal{Q}_\psi$. Namely,

Theorem 9.1 ([N11]) *Let K be a knot, and $\Delta_K \in \mathbb{Z}[t^{\pm}]/\langle t^{\pm n}\rangle$ be as above, and M be the quotient module $\mathbb{Z}[t^{\pm 1}]/(\Delta_K)$. Define $\psi_0 : M \otimes M \to \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$ by $\psi_0(x,y) = \bar{x}y$. Then, there is a $\mathbb{Z}[t^{\pm 1}]$-module isomorphism*

$$H_1(S^3 \setminus K;\mathbb{Z}[t^{\pm 1}]) \cong H^1(S^3 \setminus K, \partial(S^3 \setminus K);M)$$

such that the following equality holds for any $x,y \in H_1(S^3 \setminus K;\mathbb{Z}[t^{\pm 1}])$:

$$\mathcal{Q}_{\psi_0}(x,y) = \frac{1+t}{1-t}\cdot \mathrm{Bl}_K(x,y) \in \mathbb{Z}[t^{\pm 1}]/(\Delta_K).$$

We note that $1-t$ is invertible in $\mathbb{Z}[t^{\pm 1}]/(\Delta_K)$ because of $\Delta_K(1) = \pm 1$.

Proof (Rough description of the idea). Theorem 7.28 says that $H_1(Y_K;\Lambda)$ is identified with the cokernel of $tV - V' : \Lambda^{2g} \to \Lambda^{2g}$, where V' is the transpose of the Seifert matrix V. Then, the inverse of the Bockstein map β^{-1} is known to be presented by $(tV - V')^{-1}$; see [Tro]. Hence, Bl_K has the matrix presentation $(1-t)(tV-V')^{-1}$; see [Kea] or [Tro] for the details.

On the other hand, we notice the Leibniz rule of the Bockstein map:

$$\beta^*(x \smile y) = \beta^*(x) \smile y - x \smile \beta^*(y) \ \text{ for any } x,y \in H^1(Y_K,\partial Y_K;\Lambda/(\Delta_K)).$$

Here, the point of this proof is to find a 3-class $\nu \in H_3(Y_K,\partial Y_k : \mathbb{Z})$ which roughly satisfies $\beta_*(\nu) = \mu$. Then, we compute $\mathcal{Q}_\psi(x,y) = \psi_0\langle x \smile y, [\Sigma]\rangle$ as

$$\psi_0\langle x \smile y, \mu\rangle = \psi_0\langle x \smile y, \beta_*(\nu)\rangle = \tilde{\psi}_1\langle \beta^*(x) \smile y, \nu\rangle - \tilde{\psi}_2\langle x \smile \beta^*(y), \nu\rangle,$$

where we define $\tilde{\psi}_1 : \Lambda \otimes \Lambda/(\Delta_K) \to \Lambda/(\Delta_K)$ and $\tilde{\psi}_2 : \Lambda/(\Delta_K) \otimes \Lambda \to \Lambda/(\Delta_K)$ as the canonical lifts of ψ_0. Since the cup product between H^1 and H^2 is the canonical inner product (see the end of §7.4.3), this pairing has a matrix presentation:

$$\big(\overline{x \cdot (t^{-1}V - V')}\big) \cdot y' - \bar{x} \cdot \big(y \cdot (t^{-1}V - V')\big)',$$

which turns out to be $(1+t)\bar{x}(V - t^{-1}V')y'$. In comparison with the matrix presentation of Bl_K, we have the conclusion $\mathscr{Q}_{\psi_0}(x,y) = \psi_0\langle x \smile y, [\Sigma]\rangle = (1-t)^{-1}(1+t) \cdot \mathrm{Bl}_K(x,y)$ as required. $\square$

In conclusion, we have a diagrammatic computation of the Blanchfield pairing. Indeed, by Theorem 4.23, the right side is diagrammatically computable.

As an example, we now determine the Blanchfield pairing of the (m,n)-torus knot $T_{m,n}$. Here recall from Example 3.9 the isomorphism

$$H_1(S^3 \setminus T_{m,n}; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/(\Delta_K), \quad \text{where} \quad \Delta_K = \frac{(t^{nm}-1)(t-1)}{\big((t^n-1)(t^m-1)\big)}.$$

Proposition 9.2 ([N11]) *Fix $(n,m,a,b) \in \mathbb{Z}^4$ with $an + bm = 1$. Let $K = T_{m,n}$. Then,*

$$\mathrm{Bl}_K(y_1,y_2) = \frac{nm}{(1+t^{-1})(1-t^{bm})(1-t^{an})} \cdot \bar{y}_1 y_2 \in \mathbb{Z}[t^{\pm 1}]/(\Delta_K),$$

for $y_1,\ y_2 \in H_1(S^3 \setminus T_{m,n}; \mathbb{Z}[t^{\pm 1}]) \cong \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$.

Proof. Let $X = A = \mathbb{Z}[t^{\pm 1}]/(\Delta_K)$. By Theorem 3.8, we have $\mathrm{Col}_X(D) \cong H_1(S^3 \setminus K; A) \oplus X \cong X^2$; It is enough to compute the bilinear form $\mathscr{Q}_{\psi_0}$ with $\psi_0(y,z) = \bar{y}z$.

To this end, recall a basis of $\mathrm{Col}_X(D)$ in Example 3.9. Accordingly, the 2-form $\mathscr{Q}_{\psi_0}(y_1,y_2)$ is, by definition, formulated as

$$\sum_{i \le m,\ j \le n-1} \psi_0\Big(\, t^{an(i-2)} \frac{1-t^{bmj}}{1-t^{bm}} y_1,\ \big(\frac{1-t^{an(i-1)}}{1-t^{an}} y_2 + \delta_2\big)(1-t^{-1})\,\Big).$$

Then, using the t-invariance of ψ_0 and the equality $\sum_{i=1}^m t^{ani} = 0$, we can easily reduce the sum of the geometric progressions to $nm(1-t^{-1}) \cdot \bar{y}_1 y_2 / \big((1-t^{bm})(1-t^{an})\big)$. The last term is $(1-t)^{-1}(1+t)\mathrm{Bl}_K(y_1,y_2)$ by Theorem 9.1. So, notice that, $\Delta_K(-1) = 1$ if nm is odd, and $\Delta_K(-1) = m$ if n is even; we may divide the value $\mathscr{Q}_{\psi_0}(y_1,y_2)$ by $1+t$. In the sequel, it can be seen that, since $(1-t)^{-1} \in A$, the form $\mathscr{Q}_{\psi_0}(y_1,y_2)$ is reduced to the required formula. $\square$

9.2 From Hopf fibration to Hurwitz equivalence classes

We describe an algorithm to get invariants with respect to the Hurwitz equivalence relation; including Lefschetz fibrations over S^2 and simple surface braids.

From a general perspective, we now explain *Hurwitz equivalence problem*. Let G be a group with identity 1_G, and let a subset $Z \subset G$ be closed under conjugation. Fixing $m \in \mathbb{N}$, consider the quotient set, $\mathrm{Hur}^m(Z)$, of the set

$$\{(z_1,\dots,z_m) \in Z^m \,|\, z_1 \cdots z_m = 1_G\}$$

modulo the following relations:

$$(z_1, z_2, \dots, z_m) \sim (z_1, \dots, z_{i-1}, z_{i+1}, z_{i+1}^{-1} z_i z_{i+1}, z_{i+2}, \dots, z_m), \tag{9.4}$$

$$(z_1, z_2, \dots, z_m) \sim (w^{-1} z_1 w, w^{-1} z_2 w, \dots, w^{-1} z_m w), \tag{9.5}$$

for any $1 \leq i < m$ and $w \in Z$. Each element of this set $\mathrm{Hur}^m(Z)$ is called a *Hurwitz equivalence class*. Figuratively speaking, the set can be interpreted as the set of monodoromies over the 2-sphere with m holes, as seen in the following examples:

Example 9.3 (Lefschetz fibration over S^2) A (genus-g) Lefschetz fibration is a smooth map $\pi : E \to S^2$ from a closed smooth 4-manifold E that is a Σ_g-fiber bundle projection away from finitely many singular points. Here, Σ_g is the closed surface of genus g, and the map near the singular points is required to appear in appropriate oriented local complex coordinates as $\pi(z_1, z_2) = z_1 z_2$.

We now review theorems of Kas and Y. Matsumoto [Kas, Mats]. To describe this, let G be the mapping class group, $\mathscr{M}_g$, of the closed surface Σ_g with $g \geq 2$. Let Z be the Dehn quandle $\mathscr{D}_g$ as in (2.2). Then, given a Lefschetz fibration, we can observe that the associated monodoromy is interpreted as an m-tuple. Moreover, it follows from [Kas] and [Mats, Theorems 2.6 and 2.8] that the interpretation above gives a bijection between the Hurwitz equivalence classes $\mathrm{Hur}^m(Z)$ and fiber-isomorphism classes of Lefschetz fibrations over S^2 with m-singular fibers. To sum up,

$$\mathrm{Hur}^m(\mathscr{D}_g) \xleftrightarrow{1:1} \frac{\{\text{ Lefschetz fibrations over } S^2 \text{ with } m\text{-singular fibers}\}}{\text{fiber-isomorphisms}}.$$

Example 9.4 (Simple surface braids) Let G be the braid group B_n, and Z be the set of all elements conjugate to either σ_i or σ_i^{-1}. S. Kamada showed that the Hurwitz equivalence classes are in 1:1-correspondence with the isotopy classes of "simple surface-braids with m-branch points of degree n"; see [Kam3] for details.

$$\mathrm{Hur}^m(Z) \xleftrightarrow{1:1} \frac{\{\text{ simple surface-braids with } m\text{-branch points of degree } n\}}{\text{isotopy}}.$$

Thus, it is sensible to hope something invariant with respect to the Hurwitz relations, in general settings; as a suggestion, the author explicitly proposed an algorithm to get such invariants.

To explain this, we start with a short review of the Hopf fibration $\mu : S^3 \to S^2$. The fibration is formulated by the restriction on S^3 of the following map:

$$\mu : \mathbb{C}^2 \longrightarrow \mathbb{C} \times \mathbb{R}, \quad (z,w) \longmapsto (2z\overline{w},\ |z|^2 - |w|^2).$$

Then we can easily see, by definitions, that the preimage of m-points, $\{b_1,\dots,b_m\} \subset S^2$, is the (m,m)-torus link $T_{m,m}$. Thus, it is natural to consider a presentation of $\pi_1(S^3 \setminus T_{m,m})$; the Wirtinger presentation tells us that

$$\pi_1(S^3 \setminus T_{m,m}) \cong \langle\, a_1,\dots,a_m \mid a_1 \cdots a_m = a_2 \cdots a_m a_1 = \cdots = a_m a_1 a_2 \cdots a_{m-1} \,\rangle. \tag{9.6}$$

Here, a_i correspondences to the meridian associated with the arc α_i in Figure 9.1, and the product $a_1 \cdots a_m$ generates the summand $\mathbb{Z} \subset \mathbb{Z} \times F_{m-1} = \pi_1(S^3 \setminus T_{m,m})$.

Next, to explain Lemma 9.5 below, we now set up terminology. Let us consider the multi-set of all the Z-colorings

$$\mathfrak{Col}_{Z,G} := \left\{\, \mathscr{C} \in \mathrm{Col}_Z(D) \;\middle|\; D : \text{link diagram} \,\right\},$$

and equip $\mathfrak{Col}_{Z,G}$ with an equivalence relation by Reidemeister moves. Furthermore, we discuss the link-diagram D of $T_{m,m}$ in the left hand side of Figure 9.1. Then, given an m-tuple $(z_1,\dots,z_m) \in Z^m$ with $z_1 \cdots z_m = 1$, the assignment $\mathscr{C}_{\mathbf{z}}(a_i) = z_i$ defines a Z-coloring $\mathscr{C}_{\mathbf{z}}$ of $T_{m,m}$ according to (9.6) and Corollary 3.14. Hence we have a map

$$\mathscr{Z} : \{\, \mathbf{z} = (z_1,\dots,z_m) \in Z^m \mid z_1 \cdots z_m = 1 \,\} \longrightarrow \mathfrak{Col}_{Z,G}; \quad \mathbf{z} \longmapsto \mathscr{C}_{\mathbf{z}}.$$

Lemma 9.5 *Let $\mathscr{S}$ be a set, and take a map $\mathrm{Inv} : \mathfrak{Col}_{Z,G} \to \mathscr{S}$. Assume that Inv is invariant with respect to Reidemeister moves, and that if two homomorphisms $f_{\mathbf{z}}$ and $f'_{\mathbf{z}} : \pi_1(S^3 \setminus L) \to G$ coming from $\mathfrak{Col}_{Z,G}$ are conjugate, $\mathrm{Inv}(f_{\mathbf{z}}) = \mathrm{Inv}(f'_{\mathbf{z}})$.*

Then, the composite $\mathrm{Inv} \circ \mathscr{Z}$ induces a map $\mathrm{Hur}^m(Z) \to \mathscr{S}$.

Proof. It suffices to check the invariance with respect to the relations (9.4) and (9.5). Since the latter (9.5) is clear by assumption, we will discuss the other one (9.4). To this end, consider another diagram D' obtained from the above D by exchanging the i-th strand for the $(i+1)$-th one (see the right of Figure 9.1); Notice that D is related to D' by a finite sequence of Reidemeister moves of type II and III (see Figure 9.1). Therefore, if two m-tuples $\mathbf{z}$ and $\mathbf{z}'$ are related by (9.4), then the equality $\mathrm{Inv}(f_{\mathbf{z}}) = \mathrm{Inv}(f_{\mathbf{z}'})$ results from the assumption. Hence, we complete the proof. $\square$

Summary. Invariants of link group representations produce those of $\mathrm{Hur}^m(Z)$.

We finish this topic by giving a powerful example; precisely, we employ the bilinear form in Definition 4.18, and analyze and compute the 2-forms.

Let L be the (m,m)-torus link $T_{m,m}$ with $m \geq 2$, and let $\alpha_1,\dots,\alpha_m$ be the arcs depicted in Figure A.1. Given a homomorphism $f : \pi_L \to G$ with $f(\alpha_i) \in Z$, let us discuss X-colorings $\mathscr{C}$ over f. Here X is the quandle of the form $M \times G$. Then, concerning the relation on the ℓ-th link component, it satisfies the equation

$$\big(\cdots(\mathscr{C}(\alpha_\ell)\lhd\mathscr{C}(\alpha_{\ell+1}))\lhd\cdots\big)\lhd\mathscr{C}(\alpha_{\ell+m-1})=\mathscr{C}(\alpha_\ell),\qquad \text{for any } 1\leq \ell\leq m,$$

where we consider the indexes modulo m. Next, with notation $\mathscr{C}(\alpha_i):=(x_i,z_i)\in X$, this equation reduces to linear equations

$$(x_{\ell-1}-x_\ell)+\sum_{\ell\leq j\leq \ell+m-2}(x_j-x_{j+1})\cdot z_{j+1}z_{j+2}\cdots z_{m+\ell}=0\in M, \tag{9.7}$$

for any $1\leq \ell\leq m$. Conversely, we can easily verify that, if a map $\mathscr{C}:\{\text{arcs of } D\}\to X$ satisfies the equation (9.7), then $\mathscr{C}$ is an X-coloring. Denoting the left side in (9.7) by $\Gamma_{f,\ell}(\mathbf{x})$, consider a homomorphism

$$\Gamma_f : M^m \longrightarrow M^m;\quad (x_1,\ldots,x_m)\longmapsto(\Gamma_{f,1}(\mathbf{x}),\ldots,\Gamma_{f,m}(\mathbf{x})).$$

To conclude, the set $\mathrm{Col}_X(D_f)$ coincides with the kernel of Γ_f.

To summarize, we can obtain the bilinear form from Definition 4.18 (cf. (8.16) with every $\varepsilon_i=+1$):

Proposition 9.6 *Let $f:\pi_1(S^3\setminus T_{m,m})\to G$ be as above. Let $\psi: M\otimes M'\to A$ be a G-invariant bilinear function. For any $\ell\in\mathbb{Z}$ with $1\leq\ell\leq m$, the A-bilinear form $\mathscr{Q}_{\psi,\ell}:\mathrm{Ker}(\Gamma_f)\otimes\mathrm{Ker}(\Gamma'_f)\to A$ takes $(x_1,\ldots,x_m)\otimes(y'_1,\ldots,y'_m)$ to*

$$\sum_{k=1}^{m-1}\psi\Big(\sum_{j=1}^{k}(x_{j+\ell-1}-x_{j+\ell})\cdot z_{j+\ell}z_{j+\ell+1}\cdots z_{k+\ell-1},\ y'_{k+\ell}\cdot(1-z_{k+\ell}^{-1})\Big)\in A. \tag{9.8}$$

The paper [N13] analysed this bilinear form (9.8), and gave some examples from some Lefschetz fibrations. In addition, the author suggested an application of "the quantum representation" of $\mathscr{M}_g$.

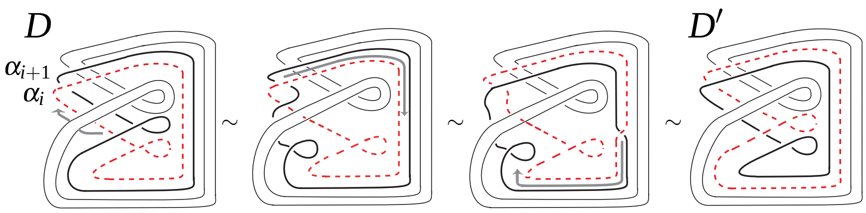

Fig. 9.1 The exchange between the i-th strand and the $(i+1)$-th one of the (m,m)-torus link. Here, any entry of the linking matrix is 1, and the band means the $(m-2)$-parallel strands. In addition, the blackline is the i-th strand and the dotted line indicates the $(i+1)$-th strand.

9.3 3-manifold (invariants) from branched 4-fold coverings

We will interpret the fundamental groups of closed 3-manifolds as some colorings (Theorem 9.10), where 3-manifolds are smooth, connected and oriented. Furthermore, we will explain an idea to get 3-manifold invariants from quandles.

To this end, let us review basic facts. Recall the famous fact of Hilden and Montesinos [Hil, Mon], which claims that every 3-manifold M is a 4-fold branched covering space of S^3 along some link L. In this situation, we have the simple homomorphism $\phi : \pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ as the monodoromy. Here such a ϕ is said to be *simple*, if ϕ is surjective and sends each meridian to a transposition in $\mathfrak{S}_4$.

So, it is sensible to consider the conjugacy quandle

$$\mathscr{S} := \{ \ (ij) \in \mathfrak{S}_4 \ | \ 1 \leq i < j \leq 4 \}$$

of order 6. Put a link diagram D of L. An $\mathscr{S}$-coloring of D whose image ($\subset \mathscr{S}$) generates $\mathfrak{S}_4$ will be called a *labeled diagram*. By Corollary 3.14, simple homomorphisms $\pi_1(S^3 \setminus L) \to \mathfrak{S}_4$ naturally correspond to labeled diagrams of D. In summary, any 3-manifold can be regarded as a labeled diagram. We often denote a labeled diagram by D_ϕ with respect to D and ϕ. Conversely, given a labeled diagram D_ϕ, we can easily construct a 3-manifold M as the resulting 4-fold cover of S^3 branched over the link L.

Further, we explain Theorem 9.7, which can completely deal with every closed 3-manifold. Namely, it is known that MI and MII moves of labeled diagrams, shown in Figure 9.2, do not change the topological type of the covering space. Conversely,

Theorem 9.7 *(Apostolakis [Apo], Bobtcheva and Piergallini [BP, Theorem 3]) Two 4-fold simple coverings of S^3 branched over links represent the same 3-manifold if and only if their associated labeled diagrams can be related by a finite sequence of MI & MII and Reidemeister moves on $\mathbb{R}^2$. To summarize,*

$$\frac{\{\text{ Closed 3-manifolds}\}}{\text{homeomorphisms}} \xleftrightarrow{1:1} \frac{\{\text{ Labeled diagrams }\}}{\text{Reidemeister, and MI, MII moves}}.$$

While the original statements were described in group theoretic terms, we use $\mathscr{S}$-coloring to state it in a little easier way. Nevertheless, whereas this theorem is beautiful, there had been a few studies to discuss 3-manifold invariants from Theorem 9.7.

However, in the work of E. Hatakenaka (and the author later), we proposed a discussion from quandle theory (Theorem 9.10): let us briefly introduce the work.

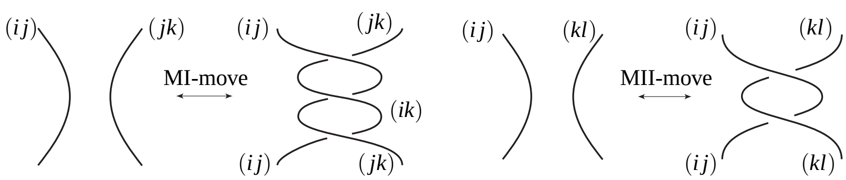

Fig. 9.2 MI, II moves of labeled diagrams. Here, the symbols $1 \leq i, j, k, l \leq 4$ mean distinct indices.

Definition 9.8 ([Hat, N3, HN]) We define a quandle starting from a group G and a central element $c \in Z(G)$. Putting $T_{12} := \{(i,j) \in \mathbb{Z}^2 | 1 \leq i,j \leq 4, \ \ i \neq j\}$, we define $\widetilde{G}_c$ to be a quotient set $G \times T_{12}/\sim$, where the relation $\sim$ is defined by

$$(g,(i,j)) \sim (g^{-1}c,(j,i)), \quad \text{for any} \ \ (i,j) \in T_{12} \ \ \text{and} \ \ g \in G.$$

Further, we define the binary operation $\lhd : \widetilde{G}_c \times \widetilde{G}_c \to \widetilde{G}_c$ by Table 9.1. Then, we can easily see that $\widetilde{G}_c$ is a quandle. Note that, if G is of finite order, then $\widetilde{G}_c$ has order $6|G|$. See [N3, HN] for some properties of $\widetilde{G}_c$.

$(g,\ t)$	$(g',\ t')$	$(g,\ t) \lhd (g',\ t')$
$(g,\ (i,j))$	$(g',\ (i,j))$	$(g'g^{-1}g',\ (i,j))$
$(g,\ (i,j))$	$(g',\ (j,k))$	$(gg',\ (i,k))$
$(g,\ (i,j))$	$(g',\ (k,l))$	$(g,\ (i,j))$

Table 9.1 The binary operation $*$ in $\widetilde{G}_c$. Here, in each line, i,j,k,l are all distinct and $t,t' \in T_{12}$.

Definition 9.9 (cf. (3.12)) Given a labeled diagram D_ϕ, we define the subset

$$\mathrm{Col}_{\widetilde{G}_c}(D_\phi) := \{ \ \mathscr{C} \in \mathrm{Col}_X(D) \mid p_{\widetilde{G}_c} \circ \mathscr{C} = \phi \ \},$$

similar to (3.12). Here, $p_{\widetilde{G}_c} : \widetilde{G}_c \to \mathscr{S}$ is a natural projection $(g,(i,j)) \mapsto (ij)$.

We see that the set $\mathrm{Col}_{\widetilde{G}_c}(D_\phi)$ is nothing but group homomorphisms. Precisely,

Theorem 9.10 ([Hat], see also [HN]) *Let (G,c) be as above, and D_ϕ a labeled diagram which presents a closed 3-manifold M. Then, there is a bijection*

$$\mathrm{Col}_{\widetilde{G}_c}(D_\phi) \simeq G^3 \times \mathrm{Hom}_{\mathrm{gr}}(\pi_1(M), G). \tag{9.9}$$

Proof (Outline). The proof is similar to Prop. 3.18 with complicated procedures. □

Thus, inspired by the quandle cocycle invariants of links, it is sensible to consider similar invariants of 3-manifolds. In fact, Hatakenaka [Hat] formulated a quandle cocycle invariant which is compatible with MI, MII-moves: Furthermore, she [Hat] and the author [HN] showed that a quandle homotopy invariant recovers the Dijkgraaf-Witten invariant (see (6.9) for the definition), and the Chern-Simons invariants of closed 3-manifolds. However, it remains a problem whether such an approach from quandles gives a stronger invariant of 3-manifolds or not.

9.4 Milnor invariant and lower central series

We explain the Milnor invariant, and observe an application from quandles.

To begin, let us fix notation. Let F be the free group of rank q. We define F_1 to be F, and F_m to be the commutator subgroup $[F_{m-1}F, F]$ by induction. Denoting by Q_m the quotient group F_{m-1}/F_m, we have an extension

$$0 \longrightarrow Q_m \longrightarrow F/F_m \xrightarrow{p_{m-1}} F/F_{m-1} \longrightarrow 0 \quad (\text{central extension}). \tag{9.10}$$

Then, as is known, Q_m is a free abelian group of finite rank.

We will review (the first non-vanishing) Milnor invariant of links. In what follows, we suppose $q \in \mathbb{Z}_{\geq 0}$ equal to the number of the link components $\#L$. Take the abelianization $f_2 : \pi_1(S^3 \setminus L) \to F/F_2 = Q_1 = \mathbb{Z}^q$. Further, by induction, we suppose

- **Assumption** $\mathscr{L}_m$. There are homomorphisms $f_k : \pi_1(S^3 \setminus L) \to F/F_k$ for k with $k \leq m$, which satisfy the commutative diagram

$$\begin{tikzcd}
\pi_1(S^3 \setminus L) \arrow[d, "f_2"] \arrow[dr, "f_3"] \arrow[drr, "f_4"] \arrow[drrrr, "f_m"] & & & & \\
F/F_2 & F/F_3 \arrow[l, "p_2"] & F/F_4 \arrow[l, "p_3"] & \cdots\cdots \arrow[l] & F/F_m. \arrow[l, "p_{m-1}"]
\end{tikzcd}$$

Here it is worth noting that, if there is another extension f'_m instead of f_m, then f_m equals f'_m up to conjugacy, by centrality. Furthermore, when $m > 1$, we can easily check that f_m sends every longitude $\mathfrak{l}_\ell$ with $\mathrm{Ab}_\ell(\mathfrak{l}_\ell) = 0$ to the central subgroup Q_m.

Definition 9.11 Suppose the assumption $\mathscr{L}_m$. Then, the (m-th) Milnor μ-invariant of L is defined as a q-tuple $\big(f_m(\mathfrak{l}_1), \ldots, f_m(\mathfrak{l}_q)\big) \in (Q_m)^q$.

In many papers, e.g, [Mil1, Mil2, Hil], the Milnor invariant is defined by using "the Magnus expansion" over the noncommutative polynomial ring $\mathbb{Z}\langle\langle X_1, \ldots, X_q \rangle\rangle$. But, this way imposes us to reformulate each longitude $\mathfrak{l}_\ell$ as a word of $\mathfrak{m}_1, \ldots, \mathfrak{m}_{\#L}$ in the group F/F_n. This reformulation with the non-commutativity (therefore, the computation of the μ-invariant) had been considered to be quite hard.

However, this section approaches the μ-invariant from a quandle and the generalized Magnus embedding [GG], and gives a computation without describing $\mathfrak{l}_\ell$.

For this, let us start by reviewing the generalized Magnus embedding. Denote by Ω_m the polynomial ring $\mathbb{Z}[\lambda_{i,j}]$ over commuting indeterminates $\lambda_{i,j}$ with $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, q\}$. Then, we define a unipotent homomorphism

$$\Upsilon_m : F \longrightarrow GL_m(\Omega_m); \qquad x_j \longmapsto \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_{1,j} & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_{2,j} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{m-2,j} & 1 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_{m-1,j} & 1 \end{pmatrix}.$$

Here are known properties of this homomorphism (see [GG] for the proof):

Lemma 9.12 *(i) $\Upsilon_m(F_m) = \{I_m\}$, and the quotient $F/F_m \to GL_m(\Omega_m)$ is injective.*

(ii) The image $\Upsilon_m(Q_m)$ restricted on Q_m is contained in the abelian subgroup consisting of matrices whose (i,j)-entry is zero for any $i \neq j$ with $(i,j) \neq (m,1)$. Moreover, the centralizer of $\Upsilon_m(x_j)$ is equal to $\Upsilon_m(Q_m) \times \{\Upsilon_m(x_j^n)\}_{n \in \mathbb{Z}}$.

(iii) For any element $y \in F$, the (s,t)-entry of $\Upsilon_m(y)$ with $s > t$ is formulated as

$$\sum_{(k_s,\dots,k_t) \in \{1,\dots,q\}^{t-s+1}} \varepsilon\Big(\frac{\partial^{t-s+1} y}{\partial x_{k_t} \cdots \partial x_{k_s}}\Big) \lambda_{s,k_s} \cdots \lambda_{t,k_t} \in \Omega_m,$$

where $\partial y / \partial x_{k_t}$ is the Fox derivation, and ε is the augmentation $\mathbb{Z}[F] \to \mathbb{Z}$.

Remark 9.13 Here, the point from (3) is that the matrix of $\Upsilon_m(y)$ is determined by only the entries on the first row (Compare them with other entries in $\Upsilon_m(y)$).

Next, we set up notations from quandles. Consider the conjugacy class of x_ℓ, i.e.,

$$X_{m,\ell} := \{\ g^{-1}\Upsilon_m(x_\ell) g \in \mathrm{Im}(\Upsilon_m) \mid g \in \mathrm{Im}(\Upsilon_m)\ \}.$$

Let X_m be the conjugacy quandle on the union $\sqcup_{\ell=1}^{\#L} X_{m,\ell}$. Then, the projection $p_m : X_{m+1} \to X_m$ is described as forgetting the $(m+1)$-column and row, and is a quandle covering as in Definition 4.15. Moreover, we should notice from Remark 9.13 that, given $A \in X_{m,\ell}$, there uniquely exists an element $\mathfrak{s}(A)$ in the image $\mathrm{Im}(\Upsilon_{m+1})$ satisfying $\mathfrak{s}(A)_{(s,t)} = A_{(s,t)}$, and $\mathfrak{s}(A)_{(1,m+1)} = 0 \in \Omega_{m+1}$, if $s < m+1$. Then, it follows from Example 8.9 that the following map is a quandle 2-cocycle

$$\psi_m : X_m \times X_m \longrightarrow \mathrm{Im}(\Upsilon_{m+1}); \quad (A,B) \mapsto \mathfrak{s}(B)^{-1}\mathfrak{s}(A)\mathfrak{s}(B)\mathfrak{s}(B^{-1}AB)^{-1}, \qquad (9.11)$$

and the image of ψ_m is contained in the kernel of $p_m : X_{m+1} \to X_m$. Notice that this kernel is abelian, since it is contained in $\Upsilon_{m+1}\big(\mathrm{Ker}(F/F_{m+1} \to F/F_m)\big) = \Upsilon_{m+1}(Q_{m+1})$.

Thus, it is sensible to consider the quandle cocycle invariant $\Phi_{\psi_m,\ell}(f_m)$ in Definition 4.15 associated with ψ_m. Then, the paper [KN] showed that this cocycle invariant completely recovers the μ-invariant as follows:

Theorem 9.14 ([KN]) *There is an isomorphism $\mathscr{I}_m : \Upsilon_m(Q_m) \to \mathrm{Ker}(p_m)$ such that*

$$\Phi_{\psi_m,\ell}(f_m) = \mathscr{I}_m \circ \Upsilon_m\big(f_m(\mathfrak{l}_\ell)\big) \in \mathrm{Ker}(p_m).$$

Proof (Outline). Consider the conjugacy action of $GL_m(\Omega_m)$ on X_m, whose stabilizer is $\Upsilon_m(Q_m)$ exactly (Why? Hint: Lemma 9.12 (ii)). Thus, we have a 2-cocycle $\phi : X_m \times X_m \to \Upsilon_m(Q_m)$ by Example 5.22, and have the associated quandle $\widetilde{X}_m$ extended over X_m. Notice from Theorem 5.23 that the cocycle invariant equals the μ-invariant. Thus, it is enough for the proof to verify that this ϕ is cohomologous to the above ψ_m. In fact, if so, we have a quandle isomorphism $\mathscr{I}_m : \widetilde{X}_m \to X_{m+1}$ such that the associated cocycle invariants are equivalent; hence, this $\mathscr{I}_m$ gives the desired equality. Since the verification is a little complicated, we omit the detail. $\square$

Exercise 20 Show a result in [Mil1] that the μ-invariant plays an obstruction for a lift of $\mathscr{L}_{m+1}$. To be precise, on the assumption $\mathscr{L}_m$, f_m admits a lift $f_{m+1} : \pi_1(S^3 \setminus L) \to F/F_{m+1}$ if and only if all the μ-invariants vanish, i.e., $f_m(\mathfrak{l}_\ell) = 0 \in Q_m$.

(Hint: compare Theorem 9.14 with Proposition 4.16).

<u>**Conclusion**</u>. This exercise gives an explicit formula of f_k by induction on k with $k \leq m$. After we get f_m, Theorem 9.14 gives a diagrammatic computation of the μ-invariant. By the definition of the cocycle invariant, we can compute the μ-invariant as a weight sum without presentations of $\mathfrak{l}_\ell$. Moreover, since the map Υ_m is over the commutative ring Ω_m, this computation is applicable for computer program. In fact, it is not so hard to compute the μ-invariants of links with small crossing number (The computation is quite easier than the previous computations). Moreover, the paper [KN] gives the first success of computing the μ-invariant of "the Milnor link".

Incidentally, the paper [KN] gives a faster computation of the μ-invariant than that of Theorem 9.14, and further discussed the higher Milnor invariant, and considered similar computations.

9.5 Bilinear forms on twisted Alexander modules of knots

In the paper [N10], the author suggested bilinear forms on twisted Alexander modules of links. This section explains the details in the knot case.

We start by briefly reviewing the twisted Alexander module associated with a linear representation $f_{\rm pre} : \pi_L \to GL_n(R)$, the ring R is assumed to be a Noetherian unique factorization domain (henceforth UFD), as a common setting (see [FV, W, Lin]). Since L is a knot, that is, $\#L = 1$, we have the abelianization $\alpha : \pi_L \to \mathbb{Z} = \langle t \rangle$. By identifying the group ring, $R[\mathbb{Z}]$ with the polynomial ring $R[t^{\pm 1}]$ and by tensoring this α with $f_{\rm pre}$, we have a linear representation

$$\alpha \otimes f_{\rm pre} : \pi_L \longrightarrow GL_n(R[t^{\pm 1}]).$$

Thus, the associated first homology $H_1(S^3 \setminus L; (R[t^{\pm 1}])^n)$ is commonly called *the twisted Alexander module* associated with $f_{\rm pre}$.

We roughly review some facts of twisted modules, and explain the idea to introduce bilinear forms on $H_1(S^3 \setminus L; (R[t^{\pm 1}])^n)$. After the concepts of twisted Alexander polynomials and modules were introduced in [Lin, W], there are the studies together with topological applications; see, e.g., a survey [FV] on twisted Alexander polynomials. However, we emphasize difficulties that it seems not so easy to define the intersection forms on the twisted homology, or homology of coverings. Several papers addressed bilinear forms on such twisted modules; For example, concerning solvable covering, we can see some pairing as in Blanchfield pairings including [COT] (See also [Pow] and references therein.)

As a solution of the difficulties, the author employed a cohomological viewpoint, and gaves an idea to construct a homomorphism

$$\mathrm{Adj} \circ \mathscr{L} : H_1(S^3 \setminus L; R[t^{\pm 1}]) \to H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_\Delta)$$

for some coefficient M_Δ. In fact, recalling the cohomology pairing $\mathscr{Q}_\psi$ on the codomain $H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_\Delta)$ in §4.4, we define a bilinear form on $H_1(S^3 \setminus L; R[t^{\pm 1}])$ by the composite $\mathscr{Q}_\psi \circ (\mathrm{Adj} \circ \mathscr{L})^{\otimes 2}$.

We will describe the idea in more details.

First, as a generalization of "localized Blanchfield pairing" (see [Hil, §2.6]), let us consider a localization (9.12) below. Notice the non-vanishing $\det(\mathrm{id} - \alpha \otimes f_{\mathrm{pre}}(\mathfrak{m})) \neq 0 \in R[t^{\pm 1}]$ for any meridian $\mathfrak{m} \in \pi_L$: Then, the assumption enables us to define the ring $A_{(\partial f)}$ obtained by inverting the determinants. Precisely, we set

$$A_{(\partial f)} := R[t^{\pm 1}, \ \det(\mathrm{id} - \alpha \otimes f_{\mathrm{pre}}(\mathfrak{m}))^{-1}]. \tag{9.12}$$

Then $A_{(\partial f)}$ has the involution $^{-}: A_{(\partial f)} \to A_{(\partial f)}$ defined by $\bar{t} = t^{-1}$.

We will intemperate the twisted Alexander module using coloring sets. From Definition 3.20, consider the quandle X of the form $M \times G$, where M is the free module $(A_{(\partial f)})^n$ and G is $GL_n(R[t^{\pm 1}])$. Choose a diagram D with α_D arcs. Then, as in Exercise 8 (1) in §3.2.2, the coloring set $\mathrm{Col}_X(D_f)$ can be regarded as the kernel of a homomorphism

$$\Gamma_{X,D} : M^{\alpha_D} \longrightarrow M^{\#\{\text{crossings of } D\}} \tag{9.13}$$

obtained from (3.11). Furthermore, let us examine the cokernel $\mathrm{Coker}(\Gamma_{X,D})$:

Lemma 9.15 *For any knot L, choose a diagram D with α_D arcs, and assume $\alpha_D = \#\{$ crossings of D $\}$. Let X be the above quandle on $M \times G$, where $M = (A_{(\partial f)})^n$ and $G = GL_n(R[t^{\pm 1}])$. Then, there is an isomorphism*

$$\mathrm{Coker}(\Gamma_{X,D}) \cong H_1(S^3 \setminus L; (A_{(\partial f)})^n) \oplus (A_{(\partial f)})^n.$$

Here the summand $(A_{(\partial f)})^n$ corresponds with the diagonal subset A_{diag} of $(A_{(\partial f)})^{n\alpha_D}$.

Proof. Recall from Proposition 7.26 that, the quotient $\mathrm{Ker}(\mathrm{id}_M \otimes \partial_1)/\mathrm{Im}(\mathrm{id}_M \otimes \partial_2)$ is isomorphic to the first homology $H_1(S^3 \setminus L; M)$ with local coefficients. From the definition of the ring $A_{(\partial f)}$ in (9.12), every $1 - \gamma_i$ is invertible in M; The map $\mathrm{id}_M \otimes \partial_1$ is a (diagonally) splitting surjection, which admits consequently a decomposition

$$\mathrm{Coker}(\mathrm{id}_M \otimes \partial_2 : M^{\alpha_D} \longrightarrow M^{\alpha_D}) \cong H_1(\pi_L; M) \oplus M.$$

Next, we prepare a commutative diagram below. With respect to a crossing τ illustrated in Figure 4.7, set up the two bijections

$$\kappa_\tau : M \longrightarrow M; \qquad m \longmapsto m - m \cdot \big(\alpha \otimes f_{\mathrm{pre}}(\alpha_\tau)\big),$$

$$\kappa'_\tau : M \longrightarrow M; \qquad m \longmapsto m - m \cdot \big(\alpha \otimes f_{\mathrm{pre}}(\gamma_\tau)\big).$$

Then, by the direct products with respect to crossings τ, we have the diagram

$$\begin{tikzcd}
\mathrm{Col}_X(D_f) \arrow[r, hook] & M^{\alpha_D} \arrow[r, "\Gamma_{X,D}"] \arrow[d, "\prod_\tau \kappa_\tau"'] & M^{\alpha_D} \arrow[r, two heads] \arrow[d, "\prod_\tau \kappa'_\tau"] & \mathrm{Coker}(\Gamma_{X,D}) & \text{(exact)} \\
& M^{\alpha_D} \arrow[r, "\mathrm{id}_M \otimes \partial_2"] & M^{\alpha_D} \arrow[r, two heads] & H_1(S^3 \setminus L; M) \oplus M & \text{(exact)}.
\end{tikzcd}$$

Examining carefully the definitions of $\kappa_\tau^{(')}$, ∂_2, and $\Gamma_{\overline{X},D}$, the diagram is commutative. Hence, the vertical maps give the desired $\mathrm{Coker}(\Gamma_{\overline{X},D}) \cong H_1(Y_L; M) \oplus M$. $\square$

Although our purpose is to construct a function on $H_1(S^3 \setminus L; (A_{(\partial f)})^n)$, there are many cases that the twisted Alexander module is a torsion $A_{(\partial f)}$-module; see, e.g., [FV, W]. Therefore, in order to get non-trivial bilinear functions from such modules, the coefficient ring shall be a quotient $A_{(\partial f)}/\mathscr{I}$ subject to an appropriate ideal $\mathscr{I}$.

One of methods to simplify such torsion is *the twisted Alexander polynomial*, Δ_f, [W, Lin], which is defined to be the $n^2(\alpha_D - 1)^2$ Jacobian of the Fox derivations (Proposition 7.26) subject to $\det(\mathrm{id} - \rho \otimes f_{\mathrm{pre}}(x_\alpha))$: To be precise,

$$\Delta_f := \det\Big(([\frac{\partial r_i}{\partial x_j}]) \otimes \mathrm{id}_{A^n_{(\partial f)}})_{1 \leq i,j \leq \alpha_D - 1} \Big) / \det(\mathrm{id} - \rho \otimes f_{\mathrm{pre}}(x_{\alpha_D})) \in A_{(\partial f)}.$$

It is shown [W] that this Δ_f is independent, up to units, of the choice of the arcs α. As is discussed in [FV], this Δ_f is almost equal to the maximal that annihilates the twisted homology $H_1(S^3 \setminus L; (R[t^{\pm 1}])^n)$. In fact, the twisted homology $H_1(S^3 \setminus L; (R[t^{\pm 1}])^n) \otimes A_{(\partial f)}$ is torsion if and only if Δ_f is not zero.

Thus, it is reasonable to consider the quotient coefficient $A_{(\partial f)}/(\Delta_f)$. Thus, from the above representation $\alpha \otimes f_{\mathrm{pre}}$ subject to (Δ_f), we can obtain two quotients

$$M_\Delta := (A_{(\partial f)}/(\Delta_f))^n, \text{ and } \quad M_{\overline{\Delta}} := (A_{(\partial f)}/(\overline{\Delta_f}))^n.$$

Similarly, we can set the quandles $X_1 := M_\Delta \times GL_n(A_{(\partial f)})$ and $X_2 := M_{\overline{\Delta}} \times GL_n(A_{(\partial f)})$ from Definition 3.20.

Moreover, as in Definition 4.18, we assume a bilinear function $\psi_{\mathrm{pre}} : R^n \times R^n \to R$ satisfying the f_{pre}-invariance

$$\psi_{\mathrm{pre}}(x, y) = \psi_{\mathrm{pre}}(x \cdot f_{\mathrm{pre}}(g),\ y \cdot f_{\mathrm{pre}}(g))$$

for any $x, y \in R^n$, and any $g \in \pi_L$. Then, we can define the map

$$\begin{aligned} \psi : (R^n \otimes_R A_{(\partial f)}/(\overline{\Delta_f})) \times (R^n \otimes_R A_{(\partial f)}/(\Delta_f)) &\longrightarrow A_{(\partial f)}/(\Delta_f); \\ (x \otimes a_1, y \otimes a_2) &\longmapsto \psi_{\mathrm{pre}}(x, y) \otimes \overline{a_1} a_2, \end{aligned} \tag{9.14}$$

for $x, y \in R^n$ and $a_1,\ a_2 \in A_{(\partial f)}$. This ψ is π_L-invariant and sesquilinear over $R[t^{\pm 1}]$.

Finally, we will explain Definition 9.16 after introducing two homomorphisms Adj and $\mathscr{L}$. Considering the decomposition $(A_{(\partial f)})^{n\alpha_D} = (A_{(\partial f)})^{n(\alpha_D - 1)} \oplus A_{\mathrm{diag}}$, we take the restriction

$$\mathrm{res}(\Gamma_{X,D}) : (A_{(\partial f)})^{n(\alpha_D-1)} \to (A_{(\partial f)})^{n(\alpha_D-1)}.$$

We should notice that the cokernel is $H_1(S^3 \setminus L; (A_{(\partial f)})^n)$ by Lemma 9.15, and the kernel subject to Δ_f is the 1-st cohomology $H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_\Delta)$ by Theorem 3.21. Since the determinant of $\mathrm{res}(\Gamma_{X,D})$ is almost equal to Δ_f, the adjugate matrix of $\mathrm{res}(\Gamma_{X,D})$ gives rise to a well-defined homomorphism

$$\mathrm{Adj} : H_1(S^3 \setminus L; (A_{(\partial f)})^n) \longrightarrow H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_\Delta). \tag{9.15}$$

Notice that the localization $R[t^{\pm 1}] \hookrightarrow A_{(\partial f)}$ gives rise to the homomorphism

$$\mathscr{L} : H_1(S^3 \setminus L; R[t^{\pm 1}]^n) \longrightarrow H_1(S^3 \setminus L; (A_{(\partial f)})^n).$$

To summarize, here is a bilinear function on the twisted Alexander module:

Definition 9.16 ([N10]) *Let R be a Noetherian UFD. Let $A_{(\partial f)}$ and G be as in Lemma 9.15. Take $M_\Delta = \big(A_{(\partial f)}/(\Delta_f)\big)^n$. Let $\psi : M_\Delta \times M_{\overline{\Delta}} \to A_{(\partial f)}/(\Delta_f)$ be the bilinear form obtained from ψ_{pre}, as in (9.14).*

Then, we define the bilinear map from the twisted Alexander module as the following composite:

$$H_1(S^3 \setminus L; R[\mathbb{Z}]^n)^{\otimes 2} \xrightarrow{\mathscr{L}^{\otimes 2}} H_1(S^3 \setminus L; (A_{(\partial f)})^n)^{\otimes 2} \xrightarrow{\mathrm{Adj}^{\otimes 2}}$$

$$\to H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_{\overline{\Delta}}) \otimes H^1(S^3 \setminus L, \partial(S^3 \setminus L); M_\Delta) \xrightarrow{\mathscr{Q}_\psi} A_{(\partial f)}/(\Delta_f).$$

By definition and Theorem 4.23, we should emphasize that it is not hard to compute the pairing from $\mathscr{Q}_\psi$.

Finally, we conclude this section by mentioning a duality. For this, we restrict on the situation. Let $m = 1$ and let R be a field of characteristic 0. Then, we can easily show the following lemma by linear algebra.

Lemma 9.17 *Assume that Δ_f is non-zero and that $\det(t \cdot \mathrm{id}_{\mathbb{F}^n} - f_{\mathrm{pre}}(\mathfrak{m})) \neq 0$ for a meridian $\mathfrak{m} \in \pi_L$ is relatively prime to Δ_f in $\mathbb{F}[t]$. Then the adjugate matrix Adj in (9.15) is an $\mathbb{F}[t]$-isomorphism.*

As seen in examples in [N10], this pairing is often degenerate in many cases, and is possible to be even zero, while the classical Blanchfield pairing is non-singular. However, the subsequent paper [N11] will show a duality theorem on the twisted pairings, under some assumptions (cf. Milnor duality [Mil4]):

Theorem 9.18 ([N11]) *Let R be a field of characteristic 0, and $\mathscr{I}$ be (Δ_f). Assume that ψ_{pre} is nondegenerate, and that Δ_f is not zero, and Δ_f is relatively prime to $\det(t \cdot \mathrm{id}_{\mathbb{F}^n} - f_{\mathrm{pre}}(\mathfrak{m})) \in \mathbb{F}[t^{\pm 1}]$ for a meridian $\mathfrak{m} \in \pi_L$.*

Then, the twisted pairing in Definition 9.16 is non-degenerate.

It is well known [Mil3] that all the (skew-)Hermitian nondegenerate bilinear forms with isometries t is completely characterised. In conclusion, if ψ is (skew-)Hermitian, we can quantitatively obtain computable information from the twisted pairing $\mathscr{Q}_\psi$.